Solve the differential equation

$$y''' + 2y'' - y' - 2y = 0$$

for the general solution.

- What is the matrix A?
- Find the first column of $\exp(At)$.

<u>Solution</u>

$$\underbrace{\begin{bmatrix} y''' \\ y'' \\ y' \end{bmatrix}}_{u'(t)} = \underbrace{\begin{bmatrix} -2 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} y'' \\ y' \\ y \end{bmatrix}}_{u(t)}$$

$\downarrow$ eigen-values & eigenvectors

$\therefore \det(A - \lambda I)$

$$= \det \begin{pmatrix} -2-\lambda & 1 & 2 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{pmatrix}$$

$$= (1-\lambda)(1+\lambda)(2+\lambda)$$

$\lambda_1 = 1 \qquad \lambda_2 = -1 \qquad \lambda_3 = -2$

$\lambda_1 =$

$$(A - I)\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 = \begin{pmatrix} -3 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$\Rightarrow a = b = c$

$x_1 = (1, 1, 1)^T$

$-\lambda_2 = -1$

$$x_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$-\lambda_3 = -2$

$$x_3 = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$$

<u>General sol</u>

$$u(t) = c_1 e^{t} x_1 + c_2 e^{-t} x_2 + c_3 e^{-2t} x_3$$

$$y(t) = c_1 e^{t} + c_2 e^{-t} + c_3 e^{-2t}.$$

(2)

$$\exp(At) = S e^{\Lambda t} S^{-1}$$

$$S = [x_1 \; x_2 \; x_3] = \begin{bmatrix} 1 & 1 & 4 \\ 1 & -1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$e^{\Lambda t} = \begin{bmatrix} e^{t} & & \\ & e^{-t} & \\ & & e^{-2t} \end{bmatrix}$$

$$\exp(At) = [e^{t} x_1 \;\; e^{-t} x_2 \;\; e^{-2t} x_3]$$

$$\begin{bmatrix} 1/6 & \cdots \\ -1/2 & \cdots \\ 1/3 & \cdots \end{bmatrix}$$

$$S^{-1} = \frac{1}{\det S} C^{T}$$

$$= \frac{1}{6} \begin{bmatrix} 1 & \cdot & 0 \\ -3 & \cdot & 0 \\ 2 & \cdot & \cdot \end{bmatrix}$$

cofactors

$\Rightarrow \therefore \exp(At)$

$$= \left[\frac{e^{t}}{6} x_1 - \frac{e^{-t}}{2} x_2 + \frac{e^{-2t}}{3} x_3 \right]$$