Proofs of 'GenHarris-ResNet: A Rotation Invariant Neural Network Based on Elementary Symmetric Polynomials'

2023

1 Matrix $\mathcal{M}_k^{(\sigma)}$

This appendix has the only purpose to give examples of matrices $\mathcal{M}_k^{(\sigma)}$.

$$\mathcal{M}_1^{(\sigma)} = G\left(\cdot, \sigma_{int}\right) * \begin{pmatrix} \left(L_x^{(\sigma)}\right)^2 & L_x^{(\sigma)} L_y^{(\sigma)} \\ L_y^{(\sigma)} L_x^{(\sigma)} & \left(L_y^{(\sigma)}\right)^2 \end{pmatrix}.$$

$$\mathcal{M}_2^{(\sigma)} = G\left(\cdot, \sigma_{int}\right) * \begin{pmatrix} \left(L_{xx}^{(\sigma)}\right)^2 & \sqrt{2} L_{xx}^{(\sigma)} L_{xy}^{(\sigma)} & L_{xx}^{(\sigma)} L_{yy}^{(\sigma)} \\ \sqrt{2} L_{xy}^{(\sigma)} L_{xx}^{(\sigma)} & 2\left(L_{xy}^{(\sigma)}\right)^2 & \sqrt{2} L_{xy}^{(\sigma)} L_{yy}^{(\sigma)} \\ L_{yy}^{(\sigma)} L_{xx}^{(\sigma)} & \sqrt{2} L_{yy}^{(\sigma)} L_{xy}^{(\sigma)} & \left(L_{yy}^{(\sigma)}\right)^2 \end{pmatrix}.$$

$$\mathcal{M}_3^{(\sigma)} = G\left(\cdot, \sigma_{int}\right) * \begin{pmatrix} \left(L_{xxx}^{(\sigma)}\right)^2 & \sqrt{3} L_{xxx}^{(\sigma)} L_{xxy}^{(\sigma)} & \sqrt{3} L_{xxx}^{(\sigma)} L_{xyy}^{(\sigma)} & L_{xxx}^{(\sigma)} L_{yyy}^{(\sigma)} \\ \sqrt{3} L_{xxy}^{(\sigma)} L_{xxx}^{(\sigma)} & 3\left(L_{xxy}^{(\sigma)}\right)^2 & 3 L_{xxy}^{(\sigma)} L_{xyy}^{(\sigma)} & \sqrt{3} L_{xxy}^{(\sigma)} L_{yyy}^{(\sigma)} \\ \sqrt{3} L_{xyy}^{(\sigma)} L_{xxx}^{(\sigma)} & 3 L_{xyy}^{(\sigma)} L_{xxy}^{(\sigma)} & 3\left(L_{xyy}^{(\sigma)}\right)^2 & \sqrt{3} L_{xyy}^{(\sigma)} L_{yyy}^{(\sigma)} \\ L_{yyy}^{(\sigma)} L_{xxx}^{(\sigma)} & \sqrt{3} L_{yyy}^{(\sigma)} L_{xxy}^{(\sigma)} & \sqrt{3} L_{yyy}^{(\sigma)} L_{xyy}^{(\sigma)} & \left(L_{yyy}^{(\sigma)}\right)^2 \end{pmatrix}.$$

2 Matrix $\mathcal{P}_k(\theta)$

This appendix aims to give a proof of Lemma 1 and Proposition 1. We first show the existence and derive the expression of $\mathcal{P}_k\left(\theta\right)$, then we present some properties of this matrix, especially its orthogonality, through existing literature [1, 2].

2.1 Orthogonality and Dyadic Product

This subsection consists in the proof of Lemma 1

Proof. Let P_{ij} be the coefficients of P, meaning that

$$P = (P_{ij})_{i\in[\![1,n]\!],j\in[\![1,n]\!]} \tag{1}$$

And let us recall that

$$P \in \mathcal{O}^n(\mathbb{R}) \iff \left(P^{-1}\right)_{ij} = P_{ji} \tag{2}$$

