Factorization into $A = LU$

Objectives

- Inverse of AB, A^T
- Product of elimination matrices.
  $A = LU$ [no row exchanges]

Inverse of product

$$AA^{-1} = I = A^{-1}A$$

$$(AB)\cdot(B^{-1}A^{-1}) = I$$

inverse

Why?

Move parantheses around

$B^{-1} A^{-1} A B = I$

Transpose of a product

$A A^{-1} = I$

$(A^{-1})^T (A)^T = I$

transpose: exchange its rows and columns

This is the inverse of $(A^T)^{-1}$

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det matrix A

$$A = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}$$

cannot put a 4 because then multiple

(: non-singular)

$$E_{21} \quad A = U$$

$$\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

$A = L$ = lower triangular U = upper triangular

$$\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

(Inverse of E_{21})

$$\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$$

$$\underbrace{E_{32}\,E_{31}\,E_{21}}_{\text{elimination.}}\,A = U \quad (\text{no row exchange})$$

$$A = \underbrace{E_{21}^{-1}\,E_{31}^{-1}\,E_{32}^{-1}}_{L = \text{product of inverses}}\,U$$

Suppose:

$$\overset{E_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}} \cdot \overset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$EA = U$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \end{bmatrix} = E$$

$$\begin{bmatrix} 10 & -5 & 1 \end{bmatrix}$$

<u>inverse</u> (reverse order)

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = L$$

$A = LU$

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↳ If no row exchanges, multiples go directly into L

- How expensive is elimination?

↳ How many operations on $m \times n$ matrix A?

($\cancel{n}$, $\cancel{n^2}$, n^3, $\cancel{n!}$) (steps)

↓ multiply + subtract

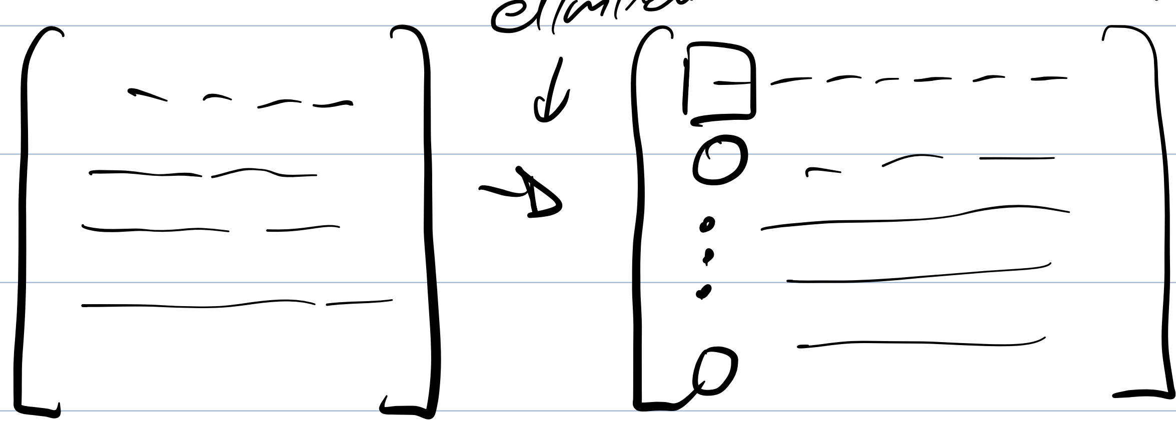
* m bigger → more data but more expensive.

$\approx (n^2 + (n-1)^2 + \dots 1^2)$

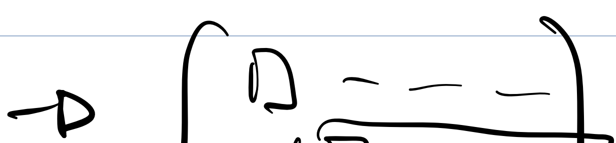
- lets say $n = 100$

cost $\approx 100^2$

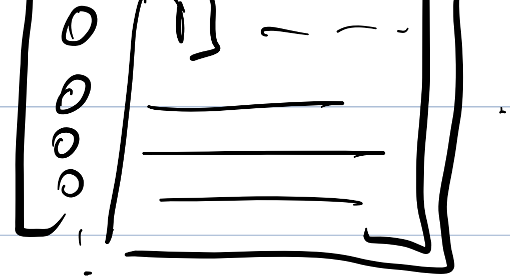
elimination



100×100 99×99



$\approx 99^2$



99x98

Count

$$n^2 + \dots + 1^2 \approx \boxed{\frac{1}{3}n^3}$$

$$1^2 + 2^2 + \cdots + (n-1)^2 + n^2 = \sum_{i=1}^{n} i^2 \approx \int_0^n x^2\,dx = \frac{1}{3}n^3.$$

Cost of b $\Rightarrow n^2$

Row-exchange

* Permutations: (3×3) 6 P's

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ldots$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$\boxed{\text{* } \underline{\text{Fact}} \quad P^{-1} = P^{T}}$

$4 \times 4 \rightarrow 24$ P's