Objectives

- Determinants (det A) → eigenvalues
- Properties 1, 2, 3, 4-10
- $\pm$ signs-

- Determinant

↳ Is a number associated with any square matrix.

[det A or $|A|$]

→ The determinant encodes a lot of information about the Matrix

↳ Test for invertibility

→ The matrix is invertible exactly when the determinant is non-zero

- We already know that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \quad \text{(determinant)}$$

- Properties

① $\det I = 1$

② Exchange rows:
Reverse the sign of the determinant.

eg

① $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$ from Prop: 1

② $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1$ from Prop 2.

*

③

a) If we multiply 1 of the rows of matrix by t

→ The det is multiplied by t.

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

b) The det behaves like a

* <u>linear function</u> on the rows of the matrix. <u>(for each row)</u>

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

④ If two equal rows → det = 0

↳ Because property ②, exchange rule

11.

same ▽ matrix.

from elimination

⑤ Subtract (ℓ x row i) from (row k)

* ⮡ Det does <u>NOT</u> change. *

$$= \begin{vmatrix} a & b \\ c-\ell a & d-\ell b \end{vmatrix}$$

Prop ③

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ -\ell a & -\ell b \end{vmatrix}$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - \ell \begin{vmatrix} a & b \\ a & b \end{vmatrix} = \phi$$

(6) Row of zeros → det $A = 0$.
↳ from property (3)(a) with $t = 0$

(prop 5, 3a, 1)
(7) The det of triangular matrix

$$U = \begin{vmatrix} d_1 & ? & ? & ? \\ 0 & d_2 & ? & ? \\ 0 & 0 & \ddots & ? \\ \cdots & & & d_n \end{vmatrix}$$

→ dont mater

$$\det U = \underbrace{(d_1)(d_2)\cdots(d_n)}$$

→ product of pivots

Note that we cannot use elimination to get a diagonal matrix if one of the d_i is zero. In that case elimination will give us a row of zeros and property 6 gives us the conclusion we want.

↳ factorize

$$\rightsquigarrow d_2 d_1 \left|\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ - & - & - & - \end{matrix}\right| \Bigg\} \text{prop } 1$$

$$\Downarrow = I$$

⑧ det A exactly when A is singular

→ If A is singular

↳ use elimination to get a row of zeros

↳ prop 6 ⇒ det = 0

→ If A is not singular

↳ elimination produces a full set of pivots $\underline{d_1, d_2, d_3, \ldots, d_n}$

↳ det $\neq 0$

eg $\begin{matrix} a & b \\ c & d \end{matrix} \rightarrow \begin{matrix} a & b \\ 0 & d - \frac{c}{a} b \end{matrix}$

$= \underline{\underline{ad - bc}}$

9 $\det AB = (\det A)(\det B)$ // $\det A^{-1}$

⇒ For example:

$$\det A^{-1} = \frac{1}{\det A}$$

because $A^{-1} A = 1$

* Note: If A is singular A^{-1} does

not exist and $\det A^{-1}$ is undefined.

- $\det A^2 = (\det A)^2$
- $\det 2A = \underbrace{2^n}_{} \det A$.

? $\Rightarrow$ factoring factor 2 out of every row. (all rows ...)

↳ Similar to volume.

If we double the length, width and height of 3D box, we increase its volume by multiple of $2^3 = 8$

(10) $\det A^T = \det A$

$\begin{vmatrix} c & d \end{vmatrix} = \begin{vmatrix} b & d \end{vmatrix}$

- $|A^T| = |A|$
- $|U^T L^T| = |LU|$

$|U^T||L^T| = |L||U|$ ✓

We have one loose end to worry about. Rule 2 told us that a row exchange changes the sign of the determinant. If it's possible to do seven row exchanges and get the same matrix you would by doing ten row exchanges, then we could prove that the determinant equals its negative. To complete the proof that the determinant is well defined by properties 1, 2 and 3 we'd need to show that the result of an odd number of row exchanges (odd permutation) can never be the same as the result of an even number of row exchanges (even permutation).