MATH 260, Linear Systems and Matrices, Summer I '14
Activity 10: Introduction to Eigenvalues and Eigenvectors

KEY

## Warm-up: Matrix Multiplication

Given the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ and the vectors: $\vec{v}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$, $\vec{v}_4 = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$, $\vec{v}_5 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Find $\mathbf{A}\vec{v}_i$ for $i = 1, ..., 5$.

$$A\vec{v}_1 = \begin{bmatrix} 4 \\ 10 \end{bmatrix}$$

$$A\vec{v}_2 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$A\vec{v}_3 = \begin{bmatrix} -2 \\ -8 \end{bmatrix}$$

$$A\vec{v}_4 = \begin{bmatrix} -5 \\ -8 \end{bmatrix}$$

$$A\vec{v}_5 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

## Visualizing Eigenvectors

1) Sketch each $\vec{v}_i$ and the result of $\mathbf{A}\vec{v}_i$ from the warm-up in the $xy$-plane. You should have 5 pictures when you are done. Be sure to clearly label which was your initial, and which your resultant vector.

$\vec{V}_2$ and $\vec{V}_5$ should have

$A\vec{v}_2$ and $A\vec{v}_5$ lie on the same line.

2) Two of the $v_i$ vectors behaved differently, which ones? Describe how they behaved differently.

$\vec{V}_2$ and $\vec{V}_5$ resulted in constant multiples of themselves when multiplying by $A$. i.e. $A\vec{v}_2 = 3\vec{v}_2$

3) For the two vectors that behaved differently, you should be able to write the eigenvector equation: $\mathbf{A}\vec{v} = \lambda\vec{v}$. Find $\lambda_i$ for each pair of vectors.

$A\vec{v}_2 = 3\vec{v}_2$

$\uparrow$

$\lambda$

$A\vec{v}_5 = -\vec{v}_5$

$\lambda = -1$.

## Practice

Let's practice finding eigenvalues and eigenvectors.

4) For each matrix, take the determinant $|\mathbf{A} - \lambda\mathbf{I}|$ then set the resulting polynomial equal to 0 and solve for $\lambda$. (They should all be factorable, so even the $3 \times 3$s should not be too hard to do). Once you have found ALL the eigenvalues ($\lambda$s) find at least one eigenvector for each matrix.

$$\mathbf{W} = \begin{bmatrix} 1 & 4 \\ -4 & 11 \end{bmatrix} \qquad \mathbf{G} = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \qquad \mathbf{R} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

W: $\lambda_1 = 3$, $\lambda_2 = 9$ $\quad \vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

G: $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = -1$ $\quad \vec{v}_1 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

R: $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = 6$ $\quad \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 16 \\ 25 \\ 10 \end{pmatrix}$

5) Go back and look at $\mathbf{R}$. What do you notice about the eigenvalues you found for $\mathbf{R}$? Can you generalize this to a property of certain types of matrices?

Triangular matrices have their eigenvalues along the diagonal.

## Some Properties of Eigenvalues

*Note: We will find these for $2 \times 2$ matrices, but these properties generalize to any $n \times n$ matrix.*

6) We find eigenvalues by solving: $|\mathbf{A} - \lambda\mathbf{I}| = 0$. What happens if $\lambda = 0$? What must be true about the matrix $\mathbf{A}$ for $\mathbf{A}$ to have eigenvalues of 0?

A must be singular (not invertible).

7)a) Find $\mathbf{W}^T$.

$$W^T = \begin{pmatrix} 1 & -4 \\ 4 & 11 \end{pmatrix}$$

b) What is the polynomial from $|\mathbf{W}^T - \lambda\mathbf{I}|$ and how does it compare to the polynomial for $|\mathbf{W} - \lambda\mathbf{I}|$?

$$(1-\lambda)(11-\lambda) + 16$$

$$= \lambda^2 - 12\lambda + 27$$

it's the same!

c) What can you say about the eigenvalues (and vectors) of $\mathbf{A}$ and $\mathbf{A}^T$?

the same eigenvalues, eigenvectors are different

8)a) Find $\mathbf{W}^{-1}$.

$$\left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ -4 & 11 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 27 & 4 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & 4/27 & 1/27 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & 11/27 & -4/27 \\ 0 & 1 & 4/27 & 1/27 \end{array}\right)$$

b) What is the polynomial from $|\mathbf{W}^{-1} - \lambda\mathbf{I}|$? Set the polynomial equal to 0, and solve for $\lambda$

$$\left(\tfrac{11}{27} - \lambda\right)\left(\tfrac{1}{27} - \lambda\right) + \tfrac{16}{27^2} = 0$$

$$\lambda = \tfrac{1}{9}, \tfrac{1}{3}$$

c) Look carefully, how do the eigenvalues of $\mathbf{W}^{-1}$ relate to the eigenvalues of $\mathbf{W}$?

reciprocals