

GAMES 105

Fundamentals of Character Animation

Lecture 02:

# Math Background

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GAMES105 课程交流



VCL @ PKU

# Outline

- Review of Linear Algebra
  - Vector and Matrix
  - Translation, Rotation, and Transformation
- Representations of 3D rotation
  - [R] Rotation matrices
  - [E] Euler angles
  - [R] Rotation vectors/Axis angles
  - [Q] Quaternions



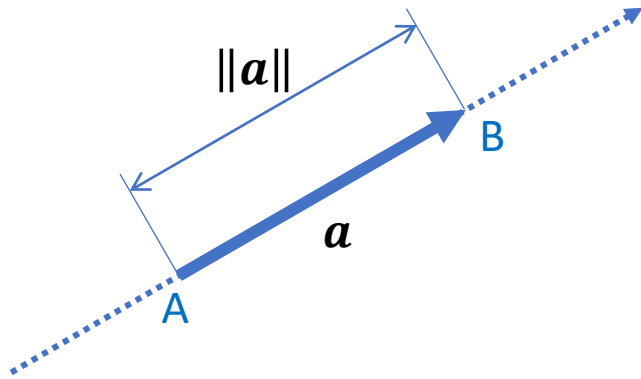
# Review of Linear Algebra

## Vectors and Matrices

\* a few slides were modified from GAMES-101 and GAMES-103

# Vector

- A quantity having both magnitude and direction



vector  $\mathbf{a}$  , written in **bold** letter

magnitude/length/norm:  $\|\mathbf{a}\|$

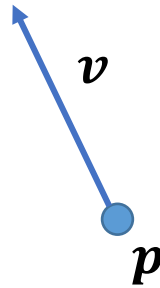
direction:  $\frac{\mathbf{a}}{\|\mathbf{a}\|}$

$\|\mathbf{a}\| = 1 \rightarrow \mathbf{a}$  is a **unit vector**

$\frac{\mathbf{a}}{\|\mathbf{a}\|} \rightarrow$  normalize  $\mathbf{a}$

# Vector

- A quantity having both magnitude and direction
- Representing a location/velocity/abstract feature.....

