CMPSC 465
Data Structures and Algorithms
Fall 2024

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November 15, 2024

Linear Programming
(Textbook, Section 7.1)

Background

Optimization: we want to maximize some objective function $f(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^n$, subject to constraints

$$C(\mathbf{x}) \leq \mathbf{b}, \text{ for } \mathbf{b} \in \mathbb{R}^n$$

- If no structures of f or C are known: general purpose constraint optimization
- Simplest non-trivial (but still powerful) case: f and C are linear functions, e.g., $f(\mathbf{x}) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$
  — **Linear Programming**

Example

Resource allocation: 168 hours in a week

S: study time; P: fun/party time; E: everything else

- to survive: $E \geq 56$
- to pass classes: $S \geq 60$
- to stay sane: $P + E \geq 70$
- $2S + E - 3P \geq 150$: need more study time if had too much fun or not enough sleep
- happiness: $2P + E$ **objective function**
  i.e., $f(S, P, E) = 2P + E$

How to allocate your time?

LP formulation

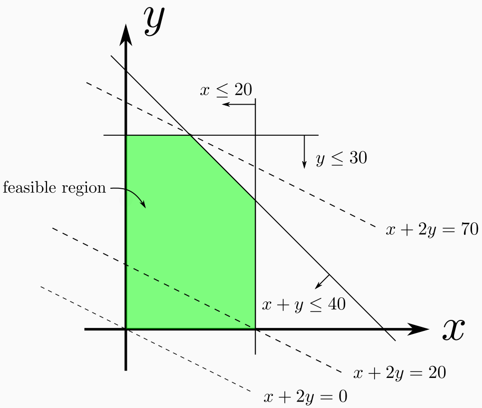
Maximize happiness: LP formulation:

$$
\begin{array}{lrcl}
\text{maximize} & 2P + E & & \\
\text{subject to} & E & \geq & 56 \\
 & S & \geq & 60 \\
 & 2S + E - 3P & \geq & 150 \\
 & S, P, E & \geq & 0 \\
 & S + P + E & \leq & 168
\end{array}
$$

How to solve an LP

Consider a simpler LP:

$$
\begin{array}{lrcl}
\text{maximize} & x + 2y & & \\
\text{subject to} & x & \leq & 20 \\
& y & \leq & 30 \\
& x + y & \leq & 40 \\
& x, y & \geq & 0
\end{array}
$$



Optimal solution: $x + 2y = 70$

Algorithm for solving LP

Observation: (search for an optimal solution)
Objective function is linear, and feasible region is convex. So a unique direction of maximal increase of objective function exists. Follow it and you will run into the boundary. At the boundary, moving in any direction will

(a) Decrease objective function $\rightarrow$ don't go this way

(b) Increase objective function $\rightarrow$ follow to a vertex

(c) Objective function stays constant $\rightarrow$ follow to a vertex

Theorem

For an LP with bounded, nonempty feasible region, the maximum value will be attained at some vertex of the feasible region

Algorithm idea

The hill climbing approach (the simplex method)

Start at a vertex, look at adjacent vertices, move in the direction of largest increase to the objective function

$$\begin{array}{lrcl} \text{maximize} & x + 2y & & \\ \text{subject to} & x & \leq & 20 \\ & y & \leq & 30 \\ & x + y & \leq & 40 \\ & x, y & \geq & 0 \end{array}$$

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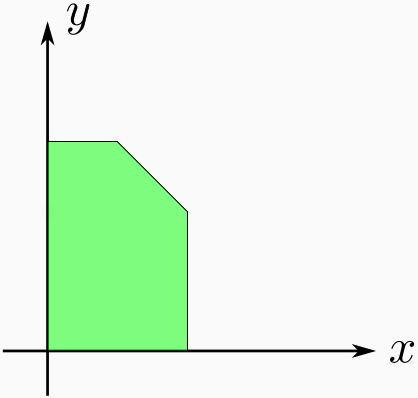

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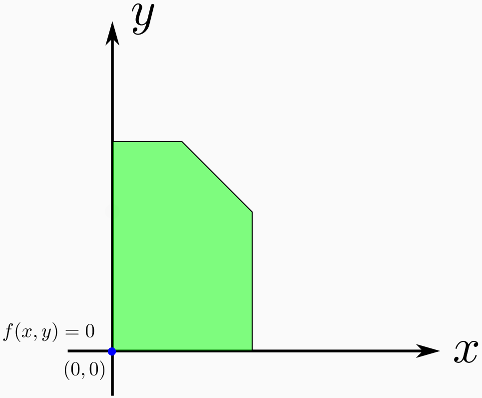

