Objectives

- 4 subspaces
- left-inverses
- right-inverses
- Pseudo-inverses

- 2-sided inverses

$AA^{-1} = I = A^{-1}A$

$r = m = n$ } full rank.

• left-inverse

full column rank $\Rightarrow r = n$

nullspace $= \{0\}$

b either has exactly one solution x or is not solvable.

The matrix A^TA is an invertible n by n symmetric matrix, so $(A^TA)^{-1}A^TA = I$. We say $A^{-1}_{\text{left}} = (A^TA)^{-1}A^T$ is a *left inverse* of A. (There may be other left inverses as well, but this is our favorite.) The fact that A^TA is invertible when A has full column rank was central to our discussion of least squares.

Note that AA^{-1}_{left} is an m by m matrix which only equals the identity if $m = n$. A rectangular matrix can't have a two sided inverse because either that matrix or its transpose has a nonzero nullspace.

- right-inverse

- full row rank $\Rightarrow r = m < n$

- $N(A^T) = \{0\}$, independent rows.

$\hookrightarrow n-m$; free variables

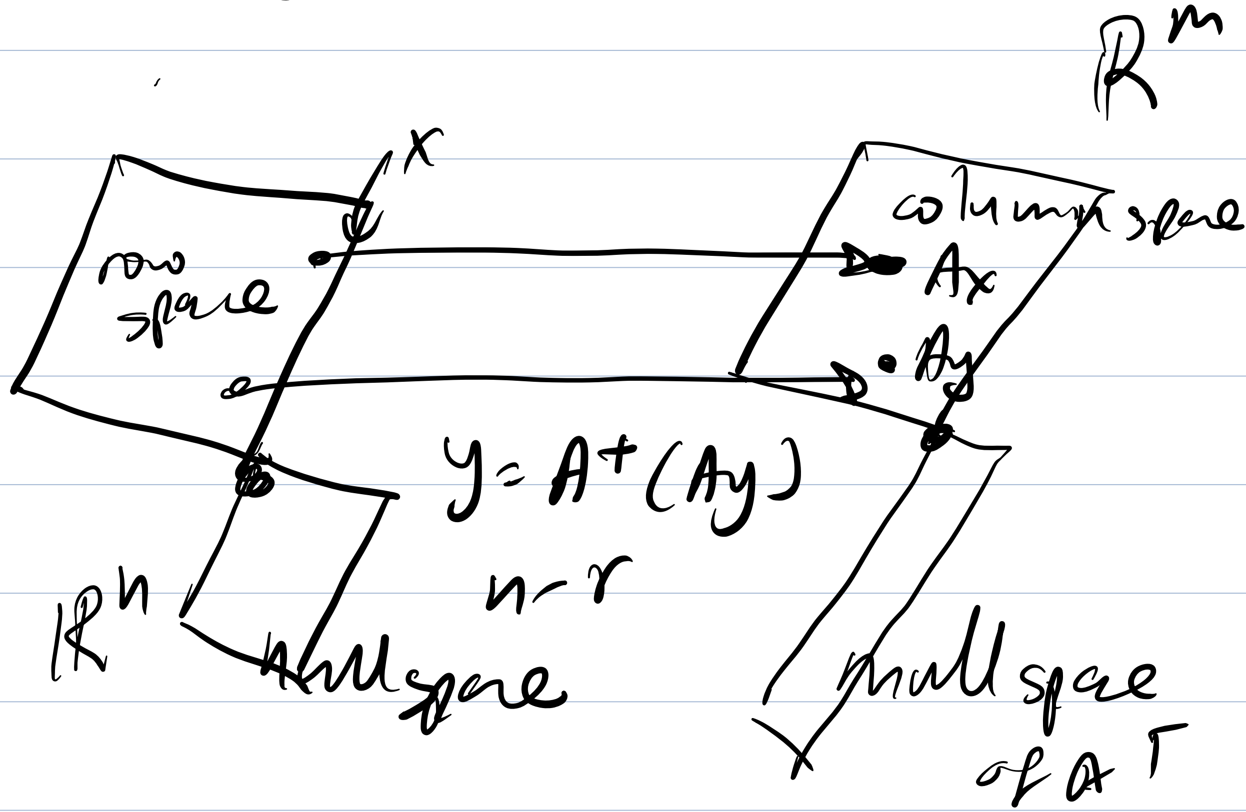
$$A\underbrace{A^T(AA^T)^{-1}} = I$$

$$A A^{-1}_{\text{right}} = I$$

- Pseudoinverses

· If x, y is in row space then

$Ax \neq Ay$.



Proof

Suppose $Ax = Ay$

in nullspace $A(x-y)=0$.

so $x-y$ is in the nullspace of A

Finding the pseudoinverse A^+

The *pseudoinverse* A^+ of A is the matrix for which $\mathbf{x} = A^+A\mathbf{x}$ for all $\mathbf{x}$ in the row space of A. The nullspace of A^+ is the nullspace of A^T.

We start from the singular value decomposition $A = U\Sigma V^T$. Recall that Σ is a m by n matrix whose entries are zero except for the singular values $\sigma_1, \sigma_2, ..., \sigma_r$ which appear on the diagonal of its first r rows. The matrices U and V are orthonormal and therefore easy to invert. We only need to find a pseudoinverse for Σ.

The closest we can get to an inverse for Σ is an n by m matrix Σ^+ whose first r rows have $1/\sigma_1, 1/\sigma_2, ..., 1/\sigma_r$ on the diagonal. If $r = n = m$ then $\Sigma^+ = \Sigma^{-1}$. Always, the product of Σ and Σ^+ is a square matrix whose first r diagonal entries are 1 and whose other entries are 0.

If $A = U\Sigma V^T$ then its pseudoinverse is $A^+ = V\Sigma^+\mathbf{U}^T$. (Recall that $Q^T = Q^{-1}$ for orthogonal matrices U, V or Q.)

We would get a similar result if we included non-zero entries in the lower right corner of Σ^+, but we prefer not to have extra non-zero entries.