Free particle, $H = \frac{\hat{p}^2}{2m}$ → have degeneracy

- If two eigenstates of parity →

$$P|\psi_1\rangle = \lambda_1 |\psi_1\rangle \quad \Rightarrow \lambda_1, \lambda_2 = \pm 1$$
$$P|\psi_2\rangle = \lambda_2 |\psi_2\rangle$$

$\hat{n}$ → odd under parity

Matrix element $\langle \psi_2 | \hat{n} | \psi_1 \rangle = \langle \psi_2 | P^{-1} P \, n P^{-1} P | \psi_1 \rangle$

$$= -\lambda_1 \lambda_2 \langle \psi_2 | n | \psi_1 \rangle$$

So unless $\lambda_1 \lambda_2 = -1$,
$\langle \psi_2 | \hat{n} | \psi_1 \rangle = 0$

Recall for stark effect is $\langle 1s | E_n | 1s \rangle = 0$

19.07 Rotational invariance and Wigner Eckart theorem:

Need to define vector operators & spherical tensors.

Real vector spaces (3 dimensions)

If vector $\bar{v}(v_1, v_2, v_3)$, $|v| = \sqrt{\sum_i v_i^2}$ transforms under rotation, $\bar{v}' = \left[\, \overline{\overline{R}} \, \right] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

Orthogonal def^n → $R^T = R^{-1}$

So a vector is something which transforms as $v_i' = \sum_j R_{ij} v_j$

use this def^n to define vector operators.

generalise to tensors.

$$\hat{v_i} = (\hat{v_1}, \hat{v_2}, \hat{v_3}) \, ; \quad \langle \psi | v_i | \psi \rangle$$

Rotation operator $|\tilde{\psi}\rangle = \underbrace{\mathcal{D}(R)}_{\text{unitary operator corresponding to } R} |\psi\rangle$

denotes rotation

We know, $\mathcal{D}(R) = \exp\left(-\frac{i\,\bar{J}\cdot\hat{n}}{\hbar}\theta\right)$; $\mathcal{D}^{+}(R) = \mathcal{D}^{-1}(R)$

Unitary

$$\langle\tilde{\psi}|v_i|\tilde{\psi}\rangle = \langle\psi|\mathcal{D}^{+}(R)\, v_i\, \mathcal{D}(R)|\psi\rangle$$

We define a vector operator, if for any state $|\psi\rangle$

$$\langle\psi|\mathcal{D}^{+}(R)\, v_i\, \mathcal{D}(R)|\psi\rangle = \sum_j R_{ij}\langle\psi|v_j|\psi\rangle$$

Because it is true for all states $\rightarrow$

$$\Rightarrow \boxed{\mathcal{D}^{+}(R)\,\hat{v}_i\,\mathcal{D}(R) = \sum_j R_{ij}\,\hat{v}_j}$$

Infinitesimal rotations $\rightarrow$ $\mathcal{D}(R) \simeq \hat{1} - \frac{i}{\hbar}(\hat{\bar{J}}\cdot\hat{n})\,\delta\theta$

Consider an infinitesimal rotation about z-axis.

$$R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{infinitesi}} \begin{bmatrix} 1 & -\delta\theta & 0 \\ \delta\theta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(we rotate the vector in this case, not axis)

$$\mathcal{D}^{+}(R)\, v_i\, \mathcal{D}(R) = \sum_j R_{ij}\,\hat{v}_j$$

$$\Rightarrow \left(1 + \frac{i}{\hbar}J_3\,\delta\theta\right)\hat{v}_i\left(1 - \frac{i}{\hbar}J_3\,\delta\theta\right) = \left(\begin{bmatrix} 1 & -\delta\theta & 0 \\ \delta\theta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \hat{v}_3 \end{bmatrix}\right)_i$$

$$\Rightarrow \hat{v}_i + \frac{i}{\hbar}[\hat{J}_3, v_i] = \quad ''$$

$$\Rightarrow \quad \hat{V}_1 + \frac{i}{\hbar}[\hat{J}_3, \hat{V}_1] = \hat{V}_1 - \delta\theta \hat{V}_2$$

$$\hat{V}_2 + \frac{i}{\hbar}[\hat{J}_3, \hat{V}_2] = \delta\theta \hat{V}_1 + \hat{V}_2$$

$$\hat{V}_3 + \frac{i}{\hbar}[\hat{J}_3, \hat{V}_3] = \hat{V}_3$$

True for all $\delta\theta$.

$$\therefore \frac{i}{\hbar}[\hat{J}_3, \hat{V}_1] = -\delta\theta \hat{V}_2 \rightarrow [\hat{J}_3, \hat{V}_1] = i\hbar \hat{V}_2$$

$$\frac{i}{\hbar}[\hat{J}_3, \hat{V}_2] = \delta\theta \hat{V}_1 \Rightarrow [\hat{J}_3, \hat{V}_2] = -i\hbar \hat{V}_1$$

$$\frac{i}{\hbar}[\hat{J}_3, \hat{V}_3] = 0 \Rightarrow [\hat{J}_3, \hat{V}_3] = 0$$

We get similar relations for rotations abt x, y, axes

Generalisation $\rightarrow$ $\boxed{[\hat{J}_\alpha, \hat{V}_\beta] = i\hbar \epsilon_{\alpha\beta\gamma} \hat{V}_\gamma}$ $\rightarrow$ ang mom commutation relation

eg of vector operators: $\bar{r}, \bar{p}, \bar{L}, \bar{S}$

explicitly verify for $\bar{r}, \bar{p}$.

Spherical tensors:

(Read addition of ang mom).

$$L^2 |l, m\rangle = l(l+1)\hbar^2 |l, m\rangle$$

$$L_z |l, m\rangle = \hbar |l, m\rangle$$

$$l \rightarrow 0, 1, 2, 3, \ldots$$

$$m \rightarrow -l, -l+1, \ldots, 0, 1, 2, \ldots, l$$

$$Y_{lm}(\theta, \phi) = \langle \hat{n} | l, m\rangle \; ; \; \hat{n} \rightarrow (\theta, \phi)$$

Vector operators defined from behaviour of real vectors under rotation.

Similarly, spherical tensors defined from behaviour of Y_{lm} functions under rotation.

Under rotations, $\hat{n} \to \hat{n}'$, $|\hat{n}'\rangle = \mathcal{D}(R)|\hat{n}\rangle$

$$Y_{lm}(\hat{n}') = \langle \hat{n}' | l, m\rangle$$
$$= \langle \hat{n} | \mathcal{D}^\dagger(R) | l, m\rangle$$
$$= \sum_{m'} \langle \hat{n} | l m'\rangle \langle l m' | \mathcal{D}^\dagger(R) | l m\rangle$$

$$\Rightarrow \boxed{Y_{lm}(\hat{n}') = \sum_{m'} [\mathcal{D}^\dagger(R)]_{m'm} Y_{lm'}(\hat{n})}$$

compare to $\hat{V}_i = \sum_j R_{ij} \hat{V}_j$ (*)

In a completely analogous manner as in the case of vector operators, we define spherical tensors T_q^k, where k → rank

components T_q^k, $q \to -k, -k+1, \ldots, 0, 1, 2, \ldots, k$

$$\boxed{\mathcal{D}^\dagger(R)\, T_q^k\, \mathcal{D}(R) = \sum_{q'} [\mathcal{D}^\dagger(R)]_{q'q} T^k_{q'}}$$ (*)

Reading out → Consider infinitesimal rotations as before.