Objectives

- Differential Equations $\frac{du}{dt} = Au$
- Exponential e^{At} of a matrix

Example:

$$u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\frac{du_1}{dt} = -u + 2u_2$$

$$\frac{du_2}{dt} = u_1 - 2u_2$$

$$\rightarrow A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$$

(S1) Find eigenvalues & eigenvectors

$$A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$$

→ singular matrix

$[\times(-2)\ 1^{st} = \text{colm}]$

TRACE $= -3$

$=$ sum of eigenvalues

$\therefore \lambda_1 = 0$

$\therefore \lambda_2 = -3$

(S2) Eigenvectors

$\lambda_1 = 0 \qquad x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad Ax_1 = 0x_1$

$\lambda_2 = -3 \qquad x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad Ax_2 = -3x_2$

⇒ Solution:

$$u(t) = \underbrace{c_1 e^{\lambda_1 t} x_1}_{\text{pure sol}} + \underbrace{c_2 e^{\lambda_2 t} x_2}_{\text{pure sol.}}$$

Check : $\frac{du}{dt} = Au$

↳ Plug in $e^{\lambda_1 t} x_1$

$\therefore LHS = \lambda_1 \cancel{e^{\lambda_1 t}} x_1 = A\cancel{e^{\lambda_1 t}} x_1$ (RHS)

(S3) What are c_1 & c_2

$$\mathbf{u}(t) = c_1 \underbrace{e^{\lambda_1 t}}_{\lambda_1 = 0} \mathbf{x}_1 + c_2 e^{\lambda_2 t} \mathbf{x}_2 =$$

$$\Rightarrow c_1 \cdot 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \cdot e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Now, use condition $u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

At $t=0$

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\therefore c_1 = c_2 = \frac{1}{3}$

$$\Rightarrow u(t) = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3} e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

This tell us that the system starts with $u_1 = 1$ & $u_2 = 0$, but that as $t \to \infty$, u_1 decays to $2/3$ & u_2 increases to $1/3$.

$\Rightarrow$ Steady state of this system is

$$u(\infty) = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Stability

1. Stability: $\mathbf{u}(t) \to 0$ when $\mathrm{Re}(\lambda) < 0$.
2. Steady state: One eigenvalue is 0 and all other eigenvalues have negative real part.
3. Blow up: if $\mathrm{Re}(\lambda) > 0$ for any eigenvalue λ.

If a two by two matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has two eigenvalues with negative real part, its trace $a + d$ is negative. The converse is not true: $\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$ has negative trace but one of its eigenvalues is 1 and e^{1t} blows up. If A has a positive determinant and negative trace then the corresponding solutions must be stable.

eg If $\left| e^{(-3+\underbrace{6i}_{\text{img.}})t} \right| = e^{-3t}$

$|e^{6it}| = 1$ (Euler formula)

Applying S

The final step of our solution to the system $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$ was to solve:

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

In matrix form:

$$\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Set $u = Sv$

↳ eigenvector matrix.

In the equation $\frac{u\mathbf{u}}{dt} = A\mathbf{u}$, the matrix A couples the pure solutions. We set $\mathbf{u} = S\mathbf{v}$, where S is the matrix of eigenvectors of A, to get:

$$S\frac{d\mathbf{v}}{dt} = AS\mathbf{v}$$

or:

$$\frac{d\mathbf{v}}{dt} = S^{-1}AS\mathbf{v} = \Lambda\mathbf{v}.$$

This diagonalizes the system: $\frac{dv_i}{dt} = \lambda_i v_i$. The general solution is then:

$$\begin{aligned} \mathbf{v}(t) &= e^{\Lambda t}\mathbf{v}(0), \qquad \text{and} \\ \mathbf{u}(t) &= Se^{\Lambda t}S^{-1}\mathbf{v}(0) = e^{At}\mathbf{u}(0). \end{aligned}$$

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Matrix exponential e^{At}

What does e^{At} mean if A is a matrix.

$$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \cdots + \frac{(At)^n}{n!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots.$$

$\Rightarrow$ We've said that $e^{At} = Se^{\Lambda t}S^{-1}$

If A has n independent eigenvectors we can prove this from the def of e^{At} by using formula $A = S\Lambda S^{-1}$

$$\begin{aligned} e^{At} &= I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \cdots \\ &= SS^{-1} + S\Lambda S^{-1}t + \frac{S\Lambda^2 S^{-1}}{2}t^2 + \frac{S\Lambda^3 S^{-1}}{6}t^3 + \cdots \\ &= Se^{\Lambda t}S^{-1}. \end{aligned}$$

e^{At} :

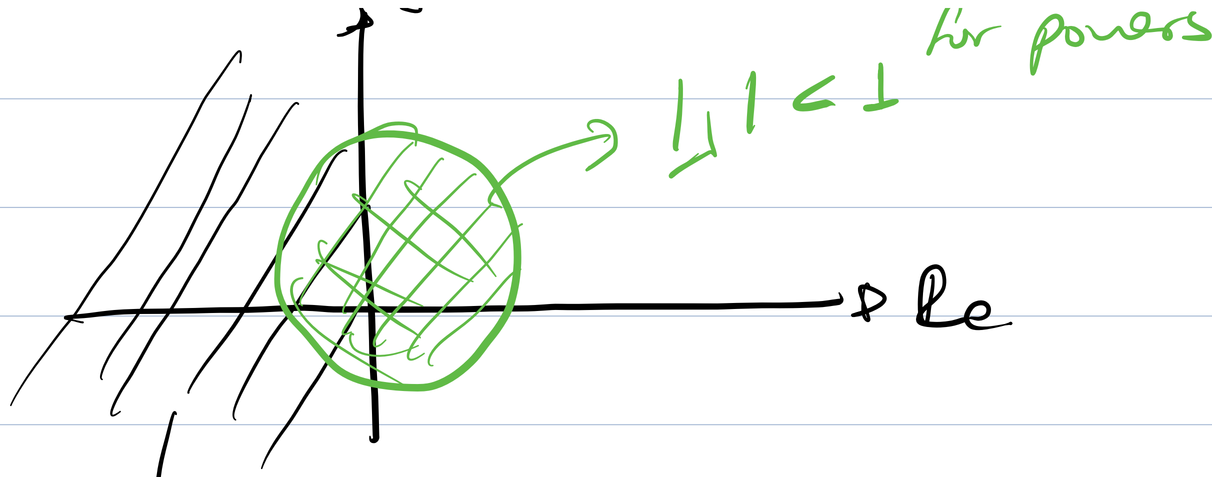
$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & \vdots & \lambda_3 & \cdots & 0 \\ 0 & \vdots & & \ddots & \\ 0 & & & & \lambda_n \end{bmatrix}$$

$\therefore$ for e^{At}

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix}$$

$\boxed{Re(\lambda < 0)}$ $\exp \to \phi$

Im



stability
$Re(\lambda) < 0)$

Second order

$$y'' + by' + ky = 0$$

$$u = \begin{bmatrix} y' \\ y \end{bmatrix}$$

$$u' = \begin{bmatrix} y'' \\ y \end{bmatrix} = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y' \\ y \end{bmatrix}$$