MATH 260, Linear Algebra, Spring '14
Activity 6: Determinants and Cramer's Rule
Honor Code: **Names:**

Directions: Everyone should work on the assignment and should fill out their paper. You are expected to make corrections based on what is presented on the board.

The worksheet is due by the end of the day (5pm).

If you need more explanations after class, you can read Chapter 3.4 of your textbook.

In-Class Learning Goals:

1. Be able to find the determinant of a matrix (both 2×2 AND nXn).
2. Be able to find the 'minor' and 'cofactor' of elements in a matrix.
3. Be able to find solutions to a system using *Cramer's Rule*.
4. (Stretch) Discover several properties of determinants including if a matrix has an inverse.

Warmup: Determinant of a 2×2

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Note: You will need to have this formula memorized!

Find the determinant of the following matrix:

$$\begin{bmatrix} -3 & 1 \\ 4 & 2 \end{bmatrix} \qquad \begin{bmatrix} 3 & 0 & -1 \\ 0 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$$

1 Some Properties and Practice

For **Activity 5** you found several inverses and transposes of matrices. Lets use them...

$$\mathbf{H} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \qquad \mathbf{H}^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{4}{3} & -\frac{1}{3} \end{bmatrix} \qquad \mathbf{H}^{T} = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \qquad \mathbf{G} = \begin{bmatrix} 3 & 0 & 3 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

a) Find $|\mathbf{H}|$.

b) Find $|\mathbf{H}^{-1}|$.

c) Find $|\mathbf{H}^{T}|$.

d) Find $|\mathbf{G}|$.

e) Are any of the numbers above the same? Closely related? Try to write each as a generalized property (check these with me).

2 Finding Minors (and some 2×2 practice)

This method is only a little harder than visiting an elementary school...

Definition: For the $n \times n$ matrix $\mathbf{A}$, the *minor* M_{ij} of a_{ij} is an $(n-1)\times(n-1)$ matrix obtained by deleting the ith row and the jth column of $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 5 & 0 & 1 & -2 \\ 4 & 0 & 1 & 0 \\ 2 & 0 & 3 & 1 \end{bmatrix} \qquad \mathbf{M}_{12} = \begin{bmatrix} \circ & \circ & \circ & \circ \\ \bullet & \circ & \bullet & \bullet \\ \bullet & \circ & \bullet & \bullet \\ \bullet & \circ & \bullet & \bullet \end{bmatrix} \to \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} = \begin{bmatrix} 5 & 1 & -2 \\ 4 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

a) Find the minor $\mathbf{M}_{32}$ of $\mathbf{A}$. To show your work, re-write $\mathbf{A}$ first...

b) Lets find some minors of minors...

(i) Find $\mathbf{M}_{21}$ of $\mathbf{M}_{12}$

(ii) Find $\mathbf{M}_{22}$ of $\mathbf{M}_{12}$

(iii) Find $\mathbf{M}_{23}$ of $\mathbf{M}_{12}$

c) Find the determinant of...

(i) $\mathbf{M}_{21}$

(ii) $\mathbf{M}_{22}$

(iii) $\mathbf{M}_{23}$

3 Co-Factors

Definition: The *co-factor* of an element a_{ij} is the scalar: $C_{ij} = (-1)^{i+j}|\mathbf{M}_{ij}|$

Notice that the co-factor is just the determinant of a minor, times ± 1. We'll use this in the next section...
a) Find C_{21} of $\mathbf{M}_{12}$

b) Find C_{22} of $\mathbf{M}_{12}$

c) Find C_{23} of $\mathbf{M}_{12}$

d) Can you find C_{12} of $\mathbf{A}$ (based on what you know now)? Why or why not?

4 Determinants of N x N matrices

Lets start by finding the determinant of the 3x3 matrix $\mathbf{M}_{12}$.
To make our notation simpler, lets call $\mathbf{M}_{12}$ matrix $\mathbf{D}$.
a) Find the following sum: $d_{21} * C_{21} + d_{22} * C_{22} + d_{23} * C_{23}$
Hint: You should get a single, scalar number.

b) Try writing the sum for (a) in summation notation (i.e. using: Σ).

Determinants of N x N matrices (cont.)

Finding the determinant of an NxN matrix is a recursive process. The definition is:

Definition: Choose a row i or column j, then $|A| = \sum_{(j \text{ or } i)}^{n} a_{ij} C_{ij} = \sum_{j \text{ or } i}^{n} a_{ij} (-1)^{i+j} |\mathbf{M}_{ij}|$

Notice that $\mathbf{M}_{ij}$ may not be a 2×2!! You might have to use this process several times.

c) Using this definition find the determinant of our original $\mathbf{A}$ matrix.
Hint: You should only need to do a few new multiplications and additions, you calculated most of it already.