ORDINARY DIFFERENTIAL EQUATIONS: BASIC CONCEPTS

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ABSTRACT. Some of the most basic concepts of ordinary differential equations are introduced and illustrated by examples.

CONTENTS

1. WHAT IS AN ORDINARY DIFFERENTIAL EQUATION?

Roughly speaking, an *ordinary differential equation* (ODE) is an equation involving a function (of one variable) and its derivatives.

Examples of ODE's are

$$y' + y = 0, \qquad \frac{\mathrm{d}x}{\mathrm{d}t} + x^2 t = \sin t, \qquad \text{and} \qquad y'' = x \cos y. \tag{1}$$

A *solution* of an ODE is a *function* that satisfies the equation. This is in contrast to algebraic equations, such as $x^2 = 3$ and $x^6 + 7x = 5$, whose solutions are *numbers*.

Example 1. Let $y(x) = e^{-x}$. Then we have

$$y'(x) = -e^{-x}, \qquad \text{and so} \qquad y'(x) + y(x) = 0. \tag{2}$$

So the function $y(x) = e^{-x}$ is a *solution* of the ODE $y' + y = 0$. Typically, ODE's have many solutions: $y(x) = 2e^{-x}$ is also a solution of the ODE $y' + y = 0$.

Example 2. The function $x(t) = t^2$ is a solution to the ODE $xx'' + tx' = 4t^2$.

When faced with an ODE, one may try to solve it *explicitly*, as was done in Example 1. However, only very special types of ODE's can be solved by formulas, so one needs to resort to *approximation methods*, that can be used to solve ODE's inexactly, and to *qualitative methods*, that can be used to extract valuable information about the solution of an ODE without actually solving it. In this course, we will focus mainly on techniques to explicitly solve ODE's, and only get a glimpse of approximation and qualitative methods. A large class of approximation methods are studied in a numerical analysis course, and qualitative methods are basically the content of a dynamical systems course.

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2. An example application: Falling bodies

ODE's have a wide range of applications in sciences and in mathematics. In this section, we will look at a simple application, and illustrate some important concepts along the way.

This example concerns the motion of a small body near Earth's surface. We assume that the body moves only in the vertical direction, and fix a coordinate system so that the y-axis is pointing upward. If we denote by $y(t)$ the y-coordinate of the particle at time moment t, and assume that there is no air resistance, then Newton's second law gives $my''(t) = -mg$, where m is the mass of the particle, and $g \approx 9.81\mathrm{m/s}^2$ is the free fall acceleration near Earth's surface. Assuming $m \neq 0$, we get

$$y'' = -g. \tag{3}$$

We note in passing that the air resistance can be modelled by

$$my'' = -mg - ky', \qquad \text{or} \qquad my'' = -mg - k(y')^2, \tag{4}$$

depending on how fast the particle moves.

Now, it is easy to see that for *any pair of numbers* A and B, the function

$$y(t) = A + Bt - \frac{1}{2}gt^2, \tag{5}$$

is a solution of (3). Conversely, if $y(t)$ is a solution of (3), then $y(t)$ must be equal to (5) for some value of the pair A and B. Expressions such as (5) are called *general solutions*, and they usually involve free parameters such as A and B. By varying the values of the parameters, one generates all possible solutions of the ODE. In contrast, a *particular solution* of an ODE is simply a solution of the ODE. For example, $y(t) = 1 - \frac{1}{2}gt^2$ is a particular solution of (3).

Remark 3. Arbitrary constants A and B arise because the equation (3) is insensitive to changing $y(t)$ by $y(t)+A+Bt$ for any pair of numbers A and B. In some sense, differentiating twice in (3) "kills" two pieces of information. Without additional input, there is no way to recover this information.

In practice, there are many ways to supply additional information about the solution $y(t)$ so that we can pin down only one solution from the large family of solutions given by (5). One approach is to specify $y(0)$ and $y'(0)$, or more generally, to specify $y(x_0)$ and $y'(x_0)$ for some x_0. Such conditions are called *initial conditions*, and an ODE together with appropriate initial conditions is called an *initial value problem.*

