Mathematics Learning Centre

The University of Sydney

Introduction to Integration Part 1: Anti-Differentiation

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Contents

1 For Reference

1.1 Table of derivatives

Function $(f(x))$	**Derivative** $\left(f'(x) \text{ i.e. } \frac{d}{dx}(f(x))\right)$	
x^n	nx^{n-1}	(n any real number)
e^x	e^x	
$\ln x$	$\frac{1}{x}$	$(x > 0)$
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	
$\tan x$	$\sec^2 x$	
$\cot x$	$-\csc^2 x$	
$\sec x$	$\sec x \tan x$	
$\csc x$	$-\csc x \cot x$	
$\sin^{-1} x$	$\dfrac{1}{\sqrt{(1-x^2)}}$	$(\lvert x\rvert < 1)$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$	

1.2 New notation

Symbol	**Meaning**
$\int f(x)dx$	The indefinite integral of $f(x)$ with respect to x i.e. a function whose derivative is $f(x)$.

Note that $\int ...dx$ acts like a pair of brackets around the function. Just as a left-hand bracket has no meaning unless it is followed by a closing right-hand bracket, the integral sign cannot stand by itself, but needs "dx" to complete it. The integral sign tells us what operation to perform and the "dx" tells us that the variable with respect to which we are integrating is x.

New terms	**Meaning**
Anti-derivative Primitive function Indefinite integral	These are all different ways of saying "a function whose derivative is ..."

2 Introduction

This booklet is intended for students who have never done integration before, or who *have* done it before, but so long ago that they feel they have forgotten it all.

Integration is used in dealing with two essentially different types of problems:

The first type are problems in which the derivative of a function, or its rate of change, or the slope of its graph, is known and we want to find the function. We are therefore required to *reverse* the process of differentiation. This reverse process is known as *anti-differentiation*, or *finding a primitive function*, or *finding an indefinite integral.*

The second type are problems which involve adding up a very large number of very small quantities, (and then taking a limit as the size of the quantities approaches zero while the number of terms tends to infinity). This process leads to the definition of the *definite integral.* Definite integrals are used for finding area, volume, centre of gravity, moment of inertia, work done by a force, and in many other applications.

This unit will deal only with problems of the first type, i.e. with indefinite integrals. The second type of problem is dealt with in *Introduction to Integration Part 2 - The Definite Integral.*

2.1 How to use this book

You will not gain much by just reading this booklet. Have pencil and paper ready to work through the examples before reading their solutions. Do *all* the exercises. It is important that you try hard to complete the exercises on your own, rather than refer to the solutions as soon as you are stuck. If you have done integration before, and want to revise it, you should skim through the text and then do the exercises for practice. If you have any difficulties with the exercises, go back and study the text in more detail.

2.2 Objectives

By the time you have worked through this unit, you should:

- Be familiar with the definition of an indefinite integral as the result of reversing the process of differentiation.
- Understand how rules for integration are worked out using the rules for differentiation (in reverse).
- Be able to find indefinite integrals of sums, differences and constant multiples of certain elementary functions.
- Be able to use the chain rule (in reverse) to find indefinite integrals of certain expressions involving composite functions.
- Be able to apply these techniques to problems in which the rate of change of a function is known and the function has to be found.

2.3 Assumed knowledge

We assume that you are familiar with the following elementary functions: polynomials, powers of x, and the trigonometric, exponential and natural logarithm functions, and are able to differentiate these. We also assume that you can recognise composite functions and are familiar with the chain rule for differentiating them.

In addition you will need to know some simple trigonometric identities: those based on the definitions of tan, cot, sec and csc and those based on Pythagoras' Theorem. These are covered in sections of the Mathematics Learning Centre booklet *Trigonometric Identities*.

Other trigonometric identities are not needed for this booklet, but will be needed in any course on integration, so if you are preparing for a course on integration you should work through the whole of *Trigonometric Identities* as well as this booklet.

Finally, knowledge of the inverse trigonometric functions, $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ and their derivatives would be a help, but is not essential.

2.4 Test yourself

To check how well you remember all the things we will be assuming, try the following questions, and check your answers against those on the next page.

