Objectives

- Diagonalizing a matrix $S^{-1}AS = \Lambda$
- Powers of A / equation $u_{k+1} = Au_k$.

* Suppose n indep. eigenvectors of A.
Put them in columns of S

$$
\begin{aligned}
AS &= A\begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1\mathbf{x}_1 & \lambda_2\mathbf{x}_2 & \cdots & \lambda_n\mathbf{x}_n \end{bmatrix} \\
&= S\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} = S\Lambda.
\end{aligned}
$$

columns of S are independent

$\Rightarrow$ Λ: is a diagonal matrix whose non-zero entries are the eigenvalues of A.

⇒ Because S are independent, S^{-1} exists and we can multiply both sides of $AS = S\Lambda$ by S^{-1}

$$\boxed{\begin{array}{c} S^{-1}AS = \Lambda \\ \parallel \\ A = S\Lambda S^{-1} \end{array}}$$

Powers of A

If. $Ax = \lambda x$

$$A^2 x = \lambda Ax = \lambda^2 x$$

⇒ ∴ Eigenvalues of A^2 are the same of the eigen values of A.

⇒ Eigenvectors of A^2 are the same as the eigenvectors of A.

Now, if we write:

$$A = S \Lambda S^{-1} \text{ then}$$

$$\underbrace{A^2 = S \Lambda S^{-1} S \Lambda S^{-1} = S \Lambda^2 S^{-1}}$$

GENERAL ⇓

$$A^k = S \Lambda^k S^{-1}$$

Theorem:

If A has n independent eigenvectors with eigenvalues λ_i

then $A^k \to 0$ as $k \to \infty$ if and only if all $|\lambda_i| < 1$.

Repeated eigenvalues

A is sure to have n independent eigenvectors
(and be diagonalizable)
if all the λ's are different
(no repeated λ's)

* If repeated eigenvalues //
may or may not have n independent
eigenvectors

$\Rightarrow$ Triangular matrix

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

eigenvalues are 2 & 2

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix}$$

$$\Rightarrow (2-\lambda)^2 \Rightarrow \lambda = 2$$

$$\lambda = 2$$

$\Rightarrow$ eigenvectors

$$A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Big\} \text{ only } 1$$

$\Rightarrow$ does not have 2 independent eigenvectors.

Difference equation $u_{k+1} = Au_k$

$\Rightarrow$ Start with given vector u_0.

$u_1 = Au_0$
$u_2 = Au_1 = AAu_0 = A^2 u_0$

$\Rightarrow \boxed{u_k = A^k u_0}$

$\Rightarrow$ To really solve:

(S1) Write $u_0 = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$

(S2) $Au_0 = c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 + \dots + c_n \lambda_n x_n$

(S3)

$u_k = A^k u_0 = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 + \dots +$

$$c_n \lambda_n^k x_n = \Lambda^k S c$$

Fibonacci sequence

Fibonacci sequence:

$0, 1, 1, 2, 3, 5, 8, 13$

$\hookrightarrow$ In general: $\left[\begin{array}{l} \boxed{F_{k+2} = F_{k+1} + F_k} \\ F_{k+1} = F_{k+1} \end{array}\right]$

Trick!

Let

$$u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

$$u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u_k$$

A = symmetric

$\Rightarrow \therefore$ eigenvalues $=$ real & eigenvectors $=$ orthogonal

- Because A is $[2\times 2]$, we know that its eigenvalues sum to 1 (Trace) & their product is -1 (determinant)

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1 = 0$$

$$\lambda = \frac{1 \pm \sqrt{1+4}}{\ldots}$$

$\therefore A - \lambda I$

$$\frac{\quad}{2}$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So

$$\lambda_1 = \frac{1}{2}(1 + \sqrt{5}) \simeq 1.618 \qquad x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = \frac{1}{2}(1 - \sqrt{5}) \approx -0.618 \qquad x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

So

$$F_{100} \simeq c_1 \left(\frac{1+\sqrt{5}}{2}\right)^{100}$$

* eigenvalues controll the growth
Lo(λ_1) with absolute value
greater than 1.

Finally, $\mathbf{u}_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ tells us that $c_1 = -c_2 = \frac{1}{\sqrt{5}}$.
Because $\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = \mathbf{u}_k = c_1\lambda_1^k x_1 + c_2\lambda_2^k x_2$, we get:

$$F_k = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^k.$$

Using eigenvalues and eigenvectors, we have found a *closed form expression* for the Fibonacci numbers.

Summary: When a sequence evolves over time according to the rules of a first order system, the eigenvalues of the matrix of that system determine the long term behavior of the series. To get an exact formula for the series we find the eigenvectors of the matrix and then solve for the coefficients $c_1, c_2, \ldots$