**MATH 260, Linear Systems and Matrices, Fall '14**
**Activity 10: Introduction to Eigenvalues and Eigenvectors**

# Warm-up: Matrix Multiplication

Given the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ and the vectors: $\vec{v}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$, $\vec{v}_4 = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$, $\vec{v}_5 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Find $\mathbf{A}\vec{v}_i$ for $i = 1, ..., 5$.

# Visualizing Eigenvectors

1) Sketch each $\vec{v}_i$ and the result of $\mathbf{A}\vec{v}_i$ from the warm-up in the $xy$-plane. You should have 5 pictures when you are done. Be sure to clearly label which was your initial, and which your resultant vector.

2) Two of the $v_i$ vectors behaved differently, which ones? Describe how they behaved differently.

3) For the two vectors that behaved differently, you should be able to write the eigenvector equation: $\mathbf{A}\vec{v} = \lambda\vec{v}$. Find $\lambda_i$ for each pair of vectors.

## Practice

Let's practice finding eigenvalues and eigenvectors.

4) For each matrix, take the determinant $|\mathbf{A} - \lambda\mathbf{I}|$ then set the resulting polynomial equal to 0 and solve for $\lambda$. (They should all be factorable, so even the $3 \times 3$s should not be too hard to do). Once you have found ALL the eigenvalues ($\lambda$s) find at least one eigenvector for each matrix.

$$\mathbf{W} = \begin{bmatrix} 1 & 4 \\ -4 & 11 \end{bmatrix} \qquad \mathbf{G} = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \qquad \mathbf{R} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

5) Go back and look at $\mathbf{R}$. What do you notice about the eigenvalues you found for $\mathbf{R}$? Can you generalize this to a property of certain types of matrices?

## Some Properties of Eigenvalues

*Note: We will find these for $2 \times 2$ matrices, but these properties generalize to any $n \times n$ matrix.*

6) We find eigenvalues by solving: $|\mathbf{A} - \lambda\mathbf{I}| = 0$. What happens if $\lambda = 0$? What must be true about the matrix $\mathbf{A}$ for $\mathbf{A}$ to have eigenvalues of 0?

7)a) Find $\mathbf{W}^T$.

b) What is the polynomial from $|\mathbf{W}^T - \lambda\mathbf{I}|$ and how does it compare to the polynomial for $|\mathbf{W} - \lambda\mathbf{I}|$?

c) What can you say about the eigenvalues (and vectors) of $\mathbf{A}$ and $\mathbf{A}^T$?

8)a) Find $\mathbf{W}^{-1}$.

b) What is the polynomial from $|\mathbf{W}^{-1} - \lambda\mathbf{I}|$? Set the polynomial equal to 0, and solve for $\lambda$

c) Look carefully, how do the eigenvalues of $\mathbf{W}^{-1}$ relate to the eigenvalues of $\mathbf{W}$?