Answer to Essential Question 7.3: All of them can be negative except for the kinetic energy, which can't be negative because $K = (1/2)mv^2$ and neither the square of the speed nor the mass can be negative. What is key is how the energies change, not what the values of the energies are.

7-4 Momentum and Collisions

Let's extend our understanding of momentum by analyzing a **collision**, which is an event in which two objects interact. As we learned in Chapter 6, Newton's third law tells us that, when no net external force acts on a system, the total momentum of the system is conserved. The momenta of the individual objects can change, but the total momentum of the system does not.

Generally, when we analyze a collision, we look at the situation immediately before the collision and compare it to the situation immediately after the collision. What happens during the collision itself can be interesting, and complicated. Fortunately, by using momentum we don't have to worry about such complications. The usual starting point in analyzing a collision is to write down a conservation of momentum equation reflecting the following relation:

Total momentum before the collision = total momentum after the collision.

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$$ (Equation 7.2: **Momentum conservation**)

where the subscripts i and f stand for initial and final, and the two colliding objects are denoted by 1 and 2.

EXPLORATION 7.4 – Two carts collide…again

Two identical carts experience a collision on a horizontal track. Before the collision, cart 1 is moving at speed v to the right, directly toward cart 2, which is at rest. Immediately after the collision, cart 2 is moving with a velocity of $v/2$ to the right.

Step 1 - ***What is the velocity of cart 1 immediately after the collision?*** Let's begin, as usual, by drawing a diagram of the situation in Figure 7.9, showing the carts before and after the collision.

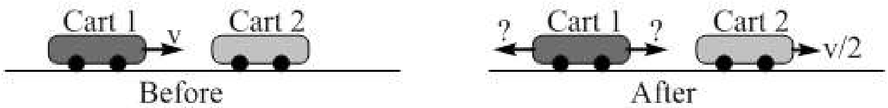

Figure 7.9: Two carts, immediately before and immediately after their collision.

The first step in applying equation 7.2 is to remember that momentum is a vector. Let's define right as the positive x-direction. We can say that each cart has a mass m, and we are given that $\vec{v}_{1i} = +v$, $\vec{v}_{2i} = 0$, and $\vec{v}_{2f} = +v/2$. Substituting all these terms into the conservation of momentum equation gives:

$$+mv = m\vec{v}_{1f} + m\frac{v}{2}$$

Dividing out a factor of m and solving for the velocity of cart 1 after the collision gives:

$$\vec{v}_{1f} = +\frac{v}{2}.$$

The two carts have the same velocity, and thus move together, after the collision. We could arrange this special case by attaching Velcro to both carts so they stick together. When the objects move together afterwards, we say that the collision is ***completely inelastic***.

Step 2 - ***Is kinetic energy conserved in this collision?*** Kinetic energy does not have to be conserved in a collision, although in certain special cases it is. Let's see what happens to the kinetic energy in this case.

Before the collision: $K_i = \frac{1}{2}mv_{1i}^2 + \frac{1}{2}mv_{2i}^2 = \frac{1}{2}mv^2$.

After the collision: $K_f = \frac{1}{2}mv_{1f}^2 + \frac{1}{2}mv_{2f}^2 = \frac{1}{2}m\left(\frac{v}{2}\right)^2 + \frac{1}{2}m\left(\frac{v}{2}\right)^2 = \frac{1}{4}mv^2 = \frac{K_i}{2}$.

In this case, only 50% of the kinetic energy from before the collision is in the system as kinetic energy after the collision. The total energy has to be conserved, but in this case, some of the kinetic energy of cart 1 before the collision is transformed to other forms of energy (such as thermal energy, which is energy associated with the motion of molecules, and sound energy) in the collision process.

