Syllabus

Daniel B. Rowe, Ph.D.

Department of Mathematics,
Statistics, and Computer Science

Department of Mathematics, Statistics, and Computer Science

Marquette University

Syllabus

Spring 2018

Course: MSCS 6020 Simulation

Time: TuTh 3:30-4:45 Cudahy Hall 120

Instructor: Daniel B. Rowe, Ph.D.

Course Description From The University Bulletin

MSCS 6020. Simulation. 3 cr. hrs.

Elements of statistical simulation and modeling with applications. Generation of random variables, Monte Carlo method, Markov chains, birth-and-death processes, queues, variance reduction, Markov chain Monte Carlo (MCMC) methods and applications, bootstrapping, validation and analysis of simulated data. Prereq: MSCS 6010 and programming competency in a high-level language.

We will be using Matlab as our programming language.

Office Hours: TuTh 2:30-3:30 pm

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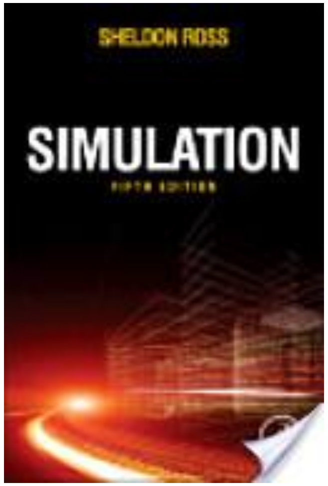
E-mail: daniel.rowe@marquette.edu



Text: Ross, Sheldon. (2012).
Simulation, Fifth edition, Academic Press.
ISBN: 0124159710

Grading: A midterm (in class and take-home) on March 8, (bi)weekly homework & class participation, and a final exam (in class and take-home) on May 10, 8:00 am – 10:00 am.

Homework & Participation (30%), Mid-Term (30%), Final (40%).

~~Chapter 2: Elements of Probability~~

~~Sample Space and Events, Axioms of Probability, Random Variables, Expectation, Discrete RVs, Continuous RVs, Conditional Expectation and Variance~~

Covered by prerequisite.

Chapter 3: Random Numbers

Number Generation, Random Numbers to Evaluate Integrals

Chapter 4: Generating Discrete RVs

Inverse Transform, Poisson RV, Binomial RV, Acceptance-Rejection, Composition Approach, Alias Method, Random Vectors

Chapter 5: Generating Continuous RVs

Inverse Transform, Rejection Polar Method for Normal RVs, Poisson Processes, Nonhomogeneous Poisson Processes, 2D Poisson Process.

Chapter 6: Multivariate Normal and Copulas

Multivariate Normal, Generating Multivariate Normal RVs, Copulas, Generating Variables from Copula Models

Chapter 7: Discrete Event Simulation

Discrete Events, Queueing Systems, Inventory Model, Insurance Risk Model, Repair Problem, Stock Option

Chapter 8: Analysis of Simulated Data

Sample Mean and Variance, Interval Estimates of Mean, Bootstrapping for Mean Square Error

Chapter 9: Variance Reduction Techniques

Antithetic Variables, Control Variates, Variance Reduction by Conditioning, Stratified Sampling, Importance Sampling, Common Random Numbers, Exotic Option

Chapter 10: Additional Variance Reduction Techniques

Conditional Bernoulli Sampling, Normalized Importance Sampling, Latin Hyper Cube Sampling

Chapter 11: Statistical Validation Techniques

Goodness of Fit Tests, Two Sample Problem, Validating Assumptions of a Nonhomogeneous Poisson Process

Chapter 12: Markov Chain Monte Carlo Methods

Markov Chains, Hastings-Metropolis Algorithm, Gibbs Sampler, Markov Chains and Queueing Loss, Simulated Annealing, Sampling Importance Resampling

Numerical Flavor

All slides are a summary of the material and do not contain all detail. Book and other sources are the ultimate authority.

Deviations form the text may occur.

Familiarize yourself with Matlab

Chapter 2: Elements of Probability

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Homework 1:

