Example Questions Chapter 3

- 3.2 - Conditional Probabilities

EXAMPLE 2a

A student is taking a one-hour-time-limit makeup examination. Suppose the probability that the student will finish the exam in less than x hours is $x/2$, for all $0 \leq x \leq 1$. Then, given that the student is still working after .75 hour, what is the conditional probability that the full hour is used?

⇒ Let L_x denote the event that the student finishes the exam in less than x hours

⇒ Let F be the event that the student uses the full hour.

$\therefore$

$$P(F) = P(L_x^c) = 1 - P(L_1) = 0.5$$

$$\Rightarrow P(F \mid L_{0.75}^c) = \frac{P(F L_{0.75}^c)}{P(L_{0.75}^c)}$$

$$= \frac{P(F)}{1 - P(L_{0.75})}$$

$$= \frac{0.5}{0.625} = \underline{\underline{0.8}}$$

EXAMPLE 2d

A total of n balls are sequentially and randomly chosen, without replacement, from an urn containing r red and b blue balls ($n \leq r + b$). Given that k of the n balls are blue, what is the conditional probability that the first ball chosen is blue?

→ Blue balls numbered from $(1 - b)$

→ Red balls numbered from $(b+1) - (b+r)$

∴ equally likely to happen.

↳ hence $\frac{k}{n}$ → P(blue ball selected)

⇒ Let B be the event of first ball chosen is blue

⇒ Let B_k be the total of k blue balls chosen.

∴

$$\mathbb{P}(B \mid B_k) = \frac{\mathbb{P}(B B_k)}{\mathbb{P}(B_k)}$$
$$= \frac{\mathbb{P}(B_k \mid B)\, \mathbb{P}(B)}{\mathbb{P}(B_k)}$$

∴ $\mathbb{P}(B_k \mid B)$ = a random choice of $(n-1)$ balls from an urn containing r red & $(b-1)$ blue results in a total $(k-1)$ blue balls being chosen.

∴

$$\mathbb{P}(B_k \mid B) = \frac{\binom{b-1}{k-1}\binom{r}{n-k}}{\binom{r+b-1}{n-1}}$$

$\therefore$ using preceding formula along with

$$\mathbb{P}(B) = \frac{b}{r+b}$$

& hypergeometric probability

$$\mathbb{P}(B_k) = \frac{\binom{b}{k}\binom{r}{n-k}}{\binom{r+b}{n}}$$

$\Leftrightarrow$ $\mathbb{P}(B \mid B_k) = \frac{k}{n}$

3.3 - BAYES'S FORMULA

Example 3a

I

An insurance company believes that people can be divided into two classes: those who are accident prone and those who are not. The company's statistics show that an accident-prone person will have an accident at some time within a fixed 1-year period with probability .4, whereas this probability decreases to .2 for a person who is not accident prone. If we assume that 30 percent of the population is accident prone, what is the probability that a new policyholder will have an accident within a year of purchasing a policy?

Let A_1 → event that the holder will have an accident within a year of purchasing the policy.
Let A → prone to accident

$\therefore$

$$\mathbb{P}(A_1) = \mathbb{P}(A_1 | A)\mathbb{P}(A) + \mathbb{P}(A_1 | A^c)\mathbb{P}(A^c)$$
$$= (0.4)(0.3) + (0.2)(0.7) = \underline{\underline{0.26}}$$

The change in the probability of a hypothesis when new evidence is introduced can be expressed compactly in terms of the change in the *odds* of that hypothesis, where the concept of odds is defined as follows.

Definition

The odds of an event A are defined by

$$\frac{P(A)}{P(A^c)} = \frac{P(A)}{1 - P(A)}$$

That is, the odds of an event A tell how much more likely it is that the event A occurs than it is that it does not occur. For instance, if $P(A) = \frac{2}{3}$, then $P(A) = 2P(A^c)$, so the odds are 2. If the odds are equal to α, then it is common to say that the odds are "α to 1" in favor of the hypothesis.

