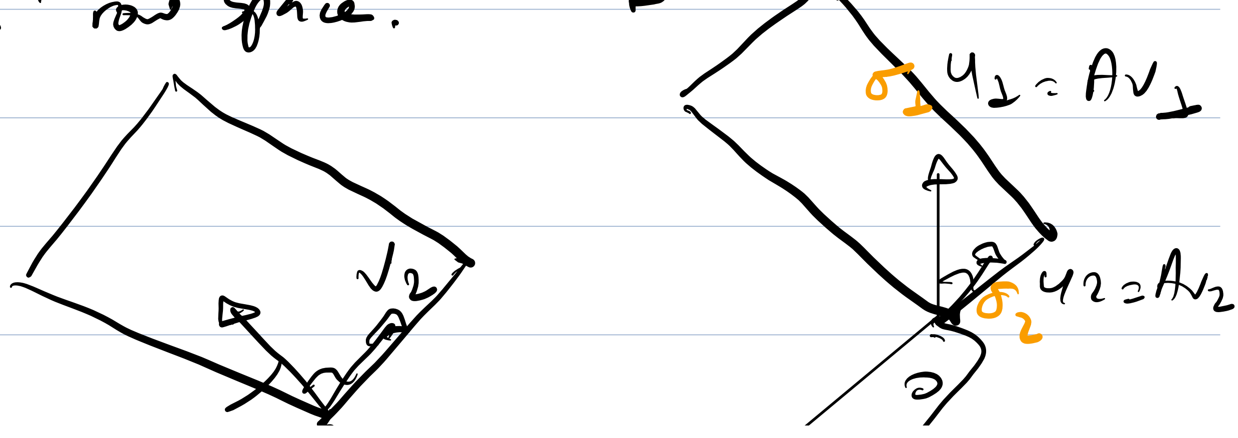
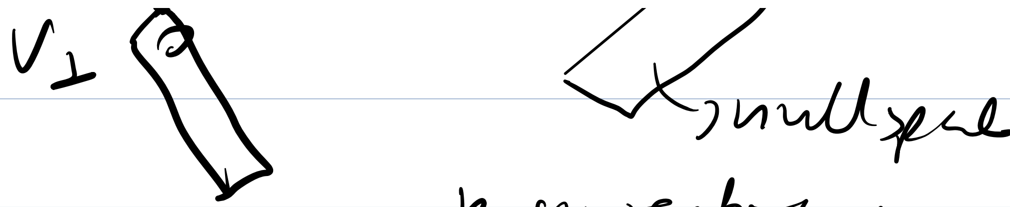
Objectives

- Singular Value Decomposition = SVD //
- $A = U \Sigma V^T$ // Σ : diagonal
  U, V = orthogonal.

- We know that if A is symmetric positive definite → eigenvectors are orthogonal $\therefore A = Q \Lambda Q^T$
  ↳ Special case:
  SVD, $U = V = Q$.

R^n row space. R^m column space





basis vectors in row space

$$A\begin{bmatrix} v_1 & v_2 & \cdots & v_r \end{bmatrix} = \begin{bmatrix} \sigma_1 u_1 & \sigma_2 u_2 & \cdots & \sigma_r u_r \end{bmatrix}$$

$$= \begin{bmatrix} u_1 & u_2 & \cdots & u_r \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{bmatrix}$$

basis vectors column space

multiplying factors

$$\Rightarrow AV = U\Sigma$$

<u>Example:</u>

[orthonormal]

v_1, v_2 in row space $\mathbb{R}^2$

$$A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$$

u_1, u_2 in col. space $\mathbb{R}^2$

$\sigma_1 > 0 \qquad \sigma_2 > 0$

$\sigma_1 \neq 0 \quad , \sigma_2 \neq 0$

$$\left\{ \begin{array}{l} Av_1 = \sigma_1 u_1 \\ Av_2 = \sigma_2 u_2 \end{array} \right\}$$

↳ This is a big step towards finding orthonormal matrices V & U and a diagonal matrix Σ for which:

$$AV = U\Sigma$$

Since V is orthogonal, we can multiply both sides by $V^{-1} = V^T$ to get

$$A = U\Sigma V^{-1} = U\Sigma V^T$$

(make U disappear)

$\therefore$

$$A^T A = V\Sigma^T \underbrace{\cancel{U^T U}}_{I} \Sigma V^T$$

$A^T A = V \Sigma^2 V^T$

↳ diagonal matrices :.

