Away from Orthonormal Basis: Sparsity Meets Redundancy

Mathematical Models and Methods for Image Processing

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Assignment

The limitations of sparsity

Generate a sparse 1D signal w.r.t. D

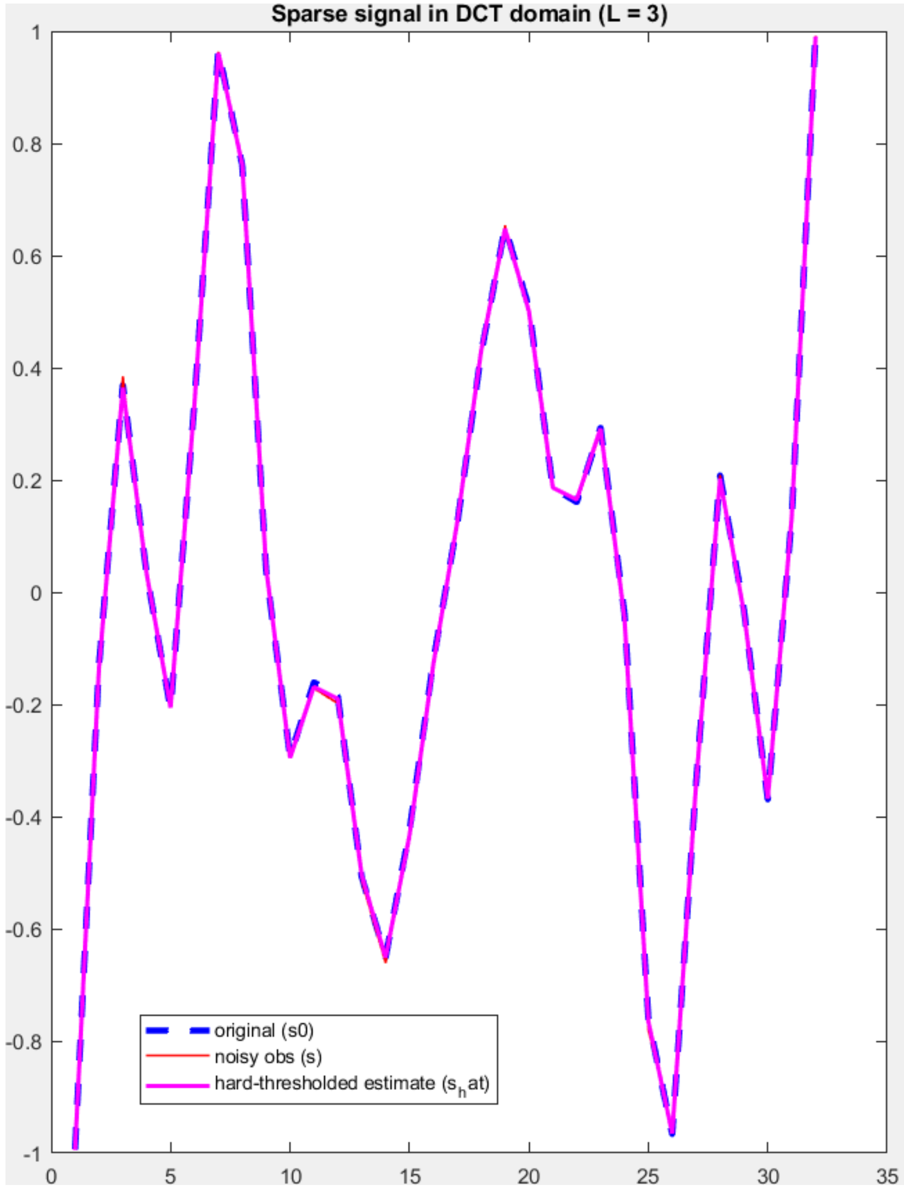
Idea:

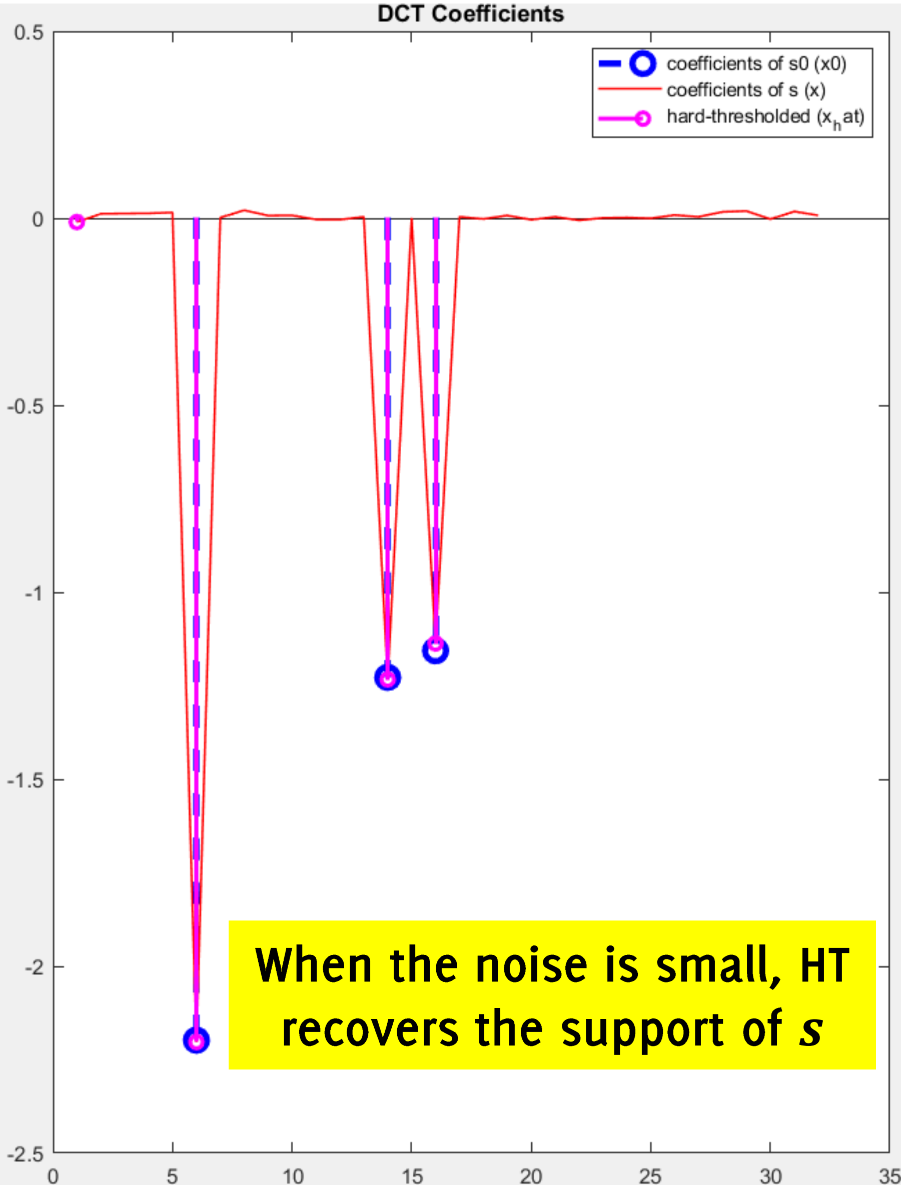
1. Randomly define sparse coefficients x_0 of size M
2. Synthesis w.r.t. a DCT dictionary, i.e. compute $s_0 = Dx_0$
3. Add white Gaussian noise η: $s = s_0 + \eta$

Rmk:

s might not look very realistic, but this is truly sparse w.r.t. D

Generate a truly sparse signal w.r.t. D

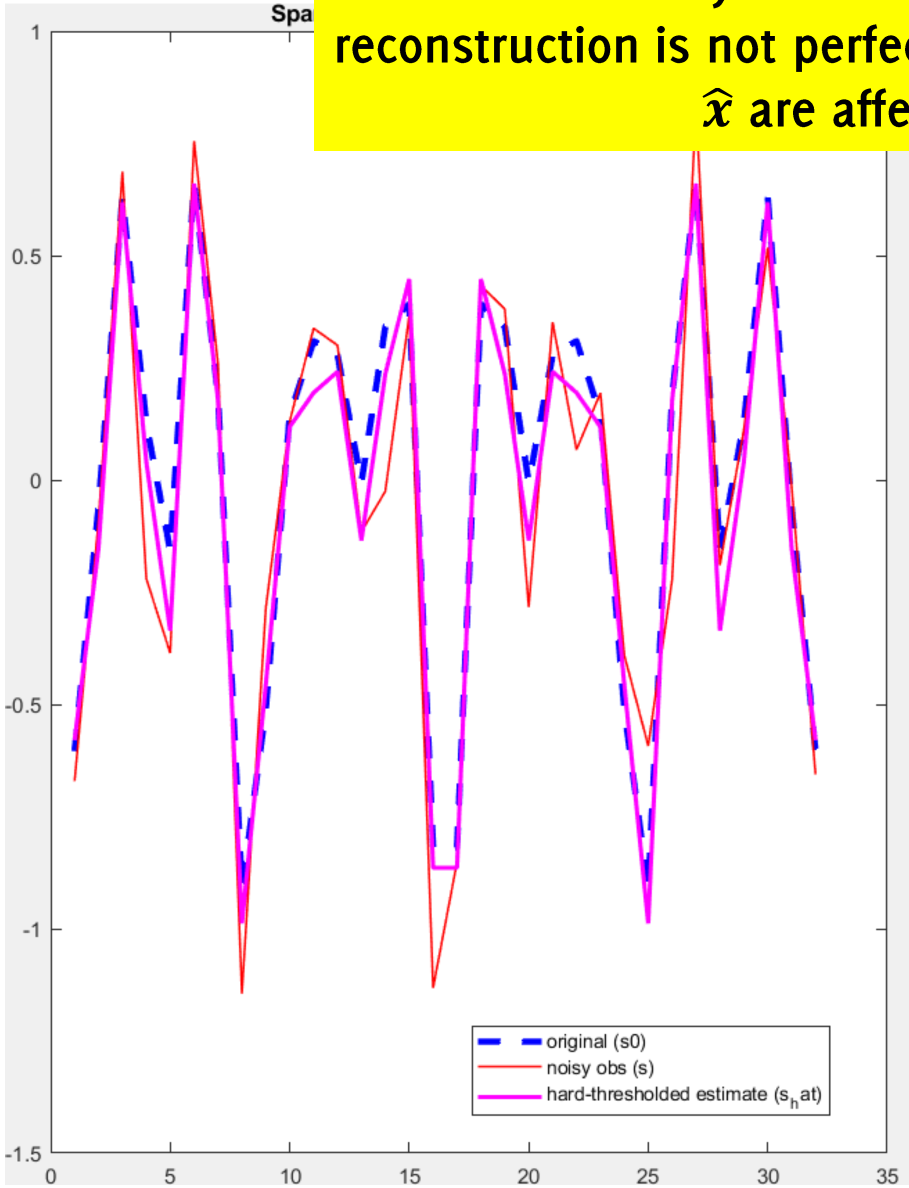