Step 3 - ***What is the velocity of the system's center of mass before the collision?*** By dividing both sides of Equation 6.4, for the position of the center of mass, by a time interval, Δt, and using the definition of velocity, we can obtain an equation for the velocity of the center of mass:

$$\vec{v}_{CM} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \ldots}{m_1 + m_2 + m_3 + \ldots}$$ (Equation 7.3: **Velocity of the center of mass**)

The *m*'s represent the masses of the various pieces of the object or system. The terms in the numerator on the right represent the momenta of the individual parts of the system, so the equation really says that the total momentum of the system is the vector sum of the momenta of its parts, which seems sensible.

Applying the equation to the two-cart system before the collision gives:

$$\vec{v}_{CM,i} = \frac{+mv + m \times 0}{m+m} = +\frac{v}{2}.$$

This result makes sense because the center of mass is halfway between the carts, so the center of mass covers half the distance as cart 1 does in the same time.

Step 4 - ***What is the velocity of the system's center of mass after the collision?*** Applying equation 7.3 to the system after the collision gives:

$$\vec{v}_{CM,f} = \frac{+m\frac{v}{2} + m\frac{v}{2}}{m+m} = +\frac{v}{2}.$$

It should come as no surprise that the velocity of the center of mass after the collision is the same as the velocity of the center of mass before the collision. Rather, this result is expected as a consequence of momentum conservation. In short, the center of mass does not even register that a collision has taken place.

Key ideas: In a collision, in general, the system's momentum is conserved while the system's kinetic energy is not necessarily conserved. In addition, in general, the motion of the system's center of mass is unaffected by the collision. **Related End-of-Chapter Exercises: 25 – 28.**

Essential Question 7.4: Under what condition is the momentum of a system conserved in a collision?

Answer to Essential Question 7.4: For momentum to be conserved, either no net force is acting on the system, or the net force must act over such a small time interval that it has a negligible effect on the momentum of the system.

7-5 Classifying Collisions

If we attach Velcro to our colliding carts, and the carts stick together after the collision, as in Exploration 7.4, the collision is **completely inelastic**. If we remove the Velcro, so the carts do not stick together, we can set up a collision with the same initial conditions (cart 1 moving toward cart 2, which is stationary) and get a variety of outcomes. We generally classify these outcomes into four categories, depending on what happens to the kinetic energy in the collision.

We can also define a parameter k called the ***elasticity***. Elasticity is the ratio of the relative velocity of the two colliding objects after the collision to the negative of their relative velocity before the collision. By this definition, the elasticity should always be positive:

$$k = \frac{\vec{v}_{2f} - \vec{v}_{1f}}{\vec{v}_{1i} - \vec{v}_{2i}} \qquad \text{(Equation 7.4: \textbf{Elasticity})}$$

The four categories of collisions can also be defined in terms of the elasticity.

Type of Collision	Kinetic Energy	Elasticity	Example
Super-elastic	$K_f > K_i$	$k > 1$	Carts are initially stationary, then pushed apart by a spring-loaded piston, as in Exploration 6.4. An explosion.
Elastic	$K_f = K_i$	$k = 1$	Carts with repelling magnets.
Inelastic	$K_f < K_i$	$k < 1$	Describes most collisions, such as two cars that make contact when colliding but that don't stick together.
Completely inelastic	$K_f < K_i$, and the objects stick together	$k = 0$	Carts with Velcro, as in Exploration 7-3, or chewing gum hitting the sidewalk.

Table 7.1: Collisions can be classified in terms of what happens to the kinetic energy or in terms of the elasticity. Note that, in an elastic collision, the fact that $k = 1$ can be obtained by combining the momentum conservation equation with the conservation of kinetic energy equation.

EXAMPLE 7.5 – An assist from gravity

Sending a space probe from Earth to another planet requires a great deal of energy. In many cases, a significant fraction of the probe's kinetic energy can be provided by a third planet, through a process known as a ***gravitational assist***. For instance, the Cassini-Huygens space probe launched on October 15, 1997, used four gravitational assists, two from Venus, one from Earth, and one from Jupiter, to speed it on its more than 1 billion km trip to Saturn, arriving there on July 1, 2004. We can treat a gravitational assist as an elastic collision, because the long-range interaction of the probe and the planet provides no mechanism for a loss of mechanical energy.

