Sequences and Series:

An Introduction to Mathematical Analysis

by Malcolm R. Adams

Contents

Chapter 1

Sequences

1.1 The general concept of a sequence

We begin by discussing the concept of a sequence. Intuitively, a sequence is an ordered list of objects or events. For instance, the sequence of events at a crime scene is important for understanding the nature of the crime. In this course we will be interested in sequences of a more mathematical nature; mostly we will be interested in sequences of numbers, but occasionally we will find it interesting to consider sequences of points in a plane or in space, or even sequences of sets.

Let's look at some examples of sequences.

Example 1.1.1

Emily flips a quarter five times, the sequence of coin tosses is HTTHT where H stands for "heads" and T stands for "tails".

As a side remark, we might notice that there are $2^5 = 32$ different possible sequences of five coin tosses. Of these, 10 have two heads and three tails. Thus the probability that in a sequence of five coin tosses, two of them are heads and three are tails is 10/32, or 5/16. Many probabilistic questions involve studying sets of sequences such as these.

Example 1.1.2

John picks colored marbles from a bag, first he picks a red marble, then a blue one, another blue one, a yellow one, a red one and finally a blue one. The sequence of marbles he has chosen could be represented by the symbols RBBYRB.

Example 1.1.3

Harry the Hare set out to walk to the neighborhood grocery. In the first ten minutes he walked half way to the grocery. In the next ten minutes he walked half of the remaining distance, so now he was 3/4 of the way to the grocery. In the following ten minutes he walked half of the remaining distance again, so now he has managed to get 7/8 of the way to the grocery. This sequence of events continues for some time, so that his progress follows the pattern 1/2, 3/4, 7/8, 15/16, 31/32, and so on. After an hour he is 63/64 of the way to the grocery. After two hours he is 4095/4096 of the way to the grocery. If he was originally one mile from the grocery, he is now about 13 inches away from the grocery. If he keeps on at this rate will he ever get there? This brings up some pretty silly questions; For instance, if Harry is 1 inch from the grocery has he reached it yet? Of course if anybody manages to get within one inch of their goal we would usually say that they have reached it. On the other hand, in a race, if Harry is 1 inch behind Terry the Tortoise he has lost the race. In fact, at Harry's rate of decelleration, it seems that it will take him forever to cross the finish line.

Example 1.1.4

Harry's friend Terry the Tortoise is more consistent than Harry. He starts out at a slower pace than Harry and covers the first half of the mile in twenty minutes. But he covers the next quarter of a mile in 10 minutes and the next eighth of a mile in 5 minutes. By the time he reaches 63/64 of the mile it has taken less than 40 minutes while it took Harry one hour. Will the tortoise beat the hare to the finish line? Will either of them ever reach the finish line? Where is Terry one hour after the race begins?

Example 1.1.5

Build a sequence of numbers in the following fashion. Let the first two numbers of the sequence be 1 and let the third number be $1 + 1 = 2$. The fourth number in the sequence will be $1 + 2 = 3$ and the fifth number is $2 + 3 = 5$. To continue the sequence, we look for the previous two terms and add them together. So the first ten terms of the sequence are:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55$$

This sequence continues forever. It is called the *Fibonnaci* sequence. This sequence is said to appear ubiquitously in nature. The volume of the chambers of the nautilus shell, the number of seeds in consecutive rows of a sunflower, and many natural ratios in art and architecture are purported to progress by this natural sequence. In many cases the natural or biological reasons for this progression are not at all straightforward.

The sequence of natural numbers,

$$1, 2, 3, 4, 5, ...$$

and the sequence of odd natural numbers,

$$1, 3, 5, 7, 9, ...$$

are other simple examples of sequences that continue forever. The symbol ... (called ellipses) represents this infinite continuation. Such a sequence is called an *infinite sequence*. In this book most of our sequences will be infinite and so from now on when we speak of sequences we will mean infinite sequences. If we want to discuss some particular *finite sequence* we will specify that it is finite.

Since we will want to discuss general sequences in this course it is necessary to develop some notation to represent sequences without writing down each term explicitly. The fairly concrete notation for representing a general infinite sequence is the following:

$$a_1, a_2, a_3, ...$$

where a_1 represents the first number in the sequence, a_2 the second number, and a_3 the third number, etc. If we wish to discuss an entry in this sequence without specifying exactly which entry, we write a_i or a_j or some similar term.

To represent a finite sequence that ends at, say, the 29^{th} entry we would write

$$a_1, a_2, ..., a_{29}.$$

Here the ellipses indicate that there are several intermediate entries in the sequence which we don't care to write out explicitly. We may also at times need to represent a series that is finite but of some undetermined length; in this case we will write

$$a_1, a_2, ..., a_N$$

where N represents the fixed, but not explicitly specified length.

A slightly more sophisticated way of representing the abstract sequence $a_1, a_2, ...$ is with the notation:

$$\{a_i\}_{i=1}^{\infty}.$$

The finite sequence $a_1, a_2, ..., a_N$ is similarly represented by:

$$\{a_i\}_{i=1}^{N}.$$

Since in this text we study mostly infinite sequences, we will often abbreviate $\{a_i\}_{i=1}^{\infty}$ with simply $\{a_i\}$. Although this looks like set notation you should be careful not to confuse a sequence with the set whose elements are the entries of the sequence. A set has no particular ordering of its elements but a sequence certainly does. For instance, the sequence $1, 1, 1, 1, ...$ has infinitely many terms, yet the set made of these terms has only one element.

When specifying any particular sequence, it is necessary to give some description of each of its terms. This can be done in various ways. For a (short) finite sequence, one can simply list the terms in order. For example, the sequence $3, 1, 4, 1, 5, 9$ has six terms which are easily listed. On the other hand, these are the first six terms of the decimal expansion of π, so this sequence can be extended to an infinite sequence, $3, 1, 4, 1, 5, 9, ...$, where it is understood from the context that we continue this sequence by computing further terms in the decimal expansion of π. Here are a few other examples of infinite sequences which can be inferred by listing the first few terms:

$$1, 2, 3, 4, ...$$
$$2, 4, 6, 8, ...$$
$$5, 10, 15, 20, ...$$
$$1, 1, 2, 3, 5, 8, 13, ...$$

Well maybe it is not so obvious how to extend this last sequence unless you are familiar with the Fibonacci sequence discussed in Example 1.1.5 . This last example demonstrates the drawback of determining a sequence by inference, it leaves it to the reader to discover what method you used to determine the next term.

A better method of describing a sequence is to state how to determine the n^{th} term with an *explicit formula* . For example, the sequence $1, 2, 3, 4, ...$ is easily specified by saying $a_n = n$. Formulas for the second and third sequence above can be specified with the formulas $a_n = 2n$ and $a_n = 5n$ respectively. An explicit formula for the n^{th} term of the Fibonacci sequence, or the n^{th} term in the decimal expansion of π is not so easy to find. In exercise 1.2.17 we will find an explicit formula for the Fibonacci sequence, but there is no such explicit formula for the n^{th} term in the decimal expansion of π.

