Chapter 4: Generating Discrete Random Variables

Daniel B. Rowe, Ph.D.

Department of Mathematics,
Statistics, and Computer Science

Agenda
4.1 The Inverses Transform Method

4.2 Generating a Poisson Random Variable

4.3 Generating Binomial Random Variables

4.4 The Acceptance-Rejection Technique

4.1 The Inverse (CDF) Transform Method

To generate discrete RVs with PMF

$$P\{X = x_j\} = p_j, \quad j = 0,1,\ldots \quad \sum_j p_j = 1$$

generate a random U that is uniformly distributed in (0,1) ,
and set

$$X = \begin{cases} x_0 & \text{If} \quad 0 < U < p_0 \\ x_1 & \text{If} \quad p_0 < U < p_0 + p_1 \\ \vdots & \\ x_j & \text{If} \quad \sum_{i=0}^{j-1} p_i < U < \sum_{i=0}^{j} p_i \\ \vdots & \end{cases}$$

then X has the desired PMF.

4.1 The Inverse (CDF) Transform Method

This is called the inverse method because two are the same!

$$X = \begin{cases} x_0 & \text{If} \quad 0 < U < p_0 \\ x_1 & \text{If} \quad p_0 < U < p_0 + p_1 \\ \vdots & \\ x_j & \text{If} \quad \sum_{i=0}^{j-1} p_i < U < \sum_{i=0}^{j} p_i \\ \vdots & \end{cases} \leftrightarrow X = \begin{cases} x_0 & \text{If} \quad 0 < U < F(x_0) \\ x_1 & \text{If} \quad F(x_0) \le U < F(x_1) \\ \vdots & \\ x_j & \text{If} \quad F(x_{j-1}) \le U < F(x_j) \\ \vdots & \end{cases}$$

Recall:

$$F(x_{j-1}) = \sum_{i=0}^{j-1} P(X = x_i) \qquad F(x_j) = \sum_{i=0}^{j} P(X = x_i)$$

4.1 The Inverse (CDF) Transform Method

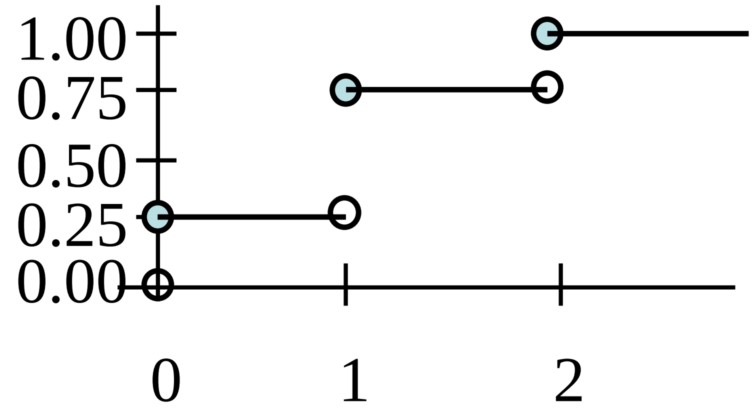
Example: Tossing a coin twice

x	0	1	2
P(X=x)	1/4	1/2	1/4

$$F(x)=\begin{cases} 0/4 & -\infty < x < 0 \\ 1/4 & 0 \le x < 1 \\ 3/4 & 1 \le x < 2 \\ 4/4 & 2 \le x < \infty \end{cases}$$

Generate U, then

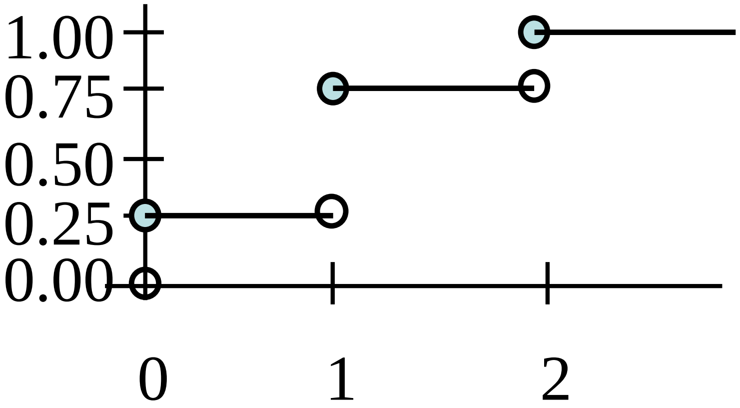
$$X=\begin{cases} 0 & \text{If} \quad 0 \le U < 1/4 \\ 1 & \text{If} \quad 1/4 \le U < 3/4 \\ 2 & \text{If} \quad 3/4 \le U < 1 \end{cases} \quad \longleftrightarrow$$