Example 4. Suppose that

$$y(0) = 10, \qquad \text{and} \qquad y'(0) = 0, \tag{6}$$

and we want to find solutions of (3) satisfying these initial conditions. From (5) we have

$$y'(t) = B - gt, \tag{7}$$

and hence $y(0) = A$ and $y'(0) = B$. This immediately gives $A = 10$ and $B = 0$, so the only solution of (3) satisfying (6) is

$$y(t) = 10 - \frac{1}{2}gt^2. \tag{8}$$

Example 5. Consider the initial value problem

$$\begin{cases} y'' = -10, \\ y(1) = 10 \\ y'(1) = 0. \end{cases} \tag{9}$$

The general solution of the ODE $y'' = -10$ is given by (5) with $g = 10$, that is, for any pair of real numbers A and B, the function

$$y(t) = A + Bt - 5t^2, \tag{10}$$

satisfies $y'' = -10$. From this and (7) with $g = 10$, we get $y(1) = A+B-5$ and $y'(1) = B-10$. Imposing $y'(1) = 0$ on the latter gives $B = 10$, and plugging this into the former, and taking into account the condition $y(1) = 10$, we infer $A = 5$. To conclude, the function

$$y(t) = 5 + 10t - 5t^2, \tag{11}$$

is a solution of the initial value problem (9). Moreover, this is the only solution, because we know that any solution of $y'' = -10$ must satisfy (10) for some real numbers A and B, and by our derivation, the only function of the form (10) that satisfies the conditions $y(1) = 10$ and $y'(1) = 0$ is (11).

3. The simplest ordinary differential equation

Apart from the trivial ones, arguably the simplest ODE is

$$y' = f(x), \tag{12}$$

where f is a given function. For example, if $f(x) = x^2$, then (12) says that the derivative of the unknown function $y(x)$ is x^2, and we know that all such functions are given by

$$y(x) = \frac{1}{3}x^3 + C, \tag{13}$$

with an arbitrary constant C. Basically, solving (12) is the problem of integrating the given function f. In this sense, solving ODE's generalizes integration, much as solving a polynomial equation $a_n x^n + \ldots + a_1 x + a_0 = 0$ generalizes taking roots $\sqrt[k]{x}$.

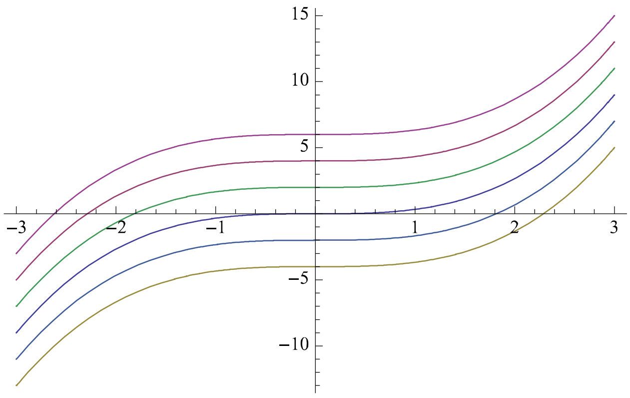

Figure 1. The graphs of (13) for different values of C. Any one of these curves represents a solution of (12) with $f(x) = x^2$.

In what follows, we present two different chains of reasoning leading to the general solution of the equation (12).

i) The equation (12) is equivalent to saying that y is an antiderivative (or a primitive) of the function f. If F is an antiderivative of f, any other antiderivative of f is given by $F(x) + C$, with some constant C. Hence, the general solution of (12) is

$$y(x) = F(x) + C, \tag{14}$$

where C is an arbitrary constant. Another way to write this is

$$y(x) = \int f(x)\,\mathrm{d}x + C. \tag{15}$$

ii) The fundamental theorem of calculus tells us

$$y(x) - y(x_0) = \int_{x_0}^{x} y'(t)\,\mathrm{d}t, \tag{16}$$

for any values x and x_0. If y satisfies (12), then

$$y(x) - y(x_0) = \int_{x_0}^{x} y'(t)\,\mathrm{d}t = \int_{x_0}^{x} f(t)\,\mathrm{d}t, \tag{17}$$

or in other words,

$$y(x) = y(x_0) + \int_{x_0}^{x} f(t)\,\mathrm{d}t. \tag{18}$$

This formula gives a way to recover the solution y, provided that we know the value $y(x_0)$ for some x_0. By choosing different values of $y(x_0)$, we can generate all possible solutions of (12), hence (18) is the general solution of (12), with $y(x_0)$ playing the role of an arbitrary constant. The relation between the formulas (18) and (14) is

$$y(x) = y(x_0) + \int_{x_0}^{x} f(t)\,\mathrm{d}t = y(x_0) + F(x) - F(x_0), \tag{19}$$

where we have used the fundamental theorem of calculus, i.e., the fact that

$$\int_{x_0}^{x} f(t)\,\mathrm{d}t = F(x) - F(x_0). \tag{20}$$

So the constant C in (14) is equal to $y(x_0) - F(x_0)$. Note that the meaning of the constant C in (14) is *a priori* not clear, whereas the meaning of the constant $y(x_0)$ in (18) is obviously the value of the unknown solution at the chosen point x_0. Thus, the formula (18) may be handy if we want to impose the initial condition $y(x_0) = a$.

