HOMEWORK - 3

Exercise 1:

Show that

$P(X = x) = F(x^+) - F(x^-)$

$\Rightarrow$ eg. $F(x_2) - F(x_1)$

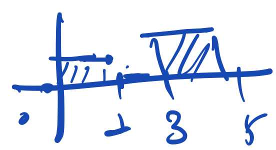
$= P(X \leq x_2) - P(x \leq x_1)$

* This can also be proove by definition

Exercise 4: (Basic question) *

Let X have probability density function:

$$f_X(x) = \begin{cases} 1/4 & 0 < x < 1 \\ 3/8 & 3 < x < 5 \\ 0 & \text{otherwise} \end{cases}$$



a) Find CDF (need to integrate)

$\therefore$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x/4 & 1 < x < 1 \\ 1/4 & 1 < x \leq 3 \\ 1/4 + \frac{3}{8}(x-3) & 3 < x \leq 5 \\ 1 & x > 5 \end{cases}$$

(b) Let $Y = \frac{1}{X}$. Find the probability density function (PDF) $f_Y(y)$ for Y.

→ Hint: Consider 3 cases

- $\frac{1}{5} \le y \le \frac{1}{3}$

- $\frac{1}{3} \le y \le 1$

- $y \geq 1$

$\therefore F_Y(y) = P(Y \le y) = P(\frac{1}{X} \le y)$

$= \underline{P(X \geq 1/y)}$

Limits for y.

Limits for x: $\left.\begin{array}{l} 0 \le x \le 1 \\ 1 \le x \le 3 \\ 3 \le x \le 5 \end{array}\right\} \rightarrow \begin{array}{l} 1 \le y \le \infty \\ \frac{1}{3} \le y \le 1 \\ \frac{1}{5} \le y \le \frac{1}{3} \end{array}$

$$\therefore \quad F_Y(y) = \mathbb{P}(X > 1/y) = \int_{1/y}^{5} f_X(x)\,dx$$

$$= \begin{cases} 0 & y < 0 \\ \frac{3}{8}(5 - 1/y) & \frac{1}{5} \le y \le 1/3 \\ 3/4 & \frac{1}{3} \le y \le 1 \\ \frac{3}{4} + \frac{1}{4}\left(1 - \frac{1}{y}\right) & y \ge 1 \end{cases} \quad \begin{matrix} \\ 3 \le x \le 5 \\ 1 \le x \le 3 \\ 0 \le x \le 1 \end{matrix}$$

Note: Try to simulate ex 4 using R and Python.

Exercise 5:

5. Let X and Y be discrete random variables. Show that X and Y are independent if and only if $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ for all x and y.

X & Y $\Rightarrow$ discrete random variables

independent if & only if

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$
$$= \mathbb{P}(X=x)\,\mathbb{P}(Y=y)$$

if independent then

$$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\,\mathbb{P}(Y \in B)$$

for all events A, B

$\therefore$

$$\mathbb{P}(X \in A, Y \in B) = \sum_{x \in A} \sum_{y \in B} f(x,y)$$

$$= \sum_{x \in A} \sum_{x \in B} f_x(X) f_y(Y)$$

which can be broken down

to

$$\sum_{x \in A} f_X(x) \sum_{y \in B} f_Y(y) = \mathbb{P}(X \in A)\,\mathbb{P}(Y \in B)$$

hence independent

7. Let X and Y be independent and suppose that each has a Uniform$(0,1)$ distribution. Let $Z = \min\{X, Y\}$. Find the density $f_Z(z)$ for Z. Hint: It might be easier to first find $\mathbb{P}(Z > z)$.

Let X & Y be independent

$\Rightarrow X \sim$ Uniform$(0,1)$

$\Rightarrow Y \sim$ Uniform$(0,1)$

Let $Z = \min\{X, Y\}$

- Firstly:

$$\mathbb{P}(Z > z) = \mathbb{P}(X > z, Y > z)$$

$$= [1 - F_X(z)][1 - F_Y(z)]$$

Since both F_X & F_Y are CDF of Uniform then:

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x \leq 1 \\ 1 & x > 1 \end{cases}$$

$$F_Y(y) = \begin{cases} 0 & y \leq 0 \\ y & 0 < y \leq 1 \\ 1 & \text{otherwise} \end{cases}$$

$\therefore$

$$F_Z(z) = \begin{cases} 0 & z \leq 0 \\ 2z - z^2 & 0 < z < 1 \\ 1 & z > 1 \end{cases}$$

[CDF]

$$f_Z(z) = F_Z'(z)$$

$$f_Z(z) = \begin{cases} 0 & z \leq 0 \\ 2 - 2z & 0 < z < 1 \\ 0 & z > 1 \end{cases}$$

