Objectives

- Eigenvalues and eigenvectors
- $\det[A - \lambda I] = 0$

TRACE $= \lambda_1 + \lambda_2 + \dots + \lambda_n$

→ What is an eigenvector?

↳ What does a matrix do

⇒ Acts on vectors / multiplies x

{ Ax parallel to x } ⇒ eigenvectors.

Other words, $\boxed{Ax = \lambda x}$

[same direction, allowing also $-$ve λ].

* If Eigenvalue $= 0$

$\Rightarrow Ax = 0x = 0$ $\therefore$ Vectors with eigenvalue 0 makes up nullspace.

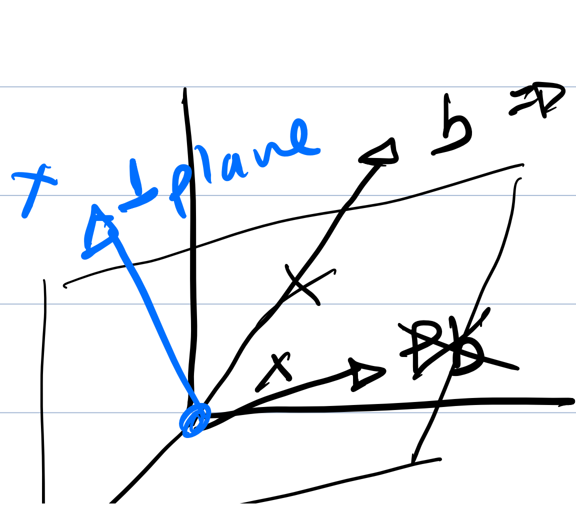
; If A is singular, then $\lambda = 0$ is an eigenvalue of A

- How do we find the values of λ?

Example:

- eg projection matrix.

↳ what are x's and λ's for projection matrix?



$\Rightarrow$ Not an eigenvector. since Pb is in different direction.

* Any x in plane: $Px = x$
x is an eigenvector $\underline{\lambda = 1}$.

* Any $x \perp$ plane: $Px = 0x$
eigenvalue $\underline{\lambda = 0}$

<u>FACT!</u>

Sum of λ's = sum down diagonal of matrix.
$[a_{11} + a_{22} + \ldots + a_{nn}]$

- How to solve $Ax = \lambda x$
- Rewrite: $\underline{(A - \lambda I)}\, x = 0$

In order for

λ to be eigenvector $\Rightarrow$ SINGULAR :

$$\boxed{\det(A - \lambda I) = 0.}$$

characteristic equation !

- [Find λ first] *Note : if λ is not distinct we have one or more repeated eigenvalues !

$\hookrightarrow$ Once, we've found an eigenvalue λ, we can use elimination to find the nullspace of $A - \lambda I$.
The vectors in that nullspace are eigenvectors of A with eigenvalue λ.

Example :

Let $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

Then $\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix}$

$$= (3-\lambda)^2 - 1$$

$$= \lambda^2 - 6\lambda + 8$$

$\Rightarrow \lambda^2 - 6\lambda + 8 = 0$ $\Big\}$ $[2 \times 2]$

8 $\Rightarrow$ determinant.

6 $\Rightarrow$ TRACE

$(\lambda - 4)(\lambda - 2)$

$\lambda = 4, \; \lambda = 2 \Rightarrow$ eigenvalues.

$\therefore A - 4I = \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix}$ $\Big\}$ Singular? ✓

$\therefore X = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \lambda_1 = 4.$

$\therefore A - 2I = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$ Singular? ✓

$x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \lambda_2 = 2.$

- A caution! $(A+B$, or $A \cdot B)$ ☆

IF

$Ax = \lambda x$ & B has eigenvalues α_1,

$By = \alpha y.$

$Bx = \alpha x$

~~$(A+B)x = (\lambda+\alpha)x$~~ × Wrong!

☆ Eigenvalues are not linear & do not multiply!

Complex eigenvalues

The matrix $Q = \begin{array}{c} \begin{array}{cc} \cos & -\sin \end{array} \\ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ \begin{array}{cc} \sin & \cos \end{array} \end{array}$

rotates every vector in the plane by 90°

→ Eigenvalues & Eigenvectors ?

trace $= 0 + 0 = \lambda_1 + \lambda_2$

det $= 1 = \lambda_1 \cdot \lambda_2$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1$$

$\det(A - \lambda I) = 0$

$\therefore \lambda_1 = i \ , \lambda_2 = -i$ } complex.

* If a matrix has complex eigenvalues $a + bi$ then the complex conjugate $a - bi$ is also an eigenvalue of that matrix.

⇒ Symmetric matrices have real eigenvalues.

⇒ Antisymmetric matrices like Q, for which $A^T = -A$, all eigenvalues are imaginary ($\lambda = bi$)

Triangular matrices and repeated eigenvalues

eg

$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ Triangular

$\therefore$

$\det(A - \lambda \det I) = \begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix}$

$= (3-\lambda)(3-\lambda) \}$ repeated!

$\lambda_1 = 3 \qquad \lambda_2 = 3$

$\therefore \quad (A - \lambda I)x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x = 0$

To get $x_1 = [\ 1\ ]$

- There is no independent eigenvector x_2