Solving $Ax=b$

Find all solutions, depending on b_1, b_2, b_3:

$$x - 2y - 2z = b_1$$
$$2x - 5y - 4z = b_2$$
$$4x - 9y - 8z = b_3$$

Solution:

$$A = \left[\begin{array}{ccc|c} 1 & -2 & -2 & b_1 \\ 2 & -5 & -4 & b_2 \\ 4 & -9 & -8 & b_3 \end{array}\right] \quad (-2) \quad (-4)$$

Elimination:

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & -2 & b_1 \\ 0 & -1 & 0 & -2b_1 + b_2 \\ 0 & -1 & 0 & -4b_1 + b_3 \end{array}\right]$$

$\Rightarrow$ Further elimination.

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & b_1 \\ 0 & -1 & 0 & -2b_1 + b_2 \\ 0 & 0 & 0 & -2b_1 - b_2 + b_3 \end{array}\right]$$

use for elim

$\times(-1)$

* If $-2b_1 - b_2 + b_3 \neq 0$

$\hookrightarrow$ NO SOLUTIONS.

* If $-2b_1 - b_2 + b_3 = 0$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 5b_1-2b_2 \\ 0 & 1 & 0 & 2b_1-b_2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

pivot variables (x, y)

free variable z

→ Particular solution:

- $Ax = b$
- $z = 0 \Rightarrow x_p = \begin{bmatrix} 5b_1 - 2b_2 \\ 2b_2 - b_2 \\ 0 \end{bmatrix}$

→ Special solution.

- $Ax = 0$
- $z = 1$ (only 1 free variable)

$$x_s = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

All solutions

$$\vec{x} = \vec{x_p} + c\,\vec{x_s}$$