Problem 23.1

(6.3 #14a)

Skew-Symmetric matrix. $(A^T = -A)$

$$\frac{du}{dt} = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} u$$

or

$$u_1' = cu_2 - bu_3$$
$$u_2' = au_3 - cu_1$$
$$u_3' = bu_1 - au_2$$

Find the derivative of $||u(t)||^2$ using definition:

$$\|u(t)\|^2 = u_1^2 + u_2^2 + u_3^2$$

$\therefore$

$$\frac{d\|u(t)\|^2}{dt} = \frac{d(u_1^2+u_2^2+u_3^2)}{dt}$$

$$= 2u_1u_1' + 2u_2u_2' + 2u_3u_3'$$

$$= 2u_1(cu_2 - bu_3) + 2u_2(au_3 - cu_1) + 2u_3(bu_1 - au_2)$$

$$= 0$$

$\therefore$

This means that $\|u(t)\|^2 = \|u(0)\|^2$

Because $u(t)$ never changes length, it's always on the circufere of a circle of radius $\|u(0)\|$

Problem 23.2

(6.3 # 24).

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$$

Eigenvalues of A are: $\lambda_1 = 1$

$\lambda_2 = 3$

Eigenvectors: $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$x_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\Rightarrow S = \begin{bmatrix} 1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 0 & 2 \end{bmatrix}$$

$$\Rightarrow \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\Rightarrow S^{-1} = \begin{bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{bmatrix}$$

$$\Rightarrow \vec{x}_0 \, e^{At}$$

$$S e^{\Lambda t} S^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} e^{t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} e^{t} & .5e^{2t} - .5e^{t} \\ 0 & e^{t} \end{bmatrix} = e^{At}.$$

Check

$$e^{At} = \begin{bmatrix} e^{t} & .5e^{3t} - .5e^{t} \\ 0 & e^{3t} \end{bmatrix}$$

equals to I, when $t=0$ ✓

$$\frac{de^{At}}{dt} = \begin{bmatrix} e^{t} & 1.5e^{3t} - .5e^{t} \\ 0 & 3e^{3t} \end{bmatrix}$$

$$\left.\frac{de^{At}}{dt}\right|_{t=0} = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} = A$$ ✓