MATH 260, Linear Systems and Matrices, Fall '14
Activity 9: Null & Column Spaces

Warm-up: In the Null Space?

Recall that the null space is the set of all vectors $\vec{x}$ such that $\mathbf{A}\vec{x} = \vec{0}$. Given the matrix:

$$\mathbf{G} = \begin{bmatrix} 2 & 3 & -1 \\ -1 & 4 & 6 \\ 1 & 7 & 5 \end{bmatrix}$$

1a) Determine which of the following vectors are in the null space of $\mathbf{G}$.

$$\vec{w}_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \qquad \vec{w}_2 = \begin{bmatrix} 8 \\ -4 \\ 4 \end{bmatrix} \qquad \vec{w}_3 = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$$

1b) Find the actual null space of $\mathbf{G}$. Do your answers above make sense?

Null Space

Given the matrix $\mathbf{F} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$.

2) Give a basis for the null space of $\mathbf{F}$.

Column Space

We know that the column space ($span(\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\})$) is a subspace. However, generally in mathematics we want to describe things in as simple terms as possible. For spaces, this means giving a *basis* only for a space.

Recall that a set is a **basis** of a vector space if it has the properties:

(i) The set is linearly independent (ii) The span of the set covers the entire vector space.

3) The column space of $\mathbf{G}$ could be given as $span\left(\left\{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 6 \\ 5 \end{bmatrix}\right\}\right)$. Does the set $S = \left\{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 6 \\ 5 \end{bmatrix}\right\}$ form a basis for the column space? (Show why or why not)

We already have all the tools and information to define a valid basis. Let's do so...

4) Which columns in the RREF of $\mathbf{G}$ are pivot columns?

5) Take the columns from the original $\mathbf{G}$ that you identified as the pivot columns. Form a new set S_B from these columns. These should be 3×1 vectors with non-zero values in each row.

5a) Is this new set S_B linearly independent?

5b) Does it span the entire column space? (*Hint: Does* $span(S) = span(S_B)$)

5c) Is the set S_B a basis for the column space? Explain.

6) We defined the **rank** of a matrix as the number of pivot columns in RREF. How does the rank of $\mathbf{G}$ relate to the column space of $\mathbf{G}$? (*Hint: What property of a space gives a single value out?*)

Note this relationship between rank and column spaces is actually true for any matrix.

7) Give a basis for the column space of $\mathbf{F}$ (from problem 2).

Rank-Nullity Theorem

8) You should have already found a basis for each of the column space and the null space of $\mathbf{F}$ (problems 2 and 7).

9a) What are the dimensions of the null space and column space for $\mathbf{F}$?

9b) How do the dimensions of $\mathbf{F}$ relate to the sum: dim(column space of $\mathbf{F}$) + dim(null space of $\mathbf{F}$)?

10a) State the dimension of the null space (the *nullity* of $\mathbf{G}$) and the dimension of the column space for matrix $\mathbf{G}$.

10b) How do the dimensions of $\mathbf{G}$ relate to the sum: dim(column space of $\mathbf{G}$) + dim(null space of $\mathbf{G}$)?

11) The **Rank-Nullity Theorem** generalizes the above results for the $m \times n$ matrix $\mathbf{A}$. Based on your results, what do you think the **Rank-Nullity Theorem** is?