Multiplication & Inverse Matrices

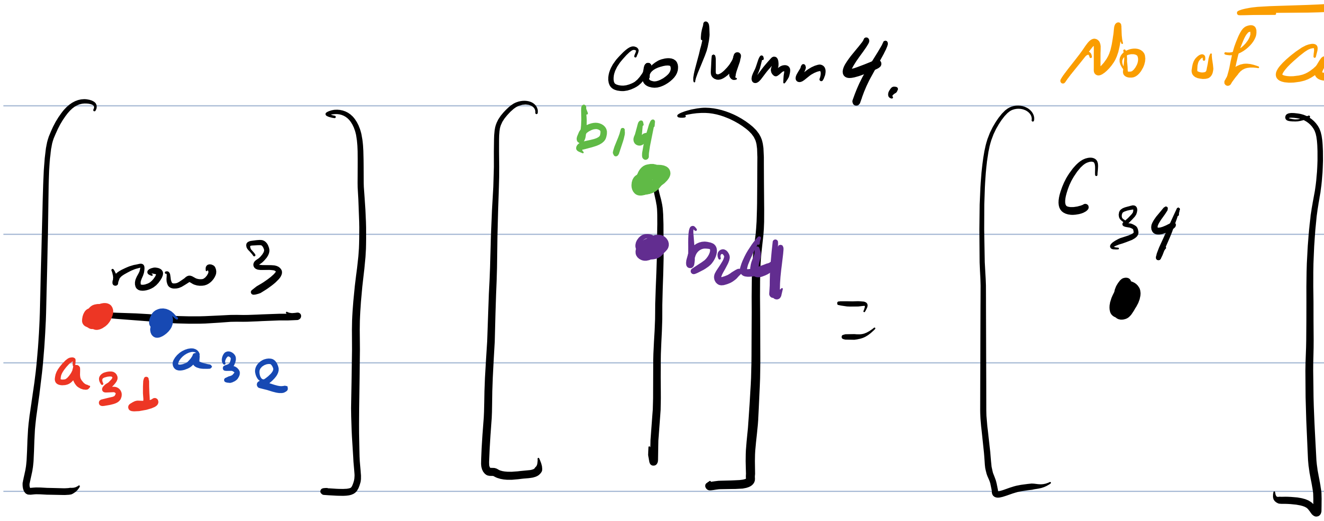
Objectives:

- Matrix Multiplication (4 ways)
- Inverse of A, AB, A^T
- Gauss-Jordan / find A^{-1}

[row x column]

Matrix Multiplication ①

* No of rows of B has to match No of columns A



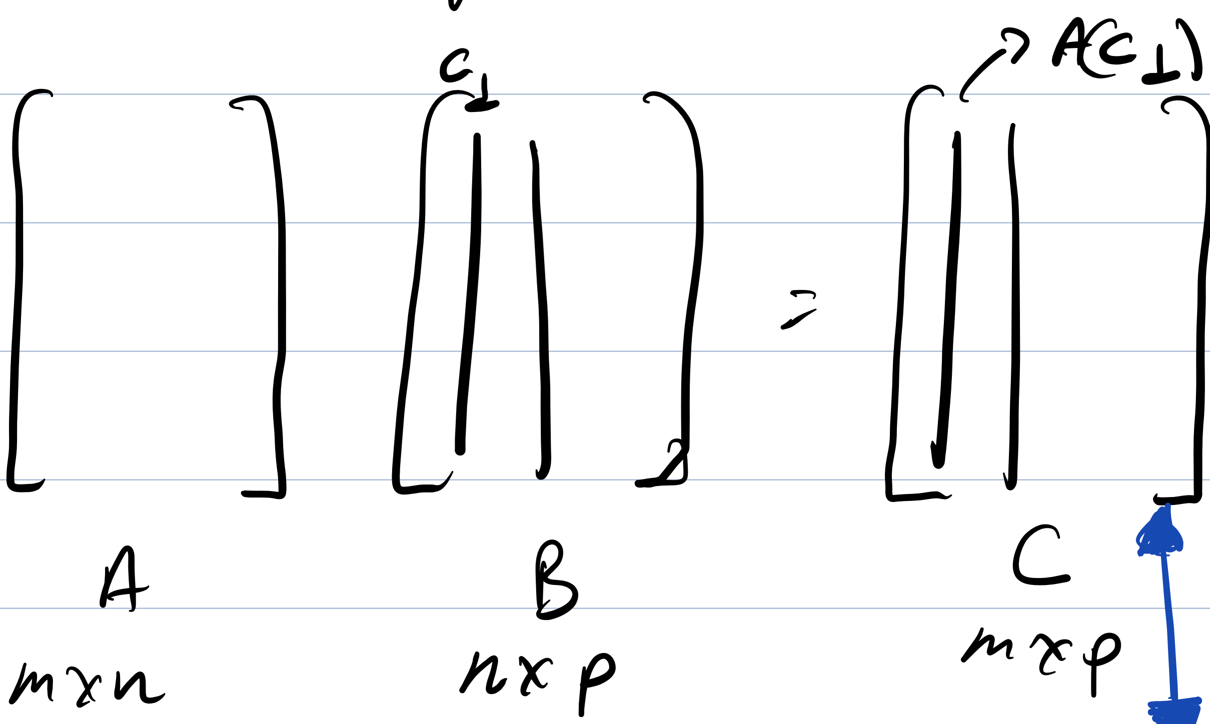
$A = m \times n$ $\quad B = n \times p$ $\quad C = AB = m \times p$

