Objectives

- Change of Basis
- Compression of Images
- Transformation $\leftrightarrow$ Matrix

Compression of images

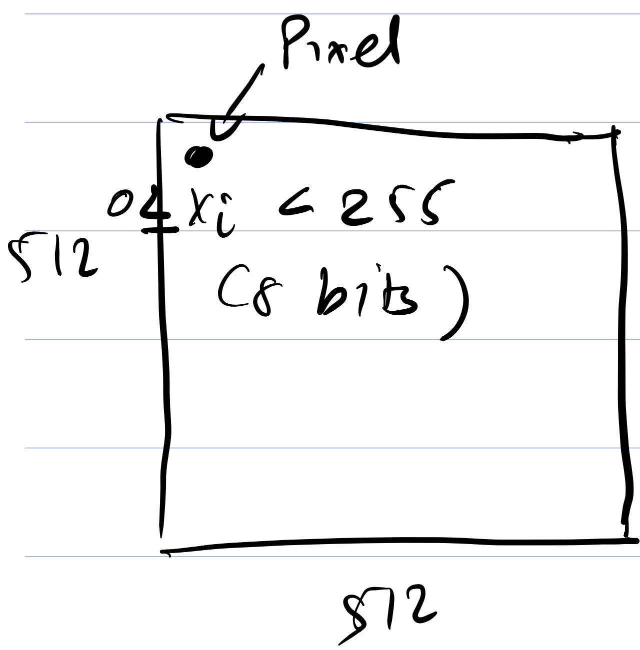


Image $x \in \mathbb{R}^{n}$, $n \to (512)^2$

$\to$ JPEG compression $\rightsquigarrow$ (change of basis)

Standard basis:

every pixel gives a value.

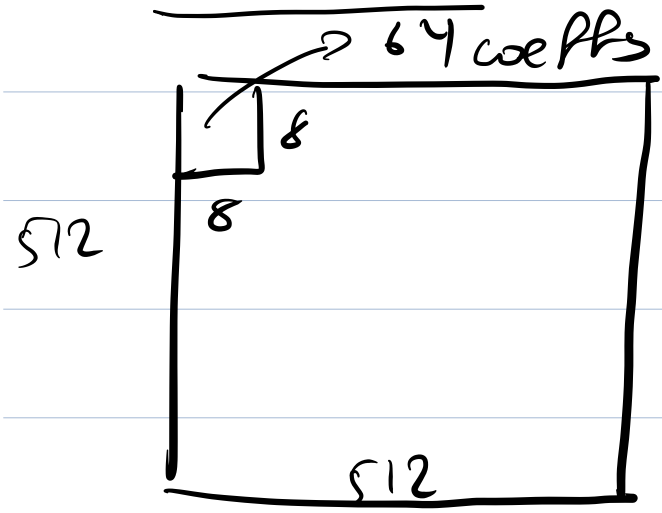
$$x = \begin{array}{c} 512 \\ \begin{bmatrix} 121 \\ 124 \\ \vdots \end{bmatrix} \end{array}$$

$$\left.\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \cdots\cdots \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}\right\} \text{standard}$$

Better basis:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ -1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ \vdots \end{bmatrix}$$

Fourier basis (JPEG) (8x8)



$$
\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix},
\begin{bmatrix}1\\\omega\\\omega^2\\\omega^3\\\omega^4\\\omega^5\\\omega^6\\\omega^7\end{bmatrix},
\begin{bmatrix}1\\\omega^2\\\omega^4\\\omega^6\\\omega^8\\\omega^{10}\\\omega^{12}\\\omega^{14}\end{bmatrix},
\ldots,
\begin{bmatrix}1\\\omega^7\\\omega^{14}\\\omega^{21}\\\omega^{28}\\\omega^{35}\\\omega^{42}\\\omega^{49}\end{bmatrix}
$$

signal $\mathbf{x} \xrightarrow{\text{lossless}}$ 64 coefficients $c \xrightarrow{\text{lossy compression}} \hat{c}$ (many zeros) $\longrightarrow \hat{\mathbf{x}} = \sum \hat{c}_i \mathbf{v}_i$

In video, not only should we consider compressing each frame, we can also consider compressing sequences of frames. There's very little difference between one frame and the next. If we do it right, we only need to encode and compress the differences between frames, not every frame in its entirety.

The Haar wavelet basis is

wavelents $\mathbb{R}^8$

$$\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_8 \end{bmatrix} = \begin{bmatrix} 1\\1\\1\\1\\1\\1\\1\\1 \end{bmatrix}, \begin{bmatrix} 1\\1\\1\\1\\-1\\-1\\-1\\-1 \end{bmatrix}, \begin{bmatrix} 1\\1\\-1\\-1\\0\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\0\\1\\1\\-1\\-1 \end{bmatrix}, \begin{bmatrix} 1\\-1\\0\\0\\0\\0\\0\\0 \end{bmatrix}, \ldots, \begin{bmatrix} 0\\0\\0\\0\\0\\0\\1\\-1 \end{bmatrix}$$

$c_1 x \quad c_2 x \quad c_3 x \quad c_4 x$

standard basis.

$\Rightarrow p = \underbrace{c_1 w_1 + \cdots + c_8 w_8}$

This is just a linear comb of wavelet basis vectors.

$$\cancel{x} = W \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_8 \end{bmatrix}$$

$\hookrightarrow$ W is the matrix whose columns are the wavelets vectors

$p = Wc$

$c = W^{-1} p.$

. GOOD BASIS:

- Multiplication by the basis matrix and its inverse is fast (as in the FFT or in the wavelet basis).
- Good compression – the image can be approximated using only a few basis vectors. Most components c_i are small – safely set to zero.

Change of basis

Vectors

Let col. of W = new basis vectors

$[x] \longrightarrow [c] \Rightarrow x = Wc$

old basis.   new basis.

Transformation matrices

T with respect to $v_1, \dots, v_8$
it has a matrix $\underline{A}$.   similar matrices.
with respect to $w_1, \dots, w_8$
it has matrix $\underline{B}$.

$$\left\{ B = M^{-1} A M \right\}$$

- What is A? using $v_1 \dots v_8$
know T completely from $T(v_1)$, $T(v_2)$, ..., $T(v_8)$

Because every $x = c_1 v_1 + c_2 v_2 + \dots + c_8 v_8$

Then $T(x) = c_1 T(v_1) + c_2 T(v_2) + \dots$
$+ c_8 T(v_8)$

$\Rightarrow$ Write $T(v_1) = a_{11} v_1 + a_{21} v_2 + a_{81} v_8$
$T(v_2) = a_{12} + \dots + a_{28} v_8$

$$[A] = \begin{bmatrix} a_{11} & a_{21} \\ \vdots & \vdots \\ a_{18} & a_{28} \end{bmatrix}$$

Eigenvector basis:

$T(v_i) = \lambda_i v_i$

What is A?

$$A = \begin{bmatrix} \lambda_1 & 0 & & \\ 0 & \lambda_2 & & \\ 0 & 0 & \ddots & \\ 0 & 0 & & \lambda_n \end{bmatrix}$$

1st inputs is v_1
its output is $\lambda_1 v_1$
2nd output v_2
output $\lambda_2 v_2$

Summary

When we change bases, the coefficients of our vectors change according to the rule $\mathbf{x} = W\mathbf{c}$. Matrix entries change according to a rule $B = M^{-1}AM$.