Proportion Inference 2

Modeling a count of successes

Download the section 15.Rmd handout to `STAT240/lecture/sect15-more-proportions`.

Download the file `chimpanzee.csv` to `STAT240/data`.

A psychic claims to be able to guess the suit of a random card without looking. 200 cards were drawn, and they guessed correctly 57 times.

Model:

$$X \sim Binom(200, p)$$

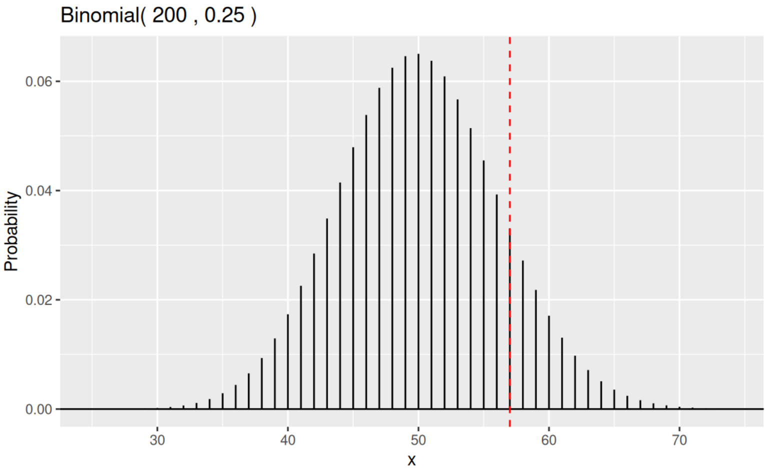
Our observed test statistic is $x_{obs} = 57$. Let's complete this test with $\alpha = 0.05$.

The problem suggests one-sided hypotheses:

$$H_0 : p \leq 0.25 \quad \text{versus} \quad H_A : p > 0.25$$

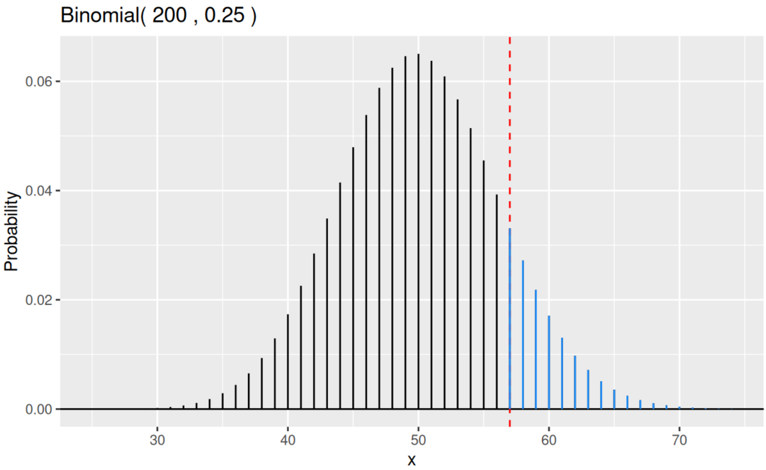
Under the null distribution, which outcomes are less likely than 57?

The distribution is not symmetric.


Binomial(200 , 0.25)
Probability
0.06
0.04
0.02
0.00
30
40
50
60
70
x

The outcomes as or less likely than x_{obs} are $X \leq 42$ and $X \geq 57$.

For $H_A : p > 0.25$, only $X \geq 57$ is relevant. We get a p-value of 0.145 and fail to reject H_0.

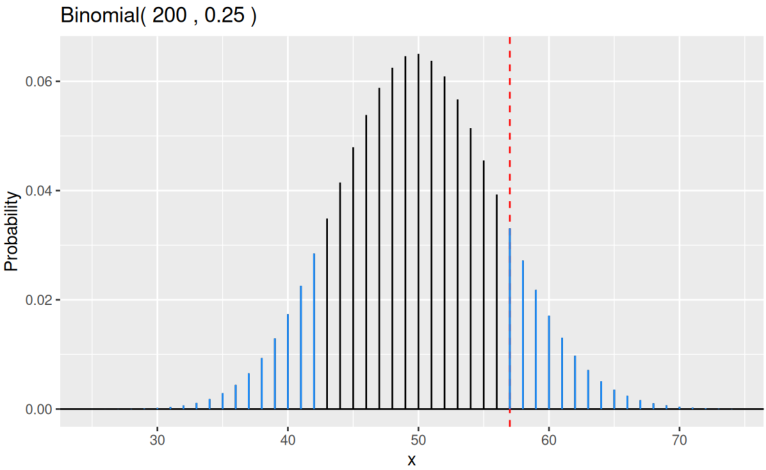

Binomial(200 , 0.25)
Probability
0.06
0.04
0.02
0.00
30
40
50
60
70
x

What if we had two-sided hypotheses?

$$H_0 : p = 0.25 \quad \text{versus} \quad H_A : p \neq 0.25$$

Now, we have to look for p being different from 0.25 in *both* directions.

