Chapter 18 Static Equilibrium

The proof of the correctness of a new rule can be attained by the repeated application of it, the frequent comparison with experience, the putting of it to the test under the most diverse circumstances. This process, would in the natural course of events, be carried out in time. The discoverer, however hastens to reach his goal more quickly. He compares the results that flow from his rule with all the experiences with which he is familiar, with all older rules, repeatedly tested in times gone by, and watches to see if he does not light on contradictions. In this procedure, the greatest credit is, as it should be, conceded to the oldest and most familiar experiences, the most thoroughly tested rules. Our instinctive experiences, those generalizations that are made involuntarily, by the irresistible force of the innumerable facts that press upon us, enjoy a peculiar authority; and this is perfectly warranted by the consideration that it is precisely the elimination of subjective caprice and of individual error that is the object aimed at.[1]

Ernst Mach

18.1 Introduction Static Equilibrium

When the vector sum of the forces acting on a point-like object is zero then the object will continue in its state of rest, or of uniform motion in a straight line. If the object is in uniform motion we can always change reference frames so that the object will be at rest. We showed that for a collection of point-like objects the sum of the external forces may be regarded as acting at the center of mass. So if that sum is zero the center of mass will continue in its state of rest, or of uniform motion in a straight line. We introduced the idea of a rigid body, and again showed that in addition to the fact that the sum of the external forces may be regarded as acting at the center of mass, forces like the gravitational force that acts at every point in the body may be treated as acting at the center of mass. However for an extended rigid body it matters where the force is applied because even though the sum of the forces on the body may be zero, a non-zero sum of torques on the body may still produce angular acceleration. In particular for fixed axis rotation, the torque along the axis of rotation on the object is proportional to the angular acceleration. It is possible that sum of the torques may be zero on a body that is not constrained to rotate about a fixed axis and the body may still undergo rotation. We will restrict ourselves to the special case in which in an inertial reference frame both the center of mass of the body is at rest and the body does not undergo any rotation, a condition that is called ***static equilibrium of an extended object.***

The two sufficient and necessary conditions for a rigid body to be in static equilibrium are:

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[1] Ernst Mach, *The Science of Mechanics: A Critical and Historical Account of Its Development*, translated by Thomas J. McCormack, Sixth Edition with Revisions through the Ninth German Edition, Open Court Publishing, Illinois.

(1) The sum of the forces acting on the rigid body is zero,

$$\vec{\mathbf{F}} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \cdots = \vec{\mathbf{0}}. \qquad (18.1.1)$$

(2) The vector sum of the torques about any point S in a rigid body is zero,

$$\vec{\boldsymbol{\tau}}_S = \vec{\boldsymbol{\tau}}_{S,1} + \vec{\boldsymbol{\tau}}_{S,2} + \cdots = \vec{\mathbf{0}}. \qquad (18.1.2)$$

18.2 Lever Law

Let's consider a uniform rigid beam of mass m_b balanced on a pivot near the center of mass of the beam. We place two objects 1 and 2 of masses m_1 and m_2 on the beam, at distances d_1 and d_2 respectively from the pivot, so that the beam is static (that is, the beam is not rotating. See Figure 18.1.) We shall neglect the thickness of the beam and take the pivot point to be the center of mass.

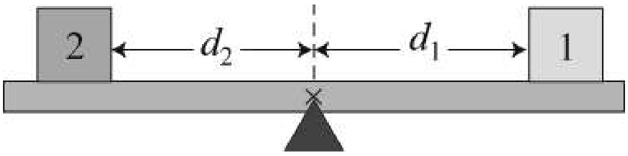

Figure 18.1 Pivoted Lever

Let's consider the forces acting on the beam. The earth attracts the beam downward. This gravitational force acts on every atom in the beam, but we can summarize its action by stating that the gravitational force $m_b\vec{\mathbf{g}}$ is concentrated at a point in the beam called the *center of gravity* of the beam, which is identical to the center of mass of the uniform beam. There is also a contact force $\vec{\mathbf{F}}_{\text{pivot}}$ between the pivot and the beam, acting upwards on the beam at the pivot point. The objects 1 and 2 exert normal forces downwards on the beam, $\vec{\mathbf{N}}_{1,b} \equiv \vec{\mathbf{N}}_1$, and $\vec{\mathbf{N}}_{2,b} \equiv \vec{\mathbf{N}}_2$, with magnitudes N_1, and N_2, respectively. Note that the normal forces are not the gravitational forces acting on the objects, but contact forces between the beam and the objects. (In this case, they are mathematically the same, due to the horizontal configuration of the beam and the fact that all objects are in static equilibrium.) The distances d_1 and d_2 are called the *moment arms* with respect to the pivot point for the forces $\vec{\mathbf{N}}_1$ and $\vec{\mathbf{N}}_2$, respectively. The force diagram on the beam is shown in Figure 18.2. Note that the pivot force $\vec{\mathbf{F}}_{\text{pivot}}$ and the force of gravity $m_b\vec{\mathbf{g}}$ each has a zero moment arm about the pivot point.

