Objectives

- Symmetric matrices * $A = A^T$ *
Eigenvalues / Eigenvectors
- START: Positive Definite Matrices

- Symmetric matrices

$A^T = A$

① eigenvalues are REAL
② eigenvectors are PERPENDICULAR (ORTHOGONAL)
(can be chosen)

- Usual case: $A = S\Lambda S^{-1}$

- Symmetric case : $A = Q \Lambda Q^{-1}$

$\Updownarrow$ $\boxed{= Q \Lambda Q^T}$

orthonormal eigenvectors $=$ column of Q.

spectral theorem

- Real eigenvalues

$$Ax = \lambda x \xRightarrow{\text{always}} \bar{A}\bar{x} = \bar{\lambda}\bar{x}$$

① $\underline{Ax = \lambda x}$ $\quad \times \bar{x}^T$ $\Rightarrow \bar{x}^T \bar{A}^T = \bar{x}^T \bar{\lambda}$

$\bar{x}^T A x = \lambda \bar{x}^T x.$ $\quad \Downarrow$ if symmetric

$= \bar{A}$

$\therefore$ Multiply both sides by x :

②

$$\underline{\bar{x} A \bar{x} = \bar{\lambda} \bar{x}^T \bar{x}}$$

Comparing the two equations we can see that

$$\bar{x}^T \bar{\jmath}\, x = \bar{x}^T \jmath\, x$$

Length square

Unless $\bar{x}^T x = 0$ THEN:

$\underline{\underline{\jmath = \bar{\jmath} \Rightarrow \text{real}}}$

How do we know that $\bar{x}^T x \neq 0$

$$\bar{x}^T x = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \dots & \bar{x}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$= \bar{x}_1 x_1 + \bar{x}_2 x_2 + \dots$

$\downarrow$

$(a-ib)(a+ib)$

$= a^2+b^2$

$\Rightarrow |x_1|^2 + |x_2|^2 + \dots + |x_n|^2$

<u>Good matrices:</u> $\longrightarrow \quad A = \bar{A}^T$

- real λ's
- Perpendiculer x's

$\rightarrow$ If real here: $A = A^T \Rightarrow$

symetric matrices.

Projection onto eigenvectors

If $A = A^T$ we can write:

$$A = Q \Lambda Q^T$$

$$= \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix}$$

$$= \lambda_1 \boxed{q_1 q_1^T} + \lambda_2 q_2 q_2^T + \cdots + \lambda_n q_n q_n^T$$

projection matrix.

- Every symmetric matrix is a combination of perp projection matrices.

Note: For symmetric matrices
$A = A^T$

number of positive pivots
=
number of positive eigenvalues.

Because the eigenvalues of $A + bI$ are just b more than the eigenvalues of A, we can use this fact to find which eigenvalues of a symmetric matrix are greater or less than any real number b. This tells us a lot about the eigenvalues of A even if we can't compute them directly.

Positive definite matrices

↳ is a symmetric matrix. A for which all eigenvalues are positive.
↳ pivots are positive.

All subdeterminants are positive.

Let $A = \begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}$

pivots of matrix are: 5 & $\frac{\det(A)}{5}$

$= 11/5$

→ Eigenvalues: $\lambda^2 - 8\lambda + 11 = 0$

$\lambda_1 = 4 + \sqrt{5}$
$\lambda_2 = 4 - \sqrt{5}$

determinant?

Not

$\begin{bmatrix} 0 & -3 \end{bmatrix}$ -ve pivots

-ve eigenvalues.