MATH 260, Linear Systems and Matrices, Summer I '14
Activity 8: Linear Independence, Basis, (Span and Dimension)

KEY

Warm-up: Linear Combinations

Let's look at the vectors: $\vec{c} =< 1, 0, 0 >$, $\vec{d} =< -1, 1, 1 >$ and $\vec{e} =< 0, 2, 2 >$.

1. Write $\vec{e}$ as a linear combination of $\vec{c}$ and $\vec{d}$.

$$\underline{2\vec{c} + 2\vec{d} = \vec{e}}$$

$$\langle 2-2, 2, 2\rangle = \langle 0, 2, 2\rangle$$

2. Can you do the same thing for each other vector? That is, write each vector as a linear combination of the other two vectors. Give the combinations, or show you cannot.

$$\vec{d} = \frac{\vec{e} - 2\vec{c}}{2} = \frac{1}{2}\vec{e} + \vec{c}$$

$$\vec{c} = \frac{\vec{e} - 2\vec{d}}{2} = \frac{1}{2}\vec{e} - \vec{d}$$

Linear Independence

Definition: A set of vectors $\{\vec{v}_1, \vec{v}_2, \cdots \vec{v}_n\}$ is *linearly independent* if no vector of the set can be written as a linear combination of the other vectors. Otherwise it is *linearly dependent*

1. a) Is the set of vectors $\{<1,0,0>, <-1,1,1>, <0,2,2>\}$ linearly independent? Explain.

No. Dependent. We found a nontrivial linear combination in the warm-up.

1. b) Write the vectors $\{<1,0,0>, <-1,1,1>, <0,2,2>\}$ as a matrix with each vector as a vertical column. Find the determinant of the matrix.

$$\det \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} = 0$$

2. a)Take the vectors $\vec{a} =< 1,2 >$ and $\vec{b} =< 1,3 >$. Write these as a matrix, then take the determinant.

$$\det \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = 3-2 = 1 \neq 0$$

2. b) For what values of c_1, c_2 is the following equation true?

$$c_1\vec{a} + c_2\vec{b} = \vec{0}$$

Only when $c_1 = c_2 = 0$.

3. a)Put each matrix into RREF, then give the number of pivot columns:

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

2 pivot col.s

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2 pivot col.s

1 pivot col.

3. b) Are the vectors used in the third matrix linearly independent? Explain why or why not.

No, $\begin{pmatrix}1\\2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2\\4\end{pmatrix}$

3. c) How do the number of pivot columns compare to the number of elements in the vectors? How does this correspond to if the vectors are linearly independent? How does the value of the determinant correspond to if vectors are linearly independent?

If the # of pivot columns is less than the # of columns (i.e. the # of vectors), then linearly dependent.

If det = 0, dep.

det ≠ 0, indep.

4. a) You now have 3 ways of testing for linear independence, what are they?

① Find that the only linear combination equal to $\vec{0}$ is trivial $(0\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 = \vec{0})$

② # of pivot columns = ~~#of~~ # of vectors

③ det ≠ 0 (only works if # of vectors = the # of entries in the vectors)

4. b) You actually know a 4th way, recall that if matrix is invertible, then $|A| \neq 0$.
i) How many (non-zero) column vectors of length 2 do you need to make a 2×3 matrix?
ii) Can you find the inverse of a 2×3 matrix? Can you find the determinant?
iii) If a set of vectors has more vectors in it than the length of the vectors, will the set be linearly independent? (i.e. if the set contains 3 vectors of length 2, or 4 vectors of length 2 or 3)

i) 3

ii) No. No.

iii) No.

Basis

A set is a **basis** of a vector space if it has the properties:
(i) The set is linearly independent
(ii) The span of the set covers the entire vector space.

1. Let's test some of the vector sets we've looked at. Which set of vectors from the previous section might constitute a basis? Explain why.

$\{\vec{a}, \vec{b}\}$ is a basis for $\mathbb{R}^2$.

It's a set of 2 linearly indep. vectors that spans all of $\mathbb{R}^2$ since $c_1\vec{a} + c_2\vec{b}$ for any c_1, c_2 will give any vector of $\mathbb{R}^2$.

2. To satisfy the 2nd property, we need the span of these vectors to cover the entire vector space of $\mathbb{R}^2$ This means we need to be able to create each of the following vectors via linear combinations:

- A vector with all positives: $< +, + >$
- A vector with all negatives: $< -, - >$
- A vector with a negative and a positive in each position: $< +, - >$ and $< -.+ >$
- The zero vector: $< 0, 0 >$

Write a linear combination of $\vec{a}$ and $\vec{b}$ which gives each of the above types of vectors.

ex! $\vec{a} + \vec{b} = \langle 2, 5 \rangle$

$-\vec{a} - \vec{b} = \langle -2, -5 \rangle$

$4\vec{a} - 3\vec{b} = \langle 1, -1 \rangle$, $-4\vec{a} + 3\vec{b} = \langle -1, 1 \rangle$

$0\vec{a} + 0\vec{b} = \langle 0, 0 \rangle$

3. a) We know the vectors from above $\vec{c}$, $\vec{d}$, $\vec{e}$ are not linearly independent. What if we just use the vectors $\vec{c} =< 1, 0, 0 >$ and $\vec{d} =< -1, 1, 1 >$ do we have a basis for $\mathbb{R}^3$?
i) Check if $\vec{c}$ and $\vec{d}$ are linearly independent
ii) Check if the span{ $\vec{c}$, $\vec{d}$ } contains all the vectors possible in $\mathbb{R}^3$. If it does not, give an example of a vector that it does not contain.

i) indep. since $\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ has 2 pivot cols & there are 2 vectors.

ii) Does not span $\mathbb{R}^3$ since no way to get $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ (or any number of other examples). Notice that $\text{span}\{\vec{c}, \vec{d}\} = \left\{ \begin{pmatrix} x \\ y \\ y \end{pmatrix} \mid x, y \in \mathbb{R} \right\}$ so elements have to have the same 2nd and 3rd entries.

3. b) Your span did not contain all of $\mathbb{R}^3$. What vector could you add to the set that makes the new set be a basis for $\mathbb{R}^3$? (show that the vector you choose is both linearly independent AND allows the span to contain all of $\mathbb{R}^3$)

include $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ to make it a spanning set, as one example.

$\det \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = 1 \neq 0$, so lin. indep. and $\text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} = \mathbb{R}^3$.