10. Let X and Y be independent. Show that $g(X)$ is independent of $h(Y)$ where g and h are functions.

X & Y are independent

$\therefore$ Show that $g(X)$ & $h(Y)$ are independent

$\Rightarrow$

$$g^{-1}(A) = \{x : g(x) \in A\}$$

$$h^{-1}(B) = \{y : h(y) \in B\}$$

$\therefore$

$$\mathbb{P}(g(X) \in A, h(Y) \in B)$$

$$= \mathbb{P}(X \in g^{-1}(A), Y \in h^{-1}(B))$$

$$= \mathbb{P}(X \in g^{-1}(A))\, \mathbb{P}(Y \in h^{-1}(B))$$

$$= \mathbb{P}((g(X) \in A)\, \mathbb{P}((h(Y)) \in B)$$

$\therefore$ independent

12. Prove Theorem 2.33. A

Suppose that the range of X and Y is a (possibly infinite) rectangle. If $f(x, y) = g(x)h(y)$ for some functions g and h (not necessarily probability density functions) then X and Y are independent.

Solution:

- Given that f is the joint PDF of X & Y

$\Rightarrow f_X(x) = \int f(x,y)\,dy = \int g(x)\,h(y)\,dy$

$= g(x) \int h(y)\,dy = \underline{H\,g(x)}$

$\rightarrow f_Y(y) = \int f(x,y)\,dx = \int g(x)\,h(y)\,dx$

$= h(y) \int g(x)\,dx = \underline{G\,h(y)}$

$\therefore f(x,y) = f_X(x)\,f_Y(y)$

$= H g(x)\, G h(y)$

$= \underline{H G f(x,y)}$

$\therefore$ Fixing $y = y_0$

$\hookrightarrow \int f_X(x) f_Y(y_0)\, dx$

$= 4G \int f(x, y_0)\, dx \Rightarrow f_X(y_0)$

$= 4G f_Y(y_0) = 4G = 1$

and so

$f_X(x) f_Y(y) = f(x, y)$

$\therefore$ independent

14. Let (X, Y) be uniformly distributed on the unit disk $\{(x,y) : x^2+y^2 \leq 1\}$. Let $R = \sqrt{X^2+Y^2}$. Find the CDF and PDF of R.

$\therefore$

$$F_R(r) = \mathbb{P}(R \leq r) = \mathbb{P}(X^2+Y^2 \leq r^2)$$

$$= \iint_{x^2+y^2} f(x,y)\,dx\,dy$$

↳ $\frac{\text{area of radius } r}{\text{area radius } 1} = \frac{\pi r^2}{\pi} = r^2$

↳ Area of circle of radius : r

$F_R(1) = 1 \quad \therefore$

CDF

$$F_R(r) = \begin{cases} 0 & r \leq 0 \\ r^2 & 0 < r \leq 1 \\ 1 & r > 1 \end{cases}$$

$$f_R(r) = \begin{cases} & r \leq 0 \\ 2r & 0 < r \leq 1 \\ 0 & r > 1 \end{cases}$$

18. Let $X \sim N(3, 16)$. Solve the following using the Normal table and using a computer package.

(a) Find $\mathbb{P}(X < 7)$.

(b) Find $\mathbb{P}(X > -2)$.

(c) Find x such that $\mathbb{P}(X > x) = .05$.

(d) Find $\mathbb{P}(0 \leq X < 4)$.

(e) Find x such that $\mathbb{P}(|X| > |x|) = .05$.

* We can also use computer package scipy.stats to compute expressions.

a) $\mathbb{P}(X<7) = \mathbb{P}\left(\frac{X-3}{\sqrt{16}} < \frac{7-3}{\sqrt{16}}\right) = \mathbb{P}(Z<1)$

$= \Phi(1) = 0.8413$ //

(b) $\mathbb{P}(X>-2) = \mathbb{P}\left(\frac{X-3}{4} > \frac{-2-3}{4}\right)$

$= \mathbb{P}(Z > -5/4) = 1 - \mathbb{P}(Z < -5/4)$

$= 1 - \Phi(-5/4) = 0.8944$ //

(c)

$P(X > x) = 0.5$

$\therefore 1 - F_X(x) = 0.5 \Rightarrow F_X(x) = 0.95$

$x = F_X^{-1}(0.95) \rightsquigarrow 9.5794$

d)

$P(0 \leq X < 4) = P\left(\frac{0-3}{4} < Z < \frac{4-3}{4}\right)$

$= \Phi(1/4) - \Phi(-3/4) = 0.3721$ ✓

e)

$P(|X| > |x|) = 0.5$ for constant $c \geq 0$

$\therefore P(|X| > c) = 1 - P(|X| < c)$

$= 1 - P(-c < X < -c)$

$= 1 - P\left(\frac{-c-3}{4} < \frac{c-3}{4}\right)$

$$= 1 - \Phi\left(\frac{c-3}{4}\right) + \Phi\left(-\frac{c-3}{4}\right)$$

$$= 2 - 2\Phi\left(\frac{c-3}{4}\right) = 0.05$$

$$\hookrightarrow c = 4\Phi^{-1}(0.975) + 3$$

$$\rightsquigarrow 10.8399 \,//$$