Example 1.1.6

The n^{th} term in a sequence is given by $a_n = (n^2 + n)/2$. The first five terms are $1, 3, 6, 10, 15$.

Example 1.1.7

The n^{th} term in the sequence $\{b_n\}$ is given by $b_n = 1 - \frac{1}{n^2}$. The first six terms of this sequence are

$$0, 3/4, 8/9, 15/16, 24/25, 35/36.$$

A third way of describing a sequence is through a *recursive formula* . A recursive formula describes the n^{th} term of the sequence in terms of previous terms in the sequence. The easiest form of a recursive formula is a description of a_n in terms of a_{n-1}. Many of our earlier examples of numerical sequences were described in this way.

Example 1.1.8

Let's return to Example 1.1.3 above. Each 10 minutes, Harry walks half of the remaining distance to the neighborhood. Let's denote the fraction of the total distance that Harry has travelled after n chunks of ten minutes by a_n. So $a_1 = 1/2, a_2 = 3/4, a_3 = 7/8$, etc. Then the fraction of the total distance that remains to be travelled after n chunks of ten minutes is $1 - a_n$. Since the distance travelled in the next ten minutes is half of this remaining distance, we see that

$$a_{n+1} = a_n + \frac{1}{2}(1 - a_n) = \frac{1}{2}(1 + a_n).$$

Notice that this formula is not enough by itself to determine the sequence $\{a_n\}$. We must also say how to start the sequence by supplying the information that

$$a_1 = 1/2.$$

Now, with this additional information, we can use the formula to determine further terms in the sequence:

$$\begin{aligned} a_2 &= \frac{1}{2}(1 + a_1) = \frac{1}{2}(1 + 1/2) = 3/4 \\ a_3 &= \frac{1}{2}(1 + a_2) = \frac{1}{2}(1 + 3/4) = 7/8 \\ a_4 &= \frac{1}{2}(1 + a_3) = \frac{1}{2}(1 + 7/8) = 15/16, \end{aligned}$$

etc.

Example 1.1.9

Let's have another look at the Fibonacci sequence from Example 1.1.5 above. Here the n^{th} term is determined by two previous terms, indeed

$$a_{n+1} = a_n + a_{n-1}.$$

Now we can't get started unless we know the first *two* steps in the sequence, namely a_1 and a_2. Since we are told that $a_1 = 1$ and $a_2 = 1$ also, we can use the recursion formula to determine

$$a_3 = a_2 + a_1 = 1 + 1 = 2.$$

And now since we have both a_2 and a_3 we can determine

$$a_4 = a_3 + a_2 = 2 + 1 = 3,$$

and similarly

$$\begin{aligned} a_5 &= a_4 + a_3 = 3 + 2 = 5, \\ a_6 &= a_5 + a_4 = 5 + 3 = 8, \\ a_7 &= a_6 + a_5 = 8 + 5 = 13, \end{aligned}$$

and so on.

To conclude this section we mention two more families of examples of sequences which often arise in mathematics, the *arithmetic* (the accent is on the third syllable!) sequences and the *geometric* sequences.

An *arithmetic* sequence has the form $a, a+b, a+2b, a+3b, ...$ where a and b are some fixed numbers. An explicit formula for this arithmetic sequence is given by $a_n = a+(n-1)b,\ n \in \mathbb{N}$, a recursive formula is given by $a_1 = a$ and $a_n = a_{n-1} + b$ for $n > 1$. Here are some examples of arithmetic sequences, see if you can determine a and b in each case:

$$\begin{gathered} 1, 2, 3, 4, 5, ... \\ 2, 4, 6, 8, 10, ... \\ 1, 4, 7, 10, 13, ... \end{gathered}$$

The distinguishing feature of an arithmetic sequence is that each term is the *arithmetic mean* of its neighbors, i.e. $a_n = (a_{n-1} + a_{n+1})/2$, (see exercise 12).

A *geometric* sequence has the form $a, ar, ar^2, ar^3, ...$ for some fixed numbers a and r. An explicit formula for this geometric sequence is given by $a_n = ar^{n-1}, n \in \mathbb{N}$. A recursive formula is given by $a_1 = a$ and $a_n = ra_{n-1}$ for $n > 1$. Here are some examples of geometric sequences, see if you can determine a and r in each case:

$$2, 2, 2, 2, 2...$$
$$2, 4, 8, 16, 32, ...$$
$$3, 3/2, 3/4, 3/8, 3/16, ...$$
$$3, 1, 1/3, 1/9, 1/27, ...$$

Geometric sequences (with positive terms) are distinguished by the fact that the n^{th} term is the *geometric mean* of its neighbors, i.e. $a_n = \sqrt{a_{n+1}a_{n-1}}$, (see exercise 13).

Example 1.1.10

If a batch of homebrew beer is inoculated with yeast it can be observed that the yeast population grows for the first several hours at a rate which is proportional to the population at any given time. Thus, if we let p_n denote the yeast population measured after n hours have passed from the inoculation, we see that there is some number $\alpha > 1$ so that

$$p_{n+1} = \alpha p_n.$$

That is, p_n forms a geometric sequence.

Actually, after a couple of days, the growth of the yeast population slows dramatically so that the population tends to a steady state. A better model for the dynamics of the population that reflects this behavior is

$$p_{n+1} = \alpha p_n - \beta p_n^2,$$

where α and β are constants determined by the characteristics of the yeast. This equation is known as the *discrete logistic equation* . Depending on the values of α and β it can display surprisingly varied behavior for the population sequence, p_n. As an example, if we choose $\alpha = 1.2, \beta = .02$ and

$p_0 = 5$, we get

$$\begin{aligned}
p_1 &= 5.5 \\
p_2 &= 5.995 \\
p_3 &= 6.475199500 \\
p_4 &= 6.931675229 \\
p_5 &= 7.357047845 \\
p_6 &= 7.745934354 \\
p_7 &= 8.095131245 \\
p_8 &= 8.403534497 \\
p_9 &= 8.671853559 \\
p_{10} &= 8.902203387.
\end{aligned}$$

Further down the road we get $p_{20} = 9.865991756, p_{30} = 9.985393020, p_{40} = 9.998743171$, and $p_{100} = 9.999999993$. Apparently the population is leveling out at about 10. It is interesting to study the behavior of the sequence of p_n's for other values of α and β (see exercise 14).

EXERCISES 1.1

1. a.) How many sequences of six coin tosses consist of three heads and three tails?

   b.) How many different sequences of six coin tosses are there altogether?

   c.) In a sequence of six coin tosses, what is the probability that the result will consist of three heads and three tails?

2. (For students with some knowledge of combinatorics.) In a sequence of $2n$ coin tosses, what is the probability that the result will be exactly n heads and n tails?

3. a.) Let $\{a_n\}$ be the sequence given explicitly by $a_n = 2n - 1$. Write out a_1, a_2, a_3, a_4, and a_5. Describe this sequence in words.

   b.) Find an explicit formula for a_{n+1} and use this to show that a recursive formula for this sequence is given by $a_{n+1} = a_n + 2$, with $a_1 = 1$.

