Objectives

- Complete solution of $\boxed{Ax = b}$
- Rank r

↳ $r = m$: solution exist.

↳ $r = n$: solution unknown.

Example:

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

→ $x_1 + 2x_2 + 2x_3 + 2x_4 = b_1$

→ $2x_1 + 4x_2 + 6x_3 + 8x_4 = b_2$

→ $3x_1 + 6x_2 + 8x_3 + 10x_4 = b_3$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right]$$

Augmented matrix = [A b]

elimination

$$\Rightarrow \left[\begin{array}{cccc|c} \boxed{1} & 2 & 2 & 2 & b_1 \\ 0 & 0 & \boxed{2} & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{array}\right]$$

pivot columns.

$$\Rightarrow \left[\begin{array}{cccc|cc} 1 & 2 & 2 & 2 & b_1 & 1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 & 3 \\ 0 & 0 & 0 & 0 & b_3 \quad b_2 - b_1 & \end{array}\right]$$

$$0 = b_3 - b_2 - b_1.$$

eg $b = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$

Solvability: Condition on b

$Ax = b$, solvable when b is in $C(A)$

If a combination of rows of A gives zero rows

Then, the same combination of entries of b must give 0.

Algorithm:

To find complete solution $Ax=b$

$x_2 = 0$
$x_4 = 0$.

(S1) x particular :- Set all free variables to zero.

- Solve $Ax=b$ for pivot variables.

$\to x_1 + 2x_3 = 1 \leadsto x_1 = -2$

$2x_3 = 3 \leadsto x_3 = 3/2$

$$x_{part} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$$

* Note: Plug in original eq. to check.

(2) X nullspace:

$X_{complete} = X_p + X_{nullspace}$.

$$A x_p = b$$
$$A x_n = 0 \quad +$$

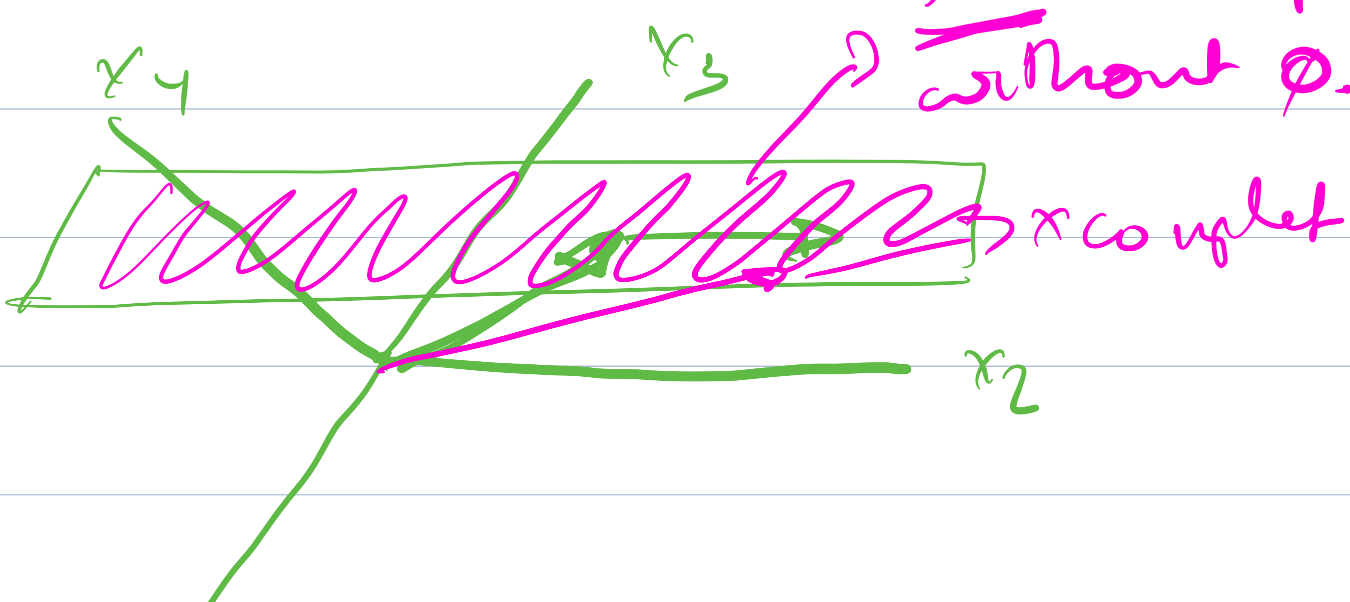
$$A(x_p + x_n) = b$$

$$\Rightarrow x_{\text{complete}} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Note:

Trying to draw the solution: $\mathbb{R}^4$

like a subspace without $\emptyset$.



$/ x_\perp$

---

m by n matrix A of rank r [# of pivots].

($\Rightarrow r \le m$, $r \le n$)

Full column rank means $\boxed{r = n}$:

no free variables

$\hookrightarrow$ Null space of A: $N(A) = \{0\}$

$\hookrightarrow$ Solution to $Ax = b$: $x = x_p$.

unique solution if it exist.

[0 or 1 solutions]

eg

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix} \qquad R = \text{(rref)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{array}{l} I \\ \\ 0 \end{array}$$

Full row rank: means $\boxed{r = m}$

I can solve $Ax = b$ for every b.

Exists

Left with $\underbrace{n - r}_{\substack{\parallel \\ (n-m)}}$ free variables.

eg

$$A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ \end{bmatrix} \qquad R = \text{rref} \begin{bmatrix} \overset{I}{1 \quad 0} & - & - \\ & F & \end{bmatrix}$$

$A = \begin{bmatrix} 3 & 1 & 1 & 1 \end{bmatrix}$, rref $\begin{bmatrix} 0 & 1 & \boxed{- \; -} \end{bmatrix}$

$r = 2$

$\boxed{r = m = n}$ } invertible matrix

$b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$

$R = I$

Summary

If R is in row reduced form with pivot columns first (rref), the table below summarizes our results.

	$r = m = n$	$r = n < m$	$r = m < n$	$r < m, r < n$
R	I	$\begin{bmatrix} I \\ 0 \end{bmatrix}$	$\begin{bmatrix} I & F \end{bmatrix}$	$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$
# solutions to $A\mathbf{x} = \mathbf{b}$	1	0 or 1	infinitely many	0 or infinitely many