Our p-value is $P(X \leq 42) + P(X \geq 57) = 0.253$.


Binomial(200 , 0.25)
Probability
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x

In general, for different hypothesis directions:

- Test statistic does not change
- Null distribution does not change

- p-value changes
- Conclusions change

What would we conclude if we HAD gotten a significant result?

An alternative to the binomial test is a **Z test**. If the null hypothesis were true,

$$\hat{p} \dot{\sim} N\left(p, \sqrt{\frac{p(1-p)}{n}}\right)$$

$$\hat{p} \dot{\sim} N\left(0.25, \sqrt{\frac{0.25(0.75)}{200}}\right)$$

By standardization,

$$Z = \frac{\hat{p} - 0.25}{\sqrt{\frac{0.25(0.75)}{200}}} \;\dot{\sim}\; N(0,1)$$

We calculate a Z test statistic and see whether it is consistent with $N(0,1)$.

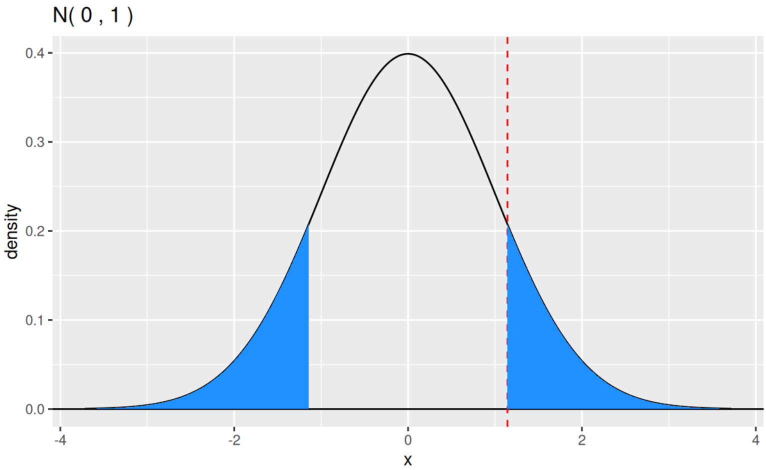
$$H_0 : p = 0.25 \quad \text{versus} \quad H_A : p \neq 0.25$$

$$z_{obs} = \frac{\frac{57}{200} - 0.25}{\sqrt{\frac{0.25(0.75)}{200}}} = 1.143$$

Is 1.143 consistent with $N(0, 1)$? For a two-sided test, we need the area outside of 1.143 in *both tails*.

$$P(Z \leq -1.143) \ + \ P(Z \geq 1.143)$$

The p-value is 0.253, just like the exact test.


N(0 , 1)
density
0.4
0.3
0.2
0.1
0.0
-4
-2
0
2
4
x

Let's return to the chimpanzee data.

What is the difference in prosocial choices made by chimpanzee C with vs without a partner?

Parameter of interest: $p_{partner} - p_{nopartner}$, or $p_1 - p_2$

$$\text{point estimate} \pm \text{critical value} \times \text{standard error}$$

- For $p_1 - p_2$, the point estimate is $\hat{p}_1 - \hat{p}_2$

What about the other parts? Consider the sampling distribution of $\hat{p}_1 - \hat{p}_2$.

We have

$$\hat{p}_1 \dot{\sim} N\left(p_1,\ \sqrt{\frac{p_1(1-p_1)}{n_1}}\right)$$

$$\hat{p}_2 \dot{\sim} N\left(p_2,\ \sqrt{\frac{p_2(1-p_2)}{n_2}}\right)$$

A difference of two independent normal RVs is also normal.

$$\hat{p}_1 - \hat{p}_2 \ \dot{\sim} \ N\Big(p_1 - p_2, \ \text{SE of difference}\Big)$$

$$\text{SE} = \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$$

Since we have a normal sampling distribution, we can build a Z CI.

