9

Applications of Integration

9.1 Area between curves

We have seen how integration can be used to find an area between a curve and the x-axis. With very little change we can find some areas between curves; indeed, the area between a curve and the x-axis may be interpreted as the area between the curve and a second "curve" with equation $y = 0$. In the simplest of cases, the idea is quite easy to understand.

EXAMPLE 9.1.1 Find the area below $f(x) = -x^2 + 4x + 3$ and above $g(x) = -x^3 + 7x^2 - 10x + 5$ over the interval $1 \le x \le 2$. In figure 9.1.1 we show the two curves together, with the desired area shaded, then f alone with the area under f shaded, and then g alone with the area under g shaded.

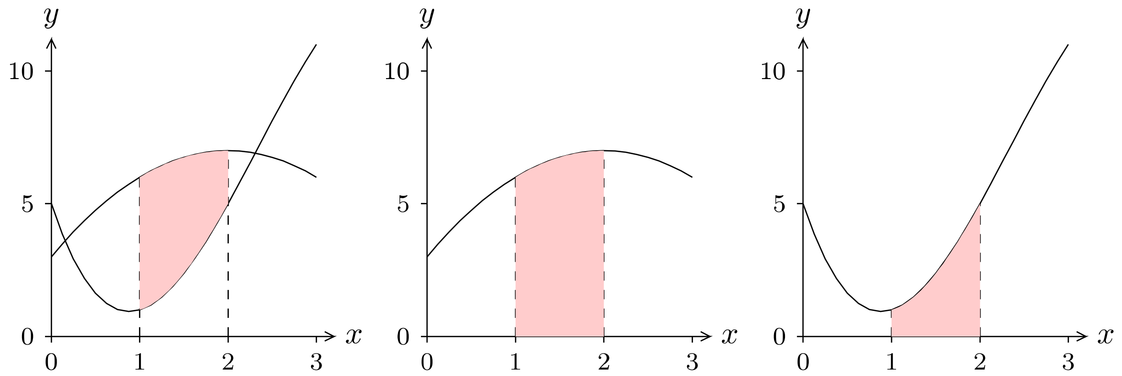

Figure 9.1.1 Area between curves as a difference of areas.

It is clear from the figure that the area we want is the area under f minus the area under g, which is to say

$$\int_1^2 f(x)\,dx - \int_1^2 g(x)\,dx = \int_1^2 f(x) - g(x)\,dx.$$

It doesn't matter whether we compute the two integrals on the left and then subtract or compute the single integral on the right. In this case, the latter is perhaps a bit easier:

$$\begin{aligned}
\int_1^2 f(x) - g(x)\,dx &= \int_1^2 -x^2 + 4x + 3 - (-x^3 + 7x^2 - 10x + 5)\,dx \\
&= \int_1^2 x^3 - 8x^2 + 14x - 2\,dx \\
&= \left.\frac{x^4}{4} - \frac{8x^3}{3} + 7x^2 - 2x\right|_1^2 \\
&= \frac{16}{4} - \frac{64}{3} + 28 - 4 - (\frac{1}{4} - \frac{8}{3} + 7 - 2) \\
&= 23 - \frac{56}{3} - \frac{1}{4} = \frac{49}{12}.
\end{aligned}$$

□

It is worth examining this problem a bit more. We have seen one way to look at it, by viewing the desired area as a big area minus a small area, which leads naturally to the difference between two integrals. But it is instructive to consider how we might find the desired area directly. We can approximate the area by dividing the area into thin sections and approximating the area of each section by a rectangle, as indicated in figure 9.1.2. The area of a typical rectangle is $\Delta x(f(x_i) - g(x_i))$, so the total area is approximately

$$\sum_{i=0}^{n-1} (f(x_i) - g(x_i))\Delta x.$$

This is exactly the sort of sum that turns into an integral in the limit, namely the integral

$$\int_1^2 f(x) - g(x)\,dx.$$

Of course, this is the integral we actually computed above, but we have now arrived at it directly rather than as a modification of the difference between two other integrals. In that example it really doesn't matter which approach we take, but in some cases this second approach is better.

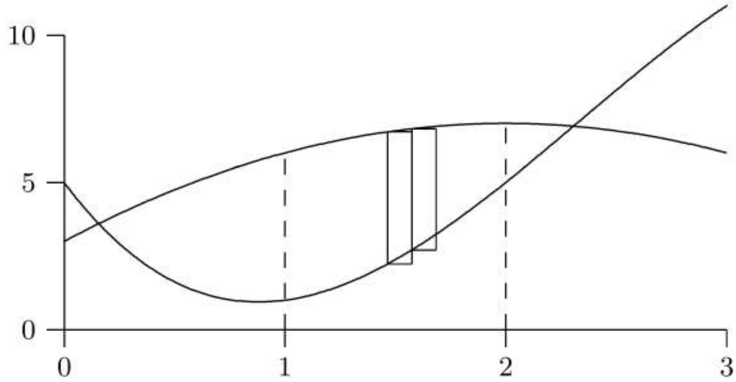

Figure 9.1.2 Approximating area between curves with rectangles.

EXAMPLE 9.1.2 Find the area below $f(x) = -x^2 + 4x + 1$ and above $g(x) = -x^3 + 7x^2 - 10x + 3$ over the interval $1 \le x \le 2$; these are the same curves as before but lowered by 2. In figure 9.1.3 we show the two curves together. Note that the lower curve now dips below the x-axis. This makes it somewhat tricky to view the desired area as a big area minus a smaller area, but it is just as easy as before to think of approximating the area by rectangles. The height of a typical rectangle will still be $f(x_i) - g(x_i)$, even if $g(x_i)$ is negative. Thus the area is

$$\int_1^2 -x^2 + 4x + 1 - (-x^3 + 7x^2 - 10x + 3)\,dx = \int_1^2 x^3 - 8x^2 + 14x - 2\,dx.$$

This is of course the same integral as before, because the region between the curves is identical to the former region—it has just been moved down by 2. □

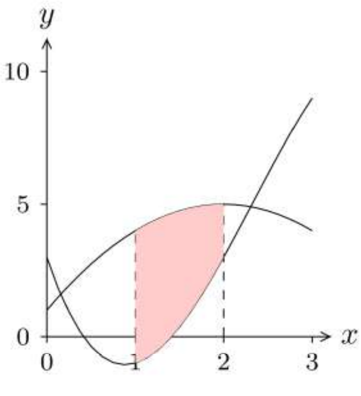

Figure 9.1.3 Area between curves.

EXAMPLE 9.1.3 Find the area between $f(x) = -x^2 + 4x$ and $g(x) = x^2 - 6x + 5$ over the interval $0 \le x \le 1$; the curves are shown in figure 9.1.4. Generally we should interpret

11. $y = x^{3/2}$ and $y = x^{2/3} \Rightarrow$

12. $y = x^2 - 2x$ and $y = x - 2 \Rightarrow$

The following three exercises expand on the geometric interpretation of the hyperbolic functions. Refer to section 4.11 and particularly to figure 4.11.2 and exercise 6 in section 4.11.

13. Compute $\int \sqrt{x^2 - 1}\,dx$ using the substitution $u = \operatorname{arccosh} x$, or $x = \cosh u$; use exercise 6 in section 4.11.

14. Fix $t > 0$. Sketch the region R in the right half plane bounded by the curves $y = x \tanh t$, $y = -x \tanh t$, and $x^2 - y^2 = 1$. Note well: t is fixed, the plane is the x-y plane.

15. Prove that the area of R is t.

9.2 Distance, Velocity, Acceleration

We next recall a general principle that will later be applied to distance-velocity-acceleration problems, among other things. If $F(u)$ is an anti-derivative of $f(u)$, then $\int_a^b f(u)\,du = F(b) - F(a)$. Suppose that we want to let the upper limit of integration vary, i.e., we replace b by some variable x. We think of a as a fixed starting value x_0. In this new notation the last equation (after adding $F(a)$ to both sides) becomes:

$$F(x) = F(x_0) + \int_{x_0}^{x} f(u)\,du.$$

(Here u is the variable of integration, called a "dummy variable," since it is not the variable in the function $F(x)$. In general, it is not a good idea to use the same letter as a variable of integration and as a limit of integration. That is, $\int_{x_0}^{x} f(x)dx$ is bad notation, and can lead to errors and confusion.)