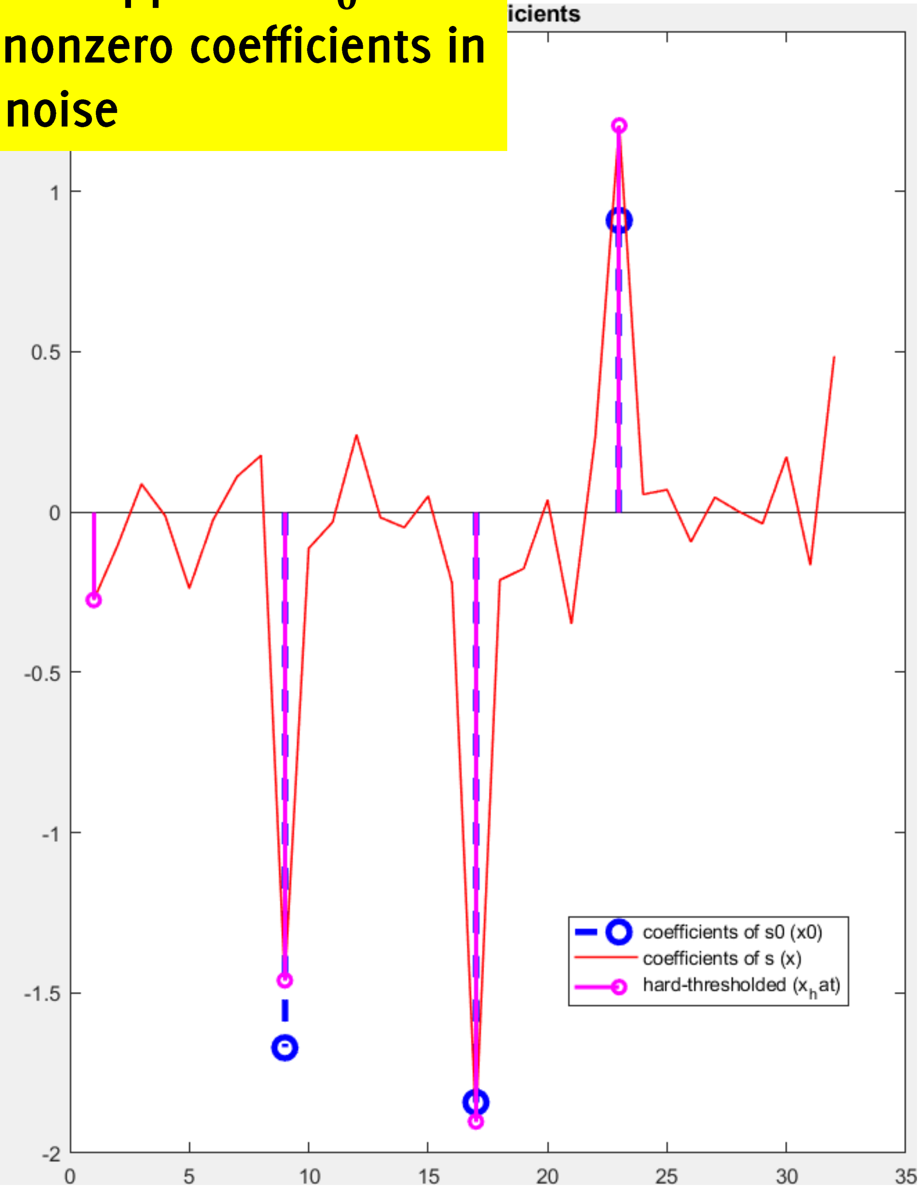


When the noise is small, HT recovers the support of s

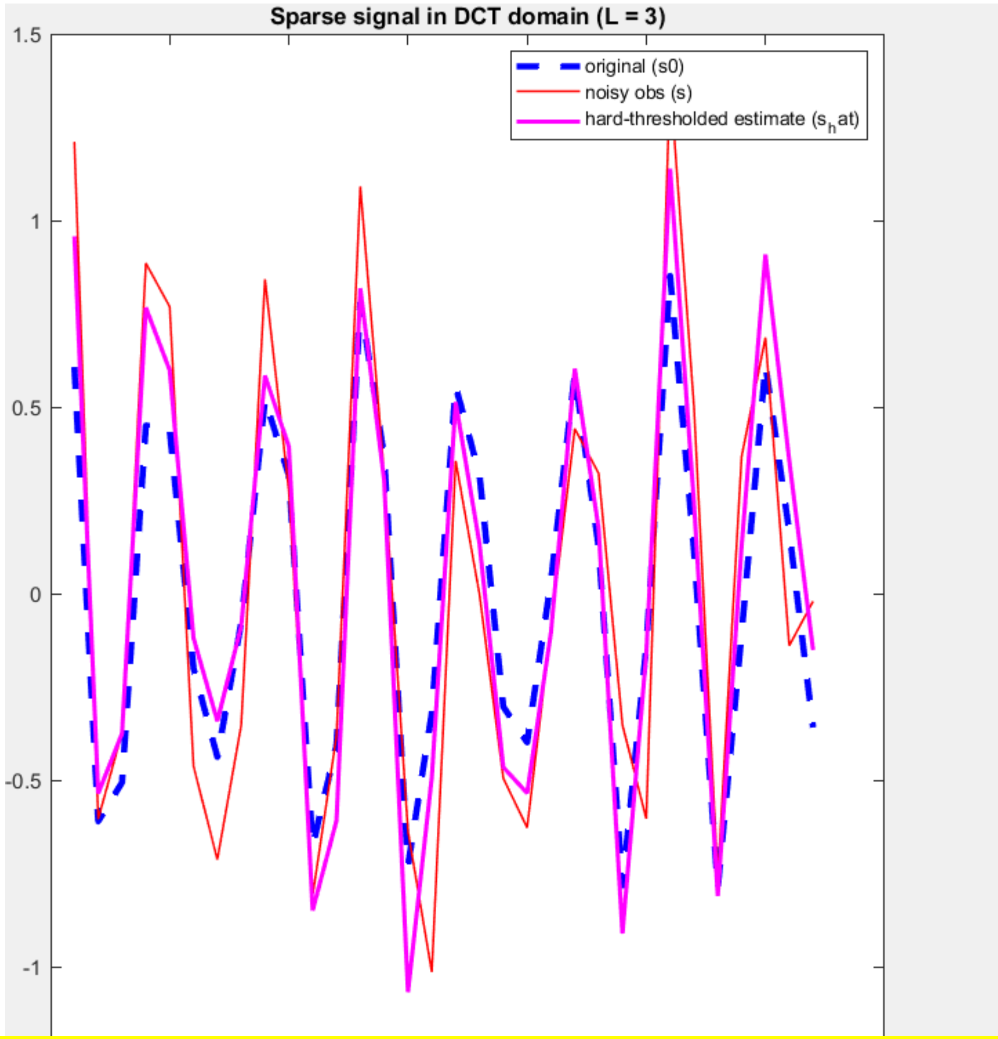
Generate a truly sparse signal w.r.t. D

