Objectives

1) Formula for A^{-1}
2) Cramers Rule for $x = A^{-1} b$
3) $|\text{Det } A|$ = Volume of box

1) Formula A^{-1}

We know that:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

product of $n-1$ entries.

$$A^{-1} = \frac{1}{\det A} \underbrace{C^T}$$: cofactor transpose matrix.

Product of n entries.

(3×3)

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} =$$

Check $\Rightarrow A C^T = (\det A) I$

$$AC^T = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix}.$$

Last time:

Cofactor Formula

$$\det A = a_{i1} C_{i1} + a_{i2} C_{i2} + \dots + a_{in} C_{in}$$

$i = 1$

→ The entry in the first row and first column of the product matrix is:

$$\sum_{j=1}^{n} a_{1j} C_{j1} = \det A$$

↳ This happens for every entry on the diagonal of AC^T

Note :

To finish proving that $AC^T = \det A\, I$ we need to check that the off diagonal entries are zero.

∴ In (2×2) case, multiplying the entries in row 1 of A by the entries in column 2 of C^T

$\Rightarrow a(-b) + b(a) = 0.$

↳ det of $A = \begin{bmatrix} a & b \\ a & b \end{bmatrix}$

- In higher dimensions, the product of the 1st row of A & last column of C^T equals the determinant of a matrix, whose first & last rows are identical.

↳ This happens with all the off diagonal matrices, which confirms that $A^{-1} = \dfrac{1}{\det A} C^T$

Cramer's Rule for $x = A^{-1}b$

We know that if $Ax = b$, and A is non-singular, then
$x = A^{-1} b$

↳ Applying the formula $A^{-1} = \frac{C^T}{\det A}$

gives us:

$$x = \frac{1}{\det A} C^T b$$

↳ Cramer's rule gives us another way of looking at this equation.

→ Break down x into components.

$$x_j = \frac{\det B_j}{\det A}$$ Because the i^{th} component of $C^T b$

$\parallel$

sum of cofactors

$\times$

some number.

where B_j is the matrix created by starting with A and then replacing column j with $\mathbf{b}$, so:

$$B_1 = \begin{bmatrix} \mathbf{b} & \begin{array}{c}\text{last n-1}\\ \text{columns}\\ \text{of } A\end{array} \end{bmatrix} \quad \text{and}$$

$$B_n = \begin{bmatrix} \begin{array}{c}\text{first n-1}\\ \text{columns}\\ \text{of } A\end{array} & \mathbf{b} \end{bmatrix}.$$

$|\det A| =$ volume of box

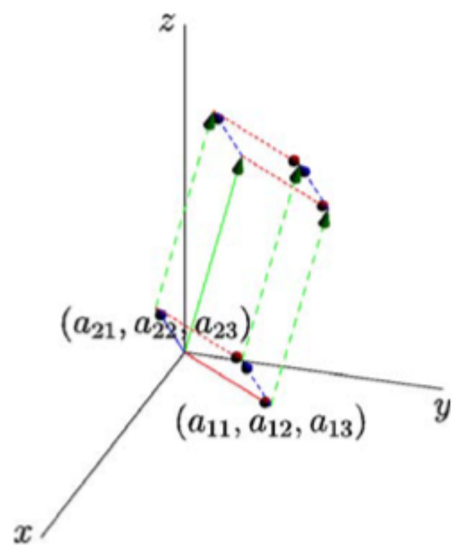

Figure 1: The box whose edges are the column vectors of A.

Claim: $|\det A|$ is the volume of the box (parallelepiped) whose edges are the column vectors of A.

If $A = I$, then the box is a unit cube and its volume is 1. Because this agrees with our claim, we can conclude that the volume obeys determinant property 1.

If $A = Q$ is an orthogonal matrix then the box is a unit cube in a different orientation with volume $1 = |\det Q|$. (Because Q is an orthogonal matrix, $Q^T Q = I$ and so $\det Q = \pm 1$.)

Swapping two columns of A does not change the volume of the box or (remembering that $\det A = \det A^T$) the absolute value of the determinant (property 2). If we show that the volume of the box also obeys property 3 we'll have proven $|\det A|$ equals the volume of the box.

$*\ \det Q = \pm 1$

$Q^T Q = I$ det on eey side

$|Q^T|\,|Q| = |I|$

$|Q|\,|Q| = 1$

$|Q|^2 = 1 \quad \therefore (\pm 1)$

If we double the length of one column of A, we double the volume of the box formed by its columns. Volume satisfies property 3(a).

Property 3(b) says that the determinant is linear in the rows of the matrix:

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}. \Big\} \text{ parallelogram}$$

Figure 2 illustrates why this should be true.

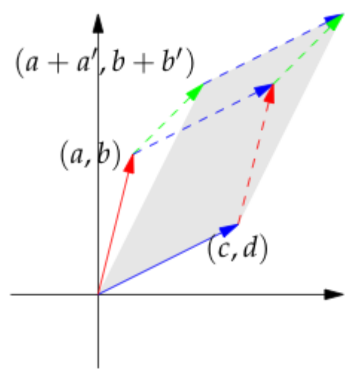

Figure 2: Volume obeys property 3(b).

Although it's not needed for our proof, we can also see that determinants obey property 4. If two edges of a box are equal, the box flattens out and has no volume.