An important application of this principle occurs when we are interested in the position of an object at time t (say, on the x-axis) and we know its position at time t_0. Let $s(t)$ denote the position of the object at time t (its distance from a reference point, such as the origin on the x-axis). Then the net change in position between t_0 and t is $s(t) - s(t_0)$. Since $s(t)$ is an anti-derivative of the velocity function $v(t)$, we can write

$$s(t) = s(t_0) + \int_{t_0}^{t} v(u)du.$$

Similarly, since the velocity is an anti-derivative of the acceleration function $a(t)$, we have

$$v(t) = v(t_0) + \int_{t_0}^{t} a(u)du.$$

EXAMPLE 9.2.1 Suppose an object is acted upon by a constant force F. Find $v(t)$ and $s(t)$. By Newton's law $F = ma$, so the acceleration is F/m, where m is the mass of the object. Then we first have

$$v(t) = v(t_0) + \int_{t_0}^{t} \frac{F}{m}\,du = v_0 + \frac{F}{m}u\Big|_{t_0}^{t} = v_0 + \frac{F}{m}(t - t_0),$$

using the usual convention $v_0 = v(t_0)$. Then

$$\begin{aligned} s(t) &= s(t_0) + \int_{t_0}^{t} \left(v_0 + \frac{F}{m}(u - t_0)\right) du = s_0 + (v_0 u + \frac{F}{2m}(u - t_0)^2)\Big|_{t_0}^{t} \\ &= s_0 + v_0(t - t_0) + \frac{F}{2m}(t - t_0)^2. \end{aligned}$$

For instance, when $F/m = -g$ is the constant of gravitational acceleration, then this is the falling body formula (if we neglect air resistance) familiar from elementary physics:

$$s_0 + v_0(t - t_0) - \frac{g}{2}(t - t_0)^2,$$

or in the common case that $t_0 = 0$,

$$s_0 + v_0 t - \frac{g}{2}t^2.$$

□

Recall that the integral of the velocity function gives the *net* distance traveled, that is, the displacement. If you want to know the *total* distance traveled, you must find out where the velocity function crosses the t-axis, integrate separately over the time intervals when $v(t)$ is positive and when $v(t)$ is negative, and add up the absolute values of the different integrals. For example, if an object is thrown straight upward at 19.6 m/sec, its velocity function is $v(t) = -9.8t + 19.6$, using $g = 9.8$ m/sec^2 for the force of gravity. This is a straight line which is positive for $t < 2$ and negative for $t > 2$. The net distance traveled in the first 4 seconds is thus

$$\int_0^4 (-9.8t + 19.6)dt = 0,$$

while the total distance traveled in the first 4 seconds is

$$\int_0^2 (-9.8t + 19.6)dt + \left|\int_2^4 (-9.8t + 19.6)dt\right| = 19.6 + |-19.6| = 39.2$$

meters, 19.6 meters up and 19.6 meters down.

7. An object is shot upwards from ground level with an initial velocity of 100 meters per second; it is subject only to the force of gravity (no air resistance). Find its maximum altitude and the time at which it hits the ground. $\Rightarrow$

8. An object moves along a straight line with acceleration given by $a(t) = -\cos(t)$, and $s(0) = 1$ and $v(0) = 0$. Find the maximum distance the object travels from zero, and find its maximum speed. Describe the motion of the object. $\Rightarrow$

9. An object moves along a straight line with acceleration given by $a(t) = \sin(\pi t)$. Assume that when $t = 0$, $s(t) = v(t) = 0$. Find $s(t)$, $v(t)$, and the maximum speed of the object. Describe the motion of the object. $\Rightarrow$

10. An object moves along a straight line with acceleration given by $a(t) = 1 + \sin(\pi t)$. Assume that when $t = 0$, $s(t) = v(t) = 0$. Find $s(t)$ and $v(t)$. $\Rightarrow$

11. An object moves along a straight line with acceleration given by $a(t) = 1 - \sin(\pi t)$. Assume that when $t = 0$, $s(t) = v(t) = 0$. Find $s(t)$ and $v(t)$. $\Rightarrow$

9.3 Volume

We have seen how to compute certain areas by using integration; some volumes may also be computed by evaluating an integral. Generally, the volumes that we can compute this way have cross-sections that are easy to describe.

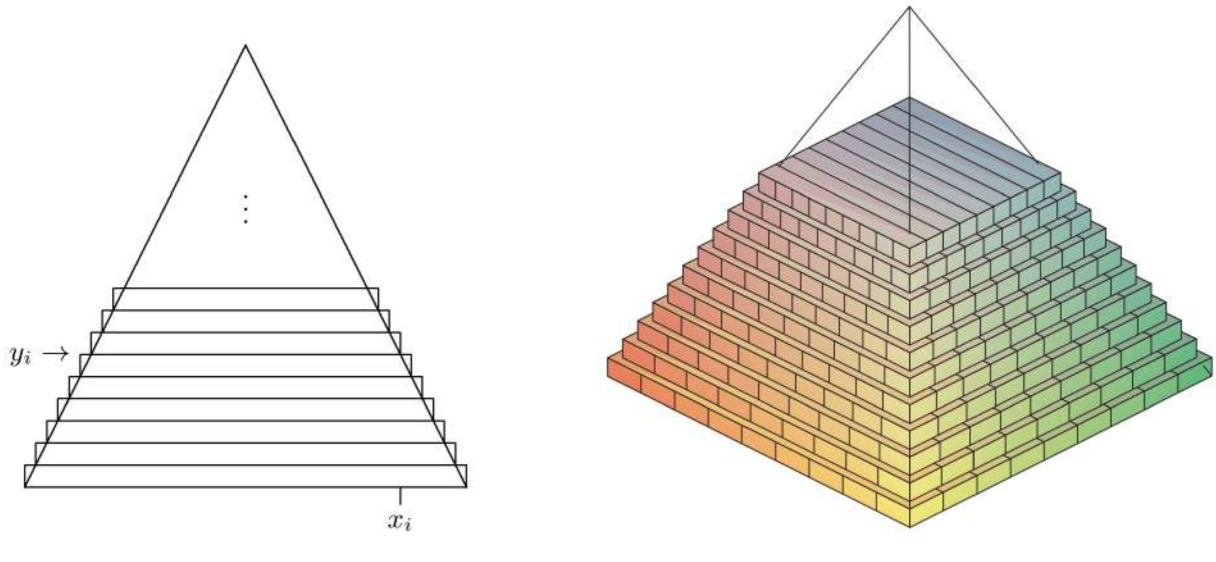

Figure 9.3.1 Volume of a pyramid approximated by rectangular prisms. (AP)

EXAMPLE 9.3.1 Find the volume of a pyramid with a square base that is 20 meters tall and 20 meters on a side at the base. As with most of our applications of integration, we begin by asking how we might approximate the volume. Since we can easily compute the volume of a rectangular prism (that is, a "box"), we will use some boxes to approximate the volume of the pyramid, as shown in figure 9.3.1: on the left is a cross-sectional view, on the right is a 3D view of part of the pyramid with some of the boxes used to approximate the volume.

Each box has volume of the form $(2x_i)(2x_i)\Delta y$. Unfortunately, there are two variables here; fortunately, we can write x in terms of y: $x = 10 - y/2$ or $x_i = 10 - y_i/2$. Then the total volume is approximately

$$\sum_{i=0}^{n-1} 4(10 - y_i/2)^2 \Delta y$$

and in the limit we get the volume as the value of an integral:

$$\int_0^{20} 4(10 - y/2)^2\,dy = \int_0^{20} (20 - y)^2\,dy = -\frac{(20-y)^3}{3}\bigg|_0^{20} = -\frac{0^3}{3} - -\frac{20^3}{3} = \frac{8000}{3}.$$

As you may know, the volume of a pyramid is $(1/3)$(height)(area of base) $= (1/3)(20)(400)$, which agrees with our answer. □

EXAMPLE 9.3.2 The base of a solid is the region between $f(x) = x^2 - 1$ and $g(x) = -x^2 + 1$, and its cross-sections perpendicular to the x-axis are equilateral triangles, as indicated in figure 9.3.2. The solid has been truncated to show a triangular cross-section above $x = 1/2$. Find the volume of the solid.

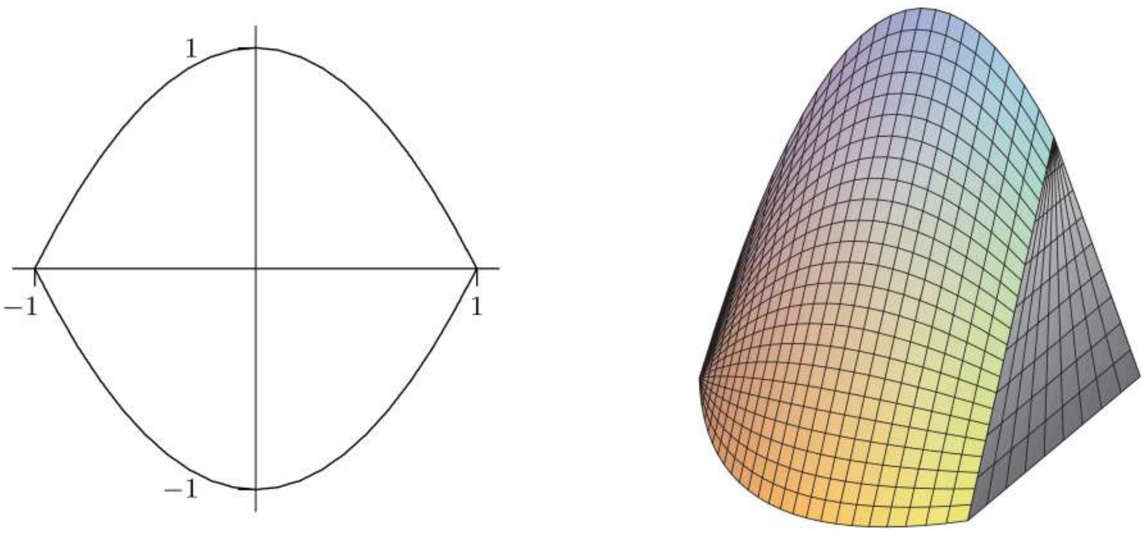

Figure 9.3.2 Solid with equilateral triangles as cross-sections. (AP)

A cross-section at a value x_i on the x-axis is a triangle with base $2(1 - x_i^2)$ and height $\sqrt{3}(1 - x_i^2)$, so the area of the cross-section is

$$\frac{1}{2}(\text{base})(\text{height}) = (1 - x_i^2)\sqrt{3}(1 - x_i^2),$$

and the volume of a thin "slab" is then

$$(1 - x_i^2)\sqrt{3}(1 - x_i^2)\Delta x.$$

Thus the total volume is

$$\int_{-1}^{1} \sqrt{3}(1 - x^2)^2\, dx = \frac{16}{15}\sqrt{3}.$$

□

One easy way to get "nice" cross-sections is by rotating a plane figure around a line. For example, in figure 9.3.3 we see a plane region under a curve and between two vertical lines; then the result of rotating this around the x-axis, and a typical circular cross-section.

