Column space and nullspace

Objectives:

- Vector Spaces & Subspaces
- Column space of A : Solving
- Null space of A $\qquad Ax = b$

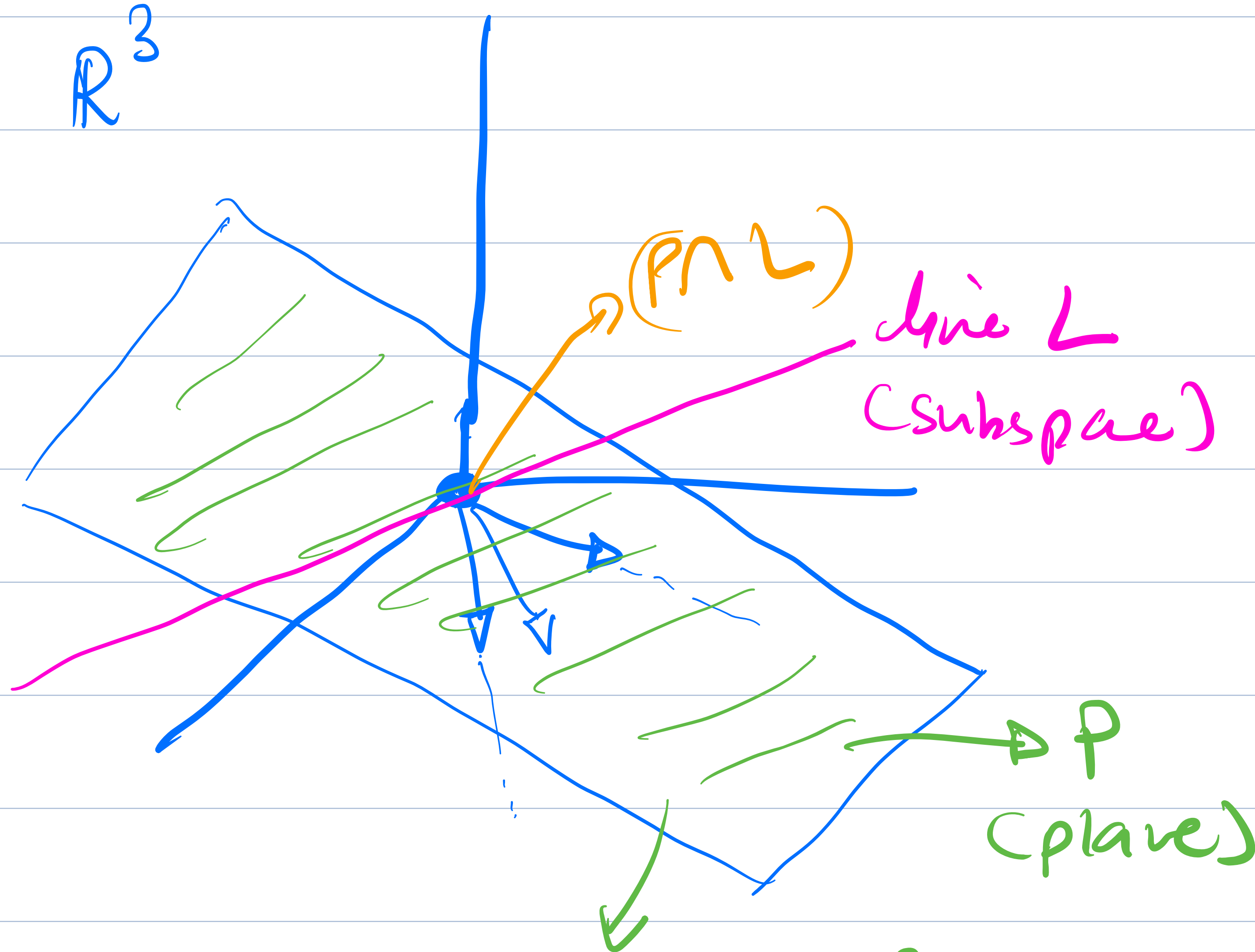
Reminder:

- What is a vector space?
  - Add 2 vectors together
  - Multiply any vector in space by scalar

$\underbrace{\hspace{10em}}$

Need to still remain in space

eg



plane through $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is subspace

* Have to contain origin

Now I have 2 subspaces:
P & L

$P \cup L$ = all vectors in P or L or both

- ↳ This (is) or (is not) subspace?
  NO, normally outside the union.

- ↳ $P \cap L$ = all vectors that are in both P & L
  Yes

- General

Subspaces S & T
↳ intersection $S \cap T$
is a subspace.

Column Space of A is a subspace

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \to \mathbb{R}^4$$

of $\mathbb{R}^4$

$[4 \times 3]$

↳ all linear comb of the columns.

→ $C(A)$

- Does $Ax = b$ have a solution for every b?

↳ (4 equations, 3 unknowns)

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

* Combination of columns do not fill the whole 4-D space

↳ which b's allow this system to be solved ??