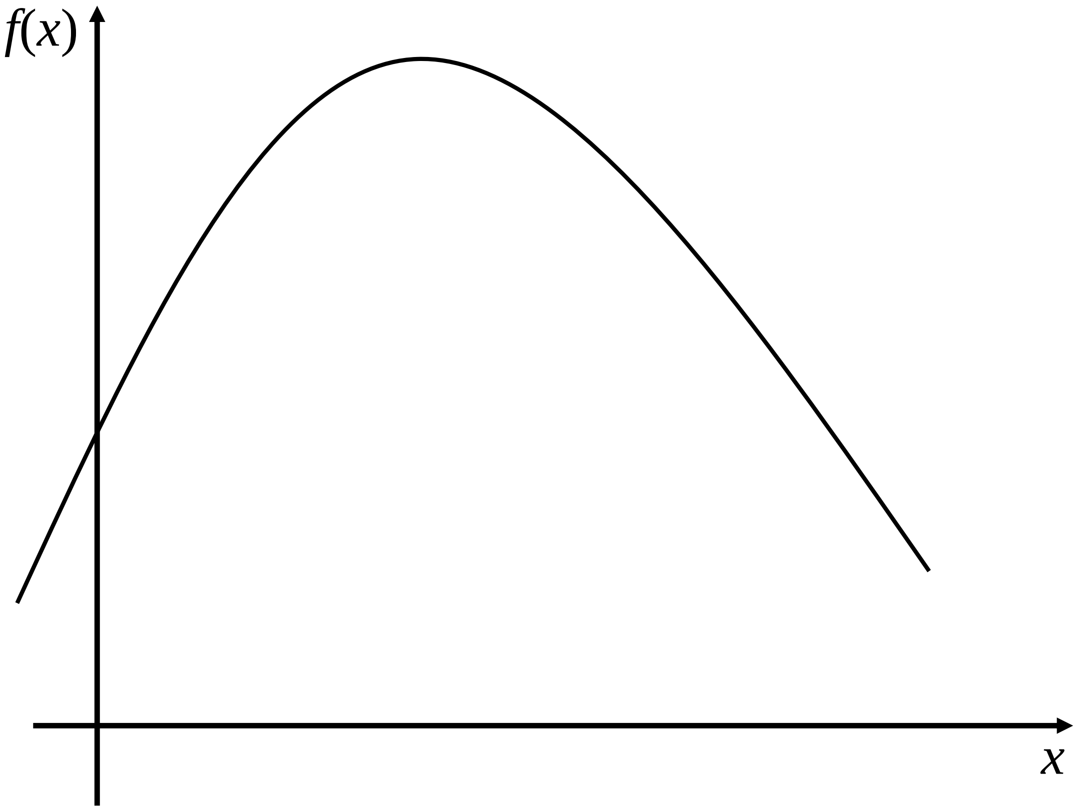
Chapter 2: # 7, 13, 29, 31, 36, 37.

Numerical Integration

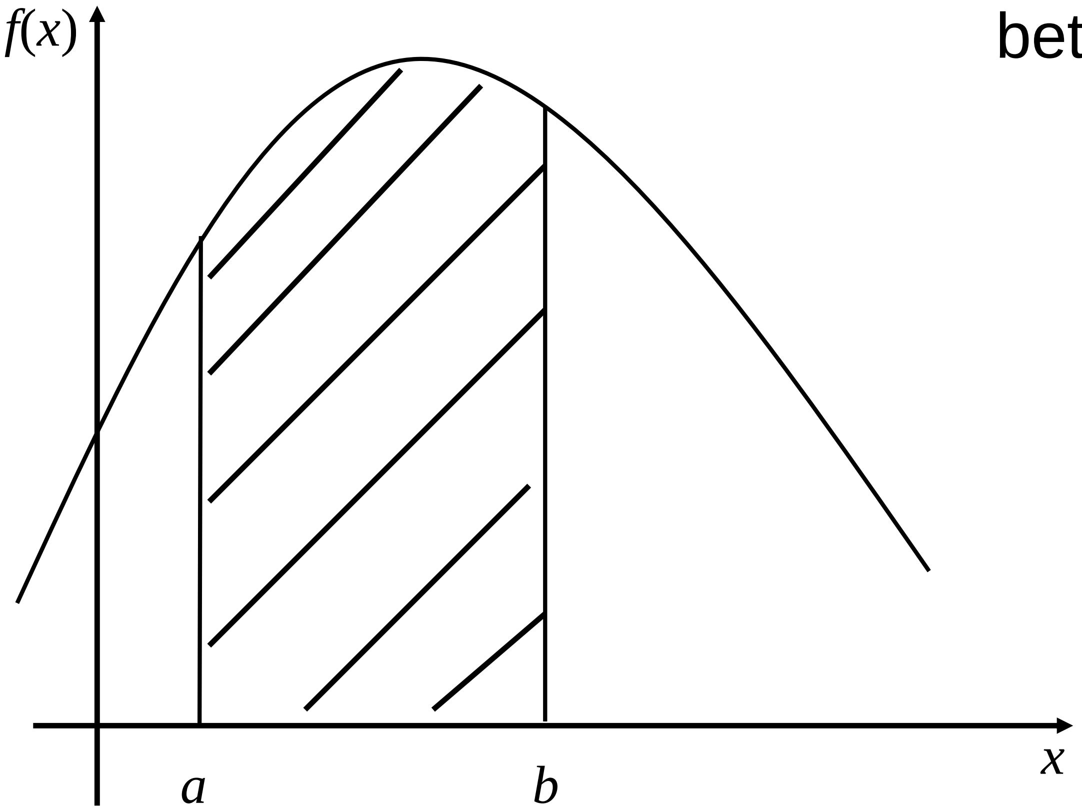
Daniel B. Rowe, Ph.D.