. Then we have:

$$\begin{aligned}
\left(P\left(v\odot v\right)P^{-1}\right)_{i,j} &= \left(\sum_{l=1}^{n}\left(\sum_{k=1}^{n}P_{ik}v_k v_l\right)\left(P^{-1}\right)_{lj}\right)_{i,j} = \left(\sum_{l=1}^{n}\sum_{k=1}^{n}P_{ik}\left(P^{-1}\right)_{lj}v_k v_l\right)_{i,j} \\
P \in \mathcal{O}^n(\mathbb{R}) \to &= \left(\sum_{l=1}^{n}\sum_{k=1}^{n}P_{ik}P_{jl}v_k v_l\right)_{i,j} = \left(\left(\sum_{k=1}^{n}P_{ik}v_k\right)\left(\sum_{l=1}^{n}P_{jl}v_l\right)\right)_{i,j} \\
&= (u\odot u)_{i,j}
\end{aligned} \tag{3}$$

□

2.2 Existence and Expression

Exactly like in section 3, we will place ourselves in the real two-dimensional space $\mathbb{R}^2$ with the system of coordinates (x_1, x_2). And we will use apostrophe to distinct entities expressed in the rotated system of coordinates.

We define Ω_k to be the set of all possible outcomes of the Binomial experiment with k repetitions:

$$\Omega_k = \{(i_1, ..., i_k) \mid \forall j \in [\![1,k]\!],\ i_j \in \{1;2\}\}.$$

Ω_k can be used to describe all the possible Gaussian partial derivatives of order k as each of them can be associated to a given element $\omega = (\omega_1, ..., \omega_k)$ of Ω_k in the following way:

$$L_\omega^{(\sigma)} = \frac{\partial^k G(.,\sigma)}{\partial x_{\omega_1}\dots\partial x_{\omega_k}} * I.$$

The existence of $\mathcal{P}_k(\theta)$ comes from equation (43) in [3] giving the transformation law of the tensor composed with all the partial derivatives of a given order. This gives us the following expression

$$\forall (j_1, ..., j_k) \in \Omega_k,\quad L'^{(\sigma)}_{j_1,\dots,j_k} = \sum_{\omega\in\Omega_k}\left(\prod_{l=1}^{k}R_{j_l,\omega_l}\right)L_\omega^{(\sigma)} \tag{4}$$

with $R_{11}(\theta) = \cos(\theta)$, $R_{12}(\theta) = \sin(\theta)$, $R_{21}(\theta) = -\sin(\theta)$ and $R_{22}(\theta) = \cos(\theta)$.

We define now $\nu_{p,k}$ to be a particular element of Ω_k given by:

$$\nu_{p,k} = (\underbrace{1,...,1}_{k-p \text{ times}}, \underbrace{2,...,2}_{p \text{ times}}).$$

In this case, we find the element $L^{(\sigma)}_{k-p,p}$ of the local N-jet $j_N(I,\sigma)$ recalled in section 3. Now, let $S_{p,k}$ be the set of all distinct permutations of $\nu_{p,k}$ such that $\forall s_1, s_2 \in S_{p,k}$, if $\forall l \in [\![1,k]\!]$ $(s_1(\nu_{p,k}))_l = (s_2(\nu_{p,k}))_l$ then $s_1 = s_2$. As an example, for $k \geq k-p \geq 2$, the permutation only interchanging the two first elements of $\nu_{p,k}$ (i.e interchanging the first two ones) is considered to be equal to Id. Then, we have

$$\Omega_k = \{s(\nu_{p,k}) \mid \forall s \in S_{p,k},\ \forall p \in [\![0,k]\!]\}.$$

We can rewrite equation 4 in the following way: $\forall (j_1,...,j_k) \in \Omega_k$, we have

$$L'^{(\sigma)}_{j_1,...,j_k} = \textstyle\sum_{p=0}^{k} \sum_{s\in S_{p,k}} \left(\prod_{l=1}^{k} R_{j_l, s(\nu_{p,k})_l}\right) L^{(\sigma)}_{s(\nu_{p,k})}.$$