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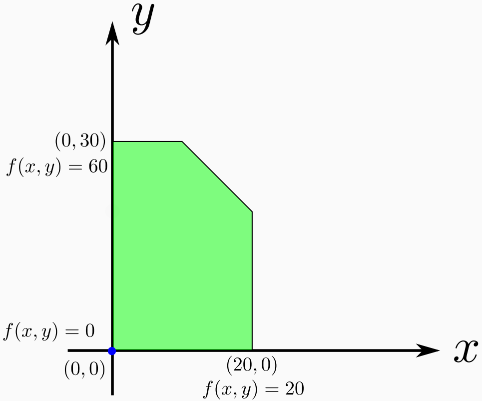

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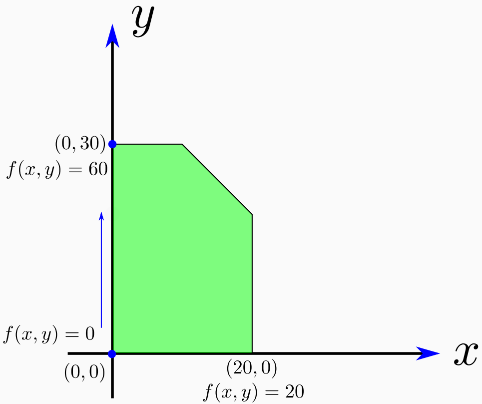

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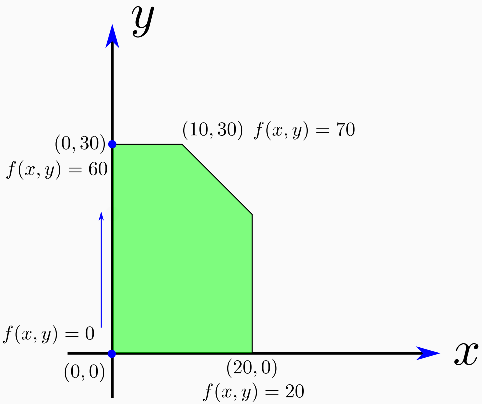

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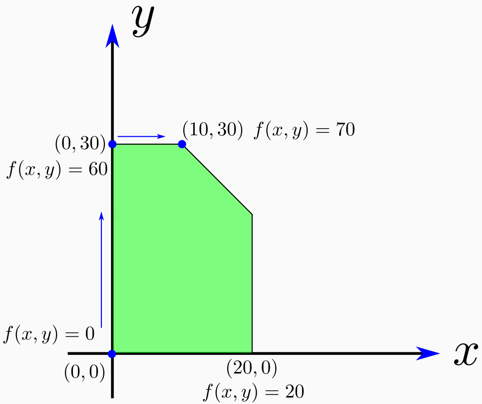

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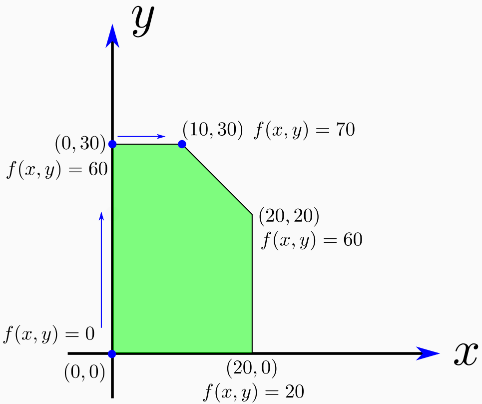

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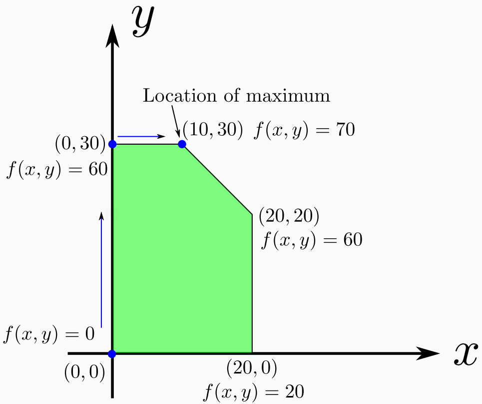


Standard forms

LP solvers, such as MOSEK, Gurobi, CVX, and COIN are implementations of the simplex method. They require the LP to be in certain standard form

Standard form 1

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \\ & \mathbf{x}, \mathbf{c}, \mathbf{b} \in \mathbb{R}^n, A \in \mathbb{R}^{m\times n} \end{aligned}$$