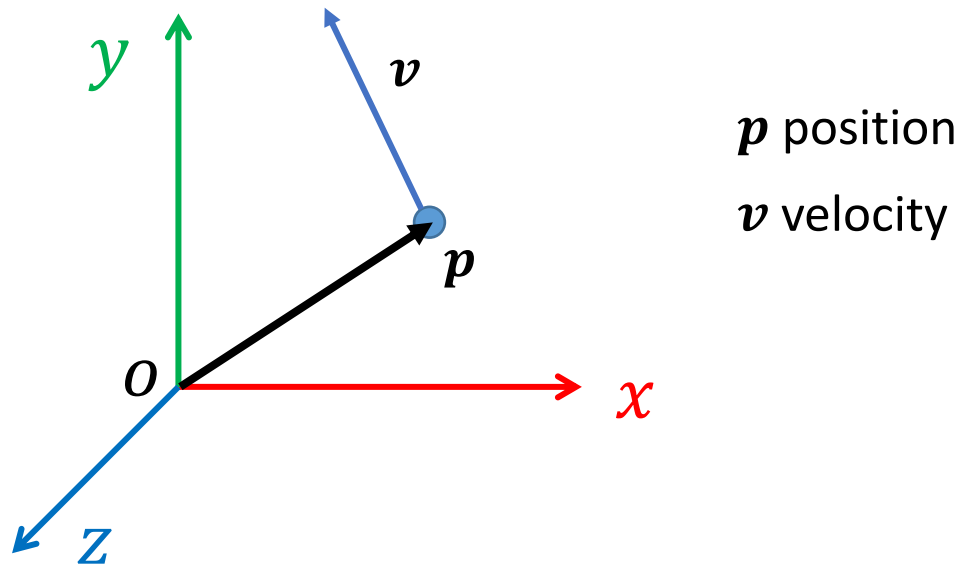


$p$  position

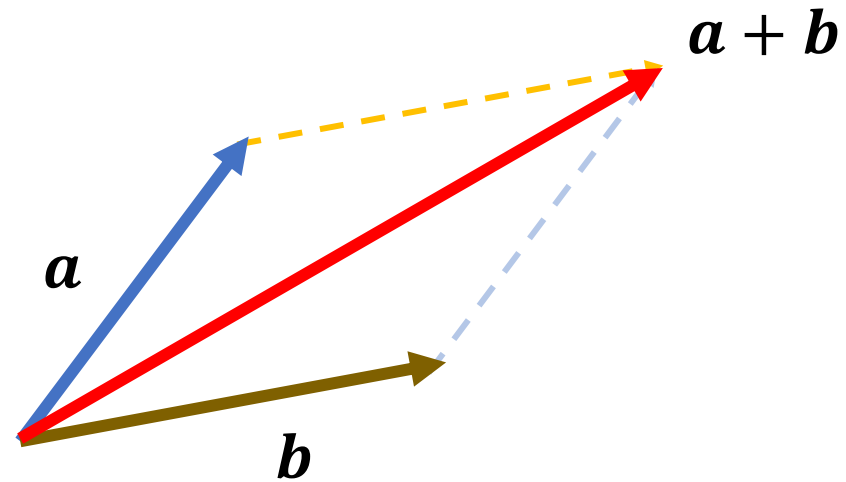
$v$  velocity

# Vector

- A quantity having both magnitude and direction
- Representing a location/velocity/abstract feature.....