Considering the symmetry of the derivative tensor, one has:

$$\forall s \in S_{p,k},\ L^{(\sigma)}_{s(\nu_{p,k})} = L^{(\sigma)}_{\nu_{p,k}}$$

and more precisely:

$$\begin{aligned} L'^{(\sigma)}_{\nu_{q,k}} &= \sum_{p=0}^{k} \sum_{s\in S_{p,k}} \left(\prod_{l=1}^{k} R_{(\nu_{q,k})_l, s(\nu_{p,k})_l}\right) L^{(\sigma)}_{\nu_{p,k}} \\ &= \sum_{p=0}^{k} \sum_{s\in S_{p,k}} \left(\prod_{l=1}^{k-q} R_{1, s(\nu_{p,k})_l}\right) \left(\prod_{l=1}^{q} R_{2, s(\nu_{p,k})_{k-q+l}}\right) L^{(\sigma)}_{\nu_{p,k}}. \end{aligned} \tag{5}$$

In order to have a better description of $S_{p,k}$, we introduce the following cardinal subsets

$$\begin{aligned} \#^{i,-,1}_{p,k}(s) &= \#\left(\{l \in [\![1,i]\!] \mid s(\nu_{p,k})_l = 1\}\right) \\ \#^{i,-,2}_{p,k}(s) &= \#\left(\{l \in [\![1,i]\!] \mid s(\nu_{p,k})_l = 2\}\right) \\ \#^{i,+,1}_{p,k}(s) &= \#\left(\{l \in [\![i+1,k]\!] \mid s(\nu_{p,k})_l = 1\}\right) \\ \#^{i,+,2}_{p,k}(s) &= \#\left(\{l \in [\![i+1,k]\!] \mid s(\nu_{p,k})_l = 2\}\right). \end{aligned}$$

We have the following relations :

$$\begin{aligned} \forall s\ S_{p,k}\ ;\ \#^{i,-,1}_{p,k}(s) + \#^{i,-,2}_{p,k}(s) + \#^{i,+,1}_{p,k}(s) + \#^{i,+,2}_{p,k}(s) &= k \\ \#^{i,-,1}_{p,k}(s) + \#^{i,-,2}_{p,k}(s) &= i \\ \#^{i,+,1}_{p,k}(s) + \#^{i,+,2}_{p,k}(s) &= k-i \\ \#^{i,-,1}_{p,k}(s) + \#^{i,+,1}_{p,k}(s) &= k-p \\ \#^{i,-,2}_{p,k}(s) + \#^{i,+,2}_{p,k}(s) &= p. \end{aligned}$$

We denote by $S_{p,k}|^{k-q,+,2}_{r}$ the set defined as

$$S_{p,k}|_r^{k-q,+,2} = \{s \in S_{p,k} \mid \#_{p,k}^{k-q,+,2}(s) = r\}.$$

We have

$$\#\left(S_{p,k}|_r^{k-q,+,2}\right) = \tbinom{q}{r}\tbinom{k-q}{p-r},$$

and

$$S_{p,k} = \textstyle\coprod_{r=max(0,q+p-k)}^{min(q,p)} S_{p,k}|_r^{k-q,+,2}.$$

where $\coprod$ denotes the disjoint union. equation 5 becomes

$$\begin{aligned} L'^{(\sigma)}_{\nu_{q,k}} &= \sum_{p=0}^{k}\left(\sum_{r=max(0,p+q-k)}^{min(p,q)}\sum_{s\in S_{p,k}|_r^{k-q,+,2}}\left(\prod_{l=1}^{k-q} R_{1,s\left(\nu_{p,k}\right)_l}\right)\left(\prod_{l=1}^{q} R_{2,s\left(\nu_{p,k}\right)_{k-q+l}}\right)\right) L^{(\sigma)}_{\nu_{p,k}} \\ &= \sum_{p=0}^{k}\underbrace{\left(\sum_{r=max(0,p+q-k)}^{min(p,q)}\binom{q}{r}\binom{k-q}{p-r}(-1)^{q-r}\,cos^{k-p-q+2r}(\theta)\,sin^{p+q-2r}(\theta)\right)}_{\left(P_k(\theta)\right)_{p,q}} L^{(\sigma)}_{\nu_{p,k}} \end{aligned} \tag{6}$$

as $\forall s \in S_{p,k}|_r^{k-q,+,2}$, we have

$$\begin{aligned} \left(\prod_{l=1}^{k-q} R_{1,s\left(\nu_{p,k}\right)_l}\right) &= cos^{k-q-p+r}(\theta)\,sin^{p-r}(\theta) \\ \left(\prod_{l=1}^{q} R_{2,s\left(\nu_{p,k}\right)_{k-q+l}}\right) &= cos^{r}(\theta)\,(-1)^{q-r}\,sin^{q-r}(\theta) \end{aligned}$$

