8

Techniques of Integration

Over the next few sections we examine some techniques that are frequently successful when seeking antiderivatives of functions. Sometimes this is a simple problem, since it will be apparent that the function you wish to integrate is a derivative in some straightforward way. For example, faced with

$$\int x^{10}\,dx$$

we realize immediately that the derivative of x^{11} will supply an x^{10}: $(x^{11})' = 11x^{10}$. We don't want the "11", but constants are easy to alter, because differentiation "ignores" them in certain circumstances, so

$$\frac{d}{dx}\frac{1}{11}x^{11} = \frac{1}{11}11x^{10} = x^{10}.$$

From our knowledge of derivatives, we can immediately write down a number of antiderivatives. Here is a list of those most often used:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad \text{if } n \neq -1$$

$$\int x^{-1}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \sec x \tan x\,dx = \sec x + C$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

8.1 Substitution

Needless to say, most problems we encounter will not be so simple. Here's a slightly more complicated example: find

$$\int 2x\cos(x^2)\,dx.$$

This is not a "simple" derivative, but a little thought reveals that it must have come from an application of the chain rule. Multiplied on the "outside" is $2x$, which is the derivative of the "inside" function x^2. Checking:

$$\frac{d}{dx}\sin(x^2) = \cos(x^2)\frac{d}{dx}x^2 = 2x\cos(x^2),$$

so

$$\int 2x\cos(x^2)\,dx = \sin(x^2) + C.$$

Even when the chain rule has "produced" a certain derivative, it is not always easy to see. Consider this problem:

$$\int x^3\sqrt{1-x^2}\,dx.$$

There are two factors in this expression, x^3 and $\sqrt{1-x^2}$, but it is not apparent that the chain rule is involved. Some clever rearrangement reveals that it is:

$$\int x^3\sqrt{1-x^2}\,dx = \int (-2x)\left(-\frac{1}{2}\right)(1-(1-x^2))\sqrt{1-x^2}\,dx.$$

This looks messy, but we do now have something that looks like the result of the chain rule: the function $1-x^2$ has been substituted into $-(1/2)(1-x)\sqrt{x}$, and the derivative of $1-x^2$, $-2x$, multiplied on the outside. If we can find a function $F(x)$ whose derivative is $-(1/2)(1-x)\sqrt{x}$ we'll be done, since then

$$\begin{aligned}\frac{d}{dx}F(1-x^2) = -2xF'(1-x^2) &= (-2x)\left(-\frac{1}{2}\right)(1-(1-x^2))\sqrt{1-x^2}\\ &= x^3\sqrt{1-x^2}\end{aligned}$$

But this isn't hard:

$$\begin{aligned}\int -\frac{1}{2}(1-x)\sqrt{x}\,dx &= \int -\frac{1}{2}(x^{1/2}-x^{3/2})\,dx \qquad (8.1.1)\\ &= -\frac{1}{2}\left(\frac{2}{3}x^{3/2}-\frac{2}{5}x^{5/2}\right)+C\\ &= \left(\frac{1}{5}x-\frac{1}{3}\right)x^{3/2}+C.\end{aligned}$$

So finally we have

$$\int x^3\sqrt{1-x^2}\,dx = \left(\frac{1}{5}(1-x^2)-\frac{1}{3}\right)(1-x^2)^{3/2}+C.$$

So we succeeded, but it required a clever first step, rewriting the original function so that it looked like the result of using the chain rule. Fortunately, there is a technique that makes such problems simpler, without requiring cleverness to rewrite a function in just the right way. It sometimes does not work, or may require more than one attempt, but the idea is simple: guess at the most likely candidate for the "inside function", then do some algebra to see what this requires the rest of the function to look like.

One frequently good guess is any complicated expression inside a square root, so we start by trying $u = 1-x^2$, using a new variable, u, for convenience in the manipulations that follow. Now we know that the chain rule will multiply by the derivative of this inner function:

$$\frac{du}{dx} = -2x,$$

so we need to rewrite the original function to include this:

$$\int x^3\sqrt{1-x^2} = \int x^3\sqrt{u}\frac{-2x}{-2x}\,dx = \int \frac{x^2}{-2}\sqrt{u}\,\frac{du}{dx}\,dx.$$

Recall that one benefit of the Leibniz notation is that it often turns out that what looks like ordinary arithmetic gives the correct answer, even if something more complicated is going on. For example, in Leibniz notation the chain rule is

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}.$$

The same is true of our current expression:

$$\int \frac{x^2}{-2}\sqrt{u}\,\frac{du}{dx}\,dx = \int \frac{x^2}{-2}\sqrt{u}\,du.$$

