MATH 260, Linear Systems and Matrices, Summer I '14
Activity 9: Null & Column Spaces

KEY

Warm-up: In the Null Space?

Recall that the null space is the set of all vectors $\vec{x}$ such that $\mathbf{A}\vec{x} = \vec{0}$. Given the matrix:

$$\mathbf{G} = \begin{bmatrix} 2 & 3 & -1 \\ -1 & 4 & 6 \\ 1 & 7 & 5 \end{bmatrix}$$

1a) Determine which of the following vectors are in the null space of $\mathbf{G}$.

$$\vec{w}_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \qquad \vec{w}_2 = \begin{bmatrix} 8 \\ -4 \\ 4 \end{bmatrix} \qquad \vec{w}_3 = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$$

$G\vec{\omega}_1 = \begin{pmatrix} 11 \\ 11 \\ 22 \end{pmatrix} \neq \vec{0}$ $\qquad G\vec{\omega}_2 = \vec{0}$ $\qquad G\vec{\omega}_3 = \vec{0}$

So $\vec{\omega}_2, \vec{\omega}_3 \in NS(G)$ (in the null space of G)

while $\vec{\omega}_1$ is not.

1b) Find the actual null space of $\mathbf{G}$. Do your answers above make sense?

(Solve $[G|\vec{0}]$)

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ -1 & 4 & 6 & 0 \\ 1 & 7 & 5 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 7 & 5 & 0 \\ -1 & 4 & 6 & 0 \\ 2 & 3 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 7 & 5 & 0 \\ 0 & 11 & 11 & 0 \\ 0 & -11 & -11 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 7 & 5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So $x - 2z = 0 \to x = 2z$
$y + z = 0 \quad y = -z$

So all elements of $NS(G)$ look like $\begin{pmatrix} 2z \\ -z \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} z$

$= \text{span}\left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right\}.$

Notice that
$\vec{\omega}_3 = -1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and
$\vec{\omega}_2 = 4 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

Null Space

Given the matrix $\mathbf{F} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$.

2) Give a basis for the null space of $\mathbf{F}$.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 2 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \qquad \begin{aligned} x+2y&=0 \\ z&=0 \end{aligned}$$

$NS(F) = \text{span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right\}$ so a basis of $NS(G)$ is $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right\}$

Column Space

We know that the column space ($span(\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\})$) is a subspace. However, generally in mathematics we want to describe things in as simple terms as possible. For spaces, this means giving a *basis* only for a space.

Recall that a set is a **basis** of a vector space if it has the properties:

(i) The set is linearly independent (ii) The span of the set covers the entire vector space.

3) The column space of $\mathbf{G}$ could be given as $span\left(\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 6 \\ 5 \end{bmatrix} \right\}\right)$. Does the set $S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 6 \\ 5 \end{bmatrix} \right\}$ form a basis for the column space? (Show why or why not)

(we know this set spans the space)

Check the linear independence of the set.

$\det \begin{pmatrix} 2 & 3 & -1 \\ -1 & 4 & 6 \\ 1 & 7 & 5 \end{pmatrix} = 0$, so the columns are linearly dependent, so S is <u>not</u> a basis of $CS(G)$.

We already have all the tools and information to define a valid basis. Let's do so...

4) Which columns in the RREF of $\mathbf{G}$ are pivot columns?

Columns 1 and 2 (see 1b for the RREF of G)

5) Take the columns from the original **G** that you identified as the pivot columns. Form a new set S_B from these columns. These should be 3×1 vectors with non-zero values in each row.

5a) Is this new set S_B linearly independent?

$S_B = \left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix} \right\}$

yes, linearly independent since the vectors ~~are~~ not linear combinations of each other.

$$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \neq k \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix}$$

5b) Does it span the entire column space? (*Hint: Does* $span(S) = span(S_B)$)

Yes, since column 3 is a linear combination of columns 1 & 2.

$$-2\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \\ 5 \end{pmatrix}$$

5c) Is the set S_B a basis for the column space? Explain.

Yes, it's a linearly independent spanning set.

6) We defined the **rank** of a matrix as the number of pivot columns in RREF. How does the rank of **G** relate to the column space of **G**? (*Hint: What property of a space gives a single value out?*)

$$\text{rank}(G) = \dim(CS(G))$$

the rank of a matrix is the dimension of its column space.

Note this relationship between rank and column spaces is actually true for any matrix.

7) Give a basis for the column space of **F** (from problem 2).

$$F = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

↑ ↑ pivot columns

so $CS(F) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ (the pivot columns)

So a basis of $CS(F)$ is

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

Rank-Nullity Theorem

8) You should have already found a basis for each of the column space and the null space of **F** (problems 2 and 7).

9a) What are the dimensions of the null space and column space for **F**?

$\dim(NS(F)) = 1$, $\dim(CS(F)) = 2$

9b) How do the dimensions of **F** relate to the sum: dim(column space of **F**) + dim(null space of **F**)?

F is 2×3 and the sum is 3, so it's the number of columns of F.

10a) State the dimension of the null space (the *nullity* of **G**) and the dimension of the column space for matrix **G**.

$\dim(NS(G)) = 1$, $\dim(CS(G)) = 2$

10b) How do the dimensions of **G** relate to the sum: dim(column space of **G**) + dim(null space of **G**)?

G is 3×3 and the sum is 3, so again it's the number of columns of G.

11) The **Rank-Nullity Theorem** generalizes the above results for the $m \times n$ matrix **A**. Based on your results, what do you think the **Rank-Nullity Theorem** is?

$\text{rank}(A) + \text{nullity}(A) = n$