It can be highlighted that the matrix $P_k(\theta)$ is quite close to the matrix $\left(A_n(m,k,\theta)\right)_{n,m\in[\![1,k]\!]}$ defined by equation (35) of [2], also obtained from the steerable property of Gaussian the $2D$ Gaussian Derivatives but written in a different ordering. More precisely there is the following relationship:

$$\forall k \in \mathbb{N}\ ;\ A_{.}(.,k,\theta) = P_k^{\lhd}(\theta)$$

where $\forall M \in \mathcal{M}_n(\mathbb{R})\ ;\ \left(M^{\lhd}\right)_{i,j} = M_{i,n-j}$. The authors of [2] also showed that $A_{.}(.,k,\theta)$ is its own inverse: $A_{.}(.,k,\theta)\,A_{.}(.,k,\theta) = I_{k+1}$.

The last step to obtain the expression of $\mathcal{P}_k(\theta)$ is to take into account the square roots of the binomial coefficients. These additional coefficients play an important role in the orthogonality property. The transformation matrix resulting from equation 4 considering the tensors in a vectorized form is already orthogonal because all the interchangeable elements of Ω_k are taken into account. Based on the symmetry, when one wants to summarize them in a single element to reduce the dimension of the studied vectors and matrix, in our case through the use of $\nu_{p,k}$, the orthogonality doesn't hold without incorporating their multiplicity. The square roots of the binomial coefficients compensate this loss of multiplicity.equation 6 becomes

$$\sqrt{\binom{k}{q}} L'^{(\sigma)}_{\nu_{q,k}} = \sum_{p=0}^{k} \underbrace{\frac{\sqrt{\binom{k}{q}}}{\sqrt{\binom{k}{p}}} \left(P_k\left(\theta\right)\right)_{p,q}}_{\left(\mathcal{P}_k(\theta)\right)_{p,q}} \sqrt{\binom{k}{p}} L^{(\sigma)}_{\nu_{p,k}}, \tag{7}$$

giving the expected relation:

$$\mathcal{L}'^{(\sigma)}_{k} = \mathcal{P}_k\left(\theta\right) \mathcal{L}^{(\sigma)}_{k}.$$

2.3 Orthogonality

The orthogonality of $\mathcal{P}_k(\theta)$ has already been proven in [1]. But the authors, instead of giving a closed form of the matrix, gave recurrence formulas with the exact form of only the zeroth, first and second orders. The goal of this section is to show that the matrices $\mathcal{P}_k(\theta)$ respect the mentioned recurrence formulas. We first recall the exact expressions of the first orders. From equation 6 and equation 7 it is easy to find $\mathcal{P}_1(\theta)$ and $\mathcal{P}_2(\theta)$

$$\begin{aligned}
\mathcal{P}_1\left(\theta\right) &= \begin{pmatrix} \cos\left(\theta\right) & \sin\left(\theta\right) \\ -\sin\left(\theta\right) & \cos\left(\theta\right) \end{pmatrix} \\
\mathcal{P}_2\left(\theta\right) &= \begin{pmatrix} \cos^2\left(\theta\right) & \sqrt{2}\sin\left(\theta\right)\cos\left(\theta\right) & \sin^2\left(\theta\right) \\ -\sqrt{2}\sin\left(\theta\right)\cos\left(\theta\right) & \cos^2\left(\theta\right)-\sin^2\left(\theta\right) & \sqrt{2}\sin\left(\theta\right)\cos\left(\theta\right) \\ \sin^2\left(\theta\right) & -\sqrt{2}\sin\left(\theta\right)\cos\left(\theta\right) & \cos^2\left(\theta\right) \end{pmatrix}.
\end{aligned}$$