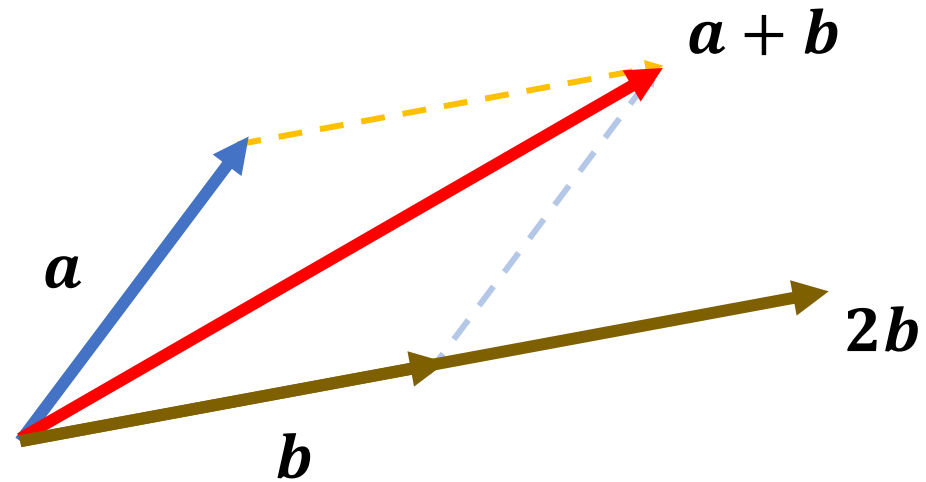
# Vector Arithmetic



$$a + b = b + a$$

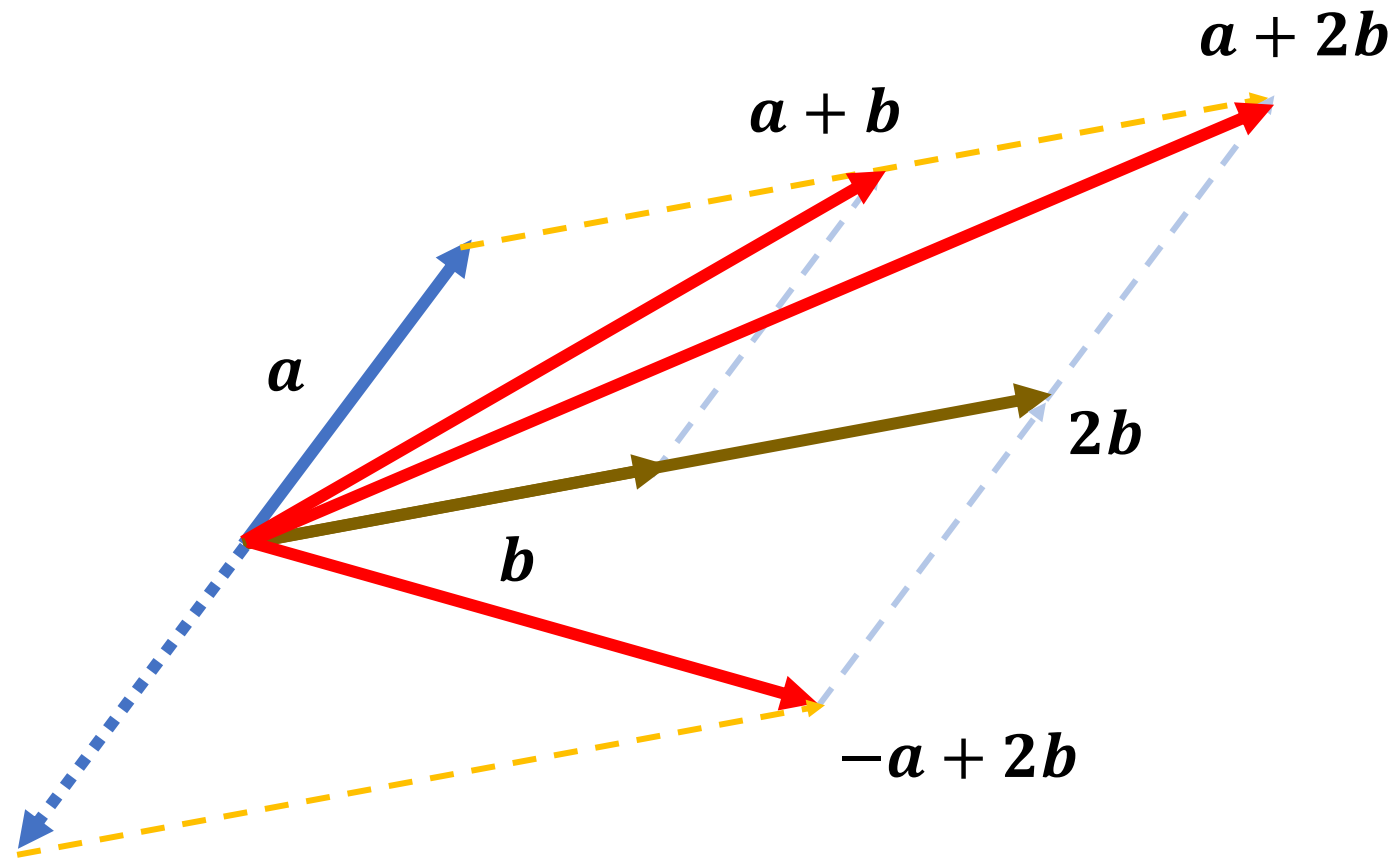
\*commutative

# Vector Arithmetic



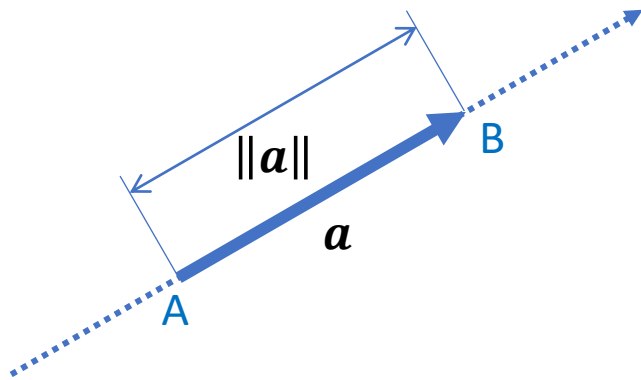


# Vector Arithmetic



# Vector Representation

- A vector can be represented as a [column] of numbers



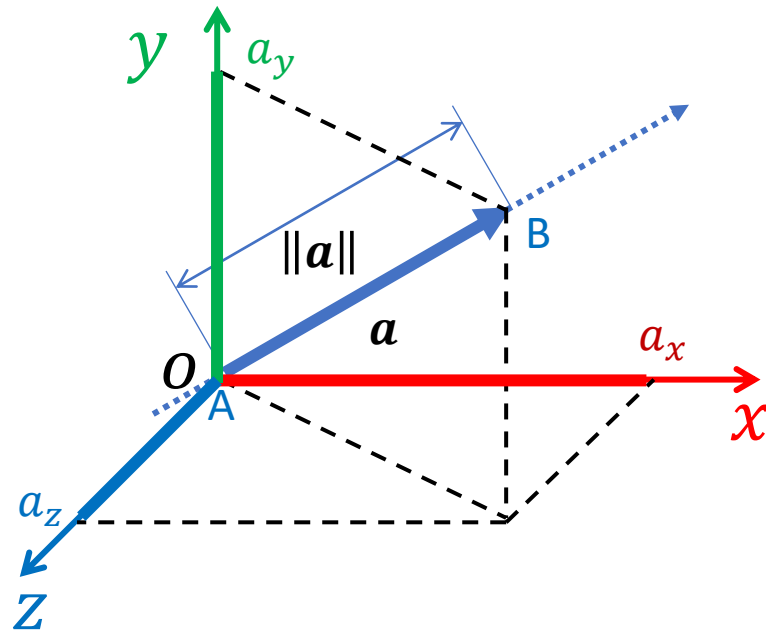
