Objectives

- Projection
- Least Square & best straight line

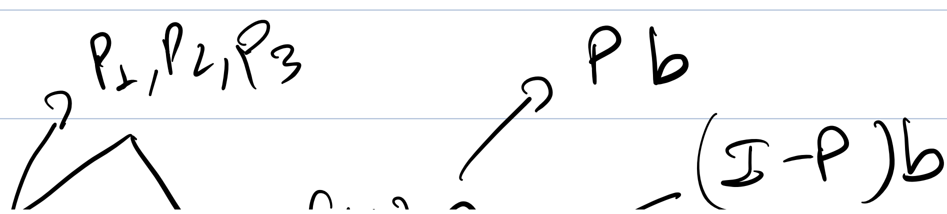
Reminder

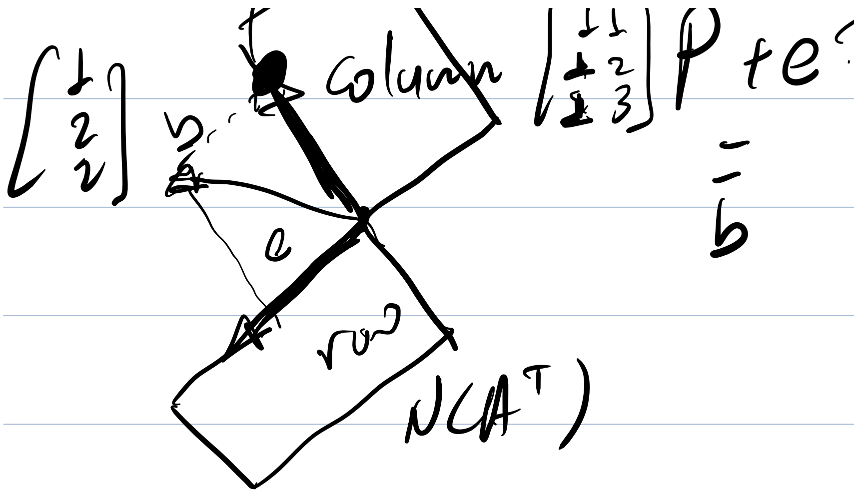
Projection MATRIX :

$P = A(A^T A)^{-1} A^T$

Ax

- If b in column space, $Pb = b$
- If $b \perp$ column space $Pb = 0$





$\bar{b}$ proj onto ⊥ space

Least squares

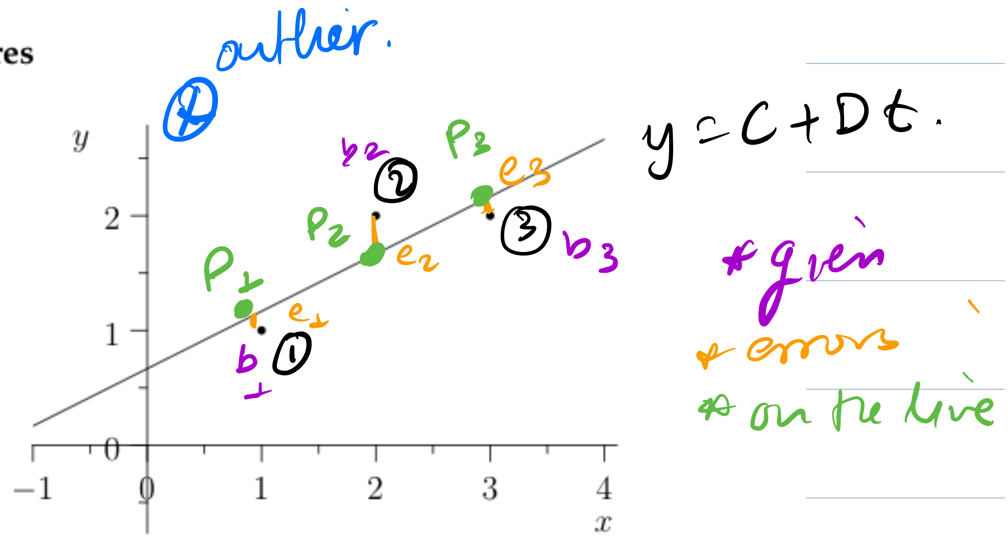


Figure 1: Three points and a line close to them.

* Trying to find the best fit line:

① $C + D = 1$

② $C + 2D = 2$

③ $C + 3D = 2$

Note:

I can make an error in any of the b's

• sum & square the errors

least squares

⇒ Best solution:

$$Ax = b$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Minimize: $\|Ax - b\|^2 = \|e\|^2$ *

In order to solve: ← Using Calculus

partial derivatives

$$= e_1^2 + e_2^2 + e_3^2 \Rightarrow (C+D-1)^2 + (C+2D-2)^2 + (C+3D-2)^2$$

Linear regression

*Note (^) hat represents estimates

- Find $\hat{x} = \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix}$, P

$$\boxed{A^T A \hat{x} = A^T b}$$

* Used for estimation / fitting lines

$\Rightarrow \boxed{p = Ax}$

$$A^TA = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 2 \end{array}\right] = \left[\begin{array}{cc|c} 3 & 6 & 5 \\ 6 & 14 & 11 \end{array}\right]$$

* symmetric

* invertible

$$\Rightarrow \begin{aligned} 3C + 6D &= 5 \\ 6C + 14D &= 11 \end{aligned} \Big\} \text{ normal equations!}$$

$$2D = 1$$

$$D = 1/2, \quad C = 2/3$$

Either way, we end up solving a system of linear equations to find that the closest line to our points is $b = \frac{2}{3} + \frac{1}{2}t$.

This gives us:

i	p_i	e_i
1	7/6	−1/6
2	5/3	1/3
3	13/6	−1/6

} see graph above.

$\Rightarrow b = p + e$

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 7/6 \\ 10/6 \\ 13/6 \end{bmatrix} + \begin{bmatrix} -1/6 \\ 2/6 \\ -1/6 \end{bmatrix}$$

- If A has independent columns, then A^TA is <u>invertible</u>. *

  ↳ Proof that x must be 0.

<u>Proof:</u>

Suppose $A^TAx = 0$

<u>IDEA:</u>

$$\underbrace{x^TA^T}\,\underbrace{Ax} = 0$$

conclude.

$$\underbrace{(Ax)^T(Ax)}_{\text{denaMol}} = 0 \Rightarrow \underline{\underline{Ax = 0}} \Rightarrow \underline{x = 0}$$

∝ <u>independent</u>

$(\text{vector})^2$

As long as the columns of A are independent, we can use linear regression to find approximate solutions to unsolvable systems of linear equations. The columns of A are guaranteed to be independent if they are *orthonormal*, i.e. if they are perpendicular unit vectors like $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, or like $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$.

unit vectors

Note:

columns definetly independent if they are perpendicular unit vectors (orthonormal vectors)