Professor
Department of Mathematics,
Statistics, and Computer Science

Integration - Area Under Curve



Integration - Area Under Curve

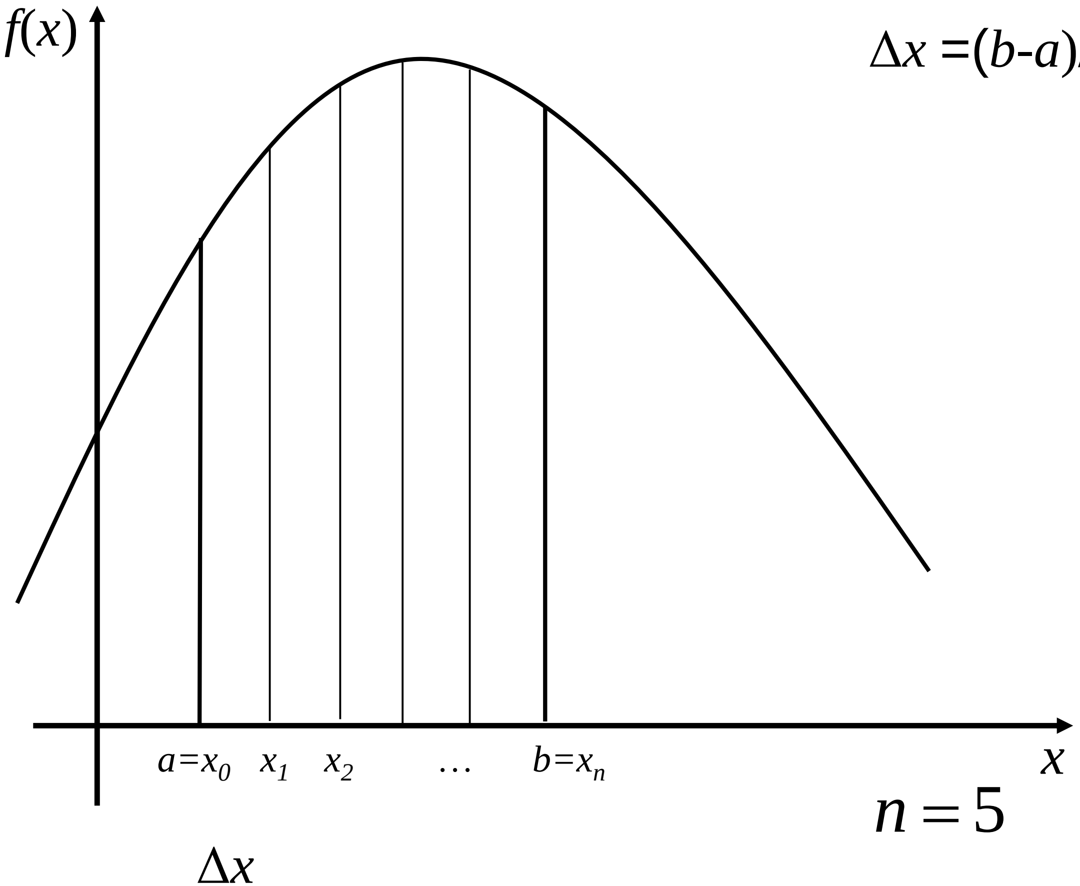


Area under curve between a and b.