13. a.) Let a and r be positive real numbers and define a geometric sequence by the recursive formula $a_1 = a$ and $a_n = ra_{n-1}, n > 1$. Show that $a_n = \sqrt{a_{n+1}a_{n-1}}$ for all $n > 1$.

    b.) Again with a and r positive real numbers define a geometric sequence by the explicit formula $a_n = ar^{n-1}$. Here also, show that $a_n = \sqrt{a_{n+1}a_{n-1}}$ for all $n > 1$.

14. (Calculator needed.) Find the first 20 terms of the sequences given by $p_{n+1} = \alpha p_n - \beta p_n^2$ where α, β, and p_0 are given below. In each case write a sentence or two describing what you think the long term behavior of the population will be.

    a.) $\alpha = 2,\ \beta = 0.1,\ p_0 = 5$.

    b.) $\alpha = 2.8,\ \beta = 0.18,\ p_0 = 5$.

    c.) $\alpha = 3.2,\ \beta = 0.22,\ p_0 = 5$.

    d.) $\alpha = 3.8,\ \beta = 0.28,\ p_0 = 5$.

    If you have MAPLE available you can explore this exercise by changing the values of a and b in the following program:

```
[> restart:
[> a:=1.2; b:= 0.02;
[> f:=x->a*x - b*x^2;
[> p[0]:=5;
[> for j from 1 to 20 do p[j]:=f(p[j-1]); end do;
```

1.2 The sequence of natural numbers

A very familiar and fundamental sequence is that of the *natural numbers*, $\mathbb{N} = \{1, 2, 3, ...\}$. As a sequence, we might describe the natural numbers by the explicit formula

$$a_n = n$$

but this seems circular. (It certainly does not give a definition of the natural numbers.) Somewhat more to the point is the recursive description,

$$a_{n+1} = a_n + 1, \quad a_1 = 1$$

but again this is not a definition of the natural numbers since the use of the sequence notation implicitly refers to the natural numbers already. In fact, the nature of the existence of the natural numbers is a fairly tricky issue in the foundations of mathematics which we won't delve into here, but we do want to discuss a defining property of the natural numbers that is extremely useful in the study of sequences and series:

The Principle of Mathematical Induction
Let S be a subset of the natural numbers, $\mathbb{N}$, which satisfies the following two properties:
1.) $1 \in S$
2.) if $k \in S$, then $k+1 \in S$.
Then S must be the entire set of natural numbers, i.e. $S = \mathbb{N}$.

Example 1.2.1

a.) Let $S = \{1, 2, 3, 4, 5\}$. Then S satisfies property 1.) but not property 2.) since 5 is an element of S but 6 is not an element of S.

b.) Let $S = \{2, 3, 4, 5, ...\}$, the set of natural numbers greater than 1. This S satisfies property 2.) but not property 1.).

c.) Let $S = \{1, 3, 5, 7, ...\}$, the set of odd natural numbers. Then S satisfies property 1.) but not property 2.).

The Principle of Mathematical Induction (which we shall henceforth abbreviate by PMI) is not only an important defining property of the natural numbers, it also gives a beautiful method of proving an infinite sequence of statements. In the present context, we will see that we can use PMI to verify explicit formulae for sequences which are given recursively.

Example 1.2.2

Consider the sequence given recursively by

$$a_{n+1} = a_n + (2n+1), \quad a_1 = 1.$$

The n^{th} term, a_n, is the sum of the first n odd natural numbers. Writing out the first 5 terms we see

$$\begin{aligned}
a_1 &= 1, \\
a_2 &= a_1 + 3 = 4, \\
a_3 &= a_2 + 5 = 9, \\
a_4 &= a_3 + 7 = 16, \\
a_5 &= a_4 + 9 = 25.
\end{aligned}$$

Noticing a pattern here we might conjecture that, in general, $a_n = n^2$. Here is how we can use PMI to prove this conjecture:

Let S be the subset of natural numbers for which it is true that $a_n = n^2$, i.e.

$$S = \{n \in \mathbb{N} \mid \ a_n = n^2\}.$$

We know that $a_1 = 1$, which is equal to 1^2, so $1 \in S$. Thus S satisfies the first requirement in PMI. Now, let k be some arbitrary element of S. Then, by the description of S we know that k is some particular natural number such that $a_k = k^2$. According to the definition of the sequence,

$$a_{k+1} = a_k + (2k+1),$$

so, since $a_k = k^2$, we can conclude that

$$a_{k+1} = k^2 + (2k+1).$$

However, since $k^2 + (2k+1) = (k+1)^2$, we conclude that

$$a_{k+1} = (k+1)^2,$$

i.e. a_{k+1} *is an element of* S *as well.* We have just shown that if $k \in S$ then $k+1 \in S$, i.e. S satisfies the second requirement of PMI. Therefore we can conclude that $S = \mathbb{N}$, i.e. the explicit formula $a_n = n^2$ is true for every natural number n.

Comments on the Subscript k:

In general, letters like i, j, k, l, m, and n are used to denote arbitrary or unspecified natural numbers. In different contexts these letters can represent either an *arbitrary* natural number or a *fixed but unspecified* natural number. These two concepts may seem almost the same, but their difference is an important issue when studying PMI. To illustrate this here we are using two different letters to represent the different concepts. Where the subscript n is used we are talking about an arbitrary natural number, i.e. we are claiming that for this sequence, $a_n = n^2$ for any natural number you choose to select. On the other hand, the subscript k is used above to denote a fixed but unspecified natural number, so we assume $a_k = k^2$ for that particular k and use what we know about the sequence to prove that the general formula holds for the *next* natural number, i.e. $a_{k+1} = (k+1)^2$.

Comments on Set Notation:

Although the issue of defining the notion of a *set* is a fairly tricky subject, in this text we will be concerned mostly with describing subsets of some given set (such as the natural numbers $\mathbb{N}$, or the real numbers $\mathbb{R}$) which we will accept as being given. Generally such subsets are described by a condition, or a collection of conditions. For example, the even natural numbers, let's call them E, are described as the subset of natural numbers which are divisible by 2. The notation used to describe this subset is as follows:

$$E = \{n \in \mathbb{N} \mid 2 \text{ divides n}\}.$$

There are four separate components to this notation. The brackets $\{,\}$ contain the set, the first entry (in this example $n \in \mathbb{N}$) describes the set from which we are taking a subset, the vertical line $\mid$ separates the first entry from the conditions (and is often read as "such that"), and the final entry gives the conditions that describe the subset (if there is more than one condition then they are simply all listed, with commas separating them). Of course sometimes a set can be described by two different sets of conditions, for instance an even natural number can also be described as twice another natural number. Hence we have

$$E = \{n \in \mathbb{N} \mid n = 2k \text{ for some k} \in \mathbb{N}\}.$$

Example 1.2.3

Let $\{a_n\}$ be the sequence defined recursively by $a_{n+1} = a_n + (n+1)$, and $a_1 = 1$. Thus a_n is the sum of the first n natural numbers. Computing the first few terms, we see $a_1 = 1, a_2 = 3, a_3 = 6, a_4 = 10$, and $a_5 = 15$. Let's now use PMI to prove that in general, $a_n = \frac{n(n+1)}{2}$. (You might want to check that this formula works in the above five cases.)