Now we're almost there: since $u = 1 - x^2$, $x^2 = 1 - u$ and the integral is

$$\int -\frac{1}{2}(1-u)\sqrt{u}\,du.$$

It's no coincidence that this is exactly the integral we computed in (8.1.1), we have simply renamed the variable u to make the calculations less confusing. Just as before:

$$\int -\frac{1}{2}(1-u)\sqrt{u}\,du = \left(\frac{1}{5}u - \frac{1}{3}\right)u^{3/2} + C.$$

Then since $u = 1 - x^2$:

$$\int x^3\sqrt{1-x^2}\,dx = \left(\frac{1}{5}(1-x^2) - \frac{1}{3}\right)(1-x^2)^{3/2} + C.$$

To summarize: if we suspect that a given function is the derivative of another via the chain rule, we let u denote a likely candidate for the inner function, then translate the given function so that it is written entirely in terms of u, with no x remaining in the expression. If we can integrate this new function of u, then the antiderivative of the original function is obtained by replacing u by the equivalent expression in x.

Even in simple cases you may prefer to use this mechanical procedure, since it often helps to avoid silly mistakes. For example, consider again this simple problem:

$$\int 2x\cos(x^2)\,dx.$$

Let $u = x^2$, then $du/dx = 2x$ or $du = 2x\,dx$. Since we have exactly $2x\,dx$ in the original integral, we can replace it by du:

$$\int 2x\cos(x^2)\,dx = \int \cos u\,du = \sin u + C = \sin(x^2) + C.$$

This is not the only way to do the algebra, and typically there are many paths to the correct answer. Another possibility, for example, is: Since $du/dx = 2x$, $dx = du/2x$, and then the integral becomes

$$\int 2x\cos(x^2)\,dx = \int 2x\cos u\,\frac{du}{2x} = \int \cos u\,du.$$

The important thing to remember is that you must eliminate all instances of the original variable x.

EXAMPLE 8.1.1 Evaluate $\int (ax+b)^n\,dx$, assuming that a and b are constants, $a \neq 0$, and n is a positive integer. We let $u = ax + b$ so $du = a\,dx$ or $dx = du/a$. Then

$$\int (ax+b)^n\,dx = \int \frac{1}{a}u^n\,du = \frac{1}{a(n+1)}u^{n+1} + C = \frac{1}{a(n+1)}(ax+b)^{n+1} + C.$$

□

EXAMPLE 8.1.2 Evaluate $\int \sin(ax+b)\,dx$, assuming that a and b are constants and $a \neq 0$. Again we let $u = ax + b$ so $du = a\,dx$ or $dx = du/a$. Then

$$\int \sin(ax+b)\,dx = \int \frac{1}{a}\sin u\,du = \frac{1}{a}(-\cos u) + C = -\frac{1}{a}\cos(ax+b) + C.$$

□

EXAMPLE 8.1.3 Evaluate $\int_2^4 x\sin(x^2)\,dx$. First we compute the antiderivative, then evaluate the definite integral. Let $u = x^2$ so $du = 2x\,dx$ or $x\,dx = du/2$. Then

$$\int x\sin(x^2)\,dx = \int \frac{1}{2}\sin u\,du = \frac{1}{2}(-\cos u) + C = -\frac{1}{2}\cos(x^2) + C.$$

Now

$$\int_2^4 x\sin(x^2)\,dx = -\frac{1}{2}\cos(x^2)\Big|_2^4 = -\frac{1}{2}\cos(16) + \frac{1}{2}\cos(4).$$

A somewhat neater alternative to this method is to change the original limits to match the variable u. Since $u = x^2$, when $x = 2$, $u = 4$, and when $x = 4$, $u = 16$. So we can do this:

$$\int_2^4 x\sin(x^2)\,dx = \int_4^{16} \frac{1}{2}\sin u\,du = -\frac{1}{2}(\cos u)\Big|_4^{16} = -\frac{1}{2}\cos(16) + \frac{1}{2}\cos(4).$$

An incorrect, and dangerous, alternative is something like this:

$$\int_2^4 x\sin(x^2)\,dx = \int_2^4 \frac{1}{2}\sin u\,du = -\frac{1}{2}\cos(u)\Big|_2^4 = -\frac{1}{2}\cos(x^2)\Big|_2^4 = -\frac{1}{2}\cos(16) + \frac{1}{2}\cos(4).$$

This is incorrect because $\int_2^4 \frac{1}{2}\sin u\,du$ means that u takes on values between 2 and 4, which is wrong. It is dangerous, because it is very easy to get to the point $-\frac{1}{2}\cos(u)\Big|_2^4$ and forget

8.2 Powers of sine and cosine

Functions consisting of products of the sine and cosine can be integrated by using substitution and trigonometric identities. These can sometimes be tedious, but the technique is straightforward. Some examples will suffice to explain the approach.

