Objectives

- Linear independence
- Spanning a space
- BASIS and dimension.

(more unknown x's than equations)

Suppose A is m by n with $m < n$
Then there are non zero solutions to $Ax = 0$

Reason: There will be free variables!.

- Independence

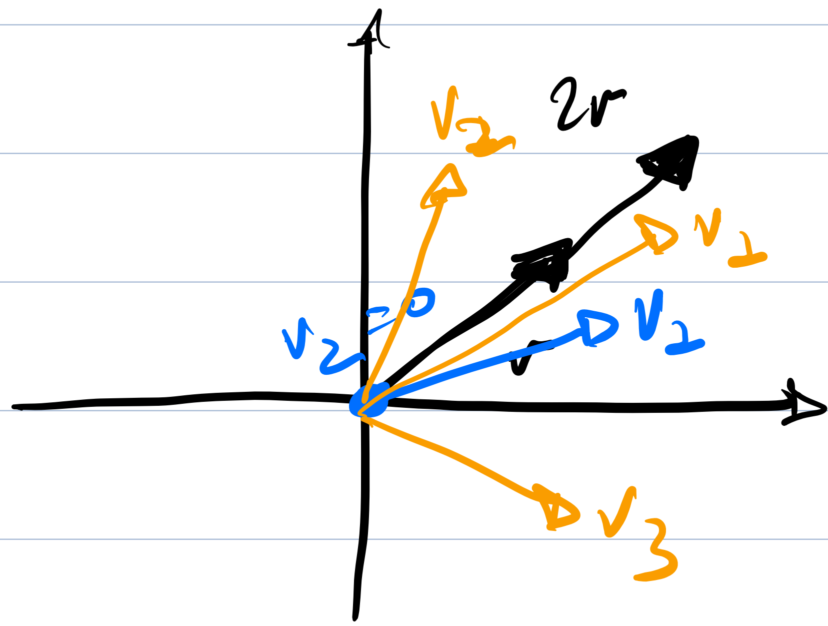
When vectors $x_1, x_2, \ldots, x_n$ are independent if.

↳ if no combination gives zero vector (except the zero comb).

↳ when all $c_i = 0$)

$c_1 x_1 + c_2 x_2 + \dots + c_n x_n \neq 0$

eg



$2v_1 - v_2 = 0$

(∴ dependent)

$0v_1 + b v_2 = 0$

$$\begin{array}{cc} & \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ A = & \begin{bmatrix} 2 & 1 & 2.5 \\ 1 & 2 & -1 \end{bmatrix} \end{array} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

→ Repeat when $v_1, \dots, v_n$ are column of A.

↳ They are independent if nullspace of A is $\{\text{zero vector}\}$ no free variables → rank $= n$

$N(A) = 0$

↳ They are dependent if $Ac = 0$ for some non-zero c → rank $< n$

Yes, free variables

· Spanning a space

Vectors $v_1, \dots, v_\ell$ span a space means:

The space consist of all combs. of those vectors.

(In other words, instead of saying
"Take all linear combinations and
put them in a space)

• <u>Basis</u> for a space is a sequence
of vectors $v_1, v_2, \dots, v_d$
with 2 properties:

- independent
- span the vector space.

<u>Example</u>:

Space is $\mathbb{R}^3$

- One basis is: $\begin{bmatrix} 1 \\ \end{bmatrix} \begin{bmatrix} 0 \\ \end{bmatrix} \begin{bmatrix} 0 \\ \end{bmatrix}$

* standard $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- Another basis:

$$\begin{array}{ccc} ① & ② & \\ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, & \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}, & \begin{bmatrix} 3 \\ 3 \\ 8 \end{bmatrix} \end{array}$$

Note:
* Not independent
→ 2 equal rows
∴ invertible

* Note: $\begin{bmatrix} 3 \\ 3 \\ 7 \end{bmatrix}$ is not correct

combination of ① + ②

∴ dependent

⇒ R^n ∴ n vectors give basis if the n×n matrix with those columns is invertible

Given a space : cols

Every basis for a space, has the same $\boxed{\text{number}}$ of vectors

Definition:

Dimension of the space

Eg

Space is $C(A)$

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \qquad R = 2$$

$N(A) = [Ax = 0]$ or combine columns to produce ϕ

$$\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- Find basis of matrix

- rank(A) = # of pivot columns
  = dimension of the $C(A)$

- another basis for $C(A)$ => $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ $\begin{bmatrix} 7 \\ 5 \\ 7 \end{bmatrix}$

  eg

- dim $C(A) = r$
- dim $N(A)$ = # free variables
  $[n-r]$