

## The four fundamental subspaces

In this lecture we discuss the four fundamental spaces associated with a matrix and the relations between them.

### Four subspaces

Any  $m$  by  $n$  matrix  $A$  determines four subspaces (possibly containing only the zero vector):

#### Column space, $C(A)$

$C(A)$  consists of all combinations of the columns of  $A$  and is a vector space in  $\mathbb{R}^m$ .

#### Nullspace, $N(A)$

This consists of all solutions  $\mathbf{x}$  of the equation  $A\mathbf{x} = \mathbf{0}$  and lies in  $\mathbb{R}^n$ .

#### Row space, $C(A^T)$

The combinations of the row vectors of  $A$  form a subspace of  $\mathbb{R}^n$ . We equate this with  $C(A^T)$ , the column space of the transpose of  $A$ .

#### Left nullspace, $N(A^T)$

We call the nullspace of  $A^T$  the *left nullspace* of  $A$ . This is a subspace of  $\mathbb{R}^m$ .

## Basis and Dimension

### Column space

The  $r$  pivot columns form a basis for  $C(A)$

$$\dim C(A) = r.$$

### Nullspace

The special solutions to  $A\mathbf{x} = \mathbf{0}$  correspond to free variables and form a basis for  $N(A)$ . An  $m$  by  $n$  matrix has  $n - r$  free variables:

$$\dim N(A) = n - r.$$

### Row space

We could perform row reduction on  $A^T$ , but instead we make use of  $R$ , the row reduced echelon form of  $A$ .

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} = R$$

Although the column spaces of  $A$  and  $R$  are different, the row space of  $R$  is the same as the row space of  $A$ . The rows of  $R$  are combinations of the rows of  $A$ , and because reduction is reversible the rows of  $A$  are combinations of the rows of  $R$ .

The first  $r$  rows of  $R$  are the "echelon" basis for the row space of  $A$ :

$$\dim C(A^T) = r.$$

### Left nullspace

The matrix  $A^T$  has  $m$  columns. We just saw that  $r$  is the rank of  $A^T$ , so the number of free columns of  $A^T$  must be  $m - r$ :

$$\dim N(A^T) = m - r.$$

The left nullspace is the collection of vectors  $y$  for which  $A^T y = 0$ . Equivalently,  $y^T A = 0$ ; here  $y$  and  $0$  are row vectors. We say "left nullspace" because  $y^T$  is on the left of  $A$  in this equation.

To find a basis for the left nullspace we reduce an augmented version of  $A$ :

$$\begin{bmatrix} A_{m \times n} & I_{m \times n} \end{bmatrix} \longrightarrow \begin{bmatrix} R_{m \times n} & E_{m \times n} \end{bmatrix}.$$

From this we get the matrix  $E$  for which  $EA = R$ . (If  $A$  is a square, invertible matrix then  $E = A^{-1}$ .) In our example,

$$EA = \begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R.$$

The bottom  $m - r$  rows of  $E$  describe linear dependencies of rows of  $A$ , because the bottom  $m - r$  rows of  $R$  are zero. Here  $m - r = 1$  (one zero row in  $R$ ).

The bottom  $m - r$  rows of  $E$  satisfy the equation  $y^T A = 0$  and form a basis for the left nullspace of  $A$ .

### New vector space

The collection of all  $3 \times 3$  matrices forms a vector space; call it  $M$ . We can add matrices and multiply them by scalars and there's a zero matrix (additive identity). If we ignore the fact that we can multiply matrices by each other, they behave just like vectors.

Some subspaces of  $M$  include:

- all upper triangular matrices
- all symmetric matrices
- $D$ , all diagonal matrices

$D$  is the intersection of the first two spaces. Its dimension is 3; one basis for  $D$  is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 7 \end{bmatrix}.$$

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