MATH 260, Linear Systems and Matrices, Summer I '14
Activity 5: Matrix Inverses

KEY

Warmup: RREF Review

Find the RREF of the following matrix:

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 4 & 1 & 0 & 1 \end{array}\right]$$

$-4R_1 + R_2 \to R_2$

$$\longrightarrow \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -3 & -4 & 1 \end{array}\right]$$

$-\frac{1}{3}R_2 \to R_2$ $$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 4/3 & -1/3 \end{array}\right]$$

$-R_2 + R_1 \to R_1$ $$\left[\begin{array}{cc|cc} 1 & 0 & -1/3 & 1/3 \\ 0 & 1 & 4/3 & -1/3 \end{array}\right]$$

Congratulations! You've found your first matrix inverse.

So $A^{-1} = \begin{bmatrix} -1/3 & 1/3 \\ 4/3 & -1/3 \end{bmatrix}$

1 The Identity Matrix and Inverses

Notice that the right-hand side (RHS) of the warmup started as the identity matrix. The warmup exercise could be stated as: $[\mathbf{A}|\mathbf{I}]$. Reducing this combined matrix into RREF is the simplest way to find the inverse of a matrix. The inverse of the matrix $\mathbf{A}$ is the RHS after getting the left-side into RREF. That is, you start with $[\mathbf{A}|\mathbf{I}]$ then row reduce to $[\mathbf{I}|\mathbf{A}^{-1}]$.

a) One trait of the inverse is that the statement: $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$. Verify this is true for the warmup.

$$\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} -1/3 & 1/3 \\ 4/3 & -1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

b) Another trait if a matrix has an inverse, then its transpose has an inverse. Let's check that:

(i) Find $\mathbf{A}^T$. Recall that the transpose is defined as: $[a_{ij}]^T = [a_{ji}]$

$$A^T = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$$

(ii) Find the inverse of $\mathbf{A}^T$. *Hint: start with the matrix* $[A^T|I]$

It ends up $(A^{-1})^T = \begin{bmatrix} -1/3 & 4/3 \\ 1/3 & -1/3 \end{bmatrix}$

c) Another practice. Obtain the inverse of the following matrix using row operations and RREF.

$$\left[\begin{matrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{matrix}\middle|\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right]$$

$$\xrightarrow{-R_1+R_2 \to R_3} \left[\begin{matrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & -1 & 0 \end{matrix}\middle|\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{matrix}\right]$$

$$\xrightarrow[-\frac{1}{2}R_2+R_1 \to R_1]{\frac{1}{2}R_2+R_3 \to R_3} \left[\begin{matrix} 1 & 0 & 1/2 \\ 0 & 2 & 1 \\ 0 & 0 & 1/2 \end{matrix}\middle|\begin{matrix} 1 & -1/2 & 0 \\ 0 & 1 & 0 \\ -1 & 1/2 & 1 \end{matrix}\right]$$

$$\to \left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{matrix}\middle|\begin{matrix} 2 & -1 & -1 \\ 0 & 1/2 & 0 \\ -2 & 1 & 2 \end{matrix}\right]$$

$$\to \left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\middle|\begin{matrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -2 & 1 & 2 \end{matrix}\right]$$

So $A^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -2 & 4 & 2 \end{bmatrix}$

2 A Use for the Inverse

a) How did we use RREF to find solutions to the matrix-vector equation: $\mathbf{A}\vec{x} = \vec{b}$?

augment $[A \mid \vec{b}]$ then get RREF so $[I \mid \vec{x}]$

b) If we do the same operations to find a RREF, can't we use the inverse to find an answer too...?

(i) Solve the following system by finding the RREF of the ***augmented*** matrix:

$$\begin{array}{rcrcr} x & + & y & = & -1 \\ 4x & + & y & = & -7 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 1 & -1 \\ 4 & 1 & -7 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 1 \end{array}\right]$$

(ii) Now multiply the inverse you found in the warmup by $\vec{b} = \begin{bmatrix} -1 \\ -7 \end{bmatrix}$

$$\begin{bmatrix} -1/3 & 1/3 \\ 4/3 & -1/3 \end{bmatrix} \begin{bmatrix} -1 \\ -7 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

c) Consider your results and write down observations. Here's a little guidance... Left multiply each side of the equation $\mathbf{A}\vec{x} = \vec{b}$ by $\mathbf{A}^{-1}$ and compare to answers you've obtained so far...

$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

So you can solve a system by finding A^{-1} then multiply by right hand side vector $\vec{b}$.

3 More Solutions...

a) Use the algorithm we discussed to find the inverse of the matrix:

$$M = \begin{bmatrix} 3 & 0 & 3 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

doesn't reduce to $[I \mid A^{-1}]$

gets $\left[\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{matrix} \middle| \text{stuff} \right]$

b) What happened? Based on your exploration in Section 2, if this matrix described a system, do you think it has a unique answer? Justify your answer.

Since can't reduce left to I there isn't a unique solution. If there is a unique solution, $\vec{x} = M^{-1}\vec{b}$. So the system must have either infinitely many or no solutions.

4 Challenge

For what value(s) a, b and c make each of the following matrices invertible. If no such constant exists, say so and explain why.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ b & b & b \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & c & 0 \end{bmatrix}$$

a can be any #; can use row op. $-aR_3 + R_1 \to R_1$ to reduce to I.

b can't be anything, since row op.s
$-bR_1 + R_3 \to R_3$
$-bR_2 + R_3 \to R_3$
makes bottom row all 0's.

$c \neq 1$, otherwise 1st and 3rd row cancel and you can't get I.