Problem:

S is spanned by $(1\ 2\ 2\ 3)$ and $(1\ 3\ 3\ 2)$

i) Find a basis for $S^{\perp}$

ii) Can every v in $\mathbb{R}^4$ be written uniquely in terms of S and $S^{\perp}$?

(i)

- If x in $S^{\perp}$

$$(1\quad 2\quad 2\quad 3)\, x = 0$$

$$(1\quad 3\quad 3\quad 2)\, x = 0$$

$$\Updownarrow$$

$$\begin{pmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{pmatrix} x = 0$$

$\Rightarrow$ row reduce

$$\begin{pmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 1 & -1 \end{pmatrix} \underline{x} = 0$$

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \qquad \text{let } x_4 = b, \quad x_3 = a$$

$$x_2 = -x_3 + x_4 = \underline{-a + b}$$

$$x_1 = -2x_2 - 2x_3 - 3x_4$$

$$= -2(\cancel{-a}+b) - \cancel{2a} - 3b$$

$$= \underline{-5b}$$

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -5b \\ -a+b \\ a \\ b \end{pmatrix}$$

$$= \underbrace{a\begin{pmatrix}0\\-1\\1\\0\end{pmatrix} + b\begin{pmatrix}-5\\1\\0\\1\end{pmatrix}}_{S^{\perp}}$$

ii) Yes,

$$v = \overbrace{c_1\begin{pmatrix}1\\2\\2\\3\end{pmatrix} + c_2\begin{pmatrix}1\\3\\3\\2\end{pmatrix}}^{S} +$$

$$\overbrace{c_3\begin{pmatrix}0\\-1\\1\\0\end{pmatrix} + c_4\begin{pmatrix}-5\\1\\0\\1\end{pmatrix}}^{S^{\perp}}$$

$$\Rightarrow \begin{pmatrix}1 & 1 & 0 & -5\\2 & 3 & -1 & 1\\2 & 3 & 1 & 0\\3 & 2 & 0 & 1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\\c_4\end{pmatrix} = v$$