HOMEWORK - 1

Exercise 1:
Proof monotone decreasing

↳ Can be represented as the compliment of [Theorem 1.8] ⇝ Found in Notes

$\therefore$

$\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$

$\lim_{n \to \infty} \mathbb{P}(A_n^c) = \mathbb{P}(A^c)$

$\lim_{n \to \infty} 1 - \mathbb{P}(A_n^c) = 1 - \mathbb{P}(A^c)$

$\lim_{n \to \infty} \mathbb{P}(A_n) = \mathbb{P}(A)$

* Monotone decreasing case can be represented by complementary series $A_1^c, A_2^c, A_3^c \dots A_n^c$

Exercise 2:
Proof of probability properties

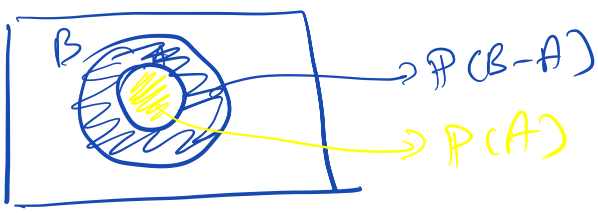
• $\mathbb{P}(\emptyset) = 0$

↳ By partitioning into disjoint event space Ω

$\mathbb{P}(\Omega) + \mathbb{P}(\emptyset) = \mathbb{P}(\Omega) \quad \therefore \underline{\mathbb{P}(\emptyset) = 0}$

• $A \subset B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$

$\therefore \mathbb{P}(A) + \mathbb{P}(B - A) = \mathbb{P}(B)$



• $0 \leq \mathbb{P}(A) \leq 1$

From AXIOM 1: $\mathbb{P}(A) \geq 0 \quad \therefore$

$\mathbb{P}(A) + \mathbb{P}(A^c) = \mathbb{P}(\Omega) = 1 \therefore \mathbb{P}(A) \leq 1$

• $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$

$\mathbb{P}(A) + \mathbb{P}(A^c) = \mathbb{P}(\Omega) = 1$

$\therefore \mathbb{P}(A^c) = 1 - \mathbb{P}(A)$

• $A \cap C = \emptyset \Rightarrow \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

If events are disjoint $\Rightarrow \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

} no intersections

$\therefore \Omega = A + B$

Exercise 3:

a) let Ω be sample space

let $A_1, A_2 \dots$ be events

$B_n = \bigcup_{i=n}^{\infty} A_i$ → nonotone increasing

$C_n = \bigcap_{i=n}^{\infty} A_i$ → monotone decreasing

Note: $A \subset B$
A is a subset of B

∴ show: $B_1 \supset B_2 \supset B_3 \dots$
$C_1 \subset C_2 \subset C_3$

sample space A_1, A_2, A_3, A_4

$B_{n+1} \supset B_n$ ✓ $C_{n+1} \subset C_n$

b) Show that $\omega \in \bigcap_{n=1}^{\infty} B_n$ if and only if ω belongs to an infinite number of events $A_1, A_2 \ldots$

- $B_n = \bigcup_{i=n}^{\infty} A_i$

- Firstly no $\omega \in A_j \quad j \in J(\omega)$
For every n there is an $m \geq n$, $\therefore m \in J(\omega)$
$\omega \in B_n$ for every n.
so $\omega \in \bigcap_{n=1}^{\infty} B_n$

- Since $\omega \in \bigcap_{n=1}^{\infty} B$, then $\omega \in B_n$, $m \geq n$
such that $\omega \in A_m$ $\exists$ infinite number of events A_m

(c) Show that $w \in \bigcup_{n\geq 1}^{\infty} C_n$ if and only if w belongs to all events $A_1, A_2, \dots,$ except possibly a finite number of those events

compliment of ex (b)

$C_n = \bigcap_{i=n}^{\infty} A_i$

- Firstly let $w \in A_j^c$ $j \in J(w)$ then for every n, $m \geq n$ $\therefore$ $w \in A_m^c$ $\therefore$ w is <u>not</u> in none of the C_n

- Since w is not in the union of C_n $\therefore$ w is not in any events of C_n, therefore for every n, there is an $m \geq n$, such that w is not in A_m

↳ This implies that there is an <u>infinite</u> number of such events A_m.

HOMEWORK - 2

Exercise: 5

Suppose we toss a fair coin until we get exactly two heads. Describe the sample space S. What is the probability that exactly k tosses are required?

fair coin : either H or T

Ω : finite sequence $\{\omega_1, \omega_2, \omega_3 \cdots \omega_k\}$

such that $\omega_k = H$ & $\omega_{k-1} = H$

Lorest of them are tails (T)

Let A : events

k : # of tosses

$$\underbrace{\omega_1, \omega_2 \cdots, \omega_{k-1}}_{T, \& \ H \text{ (one)}} \quad \underbrace{\omega_k}_{H}$$

$\therefore$ $P(H) = 1/2$

$P(T) = 1/2$

$\Rightarrow P(\text{one Head}) = \frac{1}{2^k}$

There are $\underbrace{\binom{k-1}{1}}_{\text{possible outcomes}} = k-1$

$$\boxed{\frac{k-1}{2^k}}$$

Exercise: 6

Let Ω = {0, 1,...,}. Prove that there does not exist a uniform distribution on Ω (i.e., if P(A)=P(B) whenever |A| = |B|, then P cannot satisfy the axioms of probability).