Example 6. Let us reconsider the example

$$y' = x^2. \tag{21}$$

Pick $F(x) = \frac{1}{3}x^3$ as an antiderivative of x^2. Then the first approach gives

$$y(x) = \frac{1}{3}x^3 + C, \tag{22}$$

as the general solution. The second approach gives

$$y(x) = y(x_0) + \int_{x_0}^{x} x^2\,\mathrm{d}x = y(x_0) + \frac{1}{3}(x^3 - x_0^3), \tag{23}$$

where x_0 is understood to be a fixed number.

Example 7. Consider the initial value problem

$$\begin{cases} y' = x^2, \\ y(1) = 0. \end{cases} \tag{24}$$

We know that the general solution of $y' = x^2$ is given by (22), or alternatively, by (23). The formula (22) implies $y(1) = \frac{1}{3} + C$, and so $C = -\frac{1}{3}$ by the initial condition $y(1) = 0$. Hence the (one and only one) solution to the initial value problem (24) is

$$y(x) = \frac{1}{3}(x^3 - 1). \tag{25}$$

Alternatively, we could have derived this solution by putting $x_0 = 1$ and $y(1) = 0$ into (23).

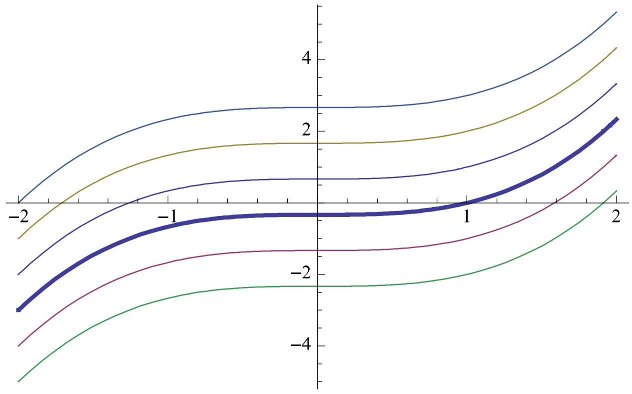

FIGURE 2. The graphs of $y(x) = \frac{1}{3}x^3 + C$ for different values of C. Any one of these curves represents a solution of $y' = x^2$. Among those solutions, $y(x) = \frac{1}{3}(x^3 - 1)$ is the only one that satisfies the initial condition $y(1) = 0$, which is represented by the thick curve.

Example 8. The general solution to

$$y' = e^{-x^2}, \tag{26}$$

is given by

$$y(x) = \int e^{-x^2}\, \mathrm{d}x + C = \frac{\sqrt{\pi}}{2}\mathrm{erf}(x) + C. \tag{27}$$

Here $\mathrm{erf}(x)$ is called the *error function*, and it is defined by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, \mathrm{d}t. \tag{28}$$

This function is simply *defined* by the integral we are trying to compute, because the anti-derivative of e^{-x^2} cannot be written in terms of a finite combination of elementary functions.

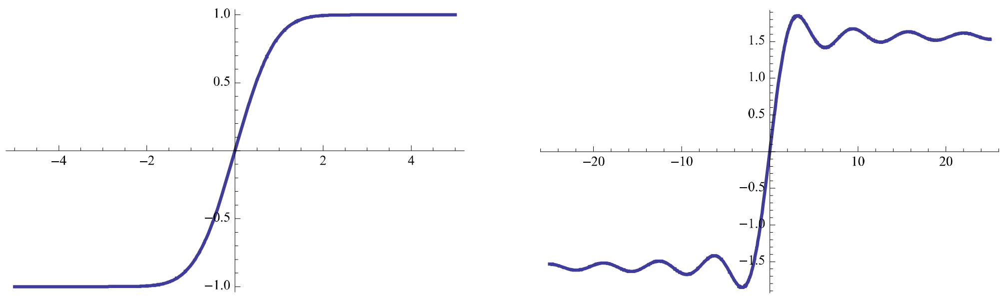

FIGURE 3. The graphs of the error function $\mathrm{erf}(x)$ and the sine integral $\mathrm{Si}(x)$.

