MATH 260, Linear Systems and Matrices, Summer '14
Activity 12: Diagonalization

KEY

Warm-up: Eigenvectors and Spaces of Repeated Roots

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 3 & 0 \\ -4 & 2 & 1 \end{bmatrix}$$

The eigenvalues of $\mathbf{G}$ are $\lambda_{1,2} = 1$ and $\lambda_3 = 3$.

1) If I had asked you to find the eigenvalues of $\mathbf{G}$, you would NOT need to calculate $\det(\mathbf{G} - \lambda\mathbf{I}) = 0$. Explain why.

G is triangular

2) Find the three eigenvectors associated with the given eigenvalues.

$\lambda = 1$: $\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ ← could also be a 1.

(or any constant multiples of these)

$\lambda = 3$: $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

3) What is the dimension of each corresponding eigenspace (recall, these are the geometric multiplicities of each eigenvalue)? What is the sum of the eigenspace dimensions for $\mathbf{G}$?

$\dim(E_1) = 2$

$\dim(E_3) = 1$

$2 + 1 = 3 = \dim(\mathbb{R}^3)$!

Diagonalized Equations

Define two matrices: $\mathbf{P} = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & & | \end{bmatrix}$ $\quad\quad$ $\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$

We are going to check some matrix equations using $\mathbf{P}$, $\mathbf{P}^{-1}$ and $\mathbf{D}$. For each of the following be sure to show some intermediate steps, since we already know what the final result should be!

For # 4-8 use the matrix: $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ with eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -1$ with corresponding eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ respectively. $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ $P = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$ $D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$

4) Find $\mathbf{AP}$ and $\mathbf{PD}$. How do they compare?

$$AP = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 6 & 2 \end{bmatrix}$$

Same!

$$PD = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 6 & 2 \end{bmatrix}$$

5) Find $\mathbf{PDP}^{-1}$. What does it equal that we have already defined?
Hint: You already calculated $\mathbf{PD}$ you just need to find $\mathbf{P}^{-1}$, and do one multiplication

$$PDP^{-1} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} = A$$

$$\left(P^{-1} = \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & -1/4 \end{bmatrix} \right)$$

6) Find $\mathbf{P}^{-1}\mathbf{AP}$. What does it equal that we have already defined?
Hint: You only need to do ONE new multiplication here!

$$P^{-1}AP = \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & -1/4 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} = D$$

7) Write all three of the equations you have calculated above here, using the variables **A**, **P**, **D**, and $\mathbf{P}^{-1}$.

$AP = PD$

$PDP^{-1} = A$

$P^{-1}AP = D$

8)a) Define the matrix $\mathbf{F} = \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$, and use the same **P** as above. Find $\mathbf{PFP}^{-1}$.

$PFP^{-1} = \begin{bmatrix} 1 & -1 \\ -4 & 1 \end{bmatrix} \neq A$

b) Does it matter to the equations you wrote in Problem 7 what order the eigenvalues and eigenvectors are in **P** and D? What relationship is required between **P** and D?

Yes! The eigenvalues and -vectors must line up in D and P.

Diagonalizing with Repeated Eigenvalues

9) In the warm-up you found the eigenvectors for a matrix with a repeated eigenvalue, but which had 3 linearly independent eigenvectors. Pick your favorite equation from #7. Is the equation valid for **G**?

They're all valid

10) The matrix $\begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix}$ has a repeated eigenvalue of 4, with one eigenvector of $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
If we let $\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$. Does the equation $\mathbf{A} = \mathbf{PDP^{-1}}$ still hold?
Hint: Think about $\mathbf{P}$. *What is* $det(\mathbf{P})$? *What does that tell you about* $\mathbf{P}^{-1}$?

$\det(P)=0$, so P is singular (not invertible), so the eqn can't hold since there is no P^{-1}.

11) If the results from # 9 and # 10 are generalizable (they are), what must be true about a matrix for us to diagonalize it?

The sum of the dimensions of the eigenspaces of the $n \times n$ matrix must be n.