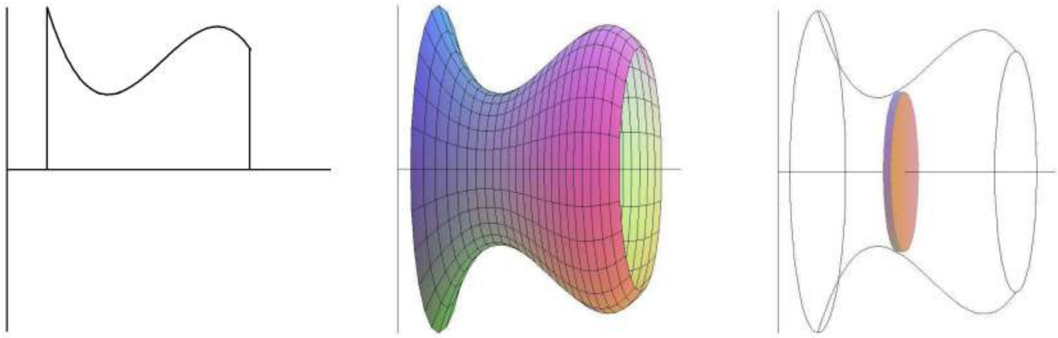

Figure 9.3.3 A solid of rotation. (AP)

Of course a real "slice" of this figure will not have straight sides, but we can approximate the volume of the slice by a cylinder or disk with circular top and bottom and straight sides; the volume of this disk will have the form $\pi r^2 \Delta x$. As long as we can write r in terms of x we can compute the volume by an integral.

EXAMPLE 9.3.3 Find the volume of a right circular cone with base radius 10 and height 20. (A right circular cone is one with a circular base and with the tip of the cone directly over the center of the base.) We can view this cone as produced by the rotation of the line $y = x/2$ rotated about the x-axis, as indicated in figure 9.3.4.

At a particular point on the x-axis, say x_i, the radius of the resulting cone is the y-coordinate of the corresponding point on the line, namely $y_i = x_i/2$. Thus the total volume is approximately

$$\sum_{i=0}^{n-1} \pi(x_i/2)^2\, dx$$

and the exact volume is

$$\int_0^{20} \pi\frac{x^2}{4}\, dx = \frac{\pi}{4}\frac{20^3}{3} = \frac{2000\pi}{3}.$$

9.4 Average value of a function

The average of some finite set of values is a familiar concept. If, for example, the class scores on a quiz are 10, 9, 10, 8, 7, 5, 7, 6, 3, 2, 7, 8, then the average score is the sum of these numbers divided by the size of the class:

$$\text{average score} = \frac{10+9+10+8+7+5+7+6+3+2+7+8}{12} = \frac{82}{12} \approx 6.83.$$

Suppose that between $t=0$ and $t=1$ the speed of an object is $\sin(\pi t)$. What is the average speed of the object over that time? We know one way to make sense of this: average speed is distance traveled divided by elapsed time. The distance traveled is $\int_0^1 \sin(\pi t)\,dt = 2/\pi \approx 0.64$, and elapsed time is 1, so the average speed is $2/\pi$. This appears to have nothing to do with the simple idea of average, as in the case of the quiz scores. We might also want to compute an average not tied to speed; for example, what is the average height of the curve $\sin(\pi t)$ over the interval $[0,1]$? Is it the same as the average speed? More generally, can we make sense of the average of $f(x)$ over an interval $[a,b]$?

To make sense of "average" in this more general context, we fall back on the idea of approximation. What is the average of $\sin(\pi t)$ over the interval $[0,1]$? We might reasonably approximate this by choosing some t values in the interval $[0,1]$, add up the corresponding values of $\sin(\pi t)$, and then divide by the number of values. If we divide $[0,1]$ into 10 equal subintervals, we get

$$\frac{1}{10}\sum_{i=0}^{9} \sin(\pi i/10) \approx \frac{1}{10}6.3 = 0.63.$$

If we compute more values of the function at more values of t, the average of these values should be closer to the "real" average. If we take the average of n values for evenly spaced values of t, we get:

$$\frac{1}{n}\sum_{i=0}^{n-1} \sin(\pi i/n).$$

Here the individual values of t are $t_i = i/n$, so rewriting slightly we have

$$\frac{1}{n}\sum_{i=0}^{n-1} \sin(\pi t_i).$$

This is almost the sort of sum that we know turns into an integral; what's apparently missing is Δt—but in fact, $\Delta t = 1/n$, the length of each subinterval. So rewriting again:

$$\sum_{i=0}^{n-1} \sin(\pi t_i)\frac{1}{n} = \sum_{i=0}^{n-1} \sin(\pi t_i)\Delta t.$$

Now this has exactly the right form, so that in the limit we get

$$\text{average} = \int_0^1 \sin(\pi t)\,dt = -\frac{\cos(\pi t)}{\pi}\bigg|_0^1 = -\frac{\cos(\pi)}{\pi} + \frac{\cos(0)}{\pi} = \frac{2}{\pi} \approx 0.64.$$

Of course, this is exactly what we computed before, but we didn't need to rely on a particular interpretation of the function. If we interpret $\sin(\pi t)$ as the height of the function, we interpret the result as the average height of $\sin(\pi t)$ over $[0,1]$.

It's not entirely obvious from this one simple example how to compute such an average in general. Let's look at a somewhat more complicated case. Suppose that the function is $16t^2+5$. What is the average between $t=1$ and $t=3$? Again we set up an approximation to the average:

$$\frac{1}{n}\sum_{i=0}^{n-1} 16t_i^2+5,$$

where the values t_i are evenly spaced between 1 and 3. Once again we are "missing" Δt, and this time $1/n$ is not the correct value. What is Δt in general? It is the length of a subinterval; in this case we take the interval $[1,3]$ and divide it into n subintervals, so each has length $(3-1)/n = 2/n = \Delta t$. Now with the usual "multiply and divide by the same thing" trick we can rewrite the sum:

$$\frac{1}{n}\sum_{i=0}^{n-1} 16t_i^2+5 = \frac{1}{3-1}\sum_{i=0}^{n-1}(16t_i^2+5)\frac{3-1}{n} = \frac{1}{2}\sum_{i=0}^{n-1}(16t_i^2+5)\frac{2}{n} = \frac{1}{2}\sum_{i=0}^{n-1}(16t_i^2+5)\Delta t.$$

In the limit this becomes

$$\frac{1}{2}\int_1^3 16t^2+5\,dt = \frac{1}{2}\frac{446}{3} = \frac{223}{3}.$$

Does this seem reasonable? Let's picture it: in figure 9.4.1 is the function $16t_i^2+5$ together with the horizontal line $y = 223/3 \approx 74.3$. Certainly the height of the horizontal line looks at least plausible for the average height of the curve.

We can interpret this result in a slightly different way. The area under $y = 16x^2+5$ above $[1,3]$ is

$$\int_1^3 16t^2+5\,dt = \frac{446}{3}.$$

The area under $y = 223/3$ over the same interval $[1,3]$ is simply the area of a rectangle that is 2 by $223/3$ with area $446/3$. So the average height of a function is the height of the horizontal line that produces the same area over the given interval.

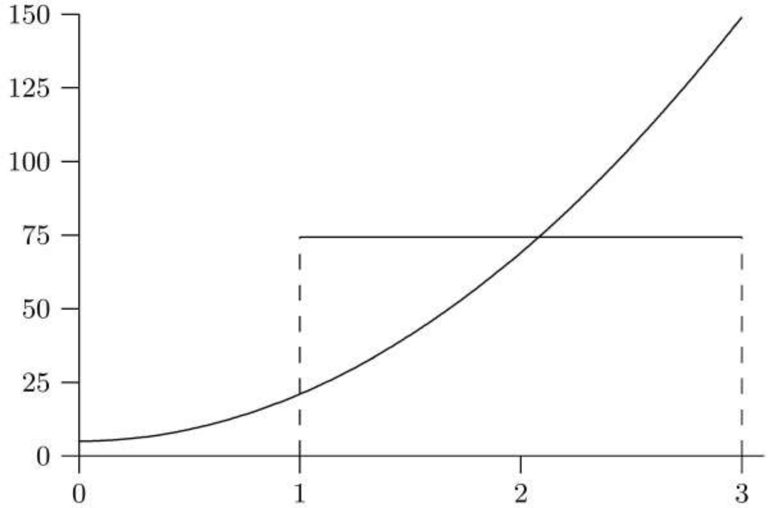

Figure 9.4.1 Average velocity.

Notice that we may interpret average speed in much the same way. If the speed of an object is $16t_i^2+5$, the average speed over the interval $[1,3]$ is $223/3$, and the object travels a distance of $446/3$ units in two seconds. If instead the object were to travel for two seconds at a constant speed of $223/3$, the distance traveled would also be $223/3 \cdot 2 = 446/3$. So average speed is the constant speed required to go the same distance in the same time.