$$\mathbf{a} = (a_1, a_2, \dots, a_n)^T = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

magnitude/length/norm:

$$\|\mathbf{a}\|_2 = \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$$

# Vector Representation

- 3D vector in Cartesian coordinates



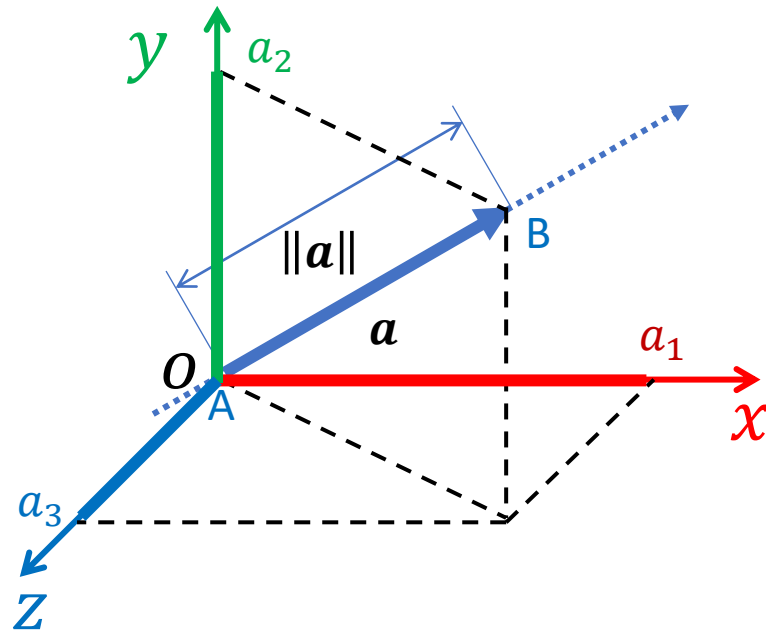
$$\mathbf{a} = (a_x, a_y, a_z)^T = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

magnitude/length/norm:

$$\|\mathbf{a}\|_2 = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

# Vector Representation

- 3D vector in Cartesian coordinates

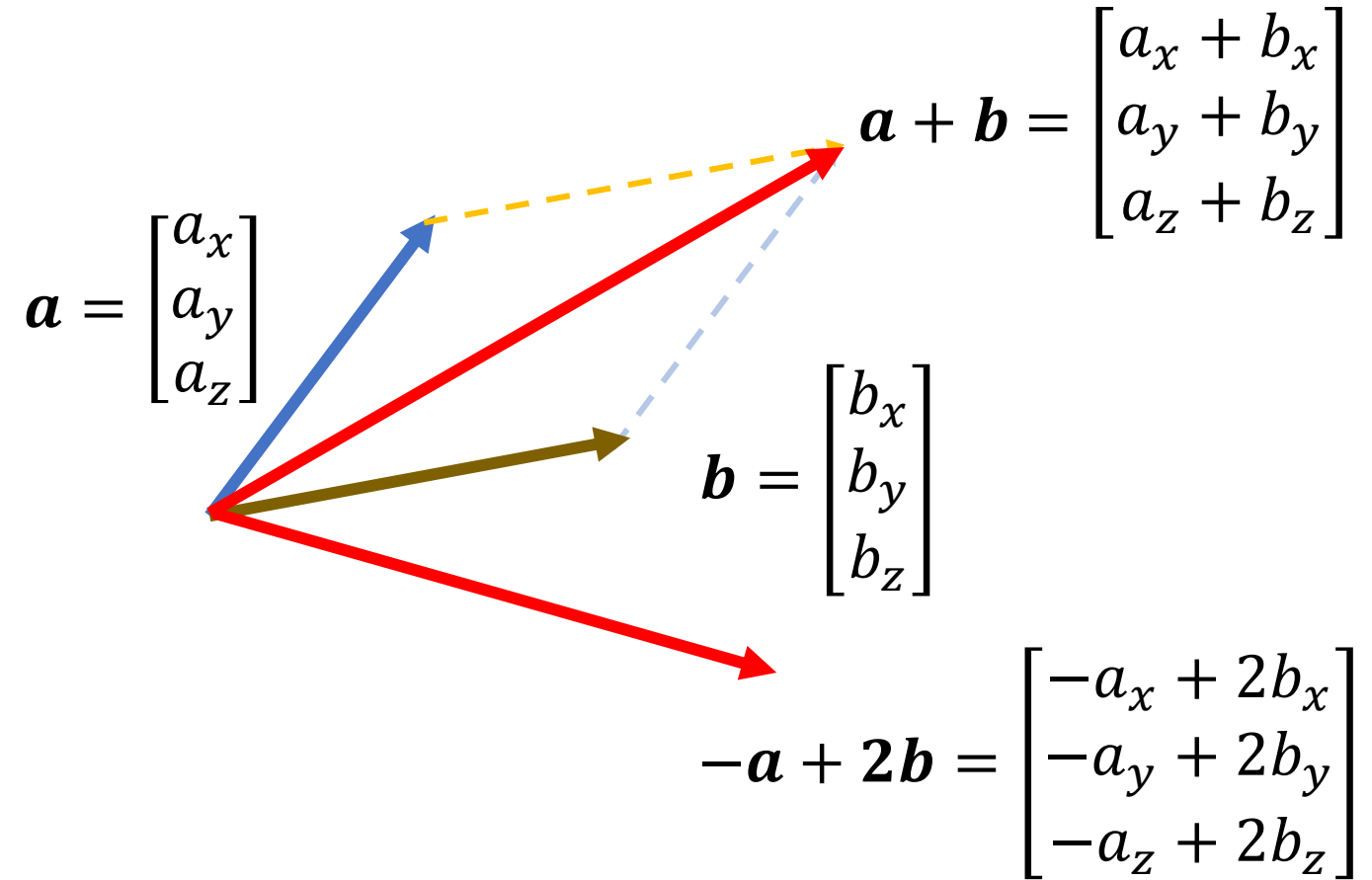


$$\mathbf{a} = (a_1, a_2, a_3)^T = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

magnitude/length/norm:

$$\|\mathbf{a}\|_2 = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

# Vector Arithmetic



# Dot Product

- Inner product/Scalar product

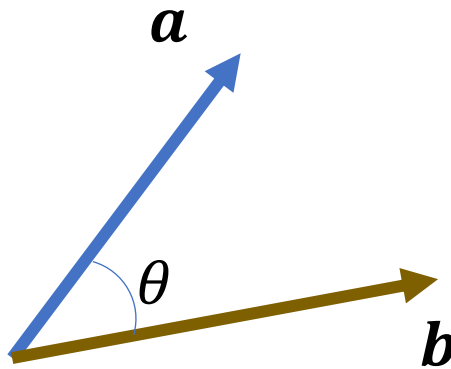
$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
- $\mathbf{a} \cdot \mathbf{a} = a_1a_1 + a_2a_2 + \cdots + a_na_n = \|\mathbf{a}\|_2^2$