Example 9. Similarly, the general solution to

$$y' = \frac{\sin x}{x}, \tag{29}$$

can be written as

$$y(x) = y(0) + \int_0^x \frac{\sin t}{t}\, \mathrm{d}t = y(0) + \mathrm{Si}(x), \tag{30}$$

where $\mathrm{Si}(x)$ is called the *sine integral*, which is defined by

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,\mathrm{d}t. \tag{31}$$

Again, the function $\frac{\sin x}{x}$ is not integrable in terms of elementary functions.

4. Functions

To minimize any terminology confusion that may arise later in the course, in the rest of these notes we want to record precise definitions of some of the most fundamental concepts. We start by reviewing the concept of a function.

Definition 10. Let U and V be sets, and suppose that to each element $x \in U$, there assigned one and only one element $y \in V$, denoted by $y = f(x)$. We call such an assignment a *function* with *domain* U and *range* V, and write $f : U \to V$.

Note that in order to fully specify a function, one must supply the domain U, the range V, and the assignment rule f.

Remark 11. In this course, we will mostly be concerned with functions $f : I \to \mathbb{R}$, where I is an interval, such as $I = (a, b)$, $I = [a, b]$, $I = [0, 1)$, $I = (0, \infty)$ and $I = (-\infty, 1)$.

One can think of a function $f : (a, b) \to \mathbb{R}$ (with $a < b$) as an infinite table

x	$f(x)$
$\ldots$	$\ldots$

Clearly, it is not feasible to provide or comprehend such tables. In the following, we list some practical methods to describe functions and illustrate them by examples.

i) We can give a function *by an explicit formula.* Example are

$$f(x) = \sin x, \qquad \text{and} \qquad f(x) = 4x^2 + e^x. \tag{32}$$

In each case, $U = V = \mathbb{R}$ is implicitly assumed. However, one must note that the following functions are *different*, because their domains are different:
- $f : (0, 1) \to \mathbb{R}$ given by $f(x) = x^2$.
- $f : (1, 2) \to \mathbb{R}$ given by $f(x) = x^2$.

So are the following functions:
- $f : (0, \infty) \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$.
- $f : (-\infty, 0) \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$.
- $f : (-\infty, 0) \cup (0, \infty) \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$.

ii) We can define a function *piece by piece*, as in

$$\theta(x) = \begin{cases} 0 & \text{for} \quad x \le 0, \\ 1 & \text{for} \quad x > 0, \end{cases} \tag{33}$$

which is called the *Heaviside step function*, and

$$f(x) = \begin{cases} 0 & \text{for} \quad x \le 0, \\ x^2 & \text{for} \quad x \ge 0. \end{cases} \tag{34}$$

In both cases, $U = V = \mathbb{R}$ is implicit.

As another example, take

$$f(x) = \begin{cases} \frac{1}{x} & \text{for} \quad x \ne 0, \\ 0 & \text{for} \quad x = 0. \end{cases} \tag{35}$$

This is different than $g(x) = \frac{1}{x}$, as the domain of f is $\mathbb{R}$.

iii) We can give a function *implicitly*, by its properties. Examples and non-examples:
- $f(x) = y$ where $x^2 + y^2 = 0$ and $y > 0$. Here $U = (-1, 1)$ is implicit.
- $f(x) = y$ where $x^2 + y^2 = 0$. This is a *non*-example, because for a given x, say from the interval $(-1, 1)$, there are two different y that satisfy $x^2 + y^2 = 0$.
- f satisfies $f'(x) = \sin x$ and $f(0) = 0$.
- f satisfies $f'(x) = \sin x$. This is a *non*-example, because for a given x, the condition $f'(x) = \sin x$ does not determine a unique value $f(x)$.

Remark 12. When we give a function implicitly, we must ensure that the conditions define a unique value $f(x)$ for each x from the intended domain.