EXAMPLE 8.2.1 Evaluate $\int \sin^5 x\,dx$. Rewrite the function:

$$\int \sin^5 x\,dx = \int \sin x \sin^4 x\,dx = \int \sin x(\sin^2 x)^2\,dx = \int \sin x(1-\cos^2 x)^2\,dx.$$

Now use $u = \cos x$, $du = -\sin x\,dx$:

$$\begin{aligned}\int \sin x(1-\cos^2 x)^2\,dx &= \int -(1-u^2)^2\,du \\ &= \int -(1-2u^2+u^4)\,du \\ &= -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + C \\ &= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C.\end{aligned}$$

□

EXAMPLE 8.2.2 Evaluate $\int \sin^6 x\,dx$. Use $\sin^2 x = (1-\cos(2x))/2$ to rewrite the function:

$$\begin{aligned}\int \sin^6 x\,dx = \int (\sin^2 x)^3\,dx &= \int \frac{(1-\cos 2x)^3}{8}\,dx \\ &= \frac{1}{8}\int 1 - 3\cos 2x + 3\cos^2 2x - \cos^3 2x\,dx.\end{aligned}$$

Now we have four integrals to evaluate:

$$\int 1\,dx = x$$

and

$$\int -3\cos 2x\,dx = -\frac{3}{2}\sin 2x$$

are easy. The $\cos^3 2x$ integral is like the previous example:

$$\begin{aligned}\int -\cos^3 2x\,dx &= \int -\cos 2x \cos^2 2x\,dx \\ &= \int -\cos 2x(1-\sin^2 2x)\,dx \\ &= \int -\frac{1}{2}(1-u^2)\,du \\ &= -\frac{1}{2}\left(u-\frac{u^3}{3}\right) \\ &= -\frac{1}{2}\left(\sin 2x - \frac{\sin^3 2x}{3}\right).\end{aligned}$$

And finally we use another trigonometric identity, $\cos^2 x = (1+\cos(2x))/2$:

$$\int 3\cos^2 2x\,dx = 3\int \frac{1+\cos 4x}{2}\,dx = \frac{3}{2}\left(x+\frac{\sin 4x}{4}\right).$$

So at long last we get

$$\int \sin^6 x\,dx = \frac{x}{8} - \frac{3}{16}\sin 2x - \frac{1}{16}\left(\sin 2x - \frac{\sin^3 2x}{3}\right) + \frac{3}{16}\left(x+\frac{\sin 4x}{4}\right) + C.$$

□

EXAMPLE 8.2.3 Evaluate $\int \sin^2 x\cos^2 x\,dx$. Use the formulas $\sin^2 x = (1-\cos(2x))/2$ and $\cos^2 x = (1+\cos(2x))/2$ to get:

$$\int \sin^2 x \cos^2 x\,dx = \int \frac{1-\cos(2x)}{2}\cdot\frac{1+\cos(2x)}{2}\,dx.$$

The remainder is left as an exercise. □

Exercises 8.2.

Find the antiderivatives.

1. $\int \sin^2 x\,dx \Rightarrow$

2. $\int \sin^3 x\,dx \Rightarrow$

3. $\int \sin^4 x\,dx \Rightarrow$

4. $\int \cos^2 x\sin^3 x\,dx \Rightarrow$

5. $\int \cos^3 x\,dx \Rightarrow$

6. $\int \sin^2 x\cos^2 x\,dx \Rightarrow$

7. $\int \cos^3 x\sin^2 x\,dx \Rightarrow$

8. $\int \sin x(\cos x)^{3/2}\,dx \Rightarrow$

9. $\int \sec^2 x\csc^2 x\,dx \Rightarrow$

10. $\int \tan^3 x\sec x\,dx \Rightarrow$

8.3 Trigonometric Substitutions

So far we have seen that it sometimes helps to replace a subexpression of a function by a single variable. Occasionally it can help to replace the original variable by something more complicated. This seems like a "reverse" substitution, but it is really no different in principle than ordinary substitution.