4.1 The Inverse (CDF) Transform Method

x	0	1	2
P(X=x)	1/4	1/2	1/4

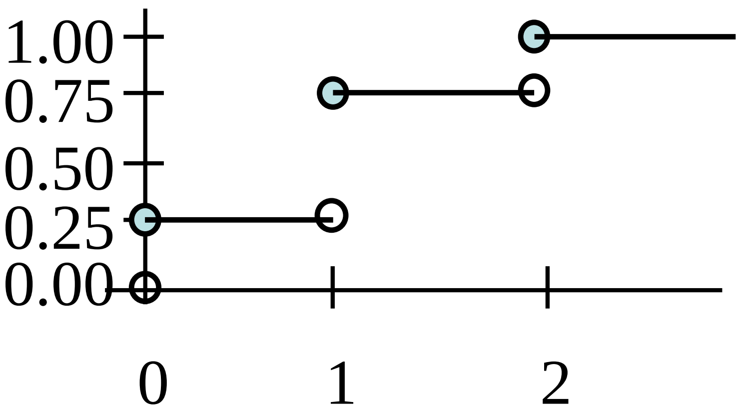
Example: Tossing a coin twice



4.1 The Inverse (CDF) Transform Method

x	**0**	**1**	**2**
P(X=*x*)	1/4	1/2	1/4

Example: Tossing a coin twice



U	*X*
0.8147	2
0.9058	2
0.1270	0
0.9134	2
0.6324	1

```
n=5;
U=rand(n,1);, X=zeros(n,1);
for j=1:n
   if   (0<=.U(j,1))&(U(j,1)<25)
       X(j,1)=0;
elseif(.25<=.U(j,1))&(U(j,1)<.75)
       X(j,1)=1;
 elseif(.75<=U(j,1))&(U(j,1)<1.0)
       X(j,1)=2;
   end
end
[U,X]
```

STEP 1: Generate a random U.

STEP 2: If $0 \le U < 0.25$, then $X=0$. Go to Step 1.

STEP 3: If $0.25 \le U < .75$, then $X=1$. Go to Step 1.

STEP 4: If $0.75 \le U < 1.00$, then $X=2$. Go to Step 1.

4.2 Generating a Poisson Random Variable

To generate discrete RVs with PMF

$$p_i = P\{X = i\} = e^{-\lambda}\frac{\lambda^i}{i!}, \quad i = 0,1,\ldots$$

use the identity that

$$p_{i+1} = \frac{\lambda}{i+1} p_i \qquad i \geq 0$$

$$\frac{p_{i+1}}{p_i} = \frac{e^{-\lambda}\dfrac{\lambda^{i+1}}{(i+1)!}}{e^{-\lambda}\dfrac{\lambda^i}{i!}}$$

STEP 1: Generate a random U.
STEP 2: i=0, $p=e^{-\lambda}=p_0$, $F=p$.
STEP 3: If $U < F$, set $X=i$ and stop.
STEP 4: $p=\lambda p/(i+1)$, $F=F+p$, $i=i+1$.
STEP 5: Go to Step 3.

then X has the desired PMF.

4.3 Generating a Binomial Random Variable

To generate discrete RVs with PMF

$$P\{X=i\}=\frac{n!}{(n-i)!i!}p^i(1-p)^{n-i},\quad i=0,1,\ldots n$$

use the identity that

$$\frac{P\{X=i+1\}}{P\{X=i\}}=\frac{\frac{n!}{(n-i-1)!(i+1)!}p^{i+1}(1-p)^{n-i-1}}{\frac{n!}{(n-i)!i!}p^i(1-p)^{n-i}}$$

$$P\{X=i+1\}=\frac{n-i}{i+1}\frac{p}{1-p}P\{X=i\}$$

4.3 Generating a Binomial Random Variable

To generate discrete RVs with PMF

$$P\{X=i\}=\frac{n!}{(n-i)!i!}p^i(1-p)^{n-i},\quad i=0,1,\ldots n$$

use the identity that

$$\frac{P\{X=i+1\}}{P\{X=i\}}=\frac{\frac{n!}{(n-i-1)!(i+1)!}p^{i+1}(1-p)^{n-i-1}}{\frac{n!}{(n-i)!i!}p^i(1-p)^{n-i}}$$

$$P\{X=i+1\}=\frac{n-i}{i+1}\frac{p}{1-p}P\{X=i\}$$

STEP 1: Generate a random U.
STEP 2: $c=p/(1-p)$, $i=0$, pr= $(1-p)^n=p_0$, F=pr.
STEP 3: If $U < F$, set $X=i$ and stop.
STEP 4: pr=$[c(n-i)/(i+1)]$pr, $F=F$+pr, $i=i+1$.
STEP 5: Go to Step 3.

then X has the desired PMF.