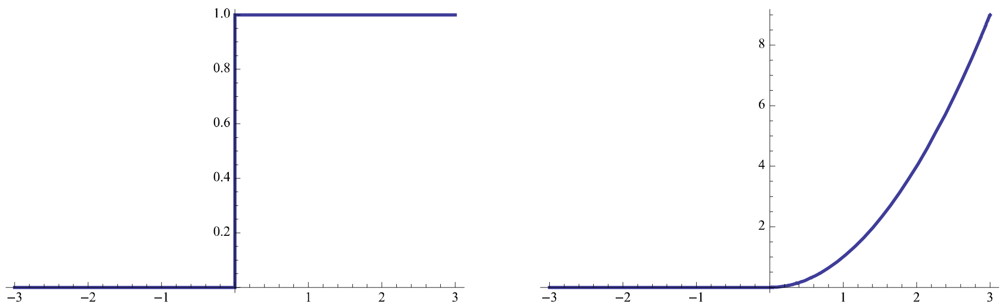

FIGURE 4. The graphs of the Heaviside theta function $\theta(x)$ given by (33), and the function $f(x)$ given by (34).

5. Ordinary differential equations and initial value problems

In this section, we make precise some notions related to ordinary differential equations, initial value problems, and their solutions.

Definition 13. An *ordinary differential equation* (ODE) is an equation of the form

$$F(x, y, y', \ldots, y^{(n)}) = 0, \tag{36}$$

where F is a given function of $n+2$ variables. Here x and y are called the *independent variable* and the *dependent variable*, respectively. If the derivative $y^{(n)}$ genuinely appears in (36), then (36) is called an *n-th order* ODE.

To clarify what we mean by the order of an ODE, the equation $y' + y = 0$ is a *first order* ODE, even though one may write it as $0 \cdot y'' + y' + y = 0$. The order of an ODE is obviously an important characteristic of that ODE.

Definition 14. Given an interval I, a *solution* (or a *particular solution*) of (36) *on the interval* I is a function $g : I \to \mathbb{R}$ satisfying

$$F(x, g(x), g'(x), \ldots, g^{(n)}(x)) = 0, \qquad \text{for all} \quad x \in I. \tag{37}$$

This definition makes it clear that when one talks about a solution of an ODE, one must keep in mind that there is also an interval is involved. It is best if one states the interval explicitly every time a solution is considered.

Example 15. Consider the ODE

$$y' = \frac{1}{x}. \tag{38}$$

The function $g : (0, \infty) \to \mathbb{R}$ defined by $g(x) = \log x$ is a solution of (38) on the interval $(0, \infty)$. Also, the function $g : (-\infty, 0) \to \mathbb{R}$ defined by $g(x) = \log(-x)$ is a solution of (38) on the interval $(-\infty, 0)$.

Remark 16. One can say that $\log|x|$ is a solution to (38) on $(-\infty,0)\cup(0,\infty)$, but we will almost exclusively consider solutions defined over an *interval*.

More generally, one can define *systems* of ODE's and their solutions. For instance, the general form of systems of two first order equations is

$$F_1(x,y_1,y_2,y_1',y_2')=0, \qquad F_2(x,y_1,y_2,y_1',y_2')=0, \tag{39}$$

where F_1 and F_2 are given functions of 5 variables. A solution of such a system would be a pair of functions y_1 and y_2 (or equivalently, a function with $\mathbb{R}^2$ as its range).

Example 17. The following is a system of first order ODE's:

$$\begin{cases} y_1' = -y_2, \\ y_2' = y_1. \end{cases} \tag{40}$$

To write down a solution to this system, observe that the vector $(-y_2, y_1)$ is exactly the vector (y_1, y_2) rotated by 90 degrees counterclockwise. Now, thinking of $(y_1(t), y_2(t))$ as a vector in the plane $\mathbb{R}^2$, with its tip at $(y_1(t), y_2(t))$ and the base at the origin $(0,0)$, the system (40) tells us that the velocity of the tip of the vector is directed perpendicularly to the vector itself. It is thus intuitively clear that if we start with some vector, and evolve it according to (40), then the tip of the vector would be tracing a circle centred at the origin. This leads us to the following guess

$$\begin{cases} y_1(t) = A\cos(kt+c), \\ y_2(t) = A\sin(kt+c), \end{cases} \tag{41}$$

where we assume that A, k and c are constants. Such a guess, that fixes a general form of the solution but leaves some parameters free, is called an *ansatz*. Note that there is no *a priori* guarantee that our guess would work. To check if it works, we differentiate (41), and get

$$y_1'(t) = -Ak\sin(kt+c), \qquad \text{and} \qquad y_2'(t) = Ak\cos(kt+c). \tag{42}$$

This shows that (41) is in fact a solution of (40), provided that we choose $k=1$.

