Maximum Flow and Minimum Cut

A *network* (also called *flow network*) models an abstracted flow (traffic, data, etc) transmits from an origin to a destination via a graphical structure. Formally, a (flow) network is a tuple $(G = (V,E), s, t, c(\cdot))$, where

1. $G = (V,E)$ is a directed graph;
2. $s \in V$ is a source vertex of G (i.e., the in-degree of s is 0), from which the flow is generated;
3. $t \in V$ is a sink vertex of G (i.e., the out-degree of t is 0), at which the flow is absorbed;
4. $c : E \to \mathbb{R}^+$, in which $c(e)$ represents the *capacity* of edge $e \in E$, which models the limit of the flow that edge e can carry.

An *s-t flow* of network $(G = (V,E), s, t, c(\cdot))$ describes how the flow is transmitted from s to t via the graph G under the capacity constraints. Formally, an *s-t* flow f is a function $f : E \to \mathbb{R}_{\geq 0}$ that satisfies the following two conditions:

1. For every $e \in E$, $0 \leq f(e) \leq c(e)$; this is called the *capacity condition*;
2. For every vertex $v \in V \setminus \{s,t\}$, $\sum_{e \in I(v)} f(e) = \sum_{e \in O(v)} f(e)$, where $I(v) := \{(u,v) \in E \mid u \in V\}$ represents the set of in-edges of v, and $O(v) := \{(v,w) \in E \mid w \in V\}$ represents the set of out-edges of v. This is called the *conservation condition*.

Intuitively, an *s-t* flow assigns an non-negative value $f(e)$ to edge e, representing the amount of flow that is carried by edge e; such amount must not exceed the capacity of e, i.e., $f(e) \leq c(e)$, for every $e \in E$. All vertices, except the source s that generates the flow and the sink that absorbs the flow, redistribute the flow (instead of repositing any flow); therefore, for each vertex $v \in V \setminus \{s,t\}$, the total amount of flow that enters v equals to the total amount of flow that leaves v. See Figure 1.

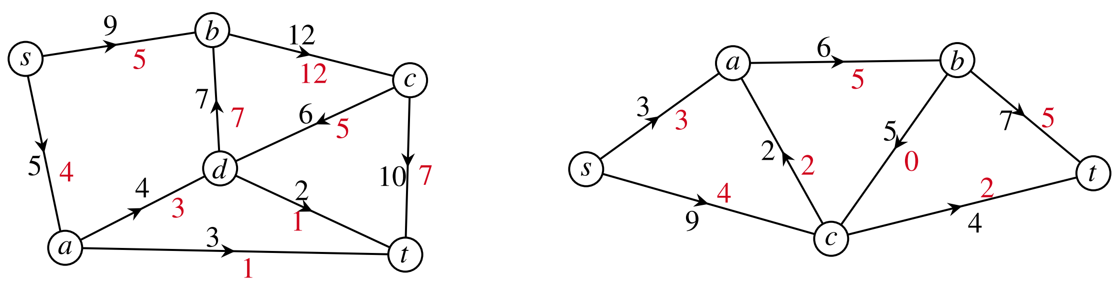


Figure 1: Two examples of networks and *s-t* flows. The capacities are marked as black numbers next to edges; the flow values are marked as red numbers next to edges. The value of the flow in the left exampe is 9; the value of the flow in the right exampe is 7;

The *value* of an *s-t* flow f, denoted as $|f|$, is defined as the total amount of flow that is generated from source vertex s: $|f| := \sum_{e \in O(s)} f(e)$. Intuitively, the total amount of flow that is generated from s eventually must be fully absorbed by sink t, as all internal vertex (i.e., $V \setminus \{s,t\}$) does not keep any flow. Therefore, we must have that $|f| = \sum_{e \in I(t)} f(e)$. This property is also a direct consequence of the conservation property. Below we formally state and prove it.

Fact 1. We have $\sum_{e \in O(s)} f(e) = \sum_{e \in I(t)} f(e)$.

Proof. According to the conservation condition: $\sum_{e \in I(v)} f(e) = \sum_{e \in O(v)} f(e)$, for every vertex $v \in V \setminus \{s,t\}$. We sum up both sides over all $v \in V \setminus \{s,t\}$: we have $\sum_{v \in V \setminus \{s,t\}} \sum_{e \in I(v)} f(e) = \sum_{v \in V \setminus \{s,t\}} \sum_{e \in O(v)} f(e)$.

We now break down the left side $L := \sum_{v \in V \setminus \{s,t\}} \sum_{e \in I(v)} f(e) = \sum_{v \in V} \sum_{e \in I(v)} f(e) - \sum_{e \in I(s)} f(e) - \sum_{e \in I(t)} f(e)$. Notice also that $\sum_{v \in V} \sum_{e \in I(v)} f(e) = \sum_{e \in E} f(e)$. Since s is a source vertex, we have $I(s) = \emptyset$, and therefore $\sum_{e \in I(s)} f(e) = 0$. Hence, $L = \sum_{e \in E} f(e) - \sum_{e \in I(t)} f(e)$.

