Exercises on factorization into $A = LU$

Problem 4.1:

What matrix E puts A into triangular form $EA = U$?
Multiply by $E^{-1} = L$ to factor A into LU

pivot

$$A = \begin{bmatrix} \boxed{1} & 3 & 0 \\ 2 & 4 & 0 \\ 2 & 0 & 1 \end{bmatrix} \xrightarrow{(\times 2)} \begin{bmatrix} 1 & 3 & 0 \\ 0 & -2 & 0 \\ \boxed{2} & 0 & 1 \end{bmatrix}$$

$(\times 2)$

$\therefore$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & \boxed{-2} & 0 \\ 0 & \boxed{-6} & 1 \end{bmatrix} (\times 3)$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$\boxed{\rightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = U}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$$\therefore E_{32}E_{31}E_{21} \} E$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix} = E$$

⇒ Now we can check that $EA = U$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

✓ correct.

To find now the inverse of E^{-1} we need to use the Gauss-Jordan elimination method

$$\left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 0 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ 4 & -3 & 1 & 0 & 0 & 1 \end{array}\right] \times 2$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} = E^{-1}$$

* check if the inverse is correct! *

$$EE^{-1} = I$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

* Finally factorise A into ALU
where $\underline{L = E^{-1}}$

$$\begin{bmatrix} 1 & 3 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 0 \\ 2 & 0 & 1 \end{bmatrix} = A = LU = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 4.2

(2.6 #13, Introduction to Linear Algebra: Strang)

- Compute L and U for the symmetric matrix

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

- Find the 4 conditions on a, b, c, d

to get $A = LU$ with 4 pivots.

$\Rightarrow$ **Solution:**

E_{21}

$$\Rightarrow \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

E_{31}

$$\begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ a & b & c & d \end{bmatrix}$$

E_{41}

$$\begin{bmatrix} a & a & a & a \\ 0 & \boxed{b-a} & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix}$$

E_{32}

$$\begin{bmatrix} a & a & a & a \\ 0 & 0 & b-a & b-a \\ 0 & 0 & \boxed{c-b} & c-b \\ 0 & 0 & c-b & d-b \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a & a & a & a \\ 0 & 0 & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix} = U$$

$\therefore$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

so 4 conditions

$$\boxed{a \neq 0,\ b \neq a,\ c \neq b,\ d \neq c}$$