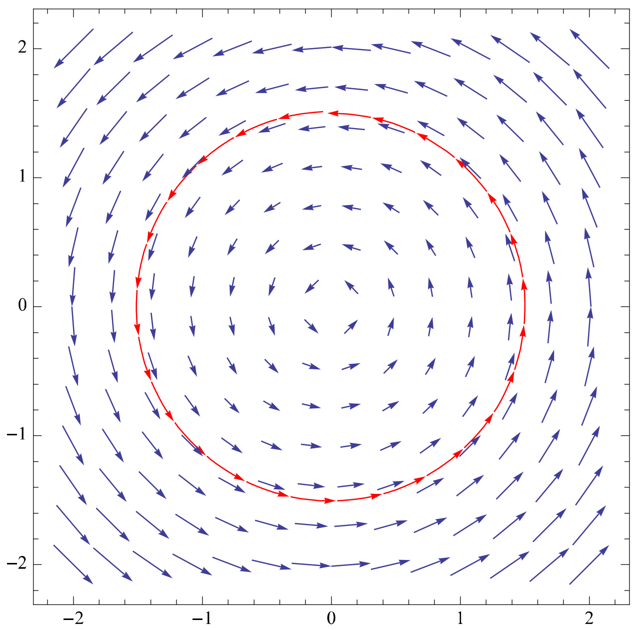

FIGURE 5. The velocity field and a solution trajectory for the system (40).

Having dealt with ODE's, we now turn to initial value problems.

Definition 18. An *initial value problem* (IVP) is given by

$$\begin{cases} F(x, y, y', \ldots, y^{(n)}) = 0, \\ y(x_0) = c_0, \\ y'(x_0) = c_1, \\ \ldots \\ y^{(n-1)}(x_0) = c_{n-1}, \end{cases} \tag{43}$$

where F is a given function of $n+2$ variables, and $c_0, c_1, \ldots, c_{n-1}$ and x_0 are given numbers.

Definition 19. Let I be an interval, and let x_0 be a point on I. Then a *solution* of the initial value problem (43) *on the interval* I is a function $g : I \to \mathbb{R}$ satisfying

$$F(x, g(x), g'(x), \ldots, g^{(n)}(x)) = 0, \qquad \text{for all } x \text{ in the } \textit{interior} \text{ of } I, \tag{44}$$

and

$$g(x_0) = c_0, \quad g'(x_0) = c_1, \ldots, \quad g^{(n-1)}(x_0) = c_{n-1}. \tag{45}$$

Recall that the *interior* of an interval I is the largest open interval contained in I. For example, $(0, 1)$ is the interior of any of the intervals $(0, 1)$, $[0, 1)$, and $[0, 1]$.

Example 20. Consider the initial value problem

$$\begin{cases} y' = \theta(x), \\ y(0) = 1, \end{cases} \tag{46}$$

where θ is the Heaviside step function defined in (33), and suppose that we are asked to solve this problem on the interval $[0, \infty)$, i.e., for $x \geq 0$. In order to solve it, first we look at the case $x > 0$. Since $\theta(x) = 1$ for $x > 0$, we have $y'(x) = 1$, which gives $y(x) = x + C$ for an arbitrary constant. Now, the function $y(x) = x + C$ is defined not only for $x > 0$ but also for all $x \in \mathbb{R}$. In particular, we can talk about $y(0)$ and hence can impose the initial condition $y(0) = 1$, resulting in the realization that $C = 1$. We conclude that

$$y(x) = x + 1, \tag{47}$$

is a solution of the initial value problem (46) on the interval $[0, \infty)$. Note that $y(0) = 1$ and that $y'(x) = 1$ for $x > 0$. It is not necessary to ensure $y'(x) = \theta(x)$ at $x = 0$, because Definition 19 requires the ODE be satisfied only in the *interior* of $[0, \infty)$, which is $(0, \infty)$.

6. Linearity and the superposition principle

Linear differential equations form the most important class of ODE's. Because of their relative simplicity, a very satisfactory theory can be developed for linear differential equations. They have rich applications, and most importantly, the theory of linear differential equations serves as a model as well as a firm basis for studying nonlinear differential equations.

Definition 21. An *n-th order linear differential equation* is an ODE of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f, \tag{48}$$

where $a_0, \ldots, a_n$ and f are given functions, with a_n not identically zero. If f is identically zero, that is, if the equation is of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0, \tag{49}$$

then the ODE is said to be *homogeneous*, and otherwise it is said to be *inhomogeneous*.