Let S be the set of natural numbers n so that it is true that $a_n = \frac{n(n+1)}{2}$, i.e.

$$S = \{n \in \mathbb{N} \mid \ a_n = \frac{n(n+1)}{2}\}.$$

We wish to show that $S = \mathbb{N}$ by means of PMI. First notice that $1 \in S$ since $a_1 = 1$ and $\frac{1(1+1)}{2} = 1$ also. That is, the formula is true when $n = 1$. Now assume that we know that some particular natural number k is an element of S. Then, by the definition of S, we know that $a_k = \frac{k(k+1)}{2}$. From the recursive definition of the sequence, we know that $a_{k+1} = a_k + (k+1)$. Substituting in $a_k = \frac{k(k+1)}{2}$, we get

$$\begin{aligned} a_{k+1} &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1)}{2} + \frac{2k+2}{2} \\ &= \frac{k^2+3k+2}{2} \\ &= \frac{(k+1)((k+1)+1)}{2}. \end{aligned}$$

But this is the explicit formula we are trying to verify in the case that $n = k+1$. Thus we have proven that whenever k is a member of S then so is $k+1$ a member of S. Therefore S satisfies the two requirements in PMI and we conclude that $S = \mathbb{N}$.

There are times when one has to be a bit tricky in re-labeling indices to apply PMI exactly as stated. Here is an example of this.

Example 1.2.4

Here we use induction to prove[1] that $n^2 + 5 < n^3$ for all $n \geq 3$. If we start off as usual by letting

$$S = \{n \in \mathbb{N} \mid \ n^2 + 5 < n^3\}$$

[1]See the end of section 1.4 for a review of the properties of inequalities.

we will be in big trouble since it is easy enough to see that 1 is not an element of S. So it can't possibly be true that $S = \mathbb{N}$. A formal way of getting around this problem is to shift the index in the inequality so that the statement to prove begins at 1 instead of 3. To do this, let $n = m + 2$, so that $m = 1$ corresponds to $n = 3$ and then substitute this into the expression we want to prove. That is, if we want to prove that $n^2 + 5 < n^3$ for all $n \geq 3$, it is equivalent to prove that $(m+2)^2+5 < (m+2)^3$ for all $m \in \mathbb{N}$. Thus we can now proceed with PMI by setting

$$\tilde{S} = \{m \in \mathbb{N} \mid \ (m+2)^2 + 5 < (m+2)^3\},$$

and showing that properties 1.) and 2.) hold for $\tilde{S}$. Actually, this formality is generally a pain in the neck. Instead of being so pedantic, we usually proceed as follows with the original set S:
$1'$.) Show that $3 \in S$.
$2'$.) Show that if $k \in S$ then so is $k + 1 \in S$.
Then we can conclude that $S = \{n \in \mathbb{N} \mid \ n \geq 3\}$. Of course, to do this we are actually applying some variation of PMI but we won't bother to state all such variations formally.

To end this example, let's see that $1'$.) and $2'$.) are actually true in this case. First, it is easy to see that $3 \in S$ since $3^2+5 = 14$ while $3^3 = 27$. Next, let us assume that some particular k is in S. Then we know that $k^2+5 < k^3$. But

$$\begin{aligned}
(k+1)^2 + 5 &= (k^2 + 2k + 1) + 5 \\
&= (k^2 + 5) + (2k + 1) \\
&< k^3 + (2k + 1) \\
&< k^3 + (2k + 1) + (3k^2 + k) \\
&= k^3 + 3k^2 + 3k + 1 \\
&= (k+1)^3,
\end{aligned}$$

where the first inequality follows from the assumption that $k^2 + 5 < k^3$, and the second inequality follows from the fact that $3k^2 + k > 0$ for $k \in \mathbb{N}$.

For the next example we will need a more substantial variation on the Principle of Mathematical Induction called the Principle of Complete Mathematical Induction (which we will abbreviate PCMI).

The Principle of Complete Mathematical Induction
Let S be a subset of the natural numbers, $\mathbb{N}$, satisfying
1.) $1 \in S$
2.) if $1, 2, 3, ..., k$ are all elements of S, then $k+1$ is an element of S as well.
Then $S = \mathbb{N}$.

Example 1.2.5

Let $\{a_n\}$ be the sequence given by the two-term recursion formula $a_{n+1} = 2a_n - a_{n-1} + 2$ for $n > 1$ and $a_1 = 3$ and $a_2 = 6$. Listing the first seven terms of this sequence, we get $3, 6, 11, 18, 27, 38, 51, ...$ Perhaps a pattern is becoming evident at this point. It seems that the n^{th} term in the sequence is given by the explicit formula $a_n = n^2 + 2$. Let's use PCMI to prove that this is the case. Let S be the subset of natural numbers n such that it is true that $a_n = n^2 + 2$, i.e.

$$S = \{n \in \mathbb{N} | \ \ a_n = n^2 + 2\}.$$

We know that $1 \in S$ since we are given $a_1 = 3$ and it is easy to check that $3 = 1^2 + 2$. Now assume that we know $1, 2, 3, ..., k$ are all elements of S. Now, as long as $k > 1$ we know that $a_{k+1} = 2a_k - a_{k-1} + 2$. Since we are assuming that k and $k-1$ are in S, we can write $a_k = k^2 + 2$ and $a_{k-1} = (k-1)^2 + 2$. Substituting these into the expression for a_{k+1} we get

$$\begin{aligned} a_{k+1} &= 2(k^2+2) - [(k-1)^2 + 2] + 2 \\ &= 2k^2 + 4 - (k^2 - 2k + 3) + 2 \\ &= k^2 + 2k + 3 \\ &= (k+1)^2 + 2. \end{aligned} \tag{1.1}$$

Thus $k + 1 \in S$ as well. We aren't quite done yet! The last argument only works when $k > 1$. What about the case that $k = 1$? If we know that $1 \in S$ can we conclude that $2 \in S$? Well, not directly, but we can check that $2 \in S$ anyway. After all, we are given that $a_2 = 6$ and it is easy to check that $6 = 2^2 + 2$. Notice that if we had given a different value for a_2, then the entire sequence changes from there on out so the explicit formula would no longer be correct. However, if you aren't careful about the subtlety at $k = 1$, you might think that you could prove the formula by induction no matter what is the value of a_2!

b.) Suppose that the sequence $\{a_n\}$ satisfies $a_n = \sqrt{a_{n+1}a_{n-1}}$ for all $n > 1$. Prove that there is an $r \in \mathbb{R}$ so that $a_n = ra_{n-1}$ for all $n > 1$. (Hint: Start with the case that $a_n \neq 0$ for all n, and use the expression $a_n = \sqrt{a_{n+1}a_{n-1}}$ to show that $a_{n+1}/a_n = a_n/a_{n-1}$ for $n > 1$ then use induction to show that $a_n/a_{n-1} = a_2/a_1$ for all $n > 1$. Don't forget to separately deal with the case that some a_n is zero.)

