Chapter - 1 - Combinatorial Analysis

1.1 - Introduction

Example:

A communication system is to consist of n seemingly identical antennas, that are to be lined up in a linear order. The resulting system will then be able to receive all incoming signals - and will be called functional - as long as no 2 consecutive antennas are defective

If it turns out that exactly m of the n antennas are defective,

What is the probability that the resulting system will be functional?

$\therefore$ $n=4$; $m=2$ → 6 possible outcomes

1 → Antenna is working

0 → Antena is <u>not</u> working.

$$\left.\begin{array}{cccc}0&1&1&0\\0&1&0&1\\1&0&1&0\end{array}\right\}\to \text{functional}$$

$$\left.\begin{array}{cccc}0&0&1&1\\1&0&0&1\\1&1&0&0\end{array}\right\}\to \text{not functional}$$

(⇒) Probability $= 3/6 = 1/2$

<u>General:</u>

Count the number of configurations that the result in the system's being functional and the divide by the total number of all possible configurations.

↳ The mathematical theory of counting is known as combinatorial analysis.

1.2 - Basic Principle of Counting

If one experiment can result in any m of possible outcomes, and if another experiment can result in any of n possible outcomes

$\Rightarrow$ then there are mn possible outcomes of the 2 experiments.

EXAMPLE 2a

A small community consists of 10 women, each of whom has 3 children. If one woman and one of her children are to be chosen as mother and child of the year, how many different choices are possible?

Solution. By regarding the choice of the woman as the outcome of the first experiment and the subsequent choice of one of her children as the outcome of the second experiment, we see from the basic principle that there are $10 \times 3 = 30$ possible choices. ■

When there are more than two experiments to be performed, the basic principle can be generalized.

The generalized basic principle of counting

If r experiments that are to be performed are such that the first one may result in any of n_1 possible outcomes; and if, for each of these n_1 possible outcomes, there are n_2 possible outcomes of the second experiment; and if, for each of the possible outcomes of the first two experiments, there are n_3 possible outcomes of the third experiment; and if ..., then there is a total of $n_1 \cdot n_2 \cdots n_r$ possible outcomes of the r experiments.

1.3 - Permutations

How many different ordered arrangements of letters a, b, c are possible?

⇝ By direct enumeration we see that there are 6

[abc, acb, bac, bca, cab, cba]

each arrangement is known as

permutation

EXAMPLE 3d

How many different letter arrangements can be formed from the letters *PEPPER*?

Solution. We first note that there are 6! permutations of the letters $P_1E_1P_2P_3E_2R$ when the 3P's and the 2E's are distinguished from each other. However, consider any one of these permutations—for instance, $P_1P_2E_1P_3E_2R$. If we now permute the P's among themselves and the E's among themselves, then the resultant arrangement would still be of the form $PPEPER$. That is, all 3! 2! permutations

$$\begin{array}{ll}
P_1P_2E_1P_3E_2R & P_1P_2E_2P_3E_1R \\
P_1P_3E_1P_2E_2R & P_1P_3E_2P_2E_1R \\
P_2P_1E_1P_3E_2R & P_2P_1E_2P_3E_1R \\
P_2P_3E_1P_1E_2R & P_2P_3E_2P_1E_1R \\
P_3P_1E_1P_2E_2R & P_3P_1E_2P_2E_1R \\
P_3P_2E_1P_1E_2R & P_3P_2E_2P_1E_1R
\end{array}$$

are of the form $PPEPER$. Hence, there are $6!/(3!\ 2!) = 60$ possible letter arrangements of the letters *PEPPER*. ■

In general, the same reasoning as that used in Example 3d shows that there are

$$\frac{n!}{n_1!\,n_2!\,\cdots\,n_r!}$$

different permutations of n objects, of which n_1 are alike, n_2 are alike, $\ldots, n_r$ are alike.

1.4 - Combinations

We are often interested in determining the number of different groups or objects that could be formed from a total of n objects.