# Geometric Meaning of Dot Product

- In Euclidean space,

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

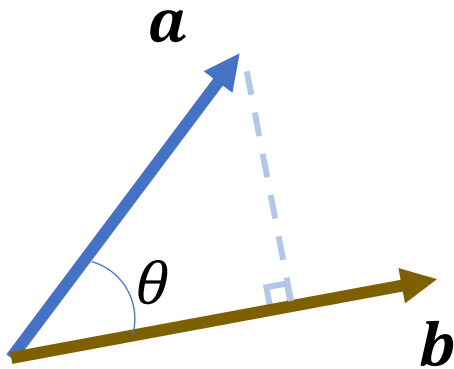


$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta$$

# Geometric Meaning of Dot Product

- In Euclidean space,

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$



$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta$$

$$\theta = \arccos \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|}$$

$$\mathbf{a} \cdot \mathbf{b} = 0$$

$$\Leftrightarrow \cos \theta = 0 \Leftrightarrow \theta = 90^\circ$$

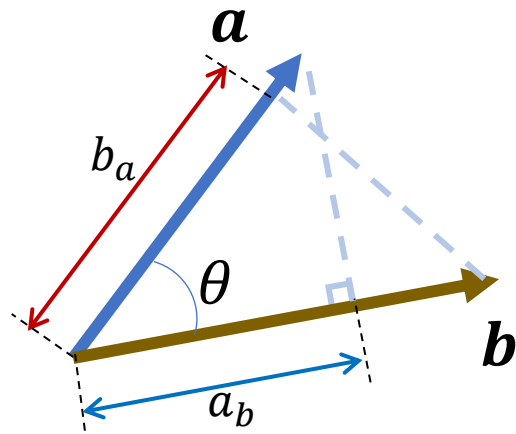
$$\Leftrightarrow \mathbf{a} \perp \mathbf{b}$$



# Geometric Meaning of Dot Product

- In Euclidean space,

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$



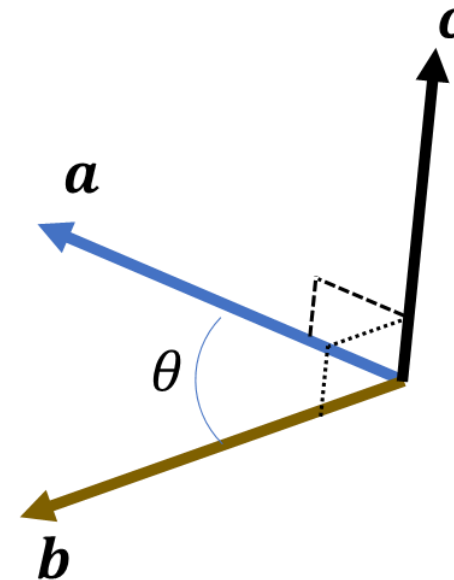
$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta$$

$$a_b = \|\mathbf{a}\| \cos \theta = \mathbf{a} \cdot \frac{\mathbf{b}}{\|\mathbf{b}\|}$$

$$b_a = \|\mathbf{b}\| \cos \theta = \mathbf{b} \cdot \frac{\mathbf{a}}{\|\mathbf{a}\|}$$