20. In this exercise we outline the proof of the equivalence of PMI and PCMI.

a.) First we assume that the natural numbers satisfy PCMI and we take a subset $T \subset \mathbb{N}$ which satisfies hypotheses 1.) and 2.) of PMI, i.e. we know:

1.) $1 \in T$

and that

2.) if $k \in T$ then $k + 1 \in T$.

Show directly that T also satisfies the hypotheses of PCMI, hence by our assumption we conclude that $T = \mathbb{N}$.

b.) This direction is a little trickier. Assume that the natural numbers satisfy PMI and take $T \subset \mathbb{N}$ which satisfies hypotheses 1.) and 2.) of PCMI. Now define S to be the subset of $\mathbb{N}$ given by $S = \{n \in T \mid 1, 2, ..., n \in T\}$. Prove by PMI that $S = \mathbb{N}$.

1.3 Sequences as functions

Let A and B be two arbitrary sets. A *function* from A to B, usually denoted by $f : A \to B$, is an assignment to each element a in A an element $f(a)$ in B. The set A is called the *domain* of f. Since the n^{th} term of a sequence, a_n, is a real number which is assigned to the natural number n, a sequence can be thought of as a function from $\mathbb{N}$ to $\mathbb{R}$, and vice versa. Thus we have

Definition 1.3.1
Let $f : \mathbb{N} \to \mathbb{R}$ be a function. The *sequence defined*[2] *by* f is the sequence

[2]More pedantically, a sequence should be defined as such a function. However, we prefer to treat the sequence and the function as separate entities, even though we never gave the notion of a sequence a separate definition.

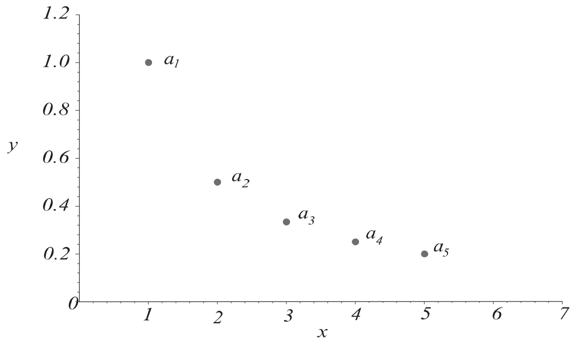

figure 1.3.1

whose n^{th} term is given by $a_n = f(n)$. On the other hand, if a sequence $\{a_n\}$ is given, then the function $f : \mathbb{N} \to \mathbb{R}$, given by $f(n) = a_n$ is called the *function defined by* $\{a_n\}$.

Example 1.3.2

a.) The function given by $f(n) = n^2$ defines the sequence with $a_n = n^2$.

b.) The sequence given by $a_n = 1/n$ defines the function $f : \mathbb{N} \to \mathbb{R}$ given by $f(n) = 1/n$.

Remark:

If a sequence is given by an explicit formula, then that formula gives an explicit formula for the corresponding function. But just as sequences may be defined in non-explicit ways, so too can functions. For instance, the consecutive digits in the decimal expansion of π define a sequence and therefore a corresponding function, $f : \mathbb{N} \to \mathbb{R}$, yet we know of no simple explicit formula for $f(n)$.

Thinking of sequences in terms of their corresponding functions gives us an important method of visualizing sequences, namely by their *graphs*. Since the natural numbers $\mathbb{N}$ are a subset of the real numbers $\mathbb{R}$ we can graph a function $f : \mathbb{N} \to \mathbb{R}$ in the plane $\mathbb{R}^2$ but the graph will consist of only isolated points whose x-coordinates are natural numbers. In figure 1.3.1 we depict a graph of the first five terms of the sequence given by $a_n = 1/n$.

The interpretation of sequences as functions leads to some of the following nomenclature. (Although we state these definitions in terms of sequences, a similar definition is meant to be understood for the corresponding functions.)

Definition 1.3.3

The sequence $\{a_n\}$ is said to be:

i.) *increasing* if $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$,

ii.) *strictly increasing* if $a_{n+1} > a_n$ for all $n \in \mathbb{N}$,

iii.) *decreasing* if $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$,

iv.) *strictly decreasing* if $a_{n+1} < a_n$ for all $n \in \mathbb{N}$.

Remark:

1.) It is important to notice that in these definitions the condition must hold for *all* $n \in \mathbb{N}$. So if a sequence sometimes increases and sometimes decreases, then it is neither increasing nor decreasing (unless it is constant).

2.) For the rest of this text we will be making extensive use of inequalities and absolute values. For this reason we have included the basic properties of these at the end of this section. Now might be a good time to review those properties.

Example 1.3.4

a.) The sequence given by $a_n = n^2$ is strictly increasing since $(n+1)^2 = n^2 + 2n + 1 > n^2$ for all $n > 0$.

b.) The sequence given by $a_n = 0$ for all n is both increasing and decreasing but it is not strictly increasing nor strictly decreasing.

c.) A strictly increasing sequence is, of course, increasing, but it is not decreasing.

d.) The sequence given by $a_n = 3 + (-1)^n/n$ is neither increasing nor decreasing.

Example 1.3.5

Consider the sequence given recursively by

$$a_{n+1} = \frac{4a_n + 3}{a_n + 2}$$

with $a_1 = 4$. We claim that this sequence is decreasing. To check this we will show that $a_{n+1} - a_n$ is negative. Now

$$\begin{aligned} a_{n+1} - a_n &= \frac{4a_n + 3}{a_n + 2} - a_n \\ &= \frac{4a_n + 3}{a_n + 2} - \frac{a_n^2 + 2a_n}{a_n + 2} \\ &= \frac{2a_n + 3 - a_n^2}{a_n + 2} \\ &= \frac{-(a_n - 3)(a_n + 1)}{a_n + 2} \end{aligned}$$

so $a_{n+1} - a_n$ is negative as long as a_n is greater than 3. But an easy induction argument shows that $a_n > 3$ for all n since

$$\begin{aligned} a_{k+1} - 3 &= \frac{4a_k + 3}{a_k + 2} - 3 \\ &= \frac{4a_k + 3}{a_k + 2} - \frac{3a_k + 6}{a_k + 2} \\ &= \frac{a_k - 3}{a_k + 2} \end{aligned}$$

which is positive as long as $a_k > 3$.

Comment on Inequalities

In the above proof we have twice used the fact that when trying to prove that $a > b$ it is often easier to prove that $a - b > 0$.

At times we need to talk about sequences that are either decreasing or increasing without specifying which, thus we have

Definition 1.3.6

The sequence $\{a_n\}$ is said to be *monotonic* if it is either increasing or decreasing.

The following equivalent condition for an increasing sequence is often useful when studying properties of such sequences. Of course there are similar equivalent conditions for decreasing sequences and strictly increasing or decreasing sequences.

Proposition 1.3.7

A sequence, $\{a_n\}$, is increasing if and only if $a_m \geq a_n$ for all natural numbers m and n with $m \geq n$.