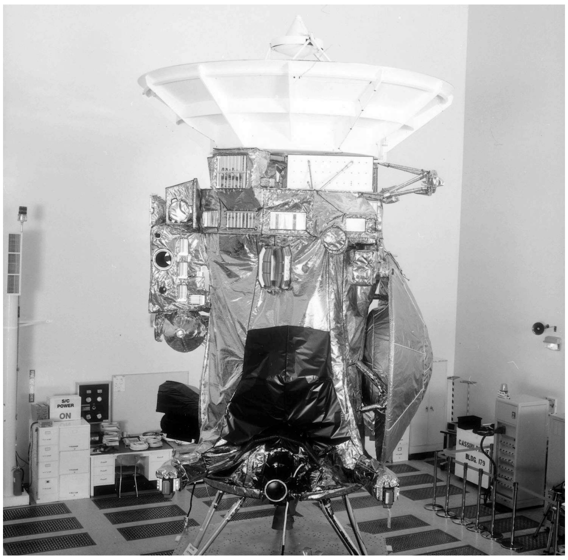

Figure 7.10: The Cassini-Huygens space probe while it was being assembled. The desk and chair at the lower left give a sense of the scale. Photo courtesy NASA/JPL-Caltech.

Answer to Essential Question 7.5: To conserve energy of the planet-probe system, the planet's speed must decrease when the probe's speed increases. Because the planet's mass is so much larger than the probe's, this decrease in speed is negligible. Thus, the assumption is reasonable.

7-6 Collisions in Two Dimensions

Momentum conservation also applies in two and three dimensions. The standard approach to a two-dimensional (or even three-dimensional) problem is to break the momentum into components and conserve momentum in both the x and y directions separately. For colliding objects, the conservation of momentum equation in the x-direction, for instance, is:

$\vec{p}_{1ix} + \vec{p}_{2ix} = \vec{p}_{1fx} + \vec{p}_{2fx}$. (Eq. 7.5: **Conserving momentum in the x-direction**)

This can be written in an equivalent form:

$m_1\vec{v}_{1ix} + m_2\vec{v}_{2ix} = m_1\vec{v}_{1fx} + m_2\vec{v}_{2fx}$ (Eq. 7.6: **Momentum conservation, *x*-direction**)

Similar equations apply in the y-direction.

EXPLORATION 7.6 – A two-dimensional collision

An object of mass m, moving in the $+x$-direction with a velocity of 5.0 m/s, collides with an object of mass $2m$. Before the collision, the second object has a velocity given by $\vec{v}_{2i} = -3.0 \text{ m/s } \hat{x} + 4.0 \text{ m/s } \hat{y}$, while, after the collision, its velocity is 3.0 m/s in the $+y$-direction. What is the velocity of the first object after the collision?

Step 1 – ***Draw a diagram of the situation.*** This is shown in Figure 7.12.

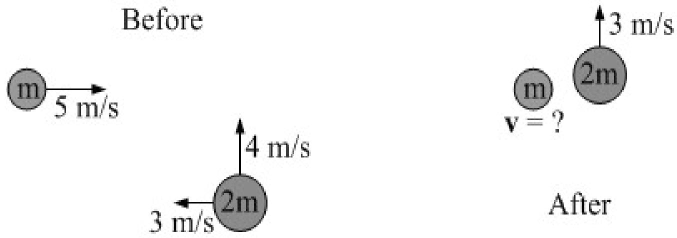

Figure 7.12: A diagram of the objects before and after they collide.

Step 2 - ***Set up a table showing the momentum components of each object before and after the collision.*** Organizing components into a table helps us keep the x-direction information separate from the y-direction information. We can combine the components into one vector at the end.

