Objectives

*Applications of eigenvalues & eigenvectors.

- Markov matrices
- Steady state
- Fourier series & Projections

Let A = markov matrix

$$A = \begin{bmatrix} .1 & .01 & .3 \\ .2 & .99 & .3 \\ .7 & .0 & .4 \end{bmatrix}$$

2 properties:

1) All entries ≥ 0
2) All columns add to 1.

Key points

1. $\lambda = 1$ is an eigenvalue
2. All other eigenvalues $|\lambda_i| < 1$

Reminder:

$$u_k = A^k u_0 = c_1 \underbrace{\lambda_1^k}_{=1} x_1 + c_2 \lambda_2^k x_2 + \ldots$$

$\lambda_2 < 1$

→ steady state: $c_1 \underbrace{x_1}_{\text{eigenvector}}$

$\underline{\underline{x_1 \geq 0}}$

- If 1 is an eigenvalue, then:

$$\begin{bmatrix} -.9 & .01 & .3 \end{bmatrix}$$

$$A - 1I = \begin{bmatrix} .2 & -.01 & .3 \\ .7 & 0 & -.6 \end{bmatrix}$$

singular?

→ All columns of A−I add to 0

↳ A−I is singular!

<u>why??</u>

↳ <u>rows</u> are dependent. (How?.)

because $(1,1,1)$ is in $N(A^T)$

(nullspace)

↳ <u>columns</u> is in $N(A) \to x_1$.

↳ eigenvector!

→ $\det(A - \lambda I) = 0$

eigenvalues of A
eigenvalues of A^T] = same.

$\rightarrow \det(A' - \lambda I) = 0$

$u_{k+1} = A u_k$ $\quad$ A is Markov.

For example:

$$\begin{bmatrix} u_{\text{Cal}} \\ u_{\text{Mass}} \end{bmatrix}_{t=k+1} = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \begin{bmatrix} u_{\text{Cal}} \\ u_{\text{Mass}} \end{bmatrix}_{t=k}$$

Initial conditions:

$$\begin{bmatrix} u_{cal} \\ u_{Mass} \end{bmatrix}_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix}$$

After one move, $u_1 = A u_0$

$$\begin{bmatrix} u_{\text{Cal}} \\ u_{\text{Mass}} \end{bmatrix}_1 = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \begin{bmatrix} 0 \\ 1000 \end{bmatrix} = \begin{bmatrix} 200 \\ 800 \end{bmatrix}.$$

To understand the long term behavior of this system, we will need the eigenvectors and eigenvalues of

$$\begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \Rightarrow \text{TRACE} = 0.9 + 0.8 = 1.7$$

↳ we already know that $\lambda_1 = 1$ is an eigenvalue.

$\therefore \text{TRACE} - \lambda_1 = .7 = \lambda_2$

<u>Eigenvectors</u>

$$A - \lambda_1 I = \begin{bmatrix} -.1 & .2 \\ .1 & -.2 \end{bmatrix} x_1 = 0,$$

$\therefore x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \underline{2/3 + 1/3}$

$\Rightarrow$ The eigenvalue $\lambda_1 = 1$ = steady state sol.

& $\lambda_2 = .7 < 1$

$\therefore$ The system approaches a limit in which 2/3 of 1000 people live in california & 1/3 in Mass.

$\Leftrightarrow$ eigenvector of $\lambda_2 = .7$:

$$A - \lambda_2 I = \begin{bmatrix} .2 & .2 \\ .1 & .1 \end{bmatrix} x_1 = 0$$

so let $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

From what we have learned about difference equations:

$$u_k = c_1 \cdot 1^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 (0.7)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

when $k=0$, we have:

$$u_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

so $c_1 = \frac{1000}{3}$, $c_2 = \frac{2000}{3}$

Markov matrices are useful in electrical engineering

Fourier series & projections

Expansion with an orthonormal basis $q_1, q_2, q_3 \ldots, q_n$

Then we can write any v as:

$v = x_1 q_1 + x_2 q_2 + \dots + x_n q_n$ where

$$q_1^T v = x_1 \cancelto{1}{q_1^T q_1} + 0 + 0 \dots$$

$$\begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ | \\ x_n \end{bmatrix} = v$$

$$Q x = v.$$

$$\therefore \; x = Q^{-1} v = Q^T v$$

$$x_1 = q_1^T v$$

Fourier series.

infinite sum

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots$$

- Fourier series is an infinite sum and the previous example was finite BUT, are related because cosines & sines in Fourier series are orthogonal.

- Since infinite dimensional vector space.

  So :. Vectors in this space are functions & the (orthogonal) basis vector are $1, \cos x, \sin x, \cos 2x, \sin 2x \ldots$

* what does orthogonal mean?

$$f^T g = \int_0^{2\pi} f(x)g(x)\,dx.$$

We integrate from 0 to 2π because Fourier series are periodic:

$$f(x) = f(x+2\pi).$$

The inner product of two basis vectors is zero, as desired. For example,

$$\int_0^{2\pi} \sin x \cos x\,dx = \frac{1}{2}(\sin x)^2\Big|_0^{2\pi} = 0.$$

How do we find a_0, a_1, etc. to find the coordinates or *Fourier coefficients* of a function in this space? The constant term a_0 is the average value of the function. Because we're working with an orthonormal basis, we can use the inner product to find the coefficients a_i.

$$\begin{aligned}\int_0^{2\pi} f(x)\cos x\,dx &= \int_0^{2\pi}(a_0 + a_1\cos x + b_1 \sin x + a_2\cos 2x + \cdots)\cos x\,dx \\ &= 0 + \int_0^{2\pi} a_1 \cos^2 x\,dx + 0 + 0 + \cdots \\ &= a_1\pi.\end{aligned}$$

We conclude that $a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x\,dx$. We can use the same technique to find any of the values a_i.