Integration - Area Under Curve

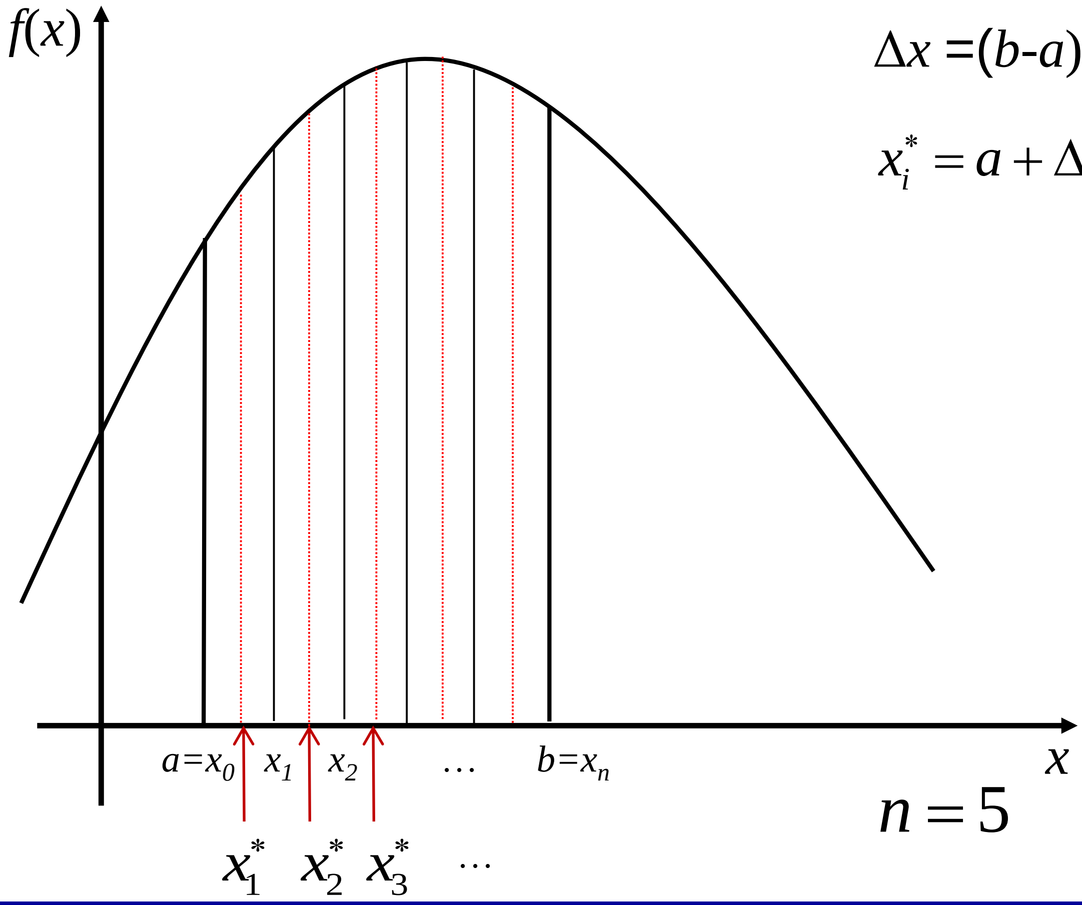
Divide into intervals: Δx small

$\Delta x = (b-a)/n$ $\qquad \Delta x = x_i - x_{i-1}$



$n = 5$

Integration - Area Under Curve



Find midpoints: Δx small

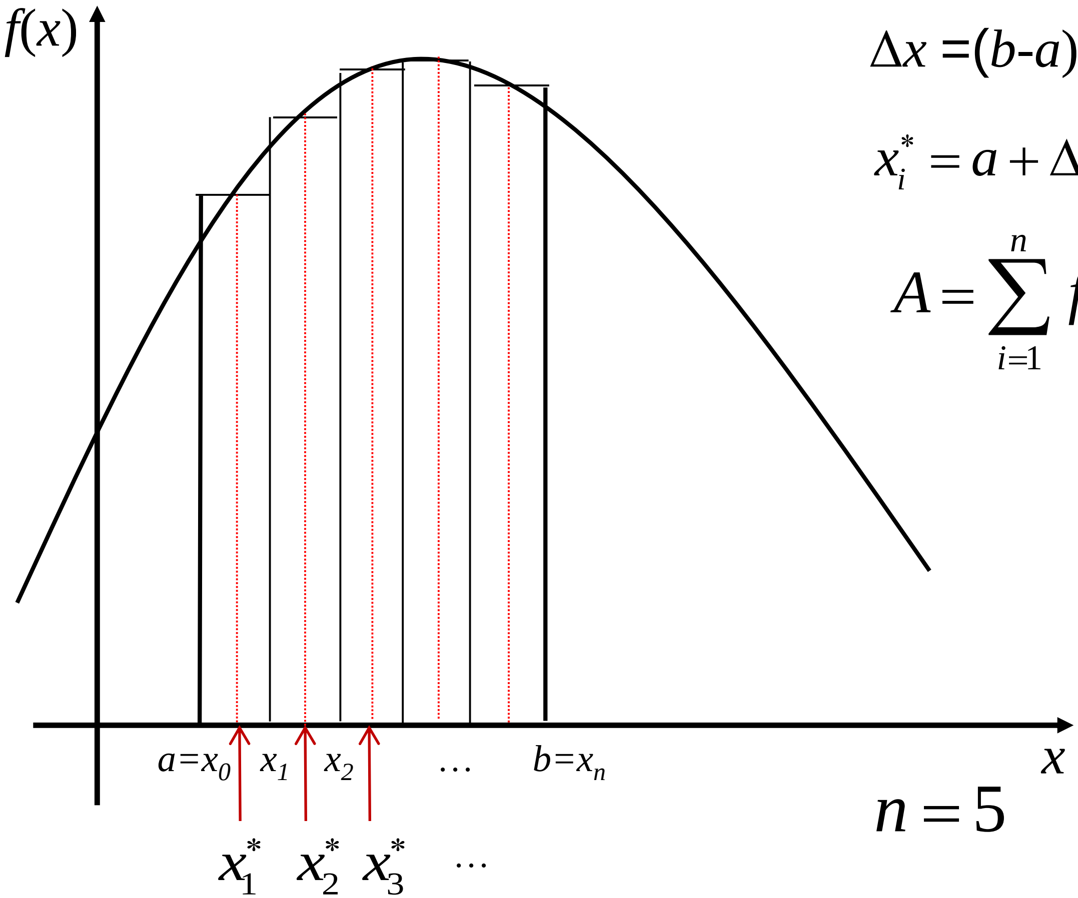
$\Delta x = (b-a)/n$ $\qquad \Delta x = x_i - x_{i-1}$