If an equation is not linear, then we say it is *nonlinear*. For example, $y' + y^2 = 0$ and $(y')^2 + x\cos y = 1$ are nonlinear ODE's.

Definition 22. Suppose that all solutions of (48) on an interval I can be written as

$$y(x) = g_0(x) + A_1 g_1(x) + \ldots + A_n g_n(x), \tag{50}$$

where $g_0, g_1, \ldots, g_n$ are (fixed) functions defined on I, and $A_1, \ldots, A_n$ are arbitrary constants. Then the expression (50) is called the *general solution* of the ODE (48) on I.

More precisely, if all solutions of (48) on I form the set

$$\{g_0 + A_1 g_1 + \ldots + A_n g_n : A_1, \ldots, A_n \in \mathbb{R}\}, \tag{51}$$

for some functions $g_0, g_1, \ldots, g_n : I \to \mathbb{R}$, then (50) is called the *general solution* of (48) on I.

Remark 23. A general solution is a *collection* of functions, and it is *not* a single function (unless in some trivial cases). Even though the set notation (51) is more appropriate in this case, it is traditional to write the general solution in the form (50) and declare that the constants $A_1, \ldots, A_n$ are "arbitrary". What this means is the following.

- When we fix a certain value for each of $A_1, \ldots, A_n$, the expression (50) produces a function. Since $A_1, \ldots, A_n$ do not depend on x, they are *constants.*
- Each of $A_1, \ldots, A_n$ can have any real number as its value, and as they take all possible combinations of their values, the expression (50) produces all possible solutions of the ODE (48). In this sense, the constants $A_1, \ldots, A_n$ are *arbitrary.*

Remark 24. General solutions can be defined for nonlinear equations, but we will talk about general solutions mostly in the context of linear equations.

When presented with Definition 22 and especially the formula (50), a natural question that might come to one's mind is: Why do we expect that the general solution depends on the constants A_1 etc. in this way? Why not in some other way, for instance, as in $y(x) = e^{A_1 x}$? The answer lies in what is called the *superposition principle*, that is unquestionably the most important property of linear equations.

Let us make a preliminary observation. If $g(x)$ is a solution of the homogeneous equation (49), then it obvious that for any constant C, the function $y(x) = Cg(x)$ is also a solution. The superposition principle for the homogeneous equation (49) says that the sum of any two solutions is again a solution. In light of the observation we have just made, this means that a linear combination of any two solutions is again a solution.

Theorem 25 (Superposition principle for homogeneous equations)**.** *Suppose that y_1 and y_2 are solutions of the homogeneous equation* (49) *on some interval I. Then for any real numbers C_1 and C_2, the function*

$$y(x) = C_1 y_1(x) + C_2 y_2(x), \tag{52}$$

is also a solution of (49) *on I.*

Proof. We will prove it only for the special case $n = 1$, since the general case involves exactly the same arguments and only requires more writing. Let us start by writing out what it means for y_1 and y_2 to be solutions of (49) on I:

$$a_1(x)y_1'(x) + a_0(x)y_1(x) = 0, \qquad a_1(x)y_2'(x) + a_0(x)y_2(x) = 0, \qquad x \in I. \tag{53}$$

Let us multiply the first equation by C_1 and the second equation by C_2, and sum the resulting two equations. After some rearranging, this gives

$$a_1(x)[C_1 y_1'(x) + C_2 y_2'(x)] + a_0(x)[C_1 y_1(x) + C_2 y_2(x)] = 0, \qquad x \in I. \tag{54}$$

Finally, taking into account the definition (52), and the related fact that $y'(x) = C_1 y_1'(x) + C_2 y_2'(x)$, we conclude

$$a_1(x)y'(x) + a_0(x)y(x) = 0, \qquad x \in I, \tag{55}$$

which is what we wanted to prove. □

Example 26. Consider the second order homogeneous linear differential equation

$$y'' + y = 0. \tag{56}$$

It is easy to check that both $y_1(x) = \sin x$ and $y_2(x) = \cos x$ are solutions of this equation on $\mathbb{R}$. Then by the superposition principle for homogeneous equations (Theorem 25), for any constants A and B, the function

$$y(x) = A \sin x + B \cos x, \tag{57}$$

is a solution of $y'' + y = 0$ on $\mathbb{R}$.

The superposition principle in the general case (48) says that one can add a solution of the homogeneous equation (49) to a solution of the general equation (48), and obtain another solution of the general equation (48).