Similarly, the right side $R := \sum_{v \in V \setminus \{s,t\}} \sum_{e \in O(v)} f(e) = \sum_{v \in V} \sum_{e \in O(v)} f(e) - \sum_{e \in O(s)} f(e) - \sum_{e \in O(t)} f(e)$. Again we have $\sum_{v \in V} \sum_{e \in O(v)} f(e) = \sum_{e \in E} f(e)$. Since t is a sink, we have $O(t) = \emptyset$, and therefore $\sum_{e \in O(t)} f(e) = 0$. Hence, $R = \sum_{e \in E} f(e) - \sum_{e \in O(s)} f(e)$.

Combining $L = R$ and noticing that $\sum_{e \in E} f(e)$ is shared, we have $\sum_{e \in O(s)} f(e) = \sum_{e \in I(t)} f(e)$, as desired. $\square$

We now define the *maximum-flow problem*: given network $(G = (V,E), s, t, c(\cdot))$, to find an s-t flow f such that $|f|$ is maximized.

See Figure 2 for an example, which shows a flow with value of 9. This flow has a larger value than the flow in Figure 1 (right panel, same network, but with a flow of value 7). Can you play with this example to try to obtain a flow with even larger value? Is this flow (of value 9) the maximum flow of the network? If it is, how can we verify it is one maximum flow? Let's answer these questions using s-t cut.

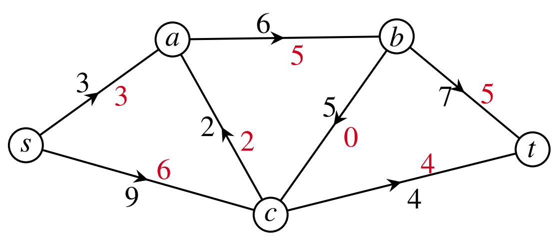


Figure 2: A flow with value 9 (same network with the right panel of Figure 1.

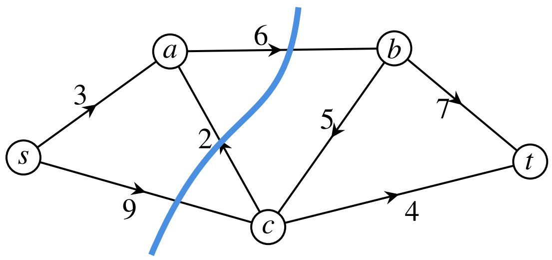
Given a network $G = (V,E)$ with source $s \in V$, sink $t \in V$, and capacity $c(\cdot)$, an *s-t cut* of the network is a pair (S,T), where $S \subset V$, $T = V \setminus S$, and that $s \in S$ and $t \in T$. In short, an s-t cut of a network is a partition of the vertices that separates the source and the sink.

An s-t cut (S,T) of a network also partitions all edges in the graph into four disjoint subsets, described below. We define the first category, i.e., $E(S,T) := \{(u,v) \in E \mid u \in S, v \in T\}$, as the *cut-edges* w.r.t. the s-t cut (S,T).

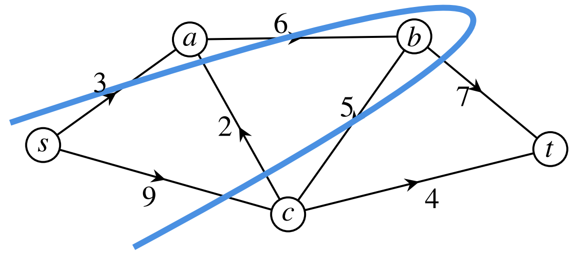
1. Edges that span the cut and point from s-side to t-side, formally $E(S,T) := \{(u,v) \in E \mid u \in S, v \in T\}$;
2. Edges that span the cut and point from t-side to s-side, formally $E(T,S) := \{(u,v) \in E \mid u \in T, v \in S\}$;
3. Edges with both end-vertices in s-side, formally $E(S,S) := \{(u,v) \in E \mid u \in S, v \in S\}$;
4. Edges with both end-vertices in t-side, formally $E(T,T) := \{(u,v) \in E \mid u \in T, v \in T\}$.

Let (S,T) be an s-t cut of a network $G = (V,E)$ with source s, sink t, and capacity $c(e)$ for any $e \in E$. We define the *capacity* of (S,T), denoted as $c(S,T)$, as the sum of the capacities of its cut-edges. Formally as $c(S,T) := \sum_{e \in E(S,T)} c(e)$. See Figure 3.

