CMPSC 465 Data Structures & Algorithms
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Recitation 1

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1. LP1: Consider the following linear program.

$$\begin{aligned} \max\ 5x + 3y \\ 5x - 2y \geq 0 \\ x + y \leq 7 \\ x \leq 5 \\ x \geq 0 \\ y \geq 0 \end{aligned}$$

   Plot the feasible region and identify the optimal solution.

   Solution:

   The optimal solution is achieved in the upper right corner of the convex feasible region, i.e. at the point $(5, 2)$, and has value $5x + 3y = 31$.

2. LP2: Duckwheat is produced in Kansas and Mexico and consumed in New York and California. Kansas produces 15 shnupells of duckwheat and Mexico 8. Meanwhile, New York consumes 10 shnupells and California 13. The transportation costs per shnupell are \$4 from Mexico to New York, \$1 from Mexico to California, \$2 from Kansas to New York, and \$3 and from Kansas to California. Write a linear program that decides the amounts of duckwheat (in shnupells and fractions of a shnupell) to be transported from each producer to each consumer, so as to minimize the overall transportation cost.

   Solution:

   Let x_{MN}, x_{MC} denote the exports of duckwheat from Mexico to Ney York and California respectively and let x_{KN}, x_{KC} be the ones from Kansas. The linear programm will be the following:

$$\begin{aligned} \min\ 4x_{MN} + x_{MC} + 2x_{KN} + 3x_{KC} \\ x_{MN} + x_{KN} = 10 \\ x_{MC} + x_{KC} = 13 \\ x_{MN} + x_{MC} = 8 \\ x_{KN} + x_{KC} = 15 \\ x_{MN}, x_{MC}, x_{KN}, x_{KC} \geq 0 \end{aligned}$$

3. LP3: A cargo plane can carry a maximum weight of 100 tons and a maximum volume of 60 cubic meters. There are three materials to be transported, and the cargo company may choose to carry any amount of each, upto the maximum available limits given below.

   1. Material 1 has density 2 tons/cubic meter, maximum available amount 40 cubic meters, and revenue \$1,000 per cubic meter.
   2. Material 2 has density 1 ton/cubic meter, maximum available amount 30 cubic meters, and revenue \$1,200 per cubic meter.
   3. Material 3 has density 3 tons/cubic meter, maximum available amount 20 cubic meters, and revenue \$12,000 per cubic meter.

   Write a linear program that optimizes revenue within the constraints.

   Solution:

   Let q_i denote the quantity (in cubic meters) of material i. The linear program will be the following:

$$\begin{aligned} \max\ 1000q_1 + 1200q_2 + 12000q_3 \\ 2q_1 + q_2 + 3q_3 &\leq 100 \\ q_1 + q_2 + q_3 &\leq 60 \\ q_1 &\leq 40 \\ q_2 &\leq 30 \\ q_3 &\leq 20 \\ q_1, q_2, q_3 &\geq 0 \end{aligned}$$

4. LP4: Convert the following linear programming to standard form 1.

$$\begin{aligned} \min\ 3x_1 - x_2 \\ -x_1 + 6x_2 - x_3 + x_4 &\geq -3 \\ 7x_2 + x_4 &= 5 \\ x_3 + x_4 &\leq 2 \\ -1 &\leq x_2 \\ x_3 &\leq 5 \\ -2 \leq x_4 &\leq 2 \end{aligned}$$

   Solution:

   Step 1:

   1. maximize $-3x_1 + x_2$
   2. $x_1 - 6x_2 + x_3 - x_4 \leq 3$

3. $7x_2 + x_4 \leq 5$ and $7x_2 + x_4 \geq 5$
4. $7x_2 + x_4 \leq 5$ and $-7x_2 - x_4 \leq -5$
5. $-2 \leq x_4$ and $x_4 \leq 2$

The LP now becomes:

$$\begin{aligned} \max\ -3x_1 + x_2 \\ x_1 - 6x_2 + x_3 - x_4 \leq 3 \\ 7x_2 + x_4 \leq 5 \\ -7x_2 - x_4 \leq -5 \\ x_3 + x_4 \leq 2 \\ x_4 \leq 2 \\ -1 \leq x_2, x_3 \leq 5, -2 \leq x_4 \end{aligned}$$

Step 2:

1. The variable x_1 is free, so we replace it by $x_1 = z_1^+ - z_1^-$ with $0 \leq z_1^+$, $0 \leq z_2^+$.
2. x_2 has a non-zero lower bound ($-1 \leq x_2$), so we replace it by $z_2 = x_2 + 1$ or $x_2 = z_2 - 1$ with $0 \leq z_2$.
3. x_3 is bounded above ($x_3 \leq 5$), so we replace it by $z_3 = 5 - x_3$ or $x_3 = 5 - z_3$ with $0 \leq z_3$.
4. x_4 is bounded below ($-2 \leq x_4$), so we replace it by $z_4 = x_4 + 2$ or $x_4 = z_4 - 2$ with $0 \leq z_4$.

Step 3:

Apply substitutions from Step 2. The LP now becomes:

$$\begin{aligned} \max\ -3z_1^+ + 3z_1^- + z_2 \\ z_1^+ - z_1^- - 6z_2 - z_3 - z_4 \leq -10 \\ 7z_2 + z_4 \leq 14 \\ -7z_2 - z_4 \leq -14 \\ -z_3 + z_4 \leq -1 \\ z_4 \leq 4 \\ 0 \leq z_1^+, z_1^-.z_2, z_3, z_4 \end{aligned}$$