MATH 260, Linear Systems and Matrices, Fall '14
Activity 11: Eigenvalues and Eigenvectors, Part 2

Warm-up: Subspaces

Recall from Lesson 7...

Vector Subspace Theorem: a nonempty subset W of a vector space V is a subspace of V if it is closed under addition and scalar multiplication:

(i) If $\vec{x}, \vec{y} \in W$, then $\vec{x} + \vec{y} \in W$ **(ii) If $\vec{x} \in W$ and $c \in \mathbb{R}$, then $c\vec{x} \in W$**

1) Sketch the lines: $y = x$ and $y = x + 1$. One line is a subspace of $\mathbb{R}^2$, one is not. Decide, then explain your reasoning.

Eigenspaces

2)a) In Activity 10 you had: $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ which had eigenvectors: $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and $\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

You also found that $\mathbf{A}\vec{v}_1 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$. Sketch $\mathbf{A}\vec{v}_1$ and $\vec{v}_1$ (on the same graph).

b) Looking back at (1), what can you say about $span\{\vec{v}_1\}$?

c) Sketch $span\{\vec{v}_2\}$. Is $span\{\vec{v}_2\}$ a subspace of $\mathbb{R}^2$?

We call the $span\{\vec{v}_i\}$ the eigenspace $\mathbb{E}_\lambda$.

3) What is the dimension of each eigenspace?

Repeated Eigenvalues

4) What are the eigenvalues and eigenvectors of: $\mathbf{H} = \begin{bmatrix} 3 & 0 \\ 5 & 3 \end{bmatrix}$

Repeated eigenvalues sometimes have more than one eigenvector though.

5) The eigenvalues for this matrix are $\lambda_{1,2} = 1$ **and** $\lambda_3 = 2$**. Find the eigenvectors associated with these eigenvalues for the matrix:** $\mathbf{R} = \begin{bmatrix} 0 & 0 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 3 \end{bmatrix}$

We know that eigenvectors always satisfy the 2nd property of vector subspaces (you should be able to explain why!). We also found above that the eigenspace from a single eigenvector is a valid subspace. However, notice that we wrote our eigenspace as $\mathbb{E}_\lambda$. But there are TWO eigenvectors with our double eigenvalue.

Let's denote the double eigenvalue, and its eigenvectors λ_1, λ_2 and $\vec{v}_1, \vec{v}_2$ respectively.

6) State all the eigenspaces and their corresponding dimension for R?

7) Give a geometric interpretation of $span\{\vec{v}_1, \vec{v}_2\}$ in $\mathbb{R}^3$ (3-d space).
Hint: this is a special shape in $\mathbb{R}^3$

The algebraic multiplicity of an eigenvalue is the number of times it is a root (e.g. a in $(\lambda - \lambda_i)^a$)
The geometric multiplicity of an eigenvalue is the dimension of its corresponding eigenspace.
8)a) Give the algebraic and geometric multiplicity for the eigenvalues of R.

b) Give the algebraic and geometric multiplicity for the eigenvalues of H.

c) How do the algebraic and geometric multiplicity compare to each other?

Distinct Eigenvalues

There is very useful theorem about distinct eigenvalues:
Distinct Eigenvalue Theorem: Let A be an $n \times n$ matrix. If $\lambda_1, \lambda_2, \ldots, \lambda_p$ are distinct eigenvalues with corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ then $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is a set of linearly independent vectors.

9)a) Find the eigenvalues and an eigenvectors for each value for: $\mathbf{W} = \begin{bmatrix} 3 & 0 \\ 5 & 2 \end{bmatrix}$

b) Show that the eigenvectors are linearly independent.

10) There's something to be careful of with this theorem though...You found the eigenvectors for R in (3). Are the eigenvectors of R linearly independent?

11) If the eigenvectors are linearly independent are you guaranteed to have distinct eigenvalues?