Consider now a hypothesis H that is true with probability $P(H)$, and suppose that new evidence E is introduced. Then the conditional probabilities, given the evidence E, that H is true and that H is not true are respectively given by

$$P(H|E) = \frac{P(E|H)P(H)}{P(E)} \qquad P(H^c|E) = \frac{P(E|H^c)P(H^c)}{P(E)}$$

Therefore, the new odds after the evidence E has been introduced are

$$\boxed{\frac{P(H|E)}{P(H^c|E)} = \frac{P(H)}{P(H^c)}\frac{P(E|H)}{P(E|H^c)}} \tag{3.3}$$

That is, the new value of the odds of H is the old value, multiplied by the ratio of the conditional probability of the new evidence given that H is true to the conditional probability given that H is not true. Thus, Equation (3.3) verifies the result of Example 3f, since the odds, and thus the probability of H, increase whenever the new evidence is more likely when H is true than when it is false. Similarly, the odds decrease whenever the new evidence is more likely when H is false than when it is true.

3.4 - Independent Events

Example 4g

A system composed of n separate components is said to be a parallel system if it functions when at least one of the components functions. (See Figure 3.2.) For such a system, if component i, which is independent of the other components, functions with probability $p_i, i = 1, \ldots, n$, what is the probability that the system functions?

Solution. Let A_i denote the event that component i functions. Then

$$\begin{aligned}
P\{\text{system functions}\} &= 1 - P\{\text{system does not function}\} \\
&= 1 - P\{\text{all components do not function}\} \\
&= 1 - P\left(\bigcap_i A_i^c\right) \\
&= 1 - \prod_{i=1}^{n}(1 - p_i) \quad \text{by independence}
\end{aligned}$$

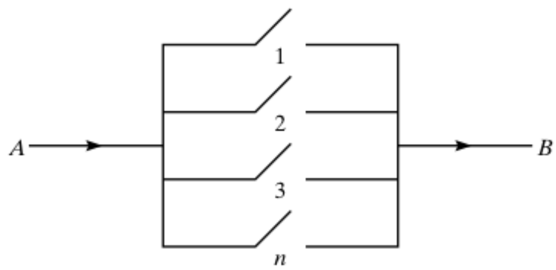

FIGURE 3.2: Parallel System: Functions if Current Flows from *A* to *B*

3.5 - P(·|F) is a Probability

EXAMPLE 3e *Laplace's rule of succession*

There are $k + 1$ coins in a box. When flipped, the ith coin will turn up heads with probability $i/k, i = 0, 1, \ldots, k$. A coin is randomly selected from the box and is then

repeatedly flipped. If the first n flips all result in heads, what is the conditional probability that the $(n + 1)$st flip will do likewise?

Solution. Let C_i denote the event that the ith coin, $i = 0, 1, \ldots, k$, is initially selected; let F_n denote the event that the first n flips all result in heads; and let H be the event that the $(n + 1)$st flip is a head. The desired probability, $P(H|F_n)$, is now obtained as follows:

$$P(H|F_n) = \sum_{i=0}^{k} P(H|F_nC_i)P(C_i|F_n)$$

Now, given that the ith coin is selected, it is reasonable to assume that the outcomes will be conditionally independent, with each one resulting in a head with probability i/k. Hence,

$$P(H|F_nC_i) = P(H|C_i) = \frac{i}{k}$$

Also,

$$P(C_i|F_n) = \frac{P(C_iF_n)}{P(F_n)} = \frac{P(F_n|C_i)P(C_i)}{\sum_{j=0}^{k} P(F_n|C_j)P(C_j)} = \frac{(i/k)^n[1/(k + 1)]}{\sum_{j=0}^{k}(j/k)^n[1/(k + 1)]}$$

Thus,

$$P(H|F_n) = \frac{\sum_{i=0}^{k}(i/k)^{n+1}}{\sum_{j=0}^{k}(j/k)^n}$$

But if k is large, we can use the integral approximations

$$\frac{1}{k}\sum_{i=0}^{k}\left(\frac{i}{k}\right)^{n+1} \approx \int_0^1 x^{n+1}\,dx = \frac{1}{n + 2}$$

$$\frac{1}{k}\sum_{j=0}^{k}\left(\frac{j}{k}\right)^{n} \approx \int_0^1 x^{n}\,dx = \frac{1}{n + 1}$$

So, for k large,

$$P(H|F_n) \approx \frac{n + 1}{n + 2}$$ ■

* Note: More examples can be found throughout this chapter