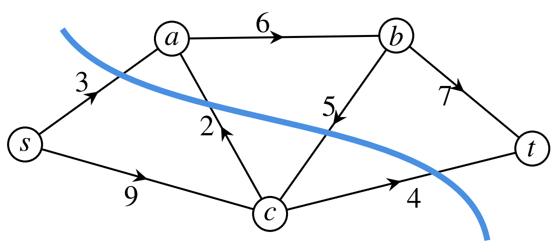
We now introduce the so-called *minimum-cut problem*: given network $G = (V,E)$ with source s, sink t and capacity $c(e)$ for any $e \in E$, we seek an s-t cut (S,T) such that its capacity $c(S,T)$ is minimized.



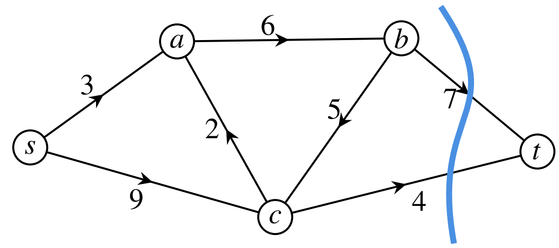
$S = \{s,a\},\ T = \{b,c,t\}$
$c(S,T) = 6+9 = 15$



$S = \{s,b\},\ T = \{a,c,t\}$
$c(S,T) = 3 + 9+5+7 = 24$



$S = \{s,c\},\ T = \{a,c,t\}$
$c(S,T) = 3+2+4 = 9$



$S = \{s,a,b,c\},\ T = \{t\}$
$c(S,T) = 7+4 = 11$

Figure 3: Four different *s*-*t* cuts and their capacities of the same network.

Note that, the definition of *s*-*t* cut, capacity of an *s*-*t* cut, and above minimum-cut problem, are completely independent of flow. Note too that, the input of the maximum-flow problem and the minimum-cut problem is the same: a network (consisting of a directed graph, source and sink, and edge capacities).

Although the maximum-flow problem and the minimum-cut problem are defined independently, they can be solved by the same algorithm (for example, the Ford-Fulkson algorithm) and are connected by an elegant theorem (i.e., the max-flow min-cut theorem). Below, we first show half of this theorem. Later on we will introduce the Ford-Fulkson algorithm and use it to prove the other other half of the theorem.

We now show that, the capacity of *any* *s*-*t* cut of a network gives an *upper bound* of the value of *any* *s*-*t* flow of the same network, formally stated below.

Claim 1. Let $(G=(V,E),s,t,c(\cdot))$ be a network. Let f be an arbitrary *s*-*t* flow and let (S,T) be an arbitrary *s*-*t* cut, of this network. We have $|f| \le c(S,T)$.

Above claim is intuitive to understand: an *s*-*t* flow f transfer $|f|$ units of flow from source s to sink t; since an *s*-*t* cut (S,T) separates s and t, so the flow, all generated from s, must cross the cut in order to reach t, and such "crossing" must use the cut-edges. Hence, the total amount of the flow that can be transferred, i.e., the value of f, is limitted by the total capacities of the cut-edges, i.e., the capacity of the *s*-*t* cut. See Figure 4.

In order to formally prove above claim, we first show that, the value of an *s*-*t* flow f can also be calculated by examining the flow carried by the "spanning edges" of any *s*-*t* cut. Formally,

Fact 2. Let f be an arbitrary *s*-*t* flow and let (S,T) be an arbitrary *s*-*t* cut, of network $(G=(V,E),s,t,c(\cdot))$. We have $|f| = \sum_{e \in E(S,T)} f(e) - \sum_{e \in E(T,S)} f(e)$.

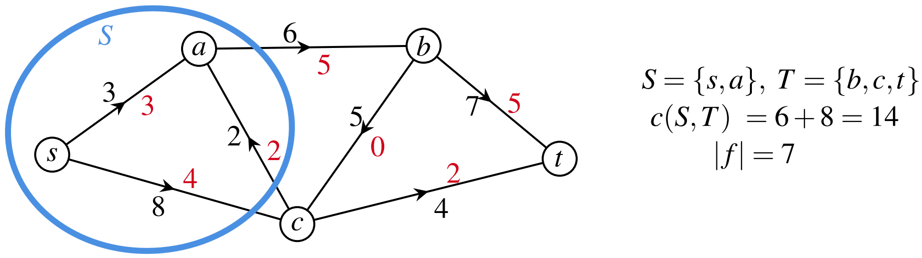


Figure 4: An example showing Claim 1 and Fact 2.

Above fact is also intuitive to understand: S is a subset that includes s but excludes t. See Figure 4. When thinking S as a whole, the total amount of flow generated by S is equal to $|f|$, as only s generates flow. On the other hand, this amount of flow emitted from S can also be calcuated by summing up the flow leaves S, i.e., $\sum_{e \in E(S,T)} f(e)$, and subtracting the amount of the flow that enters back to S, i.e., $\sum_{e \in E(T,S)} f(e)$.

