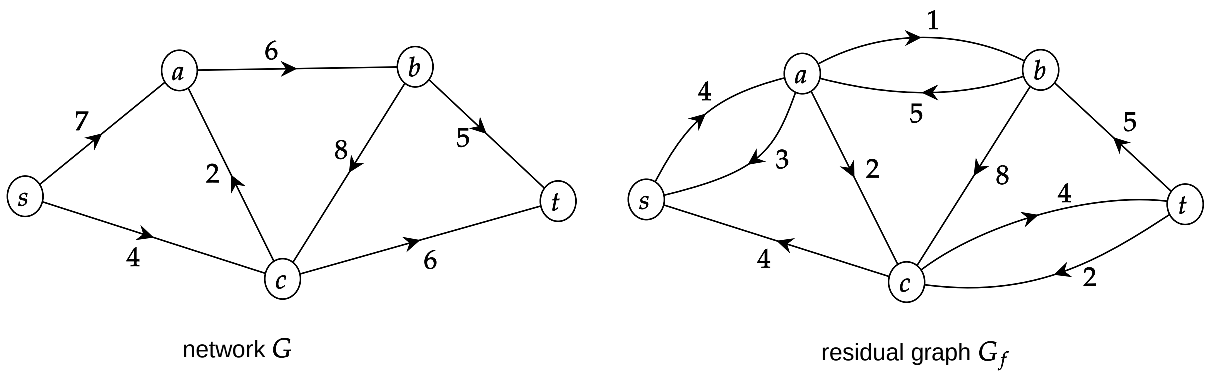
Mid-Term 1 Part 1

1. (3 pts.) The left figure shows a network G and the right figure shows the residual graph G_f with respect to a certain flow f. (The nubmers next to edges are capacities for both graphs.) What is the value of this flow f?

   A. 3
   B. 4
   C. 5
   D. 7



Ans: D

2. (3 pts.) If the weight of each edge of a graph is increased by 50, then the minimum spanning tree stays the same.

   A. True
   B. False

Ans: A

3. (3 pts.) Let C be one minimum s-t cut of a network and let $e \in C$ be one edge in it. Suppose that we increase the capacity of e by 1, then the value of the maximum flow will increase by 1.

   A. True for all networks.
   B. True for some networks but not all.
   C. False for all networks.

Ans: B

4. (3 pts.) Given a graph, Prim's and Krusakl's algorithms output the same minimum spanning tree.

   A. Always

B. Never

C. If the edge weights are unique

D. If the minimum spanning tree is unique

E. Both C and D

Ans: E

5. (3 pts.) Statement A: for a network the value of its maximum flow is always equal to the capacity of its minimum s-t cut. Statement B: for a bipartite graph the number of vertices in its minimum vertex-cover is always equal to the number of edges in its maximum matching.

A. A is true and B is true.

B. A is true and B is false.

C. A is false and B is true.

D. A is false and B is false.

Ans: A

6. (3 pts.) (111, 000, 010, 011, 100, 001) is a

A. Huffman code

B. Prefix-free code

C. Both prefix-free and Huffman code

D. None of the above

Ans: B

7. (3 pts.) Identify the valid Horn formulas

A. $\bar{x} \vee \bar{y} \vee z, \Rightarrow w$

B. $\bar{x} \vee \bar{y} \vee \bar{z} \vee \bar{w}$, $x \vee y \Rightarrow z$

C. $\bar{x} \vee \bar{y} \vee \bar{z} \vee \bar{w}$, $x \wedge \bar{y} \Rightarrow z$

D. All of the above

E. None of the above

Ans: E

8. (3 pts.) In the greedy set cover in every iteration the algorithm selects the subset that contains

A. maximum number of items

B. minimum number of items

C. maximum number of uncovered items

D. maximum number of covered items

Ans: C

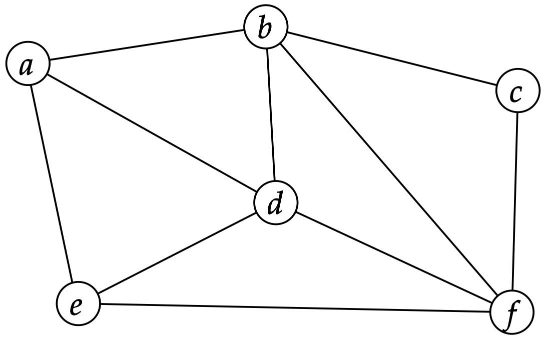
9. (3 pts.) Which one of the following is NOT true for the undirected graph G given below?

A. Maximum matching of G contains 3 edges.

B. Minimum vertex-cover of G contains 4 vertices.

C. $\{a, c, e, f\}$ is a vertex-cover of G.

D. The maximum matching of G is not unique.



Ans: C

10. (3 pts.) In Kruskal's algorithm, any intermediate solution is always a single connected component of the MST.