	***x*-direction**	***y*-direction**
Before the collision	$\vec{p}_{1ix} = m\vec{v}_{1ix} = +(5 \text{ m/s})m$	$\vec{p}_{1iy} = m\vec{v}_{1iy} = 0$
	$\vec{p}_{2ix} = 2m\vec{v}_{2ix} = -(6 \text{ m/s})m$	$\vec{p}_{2iy} = m\vec{v}_{2iy} = +(8 \text{ m/s})m$
After the collision	$\vec{p}_{1fx} = m\vec{v}_{1fx} = ?$	$\vec{p}_{1fy} = m\vec{v}_{1fy} = ?$
	$\vec{p}_{2fx} = 2m\vec{v}_{2fx} = 0$	$\vec{p}_{2fy} = 2m\vec{v}_{2fy} = +(6 \text{ m/s})m$

Table 7.2: Organizing the collision data in a table helps to keep the x-direction information separate from the y-direction information, and doing so can also help us solve the problem.

Step 3 – ***Apply conservation of momentum in the x-direction, and find the x-component of the first object's final velocity.*** Applying momentum conservation in the x-direction involves writing down the equation $\vec{p}_{1ix} + \vec{p}_{2ix} = \vec{p}_{1fx} + \vec{p}_{2fx}$.

This gives $\vec{p}_{1fx} = \vec{p}_{1ix} + \vec{p}_{2ix} - \vec{p}_{2fx} = +(5 \text{ m/s})m - (6 \text{ m/s})\, m - 0 = -(1 \text{ m/s})m$.

To get the velocity component in the x-direction we just divide by the mass, m.

$$\vec{v}_{1fx} = \frac{\vec{p}_{1fx}}{m} = \frac{-(1\text{ m/s})m}{m} = -1\text{ m/s}.$$

Step 4 – ***Use a similar process in the y-direction to find the y-component of the first object's final velocity.*** Applying momentum conservation in the y-direction involves writing down the equation $\vec{p}_{1iy} + \vec{p}_{2iy} = \vec{p}_{1fy} + \vec{p}_{2fy}$.

This equation gives $\vec{p}_{1fy} = \vec{p}_{1iy} + \vec{p}_{2iy} - \vec{p}_{2fy} = 0 + (8\text{ m/s})m - (6\text{ m/s})m = +(2\text{ m/s})m$.

To get the velocity component in the y-direction, we divide by the mass, m.

$$\vec{v}_{1fy} = \frac{\vec{p}_{1fy}}{m} = \frac{+(2\text{ m/s})m}{m} = +2\text{ m/s}.$$

Step 5 – ***Combine the x and y components to find the first object's final speed. Also, write down an expression for the first object's final velocity.*** We can use the Pythagorean theorem to find the final speed of the first object:

$$v_{1f} = \sqrt{v_{1fx}^2 + v_{1fy}^2} = \sqrt{1^2 + 2^2} = \sqrt{5}\text{ m/s}.$$

The velocity can be written in terms of components as $\vec{v}_{1f} = -1\text{ m/s}\,\hat{x} + 2\text{ m/s}\,\hat{y}$. The first ball's final velocity is shown in Figure 7.13.

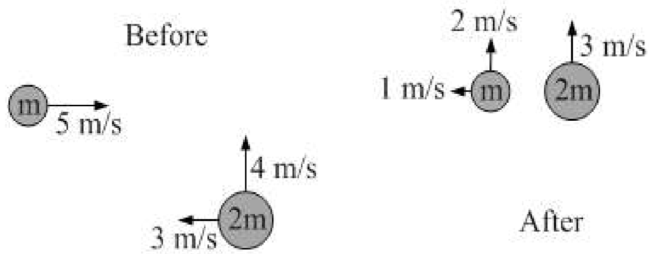

Figure 7.13: The situation Before and After the collision.

Key idea for momentum problems: We can solve a momentum problem in two dimensions with a strategy based on the independence of x and y, breaking a two-dimensional problem into two independent one-dimensional problems. **Related End-of-Chapter Exercises: 29, 57.**

Now that we've looked at a few examples, let's summarize a general method for solving a problem in which there is a collision.