$$x_i^* = a + \Delta x / 2 + (i-1)\Delta x$$

$$i = 1, \ldots, n$$

$$n = 5$$

Integration - Area Under Curve



Area by rectangles: Δx small

$\Delta x = (b-a)/n$ $\quad \Delta x = x_i - x_{i-1}$

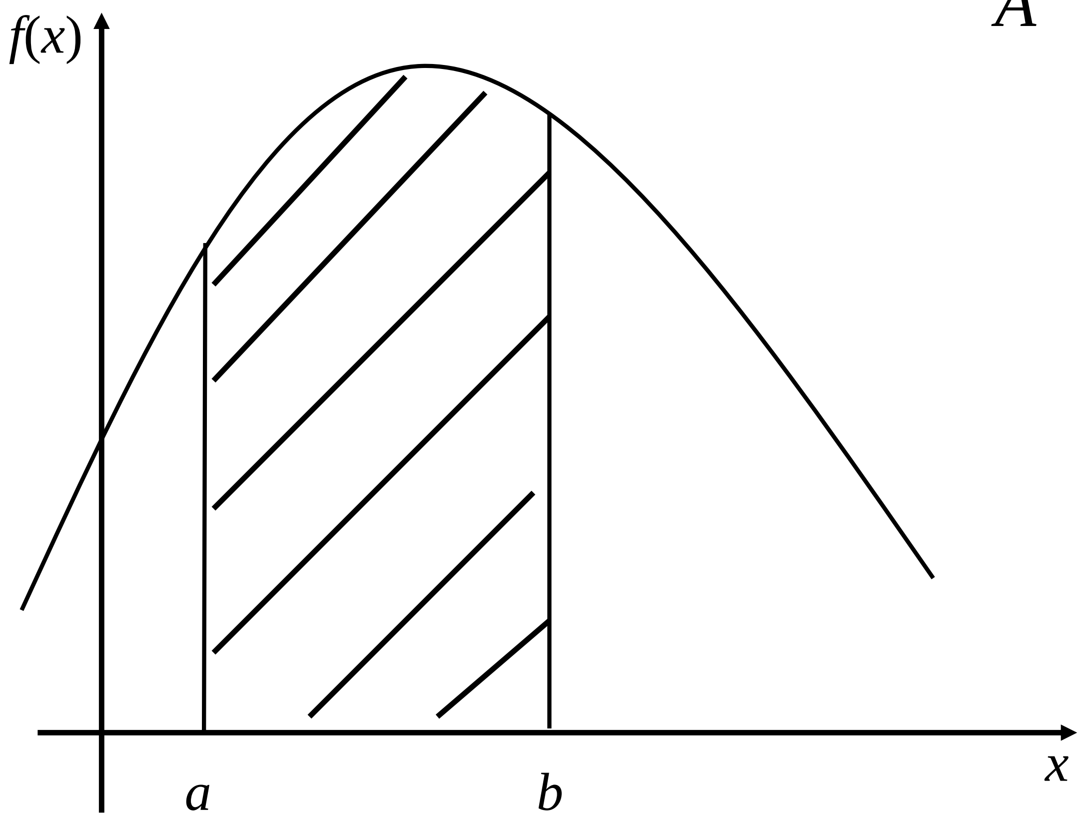
$$x_i^* = a + \Delta x / 2 + (i-1)\Delta x$$