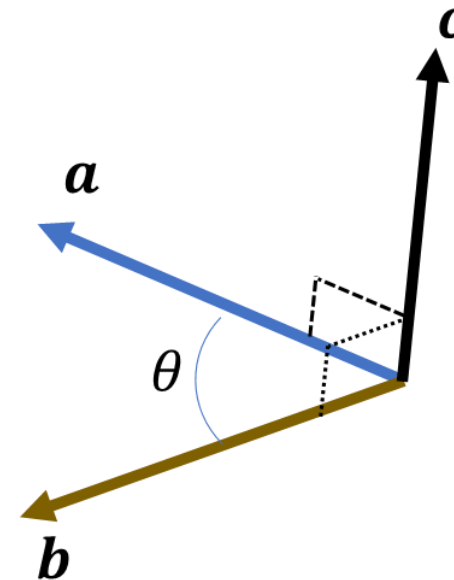
# Cross Product of 3D Vectors

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}$$



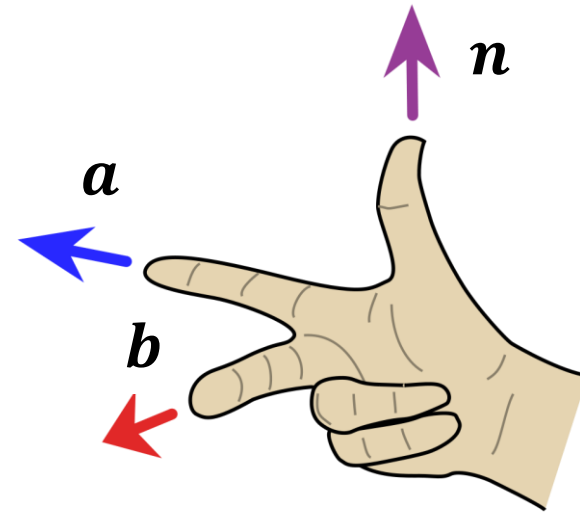
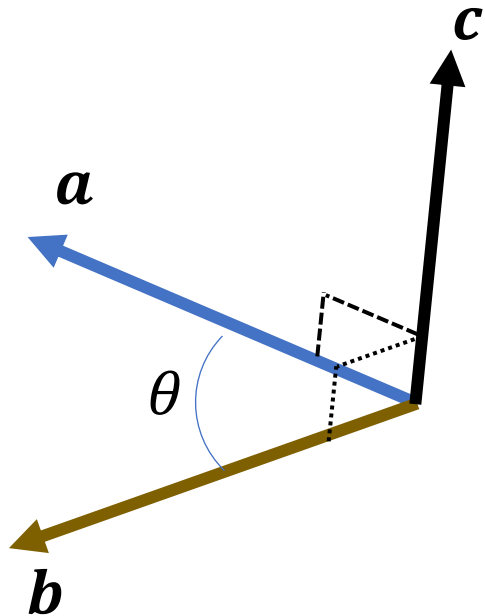
# Cross Product of 3D Vectors

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix} \quad \begin{array}{l} \text{[x]: } yz \\ \text{[y]: } zx \\ \text{[z]: } xy \end{array}$$



# Cross Product of 3D Vectors

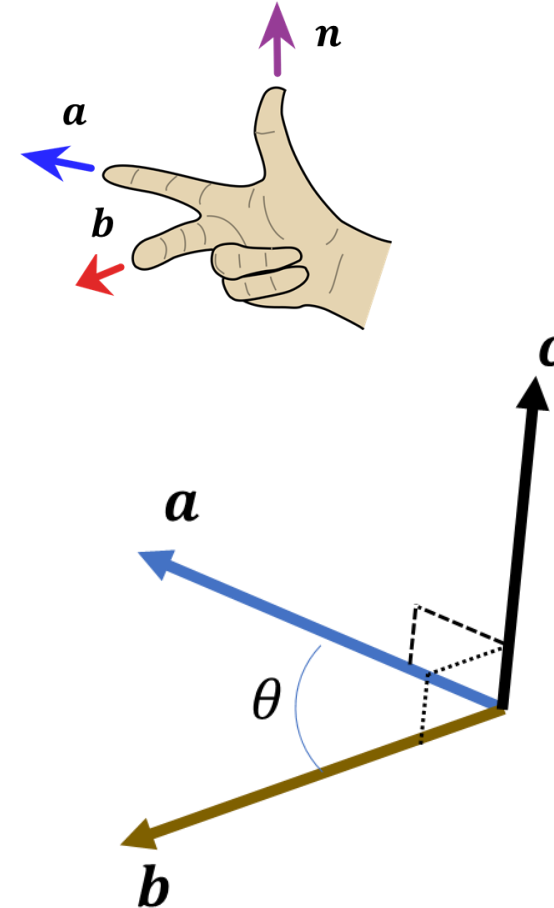
$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix} \quad \Rightarrow \quad \mathbf{c} = \mathbf{a} \times \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \sin(\theta) \mathbf{n}$$



# Cross Product of 3D Vectors

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix} \quad \begin{array}{l} [x]: yz \\ [y]: zx \\ [z]: xy \end{array}$$

