MATH 260, Linear Systems and Matrices, Summer I '14
Activity 11: Eigenvalues and Eigenvectors, Part 2

KEY

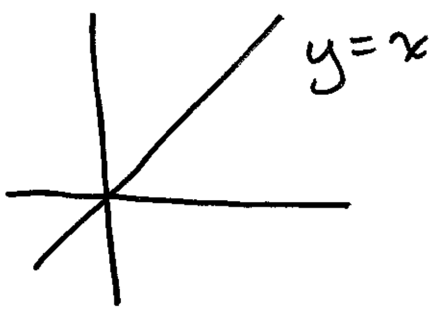
Warm-up: Subspaces

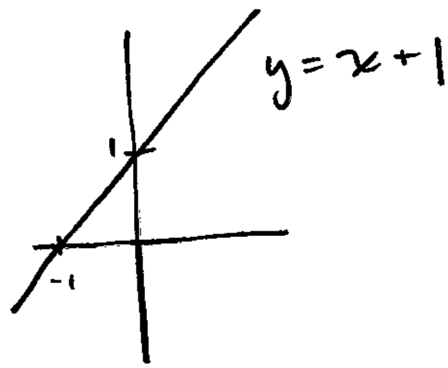
Recall from Lesson 7...

Vector Subspace Theorem: a nonempty subset W of a vector space V is a subspace of V if it is closed under addition and scalar multiplication:

(i) If $\vec{x}, \vec{y} \in W$, then $\vec{x} + \vec{y} \in W$ **(ii) If $\vec{x} \in W$ and $c \in \mathbb{R}$, then $c\vec{x} \in W$**

1) Sketch the lines: $y = x$ and $y = x + 1$. One line is a subspace of $\mathbb{R}^2$, one is not. Decide, then explain your reasoning.





subspace

includes (0,0); any vector that lies on the line is also a multiple of the others

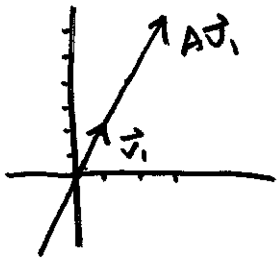
not a subspace

doesn't contain the origin

Eigenspaces

2)a) In Activity 10 you had: $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ which had eigenvectors: $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and $\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

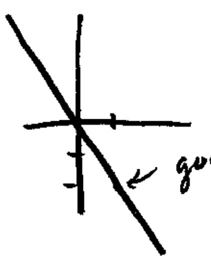
You also found that $A\vec{v}_1 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$. Sketch $A\vec{v}_1$ and $\vec{v}_1$ (on the same graph).



b) Looking back at (1), what can you say about $span\{\vec{v}_1\}$?

It's the line that $\vec{v}_1$ and $A\vec{v}_1$ lie upon.

c) Sketch $span\{\vec{v}_2\}$. Is $span\{\vec{v}_2\}$ a subspace of $\mathbb{R}^2$?



← goes through $(1,-2)$ since $\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

We call the $span\{\vec{v}_i\}$ the eigenspace $\mathbb{E}_\lambda$.

3) What is the dimension of each eigenspace?

one, since there's one vector in each basis.

Repeated Eigenvalues

4) What are the eigenvalues and eigenvectors of: $\mathbf{H} = \begin{bmatrix} 3 & 0 \\ 5 & 3 \end{bmatrix}$

e-vals: $\lambda = 3, 3$ (repeated)

e-vecs: $A - 3I = \begin{bmatrix} 0 & 0 \\ 5 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $x = 0$, y is free $\vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

only 1 linearly indep. e-vec
For a 2×2 matrix!

Repeated eigenvalues sometimes have more than one eigenvector though.

5) The eigenvalues for this matrix are $\lambda_{1,2} = 1$ and $\lambda_3 = 2$. Find the eigenvectors associated with these eigenvalues for the matrix: $\mathbf{R} = \begin{bmatrix} 0 & 0 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 3 \end{bmatrix}$

$\lambda = 1, 1, 2$

$\lambda = 1$: $\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = 2$: $\vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

We know that eigenvectors always satisfy the 2nd property of vector subspaces (you should be able to explain why!). We also found above that the eigenspace from a single eigenvector is a valid subspace. However, notice that we wrote our eigenspace as $\mathbb{E}_\lambda$. But there are TWO eigenvectors with our double eigenvalue.

Let's denote the double eigenvalue, and its eigenvectors λ_1, λ_2 and $\vec{v}_1, \vec{v}_2$ respectively.

6) State all the eigenspaces and their corresponding dimension for R?

$E_1 = \text{span}\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}\right\}$ $\quad E_2 = \text{span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$

$\dim(E_1) = 2$ $\quad \dim(E_2) = 1$

7) Give a geometric interpretation of $span\{\vec{v}_1, \vec{v}_2\}$ in $\mathbb{R}^3$ (3-d space).
Hint: this is a special shape in $\mathbb{R}^3$

A plane.

The algebraic multiplicity of an eigenvalue is the number of times it is a root (e.g. a in $(\lambda - \lambda_i)^a$)
The geometric multiplicity of an eigenvalue is the dimension of its corresponding eigenspace.
8)a) Give the algebraic and geometric multiplicity for the eigenvalues of R.

$\lambda = 1$: alg. mult. = 2
geom. mult. = 2

$\lambda = 2$: alg. mult. = 1
geom. mult. = 1

b) Give the algebraic and geometric multiplicity for the eigenvalues of H.

$\lambda = 3$: alg. mult. = 2
geom. mult = 1

c) How do the algebraic and geometric multiplicity compare to each other?

$1 \leq$ geom. mult. $\leq$ alg. mult. $\leq n$

Distinct Eigenvalues

There is very useful theorem about distinct eigenvalues:
Distinct Eigenvalue Theorem: Let A be an $n \times n$ matrix. If $\lambda_1, \lambda_2, \ldots, \lambda_p$ are distinct eigenvalues with corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ then $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is a set of linearly independent vectors.

9)a) Find the eigenvalues and an eigenvectors for each value for: $W = \begin{bmatrix} 3 & 0 \\ 5 & 2 \end{bmatrix}$

$\lambda = 3, 2$

$\lambda = 3: \ \vec{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\qquad$ $\lambda = 2: \ \vec{v}_2 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

b) Show that the eigenvectors are linearly independent.

$\det \begin{pmatrix} 0 & 1 \\ 1 & 5 \end{pmatrix} = -1 \neq 0$

10) There's something to be careful of with this theorem though...You found the eigenvectors for R in (3). Are the eigenvectors of R linearly independent?

Yes.

11) If the eigenvectors are linearly independent are you guaranteed to have distinct eigenvalues?

No, as seen in R, we don't have distinct eigenvalues as $\lambda = 1$ is a double root.