Example:

How many different groups of 3 could be selected from the 5 items (A, B, C, D, E)

→ Since there are 5 ways to select the initial term, 4 ways to then select next item . . .

Since we need groups of 3

↳ 6 different combinations

[ABC, ACB, BAC, BCA, CAB, CBA]

$$\frac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1} = 10$$

* In general, as $n(n-1)\cdots(n-r-1)$ represents the number of different ways that a group of $\underline{r}$ items could be selected from $\underline{n}$ items

$$= \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

$$\boxed{= \frac{n!}{(n-r)!\,r!}}$$

Notation and terminology

We define $\binom{n}{r}$, for $r \leq n$, by

$$\binom{n}{r} = \frac{n!}{(n - r)!\, r!}$$

and say that $\binom{n}{r}$ represents the number of possible combinations of n objects taken r at a time.†

Thus, $\binom{n}{r}$ represents the number of different groups of size r that could be selected from a set of n objects when the order of selection is not considered relevant.

EXAMPLE 4c

Consider a set of n antennas of which m are defective and $n - m$ are functional and assume that all of the defectives and all of the functionals are considered indistinguishable. How many linear orderings are there in which no two defectives are consecutive?

Solution. Imagine that the $n - m$ functional antennas are lined up among themselves. Now, if no two defectives are to be consecutive, then the spaces between the functional antennas must each contain at most one defective antenna. That is, in the $n - m + 1$ possible positions—represented in Figure 1.1 by carets—between the $n - m$ functional antennas, we must select m of these in which to put the defective antennas. Hence, there are $\binom{n-m+1}{m}$ possible orderings in which there is at least one functional antenna between any two defective ones. ■

$\wedge\, 1 \,\wedge\, 1 \,\wedge\, 1 \ldots \wedge\, 1 \,\wedge\, 1 \,\wedge$

1 = functional

$\wedge$ = place for at most one defective

FIGURE 1.1: No consecutive defectives

A useful combinatorial identity is

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r} \qquad 1 \le r \le n \tag{4.1}$$

Equation (4.1) may be proved analytically or by the following combinatorial argument: Consider a group of n objects, and fix attention on some particular one of these objects—call it object 1. Now, there are $\binom{n-1}{r-1}$ groups of size r that contain object 1 (since each such group is formed by selecting $r - 1$ from the remaining $n - 1$ objects). Also, there are $\binom{n-1}{r}$ groups of size r that do not contain object 1. As there is a total of $\binom{n}{r}$ groups of size r, Equation (4.1) follows.

The values $\binom{n}{r}$ are often referred to as *binomial coefficients* because of their prominence in the binomial theorem.

The binomial theorem

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \tag{4.2}$$

Proof :

Proof of the Binomial Theorem by Induction: When $n = 1$, Equation (4.2) reduces to

$$x + y = \binom{1}{0} x^0 y^1 + \binom{1}{1} x^1 y^0 = y + x$$

Assume Equation (4.2) for $n - 1$. Now,

$$\begin{aligned}
(x + y)^n &= (x + y)(x + y)^{n-1} \\
&= (x + y) \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-1-k} \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} x^{k+1} y^{n-1-k} + \sum_{k=0}^{n-1} \binom{n-1}{k} x^k y^{n-k}
\end{aligned}$$

Letting $i = k + 1$ in the first sum and $i = k$ in the second sum, we find that

$$\begin{aligned}
(x + y)^n &= \sum_{i=1}^{n} \binom{n-1}{i-1} x^i y^{n-i} + \sum_{i=0}^{n-1} \binom{n-1}{i} x^i y^{n-i} \\
&= x^n + \sum_{i=1}^{n-1} \left[\binom{n-1}{i-1} + \binom{n-1}{i} \right] x^i y^{n-i} + y^n \\
&= x^n + \sum_{i=1}^{n-i} \binom{n}{i} x^i y^{n-i} + y^n \\
&= \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}
\end{aligned}$$

where the next-to-last equality follows by Equation (4.1). By induction, the theorem is now proved.