• $\Omega = \{0, 1 \ldots\}$ no uniform distr. if $\mathbb{P}(A) = \mathbb{P}(B)$, whenever $|A| = |B|$

↳ This is not valid since the elements in A are included in B ∴ no intersection since $\mathbb{P}(A) = \mathbb{P}(B)$

↳ Let $c = \mathbb{P}(\{s_i\})$ if $c = 0$ then $\mathbb{P}(S) = \sum_i \mathbb{P}(\{s_i\}) = 0$ But this cannot be true since $1 = 0$

↳ if $c > 0$ ⇝ contradiction since $\sum \mathbb{P}(\{s_i\}) \to \infty$

AXIOM 3 does <u>not</u> hold

Exercise: 8

Suppose that $\mathbb{P}(A_i) = 1$ for each i:

Prove:

$$\mathbb{P}\left(\bigcap_{i=1}^{\infty} A_i\right) = 1$$

- $$\mathbb{P}\left(\bigcap_{i=1}^{\infty} A_i\right) = 1 - \mathbb{P}\left(\left(\bigcap_{i=1}^{\infty} A_i\right)^c\right)$$

$$= 1 - \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i^c\right)$$

$$\therefore \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i^c\right) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i^c) = \sum_{i=1}^{\infty} (1 - \mathbb{P}(A_i))$$

$$= \sum_{i=1}^{\infty} 0 = 0$$

so the equality holds, since a probability is non-negative.

$$\therefore \mathbb{P}\left(\bigcap_{i=1}^{\infty} A_i\right) = 1 - \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i^c\right) = 1 - 0 = 1$$

(*Note: ex 7) see pg 12

Exercise: 10 *Nice exercise.

You have probably heard it before. Now you can solve it rigorously. It is called the "Monty Hall Problem." A prize is placed at random behind one of three doors. You pick a door. To be concrete, let's suppose you always pick door 1. Now Monty Hall chooses one of the other two doors, opens it and shows you that it is empty. He then gives you the opportunity to keep your door or switch to the other unopened door. Should you stay or switch? Intuition suggests it doesn't matter. The correct answer is that you should switch. Prove it. It will help to specify the sample space and the relevant events carefully. Thus write $\Omega = \{(\omega_1,\omega_2): \omega_i \in \{1, 2, 3\}\}$ where ω_1 is where the prize is and ω_2 is the door Monty opens.

- $\Omega = \{(\omega_1, \omega_2) : \omega_i \in \{1,2,3\}\}$

where: ω_1 : where the prize is
ω_2 : the door Monty opens

- $\Omega = \{(1,2),(1,3),(2,3),(3,2)\}$

(event space)

$\mathbb{P}(\omega_2)$ = probability of opening an empty door

R = reward

ω	$\mathbb{P}$	R
$(1,2)$	$\frac{1}{3}\frac{1}{2}$	0
$(1,3)$	$\frac{1}{3}\frac{1}{2}$	0
$(2,3)$	$\frac{1}{3}1$	1
$(3,2)$	$\frac{1}{3}1$	1

$$\therefore \mathbb{P}(R \mid \omega_2 = 2) = \frac{\mathbb{P}(\{3,2\})}{\mathbb{P}(\{(3,2),(1,2)\})} = \frac{1/3 \times 1}{(1/3 \times 1) + (1/3 \cdot 1/2)}$$

$$= 2/3 = \mathbb{P}(R \mid \omega_3)$$

Exercise: 19

Suppose that 30 percent of computer owners use a Macintosh, 50 percent use Windows, and 20 percent use Linux. Suppose that 65 percent of the Mac users have succumbed to a computer virus, 82 percent of the Windows users get the virus, and 50 percent of the Linux users get the virus. We select a person at random and learn that her system was infected with the virus. What is the probability that she is a Windows user?

$M: 30\%$; $W: 50\%$; $L: 20\%$

$\downarrow$ virus: 65% $\quad\downarrow 82\%\quad\downarrow 50\%$

outcome	probability
Mac, no virus	30% * 35%
Mac, virus	30% * 65%
Windows, no virus	50% * 18%
Windows, virus	50% * 82%
Linux, no virus	20% * 50%
Linux, virus	20% * 50%

$$\therefore P(W \mid virus) = \frac{P(W, virus)}{P(virus)} =$$

$$= \frac{0.50 \times 0.82}{(0.30 \times 0.65) + (0.5 \times 0.82) + (0.2 \times 0.5)}$$

$$\approx \underline{\underline{0.5816}}$$

(ex7)

7) Let $A_1, A_2, \ldots$ be events.
Show that
$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mathbb{P}(A_n)$$
Hint: Define $B_n = A_n - \bigcup_{i=1}^{n-1} A_i$.
Then show that the B_n are disjoint and that
$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$

- Let $B_n = A_n - \bigcup_{i=1}^{n-1} A_i$

- Note that for $i<j$, B_i and B_j are disjoint since all the elements of B_i must be elements of A_i, and all elements of A_i are explicitly excluded on the definition B_j.

- Also, $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$:

$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} B_i = \bigcup_{n=1}^{\infty} B_n$, since $B_i \cup B_i = B_i$ and we can include each B_i only once in the expression.

$(\Rightarrow)$

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \mathbb{P}\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(B_n) \le \sum_{n=1}^{\infty} \mathbb{P}(A_n)$$

since

$$B_n \cup \left(\bigcup_{i=1}^{n-1} A_i\right) = A_n \quad \text{and so}$$

$$\mathbb{P}(B_n) \le \mathbb{P}(A_n) \text{ for every } n$$

$\boxed{* \quad \bigcup_{i=1}^{n-1} A_i \text{ \} represent the rings}}$

· Exercise 20 - Computer Experiment

probability p: falling heads

↳ if we flip it many times we expect the proportion of heads to be near p.

- Take $p = 0.3$ & $n = 1000$

* Plot the proportion of heads as a function of n.

+ Repeat for $p = 0.3$