Objectives

- $A^T A$ is positive definite!
- Similar Matrices A, B / Jordan Form
  $B = M^{-1} A M$

* **Reminder:**

- positive definite means
  $x^T A x > 0$ (except for $x = 0$)

$A^T A$ is positive definite

↳ Very important class of matrices.

→ appear in the form of $A^T A$ when computing least squares solutions.

→ Given a symmetric positive definite

matrix A, is its inverse also symmetric and positive definite?
Yes, because if the (positive) eigenvalues of A are $\lambda_1, \lambda_2, \dots, \lambda_d$ then the eigenvalues $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \dots, \frac{1}{\lambda_d}$

of A^{-1} are also positive.

→ If A & B are pos. def.
is $A+B$?
Use $x^T A x > 0$ & $x^T B x > 0$
To show that $x^T (A+B) x > 0$
for $x \neq 0$ and so $A+B$ is pos. def.

→ Suppose A is rectangular
$m \times n$

Not symmetric!

But,

$A^T A$, is square & symmetric.

is it pos. def.?

$$(x^T A^T)(A x) = (Ax)^T (Ax) = |Ax|^2 \geq 0$$

$\rightsquigarrow$ If A has rank n

[independent columns], then

$x^T (A^T A) x = 0$ only if $x = 0$

& A is pos. def.

Note!

Another nice feature of positive definite matrices is that you never have to do row exchanges when row reducing – there are never 0's or unsuitably small numbers in their pivot positions.

Similar matrices A & $B = M^{-1}AM$

- Two square matrices A & B are similar if $B = M^{-1}AM$ for some matrix M.

Distinct eigenvalues

Suppose A has a full set of eigenvectors.

$$S^{-1}AS = \Lambda$$

$\therefore$ A is similar to Λ

If $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, then $\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

But A is also similar to:

$$\overset{M^{-1}}{\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}} \overset{M}{\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \overset{B}{\begin{bmatrix} 2 & 9 \\ 1 & 6 \end{bmatrix}}$$

$$= \begin{bmatrix} -2 & -15 \\ 1 & 6 \end{bmatrix}$$

$\therefore$ B is similar to Λ.

* Similar matrices have same eigenvalues, λ's!! *

Some other members to the family

$$\begin{bmatrix} 3 & 7 \\ 0 & 1 \end{bmatrix} \text{ \& } \begin{bmatrix} 1 & 7 \\ 0 & 3 \end{bmatrix}.$$

<u>Proof!</u>

$$\left\{ \boxed{B = M^{-1} A M} \right\}$$

$$Ax = \lambda x$$

$$A M M^{-1} x = \lambda x$$

$$(\underbrace{M^{-1} A M}_{B}) M^{-1} x = \lambda M^{-1} x$$

$$B M^{-1} x = \lambda M^{-1} x.$$

$\therefore$ B has the same λ as an eigenvalue.
$M^{-1}x$ is the eigenvector.

* Not same eigenvectors!
↳ eigenvector of B is

M^{-1} (eigenvector of A)

* When we diagonalise A, we find a diagonal matrix Λ that is similar to A.

→ If two matrices have the same n distinct eigenvalues, they will be similar to the same diagonal matrix.

- Repeated eigenvalues (BAD CASE)

- $\lambda_1 = \lambda_2 = 4$.

One small family has only 1: $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

Big family $\begin{bmatrix} 4 & 1 \end{bmatrix}$ Jordan includes $\rightarrow \begin{bmatrix} & \\ 0 & 4 \end{bmatrix}$ } form

$\therefore$

$$M^{-1}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} M = 4M^{-1}M = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

for any invertible matrix M.

Jordan form: "Most diagonal" representative from each family of similar matrices.

- More members of family:

$$\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 17 & 4 \end{bmatrix},$$

$$\begin{bmatrix} a & b \\ (8a - a^2 - 16) & 8-a \end{bmatrix}$$

b

Trace = 8

det = 16

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Eigenvalues are 4 zeros
- Rank is 2
- $N(A)$ dimension: $4-2=2$

Now)

$$\begin{bmatrix} 0 & 1 & 7 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

- $\dim N(A) = 2$, but not nice as A

,

Now consider:

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Again rank = 2,

dim = 2

into their *Jordan blocks*:

$$A = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right], \qquad C = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

A Jordan block J_i has a repeated eigenvalue λ_i on the diagonal, zeros below the diagonal and in the upper right hand corner, and ones above the diagonal:

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & \lambda_i & 1 \\ 0 & 0 & \cdots & 0 & \lambda_i \end{bmatrix}.$$

Two matrices may have the same eigenvalues and the same number of eigenvectors, but if their Jordan blocks are different sizes those matrices can not be similar.

Jordan's theorem says that every square matrix A is similar to a Jordan matrix J, with Jordan blocks on the diagonal:

$$J = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & J_d \end{bmatrix}.$$

In a Jordan matrix, the eigenvalues are on the diagonal and there may be ones above the diagonal; the rest of the entries are zero. The number of blocks is the number of eigenvectors – there is one eigenvector per block.

To summarize:

- If A has n distinct eigenvalues, it is diagonalizable and its Jordan matrix is the diagonal matrix $J = \Lambda$.
- If A has repeated eigenvalues and "missing" eigenvectors, then its Jordan matrix will have $n - d$ ones above the diagonal.

We have not learned to compute the Jordan matrix of a matrix which is missing eigenvectors, but we do know how to diagonalize a matrix which has n distinct eigenvalues.