Combinatorial Proof of the Binomial Theorem: Consider the product

$$(x_1 + y_1)(x_2 + y_2) \cdots (x_n + y_n)$$

Its expansion consists of the sum of 2^n terms, each term being the product of n factors. Furthermore, each of the 2^n terms in the sum will contain as a factor either x_i or y_i for each $i = 1, 2, \ldots, n$. For example,

$$(x_1 + y_1)(x_2 + y_2) = x_1 x_2 + x_1 y_2 + y_1 x_2 + y_1 y_2$$

Now, how many of the 2^n terms in the sum will have k of the x_i's and $(n - k)$ of the y_i's as factors? As each term consisting of k of the x_i's and $(n - k)$ of the y_i's corresponds to a choice of a group of k from the n values $x_1, x_2, \ldots, x_n$, there are $\binom{n}{k}$ such terms. Thus, letting $x_i = x, y_i = y, i = 1, \ldots, n$, we see that

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

1.5 - Multinomial Coefficients

A set of n distinct items is to be divided into r distinct groups of respective sizes $n_1, n_2, n_3 \cdots, n_r$ where $\sum_{i=1}^{r} n_i = n$

→ How many different divisions are possible?

To answer this question we note that

there are $\binom{n}{n_1}$ possible choices for the first group; for each choice of the first group, there are $\binom{n-n_1}{n_2}$ possible choices for the second group; for each choice of the first two groups, there are $\binom{n-n_1-n_2}{n_3}$ possible choices for the third group; and so on. It then follows from the generalized version of the basic counting principle that there are

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-n_2-\cdots-n_{r-1}}{n_r}$$
$$= \frac{n!}{(n-n_1)!\,n_1!}\frac{(n-n_1)!}{(n-n_1-n_2)!\,n_2!}\cdots\frac{(n-n_1-n_2-\cdots-n_{r-1})!}{0!\,n_r!}$$
$$= \frac{n!}{n_1!\,n_2!\cdots n_r!}$$

possible divisions.

Another way to see this result is to consider the n values $1, 1, \ldots, 1, 2, \ldots, 2, \ldots, r, \ldots, r$, where i appears n_i times, for $i = 1, \ldots, r$. Every permutation of these values corresponds to a division of the n items into the r groups in the following manner: Let the permutation $i_1, i_2, \ldots, i_n$ correspond to assigning item 1 to group i_1, item 2 to group i_2, and so on. For instance, if $n = 8$ and if $n_1 = 4$, $n_2 = 3$, and $n_3 = 1$, then the permutation $1, 1, 2, 3, 2, 1, 2, 1$ corresponds to assigning items $1, 2, 6, 8$ to the first group, items $3, 5, 7$ to the second group, and item 4 to the third group. Because every permutation yields a division of the items and every possible division results from some permutation, it follows that the number of divisions of n items into r distinct groups of sizes $n_1, n_2, \ldots, n_r$ is the same as the number of permutations of n items of which n_1 are alike, and n_2 are alike, ..., and n_r are alike, which was shown in Section 1.3 to equal $\dfrac{n!}{n_1!n_2!\cdots n_r!}$.

Notation

If $n_1 + n_2 + \cdots + n_r = n$, we define $\binom{n}{n_1, n_2, \ldots, n_r}$ by

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!\, n_2! \cdots n_r!}$$

Thus, $\binom{n}{n_1, n_2, \ldots, n_r}$ represents the number of possible divisions of n distinct objects into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$.

EXAMPLE 5c

In order to play a game of basketball, 10 children at a playground divide themselves into two teams of 5 each. How many different divisions are possible?

Solution. Note that this example is different from Example 5b because now the order of the two teams is irrelevant. That is, there is no A and B team, but just a division consisting of 2 groups of 5 each. Hence, the desired answer is

$$\frac{10!/(5!\,5!)}{2!} = 126 \qquad \blacksquare$$

The proof of the following theorem, which generalizes the binomial theorem, is left as an exercise.

