Numerical Linear Algebra
A.A. 2022/2023
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Written test 13/01/2023

First Name: .... Last Name: ....

This exam has 2 questions, with subparts. You have **2 hours** to complete the exam. There are a total of 30 points available (sufficiency at 18 points). You cannot consult any notes, books or aids of any kind except for the codes implemented during the lab sessions. Write answers legibly in the additional sheets provided, and show all of your work. Please **write your name** on the exam itself and on the extra sheets. Concerning the implementations using Eigen, upload only the `.cpp` main files. For the exercises requiring LIS, report the bash commands used to perform the computations in a unique `.txt` file. **Upload the files** following the received instructions.

Exercise 1

1. Consider the following problem: find $\boldsymbol{x} \in \mathbb{R}^n$, $A\boldsymbol{x} = \boldsymbol{b}$, where $A \in \mathbb{R}^{n\times n}$ and $\boldsymbol{b} \in \mathbb{R}^n$ are given. State under which conditions the mathematical problem is well posed.

2. Describe the general form of a linear iterative method for the approximate solution of $A\boldsymbol{x} = \boldsymbol{b}$ and describe the stopping criteria.

3. State the necessary and sufficient condition for convergence.

4. State and prove the sufficient condition for convergence.

5. Describe the Generalized Minimal Residual Method (GMRES). Recall the interpretation of the scheme as a Krylov subspace method and the main theoretical results.

6. Download the matrix `A_stokes.mtx` from the Exam folder in webeep and save it on the `/shared-folder/iter_sol++` directory. Load the matrix in a new file called `exer1.cpp` using the `unsupported/Eigen/SparseExtra` module and check if the matrix is symmetric. Report on the sheet $\|A\|$ where $\|\cdot\|$ denotes the Euclidean norm.

7. Define the `Eigen` vector $\boldsymbol{b} = (1, 1, \ldots, 1)^{\mathrm{T}}$ and solve the linear system $A\boldsymbol{x} = \boldsymbol{b}$ using the GMRES method (implemented in the `gmres.hpp` template) without *restart*. Fix a maximum number of iterations which is sufficient to reduce the (relative) residual below than 10^{-9}. Use the diagonal preconditioner provided by `Eigen`. Report on the sheet the iteration counts and $\overline{x}_1$ (the first component of the obtained approximated solution $\overline{\boldsymbol{x}}$).

8. Let n be the size of the matrix loaded in the previous point and let $\tilde{A}$ be a $n \times n$ matrix

defined such that

$$\tilde{A} = A + \|A\| \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ 1 & 2 & -1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \ddots & \dots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & \dots & 0 & 1 & 2 \end{pmatrix}.$$

Report on the sheet $\|\tilde{A}_S\|$, where $\tilde{A}_S$ is the symmetric part of $\tilde{A}$, namely $2\tilde{A}_S = \tilde{A} + \tilde{A}^T$.

9. Export the matrix $\tilde{A}$ in the matrix market format (save it as `A_tilde.mtx`) and move it to the `lis-2.0.34/test` folder. Using the GMRES iterative solver available in the LIS library compute the approximate solution of the linear system $\tilde{A}\boldsymbol{x} = \boldsymbol{b}$ up to a tolerance of 10^{-9}. Explore at least two different preconditioning strategies that yield a decrease in the number of required iterations with respect to the GMRES method without preconditioning. Report on the sheet the iteration counts and the relative residual at the last iteration.

Exercise 2

1. Consider the rectangular linear system $A\boldsymbol{x} = \boldsymbol{b}$, where A is an $m \times n$ matrix, $m \geq n$. Provide the definition of the solution in the least-square sense and state under which condition the problem is well posed.

2. Describe the QR factorization of the rectangular matrix A.

3. Discuss how the QR factorization can be employed to solve the above linear system in the least-square sense.

4. What can be done if A does not have full rank?

5. Download the matrix `A_rect.mtx` from the Exam folder in webeep and save it on the `/shared-folder/iter_sol++` directory. Load the matrix in a new file called `exer2.cpp` using the `unsupported/Eigen/SparseExtra` module. Report on the sheet $\|A^T A\|$.

6. Define an `Eigen` vector $\boldsymbol{b} = A\boldsymbol{x}^*$, where $\boldsymbol{x}^* = (1, 1, \dots, 1)^{\mathrm{T}}$. Report on the sheet $\|\boldsymbol{b}\|$.

7. Use the `SparseQR` solver available in the `Eigen` library to compute the approximate solution of the least-square problem associated to $A\boldsymbol{x} = \boldsymbol{b}$. Report on the sheet the Euclidean norm of the error $\|\boldsymbol{x}_{SQR} - \boldsymbol{x}^*\|$, where $\boldsymbol{x}_{SQR}$ is the obtained approximate solution.

8. Use the `LeastSquaresConjugateGradient` solver available in the `Eigen` library to compute the approximate solution of the previous least-square problem up to a tolerance of 10^{-10}. Report on the sheet the iteration counts and the error norm $\|\boldsymbol{x}_{LSCQ} - \boldsymbol{x}^*\|$, where $\boldsymbol{x}_{LSCQ}$ is the obtained approximate solution.

9. Compare and comment the results obtained at the two previous points.