Proof: There are two implications to prove here. First we prove that if the sequence $\{a_n\}$ satisfies the condition that $a_m \geq a_n$ whenever $m \geq n$ then that sequence must be increasing. But this is easy, since if $a_m \geq a_n$ whenever $m \geq n$, then in particular we know that $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$, i.e. the sequence is increasing.

Next we must prove that if the sequence $\{a_n\}$ is known to be increasing then it must satisfy the condition $a_m \geq a_n$ whenever $m \geq n$. We will prove this by a fairly tricky induction argument. Fix a particular $n \in \mathbb{N}$ and let

$$S_n = \{l \in \mathbb{N} \mid a_{n+l} \geq a_n\}.$$

We will use PMI to show that $S_n = \mathbb{N}$. First, since the sequence is increasing we know that $a_{n+1} \geq a_n$ so $1 \in S_n$. Now suppose that some specific k is in S_n. By the definition of S_n this tells us that

$$a_{n+k} \geq a_n.$$

Since the sequence is increasing we also know that

$$a_{n+k+1} \geq a_{n+k}.$$

Putting these inequalities together yields

$$a_{n+k+1} \geq a_n,$$

i.e. $k+1 \in S_n$. This proves, by PMI that $S_n = \mathbb{N}$. In other words, for this particular value of n we now know that $a_m \geq a_n$ whenever $m \geq n$. Since n was chosen to be an arbitrary member of $\mathbb{N}$ we conclude that $a_m \geq a_n$ for all natural numbers m and n with $m \geq n$.

□

Another property of sequences that seems natural when thinking of sequences as functions is the notion of *boundedness*.

Definition 1.3.8

The sequence $\{a_n\}$ is *bounded above* if there is a real number U such that $a_n \leq U$ for all $n \in \mathbb{N}$. Any real number U which satisfies this inequality for all $n \in \mathbb{N}$ is called an *upper bound* for the sequence $\{a_n\}$.

Definition 1.3.9

The sequence $\{a_n\}$ is *bounded below* if there is a real number L such that $L \leq a_n$ for all $n \in \mathbb{N}$. Any real number L which satisfies this inequality for all $n \in \mathbb{N}$ is called a *lower bound* for the sequence $\{a_n\}$.

Example 1.3.10
It is an important property of the real numbers that *the sequence of natural numbers, $\mathbb{N}$, is not bounded above* (see appendix A). Thus, if r is a real number, there is always some natural number $n \in \mathbb{N}$ so that $n > r$. Of course, the natural numbers *are* bounded below by 0.

Example 1.3.11
a.) The sequence given by $a_n = n^2, n \in \mathbb{N}$, is bounded below by 0 but it is not bounded above.
b.) The sequence given by $a_n = -n, n \in \mathbb{N}$, is bounded above by 0 but it is not bounded below.
c.) The sequence given by $a_n = \frac{1}{n}, n \in \mathbb{N}$, is bounded below by 0 and bounded above by 1.
d.) The sequence given by $a_n = \cos(n), n \in \mathbb{N}$, is bounded below by -1 and bounded above by 1.

Definition 1.3.12
The sequence $\{a_n\}$ is *bounded* if it is both bounded above and bounded below.

In the above example, only the sequences in parts c.) and d.) are bounded. In the definition of boundedness the upper and lower bounds are not necessarily related, however, it is sometimes useful to note that they can be.

Proposition 1.3.13
If the sequence $\{a_n\}$ is bounded then there is a real number $M > 0$ such that $-M \leq a_n \leq M$ for all $n \in \mathbb{N}$.

Proof: Let U be an upper bound and L a lower bound for the bounded sequence $\{a_n\}$. Let M be the maximum of the two numbers $|L|$ and $|U|$. Then since $-|L| \leq L$ and $U \leq |U|$ we have

$$-M \leq -|L| \leq L \leq a_n \leq U \leq |U| \leq M$$

for all $n \in \mathbb{N}$.

$\square$

Given two real valued functions we can add or multiply them by adding or multiplying their values. In particular, if we have $f : \mathbb{N} \to \mathbb{R}$ and $g : \mathbb{N} \to \mathbb{R}$, we define two new functions, $(f + g)$ and fg by

$$(f + g)(n) = f(n) + g(n)$$

and

$$fg(n) = f(n)g(n).$$

If we use f and g to define sequences $\{a_n\}$ and $\{b_n\}$, i.e. $a_n = f(n)$ and $b_n = g(n)$, then the sequence given by $(f+g)$ has terms given by $a_n + b_n$ and the sequence given by fg has terms given by $a_n b_n$. These two new sequences are called the *sum* and *product* of the original sequences.

In general, one can also compose two functions as long as the set of values of the first function is contained in the domain of the second function. Namely, if $g : A \to B$ and $f : B \to C$, then we can define $f \circ g : A \to C$ by $f \circ g(a) = f(g(a))$ for each $a \in A$. Notice that $g(a) \in B$ so $f(g(a))$ makes sense. To apply the notion of composition to sequences we need to remark that the corresponding functions always have the domain given by $\mathbb{N}$. Thus, if we wish to compose these functions then the first one must take its values in $\mathbb{N}$, i.e. the first sequence must be a sequence of natural numbers. In the next few sections we will be particularly interested in the case that the first sequence is a strictly increasing sequence of natural numbers, in this case we call the composition a *subsequence* of the second sequence.

Definition 1.3.14

Let $\{a_n\}$ be the sequence defined by the function $f : \mathbb{N} \to \mathbb{R}$ and let $g : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function (with values in $\mathbb{N}$). Then the sequence $\{b_n\}$ defined by the function $f \circ g : \mathbb{N} \to \mathbb{R}$, i.e. $b_n = f(g(n))$, is called a *subsequence* of the sequence $\{a_n\}$.

Example 1.3.15

The sequence given by $b_n = (2n+1)^2$ is a subsequence of the sequence given by $a_n = n^2$. If we let $f : \mathbb{N} \to \mathbb{R}$ be given by $f(n) = n^2$ and $g : \mathbb{N} \to \mathbb{N}$ be given by $g(n) = 2n+1$, then the function corresponding to b_n is given by $f \circ g$ since $f \circ g(n) = (2n+1)^2$.

A good way to think about subsequences is the following. First list the elements of your original sequence in order:

$$a_1, a_2, a_3, a_4, a_5, ...$$

The first element, b_1, of a subsequence can by any ane of the above list, but the next element, b_2, must lie to the right of b_1 in this list. Similarly, b_3 must lie to right of b_2 in the list, b_4 must lie to the right of b_3, and so on. Of course, each of the b_j's must be taken from the original list of a_n's. Thus we can line up the b_j's under the corresponding a_n's:

$$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, ...$$

$$b_1 \qquad b_2 \qquad b_3$$

A useful result which helps us review some of the above definitions is the following.

Proposition 1.3.16
If $\{a_n\}$ is an increasing sequence, and $\{b_n\}$ is a subsequence, then $\{b_n\}$ is increasing also.