A General Method for Solving a Problem That Involves a Collision

1. Draw a diagram of the situation, showing the velocity of the objects immediately before and immediately after the collision.
2. In a two-dimensional situation, set up a table showing the components of the momentum before and after the collision for each object.
3. Use momentum conservation: $m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$. (Apply this twice, once for each direction, in a two-dimensional situation.) Account for the fact that momentum is a vector by using appropriate + and – signs.
4. If you need an additional relationship (such as in the case of an elastic collision), use the elasticity relationship or write an energy-conservation equation.

Essential Question 7.6: The strategy outlined above, which we applied in Exploration 7.6, relies on breaking vectors into components. Is there another method that we could use to solve the problem without using components?

Answer to Essential Question 7.6: A whole-vector approach, not splitting the velocity and momentum vectors into components, would also work (see End-of-Chapter Exercise 58).

7-7 Combining Energy and Momentum

To analyze some situations, we apply both energy conservation and momentum conservation in the same problem. The trick is to know when to apply energy conservation (and when not to!) and when to apply momentum conservation. Consider the following Exploration.

EXPLORATION 7.7 – Bringing the concepts together

Two balls hang from strings of the same length. Ball A, with a mass of 4.0 kg, is swung back to a point 0.80 m above its equilibrium position. Ball A is released from rest and swings down and hits ball B. After the collision, ball A rebounds to a height of 0.20 m above its equilibrium position and ball B swings up to a height of 0.050 m. Let's use $g = 10$ m/s^2 to simplify the calculations.

Step 1 – ***Sketch a diagram of the situation.*** This is shown in Figure 7.14.

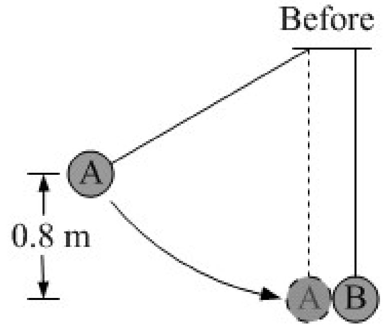

Figure 7.14: A diagram of the two balls on strings. Ball A is swung back until it is 0.80 m higher than its equilibrium point and released from rest.

Step 2 – ***Our goal is to find the mass of ball B. Can we find the mass by setting the initial gravitational potential energy of ball A equal to the sum of the final potential energy of ball A and the final potential energy of ball B? Explain why or why not.*** The answer to the question is no. We can use energy conservation to help solve the problem, but setting the mechanical energy before the collision equal to the mechanical energy after the collision is assuming too much. The balls make contact in the collision, so it is likely that some of the mechanical energy is transformed to thermal energy, for instance.

Step 3 – ***Apply energy conservation to find the speed of ball A just before the collision.*** The gravitational potential energy of ball A is transformed into kinetic energy just before the collision. We will neglect the work done by air resistance, so we can apply energy conservation before the collision. Let's start with the conservation of energy equation: $K_i + U_i + W_{nc} = K_f + U_f$.

For ball A's swing before the collision, we know that the initial kinetic energy is zero. We are assuming that non-conservative forces do no work. We can also define the zero level for gravitational potential energy to be the lowest point in the swing, just before A hits B, so $U_f = 0$. The five-term equation reduces to:

$$U_i = K_f ;$$

$$mgh = \frac{1}{2}mv_f^2 ;$$

So, $v_f = \sqrt{2gh} = \sqrt{2 \times 10 \times 0.8} = \sqrt{16} = 4.0$ m/s .