Theorem 27 (Superposition principle)**.** *Suppose that y_* is a solution of (48) on some interval I, and that y_0 is a solution of the homogeneous equation (49) on the same interval I. Then for any real number C, the function*

$$y(x) = y_*(x) + Cy_0(x), \tag{58}$$

is a solution of (48) on I.

Proof. Again, we will only prove the special case $n = 1$. The hypotheses of the theorem give

$$a_1(x)y_*'(x) + a_0(x)y_*(x) = f(x), \qquad a_1(x)y_0'(x) + a_0(x)y_0(x) = 0, \qquad x \in I. \tag{59}$$

If we multiply the second equation by C and add the result to the first equation, we get

$$a_1(x)[y_*'(x) + Cy_0'(x)] + a_0(x)[y_*(x) + Cy_0(x)] = 0, \qquad x \in I. \tag{60}$$

Taking into account the definition (58), and the fact that $y'(x) = y_*'(x) + Cy_0'(x)$, we infer

$$a_1(x)y'(x) + a_0(x)y(x) = f(x), \qquad x \in I, \tag{61}$$

which completes the proof. □

Example 28. Consider the second order *inhomogeneous* linear differential equation

$$y'' + y = 2e^x. \tag{62}$$

It is easy to guess a particular solution: $y_*(x) = e^x$. The corresponding *homogeneous* equation is $y'' + y = 0$, which has $y_0(x) = \sin x$ as one of its solutions on $\mathbb{R}$. Then the superposition principle (Theorem 27) says that for any constant A, the function

$$y(x) = e^x + A \sin x, \tag{63}$$

is a solution to (62) on $\mathbb{R}$.

We have the following converse to the superposition principle.

Theorem 29. *Suppose that y_1 and y_2 are solutions of (48) on some interval I. Then their difference $y(x) = y_1(x) - y_2(x)$ is a solution of the homogeneous equation (49) on I.*

Consequently, if y_* is a particular solution of (48) on I, and if y_0 is the general solution of the homogeneous equation (49) on I, then the general solution of (48) on I is given by

$$y(x) = y_*(x) + y_0(x), \qquad x \in I. \tag{64}$$

Proof. We will only prove the special case $n = 1$. We have

$$a_1(x)y_1'(x) + a_0(x)y_1(x) = f(x), \qquad a_1(x)y_2'(x) + a_0(x)y_2(x) = f(x), \qquad x \in I. \tag{65}$$

Subtracting the second equation from the first. we get

$$a_1(x)[y_1'(x) - y_2'(x)] + a_0(x)[y_1(x) - y_2(x)] = 0, \qquad x \in I. \tag{66}$$

This shows that $y_1 - y_2$ is a solution of the homogeneous equation (49) on I.

The second statement is straightforward. Suppose that y_* is a particular solution of (48) on I. If y is *any* solution of (48) on I, then we have $y = y_* + g$ for *some* solution g of the homogeneous equation (49) on I. Conversely, if g is *any* solution of the homogeneous equation (49) on I, then by the superposition principle, $y = y_* + g$ is a solution of (48) on I. □

The second part of the preceding theorem is so useful that it is worth reiterating: In order to find the general solution of an *inhomogeneous* (linear) equation, one just needs to find

- the general solution of the corresponding *homogeneous* equation, and
- a *particular solution* of the inhomogeneous equation.

So it basically reduces the task of solving an inhomogeneous equation into that of solving the corresponding homogeneous equation.

Example 30. Consider the first order inhomogeneous linear differential equation

$$xy' + y = 2x. \tag{67}$$

We guess one particular solution: $y_*(x) = x$, and try to find the general solution to the corresponding homogeneous equation $xy' + y = 0$. We notice that $(xy)' = xy' + y$, and by integration of $(xy)' = 0$, we get

$$xy(x) = C, \tag{68}$$

where C is an arbitrary constant. So the general solution of the homogeneous equation $xy' + y = 0$ is $y(x) = \frac{C}{x}$, and by the second part of Theorem 29, the general solution of the inhomogeneous equation (67) is $y(x) = x + \frac{C}{x}$, where C is an arbitrary constant.

Theorem 29, together with the superposition principle for homogeneous equations and the heuristic that the general solution of an n-th order equation should involve n arbitrary constants, gives a strong indication that the general solution of (48) must be of the form (50). We will come back to this question and perform a finer analysis later in the course.