Proof: Let $\{a_n\}$ define the function $f : \mathbb{N} \to \mathbb{R}$. Since $\{b_n\}$is a subsequence of $\{a_n\}$ we know that there is a strictly increasing function $g : \mathbb{N} \to \mathbb{N}$ so that $b_n = f(g(n))$. Of course we also have $b_{n+1} = f(g(n + 1))$. Now since g is strictly increasing we know that $g(n + 1) > g(n)$, also since f is increasing, we can use Proposition 1.3.6 to conclude that $f(m) \geq f(n)$ whenever $m \geq n$. Substituting $g(n+1)$ in for m and $g(n)$ in for n, we see that $f(g(n+1)) \geq f(g(n))$ for all $n \in \mathbb{N}$. That is, $b_{n+1} \geq b_n$ for all $n \in \mathbb{N}$.
$\square$

Before ending this section, we mention that there is another way of visualizing a sequence. Essentially this amounts to depicting the set of values of the corresponding function. However, if we merely show the set of values, then we have suppressed a great deal of the information in the sequence, namely, the ordering of the points. As a compromise, to retain this information one often indicates the ordering of the points by labeling a few of them. Of course, just as with a graph, any such picture will only show finitely many of the points from the sequence, but still it may give some understanding of the long term behavior of the sequence.

Example 1.3.17
Consider the sequence given by $a_n = 1/n$. A graph of the first five terms of this sequence was given in *figure 1.3.1.* The first five values of this sequence are given by the projection of this graph to the y-axis as in *figure 1.3.2.*

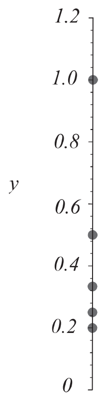

figure 1.3.2

Actually, it is customary to rotate this picture to the horizontal as in *figure 1.3.3*.

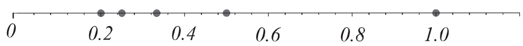

figure 1.3.3

To give an even better indication of the full sequence, we label these first five points and then indicate the location of a few more points as in *figure 1.3.4*.

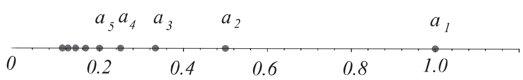

figure 1.3.4

Chapter 2

Series

2.1 Introduction to Series

In common parlance the words series and sequence are essentially synonomous, however, in mathematics the distinction between the two is that a series is the sum of the terms of a sequence.

Definition 2.1.1

Let $\{a_n\}$ be a sequence and define a new sequence $\{s_n\}$ by the recursion relation $s_1 = a_1$, and $s_{n+1} = s_n + a_{n+1}$. The sequence $\{s_n\}$ is called the *sequence of partial sums of* $\{a_n\}$.

Another way to think about s_n is that it is given by the sum of the first n terms of the sequence $\{a_n\}$, namely

$$s_n = a_1 + a_2 + ... + a_n.$$

A shorthand form of writing this sum is by using the *sigma notation*:

$$s_n = \sum_{j=1}^{n} a_j.$$

This is read as s_n *equals the sum from* j *equals one to* n *of* a *sub* j. We use the subscript j on the terms a_j (instead of n) because this is denoting an arbitrary term in the sequence while n is being used to denote how far we sum the sequence.

Example 2.1.2

Using sigma notation, the sum $1+2+3+4+5$ can be written as $\displaystyle\sum_{j=1}^{5} j$.

It can also be denoted $\displaystyle\sum_{n=1}^{5} n$, or $\displaystyle\sum_{n=0}^{4}(n+1)$. Similarly, the sum

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$$

can be written as $\displaystyle\sum_{j=2}^{6}\frac{1}{j}$, or $\displaystyle\sum_{n=2}^{6}\frac{1}{n}$, or $\displaystyle\sum_{n=1}^{5}\frac{1}{n+1}$. On the other hand, the sum

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{n}$$

can be written as $\displaystyle\sum_{j=1}^{n}\frac{1}{j}$ or $\displaystyle\sum_{k=1}^{n}\frac{1}{k}$ but *can not be written as* $\displaystyle\sum_{n=1}^{n}\frac{1}{n}$.

Definition 2.1.3

Let $\{a_n\}$ be a sequence and let $\{s_n\}$ be the sequence of partial sums of $\{a_n\}$. If $\{s_n\}$ converges we say that $\{a_n\}$ is *summable*. In this case, we denote the $\lim_{n\to\infty} s_n$ by

$$\sum_{j=1}^{\infty} a_j.$$

.

Definition 2.1.4

The expression $\sum_{j=1}^{\infty} a_j$ is called an *infinite series* (whether or not the sequence $\{a_n\}$ is summable). When we are given an infinite series $\sum_{j=1}^{\infty} a_j$ the sequence $\{a_n\}$ is called the *sequence of terms*. If the sequence of terms is summable, the infinite series is said to be *convergent*. If it is not convergent it is said to *diverge*.

Example 2.1.5

Consider the sequence of terms given by

$$a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Then

$$\begin{aligned}
s_1 &= a_1 = 1 - \frac{1}{2} = \frac{1}{2}, \\
s_2 &= a_1 + a_2 = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) \\
&= 1 + (\frac{1}{2} - \frac{1}{2}) - \frac{1}{3} \\
&= \frac{2}{3}, \\
s_3 &= a_1 + a_2 + a_3 = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) \\
&= 1 + (\frac{1}{2} - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{3}) - \frac{1}{4} \\
&= \frac{3}{4},
\end{aligned}$$

etc. Continuing this regrouping, we see that

$$\begin{aligned}
s_n &= a_1 + a_2 + \cdots + a_n = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots + (\frac{1}{n} - \frac{1}{n+1}) \\
&= 1 + (\frac{1}{2} - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{3}) + \cdots + (\frac{1}{n} - \frac{1}{n}) - \frac{1}{n+1} \\
&= \frac{n}{n+1}.
\end{aligned}$$

Therefore we see that the $\lim_{n\to\infty} s_n = 1$ and so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to 1.

Example 2.1.6

Let $a_n = \frac{1}{2^n}$. Then

$$\begin{aligned}
s_1 &= a_1 = \frac{1}{2} \\
s_2 &= a_1 + a_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \\
s_3 &= a_1 + a_2 + a_3 = \frac{3}{4} + \frac{1}{8} = \frac{7}{8},
\end{aligned}$$

etc. A straightforward induction argument shows that, in general,

$$s_n = 1 - \frac{1}{2^n}.$$

Thus $\lim s_n = 1$, and so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.

Example 2.1.7

Let $a_n = \frac{1}{n^2}$. Then the partial sum s_n is given by

$$s_n = 1 + \frac{1}{4} + \frac{1}{9} + ... + \frac{1}{n^2}.$$

In particular, $s_1 = 1, s_2 = 5/4$, and $s_3 = 49/36$. Since the a_n's are all positive the s_n's form an increasing sequence. (Indeed, $s_n - s_{n-1} = a_n = \frac{1}{n^2} > 0$.) It is not much harder to show by induction (see exercise 7) that for each n, $s_n < 2 - \frac{1}{n}$ so that the sequence $\{s_n\}$ is bounded above by 2. Thus we conclude (from the Property of Completeness) that the sequence $\{a_n\}$ is summable, i.e. the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges.

