Objectives

- Linear Transformations T
- without coordinates : no matrix
- with coordinates $\Rightarrow$ Matrix

Example 1: Projection

$T: \mathbb{R}^2 \to \mathbb{R}^2$ (mapping)

$\to$ is that every vector $\underline{v}$ is projected onto a vector $T(v)$ on the line of the projection
(Projection is a linear transformation)

Definition of linear

- A transformation T is linear if:

$$T(v+w) = T(v) + T(w)$$

&

$$T(cv) = cT(v)$$

for all vectors v & w and for all scalars.

Equivalently,

$$T(cv + dw) = cT(v) + dT(w)$$

for all vectors v & w and scalars c & d.

Note: $\boxed{T(0) = 0}$. because if not it couldn't be "true that $T(c0) = cT(0)$.

Non-example 1: Shift the whole plane

Consider the transformation $T(\mathbf{v}) = \mathbf{v} + \mathbf{v}_0$ that shifts every vector in the plane by adding some fixed vector $\mathbf{v}_0$ to it. This is *not* a linear transformation because $T(2\mathbf{v}) = 2\mathbf{v} + \mathbf{v}_0 \neq 2T(\mathbf{v})$.

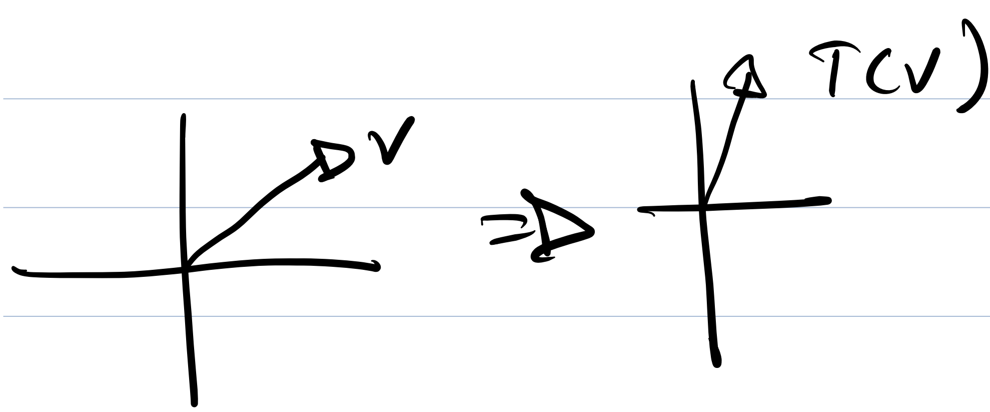
Non-example 2: $T(\mathbf{v}) = ||\mathbf{v}||$

The transformation $T(\mathbf{v}) = ||\mathbf{v}||$ that takes any vector to its length is not a linear transformation because $T(c\mathbf{v}) \neq cT(\mathbf{v})$ if $c < 0$.

We're not going to study transformations that aren't linear. From here on, we'll only use T to stand for linear transformations.

Example 2: Rotation by 45°

$T: \mathbb{R}^2 \to \mathbb{R}^2$



Example 3!! Matrix A

$$T(v) = Av$$

This is a linear transformation:

$$A(v+w) = A(v) + A(w)$$

&

$$A(cv) = cA(v)$$

Example 4:

Suppose $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

- How would we describe the transformation $T(v) = Av$?

- x component of the vector is unchanged
- y sign is reversed.

__Example 5__

start: $T: \mathbb{R}^3 \to \mathbb{R}^2$

2 by 3 matrix

$T(v) = Av$

input in $\mathbb{R}^3$

output in $\mathbb{R}^2$

---

* Information needed to know $T(v)$ for all inputs.

$T(v_1), T(v_2), \dots T(v_n)$ for any __input__ basis $v_1, \dots, v_n$

* Every $v = c_1 v_1 + \dots + c_n v_n$
· know $T(v) = c_1 T(v_1) + \dots + c_n T(v_n)$

The matrix of a linear transformation

(S1) Choose basis $, v_1, \dots, v_n$ $\mathbb{R}^n$
to give coordinates to the input vectors

