Recitation 6

Monday, Oct 14, 2024

1. Let f^* be one maximum flow of network $G = (V, E)$ with source $s \in V$ and sink $t \in V$. Let $T := \{v \in V \mid v \text{ can reach } t \text{ in } G(f^*)\}$. Prove that $(S := V \setminus T, T)$ is a minimum s-t cut.

   Solution. The proof is analogy to the proof for the correctness of Ford-Fulkerson's algorithm. We first show that (S, T) is an s-t cut, i.e., $s \in S$ and $t \in T$. It's obvious that $t \in T$ by the definition of T. Suppose coversely that $s \notin S$. Then we have $s \in T$ and therefore there exists an s-t path p in the residual graph $G(f^*)$. Hence we can use the augment procedure by augmenting f^* with this path p to obtain a new flow f' with $|f'| > |f^*|$. This is contradicting to the fact that f^* is a maximum flow.

   Now we show that (S, T) is a minimum s-t cut. Consider these edges from S to T and those edges from T to S. Again we define $E(S, T) = \{(u, v) \in E \mid u \in S, v \in T\}$, and $E(T, S) = \{(u, v) \in E \mid u \in T, v \in S\}$. For any edge $e = (u, v) \in E(S, T)$, we must have that e is saturated in f^*, i.e., $f^*(e) = c(e)$. This is because otherwise, i.e., $f^*(e) < c(e)$, there will be an edge from u to v in the residual graph $G(f^*)$, and therefore u can reach t, which contradicts to the definition of $E(S, T)$. Also, for any edge $e = (u, v) \in E(T, S)$, we must have that $f^*(e) = 0$. This is because otherwise, i.e., $f^*(e) > 0$, there will be an edge from v to u in the residual graph $G(f^*)$, and therefore v can reach t, which contradicts to the definition of $E(T, S)$. Then we compute the value of flow f^*: we proved that $|f^*| = \sum_{e \in E(S,T)} f^*(e) - \sum_{e \in E(T,S)} f^*(e)$. By using the above results, we have $|f^*| = \sum_{e \in E(S,T)} c(e) - 0 = \sum_{e \in E(S,T)} c(e)$, which is exactly the capacity of cut (S, T). As we know $c(S, T) \geq |f|$ for any cut (S, T) and any flow f, such an equation of $|f^*| = c(S, T)$ implies that (S, T) is a minimum-cut.

2. Recall that for a given network the mincut might not be unique. Given a minimum s-t cut C for a network $G = (V, E)$, let $S(C)$ denote the set of vertices that are on the same side of C as the source. Design a polynomial-time algorithm to compute $\cap_{C \in \mathcal{C}} S(C)$ and $\cup_{C \in \mathcal{C}} S(C)$, where C is the set of all minimum s-t cuts of G.

   Solution:
   The algorithm to compute $\cap_{C \in \mathcal{C}} S(C)$ and $\cup_{C \in \mathcal{C}} S(C)$ is quite simple. We use any max-flow algorithm to compute one max-flow f and its corresponding residual graph G_f. Let A be the vertices that can be reached from s in G_f, and let B be the vertices that can reach t in G_f. A and B can be computed by BFS on G_f. Then we have that $\cap_{C \in \mathcal{C}} S(C) = A$ and $\cup_{C \in \mathcal{C}} S(C) = V - B$. The running time of this algorithm is dominated by the running time of the max-flow algorithm.

   We first prove that $\cap_{C \in \mathcal{C}} S(C) = A$, and the other part can be proved symmetrically. First, we show that if $v \in \cap_{C \in \mathcal{C}} S(C)$ then we have $v \in A$. In fact, $(A, V - A)$ is a minimum s-t cut, since all the edges from A to $V - A$ are saturated by the max-flow f according to the definition of A. Thus, we have that $v \in A$, since v belongs to the s-side of any minimum s-t cut. Second, we prove that if $v \in A$ then we have $v \in \cap_{C \in \mathcal{C}}$. We prove this by contradiction.

Assume that $v \notin \cap_{C \in \mathcal{C}}$. Then there must exist one minimum s-t cut C' such that v is in the t-side of C'. Since $v \in A$, there must be one path in G_f from s to v. This path must go through some edge e of C' since v is in the t-side of C'. This means that e is not saturated by f, which is a contradition with the fact that C' is one minimum s-t cut.

Note: above proof also proves that, a network has a unique minimum s-t cut if and only if $A \cup B = V$.