1. Find the derivatives of the following functions:

i x^{10}

ii $\sqrt{x}$

iii $\dfrac{1}{x}$

iv $5x^3 - \frac{3}{x^2}$

2. Find $f'(x)$ for each of the functions $f(x)$ given:

i $f(x) = e^x$

ii $f(x) = \ln(x)$

iii $f(x) = \cos x + \sin x$

iv $f(x) = \cot x$

3. Find derivatives of:

i $(2x+1)^{12}$

ii $\sin 3x$

iii e^{x^2}

iv $\dfrac{1}{x^2 - 3}$

v $\cos(x^3)$

vi $\ln(\sin x)$

3 Definition of the Integral as an Anti-Derivative

If $\dfrac{d}{dx}(F(x)) = f(x)$ then $\int f(x)dx = F(x)$.

In words,

> If the derivative of $F(x)$ is $f(x)$, then we say that an indefinite integral of $f(x)$ with respect to x is $F(x)$.

For example, since the derivative (with respect to x) of x^2 is $2x$, we can say that an indefinite integral of $2x$ is x^2.

In symbols:

$$\frac{d}{dx}(x^2) = 2x, \qquad \text{so} \qquad \int 2x dx = x^2.$$

Note that we say *an* indefinite integral, not *the* indefinite integral. This is because the indefinite integral is not unique. In our example, notice that the derivative of $x^2 + 3$ is also $2x$, so $x^2 + 3$ is another indefinite integral of $2x$. In fact, if c is *any* constant, the derivative of $x^2 + c$ is $2x$ and so $x^2 + c$ is an indefinite integral of $2x$.

We express this in symbols by writing

$$\int 2x dx = x^2 + c$$

where c is what we call an "arbitrary constant". This means that c has no specified value, but can be given any value we like in a particular problem. In this way we encapsulate all possible solutions to the problem of finding an indefinite integral of $2x$ in a single expression.

In most cases, if you are asked to find an indefinite integral of a function, it is not necessary to add the $+c$. However, there are cases in which it is essential. For example, if additional information is given and a specific function has to be found, or if the general solution of a differential equation is sought. (You will learn about these later.) So it is a good idea to get into the habit of adding the arbitrary constant every time, so that when it is really needed you don't forget it.

The inverse relationship between differentiation and integration means that, for every statement about differentiation, we can write down a corresponding statement about integration.

For example,

$$\frac{d}{dx}(x^4) = 4x^3, \qquad \text{so} \qquad \int 4x^3 dx = x^4 + c.$$

4 Some Rules for Calculating Integrals

Rules for operating with integrals are derived from the rules for operating with derivatives. So, because

$$\frac{d}{dx}\left(cf(x)\right) = c\frac{d}{dx}\left(f(x)\right), \text{ for any constant } c,$$

we have

Rule 1

$$\int (cf(x))dx = c\int f(x)dx, \text{ for any constant } c.$$

For example $\int 10\cos x dx = 10\int \cos x dx = 10\sin x + c.$

It sometimes helps people to understand and remember rules like this if they say them in words. The rule given above says: *The integral of a constant multiple of a function is a constant multiple of the integral of the function.* Another way of putting it is *You can move a constant past the integral sign without changing the value of the expression.*

Similarly, from

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x)),$$

we can derive the rule

Rule 2

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx.$$

For example,

$$\begin{aligned}\int (e^x + 2x)dx &= \int e^x dx + \int 2x dx\\ &= e^x + x^2 + c.\end{aligned}$$

In words, *the integral of the sum of two functions is the sum of their integrals.*

We can easily extend this rule to include differences as well as sums, and to the case where there are more than two terms in the sum.

Examples

Find the following indefinite integrals:

i $\int (1 + 2x - 3x^2 + \sin x)dx$

ii $\int (3\cos x - \frac{1}{2}e^x)dx$

5 Integrating Powers of x and Other Elementary Functions

We can now work out how to integrate any power of x by looking at the corresponding rule for differentiation:

$$\frac{d}{dx}(x^n) = nx^{n-1}, \qquad \text{so} \qquad \int nx^{n-1}dx = x^n + c.$$

Similarly

$$\frac{d}{dx}\left(x^{n+1}\right) = (n+1)x^n, \qquad \text{so} \qquad \int (n+1)x^n dx = x^{n+1} + c.$$

Therefore

$$\begin{aligned} \int x^n dx &= \int \frac{1}{n+1}\cdot(n+1)x^n dx && \leftarrow \text{notice that } \tfrac{1}{n+1}\cdot(n+1) \text{ is just 1 when we cancel} \\ &= \frac{1}{n+1}\int (n+1)x^n dx && \leftarrow \text{take } \tfrac{1}{n+1} \text{ outside the } \textstyle\int \text{ sign} \\ &= \frac{1}{n+1}x^{n+1} + c. \end{aligned}$$

We should now look carefully at the formula we have just worked out and ask: for which values of n does it hold?