① $$\begin{bmatrix}1&1&2\\2&1&3\\3&1&4\\4&1&5\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$$

② $$\begin{bmatrix}1&1&2\\2&1&3\\3&1&4\\4&1&5\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

Note

Can solve $Ax = b$ when b exactly when b is in the column space.

Are those the 3 columns in A independent?

* **Note:**

we can see that the 3^{rd} column of A is the sum of the 1^{st} & 2^{nd} column

$\therefore$ 1^{st} & 2^{nd} column $\rightarrow$ pivot column

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Null space of A : all solution

$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to $Ax = 0$.

$\mathbb{R}^{3}$

in subspace $\mathbb{R}$

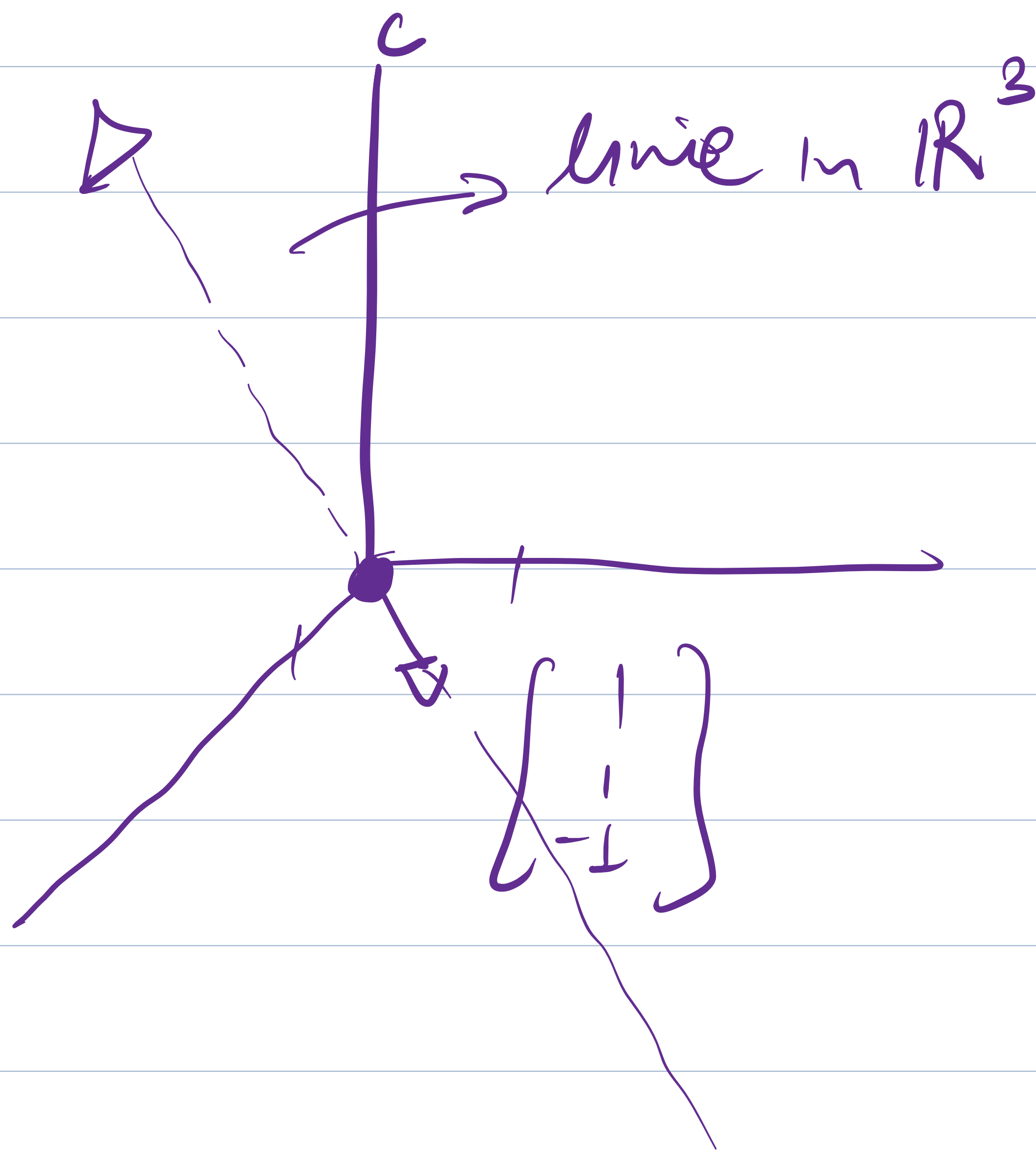
$$\Rightarrow \quad Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

* Null space

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

① $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

② $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} c \\ c \\ -c \end{bmatrix}$



- How do I know if nullspace is a <u>subspace</u>

↳ If $Av = 0$ & $Aw = 0$

then $A(v+w) = 0$

$Av + Aw = 0$

↳ then $A(12v) = 0$

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* Do I get a subspace that solve $Ax = b \neq 0$

↳ No because it does not go through the 0

∴ cannot be a vector space