$\Rightarrow C_{34} = (\text{row } 3 \text{ of } A) \cdot (\text{column } 4 \text{ of } B)$

$$= a_{31} b_{14} + a_{32} b_{24} + \cdots$$

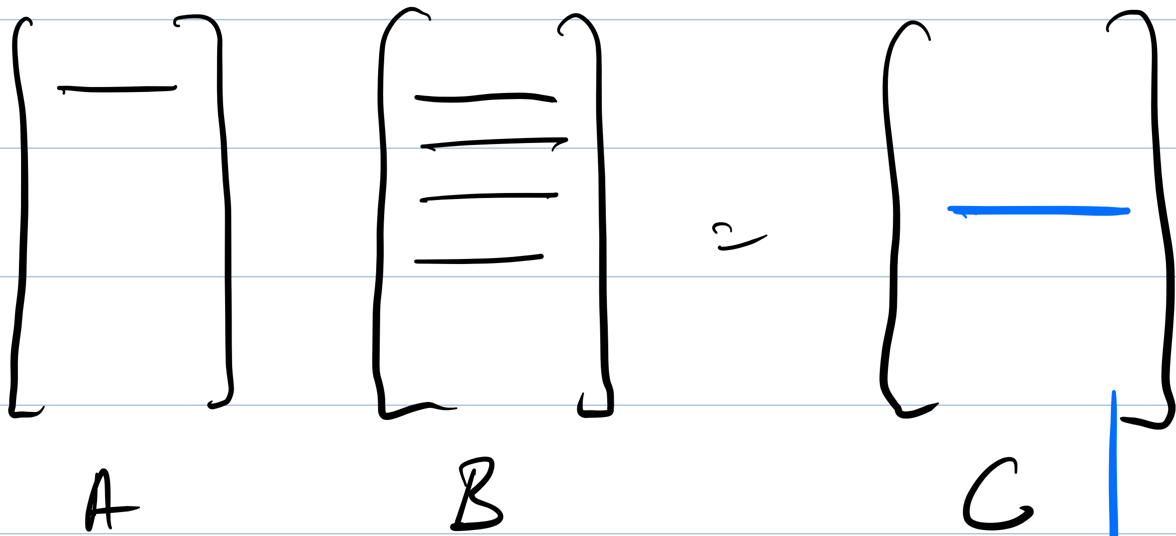
$$= \sum_{k=1}^{n} a_{3\underbrace{k}_{\text{column}}} \, b_{\underbrace{k}_{\text{row}}4}$$

Matrix Multiplication ② [Columns]



columns of C are combinations of <u>columns</u> of <u>A</u>.

Matrix Multiplication ③ (Rows)



rows of C are combinations of rows of B.

Matrix Multiplication ④

[Columns x rows]

* Column of A $\times$ row of B

$$\underbrace{(m \times 1) \times (1 \times p)}$$

$\Rightarrow C = m \times p$

eg

$\rightarrow$ multiples of $\begin{bmatrix} 1 & 6 \end{bmatrix}$

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}$$

multiples columns $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

* special matrix.
All those rows lie
on the line $[1 \ 6]$
[all in the same direction]

$\Rightarrow$ AB = sum of (columns of A)
$\times$ (rows of B)

eg

$$\begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} + \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Blocks (multiplication)

$A_1 B_1 + A_2 B_3$

$$\left[\begin{array}{c|c} A_1 & A_2 \\ \hline & \end{array}\right] \left[\begin{array}{c|c} B_1 & B_2 \\ \hline & \end{array}\right] = \left[\begin{array}{c|c} C_1 & \\ \hline & \end{array}\right]$$

$$\begin{bmatrix} A_3 & A_4 \end{bmatrix} \begin{bmatrix} B_3 & B_4 \end{bmatrix} \begin{bmatrix} & \end{bmatrix}$$

A

(is it invertible or not) ?

Inverses (square matrices)

* Not all matrices have inverses *

$$A^{-1} A = I \iff A A^{-1} = I$$

(invertible or non-singular)

↳ if this matrix exist!

Case ① : (no inverse)

(singular case / no inverse)

$$A = \begin{bmatrix} 1 & 3 \\ & \end{bmatrix}$$

$$\begin{bmatrix} 2 & 6 \end{bmatrix}$$

↳ multiple of 2^{nd} column

⇒ Why does that Matrix have no inverse?

↳

* I can find a vector x with $Ax = 0$.

$$Ax = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} \ \\ \ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

x ↓ $x \neq 0$

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

* CASE 2 – Has inverse

$$\underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a & c \\ b & d \end{bmatrix}}_{A^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I}$$

$\Rightarrow$

$\left\{ \begin{array}{c} A \times \text{column } j \text{ of } A^{-1} \\ = \\ \text{column } j \text{ of } I \end{array} \right\}$

(2 RHS instead of 1)

↳ Gauss - Jordan Elimination

[solve 2 equations at once]

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

pivot

$$\Rightarrow \left[\begin{array}{cc|cc} \boxed{1} & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] \times 2 \quad \text{(augerented matrix)}$$

$$\underset{A}{} \qquad \qquad \underset{I}{}$$

A I

$\updownarrow$

(elimination)

$\therefore$ 1st row × 2 − 2nd row

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & \boxed{1} & -2 & 1 \end{array}\right] \times 3$$

pivot

elmuated(3)

$\therefore$ 1st row

$\downarrow$

$- 3 \times (2^{nd}\ row)$

$$\left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$I \qquad \underline{\underline{A^{-1}}}$

<u>Check</u>

$$\begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1} \times A = I$$

As in the last lecture, we can write the results of the elimination method as the product of a number of elimination matrices E_{ij} with the matrix A. Letting E be the product of all the E_{ij}, we write the result of this Gauss-Jordan elimination using block matrices: $E[\,A \mid I\,] = [\,I \mid E\,]$. But if $EA = I$, then $E = A^{-1}$.