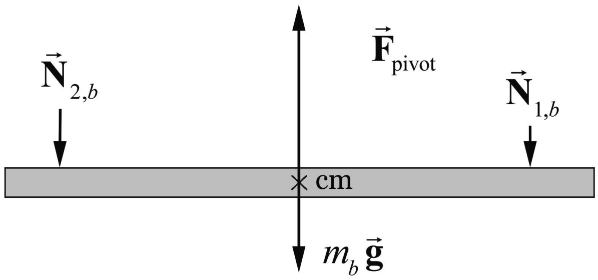

Figure 18.2 Free-body diagram on beam

Because we assume the beam is not moving, the sum of the forces in the vertical direction acting on the beam is therefore zero,

$$F_{\text{pivot}} - m_b g - N_1 - N_2 = 0\,. \tag{18.2.1}$$

The force diagrams on the objects are shown in Figure 18.3. Note the magnitude of the normal forces on the objects are also N_1 and N_2 since these are each part of an action-reaction pair, $\vec{\mathbf{N}}_{1,b} = -\vec{\mathbf{N}}_{b,1}$, and $\vec{\mathbf{N}}_{2,b} = -\vec{\mathbf{N}}_{b,2}$.

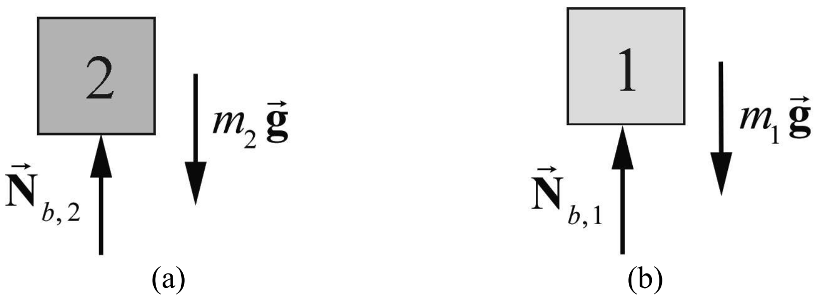

Figure 18.3 Free-body force diagrams for each body.

The condition that the forces sum to zero is not sufficient to completely predict the motion of the beam. All we can deduce is that the center of mass of the system is at rest (or moving with a uniform velocity). In order for the beam not to rotate the sum of the torques about any point must be zero. In particular the sum of the torques about the pivot point must be zero. Because the moment arm of the gravitational force and the pivot force is zero, only the two normal forces produce a torque on the beam. If we choose out of the page as positive direction for the torque (or equivalently counterclockwise rotations are positive) then the condition that the sum of the torques about the pivot point is zero becomes

$$d_2 N_2 - d_1 N_1 = 0\,. \tag{18.2.2}$$

The magnitudes of the two torques about the pivot point are equal, a condition known as the lever law.

Lever Law: *A beam of length l is balanced on a pivot point that is placed directly beneath the center of mass of the beam. The beam will not undergo rotation if the product of the normal force with the moment arm to the pivot is the same for each body,*

$$d_1 N_1 = d_2 N_2 \,. \tag{18.2.3}$$

Example 18.1 Lever Law

Suppose a uniform beam of length $l = 1.0\ \text{m}$ and mass $m_{\text{B}} = 2.0\ \text{kg}$ is balanced on a pivot point, placed directly beneath the center of the beam. We place body 1 with mass $m_1 = 0.3\ \text{kg}$ a distance $d_1 = 0.4\ \text{m}$ to the right of the pivot point, and a second body 2 with $m_2 = 0.6\ \text{kg}$ a distance d_2 to the left of the pivot point, such that the beam neither translates nor rotates. (a) What is the force $\vec{\mathbf{F}}_{\text{pivot}}$ that the pivot exerts on the beam? (b) What is the distance d_2 that maintains static equilibrium?

