The geometry of linear equations

The fundamental problem of linear algebra is to solve n linear equations in n unknowns; for example:

$$\begin{aligned} 2x - y &= 0 \\ -x + 2y &= 3. \end{aligned}$$

In this first lecture on linear algebra we view this problem in three ways.

The system above is two dimensional ($n = 2$). By adding a third variable z we could expand it to three dimensions.

Row Picture

Plot the points that satisfy each equation. The intersection of the plots (if they do intersect) represents the solution to the system of equations. Looking at Figure 1 we see that the solution to this system of equations is $x = 1, y = 2$.

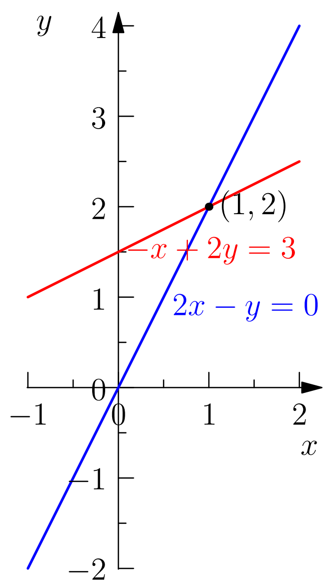

Figure 1: The lines $2x - y = 0$ and $-x + 2y = 3$ intersect at the point $(1, 2)$.

We plug this solution in to the original system of equations to check our work:

$$\begin{aligned} 2 \cdot 1 - 2 &= 0 \\ -1 + 2 \cdot 2 &= 3. \end{aligned}$$

The solution to a three dimensional system of equations is the common point of intersection of three planes (if there is one).

Column Picture

In the column picture we rewrite the system of linear equations as a single equation by turning the coefficients in the columns of the system into vectors:

$$x\begin{bmatrix} 2 \\ -1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

Given two vectors **c** and **d** and scalars x and y, the sum $x\mathbf{c} + y\mathbf{d}$ is called a *linear combination* of **c** and **d**. Linear combinations are important throughout this course.

Geometrically, we want to find numbers x and y so that x copies of vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ added to y copies of vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ equals the vector $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$. As we see from Figure 2, $x = 1$ and $y = 2$, agreeing with the row picture in Figure 2.

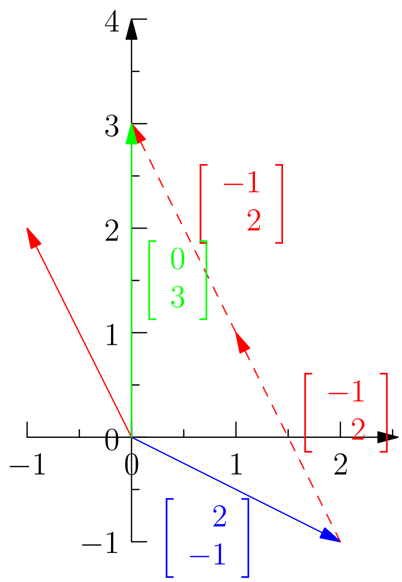

Figure 2: A linear combination of the column vectors equals the vector **b**.

In three dimensions, the column picture requires us to find a linear combination of three 3-dimensional vectors that equals the vector **b**.

Matrix Picture

We write the system of equations

$$\begin{aligned} 2x - y &= 0 \\ -x + 2y &= 3 \end{aligned}$$

as a single equation by using matrices and vectors:

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

The matrix $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ is called the *coefficient matrix*. The vector $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ is the vector of unknowns. The values on the right hand side of the equations form the vector $\mathbf{b}$:

$$A\mathbf{x} = \mathbf{b}.$$

The three dimensional matrix picture is very like the two dimensional one, except that the vectors and matrices increase in size.

Matrix Multiplication

How do we multiply a matrix A by a vector $\mathbf{x}$?

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \quad ?$$

One method is to think of the entries of $\mathbf{x}$ as the coefficients of a linear combination of the column vectors of the matrix:

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}.$$

This technique shows that $A\mathbf{x}$ is a linear combination of the columns of A.

You may also calculate the product $A\mathbf{x}$ by taking the dot product of each row of A with the vector $\mathbf{x}$:

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 + 5 \cdot 2 \\ 1 \cdot 1 + 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}.$$

Linear Independence

In the column and matrix pictures, the right hand side of the equation is a vector $\mathbf{b}$. Given a matrix A, can we solve:

$$A\mathbf{x} = \mathbf{b}$$

for every possible vector $\mathbf{b}$? In other words, do the linear combinations of the column vectors fill the xy-plane (or space, in the three dimensional case)?

If the answer is "no", we say that A is a *singular matrix*. In this singular case its column vectors are *linearly dependent*; all linear combinations of those vectors lie on a point or line (in two dimensions) or on a point, line or plane (in three dimensions). The combinations don't fill the whole space.