List of Derivative Rules

Below is a list of all the derivative rules we went over in class.

- **Constant Rule:** $f(x) = c$ then $f'(x) = 0$

- **Constant Multiple Rule:** $g(x) = c \cdot f(x)$ then $g'(x) = c \cdot f'(x)$

- **Power Rule:** $f(x) = x^n$ then $f'(x) = nx^{n-1}$

- **Sum and Difference Rule:** $h(x) = f(x) \pm g(x)$ then $h'(x) = f'(x) \pm g'(x)$

- **Product Rule:** $h(x) = f(x)g(x)$ then $h'(x) = f'(x)g(x) + f(x)g'(x)$

- **Quotient Rule:** $h(x) = \dfrac{f(x)}{g(x)}$ then $h'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

- **Chain Rule:** $h(x) = f(g(x))$ then $h'(x) = f'(g(x))g'(x)$

- **Trig Derivatives:**
  - $f(x) = \sin(x)$ then $f'(x) = \cos(x)$
  - $f(x) = \cos(x)$ then $f'(x) = -\sin(x)$
  - $f(x) = \tan(x)$ then $f'(x) = \sec^2(x)$
  - $f(x) = \sec(x)$ then $f'(x) = \sec(x)\tan(x)$
  - $f(x) = \cot(x)$ then $f'(x) = -\csc^2(x)$
  - $f(x) = \csc(x)$ then $f'(x) = -\csc(x)\cot(x)$

- **Exponential Derivatives**
  - $f(x) = a^x$ then $f'(x) = \ln(a)a^x$
  - $f(x) = e^x$ then $f'(x) = e^x$
  - $f(x) = a^{g(x)}$ then $f'(x) = \ln(a)a^{g(x)}g'(x)$
  - $f(x) = e^{g(x)}$ then $f'(x) = e^{g(x)}g'(x)$

- **Logarithm Derivatives**
  - $f(x) = \log_a(x)$ then $f'(x) = \dfrac{1}{\ln(a)x}$
  - $f(x) = \ln(x)$ then $f'(x) = \dfrac{1}{x}$
  - $f(x) = \log_a(g(x))$ then $f'(x) = \dfrac{g'(x)}{\ln(a)g(x)}$
  - $f(x) = \ln(g(x))$ then $f'(x) = \dfrac{g'(x)}{g(x)}$

THE DERIVATIVE

Summary

The slope concept usually pertains to straight lines. The definition of a straight line is a function for which the slope is constant. In other words, no matter which point we are looking at, the inclination of a line remains the same. When a function is non-linear, its slope may vary from one point to the next. We must therefore introduce the notion of derivate which allows us to obtain the slope at all points of these non-linear functions.

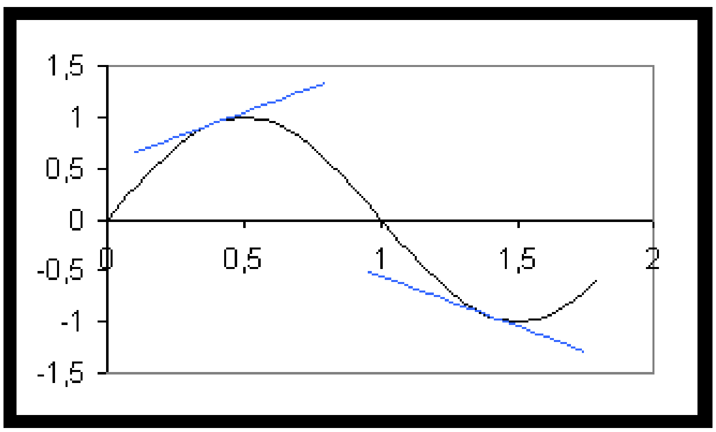
Definition

The derivative of a function f at a point x, written $f\ '(x)$, is given by:

$$f^{'}(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

if this limit exists.

Graphically, the derivative of a function corresponds to **the slope of its tangent line at one specific point.** The following illustration allows us to visualise the tangent line (in blue) of a given function at two distinct points. Note that the slope of the tangent line varies from one point to the next. The value of the derivative of a function therefore depends on the point in which we decide to evaluate it. By abuse of language, we often speak of the slope of the function instead of the slope of its tangent line.



