Objectives

- Positive Definite Matrix (Tests)
- Test for Minimum ($x^TAx > 0$)
- Ellipsoids in R^n

Positive definite matrices

Given a symmetric 2x2

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

Test if positive definite:

1. Eigenvalue test: $\lambda_1 > 0, \lambda_2 > 0$.
2. Determinants test: $a > 0, ac - b^2 > 0$.
3. Pivot test: $a > 0, \dfrac{ac - b^2}{a} > 0$.
4. $\mathbf{x}^T A\mathbf{x}$ is positive except when $\mathbf{x} = \mathbf{0}$ (this is usually the definition of positive definiteness).

eg

(det test)

$$\begin{bmatrix} 2 & 6 \\ 6 & y \end{bmatrix} \to 2y - 36 > 0$$

$\therefore = 19$ $\qquad y > 18$

if $y = 18 \to$ borderline $\therefore$
(positive semi-definite matrix)

$$\left.\begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}\right\} \text{singular} \therefore 1 \text{ pivot}$$

rank = 1

$\left(\begin{matrix} 2 \\ 6 \end{matrix} \times 3 = \begin{matrix} 6 \\ 18 \end{matrix} \right)$

$\lambda_1 = 0$
$\lambda_2 = 20 \ (2+18)$

$$x^T A x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2x_1 + 6x_2 \\ 6x_1 + 18x_2 \end{bmatrix}$$

$$= 2x_1^2 + 12x_1x_2 + 18x_2^2$$

$$= ax_1^2 + 2bx_1x_2 + cx_2^2$$

quadratic form.

$$cy^2 > 0 \ ?$$

<u>Test for minimum</u>

$$\begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix} \Big\} \ \underline{\underline{\text{not}}} \text{ positive definite.}$$

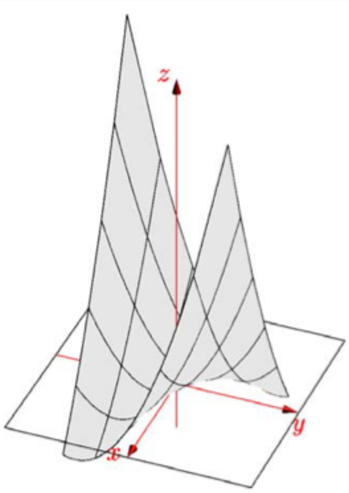

Figure 1: The graph of $f(x, y) = 2x^2 + 12xy + 7y^2$.

Now,

Matrix:

$$\begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix}$$ positive definite!

$$\left.\begin{array}{l} \det = 4 \\ \text{TRACE} = 22 \end{array}\right\}$$ so eigen values are positive.

$\therefore f(x,y) = 2x^2 + 12xy + 20y^2$

positive, except $x = y = 0$.

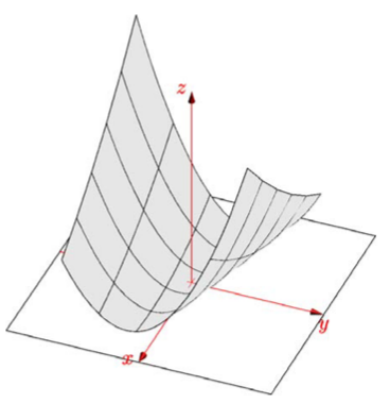

Figure 2: The graph of $f(x, y) = 2x^2 + 12xy + 20y^2$.

* The first derivatives f_x & f_y of this function are zero, so its graph is tangent to the x-y plane at $(0,0,0)$

* As in calculus: MIN $\sim \frac{d^2u}{dx^2} > 0$

$\left(\frac{du}{dx} = 0\right)$

Now, in Linear Algebra:

$f(x_1, x_2, \dots, x_n)$

MIN $\sim$ MATRIX OF 2^{nd} DERIV. is positive definite

→ We can prove that $2x^2+12xy+20y^2$ is always positive by writing it as sum of squares.

Completing the square:

$$2x^2+12xy+20y = 2(x+3y)^2+2y^2$$

Note:

$$2(x+3y)^2 = 2x^2+12xy+\underline{18}y^2$$

"borderline" between passing & failing the test for positive definiteness

When complete the square for $2x^2 + 12xy + 7y^2$ we get

$\hookrightarrow 2x^2 + 12xy + 7y^2 = 2(x+3y)^2 - 11y^2$

Which may be negative eg:

$x = -3$ & $y = 1$

→ The coefficients that appear when completing the square are exactly the entries that appear when performing <u>elimination</u> on the original matrix.

$$\begin{array}{c} A \\ \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \end{array} \xrightarrow{\text{subtract } 3\times \text{ row } 1} \begin{array}{c} U \\ \begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix} \end{array}$$

we can see the terms that appear when completing the square is:

$$\begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \quad \& \quad L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

* If the pivots are all positive then the sum of squares will always be positive. !

Hessian Matrix

The matrix of 2nd derivatives of $f(x,y)$ is:

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

- The matrix is symmetric because $f_{xy} = f_{yx}$.

- Test for minimum

$$f_{xx} f_{yy} > f_{xy}^2$$

$n \times n$ matrix

eg

A 3 by 3 example:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

$$\det[\,2\,] = 2, \qquad \det \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 5, \qquad \det \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = 4.$$

- The pivots of A are 2, 3/2 & 4/3 because the products of the pivots equal the determinants.

- The eigenvalues of A are +ve & product is 4.

$$\left(2-\sqrt{2},\ 2,\ 2+\sqrt{2}\right)$$

Ellipsoids in R^n

$$f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3.$$

Because A is positive definite, we expect $f(x)$ to be positive except when $x=0$

Its graph is sort of 4D bowl or paraboloid.

Just as an ellipse has a major and minor axis, an ellipsoid has three axes. If we write $A = Q\Lambda Q^T$, as the principal axis theorem tells us we can, the eigenvectors of A tell us the directions of the principal axes of the ellipsoid. The eigenvalues tell us the lengths of those axes.