Solution: a) By Newton's Third Law, the beam exerts equal and opposite normal forces of magnitude N_1 on body 1, and N_2 on body 2. The condition for force equilibrium applied separately to the two bodies yields

$$N_1 - m_1 g = 0\,, \tag{18.2.4}$$

$$N_2 - m_2 g = 0\,. \tag{18.2.5}$$

Thus the total force acting on the beam is zero,

$$F_{\text{pivot}} - (m_b + m_1 + m_2)g = 0\,, \tag{18.2.6}$$

and the pivot force is

$$\begin{aligned} F_{\text{pivot}} &= (m_b + m_1 + m_2)g \\ &= (2.0\ \text{kg} + 0.3\ \text{kg} + 0.6\ \text{kg})(9.8\ \text{m}\cdot\text{s}^{-2}) = 2.8 \times 10^1\ \text{N}\,. \end{aligned} \tag{18.2.7}$$

b) We can compute the distance d_2 from the Lever Law,

$$d_2 = \frac{d_1 N_1}{N_2} = \frac{d_1 m_1 g}{m_2 g} = \frac{d_1 m_1}{m_2} = \frac{(0.4\ \text{m})(0.3\ \text{kg})}{0.6\ \text{kg}} = 0.2\ \text{m}\,. \tag{18.2.8}$$

18.3 Generalized Lever Law

We can extend the Lever Law to the case in which two external forces $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ are acting on the pivoted beam at angles θ_1 and θ_2 with respect to the horizontal as shown in the Figure 18.4. Throughout this discussion the angles will be limited to the range $[0 \le \theta_1, \theta_2 \le \pi]$. We shall again neglect the thickness of the beam and take the pivot point to be the center of mass.

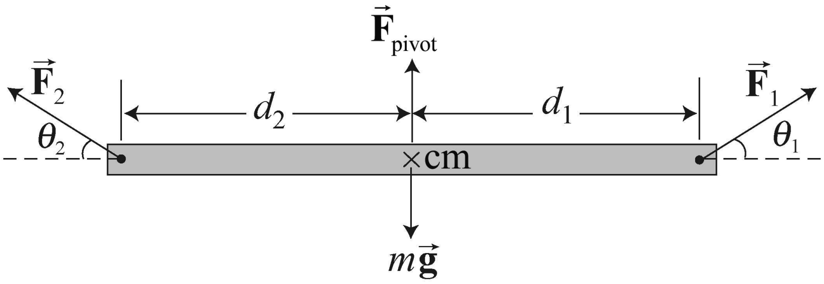

Figure 18.4 Forces acting at angles to a pivoted beam.

The forces $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ can be decomposed into separate vectors components respectively $(\vec{\mathbf{F}}_{1,\parallel}, \vec{\mathbf{F}}_{1,\perp})$ and $(\vec{\mathbf{F}}_{2,\parallel}, \vec{\mathbf{F}}_{2,\perp})$, where $\vec{\mathbf{F}}_{1,\parallel}$ and $\vec{\mathbf{F}}_{2,\parallel}$ are the horizontal vector projections of the two forces with respect to the direction formed by the length of the beam, and $\vec{\mathbf{F}}_{1,\perp}$ and $\vec{\mathbf{F}}_{2,\perp}$ are the perpendicular vector projections respectively to the beam (Figure 18.5), with

$$\vec{\mathbf{F}}_1 = \vec{\mathbf{F}}_{1,\parallel} + \vec{\mathbf{F}}_{1,\perp}, \tag{18.3.1}$$

$$\vec{\mathbf{F}}_2 = \vec{\mathbf{F}}_{2,\parallel} + \vec{\mathbf{F}}_{2,\perp}. \tag{18.3.2}$$

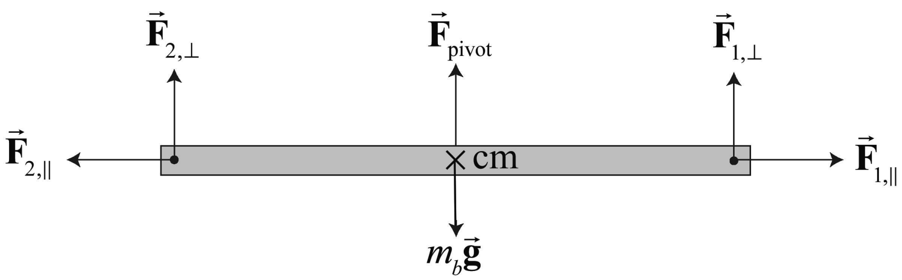

Figure 18.5 Vector decomposition of forces.