We now formally prove Fact 2. Similar to the proof of Fact 1 of Lecture 36, this proof also simply applies the conservation property.

Proof of Fact 2. According to conservation property, for any $v \in S \setminus \{s\}$, we have $\sum_{e \in I(v)} f(e) = \sum_{e \in O(v)} f(e)$. We sum up both sides over all $v \in S \setminus \{s\}$: we have $\sum_{v \in S \setminus \{s\}} \sum_{e \in I(v)} f(e) = \sum_{v \in S \setminus \{s\}} \sum_{e \in O(v)} f(e)$. Note that $I(s) = \emptyset$ as s is a source, so the left side of above equation can be rewritten as $\sum_{v \in S} \sum_{e \in I(v)} f(e)$. Note that $|f| = \sum_{e \in O(s)} f(e)$ by definition, so the right side can be rewritten as $\sum_{v \in S} \sum_{e \in O(v)} f(e) - |f|$. Combined, we have $|f| = \sum_{v \in S} \sum_{e \in O(v)} f(e) - \sum_{v \in S} \sum_{e \in I(v)} f(e)$. Now consider the 4 types of edges partitioned by (S,T) and consider if they are taken into account by the two items i.e., $\sum_{v \in S} \sum_{e \in O(v)} f(e)$ and $\sum_{v \in S} \sum_{e \in I(v)} f(e)$.

1. All edges in $E(S,T)$ appear in the first item, but not the second one.
2. All edges in $E(T,S)$ appear in the second item, but not the first one.
3. All edges in $E(S,S)$ appear in both items (so they cancel out).
4. None of the edges in $E(T,T)$ appear in either item.

Therefore, we have $|f| = \sum_{e \in E(S,T)} f(e) - \sum_{e \in E(T,S)} f(e)$. □

We now formally prove Claim 2. The proof simply combines the capacity condition (and the non-negative property) of an *s*-*t* flow, Fact 2, and the definition of the capacity of an *s*-*t* cut.

Proof of Claim 1. Since $0 \le f(e) \le c(e)$ for any $e \in E$, based on Fact 2, we can write $|f| = \sum_{e \in E(S,T)} f(e) - \sum_{e \in E(T,S)} f(e) \le \sum_{e \in E(S,T)} c(e) - 0 = c(S,T)$. □

Claim 1 states that capacity of *any* *s*-*t* cut is larger or equal to the value of *any* *s*-*t* flow. Consequently, if there exists an *s*-*t* flow f and an *s*-*t* cut (S,T) satisfies that $|f| = c(S,T)$, then this f must be a maximum-flow, and this (S,T) must be a minimum-cut. An illustration of this property is given in Figure 5, where each red

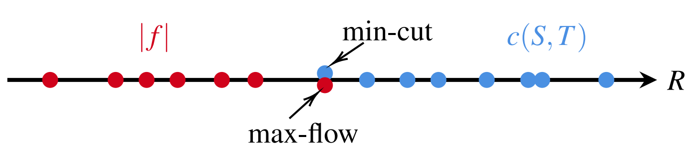


Figure 5: Illustration of the relationship between capacity of any *s*-*t* cut and value of *any* *s*-*t* flow.

point represents the value of an s-t flow while each blue point represents the capacity of an s-t cut. So all red points lie to the left side of all blue points, and if a red point and a blue point meet, they are maximum-flow and minimum-cut.

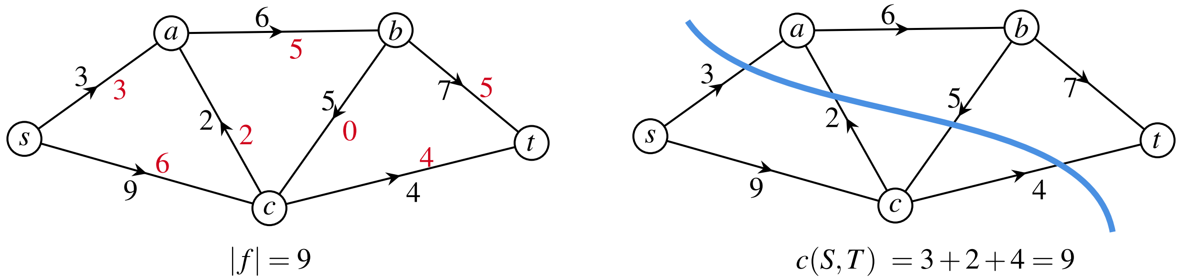


Figure 6: A maximum-flow f and a minimum cut (S,T), verified by $|f| = c(S,T)$.

In above example, we see a flow f with $|f| = 9$ and an s-t cut (S,T) with $c(S,T) = 9$. So both are optimal. See Figure 6.