Objectives

- Complex inner products
- Discrete Fast Fourier transform (FFT)

- Matrices with all real entries can have complex eigenvalues!

Complex Vectors

- Length.

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \in \mathbb{C}^n$$

$$\begin{bmatrix} z_1 \end{bmatrix}$$

$$z^T z = [z_1\ z_2 \cdots z_n] \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$$

$\hookrightarrow$ Not good!

why?

$$[1\ \ i]\begin{bmatrix}1\\ j\end{bmatrix} = 0 \Rightarrow$$ we do not want length $= 0$

$\therefore$ Correct!

$$|z|^2 = \bar{z}^T z = |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2$$

Then we have:

$$\left(\text{length}\begin{bmatrix}1\\ i\end{bmatrix}\right)^2 = [1\ \ -i]\begin{bmatrix}1\\ i\end{bmatrix} = 1+1=2$$

$\therefore$ $\rightarrow$ Hermitian!

$|z| = z^{H} z$

- <u>Inner product</u>

- Inner product or dot product of 2 complex vectors is <u>not</u> just $\underline{y^T x}$

$\Rightarrow$ Need complex conjugate of y.

$$y^H x = \bar{y}^T x = \bar{y}_1 x_1 + \bar{y}_2 x_2 + \cdots + \bar{y}_n x_n$$

<u>Hermitian matrices</u>

- Symmetric $A^T = A$

If complex

$$\bar{A}^T = A \Rightarrow \begin{bmatrix} 2 & 3+i \\ 3-i & 5 \end{bmatrix}$$ } hermitian matrices.

<u>Unitary matrices</u>

What does it mean for complex vectors $q_1, q_2, \dots, q_n$ to be perpendicular. (or orthonormal) .?

$\therefore$ Complex space to be orthonormal.

$$\bar{q}_j q_k = \begin{Bmatrix} 0 & j \neq k \\ 1 & j = k \end{Bmatrix}$$

$\therefore$

$$Q = [q_1 \ q_2 \cdots q_n]$$

$$Q^H Q = I$$

Note:

Hermitian = symmetric

unitary = orthogonal.

- A unitary matrix is a square with perpendicular column of unit length.

Discrete Fourier transform

Reminder:

A Fourier series is a way to write a periodic function or signal as a sum of functions of discrete frequencies.

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots .$$

* When working with finite dataset the discrete Fourier Transform is the key to this decomposition.

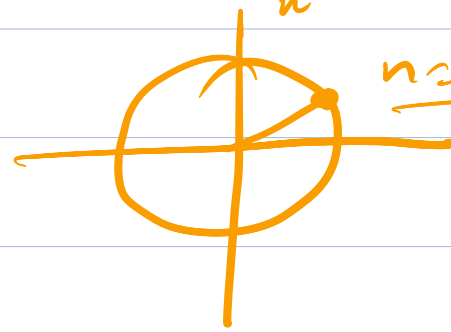
In electrical engineering and computer science, the rows and columns of a matrix are numbered starting with 0, not 1 (and ending with $n-1$, not n). We'll follow this convention when discussing the Fourier matrix:

$$\begin{array}{cc} F_n = & \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & & w^{n-1} \\ 1 & w^2 & w^4 & & w^{2(n-1)} \\ \vdots & & & \ddots & \vdots \\ 1 & w^{n-1} & w^{2(n-1)} & \cdots & w^{(n-1)^2} \end{bmatrix} . \\ & \begin{matrix} 0 & \quad 1 & \quad\quad 2 & \quad\cdots & \quad n-1 \end{matrix} \end{array}$$

$(F_n)_{ij} = (\omega)^{ij}$ $\quad : 2\pi/n$

$\Rightarrow w^n = 1 \quad \therefore w = e^{i2\pi/n}$

$i, j = 0, \ldots, n-1$ $\quad = \cos\frac{2\pi}{n} + i\sin(2\pi/n)$



$n=6$

eg

Because $w^4 = 1$ and $w = e^{2\pi i/4} = i$, our best example of a Fourier matrix is:

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}.$$

$\underbrace{1+1+1+1}$

* Columns orthogonal.

$\downarrow$

not 0

but columns are perpendicular!

$\Rightarrow$ conjugate! Here $= 0$.

$\therefore \frac{1}{4} F_4^H F_4 = I$

Fast Fourier transform

Fourier matrices can be broken down into chunks with lots of zero entries; Fourier probably didn't notice this. Gauss did, but didn't realize how significant a discovery this was.

There's a nice relationship between F_n and F_{2n} related to the fact that $w_{2n}^2 = w_n$:

$$F_{2n} = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_n & 0 \\ 0 & F_n \end{bmatrix} P,$$

where D is a diagonal matrix and P is a $2n$ by $2n$ permutation matrix:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ & & & \vdots & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ & & & \vdots & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

So, a $2n$ sized Fourier transform F times $\mathbf{x}$ which we might think would require $(2n)^2 = 4n^2$ operations can instead be performed using two size n Fourier transforms ($2n^2$ operations) plus two very simple matrix multiplications which require on the order of n multiplications. The matrix P picks out the even components $x_0, x_2, x_4, \ldots$ of a vector first, and then the odd ones – this calculation can be done very quickly.

Thus we can do a Fourier transform of size 64 on a vector by separating the vector into its odd and even components, performing a size 32 Fourier transform on each half of its components, then recombining the two halves through a process which involves multiplication by the diagonal matrix D.

$$D = \begin{bmatrix} 1 & & & & \\ & w & & & \\ & & w^2 & & \\ & & & \ddots & \\ & & & & w^{n-1} \end{bmatrix}.$$

Of course we can break each of those copies of F_{32} down into two copies of F_{16}, and so on. In the end, instead of using n^2 operations to multiply by F_n we get the same result using about $\frac{1}{2}n \log_2 n$ operations.

A typical case is $n = 1024 = 2^{10}$. Simply multiplying by F_n requires over a million calculations. The fast Fourier transform can be completed with only $\frac{1}{2}n \log_2 n = 5 \cdot 1024$ calculations. This is 200 times faster!

This is only possible because Fourier matrices are special matrices with orthogonal columns. In the next lecture we'll return to dealing exclusively with real numbers and will learn about positive definite matrices, which are the matrices most often seen in applications.