4.4 The Acceptance-Rejection Technique

Sometimes it is difficult to generate X from PMF $\{p_j, j \geq 0\}$.
If we have a technique to generate Y from PMF $\{q_j, j \geq 0\}$.
Then we can use Y to generate an X with PMF $\{p_j, j \geq 0\}$.
The PMF q_j is called the instrumental distribution

4.4 The Acceptance-Rejection Technique

Sometimes it is difficult to generate X from PMF $\{p_j, j \geq 0\}$.
If we have a technique to generate Y from PMF $\{q_j, j \geq 0\}$.
Then we can use Y to generate an X with PMF $\{p_j, j \geq 0\}$.
The PMF q_j is called the instrumental distribution

$$p_j < cq_j \qquad c > 1$$

Here cq_j is called the envelope distribution a function whose "probabilities" are all larger than p_j.
We accept the random $X=Y$ if $U < p_Y/(cq_Y)$
and X has the desired PMF $\{p_j, j \geq 0\}$.

4.4 The Acceptance-Rejection Technique

Rejection Method

STEP 1: Simulate the value of Y, having PMF q_j.
STEP 2: Generate a random U.
STEP 3: If $U < p_Y/(cq_Y)$, set $X=Y$ and stop. Otherwise go to 1.

Theorem

The acceptance-rejection algorithm generates a RV X such that

$$P\{X = j\} = p_j, \quad i = 0,1,\ldots n$$

The number of iterations needed to obtain X is a geometric RV with mean c.

4.4 The Acceptance-Rejection Technique

Example

Use rejection sampling to generate Poisson random variables.
Let λ=2 and use discrete uniform(0,N) instrumental distribution.

$$p_j = e^{-\lambda} \frac{\lambda^j}{j!}, \quad j = 0,1,\ldots$$

and

$$q_j = \frac{1}{N}, \quad j = 0,1,\ldots,N$$

$$N = \max(j) \ni p_j > 0$$

4.4 The Acceptance-Rejection Technique

Example

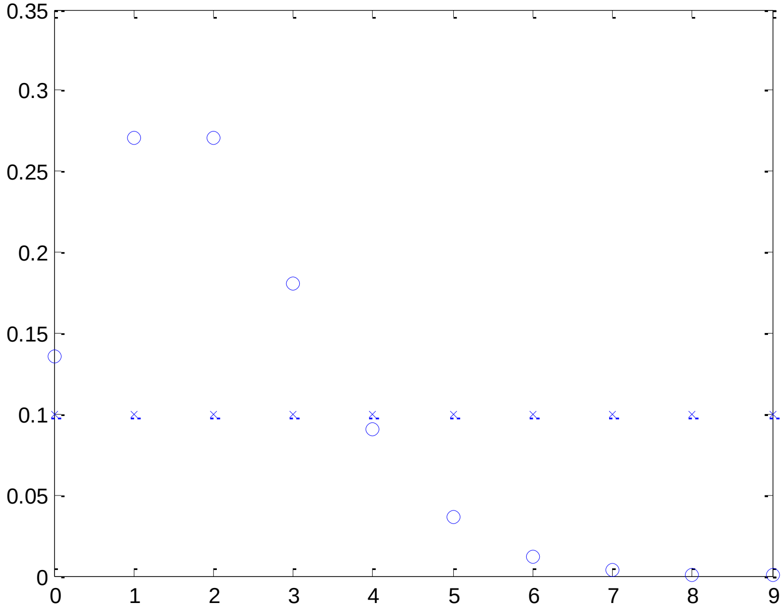
Use rejection sampling to generate Poisson random variables. Let λ=2 and use discrete uniform(0,N) instrumental distribution.

$$p_j = e^{-\lambda}\frac{\lambda^j}{j!}, \quad j = 0,1,\ldots$$

and

$$q_j = \frac{1}{N}, \quad j = 0,1,\ldots,N$$

$$N = \max(j) \ni p_j > 0$$



j	p_j	q_j
0	.1353	.1
1	.2707	.1
2	.2707	.1
3	.1804	.1
4	.0902	.1
5	.0361	.1
6	.0120	.1
7	.0034	.1
8	.0009	.1
9	.0002	.1