EXAMPLE 8.3.1 Evaluate $\int \sqrt{1-x^2}\,dx$. Let $x = \sin u$ so $dx = \cos u\,du$. Then

$$\int \sqrt{1-x^2}\,dx = \int \sqrt{1-\sin^2 u}\cos u\,du = \int \sqrt{\cos^2 u}\cos u\,du.$$

We would like to replace $\sqrt{\cos^2 u}$ by $\cos u$, but this is valid only if $\cos u$ is positive, since $\sqrt{\cos^2 u}$ is positive. Consider again the substitution $x = \sin u$. We could just as well think of this as $u = \arcsin x$. If we do, then by the definition of the arcsine, $-\pi/2 \le u \le \pi/2$, so $\cos u \ge 0$. Then we continue:

$$\begin{aligned}\int \sqrt{\cos^2 u}\cos u\,du &= \int \cos^2 u\,du = \int \frac{1+\cos 2u}{2}\,du = \frac{u}{2} + \frac{\sin 2u}{4} + C\\ &= \frac{\arcsin x}{2} + \frac{\sin(2\arcsin x)}{4} + C.\end{aligned}$$

This is a perfectly good answer, though the term $\sin(2\arcsin x)$ is a bit unpleasant. It is possible to simplify this. Using the identity $\sin 2x = 2\sin x\cos x$, we can write $\sin 2u = 2\sin u\cos u = 2\sin(\arcsin x)\sqrt{1-\sin^2 u} = 2x\sqrt{1-\sin^2(\arcsin x)} = 2x\sqrt{1-x^2}$. Then the full antiderivative is

$$\frac{\arcsin x}{2} + \frac{2x\sqrt{1-x^2}}{4} = \frac{\arcsin x}{2} + \frac{x\sqrt{1-x^2}}{2} + C.$$

□

This type of substitution is usually indicated when the function you wish to integrate contains a polynomial expression that might allow you to use the fundamental identity $\sin^2 x + \cos^2 x = 1$ in one of three forms:

$$\cos^2 x = 1 - \sin^2 x \qquad \sec^2 x = 1 + \tan^2 x \qquad \tan^2 x = \sec^2 x - 1.$$

If your function contains $1-x^2$, as in the example above, try $x = \sin u$; if it contains $1+x^2$ try $x = \tan u$; and if it contains x^2-1, try $x = \sec u$. Sometimes you will need to try something a bit different to handle constants other than one.

Exercises 8.3.

Find the antiderivatives.

1. $\int \csc x \, dx \Rightarrow$

2. $\int \csc^3 x \, dx \Rightarrow$

3. $\int \sqrt{x^2 - 1} \, dx \Rightarrow$

4. $\int \sqrt{9 + 4x^2} \, dx \Rightarrow$

5. $\int x\sqrt{1 - x^2} \, dx \Rightarrow$

6. $\int x^2\sqrt{1 - x^2} \, dx \Rightarrow$

7. $\int \frac{1}{\sqrt{1 + x^2}} \, dx \Rightarrow$

8. $\int \sqrt{x^2 + 2x} \, dx \Rightarrow$

9. $\int \frac{1}{x^2(1 + x^2)} \, dx \Rightarrow$

10. $\int \frac{x^2}{\sqrt{4 - x^2}} \, dx \Rightarrow$

11. $\int \frac{\sqrt{x}}{\sqrt{1 - x}} \, dx \Rightarrow$

12. $\int \frac{x^3}{\sqrt{4x^2 - 1}} \, dx \Rightarrow$

13. Compute $\int \sqrt{x^2 + 1} \, dx$. (Hint: make the substitution $x = \sinh(u)$ and then use exercise 6 in section 4.11.)

14. Fix $t > 0$. The shaded region in the left-hand graph in figure 4.11.2 is bounded by $y = x \tanh t$, $y = 0$, and $x^2 - y^2 = 1$. Prove that twice the area of this region is t, as claimed in section 4.11.

8.4 Integration by Parts

We have already seen that recognizing the product rule can be useful, when we noticed that

$$\int \sec^3 u + \sec u \tan^2 u \, du = \sec u \tan u.$$

As with substitution, we do not have to rely on insight or cleverness to discover such antiderivatives; there is a technique that will often help to uncover the product rule.

Start with the product rule:

$$\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x).$$

We can rewrite this as

$$f(x)g(x) = \int f'(x)g(x) \, dx + \int f(x)g'(x) \, dx,$$

and then

$$\int f(x)g'(x) \, dx = f(x)g(x) - \int f'(x)g(x) \, dx.$$

This may not seem particularly useful at first glance, but it turns out that in many cases we have an integral of the form

$$\int f(x)g'(x)\,dx$$

but that

$$\int f'(x)g(x)\,dx$$

is easier. This technique for turning one integral into another is called **integration by parts**, and is usually written in more compact form. If we let $u = f(x)$ and $v = g(x)$ then $du = f'(x)\,dx$ and $dv = g'(x)\,dx$ and

$$\int u\,dv = uv - \int v\,du.$$

To use this technique we need to identify likely candidates for $u = f(x)$ and $dv = g'(x)\,dx$.