Σ^2 on diagonals

$$= V \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_n^2 \end{bmatrix} V^T$$

This is in the form $Q \Lambda Q^T$

→ Columns of V are eigenvectors of $A^T A$ and the eigenvalues are σ_i^2

- U are the same thing with AA^T.

SVD Example

$$A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$$

- eigenvectors: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$= 32 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ 18 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\rightarrow$ To get orthonormal basis:

$v_1 = \Big[\ 1/\sqrt{2}$, $v_2 = \Big[\ 1/\sqrt{2}$

$\ldots [\, 1/\sqrt{2} \,] \quad , \ldots \quad [\, -1/\sqrt{2} \,]$

$$\underset{\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}}{A} = \underset{\begin{bmatrix} \quad & \quad \end{bmatrix}}{U} \; \underset{\begin{bmatrix} \overset{\sqrt{32}}{4\sqrt{2}} & 0 \\ 0 & \underset{\sqrt{18}}{3\sqrt{2}} \end{bmatrix}}{\Sigma} \; \underset{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}{V^T}.$$

Find u's? u_1, u_2

$$AA^T = U\Sigma^2 U^T$$

$$= \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix} \Bigg\} \text{ diagonal!}$$

eigenvector:

$$u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} : 32 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$u_2 = \begin{bmatrix} 0 \\ -18 \end{bmatrix} = 18 \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

Thus, the SVD of A is:

$$\underset{A}{\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}} = \underset{U}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}} \underset{\Sigma}{\begin{bmatrix} 4\sqrt{2} & 0 \\ 0 & 3\sqrt{2} \end{bmatrix}} \underset{V^T}{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}.$$

Example #2 - with nullspace

Let $A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$

This has 1 dimensional nullspace & 1 dimensional row & column spaces.

$\Rightarrow$ Row space: multiples of $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$\leadsto$ Column space: multiples of $\begin{bmatrix} 4 \\ 8 \end{bmatrix}$

$$v_1 = \begin{bmatrix} .8 \\ .6 \end{bmatrix}, \quad u = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

SVD

$$\underset{A}{\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}} = \underset{U}{\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}} \underset{\Sigma}{\begin{bmatrix} \sqrt{125} & 0 \\ 0 & 0 \end{bmatrix}} \underset{V^T}{\begin{bmatrix} .8 & .6 \\ .6 & -.8 \end{bmatrix}}.$$

$$\Downarrow$$

$$\sigma_1 = A^T A$$

$$A^T A = \begin{bmatrix} 4 & 8 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 80 & 60 \end{bmatrix} \quad \text{rank } 1$$

$\begin{bmatrix} 60 & 45 \end{bmatrix}$) matrix.

$\rightarrow$ eigenvalues: $0, 125$. (trace)

The singular value decomposition combines topics in linear algebra ranging from positive definite matrices to the four fundamental subspaces.

$\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_r$	is an orthonormal basis for the row space.
$\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_r$	is an orthonormal basis for the column space.
$\mathbf{v}_{r+1}, \ldots \mathbf{v}_n$	is an orthonormal basis for the nullspace.
$\mathbf{u}_{r+1}, \ldots \mathbf{u}_m$	is an orthonormal basis for the left nullspace.

These are the "right" bases to use, because $A\mathbf{v}_i = \sigma_i \mathbf{u}_i$.