# October 24

1. (a) Given an undirected graph $G = (V, E)$ and a set $E' \subset E$, briefly describe how to update Kruskal's algorithm to find the minimum spanning tree that includes all edges from $E'$. Assume $E'$ doesn't have a cycle.

   (b) Assume you are given a graph $G = (V, E)$ with positive and negative edge weights and an algorithm that can return a minimum spanning tree when given a graph with only positive edges. Describe a way to transform $G$ into a new graph $G'$ containing only positive edge weights so that the minimum spanning tree of $G$ can be easily found from the minimum spanning tree of $G'$.

   **solution**

   (a) Assuming $E'$ doesn't have a cycle, add all edges from $E'$ to the MST first, then sort $E \setminus E'$ and run Kruskal's as normal.

   (b) We create $G'$ by adding a large positive integer M to all the edge weights of $G$ so that each edge weight is positive. The minimum spanning tree of $G'$ is then the same as the minimum spanning tree of $G$.

   Unlike Dijkstra's algorithm, which is finding minimum paths which may have different numbers of edges, all spanning trees of $G$ must have precisely $|V| - 1$ edges, conserving the MST. So, if the minimum spanning tree of $G$ has weight $w$, the minimum spanning tree of $G'$ has weight $w + (|V| - 1)M$.

2. Show that a graph has a unique minimum spanning tree if all its edges have distinct weights/costs.

   **Solution:**

   For contradiction, suppose there exist two different MST $T$ and $T'$. Since they are different, the union of their edges $E(T) \cup E(T')$ has at least $n$ edges, and thus the graph $G' = (V, E(T) \cup E(T'))$ contains a cycle $C$. Let $e \in C$ be the edge with maximum cost. Since by assumption the edges have distinct weights, $e$ can be uniquely determined. By the cycle property, $e$ does not belong to any MST, but by our construction, $e$ is in either $T$ or $T'$, so there is a contradiction.