Remember that the differentiation rule $\frac{d}{dx}(x^n) = nx^{n-1}$ holds whether n is positive or negative, a whole number or a fraction or even irrational; in other words, for all real numbers n.

We might expect the integration rule to hold for all real numbers also. But there is one snag: in working it out, we divided by $n+1$. Since division by zero does not make sense, the rule will not hold when $n+1=0$, that is, when $n=-1$. So we conclude that

Rule 3

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + c$$

for all real numbers n, except $n = -1$.

When $n=-1$, $\int x^n dx$ becomes $\int x^{-1}dx = \int \frac{1}{x}dx$. We don't need to worry that the rule above doesn't apply in this case, because we already know the integral of $\frac{1}{x}$.

Since

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \qquad \text{we have} \qquad \int \frac{1}{x}dx = \ln x + c.$$

At this stage it is very tempting to give a list of standard integrals, corresponding to the list of derivatives given at the beginning of this booklet. However, you are NOT encouraged to memorise integration formulae, but rather to become VERY familiar with the list of derivatives and to practise recognising a function as the derivative of another function.

If you try memorising *both* differentiation *and* integration formulae, you will one day mix them up and use the wrong one. And there is absolutely *no need* to memorise the integration formulae if you know the differentiation ones.

It is much better to recall the way in which an integral is defined as an anti-derivative. *Every time you perform an integration* you should pause for a moment and check it by differentiating the answer to see if you get back the function you began with. This is a very important habit to develop. There is no need to write down the checking process every time, often you will do it in your head, but if you get into this habit you will avoid a lot of mistakes.

There is a table of derivatives at the front of this booklet. Try to avoid using it if you can, but refer to it if you are unsure.

Examples

Find the following indefinite integrals:

i $\int \left(e^x + 3x^{\frac{5}{2}}\right) dx$

ii $\int (5\csc^2 x + 3\sec^2 x) dx$

Solutions

i

$$\begin{aligned} \int \left(e^x + 3x^{\frac{5}{2}}\right) dx &= \int e^x dx + 3\int x^{\frac{5}{2}} dx \\ &= e^x + 3 \cdot \frac{1}{\frac{5}{2}+1} x^{\frac{5}{2}+1} + c \\ &= e^x + 3 \cdot \frac{2}{7} x^{\frac{7}{2}} + c \\ &= e^x + \frac{6}{7} x^{\frac{7}{2}} + c. \end{aligned}$$

ii

$$\begin{aligned} \int (5\csc^2 x + 3\sec^2 x) dx &= -5\int (-\csc^2 x) dx + 3\int \sec^2 x dx \\ &= -5\cot x + 3\tan x + c. \end{aligned}$$

6 Things You Can't Do With Integrals

It is just as important to be aware of what you *can't* do when integrating, as to know what you *can* do. In this way you will avoid making some serious mistakes.

We mention here two fairly common ones.

1. We know that $\int cf(x)dx = c\int f(x)dx$, where c is a constant. That is, "you are allowed to move a constant past the integral sign". It is often very tempting to try the same thing with a variable, i.e. to equate $\int xf(x)dx$ with $x\int f(x)dx$.

If we check a few special cases, however, it will become clear that this is **not correct**. For example, compare the values of

$$\int x \cdot xdx \qquad \text{and} \qquad x\int xdx.$$

$$\int x \cdot xdx = \int x^2 dx = \frac{1}{3}x^3 + c \qquad \text{while} \qquad x\int xdx = x \cdot \frac{1}{2}x^2 = \frac{1}{2}x^3 + c.$$

These expressesions are obviously different.

So the "rule" we tried to invent does not work!

It is unlikely that anybody would try to find $\int x^2dx$ in the way shown above. However, if asked to find $\int x\sin xdx$, one might very easily be tempted to write $x\int \sin xdx = -x\cos x + c$. Although we do not yet know a method for finding $\int x\sin xdx$, we can very easily show that the answer obtained above is wrong. How?