A. True

B. False

Ans: B

11. (7+5+3 pts.) In a semester the computer science department offers n courses to m students. Due to prerequisite requirements, each student has a specific set of courses the student is eligible to take. The i-th course has a maximum capacity of c_i students, $1 \leq i \leq n$. Each student can enroll in up to 4 courses. Design a polynomial-time algorithm that assigns courses to student so that the total number of course enrollments across all courses is maximized. Describe your algorithm, prove its correctness, and analyze its time complexity.

Solution (Network Flows Approach.)
Algorithm: For each student j where $1 \leq j \leq m$, let $C_j \subseteq \{1, 2, \dots n\}$ denote the set of courses which student j can take according to their prerequisites. Let (G, c) be a network defined as:

- $G = (V, E)$ is a digraph which has one vertex s_j for each student j, one vertex t_i for each course i, and two designated vertices, namely the source s and sink t. Formally,

$$V = \{s_1, ..s_m\} \cup \{t_1, \dots t_n\} \cup \{s, t\}$$

  The edges are described as follows: The source has an outgoing edge to each vertex corresponding to a student, the sink has an incoming edge from each vertex corresponding to a course and for each student j, there is an edge from s_j to t_i iff the ith course is in student j's requirements i.e. $i \in C_j$. Formally,

$$E = \{(s, s_j) \, : \, 1 \leq j \leq m\} \cup \{(t_i, t) \, : \, 1 \leq i \leq n\} \cup \{(s_j, t_i) \, : \, 1 \leq j \leq m, 1 \leq i \leq n, i \in C_j\}$$

- $c : E \to \mathbb{R}_{\geq 0}$ is the capacity function defined as: The capacity of each edge from the source to a vertex corresponding to a student is 4, the capacity of each edge from the vertex corresponding to course i to the sink is c_i, and finally the capacity of each edge from a vertex corresponding to a student to a vertex corresponding to a course is 1. Formally,

$$c(e) = \begin{cases} 4 & e = (s, s_j) \\ c_i & e = (t_i, t) \\ 1 & e = (s_j, t_i) \end{cases}$$

We run Ford Fulkerson on the network (G, c) to find the value of the maximum flow $|f^*|$, which we output as our answer for the maximum possible course enrollments.
Correctness. We show that $|f^*|$ is indeed the maximum possible course enrollment. We first identify any course enrollment as a mapping from each student to the courses they enroll in i.e. as a function E, from $\{1, \dots m\}$ to $\mathcal{P}(\{1, \dots n\})$ s.t. $E(j) \subseteq C_j$, $|E(j)| \leq 4$ and $|\{j \ : \ i \in E(j)\}| \leq c_i$. Note that the total number of course enrollments for E is given by $\sum_{j=1}^{m} |E(j)|$. Let $\mathcal{E}$ represent the set of all possible course enrollments and let $\mathcal{F}$ denote the set of all valid flows in (G, c). We then show the following lemma:

Lemma. The mapping $T : \mathcal{E} \to \mathcal{F}$ defined by $T(E) = f$ where

$$f(e) = \begin{cases} |E(j)| & e = (s, s_j) \\ |\{j \ : \ i \in E(j)\}| & e = (t_i, t) \\ 1 & e = (s_j, t_i) \wedge i \in E(j) \\ 0 & e = (s_j, t_i) \wedge i \notin E(j) \end{cases}$$

is a bijection and furthermore $|f| = \sum_{j=1}^{m} |E(j)|$.

Proof: We can easily check that for any course enrollment $E \in \mathcal{E}$, $T(E)$ is a valid flow in (G, c) by using the constraints on E.
To check that T is bijective, simply note that the function $G : \mathcal{F} \to \mathcal{E}$ defined by $G(f) = \hat{E}$ where $\hat{E}(j) := \{i \ : \ (s_j, t_i) \in E(G), f((s_j, t_i)) = 1\}$ is the inverse function to T by checking that $T \circ G = id_{\mathcal{F}}$ and $G \circ T = id_{\mathcal{E}}$.
Finally checking that $|f| = \sum_{j=1}^{m} |E(j)|$ if $T(E) = f$ follows trivially from the definition of $|f|$. $\square$
Thus the number of course enrollments for any course enrollment is the value of some flow of (G, c), and hence the maximum number of course enrollments is the value of the maximum flow $|f^*|$.
Time Complexity. Ford fulkerson takes $O(|f^*| \cdot (|V| + |E|))$. For our network $|V| = n + m + 2$, $|E| \leq nm + n + m$ and most importantly $|f^*| \leq 4m$ (if you do not notice this last statement, you only get a pseudo-polynomial time algorithm and not a polynomial algorithm) as there are m outgoing edges from the source s, each with a maximum capacity of 4. Thus the running time of this algorithm is $O(m^2 n)$.