Notation

Here, we represent the derivative of a function by a prime symbol. For example, writing $f'(x)$ represents the derivative of the function f evaluated at point x. Similarly, writing $(3x\ +\ 2)'$ indicates we are carrying out the derivative of the function $3x\ +\ 2$. The prime symbol disappears as soon as the derivative has been calculated.

1. Derivatives of usual functions

Below you will find a list of the most important derivatives. Although these formulas can be formally proven, we will only state them here. We recommend you learn them by heart.

1.1. The constant function

Let $f(x) = k$, where k is some real constant. Then

$$f'(x) = (k)' = 0$$

Examples

$(8)' = 0$

$(-5)' = 0$

$(0{,}2321)' = 0$

1.2. The identity function $f(x) = x$

Let $f(x) = x$, the identity function of x. Then

$$f'(x) = (x)' = 1$$

1.3. A function of the form x^n

Let $f(x) = x^n$, a function of x, and n a real constant. We have

$$f'(x) = (x^n)' = n\,x^{n-1}$$

Examples

$(x^4)' = 4\,x^{4-1} = 4\,x^3$

$(x^{1/2})' = 1/2\,x^{\frac{1}{2}-1} = 1/2\,x^{-1/2}$

$(\boldsymbol{x}^{-2})' = -2\boldsymbol{x}^{-2-1} = -2\,\boldsymbol{x}^{-3}$

$\left(x^{-\frac{1}{3}}\right)' = \left(-\frac{1}{3}\right)\boldsymbol{x}^{-\frac{1}{3}-1} = \left(-\frac{1}{3}\right)\boldsymbol{x}^{-\frac{4}{3}}$

Notes on the $(\boldsymbol{x}^{\boldsymbol{n}})' = \boldsymbol{n}\,\boldsymbol{x}^{\boldsymbol{n}-1}$ rule:

- The rule mentioned above applies to all types of exponents (natural, whole, fractional). It is however essential that this exponent is constant. Another rule will need to be studied for exponential functions (of type $\boldsymbol{a^x}$).
- The identity function is a particular case of the functions of form $\boldsymbol{x^n}$ (with n = 1) and follows the same derivation rule : $(x)' = (\boldsymbol{x^1})' = 1\,x^{1-1} = 1\,x^0 = 1$
- It is often the case that a function satisfies this form but requires a bit of reformulation before proceeding to the derivative. It is the case of roots (square, cubic, etc.) representing fractional exponents.

Examples

$$\sqrt{x} = x^{1/2} \rightarrow \left(\sqrt{x}\right)' = \left(x^{\frac{1}{2}}\right)' = \frac{1}{2}\,x^{(\frac{1}{2}-1)} = \frac{1}{2}x^{-1/2}$$

$$\sqrt[3]{x}=x^{1/3} \quad \rightarrow(\sqrt[3]{x})'=(x^{1/3})' = \frac{1}{3}\,x^{(\frac{1}{3}-1)} = \frac{1}{3}x^{-2/3}$$

- Beware of rational functions. For example, the function $\frac{1}{x^4}$ cannot be differentiated in the same manner as the function x^4. You must first reformulate the function so that "x" is a numerator, forcing us to change its exponent's sign.

  Examples

$$\frac{1}{x^4} = x^{-4} \qquad \rightarrow \qquad \left(\frac{1}{x^4}\right)' = (x^{-4})' = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5}$$

$$\frac{1}{x^{3/2}} = x^{-3/2} \quad \rightarrow \qquad (\frac{1}{x^{3/2}})' = (x^{-\frac{3}{2}})' = -\frac{3}{2}x^{-\frac{3}{2}-1} = -\frac{3}{2}x^{-\frac{5}{2}} = -\frac{3}{2x^{\frac{5}{2}}}$$

- Finally, a derivate can greatly be simplified by proceeding first, if possible, to an algebraic simplification.

  Example

$$\frac{x^2}{x^3\sqrt{x}} = \frac{x^2}{x^3x^{1/2}} = x^{2-3-1/2} = x^{-3/2}$$

  That is how the derivative of $\frac{x^2}{x^3\sqrt{x}}$ is greatly facilitated by carrying out the derivative of $x^{-3/2}$.

$$\left(\frac{x^2}{x^3\sqrt{x}}\right)' = \left(x^{-\frac{3}{2}}\right)' = -\frac{3}{2}x^{-\frac{3}{2}-1} = -\frac{3}{2}x^{-\frac{5}{2}}$$

1.4. An exponential function (of the form a^x with $a > 0$):

It is very easy to confuse the exponential function a^x with a function of the form x^n since both have exponents. They are, however, quite different. In an exponential function, the exponent is a variable.