To summarize, to compute the average value of $f(x)$ over $[a,b]$, compute the integral of f over the interval and divide by the length of the interval:

$$\text{average} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Exercises 9.4.

1. Find the average height of $\cos x$ over the intervals $[0,\pi/2]$, $[-\pi/2,\pi/2]$, and $[0,2\pi]$. $\Rightarrow$

2. Find the average height of x^2 over the interval $[-2,2]$. $\Rightarrow$

3. Find the average height of $1/x^2$ over the interval $[1,A]$. $\Rightarrow$

4. Find the average height of $\sqrt{1-x^2}$ over the interval $[-1,1]$. $\Rightarrow$

5. An object moves with velocity $v(t) = -t^2+1$ feet per second between $t=0$ and $t=2$. Find the average velocity and the average speed of the object between $t=0$ and $t=2$. $\Rightarrow$

6. The observation deck on the 102nd floor of the Empire State Building is 1,224 feet above the ground. If a steel ball is dropped from the observation deck its velocity at time t is approximately $v(t) = -32t$ feet per second. Find the average speed between the time it is dropped and the time it hits the ground, and find its speed when it hits the ground. $\Rightarrow$

9.5 WORK

A fundamental concept in classical physics is **work**: If an object is moved in a straight line against a force F for a distance s the work done is $W = Fs$.

EXAMPLE 9.5.1 How much work is done in lifting a 10 pound weight vertically a distance of 5 feet? The force due to gravity on a 10 pound weight is 10 pounds at the surface of the earth, and it does not change appreciably over 5 feet. The work done is $W = 10 \cdot 5 = 50$ foot-pounds. □

In reality few situations are so simple. The force might not be constant over the range of motion, as in the next example.

EXAMPLE 9.5.2 How much work is done in lifting a 10 pound weight from the surface of the earth to an orbit 100 miles above the surface? Over 100 miles the force due to gravity does change significantly, so we need to take this into account. The force exerted on a 10 pound weight at a distance r from the center of the earth is $F = k/r^2$ and by definition it is 10 when r is the radius of the earth (we assume the earth is a sphere). How can we approximate the work done? We divide the path from the surface to orbit into n small subpaths. On each subpath the force due to gravity is roughly constant, with value k/r_i^2 at distance r_i. The work to raise the object from r_i to r_{i+1} is thus approximately $k/r_i^2\Delta r$ and the total work is approximately

$$\sum_{i=0}^{n-1} \frac{k}{r_i^2}\Delta r,$$

or in the limit

$$W = \int_{r_0}^{r_1} \frac{k}{r^2}\,dr,$$

where r_0 is the radius of the earth and r_1 is r_0 plus 100 miles. The work is

$$W = \int_{r_0}^{r_1} \frac{k}{r^2}\,dr = -\left.\frac{k}{r}\right|_{r_0}^{r_1} = -\frac{k}{r_1} + \frac{k}{r_0}.$$

Using $r_0 = 20925525$ feet we have $r_1 = 21453525$. The force on the 10 pound weight at the surface of the earth is 10 pounds, so $10 = k/20925525^2$, giving $k = 4378775965256250$. Then

$$-\frac{k}{r_1} + \frac{k}{r_0} = \frac{491052320000}{95349} \approx 5150052 \quad \text{foot-pounds.}$$

Note that if we assume the force due to gravity is 10 pounds over the whole distance we would calculate the work as $10(r_1 - r_0) = 10 \cdot 100 \cdot 5280 = 5280000$, somewhat higher since we don't account for the weakening of the gravitational force. □

9. The cable in the previous problem has a 100 kilogram bucket of concrete attached to its lower end. How much work is required to lift the entire cable and bucket to the height of its top end? ⇒

10. Consider again the cable and bucket of the previous problem. How much work is required to lift the bucket 10 meters by raising the cable 10 meters? (The top half of the cable ends up at the height of the top end of the cable, while the bottom half of the cable is lifted 10 meters.) ⇒

9.6 Center of Mass

Suppose a beam is 10 meters long, and that there are three weights on the beam: a 10 kilogram weight 3 meters from the left end, a 5 kilogram weight 6 meters from the left end, and a 4 kilogram weight 8 meters from the left end. Where should a fulcrum be placed so that the beam balances? Let's assign a scale to the beam, from 0 at the left end to 10 at the right, so that we can denote locations on the beam simply as x coordinates; the weights are at $x = 3$, $x = 6$, and $x = 8$, as in figure 9.6.1.

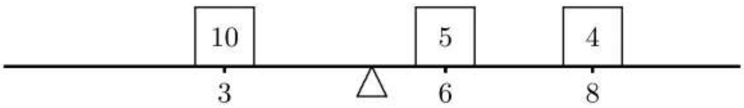

Figure 9.6.1 A beam with three masses.

Suppose to begin with that the fulcrum is placed at $x = 5$. What will happen? Each weight applies a force to the beam that tends to rotate it around the fulcrum; this effect is measured by a quantity called **torque**, proportional to the mass times the distance from the fulcrum. Of course, weights on different sides of the fulcrum rotate the beam in opposite directions. We can distinguish this by using a signed distance in the formula for torque. So with the fulcrum at 5, the torques induced by the three weights will be proportional to $(3-5)10 = -20$, $(6-5)5 = 5$, and $(8-5)4 = 12$. For the beam to balance, the sum of the torques must be zero; since the sum is $-20+5+12 = -3$, the beam rotates counter-clockwise, and to get the beam to balance we need to move the fulcrum to the left. To calculate exactly where the fulcrum should be, we let $\bar{x}$ denote the location of the fulcrum when the beam is in balance. The total torque on the beam is then $(3-\bar{x})10 + (6-\bar{x})5 + (8-\bar{x})4 = 92 - 19\bar{x}$. Since the beam balances at $\bar{x}$ it must be that $92 - 19\bar{x} = 0$ or $\bar{x} = 92/19 \approx 4.84$, that is, the fulcrum should be placed at $x = 92/19$ to balance the beam.

Now suppose that we have a beam with varying density—some portions of the beam contain more mass than other portions of the same size. We want to figure out where to put the fulcrum so that the beam balances.

m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9

Figure 9.6.2 A solid beam.

EXAMPLE 9.6.1 Suppose the beam is 10 meters long and that the density is $1 + x$ kilograms per meter at location x on the beam. To approximate the solution, we can think of the beam as a sequence of weights "on" a beam. For example, we can think of the portion of the beam between $x = 0$ and $x = 1$ as a weight sitting at $x = 0$, the portion between $x = 1$ and $x = 2$ as a weight sitting at $x = 1$, and so on, as indicated in figure 9.6.2. We then approximate the mass of the weights by assuming that each portion of the beam has constant density. So the mass of the first weight is approximately $m_0 = (1 + 0)1 = 1$ kilograms, namely, $(1 + 0)$ kilograms per meter times 1 meter. The second weight is $m_1 = (1 + 1)1 = 2$ kilograms, and so on to the tenth weight with $m_9 = (1 + 9)1 = 10$ kilograms. So in this case the total torque is

$$(0 - \bar{x})m_0 + (1 - \bar{x})m_1 + \cdots + (9 - \bar{x})m_9 = (0 - \bar{x})1 + (1 - \bar{x})2 + \cdots + (9 - \bar{x})10.$$

If we set this to zero and solve for $\bar{x}$ we get $\bar{x} = 6$. In general, if we divide the beam into n portions, the mass of weight number i will be $m_i = (1 + x_i)(x_{i+1} - x_i) = (1 + x_i)\Delta x$ and the torque induced by weight number i will be $(x_i - \bar{x})m_i = (x_i - \bar{x})(1 + x_i)\Delta x$. The total torque is then

$$\begin{aligned}
(x_0 - \bar{x})(1 + x_0)\Delta x &+ (x_1 - \bar{x})(1 + x_1)\Delta x + \cdots + (x_{n-1} - \bar{x})(1 + x_{n-1})\Delta x \\
&= \sum_{i=0}^{n-1} x_i(1 + x_i)\Delta x - \sum_{i=0}^{n-1} \bar{x}(1 + x_i)\Delta x \\
&= \sum_{i=0}^{n-1} x_i(1 + x_i)\Delta x - \bar{x}\sum_{i=0}^{n-1}(1 + x_i)\Delta x.
\end{aligned}$$

If we set this equal to zero and solve for $\bar{x}$ we get an approximation to the balance point of the beam:

$$\begin{aligned}
0 &= \sum_{i=0}^{n-1} x_i(1 + x_i)\Delta x - \bar{x}\sum_{i=0}^{n-1}(1 + x_i)\Delta x \\
\bar{x}\sum_{i=0}^{n-1}(1 + x_i)\Delta x &= \sum_{i=0}^{n-1} x_i(1 + x_i)\Delta x \\
\bar{x} &= \frac{\displaystyle\sum_{i=0}^{n-1} x_i(1 + x_i)\Delta x}{\displaystyle\sum_{i=0}^{n-1}(1 + x_i)\Delta x}.
\end{aligned}$$

The denominator of this fraction has a very familiar interpretation. Consider one term of the sum in the denominator: $(1+x_i)\Delta x$. This is the density near x_i times a short length, Δx, which in other words is approximately the mass of the beam between x_i and x_{i+1}. When we add these up we get approximately the mass of the beam.