The recurrence formulas from [1] are the following ones: $\forall k \in \mathbb{N}^* \; ; \; \forall i,j \in [\![0,k]\!]$

$$\begin{pmatrix} \left(\mathcal{P}_{k+1}\left(\theta\right)\right)_{i+1,j} \\ \left(\mathcal{P}_{k+1}\left(\theta\right)\right)_{i,j} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{i+1}} & 0 \\ 0 & \frac{1}{\sqrt{k-i+1}} \end{pmatrix} \begin{pmatrix} \cos\left(\theta\right) & -\sin\left(\theta\right) \\ \sin\left(\theta\right) & \cos\left(\theta\right) \end{pmatrix} \begin{pmatrix} \sqrt{j}\left(\mathcal{P}_k\left(\theta\right)\right)_{i,j-1} \\ \sqrt{k-j+1}\left(\mathcal{P}_k\left(\theta\right)\right)_{i,j} \end{pmatrix} \tag{8}$$

and

$$\begin{pmatrix} \left(\mathcal{P}_{k+1}\left(\theta\right)\right)_{i,j+1} \\ \left(\mathcal{P}_{k+1}\left(\theta\right)\right)_{i,j} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{j+1}} & 0 \\ 0 & \frac{1}{\sqrt{k-j+1}} \end{pmatrix} \begin{pmatrix} \cos\left(\theta\right) & \sin\left(\theta\right) \\ -\sin\left(\theta\right) & \cos\left(\theta\right) \end{pmatrix} \begin{pmatrix} \sqrt{i}\left(\mathcal{P}_k\left(\theta\right)\right)_{i-1,j} \\ \sqrt{k-i+1}\left(\mathcal{P}_k\left(\theta\right)\right)_{i,j} \end{pmatrix} \tag{9}$$

Lemma. *The following closed form for $\mathcal{P}_k(\theta)$ proposed in Proposition 1 and given by equation 7 satisfies equation 8 and equation 9*

$$\forall i,j \in [\![0,k]\!],\; \left(\mathcal{P}_k\left(\theta\right)\right)_{i,j} = \frac{\sqrt{\binom{k}{i}}}{\sqrt{\binom{k}{j}}} \sum_{r=max(0,i+j-k)}^{min(i,j)} \binom{i}{r}\binom{k-i}{j-r} \left(-1\right)^{i-r} s^{i+j-2r} c^{k+2r-i-j}$$

Proof: Let us develop the right term of equation 8, using the proposed expression for $\mathcal{P}_k(\theta)$, which gives a vector $(a,b)^T \in \mathbb{R}^2$ with

$$
\begin{aligned}
a &= \frac{1}{\sqrt{i+1}}\left(cos\left(\theta\right)\sqrt{j}\left(\mathcal{P}_k\left(\theta\right)\right)_{i,j-1} - sin\left(\theta\right)\sqrt{k-j+1}\left(\mathcal{P}_k\left(\theta\right)\right)_{i,j}\right)\\
&= \frac{1}{\sqrt{i+1}}\left(\sqrt{j}\frac{\sqrt{\binom{k}{i}}}{\sqrt{\binom{k}{j-1}}}\sum_{r=max(0,i+j-k-1)}^{min(i,j-1)}\binom{i}{r}\binom{k-i}{j-1-r}(-1)^{i-r}\, s^{i+j-1-2r}c^{k+2r-i-j+2}\right.\\
&\qquad\left.+\sqrt{k-j+1}\frac{\sqrt{\binom{k}{i}}}{\sqrt{\binom{k}{j}}}\sum_{r=max(0,i+j-k)}^{min(i,j)}\binom{i}{r}\binom{k-i}{j-r}(-1)^{i-r+1}\, s^{i+j+1-2r}c^{k+2r-i-j}\right)\\
&= \frac{\sqrt{\binom{k+1}{i+1}}}{\sqrt{\binom{k+1}{j}}}\left(\sum_{r=max(0,i+j-k-1)}^{min(i,j-1)}\binom{i}{r}\binom{k-i}{j-1-r}(-1)^{i-r}\, s^{i+j-1-2r}c^{k+2r-i-j+2}\right.\\
&\qquad\left.+\sum_{r=max(0,i+j-k)}^{min(i,j)}\binom{i}{r}\binom{k-i}{j-r}(-1)^{i-r+1}\, s^{i+j+1-2r}c^{k+2r-i-j}\right)
\end{aligned}
$$