Step 4 – ***Apply conservation of energy again to find the speed of ball A just after the collision.*** We could try applying conservation of momentum here, but there are too many unknowns. Instead, we can follow the conservation of energy method we used above. Note that we will not state that the kinetic energy immediately before the collision is equal to the kinetic energy after the collision, because that is not true. We can apply energy conservation, however, if we confine ourselves to the mechanical energy before the collision (as in step 3) or to the mechanical energy after the collision (this step). If we focus on the upswing, we have the kinetic energy of ball A, immediately after the collision, being transformed into gravitational potential energy. The conservation of energy equation reduces to:

$$K_i = U_f ;$$

$$\frac{1}{2} m v_i^2 = mgh ;$$

So, $v_i = \sqrt{2gh} = \sqrt{2 \times 10 \times 0.2} = \sqrt{4.0} = 2.0$ m/s .

Let's be clear on what we have calculated in parts 3 and 4, because the notation can be confusing. We are analyzing the motion in three separate parts. The first part of ball A's motion is the downswing, which we analyzed in step 3. The third part is the upswing, which we analyzed in step 4. The second part is the collision, which we still have to analyze. The velocity of ball A immediately before the collision, at the end of the downswing, is $\vec{v}_{Ai} = 4.0 \text{ m/s}$ to the right, while A's velocity just after the collision, at the start of the upswing, is $\vec{v}_{Af} = 2.0 \text{ m/s}$ to the left. These are the values we will use in the conservation of momentum equation in step 5.

Step 5 – ***First, apply energy conservation to find the speed of ball B after the collision. Then, apply momentum conservation to find the mass of ball B.*** We still have to find the velocity of ball B, after the collision, before we use the conservation of momentum equation to find ball B's mass. We can find B's speed immediately after the collision by following the same process we used for ball A in step 3. We get:

$$K_i = U_f$$

$$\frac{1}{2} m v_i^2 = mgh$$

So, $v_{Bf} = \sqrt{2gh_B} = \sqrt{2 \times (10 \text{ m/s}^2) \times 0.050 \text{ m}} = \sqrt{1.0 \text{m}^2/\text{s}^2} = 1.0 \text{ m/s}$.

The velocity of ball B immediately after the collision is $\vec{v}_{Bf} = 1.0 \text{ m/s}$ to the right.

Now, we can write out a conservation of momentum equation to solve for the mass of ball B. It is critical to account for the fact that momentum is a vector. In this case, we account for the vector nature of momentum by using a minus sign for the velocity of ball A after the collision to reflect that it is moving to the left, when we chose right to be the positive direction. This gives: $m_A \vec{v}_{Ai} + m_B \vec{v}_{Bi} = m_A \vec{v}_{Af} + m_B \vec{v}_{Bf}$, where $\vec{v}_{Bi} = 0$. Solving for the mass of ball B gives:

$$m_B = \frac{m_A \vec{v}_{Ai} - m_A \vec{v}_{Af}}{\vec{v}_{Bf}} = \frac{(4.0 \text{kg}) \times (+4.0 \text{ m/s}) - (4.0 \text{ kg}) \times (-2.0 \text{ m/s})}{+1.0 \text{ m/s}} = 24 \text{ kg} .$$

Key idea: In some situations, we can apply conservation of energy and conservation of momentum ideas together. In general, we apply conservation of momentum to connect the situation before the collision to the situation after the collision. We use energy conservation to learn something about the situation before the collision and/or the situation afterwards.
Related End-of-Chapter Exercises: 30 – 32.

Essential Question 7.7: Is the collision in Exploration 7.7 super-elastic, elastic, inelastic, or completely inelastic? Justify your answer in two different ways.

Answer to Essential Question 7.7: The balls don't stick together, so we know the collision is not completely inelastic. One way to classify the collision is to find the elasticity, *k* (see equation 7.4).

$$k = \frac{\vec{v}_{Bf} - \vec{v}_{Af}}{\vec{v}_{Ai} - \vec{v}_{Bi}} = \frac{1.0 \text{ m/s} - (-2.0 \text{ m/s})}{4.0 \text{ m/s} - 0} = 0.75 .$$

The fact that *k* is less than 1 means the collision is inelastic. We can confirm this result by looking at the kinetic energy before and after the collision.