Now each of the sums in the fraction has the right form to turn into an integral, which in turn gives us the exact value of $\bar{x}$:

$$\bar{x} = \frac{\displaystyle\int_0^{10} x(1+x)\,dx}{\displaystyle\int_0^{10} (1+x)\,dx}.$$

The numerator of this fraction is called the **moment** of the system around zero:

$$\int_0^{10} x(1+x)\,dx = \int_0^{10} x + x^2\,dx = \frac{1150}{3},$$

and the denominator is the mass of the beam:

$$\int_0^{10} (1+x)\,dx = 60,$$

and the balance point, officially called the **center of mass**, is

$$\bar{x} = \frac{1150}{3}\frac{1}{60} = \frac{115}{18} \approx 6.39.$$

□

It should be apparent that there was nothing special about the density function $\sigma(x) = 1+x$ or the length of the beam, or even that the left end of the beam is at the origin. In general, if the density of the beam is $\sigma(x)$ and the beam covers the interval $[a,b]$, the moment of the beam around zero is

$$M_0 = \int_a^b x\sigma(x)\,dx$$

and the total mass of the beam is

$$M = \int_a^b \sigma(x)\,dx$$

and the center of mass is at

$$\bar{x} = \frac{M_0}{M}.$$

EXAMPLE 9.6.2 Suppose a beam lies on the x-axis between 20 and 30, and has density function $\sigma(x) = x - 19$. Find the center of mass. This is the same as the previous example

Exercises 9.6.

1. A beam 10 meters long has density $\sigma(x) = x^2$ at distance x from the left end of the beam. Find the center of mass $\bar{x}$. ⇒

2. A beam 10 meters long has density $\sigma(x) = \sin(\pi x/10)$ at distance x from the left end of the beam. Find the center of mass $\bar{x}$. ⇒

3. A beam 4 meters long has density $\sigma(x) = x^3$ at distance x from the left end of the beam. Find the center of mass $\bar{x}$. ⇒

4. Verify that $\displaystyle\int 2x \arccos x\,dx = x^2 \arccos x - \frac{x\sqrt{1-x^2}}{2} + \frac{\arcsin x}{2} + C$.

5. A thin plate lies in the region between $y = x^2$ and the x-axis between $x = 1$ and $x = 2$. Find the centroid. ⇒

6. A thin plate fills the upper half of the unit circle $x^2 + y^2 = 1$. Find the centroid. ⇒

7. A thin plate lies in the region contained by $y = x$ and $y = x^2$. Find the centroid. ⇒

8. A thin plate lies in the region contained by $y = 4 - x^2$ and the x-axis. Find the centroid. ⇒

9. A thin plate lies in the region contained by $y = x^{1/3}$ and the x-axis between $x = 0$ and $x = 1$. Find the centroid. ⇒

10. A thin plate lies in the region contained by $\sqrt{x} + \sqrt{y} = 1$ and the axes in the first quadrant. Find the centroid. ⇒

11. A thin plate lies in the region between the circle $x^2 + y^2 = 4$ and the circle $x^2 + y^2 = 1$, above the x-axis. Find the centroid. ⇒

12. A thin plate lies in the region between the circle $x^2 + y^2 = 4$ and the circle $x^2 + y^2 = 1$ in the first quadrant. Find the centroid. ⇒

13. A thin plate lies in the region between the circle $x^2 + y^2 = 25$ and the circle $x^2 + y^2 = 16$ above the x-axis. Find the centroid. ⇒

9.7 Kinetic energy; improper integrals

Recall example 9.5.3 in which we computed the work required to lift an object from the surface of the earth to some large distance D away. Since $F = k/x^2$ we computed

$$\int_{r_0}^{D} \frac{k}{x^2}\,dx = -\frac{k}{D} + \frac{k}{r_0}.$$

We noticed that as D increases, k/D decreases to zero so that the amount of work increases to k/r_0. More precisely,

$$\lim_{D\to\infty} \int_{r_0}^{D} \frac{k}{x^2}\,dx = \lim_{D\to\infty} -\frac{k}{D} + \frac{k}{r_0} = \frac{k}{r_0}.$$

We might reasonably describe this calculation as computing the amount of work required to lift the object "to infinity," and abbreviate the limit as

$$\lim_{D\to\infty} \int_{r_0}^{D} \frac{k}{x^2}\,dx = \int_{r_0}^{\infty} \frac{k}{x^2}\,dx.$$

Such an integral, with a limit of infinity, is called an **improper integral**. This is a bit unfortunate, since it's not really "improper" to do this, nor is it really "an integral"—it is an abbreviation for the limit of a particular sort of integral. Nevertheless, we're stuck with the term, and the operation itself is perfectly legitimate. It may at first seem odd that a finite amount of work is sufficient to lift an object to "infinity", but sometimes surprising things are nevertheless true, and this is such a case. If the value of an improper integral is a finite number, as in this example, we say that the integral **converges**, and if not we say that the integral **diverges**.

Here's another way, perhaps even more surprising, to interpret this calculation. We know that one interpretation of

$$\int_1^D \frac{1}{x^2}\,dx$$

is the area under $y = 1/x^2$ from $x = 1$ to $x = D$. Of course, as D increases this area increases. But since

$$\int_1^D \frac{1}{x^2}\,dx = -\frac{1}{D} + \frac{1}{1},$$

while the area increases, it never exceeds 1, that is

$$\int_1^\infty \frac{1}{x^2}\,dx = 1.$$

The area of the infinite region under $y = 1/x^2$ from $x = 1$ to infinity is finite.

Consider a slightly different sort of improper integral: $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\,dx$. There are two ways we might try to compute this. First, we could break it up into two more familiar integrals:

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \int_{-\infty}^{0} xe^{-x^2}\,dx + \int_{0}^{\infty} xe^{-x^2}\,dx.$$

Now we do these as before:

$$\int_{-\infty}^{0} xe^{-x^2}\,dx = \lim_{D\to\infty} -\frac{e^{-x^2}}{2}\bigg|_{-D}^{0} = -\frac{1}{2},$$

and

$$\int_{0}^{\infty} xe^{-x^2}\,dx = \lim_{D\to\infty} -\frac{e^{-x^2}}{2}\bigg|_{0}^{D} = \frac{1}{2},$$

so

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx = -\frac{1}{2} + \frac{1}{2} = 0.$$

Alternately, we might try

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \lim_{D\to\infty}\int_{-D}^{D} xe^{-x^2}\,dx = \lim_{D\to\infty} -\frac{e^{-x^2}}{2}\bigg|_{-D}^{D} = \lim_{D\to\infty} -\frac{e^{-D^2}}{2} + \frac{e^{-D^2}}{2} = 0.$$

So we get the same answer either way. This does not always happen; sometimes the second approach gives a finite number, while the first approach does not; the exercises provide examples. In general, we interpret the integral $\int_{-\infty}^{\infty} f(x)\,dx$ according to the first method: both integrals $\int_{-\infty}^{a} f(x)\,dx$ and $\int_{a}^{\infty} f(x)\,dx$ must converge for the original integral to converge. The second approach does turn out to be useful; when $\lim_{D\to\infty}\int_{-D}^{D} f(x)\,dx = L$, and L is finite, then L is called the **Cauchy Principal Value** of $\int_{-\infty}^{\infty} f(x)\,dx$.

Here's a more concrete application of these ideas. We know that in general

$$W = \int_{x_0}^{x_1} F\,dx$$

is the work done against the force F in moving from x_0 to x_1. In the case that F is the force of gravity exerted by the earth, it is customary to make $F < 0$ since the force is "downward." This makes the work W negative when it should be positive, so typically the work in this case is defined as

$$W = -\int_{x_0}^{x_1} F\,dx.$$

Also, by Newton's Law, $F = ma(t)$. This means that

$$W = -\int_{x_0}^{x_1} ma(t)\,dx.$$

Unfortunately this integral is a bit problematic: $a(t)$ is in terms of t, while the limits and the "dx" are in terms of x. But x and t are certainly related here: $x = x(t)$ is the function that gives the position of the object at time t, so $v = v(t) = dx/dt = x'(t)$ is its velocity and $a(t) = v'(t) = x''(t)$. We can use $v = x'(t)$ as a substitution to convert the integral from "dx" to "dv" in the usual way, with a bit of cleverness along the way:

$$\begin{aligned} dv &= x''(t)\,dt = a(t)\,dt = a(t)\frac{dt}{dx}\,dx \\ \frac{dx}{dt}\,dv &= a(t)\,dx \\ v\,dv &= a(t)\,dx. \end{aligned}$$

Substituting in the integral:

$$W = -\int_{x_0}^{x_1} ma(t)\,dx = -\int_{v_0}^{v_1} mv\,dv = -\left.\frac{mv^2}{2}\right|_{v_0}^{v_1} = -\frac{mv_1^2}{2} + \frac{mv_0^2}{2}.$$

You may recall seeing the expression $mv^2/2$ in a physics course—it is called the **kinetic energy** of the object. We have shown here that the work done in moving the object from one place to another is the same as the change in kinetic energy.

We know that the work required to move an object from the surface of the earth to infinity is

$$W = \int_{r_0}^{\infty} \frac{k}{r^2}\,dr = \frac{k}{r_0}.$$

At the surface of the earth the acceleration due to gravity is approximately 9.8 meters per second squared, so the force on an object of mass m is $F = 9.8m$. The radius of the earth is approximately 6378.1 kilometers or 6378100 meters. Since the force due to gravity obeys an inverse square law, $F = k/r^2$ and $9.8m = k/6378100^2$, $k = 398665564178000m$ and $W = 62505380m$.