but

$$
\begin{aligned}
&\sum_{r=max(0,i+j-k-1)}^{min(i,j-1)}\binom{i}{r}\binom{k-i}{j-1-r}(-1)^{i-r}\, s^{i+j-1-2r}c^{k+2r-i-j+2}\\
&= \sum_{r=max(1,i+j-k)}^{min(i+1,j)}\binom{i}{r-1}\binom{k-i}{j-r}(-1)^{i-r+1}\, s^{i+j+1-2r}c^{k+2r-i-j}\\
\binom{i}{-1} = 0 \rightarrow &= \sum_{r=max(0,i+j-k)}^{min(i+1,j)}\binom{i}{r-1}\binom{k-i}{j-r}(-1)^{i-r+1}\, s^{i+j+1-2r}c^{k+2r-i-j}
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{r=max(0,i+j-k)}^{min(i,j)}\binom{i}{r}\binom{k-i}{j-r}(-1)^{i-r+1}\, s^{i+j+1-2r}c^{k+2r-i-j}\\
\binom{i}{i+1} = 0 \rightarrow &= \sum_{r=max(0,i+j-k)}^{min(i+1,j)}\binom{i}{r}\binom{k-i}{j-r}(-1)^{i-r+1}\, s^{i+j+1-2r}c^{k+2r-i-j}
\end{aligned}
$$

giving

$$
\begin{aligned}
a &= \frac{\sqrt{\binom{k+1}{i+1}}}{\sqrt{\binom{k+1}{j}}}\sum_{r=max(0,i+j-k)}^{min(i+1,j)}\left(\binom{i}{r-1}+\binom{i}{r}\right)\binom{k-i}{j-r}(-1)^{i-r+1}\, s^{i+j+1-2r}c^{k+2r-i-j}\\
&= \frac{\sqrt{\binom{k+1}{i+1}}}{\sqrt{\binom{k+1}{j}}}\sum_{r=max(0,i+j-k)}^{min(i+1,j)}\binom{i+1}{r}\binom{k-i}{j-r}(-1)^{i-r+1}\, s^{i+j+1-2r}c^{k+2r-i-j}\\
&= \left(\mathcal{P}_{k+1}\left(\theta\right)\right)_{i+1,j}.
\end{aligned}
$$

In a same manner

$$
\begin{aligned}
b &= \frac{1}{\sqrt{k-i+1}}\left(sin\left(\theta\right)\sqrt{j}\left(\mathcal{P}_k\left(\theta\right)\right)_{i,j-1} + cos\left(\theta\right)\sqrt{k-j+1}\left(\mathcal{P}_k\left(\theta\right)\right)_{i,j}\right)\\
&= \frac{1}{\sqrt{k-i+1}}\left(\sqrt{j}\frac{\sqrt{\binom{k}{i}}}{\sqrt{\binom{k}{j-1}}}\sum_{r=max(0,i+j-k-1)}^{min(i,j-1)}\binom{i}{r}\binom{k-i}{j-1-r}(-1)^{i-r}\, s^{i+j-2r}c^{k+2r-i-j+1}\right.\\
&\quad \left.+\sqrt{k-j+1}\frac{\sqrt{\binom{k}{i}}}{\sqrt{\binom{k}{j}}}\sum_{r=max(0,i+j-k)}^{min(i,j)}\binom{i}{r}\binom{k-i}{j-r}(-1)^{i-r}\, s^{i+j-2r}c^{k+2r-i-j+1}\right)\\
&= \frac{\sqrt{\binom{k+1}{i}}}{\sqrt{\binom{k+1}{j}}}\sum_{r=max(0,i+j-k-1)}^{min(i,j)}\binom{i}{r}\left(\binom{k-i}{j-1-r}+\binom{k-i}{j-r}\right)(-1)^{i-r}\, s^{i+j-2r}c^{k+2r-i-j+1}\\
&= \frac{\sqrt{\binom{k+1}{i}}}{\sqrt{\binom{k+1}{j}}}\sum_{r=max(0,i+j-k-1)}^{min(i,j)}\binom{i}{r}\binom{k+1-i}{j-r}(-1)^{i-r}\, s^{i+j-2r}c^{k+2r-i-j+1}\\
&= \left(\mathcal{P}_{k+1}\left(\theta\right)\right)_{i,j}
\end{aligned}
$$