The multinomial theorem

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{\substack{(n_1,\ldots,n_r):\\ n_1 + \cdots + n_r = n}} \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$

That is, the sum is over all nonnegative integer-valued vectors $(n_1, n_2, \ldots, n_r)$ such that $n_1 + n_2 + \cdots + n_r = n$.

The numbers $\binom{n}{n_1, n_2, \ldots, n_r}$ are known as *multinomial coefficients*.

EXAMPLE 5d

~ Important ☆

In the first round of a knockout tournament involving $n = 2^m$ players, the n players are divided into $n/2$ pairs, with each of these pairs then playing a game. The losers of the games are eliminated while the winners go on to the next round, where the process is repeated until only a single player remains. Suppose we have a knockout tournament of 8 players.

(a) How many possible outcomes are there for the initial round? (For instance, one outcome is that 1 beats 2, 3 beats 4, 5 beats 6, and 7 beats 8.)

(b) How many outcomes of the tournament are possible, where an outcome gives complete information for all rounds?

Solution. One way to determine the number of possible outcomes for the initial round is to first determine the number of possible pairings for that round. To do so, note that the number of ways to divide the 8 players into a *first* pair, a *second* pair, a *third* pair, and a *fourth* pair is $\binom{8}{2,2,2,2} = \frac{8!}{2^4}$. Thus, the number of possible pairings when there is no ordering of the 4 pairs is $\frac{8!}{2^4\,4!}$. For each such pairing, there are 2 possible choices from each pair as to the winner of that game, showing that there are $\frac{8!2^4}{2^4\,4!} = \frac{8!}{4!}$ possible results of round 1. (Another way to see this is to note that there are $\binom{8}{4}$ possible choices of the 4 winners and, for each such choice, there are 4! ways to pair the 4 winners with the 4 losers, showing that there are $4!\binom{8}{4} = \frac{8!}{4!}$ possible results for the first round.)

Similarly, for each result of round 1, there are $\frac{4!}{2!}$ possible outcomes of round 2, and for each of the outcomes of the first two rounds, there are $\frac{2!}{1!}$ possible outcomes of round 3. Consequently, by the generalized basic principle of counting, there are $\frac{8!}{4!}\frac{4!}{2!}\frac{2!}{1!} = 8!$ possible outcomes of the tournament. Indeed, the same argument can be used to show that a knockout tournament of $n = 2^m$ players has $n!$ possible outcomes.

Knowing the preceding result, it is not difficult to come up with a more direct argument by showing that there is a one-to-one correspondence between the set of possible tournament results and the set of permutations of $1, \ldots, n$. To obtain such a correspondence, rank the players as follows for any tournament result: Give the tournament winner rank 1, and give the final-round loser rank 2. For the two players who lost in the next-to-last round, give rank 3 to the one who lost to the player ranked 1 and give rank 4 to the one who lost to the player ranked 2. For the four players who lost in the second-to-last round, give rank 5 to the one who lost to player ranked 1, rank 6 to the one who lost to the player ranked 2, rank 7 to the one who lost to the player ranked 3, and rank 8 to the one who lost to the player ranked 4. Continuing on in this manner gives a rank to each player. (A more succinct description is to give the winner of the tournament rank 1 and let the rank of a player who lost in a round having 2^k matches be 2^k plus the rank of the player who beat him, for $k = 0, \ldots, m - 1$.) In this manner, the result of the tournament can be represented by a permutation $i_1, i_2, \ldots, i_n$, where i_j is the player who was given rank j. Because different tournament results give rise to different permutations, and because there is a tournament result for each permutation, it follows that there are the same number of possible tournament results as there are permutations of $1, \ldots, n$. ■

EXAMPLE 5e

$$
\begin{aligned}
(x_1 + x_2 + x_3)^2 &= \binom{2}{2,0,0} x_1^2x_2^0x_3^0 + \binom{2}{0,2,0} x_1^0x_2^2x_3^0 \\
&\quad + \binom{2}{0,0,2} x_1^0x_2^0x_3^2 + \binom{2}{1,1,0} x_1^1x_2^1x_3^0 \\
&\quad + \binom{2}{1,0,1} x_1^1x_2^0x_3^1 + \binom{2}{0,1,1} x_1^0x_2^1x_3^1 \\
&= x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3
\end{aligned}
$$
■

1.6 - The Number of integer solutions of Equations

There are r^n possible outcomes when n distinguishable balls are to be distributed into r distinguishable urns.