$i = 1, \ldots, n$

$$A = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

$n = 5$

Integration - Area Under Curve



$$A = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

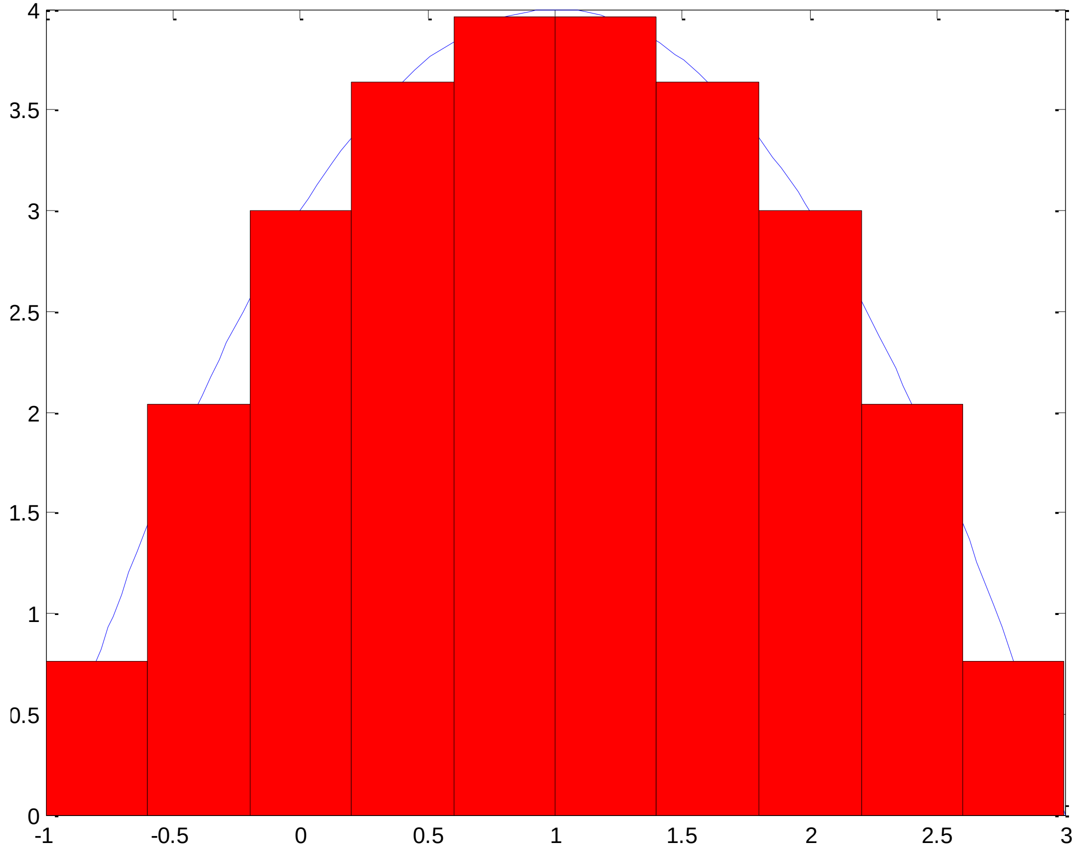
$$= \int_{x=a}^{b} f(x)dx$$

$$\Delta x = (b-a)/n$$

$$x_i^* = a + \Delta x/2 + (i-1)\Delta x$$

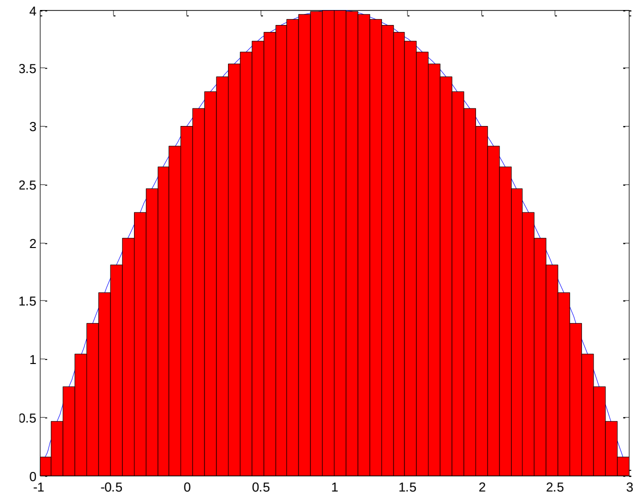
Integration - Numerical Approach

numerical



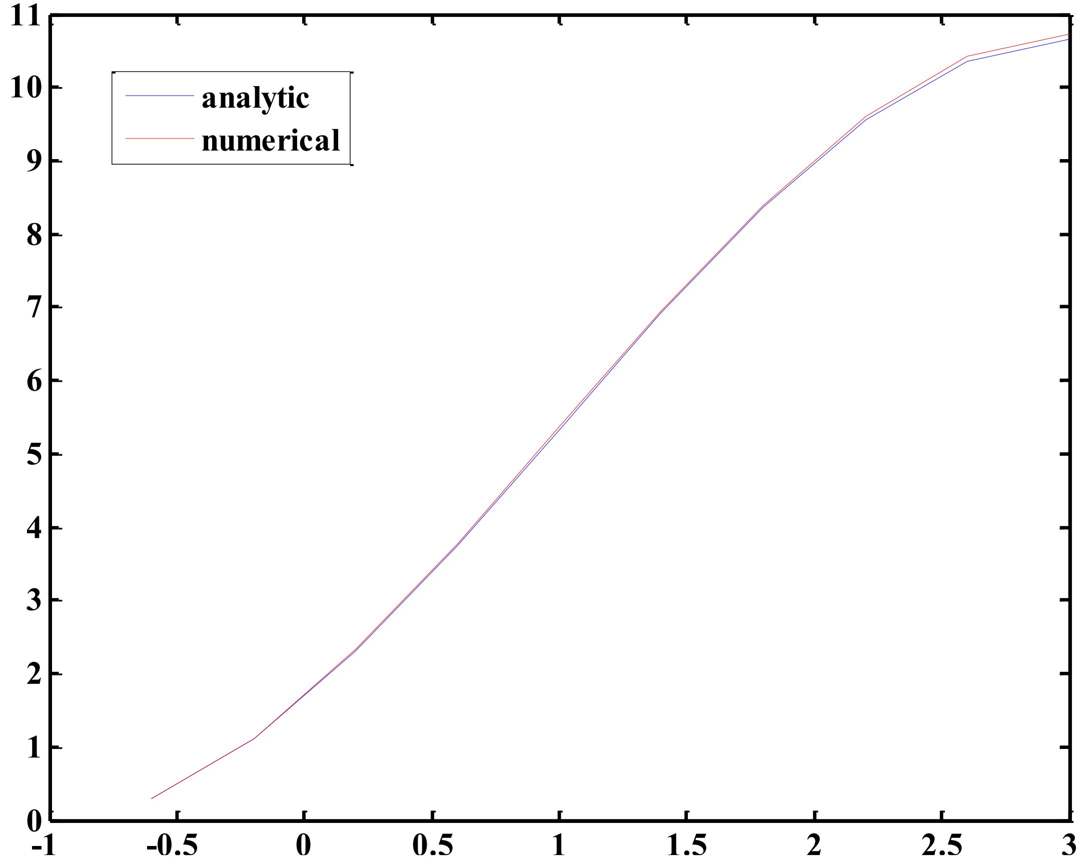
← n=10, Δx=0.4

↓ n=50, Δx=0.08



↖ analytic

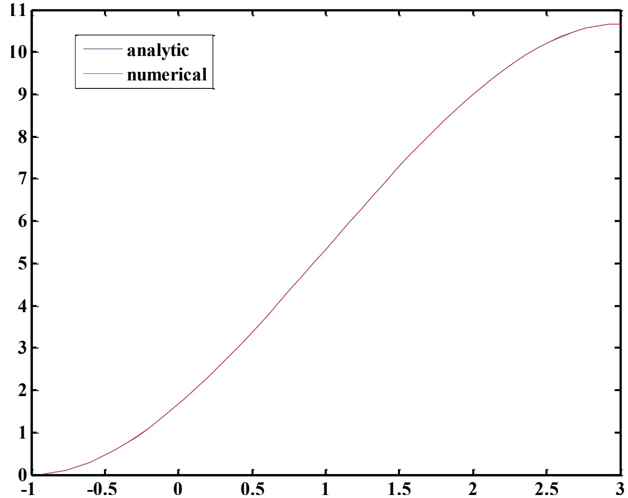
Integration - Numerical Approach



numerical

← n=10, Δx=0.4
numint=10.7200

↓ n=50, Δx=0.08, numint=10.6688



← analytic

Integration - Area Under Curve

Numerical Integration can be applied to find Expectations $E[g(x)]$

$$A = \int g(x) f(x) \; dx$$

Approximated as

$$A = \sum_{i=1}^{n} g(x_i^*) f(x_i^*) \Delta x$$

Homework:

$$f(x) = x^3, x \in \mathbb{R}$$

1) Differentiate analytically and evaluate at a=-1 and b=1.
2) Differentiate by hand numerically with Δx=0.5.
3) Write a Matlab program for 2) then let Δx=1/100.
4) Integrate analytically and evaluate from a=-1 to b=1.
5) Integrate by hand numerically using n=4.