Example:

$$\begin{aligned} \text{maximize} \quad & x + 2y \\ \text{subject to} \quad & x \leq 20 \\ & y \leq 30 \\ & x + y \leq 40 \\ & x, y \geq 0 \end{aligned} \quad \equiv \quad \begin{aligned} \text{maximize} \quad & (1, 2)\begin{pmatrix} x \\ y \end{pmatrix} \\ \text{subject to} \quad & \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} 20 \\ 30 \\ 40 \end{pmatrix} \\ & x, y \geq 0 \end{aligned}$$

Convert to the standard form

- Minimization to maximization

$$\begin{array}{lrcl} \min & \mathbf{c}^T\mathbf{x} & & \\ \text{s. t.} & A\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \end{array} \quad \equiv \quad \begin{array}{lrcl} \max & -\mathbf{c}^T\mathbf{x} & & \\ \text{s. t.} & A\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \end{array}$$

- Equality to inequality

$$\begin{array}{lrcl} \max & \mathbf{c}^T\mathbf{x} & & \\ \text{s. t.} & x_1 + x_2 & = & 7 \end{array} \quad \equiv \quad \begin{array}{lrcl} \max & \mathbf{c}^T\mathbf{x} & & \\ \text{s. t.} & x_1 + x_2 & \leq & 7 \\ & x_1 + x_2 & \geq & 7 \end{array}$$

- Wrong inequality direction

$$\begin{array}{lrcl} \max & \mathbf{c}^T\mathbf{x} & & \\ \text{s. t.} & x_1 + x_2 & \geq & 7 \end{array} \quad \equiv \quad \begin{array}{lrcl} \max & \mathbf{c}^T\mathbf{x} & & \\ \text{s. t.} & -x_1 - x_2 & \leq & -7 \end{array}$$

- Missing nonnegative constraints

$$
\begin{array}{lrcl}
\max & x_1 + 2x_2 & & \\
\text{s. t.} & x_1 & \leq & 20 \\
& x_1 + x_2 & \leq & 40 \\
& x_1 & \geq & 0
\end{array}
\quad \equiv \quad
\begin{array}{lrcl}
& \text{rewrite } x_2 = x_2^+ - x_2^- & & \\
\max & x_1 + 2(x_2^+ - x_2^-) & & \\
\text{s. t.} & x_1 & \leq & 20 \\
& x_1 + (x_2^+ - x_2^-) & \leq & 40 \\
& x_1 & \geq & 0 \\
& x_2^+ & \geq & 0 \\
& x_2^- & \geq & 0
\end{array}
$$

Another Standard form

Standard form 2

$$\begin{array}{lrcl} \text{maximize} & \mathbf{c}^T\mathbf{x} & & \\ \text{subject to} & A\mathbf{x} & = & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \\ & \mathbf{x}, \mathbf{c}, \mathbf{b} & \in & \mathbb{R}^n, A \in \mathbb{R}^{m\times n} \end{array}$$

- Inequality to equality: use **slack variables**

$$\begin{array}{lrcl} \text{maximize} & \mathbf{c}^T\mathbf{x} & & \\ \text{subject to} & x_1 & \leq & 20 \\ & x_1 & \geq & 0 \end{array} \equiv \begin{array}{lrcl} \text{maximize} & \mathbf{c}^T\mathbf{x} + 0 \cdot s & & \\ \text{subject to} & x_1 + s & = & 20 \\ & x_1 & \geq & 0 \\ & s & \geq & 0 \end{array}$$

20 is bigger than x_1 by some positive amount, call it s

The new variable s is call the *slack variable*

Application of LP — network flow

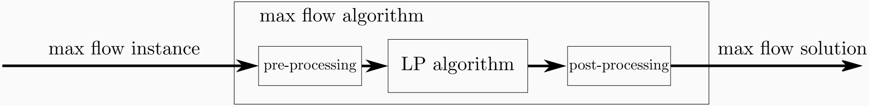
We are given $G = (V, E)$, $s, t \in V$, capacity c_e for all $e \in E$ Find a flow $f : E \to \mathrm{R}^{\geq 0}$ s.t.

- $0 \leq f(e) \leq c_e$
- $\sum_{(u,v)\in E} f(u, v) = \sum_{(v,w)\in E} f(v, w)$

LP formulation

$$
\begin{array}{lrcll}
\max & \sum_{(s,u)\in E} f_{s,u} & & & \\
\text{s.t.} & f_e & \leq & c_e & \forall e \in E \\
& \sum_{(u,v)\in E} f_{u,v} - \sum_{(v,w)\in E} f_{v,w} & = & 0 & \\
& f_e & \geq & 0 &
\end{array}
$$

We just **reduced** $\mathrm{max\_flow}$ to LP



Set Cover

We are given a set of elements $E = \{e_1, e_2, \ldots, e_n\}$, subsets $S_1, S_2, \ldots, S_m \subseteq E$ where cost of selecting S_i is w_i.