Now suppose that the initial velocity of the object, v_0, is just enough to get it to infinity, that is, just enough so that the object never slows to a stop, but so that its speed decreases to zero, i.e., so that $v_1 = 0$. Then

$$62505380m = W = -\frac{mv_1^2}{2} + \frac{mv_0^2}{2} = \frac{mv_0^2}{2}$$

so

$$v_0 = \sqrt{125010760} \approx 11181 \quad \text{meters per second,}$$

or about 40251 kilometers per hour. This speed is called the **escape velocity**. Notice that the mass of the object, m, canceled out at the last step; the escape velocity is the same for all objects. Of course, it takes considerably more energy to get a large object up to 40251 kph than a small one, so it is certainly more difficult to get a large object into deep space than a small one. Also, note that while we have computed the escape velocity for the earth, this speed would not in fact get an object "to infinity" because of the large mass in our neighborhood called the sun. Escape velocity for the sun *starting at the distance of the earth from the sun* is nearly 4 times the escape velocity we have calculated.

Exercises 9.7.

1. Is the area under $y = 1/x$ from 1 to infinity finite or infinite? If finite, compute the area. ⇒
2. Is the area under $y = 1/x^3$ from 1 to infinity finite or infinite? If finite, compute the area. ⇒
3. Does $\int_0^\infty x^2 + 2x - 1\,dx$ converge or diverge? If it converges, find the value. ⇒
4. Does $\int_1^\infty 1/\sqrt{x}\,dx$ converge or diverge? If it converges, find the value. ⇒
5. Does $\int_0^\infty e^{-x}\,dx$ converge or diverge? If it converges, find the value. ⇒
6. $\int_0^{1/2} (2x-1)^{-3}\,dx$ is an improper integral of a slightly different sort. Express it as a limit and determine whether it converges or diverges; if it converges, find the value. ⇒
7. Does $\int_0^1 1/\sqrt{x}\,dx$ converge or diverge? If it converges, find the value. ⇒
8. Does $\int_0^{\pi/2} \sec^2 x\,dx$ converge or diverge? If it converges, find the value. ⇒
9. Does $\int_{-\infty}^\infty \frac{x^2}{4+x^6}\,dx$ converge or diverge? If it converges, find the value. ⇒
10. Does $\int_{-\infty}^\infty x\,dx$ converge or diverge? If it converges, find the value. Also find the Cauchy Principal Value, if it exists. ⇒
11. Does $\int_{-\infty}^\infty \sin x\,dx$ converge or diverge? If it converges, find the value. Also find the Cauchy Principal Value, if it exists. ⇒
12. Does $\int_{-\infty}^\infty \cos x\,dx$ converge or diverge? If it converges, find the value. Also find the Cauchy Principal Value, if it exists. ⇒
13. Suppose the curve $y = 1/x$ is rotated around the x-axis generating a sort of funnel or horn shape, called **Gabriel's horn** or **Toricelli's trumpet**. Is the volume of this funnel from $x = 1$ to infinity finite or infinite? If finite, compute the volume. ⇒
14. An officially sanctioned baseball must be between 142 and 149 grams. How much work, in Newton-meters, does it take to throw a ball at 80 miles per hour? At 90 mph? At 100.9 mph? (According to the Guinness Book of World Records, at `http://www.baseball-almanac.com/recbooks/rb_guin.shtml`, "The greatest reliably recorded speed at which a baseball has been pitched is 100.9 mph by Lynn Nolan Ryan (California Angels) at Anaheim Stadium in California on August 20, 1974.") ⇒

9.8 Probability

You perhaps have at least a rudimentary understanding of **discrete probability**, which measures the likelihood of an "event" when there are a finite number of possibilities. For example, when an ordinary six-sided die is rolled, the probability of getting any particular number is $1/6$. In general, the probability of an event is the number of ways the event can happen divided by the number of ways that "anything" can happen.

For a slightly more complicated example, consider the case of two six-sided dice. The dice are physically distinct, which means that rolling a 2–5 is different than rolling a 5–2; each is an equally likely event out of a total of 36 ways the dice can land, so each has a probability of $1/36$.

Most interesting events are not so simple. More interesting is the probability of rolling a certain sum out of the possibilities 2 through 12. It is clearly not true that all sums are equally likely: the only way to roll a 2 is to roll 1–1, while there are many ways to roll a 7. Because the number of possibilities is quite small, and because a pattern quickly becomes evident, it is easy to see that the probabilities of the various sums are:

$$\begin{aligned}
P(2) = P(12) &= 1/36\\
P(3) = P(11) &= 2/36\\
P(4) = P(10) &= 3/36\\
P(5) = P(9) &= 4/36\\
P(6) = P(8) &= 5/36\\
P(7) &= 6/36
\end{aligned}$$

Here we use $P(n)$ to mean "the probability of rolling an n." Since we have correctly accounted for all possibilities, the sum of all these probabilities is $36/36 = 1$; the probability that the sum is one of 2 through 12 is 1, because there are no other possibilities.

The study of probability is concerned with more difficult questions as well; for example, suppose the two dice are rolled many times. On the average, what sum will come up? In the language of probability, this average is called the **expected value** of the sum. This is at first a little misleading, as it does not tell us what to "expect" when the two dice are rolled, but what we expect the long term average will be.

Suppose that two dice are rolled 36 million times. Based on the probabilities, we would expect about 1 million rolls to be 2, about 2 million to be 3, and so on, with a roll of 7 topping the list at about 6 million. The sum of all rolls would be 1 million times 2 plus 2 million times 3, and so on, and dividing by 36 million we would get the average:

$$\begin{aligned}
\bar{x} &= (2\cdot 10^6 + 3(2\cdot 10^6) + \cdots + 7(6\cdot 10^6) + \cdots + 12\cdot 10^6)\frac{1}{36\cdot 10^6}\\
&= 2\frac{10^6}{36\cdot 10^6} + 3\frac{2\cdot 10^6}{36\cdot 10^6} + \cdots + 7\frac{6\cdot 10^6}{36\cdot 10^6} + \cdots + 12\frac{10^6}{36\cdot 10^6}\\
&= 2P(2) + 3P(3) + \cdots + 7P(7) + \cdots + 12P(12)\\
&= \sum_{i=2}^{12} iP(i) = 7.
\end{aligned}$$

There is nothing special about the 36 million in this calculation. No matter what the number of rolls, once we simplify the average, we get the same $\sum_{i=2}^{12} iP(i)$. While the actual average value of a large number of rolls will not be exactly 7, the average should be close to 7 when the number of rolls is large. Turning this around, if the average is not close to 7, we should suspect that the dice are not fair.

A variable, say X, that can take certain values, each with a corresponding probability, is called a **random variable**; in the example above, the random variable was the sum of the two dice. If the possible values for X are $x_1, x_2, \ldots, x_n$, then the expected value of the random variable is $E(X) = \sum_{i=1}^{n} x_i P(x_i)$. The expected value is also called the **mean**.

When the number of possible values for X is finite, we say that X is a discrete random variable. In many applications of probability, the number of possible values of a random variable is very large, perhaps even infinite. To deal with the infinite case we need a different approach, and since there is a sum involved, it should not be wholly surprising that integration turns out to be a useful tool. It then turns out that even when the number of possibilities is large but finite, it is frequently easier to pretend that the number is infinite. Suppose, for example, that a dart is thrown at a dart board. Since the dart board consists of a finite number of atoms, there are in some sense only a finite number of places for the dart to land, but it is easier to explore the probabilities involved by pretending that the dart can land on any point in the usual x-y plane.

DEFINITION 9.8.1 Let $f : \mathbb{R} \to \mathbb{R}$ be a function. If $f(x) \ge 0$ for every x and $\int_{-\infty}^{\infty} f(x)\,dx = 1$ then f is a **probability density function**. □

We associate a probability density function with a random variable X by stipulating that the probability that X is between a and b is $\int_a^b f(x)\,dx$. Because of the requirement that the integral from $-\infty$ to ∞ be 1, all probabilities are less than or equal to 1, and the probability that X takes on some value between $-\infty$ and ∞ is 1, as it should be.