$$\Delta x = 0.5 \qquad \left(x_1^*, x_2^*, x_3^*, x_4^*\right) = \left(-0.75, -0.25, 0.25, 0.75\right)$$

6) Write a Matlab program for 5) then let n=100.

Chapter 3: Random Numbers

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MARQUETTE UNIVERSITY

Be The Difference.

3.1 Pseudorandom Number Generation
"Random" (0,1) numbers

Multiplicative Congruential Method

(1) Start with seed x_0

(2) Compute $x_n = a x_{n-1}$ modulo m , n=0,1,2,…

where a and m are given positive integers

$$x_n = \text{remainder}(a x_{n-1} / m) \sim U(0,1)$$

choose $m = 2^{31}-1$ and $a=7^5$ for 32 bit machines

3.1 Pseudorandom Number Generation
"Random" (0,1) numbers

Mixed Congruential Method

(1) Start with seed x_0

(2) Compute x_n=($ax_{n\text{-}1}$+c) modulo m , n=0,1,2,…

where a, c, and m are given positive integers

$$x_n = \text{remainder}((ax_{n-1} + c)/m) \sim U(0,1)$$

choose m = to be the computer's word length

3.2 Using Random Numbers to Evaluate Integrals
Monte Carlo Integration

Let $g(x)$ be a function and suppose we want

$$\theta = \int_0^1 g(x)dx$$

If U is uniformly distributed over (0,1), then

$$\theta = E(g(U))$$

Think of $f(u) = \begin{cases} 1 & \text{if } 0 \le u \le 1 \\ 0 & \text{otherwise} \end{cases}$ and $\theta = \int_0^1 g(u) f(u) du$