Given the exponential function $f(x) = a^x$ where $a > 0$. We have

$$f'(x) = (a^x)' = a^x ln(a)$$

Examples

$(3^x)' = 3^x ln(3)$

$$\left(\left(\frac{1}{2}\right)^x\right)' = \left(\frac{1}{2}\right)^x ln\left(\frac{1}{2}\right)$$

1.5. The function e^x

Let the function $f(x) = e^x$. Then

$$f'(x) = (e^x)' = e^x$$

Here is a special case of the previous rule since the function $f(x) = \boldsymbol{e^x}$ is an exponential function with $a = e$.

Therefore $f'(x) = (\boldsymbol{e^x})' = \boldsymbol{e^x} ln(e) = \boldsymbol{e^x}(1) = \boldsymbol{e^x}$

1.6. The logarithmic function $ln\, x$

Given the logarithmic function $f(x) = ln\, x$. We have

$$f'(x) = (ln\, x)' = \frac{1}{x}$$

2. Basic derivation rules

We will generally have to confront not only the functions presented above, but also combinations of these : multiples, sums, products, quotients and composite functions. We therefore need to present the rules that allow us to derive these more complex cases.

2.1. Constant multiples

Let k be a real constant and $f(x)$ any given function. Then

$$(k\, f(x))' = k\, f'(x)$$

In other words, we can forget the constant which will remain unchanged and only derive the function of x.

Examples

$(4x^2)' = 4(x^2)' = 4(2x) = 8x$

$(-5e^x)' = -5(e^x) = -5e^x$

$(12lnx)' = 12(lnx)' = 12\left(\frac{1}{x}\right) = \frac{12}{x}$

2.2. Addition and subtraction of functions

Let $f(x)$ and $g(x)$ be two functions. Then

$$\big(f(x) \pm g(x)\big)' = f'(x) \pm g'(x)$$

When we derive a sum or a subtraction of two functions, the previous rule states that the functions can be individually derived without changing the operation linking them.

Example 1

$(e^x + x^5)' = (e^x)' + (x^5)' = e^x + 5x^4$

Example 2

$$\left(lnx - \frac{1}{x^2} + 8\right)' = (lnx)' - (x^{-2})' + (8)'$$

$$= \frac{1}{x} - (-2x^{-3}) + 0$$

$$= \frac{1}{x} + \frac{2}{x^3}$$

Example 3

$$\left(3\sqrt{x} + 2x - \frac{8}{x}\right)' = \left(3x^{1/2}\right)' + (2x)' - (8x^{-1})'$$

$$= 3\left(x^{\frac{1}{2}}\right)' + 2(x)' - 8(x^{-1})'$$

$$= 3\left(\frac{1}{2}x^{-\frac{1}{2}}\right) + 2(1) - 8(-x^{-2})$$

$$= \frac{3}{2}x^{-\frac{1}{2}} + 2 + 8x^{-2}$$

2.3. Product rule

Let $f(x)$ and $g(x)$ be two functions. Then the derivate of the product

$$(f(x)\, g(x)\,)' = f'(x)\, g(x) + f(x)\, g'(x)$$

We must follow this rule religiously and not succumb to the temptation of writing $\left(f(x)g(x)\right)' = f'(x)g'(x)$; a faulty statement.

Example 1

$$(x^3e^x)' = (x^3)'e^x + x^3(e^x)'$$

$$= 3x^2e^x + x^3e^x$$

Example 2

$$(3\sqrt{x}lnx)' = (3\sqrt{x})'lnx + 3\sqrt{x}(lnx)'$$

$$= 3\left(x^{\frac{1}{2}}\right)' lnx + 3\sqrt{x}(lnx)'$$

$$= 3\left(\frac{1}{2}x^{\frac{1}{2}-1}\right) lnx + 3\sqrt{x}\frac{1}{x}$$

$$= \frac{3}{2}x^{-\frac{1}{2}}lnx + 3x^{-\frac{1}{2}}$$

2.4. Quotient rule

Let $f(x)$ and $g(x)$ be two functions. Then the derivative of the quotient

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Just as with the product rule, the quotient rule must religiously be respected.

