Chapter 3

Motion in Two and Three Dimensions

3.1 The Important Stuff

3.1.1 Position

In three dimensions, the location of a particle is specified by its **location vector, r**:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{3.1}$$

If during a time interval Δt the position vector of the particle changes from $\mathbf{r}_1$ to $\mathbf{r}_2$, the displacement $\Delta\mathbf{r}$ for that time interval is

$$\begin{aligned} \Delta\mathbf{r} &= \mathbf{r}_1 - \mathbf{r}_2 && (3.2) \\ &= (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k} && (3.3) \end{aligned}$$

3.1.2 Velocity

If a particle moves through a displacement $\Delta\mathbf{r}$ in a time interval Δt then its average velocity for that interval is

$$\overline{v} = \frac{\Delta\mathbf{r}}{\Delta t} = \frac{\Delta x}{\Delta t}\mathbf{i} + \frac{\Delta y}{\Delta t}\mathbf{j} + \frac{\Delta z}{\Delta t}\mathbf{k} \tag{3.4}$$

As before, a more interesting quantity is the *instantaneous* velocity $\mathbf{v}$, which is the limit of the average velocity when we shrink the time interval Δt to zero. It is the time derivative of the position vector $\mathbf{r}$:

$$\begin{aligned} \mathbf{v} &= \frac{d\mathbf{r}}{dt} && (3.5) \\ &= \frac{d}{dt}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) && (3.6) \\ &= \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} && (3.7) \end{aligned}$$

can be written:

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k} \tag{3.8}$$

where

$$v_x = \frac{dx}{dt} \qquad v_y = \frac{dy}{dt} \qquad v_z = \frac{dz}{dt} \tag{3.9}$$

The instantaneous velocity **v** of a particle is always tangent to the path of the particle.

3.1.3 Acceleration

If a particle's velocity changes by $\Delta\mathbf{v}$ in a time period Δt, the average acceleration $\overline{\mathbf{a}}$ for that period is

$$\overline{\mathbf{a}} = \frac{\Delta\mathbf{v}}{\Delta t} = \frac{\Delta v_x}{\Delta t}\mathbf{i} + \frac{\Delta v_y}{\Delta t}\mathbf{j} + \frac{\Delta v_z}{\Delta t}\mathbf{k} \tag{3.10}$$

but a much more interesting quantity is the result of shrinking the period Δt to zero, which gives us the instantaneous acceleration, **a**. It is the time derivative of the velocity vector **v**:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} \tag{3.11}$$

$$= \frac{d}{dt}(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}) \tag{3.12}$$

$$= \frac{dv_x}{dt}\mathbf{i} + \frac{dv_y}{dt}\mathbf{j} + \frac{dv_z}{dt}\mathbf{k} \tag{3.13}$$

which can be written:

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \tag{3.14}$$

where

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} \qquad a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} \qquad a_z = \frac{dv_z}{dt} = \frac{d^2z}{dt^2} \tag{3.15}$$

3.1.4 Constant Acceleration in Two Dimensions

When the acceleration **a** (for motion in two dimensions) is constant we have two sets of equations to describe the x and y coordinates, each of which is similar to the equations in Chapter 2. (Eqs. 2.6—2.9.) In the following, motion of the particle begins at $t = 0$; the initial position of the particle is given by

$$\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j}$$

and its initial velocity is given by

$$\mathbf{v}_0 = v_{0x}\mathbf{i} + v_{0y}\mathbf{j}$$

and the vector $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j}$ is *constant.*

$$v_x = v_{0x} + a_x t \qquad v_y = v_{0y} + a_y t \tag{3.16}$$

$$x = x_0 + v_{0x}t + \tfrac{1}{2}a_x t^2 \qquad y = y_0 + v_{0y}t + \tfrac{1}{2}a_y t^2 \tag{3.17}$$

$$v_x^2 = v_{0x}^2 + 2a_x(x - x_0) \qquad v_y^2 = v_{0y}^2 + 2a_y(y - y_0) \tag{3.18}$$

$$x = x_0 + \tfrac{1}{2}(v_{0x} + v_x)t \qquad y = y_0 + \tfrac{1}{2}(v_{0y} + v_y)t \tag{3.19}$$

Though the equations in each pair have the same *form* they are not identical because the components of $\mathbf{r}_0$, $\mathbf{v}_0$ and **a** are not the same.