EXAMPLE 9.8.2 Consider again the two dice example; we can view it in a way that more resembles the probability density function approach. Consider a random variable X that takes on any real value with probabilities given by the probability density function in figure 9.8.1. The function f consists of just the top edges of the rectangles, with vertical sides drawn for clarity; the function is zero below 1.5 and above 12.5. The area of each rectangle is the probability of rolling the sum in the middle of the bottom of the rectangle, or

$$P(n) = \int_{n-1/2}^{n+1/2} f(x)\,dx.$$

The probability of rolling a 4, 5, or 6 is

$$P(n) = \int_{7/2}^{13/2} f(x)\,dx.$$

Of course, we could also compute probabilities that don't make sense in the context of the dice, such as the probability that X is between 4 and 5.8. □

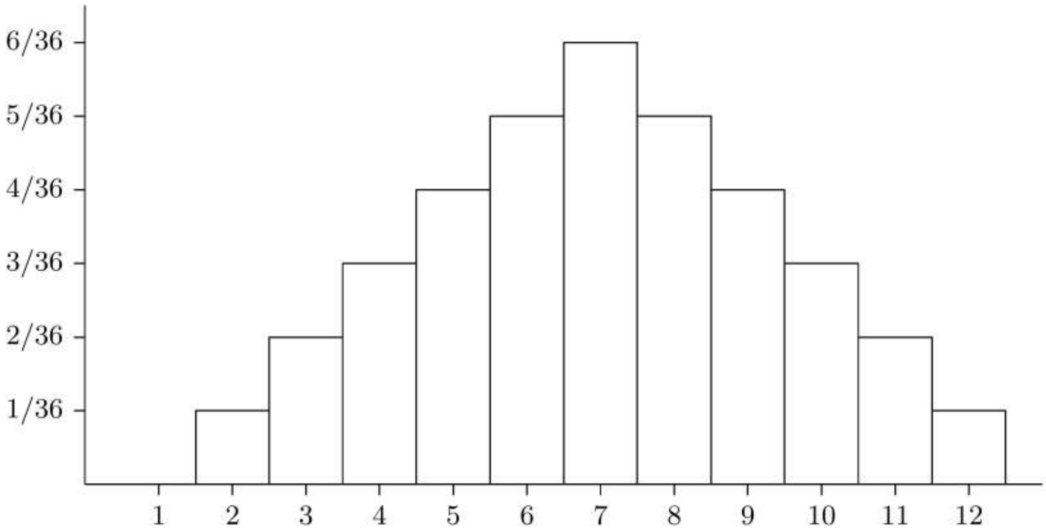

Figure 9.8.1 A probability density function for two dice.

The function

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt$$

is called the **cumulative distribution function** or simply (probability) distribution.

EXAMPLE 9.8.3 Suppose that $a < b$ and

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise.} \end{cases}$$

Then $f(x)$ is the **uniform probability density function** on $[a, b]$. and the corresponding distribution is the **uniform distribution** on $[a, b]$. □

We have shown that A is some finite number without computing it; we cannot compute it with the techniques we have available. By using some techniques from multivariable calculus, it can be shown that $A = \sqrt{2\pi}$.

EXAMPLE 9.8.5 The **exponential distribution** has probability density function

$$f(x) = \begin{cases} 0 & x < 0 \\ ce^{-cx} & x \ge 0 \end{cases}$$

where c is a positive constant. □

The mean or expected value of a random variable is quite useful, as hinted at in our discussion of dice. Recall that the mean for a discrete random variable is $E(X) = \sum_{i=1}^{n} x_i P(x_i)$. In the more general context we use an integral in place of the sum.

DEFINITION 9.8.6 The **mean** of a random variable X with probability density function f is $\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$, provided the integral converges. □

When the mean exists it is unique, since it is the result of an explicit calculation. The mean does not always exist.

The mean might look familiar; it is essentially identical to the center of mass of a one-dimensional beam, as discussed in section 9.6. The probability density function f plays the role of the physical density function, but now the "beam" has infinite length. If we consider only a finite portion of the beam, say between a and b, then the center of mass is

$$\bar{x} = \frac{\int_a^b x f(x)\,dx}{\int_a^b f(x)\,dx}.$$

If we extend the beam to infinity, we get

$$\bar{x} = \frac{\int_{-\infty}^{\infty} x f(x)\,dx}{\int_{-\infty}^{\infty} f(x)\,dx} = \int_{-\infty}^{\infty} x f(x)\,dx = E(X),$$

because $\int_{-\infty}^{\infty} f(x)\,dx = 1$. In the center of mass interpretation, this integral is the total mass of the beam, which is always 1 when f is a probability density function.

9. If you have access to appropriate software, find r so that

$$\int_{-\infty}^{10-r} f(x)\,dx + \int_{10+r}^{\infty} f(x)\,dx \approx 0.05,$$

using the function of example 9.8.9. Discuss the impact of using this new value of r to decide whether to investigate the chip manufacturing process. ⇒

9.9 Arc Length

Here is another geometric application of the integral: find the length of a portion of a curve. As usual, we need to think about how we might approximate the length, and turn the approximation into an integral.

We already know how to compute one simple arc length, that of a line segment. If the endpoints are $P_0(x_0, y_0)$ and $P_1(x_1, y_1)$ then the length of the segment is the distance between the points, $\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$, from the Pythagorean theorem, as illustrated in figure 9.9.1.

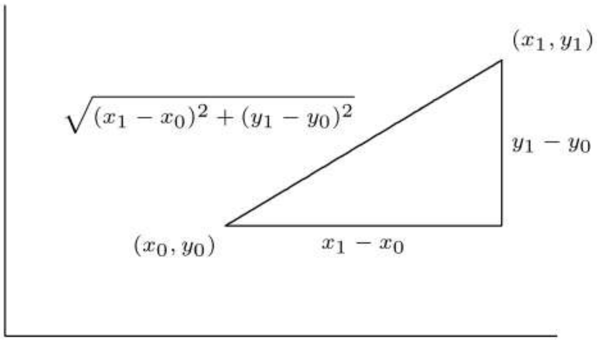

Figure 9.9.1 The length of a line segment.

Now if the graph of f is "nice" (say, differentiable) it appears that we can approximate the length of a portion of the curve with line segments, and that as the number of segments increases, and their lengths decrease, the sum of the lengths of the line segments will approach the true arc length; see figure 9.9.2.

Now we need to write a formula for the sum of the lengths of the line segments, in a form that we know becomes an integral in the limit. So we suppose we have divided the interval $[a, b]$ into n subintervals as usual, each with length $\Delta x = (b - a)/n$, and endpoints $a = x_0$, x_1, x_2, $\ldots$, $x_n = b$. The length of a typical line segment, joining $(x_i, f(x_i))$ to $(x_{i+1}, f(x_{i+1}))$, is $\sqrt{(\Delta x)^2 + (f(x_{i+1}) - f(x_i))^2}$. By the Mean Value Theorem (6.5.2), there is a number t_i in (x_i, x_{i+1}) such that $f'(t_i)\Delta x = f(x_{i+1}) - f(x_i)$, so the length of

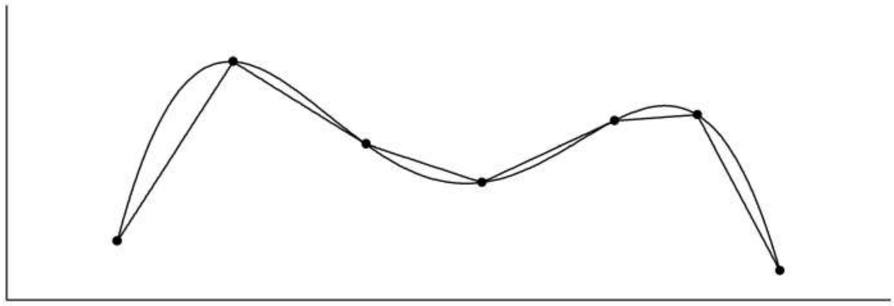

Figure 9.9.2 Approximating arc length with line segments.

the line segment can be written as

$$\sqrt{(\Delta x)^2 + (f'(t_i))^2 \Delta x^2} = \sqrt{1 + (f'(t_i))^2}\, \Delta x.$$

The arc length is then

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} \sqrt{1 + (f'(t_i))^2}\, \Delta x = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

Note that the sum looks a bit different than others we have encountered, because the approximation contains a t_i instead of an x_i. In the past we have always used left endpoints (namely, x_i) to get a representative value of f on $[x_i, x_{i+1}]$; now we are using a different point, but the principle is the same.

To summarize, to compute the length of a curve on the interval $[a, b]$, we compute the integral

$$\int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

Unfortunately, integrals of this form are typically difficult or impossible to compute exactly, because usually none of our methods for finding antiderivatives will work. In practice this means that the integral will usually have to be approximated.

EXAMPLE 9.9.1 Let $f(x) = \sqrt{r^2 - x^2}$, the upper half circle of radius r. The length of this curve is half the circumference, namely πr. Let's compute this with the arc length formula. The derivative f' is $-x/\sqrt{r^2 - x^2}$ so the integral is

$$\int_{-r}^{r} \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx = \int_{-r}^{r} \sqrt{\frac{r^2}{r^2 - x^2}}\, dx = r \int_{-r}^{r} \sqrt{\frac{1}{r^2 - x^2}}\, dx.$$

Using a trigonometric substitution, we find the antiderivative, namely $\arcsin(x/r)$. Notice that the integral is improper at both endpoints, as the function $\sqrt{1/(r^2 - x^2)}$ is undefined when $x = \pm r$. So we need to compute

$$\lim_{D\to -r^+} \int_D^0 \sqrt{\frac{1}{r^2 - x^2}}\, dx + \lim_{D\to r^-} \int_0^D \sqrt{\frac{1}{r^2 - x^2}}\, dx.$$

This is not difficult, and has value π, so the original integral, with the extra r in front, has value πr as expected. □

Exercises 9.9.