By differentiating the answer, of course!

If we don't get back to $x\sin x$, we must have gone wrong somewhere.

Notice that $x\cos x$ is a product, so we must use the product rule to differentiate it.

$$\begin{aligned}\frac{d}{dx}(-x\cos x + c) &= -x(-\sin x) + (-1)\cos x \\ &= x\sin x - \cos x.\end{aligned}$$

The first term is correct, but the second term shouldn't be there! So the method we used was wrong.

$\int xf(x)dx$ is **not** equal to $x\int f(x)dx$.

In words,

you **cannot** move a variable past the integral sign.

2. Again, we know that $\int (f(x) + g(x))\, dx = \int f(x)dx + \int g(x)dx$. That is, "the integral of a sum is equal to the sum of the integrals". It may seem reasonable to wonder whether there is a similar rule for products. That is, whether we can equate $\int f(x) \cdot g(x)dx$ with $\int f(x)dx \cdot \int g(x)dx$.

Once again, checking a few special cases will show that this is **not correct**.

Take, for example, $\int x \sin x dx$.

Now

$$\begin{aligned}\int x dx \cdot \int \sin x dx &= \frac{1}{2}x^2 \cdot (-\cos x) + c \\ &= -\frac{1}{2}x^2 \cos x + c.\end{aligned}$$

But

$$\begin{aligned}\frac{d}{dx}\left(-\frac{1}{2}x^2 \cos x + c\right) &= -\frac{1}{2}x^2 \cdot (-\sin x) + (-x)\cos x \\ &= \frac{1}{2}x^2 \sin x - x \cos x.\end{aligned}$$

product rule again!

and this is nothing like the right answer! (Remember, it ought to have been $x \sin x$.)

So we conclude that

$$\int f(x)g(x) \text{ is } \textbf{not} \text{ equal to } \int f(x)dx \cdot \int g(x)dx.$$

In words,

> the integral of the product of two functions is **not** the same as the product of their integrals.

The other important point you should have learned from this section is the value of *checking* any integration by *differentiating the answer*. If you don't get back to what you started with, you know you have gone wrong somewhere, and since differentiation is generally easier than integration, the mistake is likely to be in the integration.

Exercises 6

Explain the mistakes in the following integrations, and *prove* that the answer obtained in each case is wrong, by differentiating the answers given.

i $\int x^2 e^x dx = \frac{1}{3}x^3 e^x + c$

ii $\int \frac{x dx}{\sqrt{(1-x^2)}} = x \int \frac{1}{\sqrt{(1-x^2)}} dx = x \sin^{-1} x + c$

You will learn how to integrate these functions later.

7 Using the Chain Rule in Reverse

Recall that the Chain Rule is used to differentiate composite functions such as $\cos(x^3+1), e^{\frac{1}{2}x^2}, (2x^2+3)^{11}, \ln(3x+1)$. (The Chain Rule is sometimes called the Composite Functions Rule or Function of a Function Rule.)

If we observe carefully the answers we obtain when we use the chain rule, we can learn to recognise when a function has this form, and so discover how to integrate such functions.

Remember that, if $y = f(u)$ and $u = g(x)$

so that $\quad y = f(g(x))$, (a composite function)

then $\quad \dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}.$

Using function notation, this can be written as

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x).$$

In this expression, $f'(g(x))$ is another way of writing $\dfrac{dy}{du}$ where $y = f(u)$ and $u = g(x)$ and $g'(x)$ is another way of writing $\dfrac{du}{dx}$ where $u = g(x)$.

This last form is the one you should learn to recognise.

Examples

By differentiating the following functions, write down the corresponding statement for integration.

i $\sin 3x$

ii $(2x+1)^7$

iii e^{x^2}

Solution

i $\dfrac{d}{dx}\sin 3x = \cos 3x \cdot 3, \quad$ so $\quad \int \cos 3x \cdot 3dx = \sin 3x + c.$

ii $\dfrac{d}{dx}(2x+1)^7 = 7(2x+1)^6 \cdot 2, \quad$ so $\quad \int 7(2x+1)^6 \cdot 2dx = (2x+1)^7 + c.$

iii $\dfrac{d}{dx}\left(e^{x^2}\right) = e^{x^2} \cdot 2x, \quad$ so $\quad \int e^{x^2} \cdot 2xdx = e^{x^2} + c.$