- Let us now suppose that n balls are indistinguishable from each other.

How many different outcomes are possible?

→ As the balls are indistinguishable then n balls into r urns. can be described by vector

$(x_1, x_2, x_3, \ldots, x_r)$

x_i: denotes # of balls

distributed into the ith urn.

$\therefore$ The problem reduces to finding the number of distinct non-negative integer-valued vectors $(x_1, x_2 \cdots x_r)$ such that

$$x_1 + x_2 + x_3 + \cdots x_r = n.$$

→ To compute this number, lets consider the # of positive integers valued solution.

Imagine we have n indistinguishable integers lined up and that we want to divide them into r-nonempty groups

→ To do so we can select $r-1$ of the $n-1$ spaces between adjacent objects as our dividing point.

To compute this number, let us start by considering the number of positive integer-valued solutions. Toward that end, imagine that we have n indistinguishable objects lined up and that we want to divide them into r nonempty groups. To do so, we can select $r - 1$ of the $n - 1$ spaces between adjacent objects as our dividing points. (See Figure 1.2.) For instance, if we have $n = 8$ and $r = 3$ and we choose the 2 divisors so as to obtain

$$\text{ooo}|\text{ooo}|\text{oo}$$

$$0 \wedge 0 \wedge 0 \wedge \ldots \wedge 0 \wedge 0$$

n objects 0

Choose $r - 1$ of the spaces $\wedge$.

FIGURE 1.2: Number of positive solutions

then the resulting vector is $x_1 = 3, x_2 = 3, x_3 = 2$. As there are $\binom{n-1}{r-1}$ possible selections, we have the following proposition.

Proposition 6.1. There are $\binom{n-1}{r-1}$ distinct positive integer-valued vectors $(x_1, x_2, \ldots, x_r)$ satisfying the equation

$$x_1 + x_2 + \cdots + x_r = n \quad x_i > 0, i = 1, \ldots, r$$

To obtain the number of nonnegative (as opposed to positive) solutions, note that the number of nonnegative solutions of $x_1 + x_2 + \cdots + x_r = n$ is the same as the number of positive solutions of $y_1 + \cdots + y_r = n + r$ (seen by letting $y_i = x_i + 1,\ i = 1, \ldots, r$). Hence, from Proposition 6.1, we obtain the following proposition.

Proposition 6.2. There are $\binom{n+r-1}{r-1}$ distinct nonnegative integer-valued vectors $(x_1, x_2, \ldots, x_r)$ satisfying the equation

$$x_1 + x_2 + \cdots + x_r = n \tag{6.1}$$

Note: 1-6 * optional material
[find more information].

SUMMARY

The basic principle of counting states that if an experiment consisting of two phases is such that there are n possible outcomes of phase 1 and, for each of these n outcomes, there are m possible outcomes of phase 2, then there are nm possible outcomes of the experiment.

There are $n! = n(n-1)\cdots 3 \cdot 2 \cdot 1$ possible linear orderings of n items. The quantity 0! is defined to equal 1.

Let

$$\binom{n}{i} = \frac{n!}{(n-i)!\, i!}$$

when $0 \le i \le n$, and let it equal 0 otherwise. This quantity represents the number of different subgroups of size i that can be chosen from a set of size n. It is often called a *binomial coefficient* because of its prominence in the binomial theorem, which states that

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

For nonnegative integers $n_1, \ldots, n_r$ summing to n,

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}$$

is the number of divisions of n items into r distinct nonoverlapping subgroups of sizes $n_1, n_2, \ldots, n_r$.