3.2 Using Random Numbers to Evaluate Integrals
Monte Carlo Integration

If we generate $U_1,\ldots,U_k$ independent uniforms then $g(U_1),\ldots, g(U_k)$ will be IID with mean θ.

$$\theta = \int_0^1 g(u) f(u) du$$

$$\sum_{i=1}^{k} \frac{g(U_i)}{k} \rightarrow E[g(U)] = \theta \qquad \text{as } k \rightarrow \infty$$

3.2 Using Random Numbers to Evaluate Integrals Monte Carlo Integration

If we want $\theta = \int_a^b g(x)dx$, then we can transform x to y as y =(x -a)/(b-a), with dy =dx/(b-a)

$$\theta = \int_a^b g(x)dx \qquad dx = (b-a)dy$$

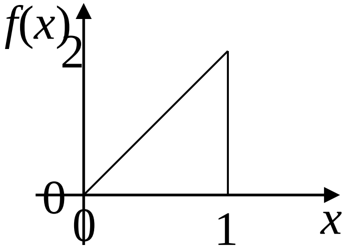
$$x = a + y[b-a]$$

$$\theta = \int_0^1 g(a+[b-a]y)(b-a)dy$$

$$\theta = \int_0^1 h(y)dy \qquad h(y) = g(a+[b-a]y)(b-a)$$

3.2 Using Random Numbers to Evaluate Integrals
Monte Carlo Integration

Integrate $\theta = \int_0^1 2x \, dx$ using $U(0,1)$ random numbers.



Analytically, we'd get $\theta = \int_0^1 2x \, dx = 2 \frac{x^2}{2} \Big|_0^1 = 1$.

Or we can generate uniform numbers $u_1, \ldots, u_n$ and

Calculate $\theta \approx \frac{1}{n} \sum_{i=1}^{n} (2u_i)$.

Matlab Code:

```
rng('default')
n=10^6;
u=rand(n,1);
theta=sum(2*u)/n
```

Matlab Output:

```
theta = 1.0006
```

3.2 Using Random Numbers to Evaluate Integrals Monte Carlo Integration

If we want $\theta=\int_0^\infty g(x)dx$, then we can transform x to y as y =1/(x +1), with dy =-dx/(x+1)^2=-y^2dx

$$\theta=\int_0^\infty g(x)dx \qquad dx=-dy/y^2 \qquad x=1/y-1$$

limits $x=0\rightarrow y=1$

$x=1\rightarrow y=0$

$$\theta=\int_0^1 g(1/y-1)dy/y^2$$

$$\theta=\int_0^1 h(y)dy$$

$$h(y)=g(1/y-1)/y^2$$

3.2 Using Random Numbers to Evaluate Integrals Monte Carlo Integration

If we want $\theta=\int_0^\infty g(x)dx$, then we can transform x to y as y =1/(x +1), with dy =-dx/(x+1)^2=-y^2dx

$$\theta=\int_0^\infty g(x)dx \qquad dx=-dy/y^2 \qquad x=1/y-1$$

limits $x=0\rightarrow y=1$

$x=1\rightarrow y=0$

$$\theta=\int_0^1 g(1/y-1)dy/y^2$$

$$\theta=\int_0^1 h(y)dy \qquad h(y)=g(1/y-1)/y^2$$

3.2 Using Random Numbers to Evaluate Integrals Monte Carlo Integration

Most useful in evaluating multiple integrals.

$$\theta = \int_0^1 \int_0^1 \dots \int_0^1 g(x_1,\dots,x_n) dx_1 \dots dx_n$$

The key is to use $\theta = E[g(U_1,\dots,U_n)]$, where $U_1,\dots,U_n$ are independent $U(0,1)$'s.

If we generate $U_1^1,\dots,U_n^1$ then $\theta = E[g(U_1,\dots,U_n)] \approx \sum_{i=1}^{k} \frac{g(U_1^i,\dots,U_n^i)}{k}$.

$$\begin{array}{c} U_1^2,\dots,U_n^2 \\ \vdots \\ U_1^k,\dots,U_n^k \end{array}$$

Homework 2:

Chapter 3: # 1*, 3, 7, 9, 11.

*Repeat generating 10^4 of these.

Compute mean, variance, and make a histogram.

Repeat using Matlab's rand() command.

Compare results from to Matlab's rand().