Example 1

$$\left(\frac{x^3}{e^x}\right)' = \frac{(x^3)'e^x - x^3(e^x)'}{(e^x)^2}$$

$$= \frac{3x^2e^x - x^3e^x}{(e^x)^2}$$

$$= \frac{x^2e^x(3-x)}{(e^x)^2}$$

$$= \frac{x^2(3-x)}{e^x}$$

Example 2

$$\left(\frac{3\sqrt{x}}{lnx}\right)' = \frac{\left(3\sqrt{x}\right)' lnx - 3\sqrt{x}(lnx)'}{(lnx)^2} = \frac{3\left(x^{\frac{1}{2}}\right)' lnx - 3\sqrt{x}(lnx)'}{(lnx)^2}$$

$$= \frac{3\left(\frac{1}{2}x^{\frac{1}{2}-1}\right)lnx - 3\sqrt{x}\frac{1}{x}}{(lnx)^2} = \frac{3x^{-\frac{1}{2}}lnx - 6x^{-\frac{1}{2}}}{2(lnx)^2}$$

$$= \frac{3x^{-\frac{1}{2}}(lnx-2)}{2(lnx)^2}$$

3. Derivative of composite functions

A composite function is a function with form $f(g(x))$.

How do we recognize a composite function?

A composite function is in fact a function that contains another function. If you have a function that can be broken down into many parts, where each part is in itself a function and where these parts are not linked by addition, subtraction, product or division, you usually have a composite function.

For example, the function $f(x) = e^{x^3}$ is a composite function. We can write it as $f(g(x))$ where $g(x) = x^3$.

Unlike the function *f(x)* = x^3e^x which is not a composite function. It is only the product of functions.

Here are a few examples of composite functions:

- $f(x) = \ln(x^2 + 2x + 1)$

  We can write this function as $f(g(x)) = ln\,(g(x))$ where $g(x) = x^2 + 2x + 1$

- $f(x) = e^{3x-5}$

  We can write this function as $f(g(x)) = e^{g(x)}$ where $g(x) = 3x - 5$

- $f(x) = (ln(x) + 3x - e^x)^4$

  We can write this function as $f(g(x)) = (g(x))^4$ where

$$g(x) = ln(x) + 3x - e^x$$

3.1. The chain rule

Let f and g be two functions. Then the derivative of the composite function $f(g(x))$ is

$$(f(g(x)))' = f'(g(x))\,g'(x)$$

or $(f(u))' = f'(u)\,u'$, where $u = g(x)$

The chain rule states that when we derive a composite function, we must first derive the external function (the one which contains all others) by keeping the internal function as is and then multiplying it with the derivative of the internal function. If the latter is also composite, the process is repeated. Be alert as the internal function could also be a product, a quotient, ...!

3.2. Chain derivatives of usual functions

In concrete terms, we can express the chain rule for the most important functions as follows :

If $u = g(x)$ represents any given function of *x*

- $(u^n)' = n\, u^{n-1}u'$
- $(a^u)' = a^u\, ln(a)u'$
- $(e^u)' = e^u u'$
- $(ln\, u)' = \frac{1}{u} \times u'$

Examples

$$[\ln(x^2+2x+1)]' = \frac{1}{x^2+2x+1}(x^2+2x+1)'$$

$$= \frac{1}{x^2+2x+1}(2x+2)$$

$$= \frac{2x+2}{x^2+2x+1}$$

$$[(lnx+3x-e^x)^4]' = 4(lnx+3x-e^x)^3(lnx+3x-e^x)'$$

$$= 4(lnx+3x-e^x)^3(\frac{1}{x}+3-e^x)$$

$$(e^{\,3x-5})' = e^{\,3x-5}(3x-5)'$$

$$= e^{\,3x-5}.3$$

Below are additional examples that demonstrate that many rules may be necessary for one derivative.

