CMPSC 465 Data Structures & Algorithms
Fall 2024 Mingfu Shao and Debarati Das

HW 9

Due November 1st (Friday), 11:59 pm

Formatting: Each problem should begin on a new page. When submitting in Gradescope, try to assign pages to problems from the rubric as much as you can. Make sure you write all your group members' names. For the full policy on assignments, consult the syllabus.

1. (20 pts.)
Let $G = (V, E)$ be an undirected graph with cost $c(e) \geq 0$ on the edges $e \in E$. Assume we are given an MST T in G. Now assume that a new edge $e = (u, v)$ is added to G and $c(e) = R$.

   1. Give an $O(|E|)$ time algorithm to test if T remains the MST for the modified graph (i.e., G with the new edge e). Analyze the running time. Can you do it in time $O(|V|)$?
   2. Suppose T is no longer the MST. Give an $O(|E|)$ time algorithm to update the tree T to the new MST.

Solution:

(a) Adding e to T, we obtain exactly one cycle $C \subseteq E(T) \cup \{e\}$. We claim e is (one of) the most expensive edge in C if and only if T remains the MST for the modified graph G'. Finding this cycle takes $O(E)$ steps, using the standard cycle detection algorithm. Moreover, if we take $(V, E(T) \cup \{e\})$ as input to the cycle detection algorithm, the running time is merely $O(|V|)$. Checking if there exists a path from u to v in T is in time $O(|V|)$.

To see ($\Leftarrow$) of the claim, if an edge $f \in E(T)$ has weight $c(f) > c(e)$, then $c(T - f + e) < c(T)$, which shows that T is not a MST for G'. Now we show ($\Rightarrow$) of the claim. By the cycle property, there exists a MST of G' that does not use e. In particular, any MST of G would still be a MST for G'.

(b) Let f be the most expensive edge on C, where C is defined as above. We return a new tree $T - f + e$ as the new MST. We show next why $T - f + e$ is a MST for the modified graph G'.

Suppose for contradiction that there is a spanning tree T^* of G' with costs $c(T^*) < c(T - f + e)$. Let $e = (u, v)$. T^* must contain e because T^* is not a MST for the initial graph G. If we remove e from T^*, the tree T^* would be disconnected as two connected components U and V such that $u \in U$ and $v \in V$. Note that $P := C \setminus \{e\}$ is a path connecting U and V and that $P \setminus E(T^*) \neq \emptyset$ since T^* does not contain the whole cycle C. Thus we can find an edge $f' \in P \setminus E(T^*)$ that re-connects U and V, and forms a new spanning tree $T_2 := T^* - e + f'$. Because T_2 does not make use of e, T_2 is in fact a spanning tree of G. However, the existence of T_2 implies that T is not a MST for G, since

$$c(T_2) = c(T^*) - c(e) + c(f') < c(T - f + e) - c(e) + c(f') \leq c(T).$$

Therefore, there is no spanning tree of G' with lower cost than that of $T - f + e$.

2. (15 pts.)
   Let T be an MST of graph G. Given a connected subgraph H of G, show that $T \cap H$ is contained in some MST of H.

   Solution:

   Let $T \cap H = \{e_1, \ldots, e_k\}$. We use the cut property repeatedly to show that there exists an MST of H containing $T \cap H$. Suppose for $i < k$, $X = \{e_1, \ldots, e_i\}$ is contained in some MST of H. Removing the edge e_{i+1} from T divides T in two parts giving a cut $(S, G \setminus S)$ and a corresponding cut $(S_1, H \setminus S_1)$ of H with $S_1 = S \cap H$. Now, e_{i+1} must be the lightest edge in G (and hence also in H) crossing the cut, else we can include the lightest and remove e_{i+1} getting a better tree. Also, no other edges in T, and hence also in X, cross this cut. We can then apply the cut property to get that $X \cup e_{i+1}$ must be contained in some MST of H. Continuing in this manner, we get the result for $T \cap H = \{e_1, \ldots, e_k\}$.

3. (15 pts.)
   Let T be a minimum spanning tree of a graph $G = (V, E)$ and V' be a subset of V. Let T' be a subgraph of T induced by V' (i.e., an edge $(u, v) \in T$ is present in T' iff both $u, v \in V'$) and G' be a subgraph of G induced by V'. Show that if T' is connected, then T' is a minimum spanning tree of G'.

   Solution:

   For contradiction assume T' is not a minimum spanning tree of G'. Let T'' be a MST of G' and thus $w(T'') < w(T')$. Let S be the set of edges that are present in T but not in T'. We show that we can then construct a spanning tree $\tilde{T}$ of G by considering the edges $S \cup T''$ such that $w(\tilde{T}) < w(T)$ and we get a contradiction as we assumed T to be a MST of G. Thus our assumption that there was a spanning tree of V' cheaper than T' must be false.

   First we argue that $\tilde{T}$ is a spanning tree of G. This is because $S \cup T'$ is a spanning tree of G, and T'' makes all the vertices in V' connected just like T' does. Next, we have $w(S \cup T'') = w(S) + w(T'') < w(s) + w(T') = w(S \cup T') = w(T)$. The second inequality follows as by our assumption $w(T'') < w(T')$. The third equality follows from the definition of S.