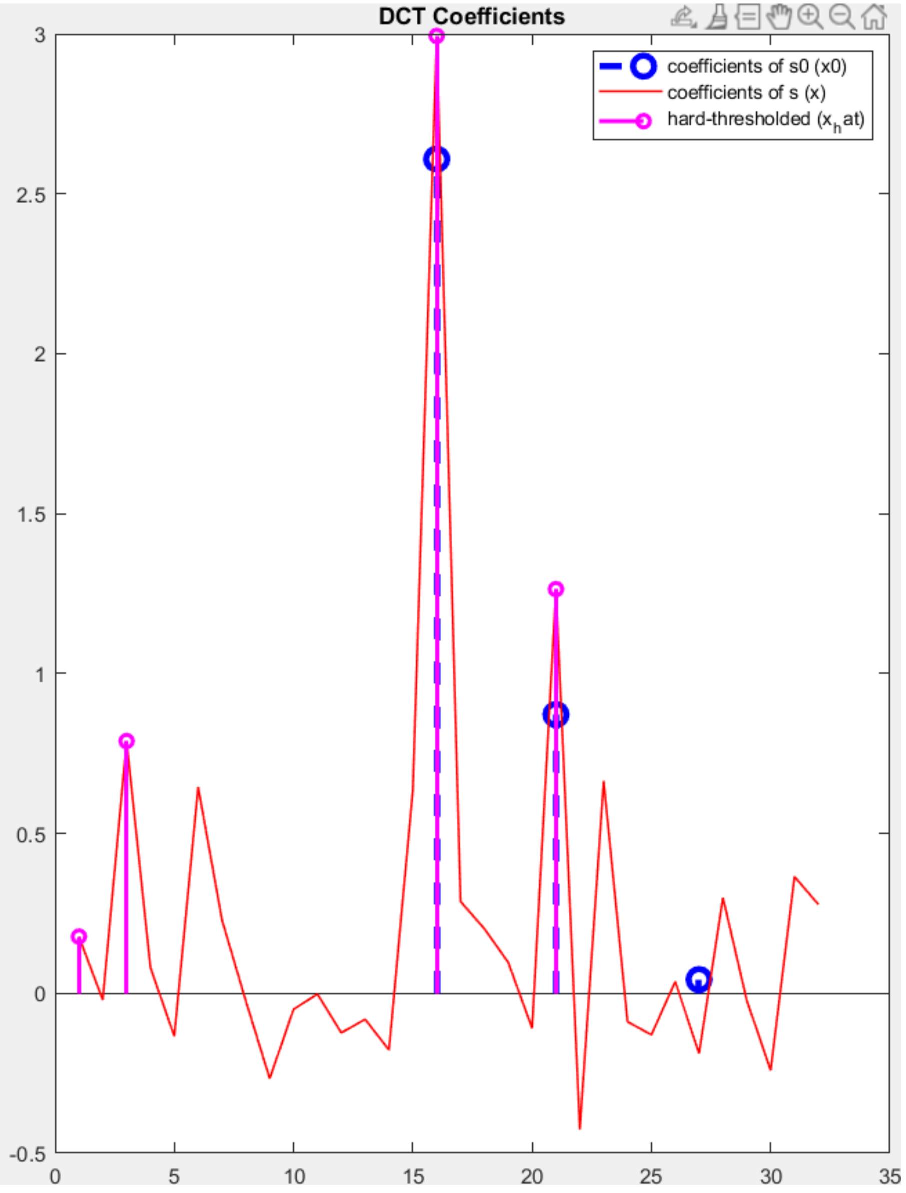
Even when you can recover the support of s_0 reconstruction is not perfect since nonzero coefficients in $\hat{x}$ are affected by noise