1. Find the arc length of $f(x) = x^{3/2}$ on $[0, 2]$. ⇒

2. Find the arc length of $f(x) = x^2/8 - \ln x$ on $[1, 2]$. ⇒

3. Find the arc length of $f(x) = (1/3)(x^2 + 2)^{3/2}$ on the interval $[0, a]$. ⇒

4. Find the arc length of $f(x) = \ln(\sin x)$ on the interval $[\pi/4, \pi/3]$. ⇒

5. Let $a > 0$. Show that the length of $y = \cosh x$ on $[0, a]$ is equal to $\int_0^a \cosh x\, dx$.

6. Find the arc length of $f(x) = \cosh x$ on $[0, \ln 2]$. ⇒

7. Set up the integral to find the arc length of $\sin x$ on the interval $[0, \pi]$; do not evaluate the integral. If you have access to appropriate software, approximate the value of the integral. ⇒

8. Set up the integral to find the arc length of $y = xe^{-x}$ on the interval $[2, 3]$; do not evaluate the integral. If you have access to appropriate software, approximate the value of the integral. ⇒

9. Find the arc length of $y = e^x$ on the interval $[0, 1]$. (This can be done exactly; it is a bit tricky and a bit long.) ⇒

9.10 Surface Area

Another geometric question that arises naturally is: "What is the surface area of a volume?" For example, what is the surface area of a sphere? More advanced techniques are required to approach this question in general, but we can compute the areas of some volumes generated by revolution.

As usual, the question is: how might we approximate the surface area? For a surface obtained by rotating a curve around an axis, we can take a polygonal approximation to the curve, as in the last section, and rotate it around the same axis. This gives a surface composed of many "truncated cones;" a truncated cone is called a **frustum** of a cone. Figure 9.10.1 illustrates this approximation.

So we need to be able to compute the area of a frustum of a cone. Since the frustum can be formed by removing a small cone from the top of a larger one, we can compute the desired area if we know the surface area of a cone. Suppose a right circular cone has base radius r and slant height h. If we cut the cone from the vertex to the base circle

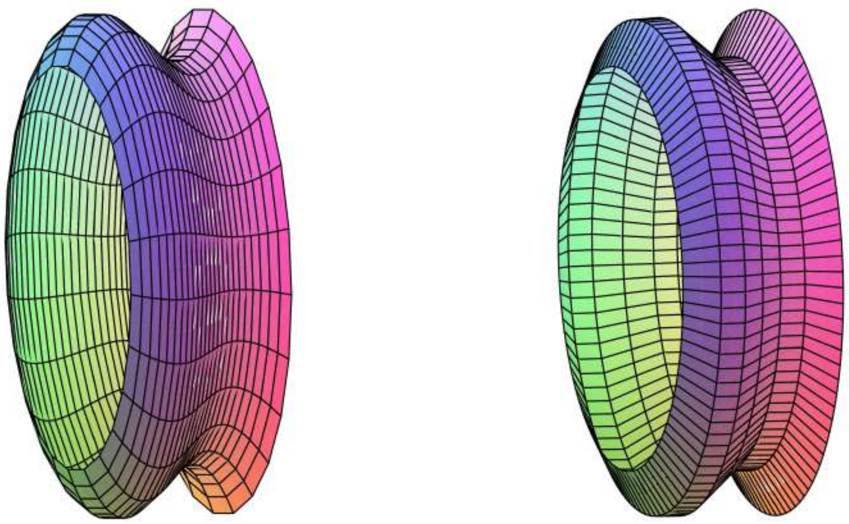

Figure 9.10.1 Approximating a surface (left) by portions of cones (right).

and flatten it out, we obtain a sector of a circle with radius h and arc length $2\pi r$, as in figure 9.10.2. The angle at the center, in radians, is then $2\pi r/h$, and the area of the cone is equal to the area of the sector of the circle. Let A be the area of the sector; since the area of the entire circle is πh^2, we have

$$\frac{A}{\pi h^2} = \frac{2\pi r/h}{2\pi}$$
$$A = \pi r h.$$

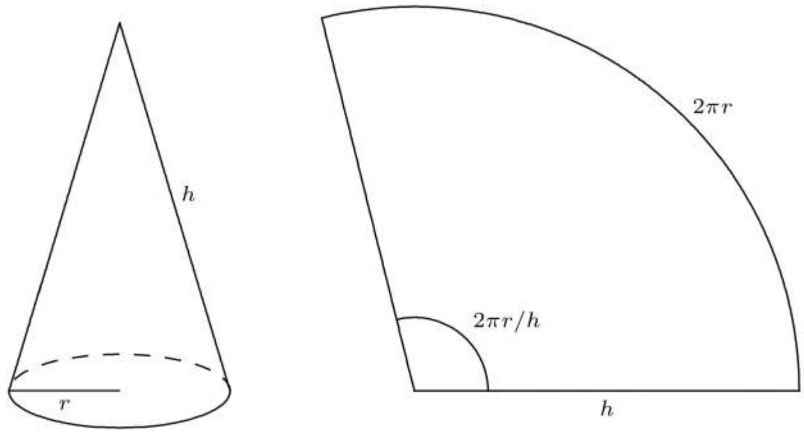

Figure 9.10.2 The area of a cone.

Now suppose we have a frustum of a cone with slant height h and radii r_0 and r_1, as in figure 9.10.3. The area of the entire cone is $\pi r_1(h_0 + h)$, and the area of the small cone is $\pi r_0 h_0$; thus, the area of the frustum is $\pi r_1(h_0+h) - \pi r_0 h_0 = \pi((r_1-r_0)h_0 + r_1 h)$. By similar triangles,

$$\frac{h_0}{r_0} = \frac{h_0+h}{r_1}.$$

With a bit of algebra this becomes $(r_1-r_0)h_0 = r_0 h$; substitution into the area gives

$$\pi((r_1-r_0)h_0 + r_1 h) = \pi(r_0 h + r_1 h) = \pi h(r_0+r_1) = 2\pi\frac{r_0+r_1}{2}h = 2\pi r h.$$

The final form is particularly easy to remember, with r equal to the average of r_0 and r_1, as it is also the formula for the area of a cylinder. (Think of a cylinder of radius r and height h as the frustum of a cone of infinite height.)

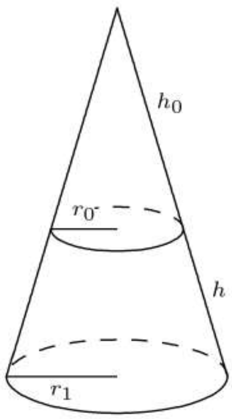

Figure 9.10.3 The area of a frustum.

Now we are ready to approximate the area of a surface of revolution. On one subinterval, the situation is as shown in figure 9.10.4. When the line joining two points on the curve is rotated around the x-axis, it forms a frustum of a cone. The area is

$$2\pi r h = 2\pi\frac{f(x_i)+f(x_{i+1})}{2}\sqrt{1+(f'(t_i))^2}\,\Delta x.$$

Here $\sqrt{1+(f'(t_i))^2}\,\Delta x$ is the length of the line segment, as we found in the previous section. Assuming f is a continuous function, there must be some x_i^* in $[x_i, x_{i+1}]$ such that $(f(x_i)+f(x_{i+1}))/2 = f(x_i^*)$, so the approximation for the surface area is

$$\sum_{i=0}^{n-1} 2\pi f(x_i^*)\sqrt{1+(f'(t_i))^2}\,\Delta x.$$

This is not quite the sort of sum we have seen before, as it contains two different values in the interval $[x_i, x_{i+1}]$, namely x_i^* and t_i. Nevertheless, using more advanced techniques than we have available here, it turns out that

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} 2\pi f(x_i^*)\sqrt{1+(f'(t_i))^2}\,\Delta x = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$

is the surface area we seek. (Roughly speaking, this is because while x_i^* and t_i are distinct values in $[x_i, x_{i+1}]$, they get closer and closer to each other as the length of the interval shrinks.)

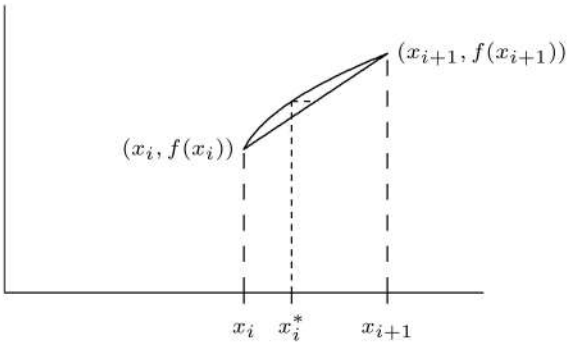

Figure 9.10.4 One subinterval.

EXAMPLE 9.10.1 We compute the surface area of a sphere of radius r. The sphere can be obtained by rotating the graph of $f(x) = \sqrt{r^2 - x^2}$ about the x-axis. The derivative f' is $-x/\sqrt{r^2-x^2}$, so the surface area is given by

$$\begin{aligned} A &= 2\pi \int_{-r}^{r} \sqrt{r^2-x^2}\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx \\ &= 2\pi \int_{-r}^{r} \sqrt{r^2-x^2}\sqrt{\frac{r^2}{r^2-x^2}}\,dx \\ &= 2\pi \int_{-r}^{r} r\,dx = 2\pi r \int_{-r}^{r} 1\,dx = 4\pi r^2 \end{aligned}$$

□

If the curve is rotated around the y axis, the formula is nearly identical, because the length of the line segment we use to approximate a portion of the curve doesn't change. Instead of the radius $f(x_i^*)$, we use the new radius $\bar{x}_i = (x_i + x_{i+1})/2$, and the surface area integral becomes

$$\int_a^b 2\pi x\sqrt{1+(f'(x))^2}\,dx.$$

EXAMPLE 9.10.2 Compute the area of the surface formed when $f(x) = x^2$ between 0 and 2 is rotated around the y-axis.