EXAMPLE 8.4.1 Evaluate $\int x \ln x\,dx$. Let $u = \ln x$ so $du = 1/x\,dx$. Then we must let $dv = x\,dx$ so $v = x^2/2$ and

$$\int x \ln x\,dx = \frac{x^2 \ln x}{2} - \int \frac{x^2}{2}\frac{1}{x}\,dx = \frac{x^2 \ln x}{2} - \int \frac{x}{2}\,dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C.$$

□

EXAMPLE 8.4.2 Evaluate $\int x \sin x\,dx$. Let $u = x$ so $du = dx$. Then we must let $dv = \sin x\,dx$ so $v = -\cos x$ and

$$\int x \sin x\,dx = -x\cos x - \int -\cos x\,dx = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C.$$

□

EXAMPLE 8.4.3 Evaluate $\int \sec^3 x\,dx$. Of course we already know the answer to this, but we needed to be clever to discover it. Here we'll use the new technique to discover the antiderivative. Let $u = \sec x$ and $dv = \sec^2 x\,dx$. Then $du = \sec x \tan x\,dx$ and $v = \tan x$

Exercises 8.4.

Find the antiderivatives.

1. $\int x \cos x \, dx \Rightarrow$

2. $\int x^2 \cos x \, dx \Rightarrow$

3. $\int x e^x \, dx \Rightarrow$

4. $\int x e^{x^2} \, dx \Rightarrow$

5. $\int \sin^2 x \, dx \Rightarrow$

6. $\int \ln x \, dx \Rightarrow$

7. $\int x \arctan x \, dx \Rightarrow$

8. $\int x^3 \sin x \, dx \Rightarrow$

9. $\int x^3 \cos x \, dx \Rightarrow$

10. $\int x \sin^2 x \, dx \Rightarrow$

11. $\int x \sin x \cos x \, dx \Rightarrow$

12. $\int \arctan(\sqrt{x}) \, dx \Rightarrow$

13. $\int \sin(\sqrt{x}) \, dx \Rightarrow$

14. $\int \sec^2 x \csc^2 x \, dx \Rightarrow$

8.5 Rational Functions

A **rational function** is a fraction with polynomials in the numerator and denominator. For example,

$$\frac{x^3}{x^2+x-6}, \qquad \frac{1}{(x-3)^2}, \qquad \frac{x^2+1}{x^2-1},$$

are all rational functions of x. There is a general technique called "partial fractions" that, in principle, allows us to integrate any rational function. The algebraic steps in the technique are rather cumbersome if the polynomial in the denominator has degree more than 2, and the technique requires that we factor the denominator, something that is not always possible. However, in practice one does not often run across rational functions with high degree polynomials in the denominator for which one has to find the antiderivative function. So we shall explain how to find the antiderivative of a rational function only when the denominator is a quadratic polynomial ax^2+bx+c.

We should mention a special type of rational function that we already know how to integrate: If the denominator has the form $(ax+b)^n$, the substitution $u = ax+b$ will always work. The denominator becomes u^n, and each x in the numerator is replaced by $(u-b)/a$, and $dx = du/a$. While it may be tedious to complete the integration if the numerator has high degree, it is merely a matter of algebra.

EXAMPLE 8.5.1 Find $\displaystyle\int \frac{x^3}{(3-2x)^5}\,dx$. Using the substitution $u = 3 - 2x$ we get

$$\begin{aligned}
\int \frac{x^3}{(3-2x)^5}\,dx &= \frac{1}{-2}\int \frac{\left(\frac{u-3}{-2}\right)^3}{u^5}\,du = \frac{1}{16}\int \frac{u^3 - 9u^2 + 27u - 27}{u^5}\,du \\
&= \frac{1}{16}\int u^{-2} - 9u^{-3} + 27u^{-4} - 27u^{-5}\,du \\
&= \frac{1}{16}\left(\frac{u^{-1}}{-1} - \frac{9u^{-2}}{-2} + \frac{27u^{-3}}{-3} - \frac{27u^{-4}}{-4}\right) + C \\
&= \frac{1}{16}\left(\frac{(3-2x)^{-1}}{-1} - \frac{9(3-2x)^{-2}}{-2} + \frac{27(3-2x)^{-3}}{-3} - \frac{27(3-2x)^{-4}}{-4}\right) + C \\
&= -\frac{1}{16(3-2x)} + \frac{9}{32(3-2x)^2} - \frac{9}{16(3-2x)^3} + \frac{27}{64(3-2x)^4} + C
\end{aligned}$$

□

We now proceed to the case in which the denominator is a quadratic polynomial. We can always factor out the coefficient of x^2 and put it outside the integral, so we can assume that the denominator has the form $x^2 + bx + c$. There are three possible cases, depending on how the quadratic factors: either $x^2 + bx + c = (x-r)(x-s)$, $x^2 + bx + c = (x-r)^2$, or it doesn't factor. We can use the quadratic formula to decide which of these we have, and to factor the quadratic if it is possible.