The proposed closed form for $\mathcal{P}_k\left(\theta\right)$ respects equation 8. Concerning equation 9, exactly like equation 8, we develop the right term to obtain a vector $\left(a,b\right)^T \in \mathbb{R}^2$

$$
\begin{aligned}
a &= \frac{1}{\sqrt{j+1}}\left(cos\left(\theta\right)\sqrt{i}\left(\mathcal{P}_k\left(\theta\right)\right)_{i-1,j} + sin\left(\theta\right)\sqrt{k-i+1}\left(\mathcal{P}_k\left(\theta\right)\right)_{i,j}\right)\\
&= \frac{1}{\sqrt{j+1}}\left(\sqrt{i}\frac{\sqrt{\binom{k}{i-1}}}{\sqrt{\binom{k}{j}}}\sum_{r=max(0,i+j-k-1)}^{min(i-1,j)}\binom{i-1}{r}\binom{k-i+1}{j-1-r}(-1)^{i-r-1}\, s^{i+j-2r-1}c^{k+2r-i-j+2}\right.\\
&\quad \left.+\sqrt{k-i+1}\frac{\sqrt{\binom{k}{i}}}{\sqrt{\binom{k}{j}}}\sum_{r=max(0,i+j-k)}^{min(i,j)}\binom{i}{r}\binom{k-i}{j-r}(-1)^{i-r}\, s^{i+j-2r+1}c^{k+2r-i-j}\right)\\
&= \frac{\sqrt{\binom{k+1}{i}}}{\sqrt{\binom{k+1}{j+1}}}\sum_{r=max(0,i+j-k)}^{min(i,j+1)}\binom{i}{r}\binom{k-i+1}{j-r+1}(-1)^{i-r}\, s^{i+j+1-2r}c^{k+2r-i-j}\\
&= \left(\mathcal{P}_{k+1}\left(\theta\right)\right)_{i,j+1}
\end{aligned}
$$

and

$$
\begin{aligned}
b &= \frac{1}{\sqrt{k-j+1}}\left(-sin\left(\theta\right)\sqrt{i}\left(\mathcal{P}_k\left(\theta\right)\right)_{i-1,j}+cos\left(\theta\right)\sqrt{k-i+1}\left(\mathcal{P}_k\left(\theta\right)\right)_{i,j}\right)\\
&= \frac{1}{\sqrt{k-j+1}}\left(\sqrt{i}\frac{\sqrt{\binom{k}{i-1}}}{\sqrt{\binom{k}{j}}}\sum_{r=max(0,i+j-k-1)}^{min(i-1,j)}\binom{i-1}{r}\binom{k-i+1}{j-1-r}\left(-1\right)^{i-r}s^{i+j-2r}c^{k+2r-i-j+1}\right.\\
&\qquad\left.+\sqrt{k-i+1}\frac{\sqrt{\binom{k}{i}}}{\sqrt{\binom{k}{j}}}\sum_{r=max(0,i+j-k)}^{min(i,j)}\binom{i}{r}\binom{k-i}{j-r}\left(-1\right)^{i-r}s^{i+j-2r}c^{k+2r-i-j+1}\right)\\
&= \frac{1}{\left(k-j+1\right)}\frac{\sqrt{\binom{k+1}{i}}}{\sqrt{\binom{k+1}{j}}}\left(\sum_{r=max(0,i+j-k-1)}^{min(i,j)}\binom{i}{r}\binom{k-i+1}{j-r}\left(i-r\right)\left(-1\right)^{i-r}s^{i+j-2r}c^{k+2r-i-j+1}\right.\\
&\qquad\left.+\sum_{r=max(0,i+j-k-1)}^{min(i,j+1)}\binom{i}{r}\binom{k-i+1}{j-r}\left(k-i+1-j+r\right)\left(-1\right)^{i-r}s^{i+j-2r}c^{k+2r-i-j+1}\right)\\
&= \frac{\sqrt{\binom{k+1}{i}}}{\sqrt{\binom{k+1}{j}}}\sum_{r=max(0,i+j-k-1)}^{min(i,j+1)}\binom{i}{r}\binom{k-i+1}{j-r}\left(-1\right)^{i-r}s^{i+j-2r}c^{k+2r-i-j+1}\\
&= \left(\mathcal{P}_{k+1}\left(\theta\right)\right)_{i,j}
\end{aligned}
$$

The proposed closed form for $\mathcal{P}_k\left(\theta\right)$ respects equation 9. The orthogonality property proved in [1] applies to $\mathcal{P}_k\left(\theta\right)$.

References

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