Objectives

- Orthogonal basis $q_1, \dots, q_n$
- Orthogonal matrix Q : square
- Gram-Schmidt $A \to Q$

Orthonormal vectors

$$q_i^T q_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

(length square)

- In other words, they all have (normal) length 1 and are perpendicular (ortho) to each other.

- Orthonormal vectors are always independent.

- Orthonormal matrix

If the columns of

$$Q = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix}$$

Then

$$Q^T Q = \begin{bmatrix} - & q_1^T & - \\ & \vdots & \\ - & q_n & - \end{bmatrix} \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ & \vdots & \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \boxed{Q^T Q = I}$$

- If Q is squared then $Q^T Q = I$ tells us $Q^T = Q^{-1}$

<u>Examples</u>.

① perm $Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$Q^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$\times = I$

②

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$Q = 1 \begin{bmatrix} 1 & 1 \end{bmatrix}$

$$\sqrt{2} \quad 1 \quad -1$$

③

$$Q = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

* Hadamard matrices *

④ Rectangular matrix with orthonormal columns

$$Q = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$$

Orthonormal columns are good

Suppose Q has orthonormal columns.
The matrix that projects onto the column space of Q is:

$$P = Q^T (Q^T Q)^{-1} Q^T$$

- If the columns of Q are orthogonal, then $Q^T Q = I$ & $P = QQ^T$.

- If Q is square, then $P = I$ because the columns of Q span the entire space.

- Many equations become trivial

when using a matrix with orthonormal columns.

eg $A^TA\hat{x}=A^Tb$; now $A=Q$

$$\underbrace{Q^TQ}_{I}\hat{x}=Q^Tb \quad \Big\} \quad \hat{x}=Q^Tb$$

In other words $\boxed{\hat{x}_i=q_i^Tb}$

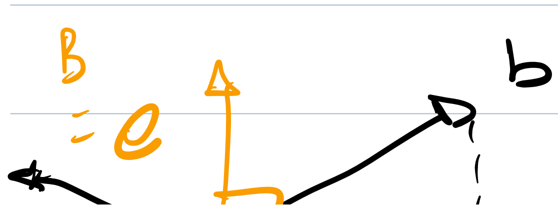
• **Gram-Schmidt**

- We start with 2 independent vectors a, b

projection

$$B=b-p$$

$$B=b-\frac{A^Tb}{A^TA}A$$



$A \perp B$

- Want to find orthogonal vectors A, B
- Want to find orthonormal vectors

$$q_1 = \frac{A}{\|A\|}, \quad q_2 = \frac{B}{\|B\|}$$

- Check $\perp$: multiply by $A^T = 0$

$\therefore$

$$A^T B = A^T\left(b - \frac{A^T b}{A^T A} A\right) = 0$$

→ If we had $A, B, \underline{C}$

add $q_3 = \frac{C}{\|C\|} = ??$

$$c - \underbrace{\frac{A^T c}{A^T A} A} - \underbrace{\frac{B^T c}{B^T B} B}$$

component in A dimension.

Component in B dimension

$\Rightarrow C \perp A$

$C \perp B$

Example:

$a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

Then $A = a$ &

$$B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{A^T b}{A^T A} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

b $\quad A \perp B$

$$= \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Normalizing we get

$$Q = \begin{bmatrix} | & | \\ q_1 & q_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$$

- The column space of Q is the plane spanned by a & b

Note

When we studied elimination, we wrote the process in terms of matrices and found $A = LU$

- A similar equation $\underline{A = QR}$ relates our starting matrix A to the result Q of Gram-Schmidt

Suppose:

$$A = \begin{array}{cc} a & b \\ \multicolumn{2}{c}{\begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix}} \end{array}$$

$$= \overset{\underline{Q}}{\begin{bmatrix} q_1 & q_2 \end{bmatrix}} \overset{\underline{R}}{\begin{bmatrix} a_1^T q_1 & a_2^T q_1 \\ \underbrace{a_1^T q_2}_{\text{why } = 0?} & a_2^T q_2 \end{bmatrix}}$$

When we studied elimination, we wrote the process in terms of matrices and found $A = LU$. A similar equation $A = QR$ relates our starting matrix A to the result Q of the Gram-Schmidt process. Where L was lower triangular, R is upper triangular.

Suppose $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$. Then:

$$\overset{A}{\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}} = \overset{Q}{\begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix}} \overset{R}{\begin{bmatrix} \mathbf{a}_1^T\mathbf{q}_1 & \mathbf{a}_2^T\mathbf{q}_1 \\ \mathbf{a}_1^T\mathbf{q}_2 & \mathbf{a}_2^T\mathbf{q}_2 \end{bmatrix}}.$$

If R is upper triangular, then it should be true that $\mathbf{a}_1^T\mathbf{q}_2 = 0$. This must be true because we chose $\mathbf{q}_1$ to be a unit vector in the direction of $\mathbf{a}_1$. All the later $\mathbf{q}_i$ were chosen to be perpendicular to the earlier ones.

Notice that $R = Q^T A$. This makes sense; $Q^T Q = I$.