EXAMPLE 8.5.2 Determine whether $x^2 + x + 1$ factors, and factor it if possible. The quadratic formula tells us that $x^2 + x + 1 = 0$ when

$$x = \frac{-1 \pm \sqrt{1-4}}{2}.$$

Since there is no square root of -3, this quadratic does not factor. □

EXAMPLE 8.5.3 Determine whether $x^2 - x - 1$ factors, and factor it if possible. The quadratic formula tells us that $x^2 - x - 1 = 0$ when

$$x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

Therefore

$$x^2 - x - 1 = \left(x - \frac{1+\sqrt{5}}{2}\right)\left(x - \frac{1-\sqrt{5}}{2}\right).$$

□

Exercises 8.5.

Find the antiderivatives.

1. $\displaystyle\int \frac{1}{4-x^2}\,dx \Rightarrow$

2. $\displaystyle\int \frac{x^4}{4-x^2}\,dx \Rightarrow$

3. $\displaystyle\int \frac{1}{x^2+10x+25}\,dx \Rightarrow$

4. $\displaystyle\int \frac{x^2}{4-x^2}\,dx \Rightarrow$

5. $\displaystyle\int \frac{x^4}{4+x^2}\,dx \Rightarrow$

6. $\displaystyle\int \frac{1}{x^2+10x+29}\,dx \Rightarrow$

7. $\displaystyle\int \frac{x^3}{4+x^2}\,dx \Rightarrow$

8. $\displaystyle\int \frac{1}{x^2+10x+21}\,dx \Rightarrow$

9. $\displaystyle\int \frac{1}{2x^2-x-3}\,dx \Rightarrow$

10. $\displaystyle\int \frac{1}{x^2+3x}\,dx \Rightarrow$

8.6 Numerical Integration

We have now seen some of the most generally useful methods for discovering antiderivatives, and there are others. Unfortunately, some functions have no simple antiderivatives; in such cases if the value of a definite integral is needed it will have to be approximated. We will see two methods that work reasonably well and yet are fairly simple; in some cases more sophisticated techniques will be needed.

Of course, we already know one way to approximate an integral: if we think of the integral as computing an area, we can add up the areas of some rectangles. While this is quite simple, it is usually the case that a large number of rectangles is needed to get acceptable accuracy. A similar approach is much better: we approximate the area under a curve over a small interval as the area of a trapezoid. In figure 8.6.1 we see an area under a curve approximated by rectangles and by trapezoids; it is apparent that the trapezoids give a substantially better approximation on each subinterval.

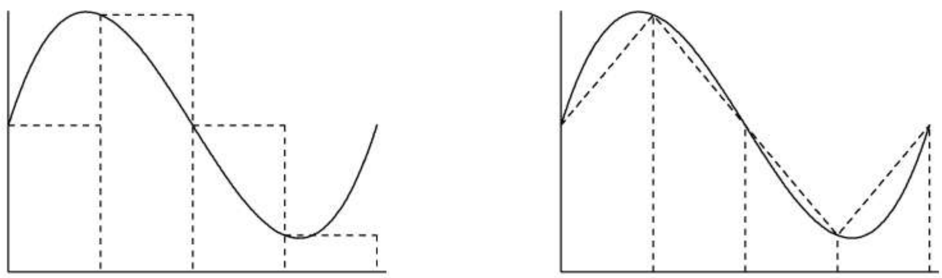

Figure 8.6.1 Approximating an area with rectangles and with trapezoids. (AP)

As with rectangles, we divide the interval into n equal subintervals of length Δx. A typical trapezoid is pictured in figure 8.6.2; it has area $\dfrac{f(x_i)+f(x_{i+1})}{2}\Delta x$. If we add up the areas of all trapezoids we get

$$\frac{f(x_0)+f(x_1)}{2}\Delta x + \frac{f(x_1)+f(x_2)}{2}\Delta x + \cdots + \frac{f(x_{n-1})+f(x_n)}{2}\Delta x =$$
$$\left(\frac{f(x_0)}{2} + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{f(x_n)}{2}\right)\Delta x.$$

This is usually known as the **Trapezoid Rule**. For a modest number of subintervals this is not too difficult to do with a calculator; a computer can easily do many subintervals.