Remark: It is important to note here that although we have proven that this series converges, we do not know the value of the sum.

Example 2.1.8

Let $a_n = \frac{1}{n}$. Then the partial sum s_n is given by

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}.$$

In particular, $s_1 = 1, s_2 = 3/2$, and $s_3 = 11/6$. Again, notice that the s_n's are increasing. However, we will show that in this case they are not bounded above, hence the *harmonic* series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. To see that the partial sums are not bounded above we focus our attention on the subsequence $\sigma_n = s_{g(n)}$ where $g(n) = 2^{n-1}$. So we have

$$\sigma_1 = s_1 = 1,$$
$$\sigma_2 = s_2 = \frac{3}{2},$$
$$\sigma_3 = s_4 = \frac{25}{12},$$

etc. We will show that $\sigma_n \geq \sigma_{n-1}+1/2$. From this it follows by induction that $\sigma_n > n/2$ and so the sequence $\{\sigma_n\}$ is unbounded. To see that $\sigma_n \geq \sigma_{n-1}+1/2$ notice that $\sigma_n - \sigma_{n-1}$ is given by adding up the a_j's up to $a_{2^{n-1}}$ and then subtracting all of them up to $a_{2^{n-2}}$ so the net result is that

$$\sigma_n - \sigma_{n-1} = a_{2^{n-2}+1} + a_{2^{n-2}+2} + ... + a_{2^{n-1}}.$$

Now there are exactly $2^{n-1} - 2^{n-2} = 2^{n-2}$ terms in the sum on the right hand side and each of them is larger than or equal to $a_{2^{n-1}} = \frac{1}{2^{n-1}}$. Thus the $\sigma_n - \sigma_{n-1}$ is larger than or equal to $\frac{2^{n-2}}{2^{n-1}} = \frac{1}{2}$.

The following picture might be helpful in understanding the above argument;

$$1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + ...$$
$$> \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + ...$$

Example 2.1.9

Let $\{b_n\}$ be a given sequence. We can build a sequence $\{a_n\}$ whose sequence of partial sums is given by $\{b_n\}$ in the following way: Let $a_1 = b_1$ and for $n > 1$ let $a_n = b_n - b_{n-1}$. Then we have $a_1 + a_2 = b_1 + (b_2 - b_1) = b_2, a_1 + a_2 + a_3 = b_1 + (b_2 - b_1) + (b_3 - b_2) = b_3$, etc. Thus the sequence $\{b_n\}$ converges if and only if the sequence $\{a_n\}$ is summable.

The above example shows that a sequence is convergent if and only if a related sequence is summable. Similarly, a sequence is summable if and only if the sequence of partial sums converges. However it should be kept in mind that the sequence given by $a_n = 1/n$ converges but is *not* summable. On the other hand we have

Proposition 2.1.10

If the sequence $\{a_n\}$ is summable then $\lim a_n = 0$.

Proof: Let $\{s_n\}$ denote the sequence of partial sums of $\{a_n\}$. Then we know that $\lim s_n$ exists, call it S. Notice that $a_n = s_n - s_{n-1}$ so by the algebra of limits

$$\lim a_n = \lim s_n - \lim s_{n-1} = S - S = 0.$$

□

We should also mention that, as with sequences, convergent series behave well with respect to sums and multiplication by a fixed real number (scalar multiplication). Of course multiplication of two series is more complicated, since even for finite sums, the product of two sums is not simply the sum of the products. We will return to a discussion of products of series in section 2.3. For now we state the result for sums and scalar multiplication of series, leaving the proofs to exercises 2.1.12 and 2.1.13.

Proposition 2.1.11
Let $\{a_n\}$ and $\{b_n\}$ be summable sequences and let r be a real number. Define two new sequences by $c_n = a_n + b_n$ and $d_n = ra_n$. Then $\{c_n\}$ and $\{d_n\}$ are both summable and

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$
$$\sum_{n=1}^{\infty} d_n = r \sum_{n=1}^{\infty} a_n.$$

We conclude this section with an important example called the *geometric series.*

Definition 2.1.12
Let $r \in \mathbb{R}$. The series $\sum_{n=1}^{\infty} r^n$ is called a *geometric series.*

Proposition 2.1.13
The geometric series $\sum_{n=1}^{\infty} r^n$ converges if and only if $|r| < 1$. In the case that $|r| < 1$ we have

$$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$$

Proof: First note that the terms of this series are given by $a_n = r^n$ and this sequence converges to zero if and only if $|r| < 1$. Hence if $|r| \geq 1$ we conclude from proposition 2.1.7 that the series diverges.

Now if $|r| < 1$ consider the partial sum

$$s_n = r + r^2 + r^3 + ... + r^n.$$

Notice that

$$rs_n = r^2 + r^3 + r^4 ... + r^{n+1} = s_n - r + r^{n+1}.$$

Solving for s_n yields

$$s_n = \frac{r - r^{n+1}}{1-r} = \frac{r}{1-r} - \frac{r^{n+1}}{1-r}.$$

Now if $|r| < 1$ we know that $\lim r^n = 0$ and hence $\lim \frac{r^{n+1}}{1-r} = 0$. Thus $\lim s_n = \frac{r}{1-r}$.

$\square$

Example 2.1.14

a.) The sum $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n}$ converges to $\displaystyle\frac{1/5}{1-1/5} = \frac{1/5}{4/5} = 1/4$.

b.) The sum $\displaystyle\sum_{n=1}^{\infty} \frac{3}{5^n}$ equals $\displaystyle 3\sum_{n=1}^{\infty} \frac{1}{5^n}$ and so converges to $3/4$.

c.) The sum $\displaystyle\sum_{n=3}^{\infty} \frac{1}{5^n}$ equals $\displaystyle(\sum_{n=1}^{\infty} \frac{1}{5^n}) - (\frac{1}{5} + \frac{1}{5^2})$ and so converges to $1/4 - 1/5 - 1/25 = 19/100$.

EXERCISES 2.1

1. Consider the sequence given by $a_n = \frac{1}{2^n}$. Compute the first five partial sums of this sequence.

2. Rewrite the following sums using sigma notation:

   a.) $(1 + 4 + 9 + 16 + 25 + 36 + 49)$

   b.) $(5 + 6 + 7 + 8 + 9 + 10)$

   c.) $(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + ... + \frac{1}{28})$

   d.) $(2 + 2 + 2 + 2 + 2 + 2 + 2 + 2)$

3. Evaluate the following finite sums:

   a.) $\displaystyle\sum_{k=1}^{n} 1$ b.) $\displaystyle\sum_{k=1}^{n} 1/n$ c.) $\displaystyle\sum_{k=1}^{n} k$

   d.) $\displaystyle\sum_{k=1}^{2n} k$ e.) $\displaystyle\sum_{k=1}^{n} k^2$