CMPSC 465 — Data Structures & Algorithms
Fall 2024 — Mingfu Shao and Debarati Das

HW 13

Due December 6th (Friday), 11:59 pm

Formatting: Each problem should begin on a new page. When submitting in Gradescope, try to assign pages to problems from the rubric as much as you can. Make sure you write all your group members' names. For the full policy on assignments, consult the syllabus.

1. (20 pts.) Consider a scenario where C is the set of clients and S is the set of servers. Each client i in set C can be served by server j in set S with a service cost c_{ij}. Because of the emerging issue of carbon emission and global warming at most k servers can stay functional at a time. Our objective is to find the optimal set of k servers so that each client can be served while minimizing the total service cost. Let OPT be the sum of the service costs over all clients for the optimal solution.

   1. Design an LP representing the problem. [clearly describe in language the variables, describe the constraints and objective function]
   2. Write the dual LP.

   Solution:

   (1) The problem is modeled by the following ILP:

$$
\begin{aligned}
\text{minimize} \quad & \sum_{j \in S, i \in C} c_{ij} x_{ij} \\
\text{subject to} \quad & \sum_{j \in S} x_{ij} = 1, \qquad \forall i \in C \\
& \sum_{j \in S} y_j \leq k \\
& x_{ij} \leq y_j \qquad\quad \forall j \in S, i \in C \\
& x_{ij}, y_j \in \{0,1\}, \forall j \in S, i \in C,
\end{aligned}
$$

   where $x_{ij} = 1$ iff the client i is served by the server j and $y_j = 1$ iff server j is functional.

   Whenever j is serving i, then c_{ij} is added to the objective, meaning that the objective is the total cost of servers serving clients. The first constraint ensures that each client is served by exactly one server. The second constraint ensures that there are at most k servers in the solution. The next constraint ensures that a client cannot be served by a server that is not in the solution.

   We get the LP relaxation of this problem by replacing $\in \{0, 1\}$ by ≥ 0.

(2) The dual LP is

$$\begin{array}{ll}\text{maximize} & \sum_{i\in C} a_i - kq \\ \text{subject to} & a_i - b_{ij} \le c_{ij},\ \forall j \in S, i \in C \\ & \sum_{i\in C} b_{ij} - q \le 0 \forall j \in S \\ & q \ge 0 \\ & a_i \ge 0 \qquad\qquad \forall i \in C \\ & b_{ij} \ge 0, \qquad\quad \forall j \in S, i \in C.\end{array}$$

2. (10 pts.) Convert the following LP into standard form form 1:

$$\begin{array}{ll}\text{maximize} & 2x_1 + 7x_2 + x_3 \\ \text{subject to} & x_1 - x_3 = 7 \\ & 3x_1 + x_2 \ge 24 \\ & x_2 \ge 0 \\ & x_3 \le 0.\end{array}$$

Solution:

Set $x_3 = -x_4$ and $x_1 = x_1^+ - x_1^-$

$$\begin{array}{ll}\text{maximize} & 2x_1^+ - 2x_1^- + 7x_2 - x_4 \\ \text{subject to} & -x_1^+ + x_1^- - x_4 \le -7 \\ & x_1^+ - x_1^- + x_4 \le 7 \\ & -3x_1^+ + 3x_1^- - x_2 \le -24 \\ & x_1^+, x_1^-, x_2, x_4 \ge 0\end{array}$$

3. (15 pts.)

A film producer wants to make a motion picture. For this she needs to choose among n available actors. Actor i demands a payment of s_i dollars to participate in the picture. The funding of the picture will come from m investors. The k_{th} investor will pay the producer p_k dollars, but only under the following condition. The investor has a list of actors $L_k \subseteq \{1, \ldots, n\}$, and he will only invest iff all the actors on his list appear in the picture. The profit of the producer is the sum of payments from the investors that she agrees to take funding from, minus the sum of payments she makes to the actors that appear in the picture. The goal is to maximize the producer's profit. Give an integer linear programming to model this problem.

Solution:

We write the integer linear program for the problem. The variables are x_i, $1 \le i \le n$, one per actor, and $y_k, 1 \le k \le m$, one per investor. The intention is to have $x_i = 1$ if actor i plays in the movie and 0 otherwise, and $y_k = 1$ if investor k signs up for funding and 0 otherwise. With this in mind, we write the following linear program:

$$\begin{array}{lll}\text{maximize} & \sum_{k=1}^{m} p_k y_k - \sum_{i=1}^{n} s_i x_i & \\ \text{subject to} & x_i \ge y_k & \text{for all } k \text{ and } i \in L_k \\ & x_i, y_k \in \{0, 1\} & \text{for all } i, k\end{array}$$

The constraint says that whenever $i \in L_k$, choosing investor k (setting $y_k = 1$) implies choosing actor i (that is, $x_i = 1$).