The horizontal components of the forces are

$$F_{1,\parallel} = F_1 \cos\theta_1, \tag{18.3.3}$$

$$F_{2,\parallel} = -F_2 \cos\theta_2, \tag{18.3.4}$$

where our choice of positive horizontal direction is to the right. Neither horizontal force component contributes to possible rotational motion of the beam. The sum of these horizontal forces must be zero,

$$F_1 \cos\theta_1 - F_2 \cos\theta_2 = 0 . \tag{18.3.5}$$

The perpendicular component forces are

$$F_{1,\perp} = F_1 \sin\theta_1 , \tag{18.3.6}$$

$$F_{2,\perp} = F_2 \sin\theta_2 , \tag{18.3.7}$$

where the positive vertical direction is upwards. The perpendicular components of the forces must also sum to zero,

$$F_{\text{pivot}} - m_b g + F_1 \sin\theta_1 + F_2 \sin\theta_2 = 0 . \tag{18.3.8}$$

Only the vertical components $F_{1,\perp}$ and $F_{2,\perp}$ of the external forces are involved in the lever law (but the horizontal components must balance, as in Equation (18.3.5), for equilibrium). Then the Lever Law can be extended as follows.

> **Generalized Lever Law** *A beam of length l is balanced on a pivot point that is placed directly beneath the center of mass of the beam. Suppose a force $\vec{\mathbf{F}}_1$ acts on the beam a distance d_1 to the right of the pivot point. A second force $\vec{\mathbf{F}}_2$ acts on the beam a distance d_2 to the left of the pivot point. The beam will remain in static equilibrium if the following two conditions are satisfied:*
>
> *1) The total force on the beam is zero,*
>
> *2) The product of the magnitude of the perpendicular component of the force with the distance to the pivot is the same for each force,*

$$d_1 \left| F_{1,\perp} \right| = d_2 \left| F_{2,\perp} \right| . \tag{18.3.9}$$

The Generalized Lever Law can be stated in an equivalent form,

$$d_1 F_1 \sin\theta_1 = d_2 F_2 \sin\theta_2 . \tag{18.3.10}$$

We shall now show that the generalized lever law can be reinterpreted as the statement that the vector sum of the torques about the pivot point S is zero when there are just two forces $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ acting on our beam as shown in Figure 18.6.

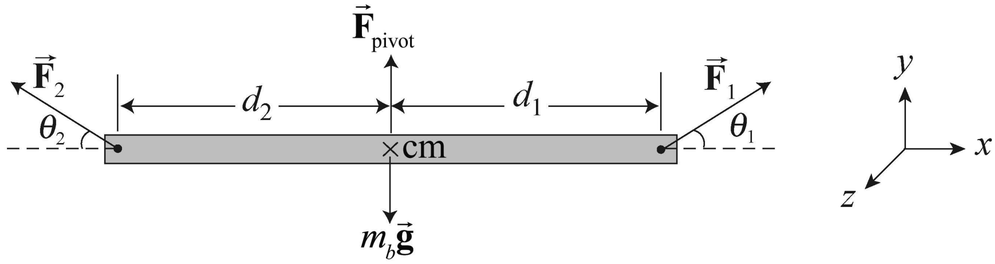

Figure 18.6 Force and torque diagram.

Let's choose the positive z-direction to point out of the plane of the page then torque pointing out of the page will have a positive z-component of torque (counterclockwise rotations are positive). From our definition of torque about the pivot point, the magnitude of torque due to force $\vec{\mathbf{F}}_1$ is given by

$$\tau_{S,1} = d_1 F_1 \sin\theta_1 . \tag{18.3.11}$$

From the right hand rule this is out of the page (in the counterclockwise direction) so the component of the torque is positive, hence,

$$(\tau_{S,1})_z = d_1 F_1 \sin\theta_1 . \tag{18.3.12}$$

The torque due to $\vec{\mathbf{F}}_2$ about the pivot point is into the page (the clockwise direction) and the component of the torque is negative and given by