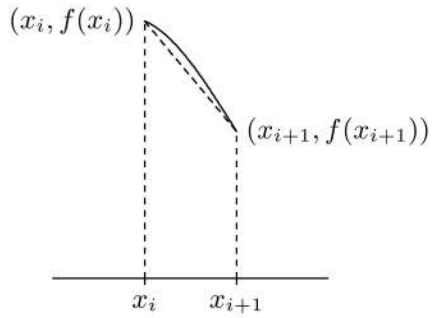

Figure 8.6.2 A single trapezoid.

In practice, an approximation is useful only if we know how accurate it is; for example, we might need a particular value accurate to three decimal places. When we compute a particular approximation to an integral, the error is the difference between the approximation and the true value of the integral. For any approximation technique, we need an **error estimate**, a value that is guaranteed to be larger than the actual error. If A is an approximation and E is the associated error estimate, then we know that the true value of the integral is between $A - E$ and $A + E$. In the case of our approximation of the integral, we want $E = E(\Delta x)$ to be a function of Δx that gets small rapidly as Δx gets small. Fortunately, for many functions, there is such an error estimate associated with the trapezoid approximation.

THEOREM 8.6.1 Suppose f has a second derivative f'' everywhere on the interval $[a, b]$, and $|f''(x)| \le M$ for all x in the interval. With $\Delta x = (b-a)/n$, an error estimate for the trapezoid approximation is

$$E(\Delta x) = \frac{b-a}{12} M(\Delta x)^2 = \frac{(b-a)^3}{12n^2} M.$$

■

Let's see how we can use this.

EXAMPLE 8.6.2 Approximate $\int_0^1 e^{-x^2}\,dx$ to two decimal places. The second derivative of $f = e^{-x^2}$ is $(4x^2-2)e^{-x^2}$, and it is not hard to see that on $[0,1]$, $|(4x^2-2)e^{-x^2}| \le 2$. We begin by estimating the number of subintervals we are likely to need. To get two decimal places of accuracy, we will certainly need $E(\Delta x) < 0.005$ or

$$\begin{aligned} \frac{1}{12}(2)\frac{1}{n^2} &< 0.005 \\ \frac{1}{6}(200) &< n^2 \\ 5.77 \approx \sqrt{\frac{100}{3}} &< n \end{aligned}$$

With $n = 6$, the error estimate is thus $1/6^3 < 0.0047$. We compute the trapezoid approximation for six intervals:

$$\left(\frac{f(0)}{2} + f(1/6) + f(2/6) + \cdots + f(5/6) + \frac{f(1)}{2}\right)\frac{1}{6} \approx 0.74512.$$

So the true value of the integral is between $0.74512 - 0.0047 = 0.74042$ and $0.74512 + 0.0047 = 0.74982$. Unfortunately, the first rounds to 0.74 and the second rounds to 0.75, so we can't be sure of the correct value in the second decimal place; we need to pick a larger n. As it turns out, we need to go to $n = 12$ to get two bounds that both round to the same value, which turns out to be 0.75. For comparison, using 12 rectangles to approximate the area gives 0.7727, which is considerably less accurate than the approximation using six trapezoids.

In practice it generally pays to start by requiring better than the maximum possible error; for example, we might have initially required $E(\Delta x) < 0.001$, or

$$\begin{aligned} \frac{1}{12}(2)\frac{1}{n^2} &< 0.001 \\ \frac{1}{6}(1000) &< n^2 \\ 12.91 \approx \sqrt{\frac{500}{3}} &< n \end{aligned}$$

Had we immediately tried $n = 13$ this would have given us the desired answer. □

The trapezoid approximation works well, especially compared to rectangles, because the tops of the trapezoids form a reasonably good approximation to the curve when Δx is fairly small. We can extend this idea: what if we try to approximate the curve more closely, by using something other than a straight line? The obvious candidate is a parabola: if we can approximate a short piece of the curve with a parabola with equation $y = ax^2+bx+c$, we can easily compute the area under the parabola.