3.1.5 Projectile Motion

When a particle moves in a vertical plane during free–fall its acceleration is constant; the acceleration has magnitude $9.80\,\frac{\mathrm{m}}{\mathrm{s}^2}$ and is directed downward. If its coordinates are given by a horizontal x axis and a vertical y axis which is directed upward, then the acceleration of the **projectile** is

$$a_x = 0 \qquad a_y = -9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2} = -g \tag{3.20}$$

For a projectile, the horizontal acceleration a_x is zero!!!

Projectile motion is a special case of constant acceleration, so we simply use Eqs. 3.16–3.19, with the proper values of a_x and a_y.

3.1.6 Uniform Circular Motion

When a particle is moving in a circular path (or part of one) at *constant speed* we say that the particle is in **uniform circular motion**. Even though the speed is not changing, *the particle is accelerating* because its velocity $\mathbf{v}$ is changing *direction*.

The acceleration of the particle is directed toward the center of the circle and has magnitude

$$a = \frac{v^2}{r} \tag{3.21}$$

where r is the radius of the circular path and v is the (constant) speed of the particle. Because of the direction of the acceleration (i.e. toward the center), we say that a particle in uniform circular motion has a **centripetal acceleration**.

If the particle repeatedly makes a complete circular path, then it is useful to talk about the time T that it takes for the particle to make one complete trip around the circle. This is called the **period** of the motion. The period is related to the speed of the particle and radius of the circle by:

$$T = \frac{2\pi r}{v} \tag{3.22}$$

3.1.7 Relative Motion

The velocity of a particle depends on who is doing the measuring; as we see later on it is perfectly valid to consider "moving" observers who carry their own clocks and coordinate systems with them, i.e. they make measurements according to their own **reference frame**; that is to say, a set of Cartesian coordinates which may be in motion with respect to another set of coordinates. Here we will assume that the axes in the different system remain parallel to one another; that is, one system can move (translate) but not *rotate* with respect to another one.

Suppose observers in frames A and B measure the position of a point P. Then then if we have the definitions:

$$\mathbf{r}_{PA} = \text{ position of } P \text{ as measured by } A$$

$$\mathbf{r}_{PB} = \text{ position of } P \text{ as measured by } B$$

$$\mathbf{r}_{BA} = \text{ position of } B\text{'s origin, as measured by } A$$

with $\mathbf{v}$'s and $\mathbf{a}$'s standing for the appropriate time derivatives, then we have the relations:

$$\mathbf{r}_{PA} = \mathbf{r}_{PB} + \mathbf{r}_{BA} \tag{3.23}$$

$$\mathbf{v}_{PA} = \mathbf{v}_{PB} + \mathbf{v}_{BA} \tag{3.24}$$

For the purposes of doing physics, it is important to consider reference frames which move at *constant velocity* with respect to one another; for these cases, $\mathbf{v}_{BA} = 0$ and then we find that point P has the same acceleration in these reference frames:

$$\mathbf{a}_{PA} = \mathbf{a}_{PB}$$

Newton's Laws (next chapter!) apply to such a set of **inertial reference frames**. Observers in each of these frames agree on the value of a particle's acceleration.

Though the above rules for translation between reference frames seem very reasonable, it was the great achievement of Einstein with his theory of **Special Relativity** to understand the more subtle ways that we must relate measured quantities between reference frames. The trouble comes about because time (t) is *not* the same absolute quantity among the different frames.

Among other places, Eq. 3.24 is used in problems where an object like a plane or boat has a known velocity in the frame of (with respect to) a medium like air or water which *itself* is moving with respect to the stationary ground; we can then find the velocity of the plane or boat with respect to the *ground* from the *vector sum* in Eq. 3.24.

3.2 Worked Examples

3.2.1 Velocity

1. The position of an electron is given by $\mathbf{r} = 3.0t\mathbf{i} - 4.0t^2\mathbf{j} + 2.0\mathbf{k}$ (where t is in seconds and the coefficients have the proper units for $\mathbf{r}$ to be in meters). (a) What is $\mathbf{v}(t)$ for the electron? (b) In unit–vector notation, what is $\mathbf{v}$ at $t = 2.0\,\mathrm{s}$? (c) What are the magnitude and direction of $\mathbf{v}$ just then? [HRW5 4-9]

(a) The velocity vector $\mathbf{v}$ is the time–derivative of the position vector $\mathbf{r}$:

$$\begin{aligned} \mathbf{v} &= \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(3.0t\mathbf{i} - 4.0t^2\mathbf{j} + 2.0\mathbf{k}) \\ &= 3.0\mathbf{i} - 8.0t\mathbf{j} \end{aligned}$$

where we mean that when t is in seconds, $\mathbf{v}$ is given in $\frac{\mathrm{m}}{\mathrm{s}}$.