Diagonalise A by constructing its eigenvalue matrix Λ and eigenvector matrix S

$$A = \begin{pmatrix} 2 & 1-i \\ 1+i & 3 \end{pmatrix} = \bar{A}^T = A^H$$

$$\det(A - \lambda I) = 0$$

$$\det \begin{pmatrix} 2-\lambda & 1-i \\ 1+i & 3-\lambda \end{pmatrix} = 0$$

$$(2-\lambda)(3-\lambda) - (1+i)(1-i) = 0$$

$$6 - 5\lambda + \lambda^2 - 2 = 0$$

$$\lambda^2 - 5\lambda + 4 = 0$$

$$(\lambda - 1)(\lambda - 4) = 0$$

$$\downarrow \qquad \downarrow$$

$$\underline{\lambda_1 = 1} \qquad \underline{\lambda_2 = 4}$$

* Hermitian matrices always have real eigenvalues

(S2) Find eigenvectors

$\underline{\lambda_1 = 1}$:

$$(A - I)v = 0$$

$$\begin{pmatrix} 1 & 1-i \\ 1+i & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\left.\begin{aligned} v_1 &= (1-i) \\ v_2 &= -1 \end{aligned}\right\} \quad v = \begin{pmatrix} 1-i \\ -1 \end{pmatrix}$$

$\underline{\lambda_2 = 4}$:

$$\begin{pmatrix} -2 & 1-i \\ 1+i & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

$$u = \begin{pmatrix} 1 \\ 1+i \end{pmatrix} \Rightarrow |1|^2 + |i+1|^2 = 1 + \sqrt{2}$$

S3

$$\sqrt{1^2 + (\sqrt{2})^2} = \sqrt{1+2} = \sqrt{3}$$

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \qquad S = \frac{1}{\sqrt{3}} \begin{pmatrix} 1-i & 1 \\ -1 & 1+i \end{pmatrix}$$

normalised !

$$\bar{v}^T u = \left[(1+i) \quad -1 \right] \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$$

$S^{-1} = \bar{S}^T$ (unitary!)

$$A = S \Lambda S^{-1} = S \Lambda \bar{S}^T$$

$$A = \frac{1}{\sqrt{3}} \begin{pmatrix} 1-i & 1 \\ -1 & 1+i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

$$\frac{1}{\sqrt{3}} \begin{pmatrix} 1+i & -1 \\ 1 & 1-i \end{pmatrix}$$

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