Objective: Find $I \subseteq \{1, 2, \ldots, m\}$, $\cup_{i \in I} S_i = E$ while minimizing the cost $\sum_{i \in I} w_i$.

ILP formulation

$$
\begin{array}{lrl}
\text{min} & \sum_{i=1}^{m} x_i w_i & \\
\text{s.t.} & \sum_{i: e_j \in S_i} x_i \geq 1 & \forall e_j \in E \\
 & x_i \in \{0, 1\} & \forall i \in \{1, \ldots, m\}
\end{array}
$$

Set Cover

We are given a set of elements $E = \{e_1, e_2, \ldots, e_n\}$, subsets $S_1, S_2, \ldots, S_m \subseteq E$ where cost of selecting S_i is w_i.

Objective: Find $I \subseteq \{1, 2, \ldots, m\}$, $\cup_{i \in I} S_i = E$ while minimizing the cost $\sum_{i \in I} w_i$.

LP formulation

$$
\begin{array}{lrl}
\min & \sum_{i=1}^{m} x_i w_i & \\
\text{s.t.} & \sum_{i: e_j \in S_i} x_i \geq 1 & \forall e_j \in E \\
 & x_i \geq 0 & \forall i \in \{1, \ldots, m\}
\end{array}
$$

Duality of LP (I)

Consider

$$
\begin{aligned}
\text{maximize} \quad & x_1 + 2x_2 \\
\text{subject to} \quad & x_1 \leq 20 \\
& x_2 \leq 30 \\
& x_1 + x_2 \leq 40 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Can we show the optimal solution is at least 60? Check (0, 30)

Can we show that the optimal solution is at most 90? Use linear combination constraints

Duality of LP (II)

Define a variable for each constraint

$$\begin{aligned} \text{maximize} \quad & x_1 + 2x_2 & \\ \text{subject to} \quad & x_1 \le 20 & \color{red}{y_1} \\ & x_2 \le 30 & \color{red}{y_2} \\ & x_1 + x_2 \le 40 & \color{red}{y_3} \\ & x_1, x_2 \ge 0 & \end{aligned}$$

Adding them together:
$(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \le 20y_1 + 30y_2 + 40y_3$
We let $y_1 + y_3 \ge 1$ and $y_2 + y_3 \ge 2$ to get an upper bound on $x_1 + 2x_2$:
$x_1 + 2x_2 \le (y_1 + y_3)x_1 + (y_2 + y_3)x_2 \le 20y_1 + 30y_2 + 40y_3$

Duality of LP (III)

Primal LP

$$\begin{aligned} \text{maximize} \quad & x_1 + 2x_2 \\ \text{subject to} \quad & x_1 \leq 20 \\ & x_2 \leq 30 \\ & x_1 + x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Optimal solution: $(x_1, x_2) = (10, 30) \implies x_1 + 2x_2 = 70$

Dual LP

$$\begin{aligned} \text{minimize} \quad & 20y_1 + 30y_2 + 40y_3 \\ \text{subject to} \quad & y_1 + y_3 \geq 1 \\ & y_2 + y_3 \geq 2 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

Optimal solution:
$(y_1, y_2, y_3) = (0, 1, 1) \implies 20y_1 + 30y_2 + 40y_3 = 70$

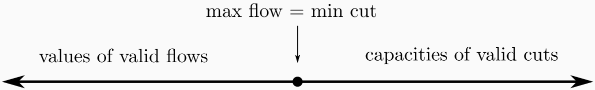
Duality of LP(IV)

More generally

Primal LP	Dual LP
max $c_1x_1 + c_2x_2 + \cdots + c_nx_n$	min $b_1y_1 + b_2y_2 + \cdots + b_my_m$
s.t. $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1$	s.t. $a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \ge c_1$
$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2$	$a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \ge c_2$
$\vdots$	$\vdots$
$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m$	$a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \ge c_n$
$x_1, x_2, \ldots, x_n \ge 0$	$y_1, y_2, \ldots, y_m \ge 0$

Duality of LP (V)

Duality of flow and cut



For LP we have:

Theorem (Weak Duality)

A feasible solution to the dual LP is an upper bound on any feasible solution to the primal LP

Theorem (Strong Duality)

The optimal solution to the dual LP is equal to the optimal solution to the primal LP