$$K_i = \frac{1}{2} m_A v_{Ai}^2 + \frac{1}{2} m_B v_{Bi}^2 = \frac{1}{2} \times (4.0 \text{ kg}) \times (4.0 \text{ m/s})^2 + 0 = 32 \text{ J} .$$

$$K_f = \frac{1}{2} m_A v_{Af}^2 + \frac{1}{2} m_B v_{Bf}^2 = \frac{1}{2} \times (4.0 \text{ kg}) \times (2.0 \text{ m/s})^2 + \frac{1}{2} \times (24.0 \text{ kg}) \times (1.0 \text{ m/s})^2 ;$$

$$K_f = 8.0 \text{ J} + 12 \text{ J} = 20 \text{ J} .$$

The kinetic energy in the system after the collision is less than it is before the collision, so we have an inelastic collision.

Chapter Summary

Essential Idea about Conservation Laws

Many physical situations can be analyzed using forces, which we learned about in previous chapters, and/or by applying the fundamental concepts of conservation of momentum and conservation of energy, which we learned about in this chapter.

Comparing the Energy and Force Methods

The primary methods we use to analyze situations are to use forces and Newton's Laws, or to use energy conservation. Let's compare these two methods.

- The energy approach can be very effective, because we often just have to deal with the initial and final states and we don't have to account for the path taken by the system in going from one state to another, as we do with the force approach.
- The energy approach, by itself, does not give us any information about time, such as about how long it takes a system to move from one state to another. If you need to know about time, use a force analysis.
- Energy is a scalar. Thus energy, by itself, tells us nothing about direction. Force is a vector, and this it can give us information about direction.
- If W_{nc}, the work done by non-conservative forces, is zero, then the total ***mechanical energy*** (the sum of the kinetic and potential energies) is conserved.

A General Method for Solving a Problem Involving Energy Conservation

1. Draw a diagram of the situation. Usually, we use energy to relate a system at one point, or instant in time, to the system at a different point, or a different instant.
2. Apply energy conservation: $K_i + U_i + W_{nc} = K_f + U_f$. (Eq. 7.1)
3. Choose a level to be the zero for gravitational potential energy. Setting the zero level so that either U_i or U_f (or both) is zero is often best.
4. Identify the terms in the equation that are zero.
5. Take the remaining terms and solve.

Collisions and Momentum Conservation

In general, the momentum of a system is conserved in a collision, but the system's kinetic energy is often not conserved in a collision. In fact, one of the two ways in which we classify collisions is based on how the kinetic energy before the collision compares to that afterwards. The second way collisions can be classified is in terms of the ***elasticity***, k, which is the ratio of the relative speed of the colliding objects after the collision to their relative speed after the collision:

$$k = \frac{\vec{v}_{2f} - \vec{v}_{1f}}{\vec{v}_{1i} - \vec{v}_{2i}} \quad . \qquad \text{(Equation 7.4: \textbf{Elasticity})}$$

This equation is particularly useful when the collision is elastic and the relative velocity of the objects has the same magnitude before and after the collision.

The four collision categories are:

Type of Collision	Kinetic Energy	Elasticity
Super-elastic	$K_f > K_i$	$k > 1$
Elastic	$K_f = K_i$	$k = 1$
Inelastic	$K_f < K_i$	$k < 1$
Completely inelastic	$K_f < K_i$, and the objects stick together	$k = 0$

A General Method for Solving a Problem Involving a Collision

1. Draw a diagram of the situation, showing the velocity of the objects immediately before and immediately after the collision.
2. In a two-dimensional situation, set up a table showing the components of the momentum before and after the collision for each object.
3. Use momentum conservation: $m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$. (Eq. 7.2)

   Apply equation 7.2 twice, once for each direction, in a two-dimensional situation. Account for the fact that momentum is a vector with + and – signs.
4. If you require an additional relationship (such as in the case of an elastic collision) use the elasticity relationship or write an energy-conservation equation.