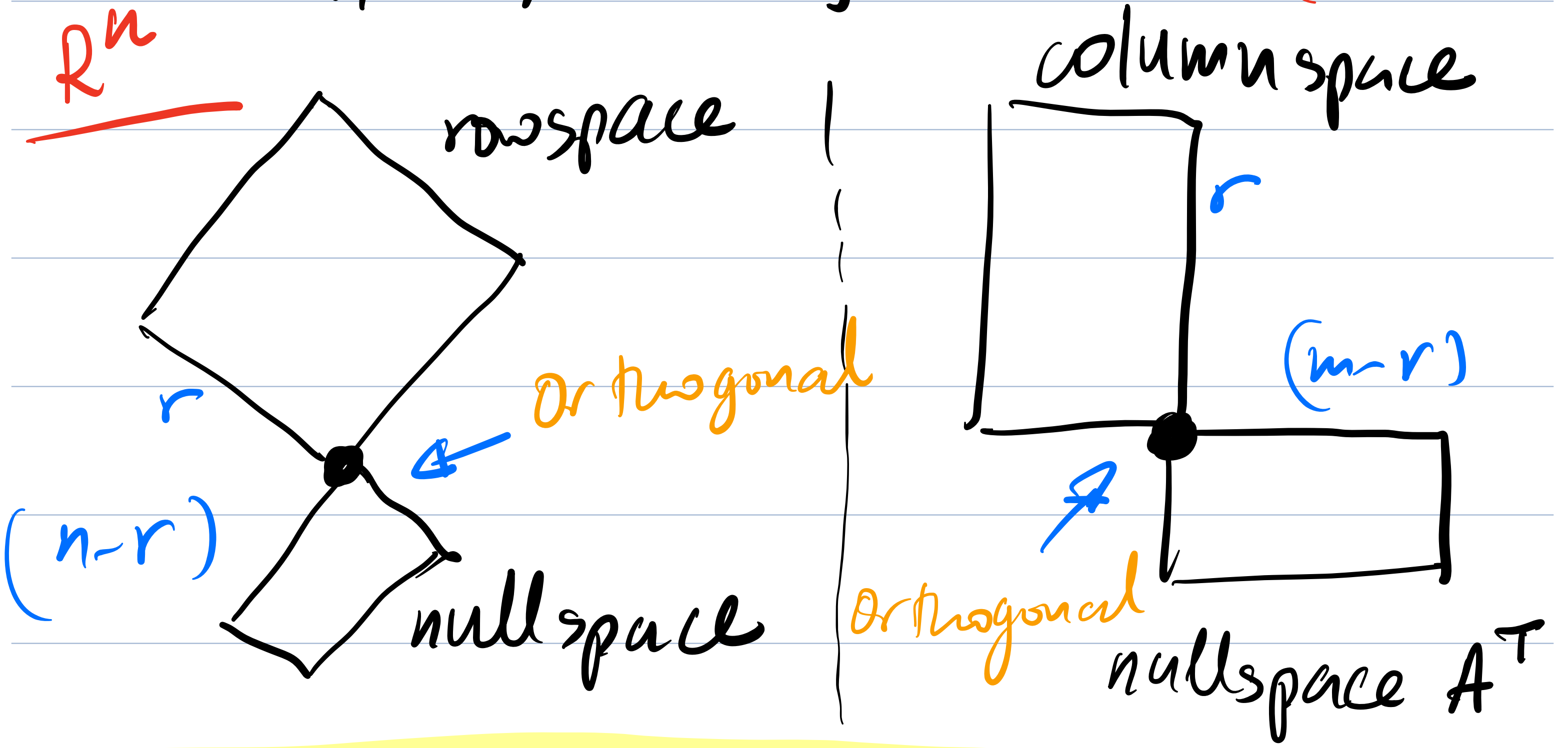
Lecture #14 – Orthogonal vectors & subspaces

Objectives

- Orthogonal vectors & subspaces
- nullspace ⊥ row space

$$N(A^T A) = N(A)$$



- Orthogonal Vectors: perpendicular

↳ condition: 2 vectors are perpendicular if the angle between them is 90°

↳ $\boxed{x^T y = y^T x = 0}$ [Dot product]

$$\|x\|^2 + \|y\|^2 = \|x+y\|^2$$

$$\cancel{x^Tx} + \cancel{y^Ty} = (x+y)^T(x+y)$$

$$= \cancel{x^Tx} + \cancel{y^Ty} + x^Ty + y^Tx$$

$$0 = \Rightarrow 2x^Ty.$$

Let $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $y = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

$\|x\|^2 = 14$, $\|y\|^2 = 5$

$x+y = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$, $\|x+y\|^2 = 19$

Orthogonal subspace

Subspace S is orthogonal to subspace T

↳ every vector in S is orthogonal to every vector in T

→ In the plane, the space containing only the zero vector and any line through the origin are orthogonal subspaces.

→ A line through the origin and the whole plane are never orthogonal subspaces

→ Two lines through the origin are orthogonal subspaces if they meet at right angles.

Nullspace is perpendicular to row space

→ The row space is orthogonal to nullspace {x in nullspace}

Why? $Ax = 0$

$$\begin{bmatrix} \text{row 1 of } A \\ \text{row 2 of } A \\ \vdots \\ \text{row } n \text{ of } A \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

⇒ x is orthogonal to all rows

$$\left.\begin{aligned} c_1 \, (\text{row } 1)^T x &= 0 \\ + \; c_2 \, (\text{row } 2)^T x &= 0 \end{aligned}\right\} \; c_1 \text{row } 1 + c_2 \text{row } 2 + \cdots = 0$$

eg

$n = 3, r = 1$

$$A = \begin{bmatrix} 1 & 2 & 5 \\ \cancel{2} & \cancel{4} & \cancel{10} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \cancel{0} \end{bmatrix}$$

dim rowspace = 1

dim N(A) = 2

- nullspace and rowspace are orthogonal

are orthogonal complements in $\mathbb{R}^n$

$\therefore$ Nullspace contains all vectors

$\perp$

row space

Note :

We could say that this is part two of the fundamental theorem of linear algebra. Part one gives the dimensions of the four subspaces, part two says those subspaces come in orthogonal pairs, and part three will be about orthogonal bases for these subspaces.

Coming: $Ax = b$ (when there is no solution?)

$$\downarrow$$

"Solve" $A^T A \hat{x} = A^T b$ (b not in column space)

$m > n$ = # of unknowns

The matrix $\underline{A^T A}$ → symmetric

$n \times m \quad m \times n$

$\underline{\underline{n \times n}}$

$\rightarrow (A^T A)^T = A^T \underline{A^{TT}} \Rightarrow A = A^T A$

eg

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$r = 2$

Cu-dependent col.

* Can only be solves if b is in the column space.

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 3 & 8 \\ 8 & 30 \end{bmatrix}}}$$

invertible

rank = 1

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 3 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 9 & 27 \end{bmatrix}$$

↳ not invertible

$\therefore N(A^TA) = N(A)$

rank of A^TA = rank A

We conclude that A^TA is <u>invertible</u> exactly if A has independent columns.

[illegible]