Generate a truly sparse signal w.r.t. D





When the noise is large, HT might fail even at recovering the support of x_0

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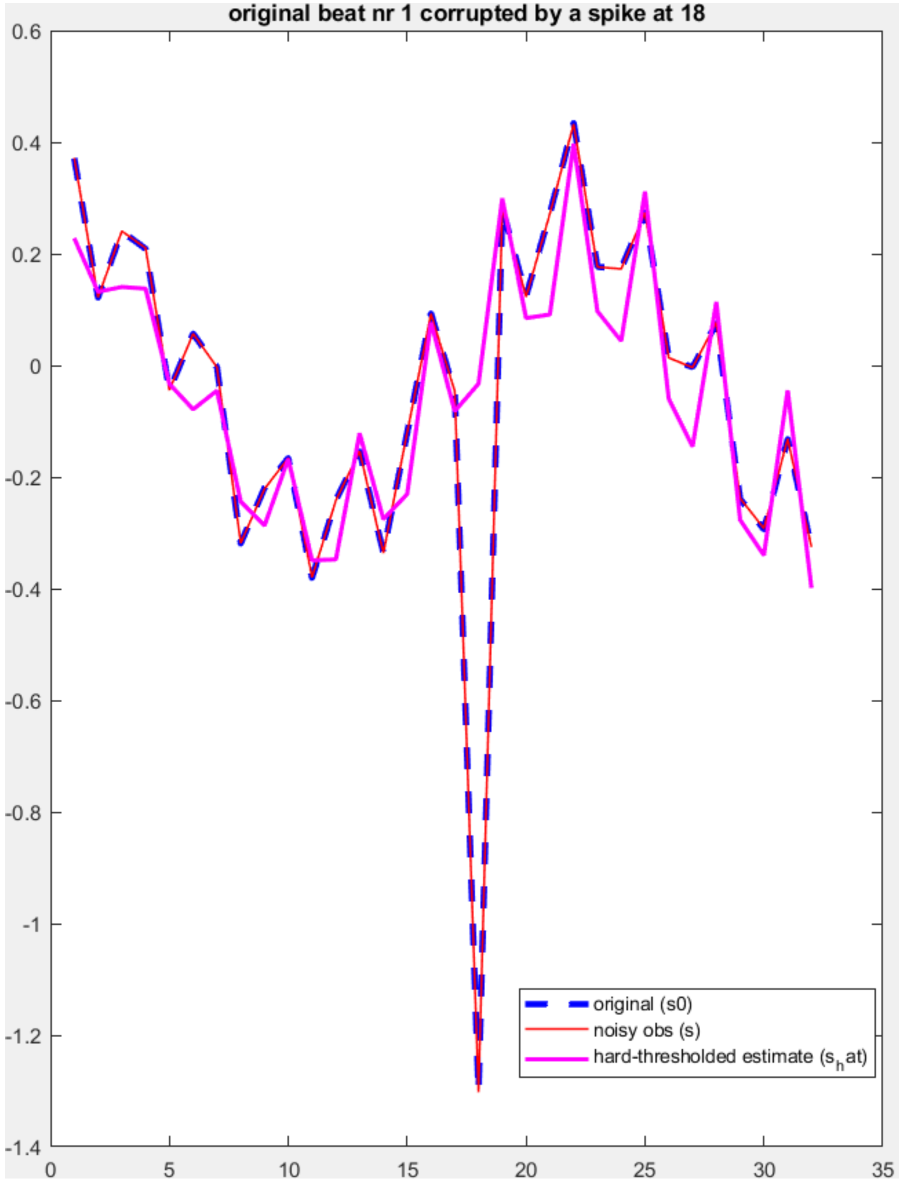
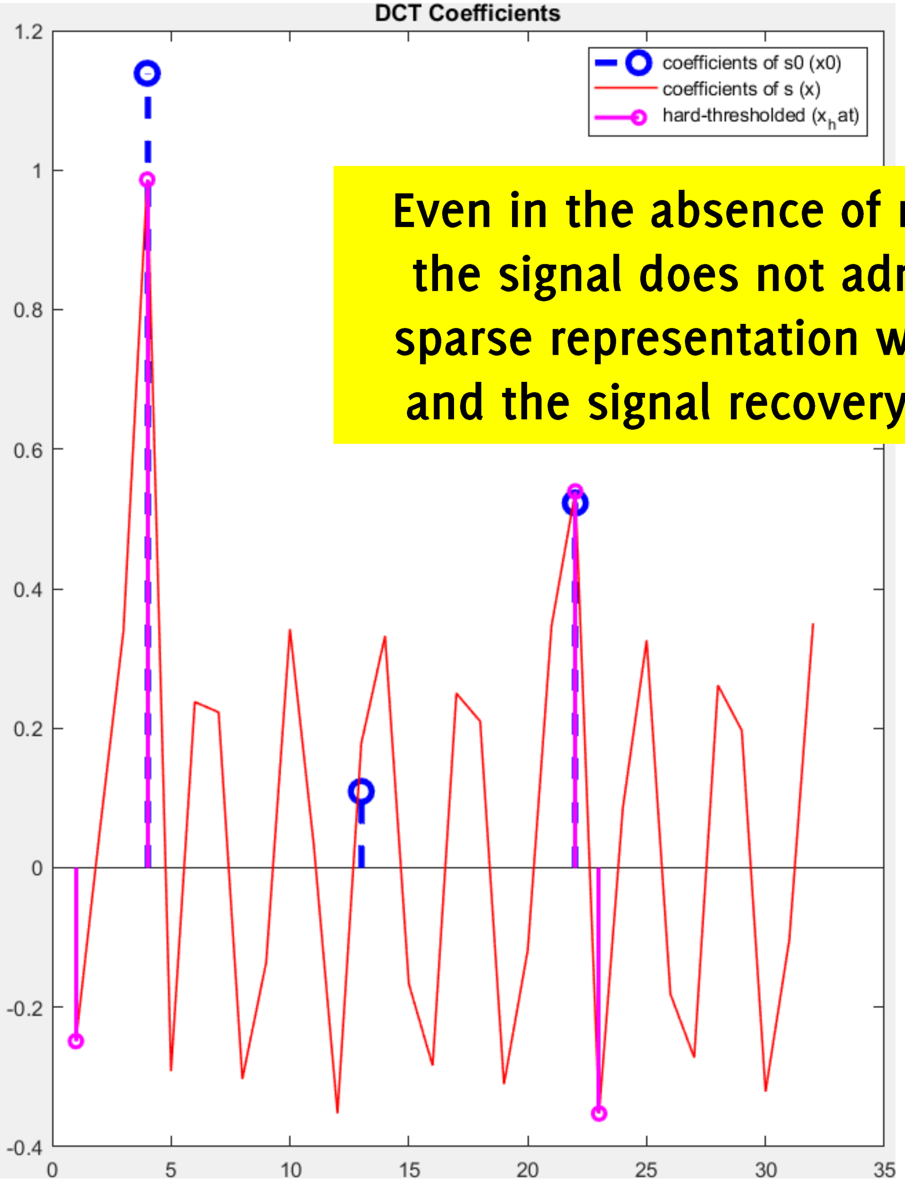
Now, assume your signal is sparse w.r.t. $[D, C]$

Idea:

1. Randomly define sparse coefficients x_0
2. Synthesis w.r.t. a DCT dictionary, i.e. compute $s_0 = Dx_0$
3. Add a spike δ_c at location c, which is a sparse element w.r.t. C
$$s_0 = s_0 + \lambda\delta_c$$
   where λ and c are randomly defined
4. Add noise: $s = s_0 + \eta$

Truly sparse signals w.r.t. $[D, C]$





Even in the absence of noise, the signal does not admit a sparse representation w.r.t. D and the signal recovery fails

Assignment

Uniqueness of Representation

A Simple Proof

Proof that if a set of vectors $\{\boldsymbol{e_i}\}$, $\boldsymbol{e}_i \in \mathbb{R}^M$ are linearly independent

and if

$$\boldsymbol{v} = \sum_i x_i \boldsymbol{e_i} \ , x_i \in \mathbb{R}$$

Then the representation $\{x_i\}$ is unique