[coordinates come from a basis
" of $v = c_1 v_1 + \dots + c_n v_n$

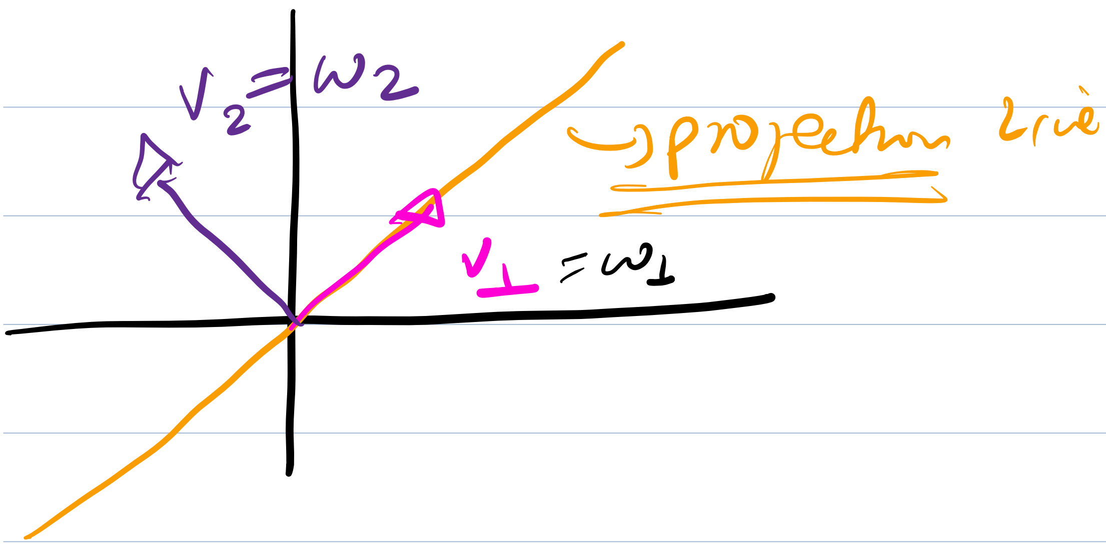
* Construct matrix A that
represents lin. tr. T
$T: \mathbb{R}^n \to \mathbb{R}^m$

- Choose a basis $v_1, \dots, v_n$ for
inputs $\mathbb{R}^n$

" —— " —— " $w_1, \dots, w_n$ for outputs $\mathbb{R}^m$.

WANT!

Matrix A



$v = c_1 v_1 + c_2 v_2$

$T(v) = c_1 v_1$

(c_1, c_2)

$\downarrow$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$$

A · input coordintes　　output coordinates

- eigen vector basis
leads to diagonal matrix Λ

- Proj onto $45°$ line
use standard $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = w_1$

$v_2 = \begin{bmatrix} 0 \\ \cdot \end{bmatrix} = w_2$

Matrix $P = \dfrac{aa^T}{a^Ta} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

- Rule to find A.

Given bases $v_1 - v_n$

$w_1 - w_m$

- 1^{st} column of A:

write $T(v_1) = \underline{a_{11}} w_1 + \underline{a_{21}} w_2 + \cdots$
$+ \underline{a_{m1}} w_m$

- 2^{nd} column of A:

$T(v_2) = a_{12} w_1 + \cdots + a_{m2} w_m$

$A \begin{pmatrix} \text{input c} \\ \text{coordinates} \end{pmatrix} = \begin{pmatrix} \text{output} \\ \text{coordinates} \end{pmatrix}$

Example 6: $T = \frac{d}{dx}$

Example 6: $T = \dfrac{d}{dx}$

Let T be a transformation that takes the derivative:

$$T(c_1 + c_2x + c_3x^2) = c_2 + 2c_3x. \tag{1}$$

The input space is the three dimensional space of quadratic polynomials $c_1 + c_2x + c_3x^2$ with basis $\mathbf{v}_1 = 1$, $\mathbf{v}_2 = x$ and $\mathbf{v}_3 = x^2$. The output space is a two dimensional subspace of the input space with basis $\mathbf{w}_1 = \mathbf{v}_1 = 1$ and $\mathbf{w}_2 = \mathbf{v}_2 = x$.

This is a linear transformation! So we can find $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and write the transformation (1) as a matrix multiplication (2):

$$T\left(\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}\right) = A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_2 \\ 2c_3 \end{bmatrix}. \tag{2}$$

Conclusion

For any linear transformation T we can find a matrix A so that $T(\mathbf{v}) = A\mathbf{v}$. If the transformation is invertible, the inverse transformation has the matrix A^{-1}. The product of two transformations $T_1 : \mathbf{v} \mapsto A_1\mathbf{v}$ and $T_2 : \mathbf{w} \mapsto A_2\mathbf{w}$ corresponds to the product A_2A_1 of their matrices. This is where matrix multiplication came from!