4.4 The Acceptance-Rejection Technique

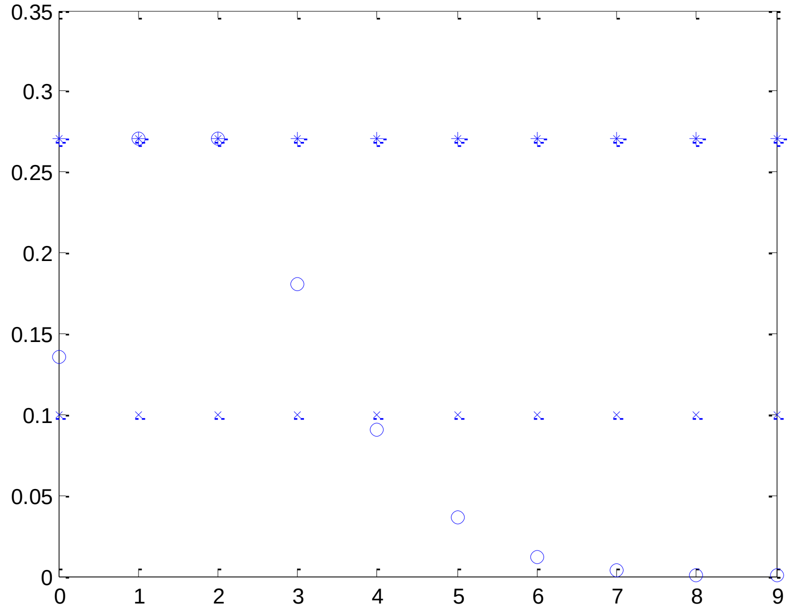
Example

Use rejection sampling to generate Poisson random variables. Let λ=2 and use discrete uniform(0,N) instrumental distribution.

Get envelope "distribution"

cq_j, where c=max(p_j/q_j).

c= (0.2707/0.1)=2.707



j	p_j	q_j
0	.1353	.1
1	.2707	.1
2	.2707	.1
3	.1804	.1
4	.0902	.1
5	.0361	.1
6	.0120	.1
7	.0034	.1
8	.0009	.1
9	.0002	.1

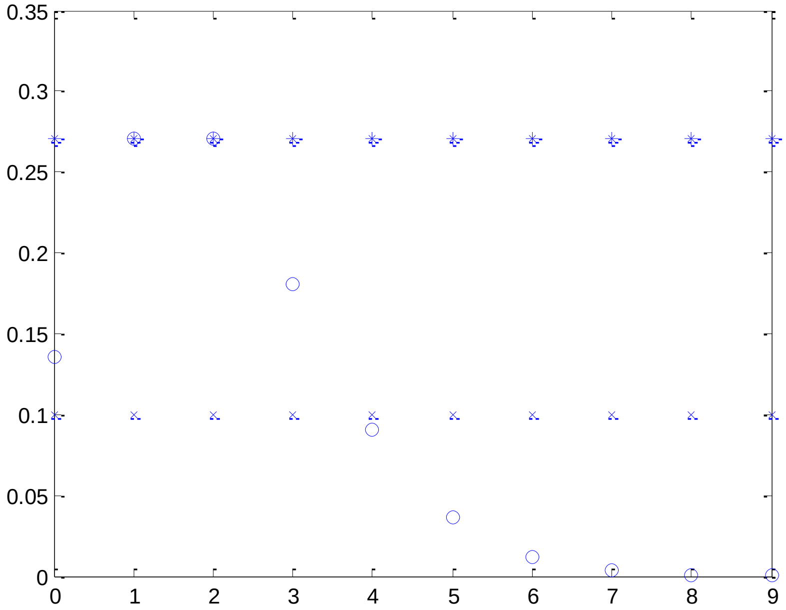
4.4 The Acceptance-Rejection Technique

Example

Use rejection sampling to generate Poisson random variables.
Let λ=2 and use discrete uniform(0,N) instrumental distribution.

c= (0.2707/0.1)=2.707

STEP 1: Generate Y from q_j.
STEP 2: Generate a U.
STEP 3: If $U < p_Y/(cq_Y)$,
set X=Y and stop.
Otherwise go to 1.



j	p_j	q_j	p_j/cq_j
0	.1353	.1	.5000
1	.2707	.1	1.000
2	.2707	.1	1.000
3	.1804	.1	.6667
4	.0902	.1	.3333
5	.0361	.1	.1333
6	.0120	.1	.0444
7	.0034	.1	.0127
8	.0009	.1	.0032
9	.0002	.1	.0007

4.4 The Acceptance-Rejection Technique

```
clear all
close all

N=10;
x = (0:N-1)';
p = poisspdf(x,2);

figure;
plot(x,p,'o')

q=1/N*ones(N,1);
c=max(p./q)

hold on
plot(x,c*q,'+')
```

```
n=10;, nn=100;
X=zeros(n,1);
count=0; Y=zeros(n,1);
R=p./(c*q)
for j=1:nn
    U1=rand(1,1);
    Y(j,1)=round((N-1)*U1)+1;
    U2=rand(1,1);
    if (U2<=p(Y(j,1),1)/(c*q(Y(j,1),1)))
        count=count+1;
        X(count,1)=Y(j,1);
    end
    if (count==n)
        return;
    end
end
```

Homework 3:

Chapter 4: # 1*, 3,12,15,17.

*Repeat using Matlab’s binornd() command.
Compare results to Matlab’s binornd().