We need to estimate the standard error. The **Agresti-Coffe** adjustment works like Agresti-Coull.

2 successes and 2 failures are distributed across *both* groups.

$$\hat{p}_{1AC} = \frac{X_1 + 1}{n_1 + 2}, \quad \hat{p}_{2AC} = \frac{X_2 + 1}{n_2 + 2}$$

For chimpanzee C:

$$\hat{p}_{1AC} = \frac{57+1}{90+2} = 0.63$$

$$\hat{p}_{2AC} = \frac{17+1}{30+2} = 0.56$$

The 95% AC interval is (-0.13, 0.27).

We have not done inference on p_1 or p_2, just the difference.

We are 95% confident that the difference in the % of prosocial choices is between (-0.13, 0.27).

- Negative: More prosocial without a partner
- Positive: More prosocial with a partner
- Ours covers 0

Build and interpret a 95% CI for the difference in prosocial behavior for Chimpanzee B.

- $\hat{p}_1$: proportion of prosocial choices with a partner
- $\hat{p}_2$: proportion of prosocial choices without a partner

Make sure to use the A-C adjustment.

We can also use the normal distribution to test a difference in proportions.

Is the probability of chimpanzee C making the prosocial choice higher when there is a partner?

We have hypotheses

$$H_0 : p_1 \leq p_2 \quad \text{versus} \quad H_A : p_1 > p_2$$

$$H_0 : p_1 - p_2 \leq 0 \quad \text{versus} \quad H_A : p_1 - p_2 > 0$$

We build a test statistic based on $\hat{p}_1 - \hat{p}_2$.

From before:

$$\hat{p}_1 - \hat{p}_2 \ \dot{\sim}\ N\Big(p_1 - p_2,\ \text{SE of difference}\Big)$$

$$\text{SE} \ = \ \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$$

We will standardize this to create a Z test statistic. But, we can simplify things.

$p_1 = p_2$ implies that there is a **common proportion** p. This is the overall rate of prosocial choices.

Under H_0,

$$\hat{p}_1 - \hat{p}_2 \ \dot{\sim} \ N\left(0, \ \sqrt{p(1-p)\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}\right)$$

Estimate p with the overall observed prosocial rate for C:

$$\hat{p} \ = \ \frac{X_1 + X_2}{n_1 + n_2} \ = \ \frac{57 + 17}{90 + 30}$$

When finding standard error, we just use $\hat{p}$ rather than the AC adjustment. We have test statistic

$$Z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

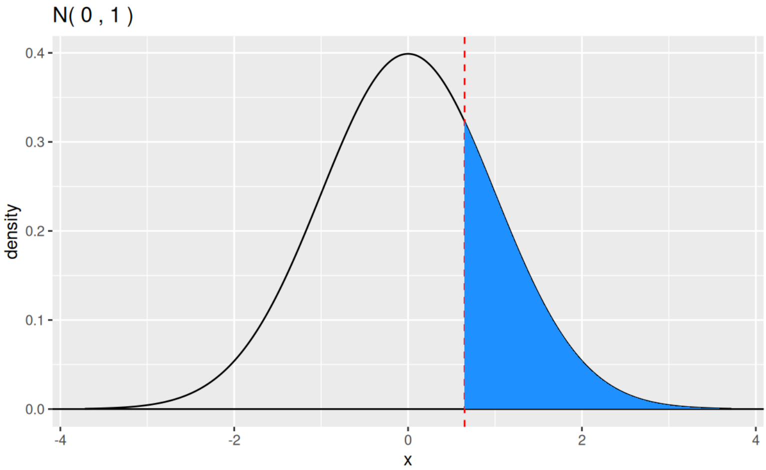
which has value $z_{obs} = 0.65$.

Since we have one-sided hypotheses

$$H_0 : p_1 - p_2 \leq 0 \quad \text{versus} \quad H_A : p_1 - p_2 > 0$$

our p-value is the area above 0.65 on $N(0, 1)$.

We get a large p-value of 0.258 and fail to reject the null.


N(0 , 1)
density
x
0.4
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4

Perform a hypothesis test of

$$H_0 : p_1 - p_2 = 0 \quad \text{versus} \quad H_A : p_1 - p_2 \neq 0$$

for chimpanzee B.

- Test stat and null distribution are the same
- Now we have *two-sided* hypotheses.

General test procedure:

1. Define a model
2. Write hypotheses
3. Find test statistic
4. Find relevant outcomes on null dist
5. Calculate p-value
6. Interpret results