$$(\tau_{S,2})_z = -d_2 F_2 \sin\theta_2 . \tag{18.3.13}$$

The z-component of the torque is the sum of the z-components of the individual torques and is zero,

$$(\tau_{S,\text{total}})_z = (\tau_{S,1})_z + (\tau_{S,2})_z = d_1 F_1 \sin\theta_1 - d_2 F_2 \sin\theta_2 = 0 , \tag{18.3.14}$$

which is equivalent to the Generalized Lever Law, Equation (18.3.10),

$$d_1 F_1 \sin\theta_1 = d_2 F_2 \sin\theta_2 .$$

18.4 Worked Examples

Example 18.2 Suspended Rod

A uniform rod of length $l = 2.0\text{ m}$ and mass $m = 4.0\text{ kg}$ is hinged to a wall at one end and suspended from the wall by a cable that is attached to the other end of the rod at an

Appendix 18A The Torques About Any Two Points are Equal for a Body in Static Equilibrium

When the net force on a body is zero, the torques about any two points are equal. To show this, consider any two points A and B. Choose a coordinate system with origin O and denote the constant vector from A to B by $\vec{\mathbf{r}}_{A,B}$. Suppose a force $\vec{\mathbf{F}}_i$ is acting at the point $\vec{\mathbf{r}}_{O,i}$. The vector from the point A to the point where the force acts is denoted by $\vec{\mathbf{r}}_{A,i}$, and the vectors from the point B to the point where the force acts is denoted by $\vec{\mathbf{r}}_{B,i}$.

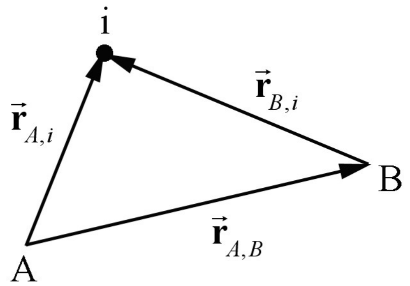

Figure 18A.1 Location of body i with respect to the points A and B.

In Figure 18A.1, the position vectors satisfy

$$\vec{\mathbf{r}}_{A,i} = \vec{\mathbf{r}}_{A,B} + \vec{\mathbf{r}}_{B,i}. \tag{18.A.1}$$

The sum of the torques about the point A is given by

$$\vec{\boldsymbol{\tau}}_A = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{A,i} \times \vec{\mathbf{F}}_i. \tag{18.A.2}$$

The sum of the torques about the point B is given by

$$\vec{\boldsymbol{\tau}}_B = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{B,i} \times \vec{\mathbf{F}}_i. \tag{18.A.3}$$

We can now substitute Equation (18.A.1) into Equation (18.A.2) and find that

$$\vec{\boldsymbol{\tau}}_A = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{A,i} \times \vec{\mathbf{F}}_i = \sum_{i=1}^{i=N} (\vec{\mathbf{r}}_{A,B} + \vec{\mathbf{r}}_{B,i}) \times \vec{\mathbf{F}}_i = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{A,B} \times \vec{\mathbf{F}}_i + \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{B,i} \times \vec{\mathbf{F}}_i. \tag{18.A.4}$$

In the next-to-last term in Equation (18.A.4), the vector $\vec{\mathbf{r}}_{A,B}$ is constant and so may be taken outside the summation,

$$\sum_{i=1}^{i=N} \vec{\mathbf{r}}_{A,B} \times \vec{\mathbf{F}}_i = \vec{\mathbf{r}}_{A,B} \times \sum_{i=1}^{i=N} \vec{\mathbf{F}}_i \,. \tag{18.A.5}$$

We are assuming that there is no net force on the body, and so the sum of the forces on the body is zero,

$$\sum_{i=1}^{i=N} \vec{\mathbf{F}}_i = \vec{\mathbf{0}} \,. \tag{18.A.6}$$

Therefore the torque about point A, Equation (18.A.2), becomes

$$\vec{\boldsymbol{\tau}}_A = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{B,i} \times \vec{\mathbf{F}}_i \; = \vec{\boldsymbol{\tau}}_B \,. \tag{18.A.7}$$

For static equilibrium problems, the result of Equation (18.A.7) tells us that it does not matter which point we use to determine torques. In fact, note that the position of the chosen origin did not affect the result at all. Choosing the point about which to calculate torques (variously called "A", "B", "S" or sometimes "O") so that unknown forces do not exert torques about that point may often greatly simplify calculations.