Example 1

$$([\ln(3x^3 - 9e^x]^3)' = 3[\ln(3x^3 - 9e^x)]^2.[\ln(3x^3 - 9e^x)]'$$

$$= 3[\ln(3x^3 - 9e^x)]^2.\frac{1}{3x^3 - 9e^x}.(3x^3 - 9e^x)'$$

$$= 3[\ln(3x^3 - 9e^x)]^2.\frac{1}{3x^3 - 9e^x}.(9x^2 - 9e^x)$$

Example 2

$$[e^{xlnx}]' = e^{xlnx}.(xlnx)'$$

$$= e^{xlnx}[(x)'lnx + x(lnx)']$$ (product rule)

$$= e^{xlnx}\left(1.lnx + x.\frac{1}{x}\right)$$

$$= e^{xlnx}(lnx + 1)$$

Example 3

$$\left[\frac{x^2+1}{(2x+1)^{\frac{1}{2}}}\right]' = \frac{(x^2+1)'.(2x+1)^{\frac{1}{2}} - (x^2+1).\left[(2x+1)^{\frac{1}{2}}\right]'}{\left[(2x+1)^{\frac{1}{2}}\right]^2}$$ (quotient rule)

$$= \frac{2x.(2x+1)^{\frac{1}{2}} - (x^2+1).\left[(2x+1)^{\frac{1}{2}}\right]'}{2x+1}$$

$$= \frac{2x.(2x+1)^{\frac{1}{2}} - (x^2+1).\frac{1}{2}(2x+1)^{-\frac{1}{2}}.(2x+1)'}{2x+1}$$

$$= \frac{2x.(2x+1)^{\frac{1}{2}} - (x^2+1).\frac{1}{2}(2x+1)^{-\frac{1}{2}}.2}{2x+1}$$

$$= \frac{2x.(2x+1)^{\frac{1}{2}} - (x^2+1).(2x+1)^{-\frac{1}{2}}}{2x+1}$$

4. Evaluation of the slope of the tangent at one point

As we mentioned at the very beginning, the derivative function $f'(x)$ represents the slope of the tangent line at $f(x)$ at all points x. We will often have to evaluate this slope at a specific point.

To evaluate the slope of the tangent of the function $f(x)$ at the point $x = 1$ for example, we most certainly cannot calculate $f(1)$ and derive this value... we would then obtain a slope of 0 since $f(1)$ is a constant. Instead, we need to find the derivative $f'(x)$ at all points and then evaluate it at $x = 1$. We will use the notation $f'(a)$ to represent the derivative of the function f evaluated at the point $x = a$.

Example

Evaluate the slope of the function $f(x) = x^3e^x$ at the point $x = 0$.

We are looking to calculate $f'(0)$. We must first find the derivative at all points, $f'(x)$. Yet earlier we demonstrated that $f'(x) = (x^3e^x)' = 3x^2e^x + x^3e^x$

Evaluated at $x = 0$, we obtain $f'(0) = 3.0^2e^0 + 0^3e^0 = 0$. The slope of the function $f(x) = x^3e^x$ is therefore zero at $x = 0$. We will let you verify that this is not the case at point $x = 1$.

5. Increasing and decreasing functions

There is a direct relationship between the growth and decline of a function and the value of its derivative at one point.

- If the value of the derivative is negative at a given point, this indicates that the function is decreasing at that point.
- If the value of the derivative is positive at a given point, this indicates that the function is increasing at that point.

Example

- Find the derivative of the function $f(x) = (x^2 - 4)^3$.
- What is the slope of the tangent of $f(x)$ at the point $= 1$?
- Is the function $f(x)$ increasing or decreasing at the point $x = 1$?
- Find all points where the slope of $f(x)$ is 0.

Solution

- We need to derive the composite function u^3, where $u = x^2 - 4$. Consequently, we need to use the chain derivative.

  $f^{'}(x) = [(x^2 - 4)^3]'$

  $= 3(x^2 - 4)^2 . (x^2 - 4)^{'}$

  $= 3(x^2 - 4)^2 . 2x$

  $= 6x(x^2 - 4)^2$

- At point $x = 1$, the slope of the tangent of the function f is

  $f'(1) = 6(1)(1^2 - 4)^2 = 6(1)(-3)^2 = 54$

- Since the slope is positive at $x = 1$, the function $f(x)$ is increasing at this point.

- The slope is 0 at points like $f'(x) = 0$. We therefore need to find the values of *x* so that

  $6x(x^2 - 4)^2 = 0$

  $x = 0, x = -2$ and $x = 2$ are the values sought.