There are an infinite number of parabolas through any two given points, but only one through three given points. If we find a parabola through three consecutive points $(x_i, f(x_i))$, $(x_{i+1}, f(x_{i+1}))$, $(x_{i+2}, f(x_{i+2}))$ on the curve, it should be quite close to the curve over the whole interval $[x_i, x_{i+2}]$, as in figure 8.6.3. If we divide the interval $[a, b]$ into an even number of subintervals, we can then approximate the curve by a sequence of parabolas, each covering two of the subintervals. For this to be practical, we would like a simple formula for the area under one parabola, namely, the parabola through $(x_i, f(x_i))$, $(x_{i+1}, f(x_{i+1}))$, and $(x_{i+2}, f(x_{i+2}))$. That is, we should attempt to write down the parabola $y = ax^2 + bx + c$ through these points and then integrate it, and hope that the result is fairly simple. Although the algebra involved is messy, this turns out to be possible. The algebra is well within the capability of a good computer algebra system like Sage, so we will present the result without all of the algebra; you can see how to do it in this Sage worksheet.

To find the parabola, we solve these three equations for a, b, and c:

$$\begin{aligned}
f(x_i) &= a(x_{i+1} - \Delta x)^2 + b(x_{i+1} - \Delta x) + c \\
f(x_{i+1}) &= a(x_{i+1})^2 + b(x_{i+1}) + c \\
f(x_{i+2}) &= a(x_{i+1} + \Delta x)^2 + b(x_{i+1} + \Delta x) + c
\end{aligned}$$

Not surprisingly, the solutions turn out to be quite messy. Nevertheless, Sage can easily compute and simplify the integral to get

$$\int_{x_{i+1}-\Delta x}^{x_{i+1}+\Delta x} ax^2 + bx + c\,dx = \frac{\Delta x}{3}(f(x_i) + 4f(x_{i+1}) + f(x_{i+2})).$$

Now the sum of the areas under all parabolas is

$$\begin{gathered}
\frac{\Delta x}{3}(f(x_0)+4f(x_1)+f(x_2)+f(x_2)+4f(x_3)+f(x_4)+\cdots+f(x_{n-2})+4f(x_{n-1})+f(x_n)) = \\
\frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)).
\end{gathered}$$

This is just slightly more complicated than the formula for trapezoids; we need to remember the alternating 2 and 4 coefficients; note that n must be even for this to make sense. This approximation technique is referred to as **Simpson's Rule**.

As with the trapezoid method, this is useful only with an error estimate:

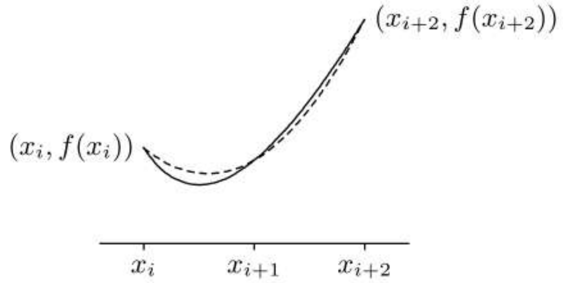

Figure 8.6.3 A parabola (dashed) approximating a curve (solid). (AP)

THEOREM 8.6.3 Suppose f has a fourth derivative $f^{(4)}$ everywhere on the interval $[a,b]$, and $|f^{(4)}(x)| \le M$ for all x in the interval. With $\Delta x = (b-a)/n$, an error estimate for Simpson's approximation is

$$E(\Delta x) = \frac{b-a}{180} M(\Delta x)^4 = \frac{(b-a)^5}{180n^4} M.$$

$\blacksquare$

EXAMPLE 8.6.4 Let us again approximate $\int_0^1 e^{-x^2}\,dx$ to two decimal places. The fourth derivative of $f = e^{-x^2}$ is $(16x^4 - 48x^2 + 12)e^{-x^2}$; on $[0,1]$ this is at most 12 in absolute value. We begin by estimating the number of subintervals we are likely to need. To get two decimal places of accuracy, we will certainly need $E(\Delta x) < 0.005$, but taking a cue from our earlier example, let's require $E(\Delta x) < 0.001$:

$$\begin{aligned} \frac{1}{180}(12)\frac{1}{n^4} &< 0.001 \\ \frac{200}{3} &< n^4 \\ 2.86 \approx \sqrt[4]{\frac{200}{3}} &< n \end{aligned}$$

So we try $n = 4$, since we need an even number of subintervals. Then the error estimate is $12/180/4^4 < 0.0003$ and the approximation is

$$(f(0) + 4f(1/4) + 2f(1/2) + 4f(3/4) + f(1))\frac{1}{3 \cdot 4} \approx 0.746855.$$

So the true value of the integral is between $0.746855 - 0.0003 = 0.746555$ and $0.746855 + 0.0003 = 0.7471555$, both of which round to 0.75. $\square$