- $\mathbf{c} \cdot \mathbf{a} = \mathbf{c} \cdot \mathbf{b} = 0$ 
  - $\mathbf{c} \perp \mathbf{a}, \mathbf{c} \perp \mathbf{b}$
- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
- $\mathbf{a} \times (\mathbf{b} + \mathbf{d}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{d}$
- $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$

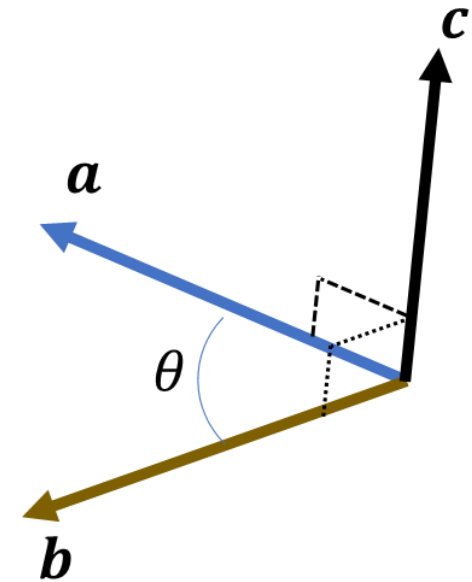
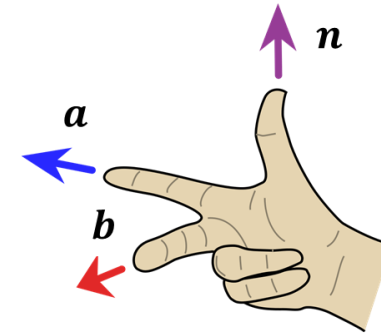


# Cross Product of 3D Vectors

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix} \quad \begin{array}{l} \text{[x]: } yz \\ \text{[y]: } zx \\ \text{[z]: } xy \end{array}$$

- Find a direction  $\mathbf{n}$  perpendicular to both  $\mathbf{a}$  and  $\mathbf{b}$

$$\mathbf{n} = \frac{\mathbf{a}}{\|\mathbf{a}\|} \times \frac{\mathbf{b}}{\|\mathbf{b}\|}$$

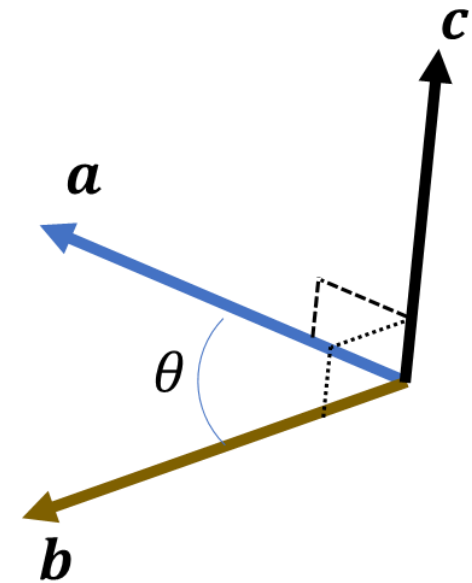
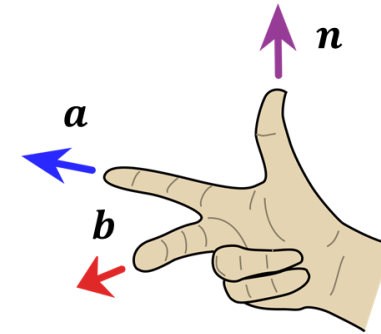


# Cross Product of 3D Vectors

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix} \quad \begin{array}{l} [x]: yz \\ [y]: zx \\ [z]: xy \end{array}$$

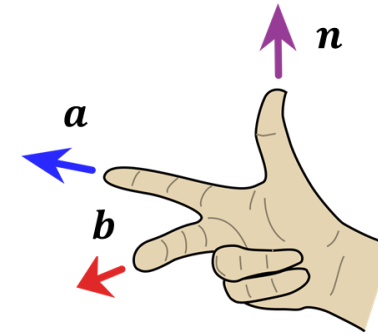
- Find a direction  $\mathbf{n}$  perpendicular to both  $\mathbf{a}$  and  $\mathbf{b}$

$$\mathbf{n} = \frac{\mathbf{a} \times \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|}$$



# Cross Product of 3D Vectors

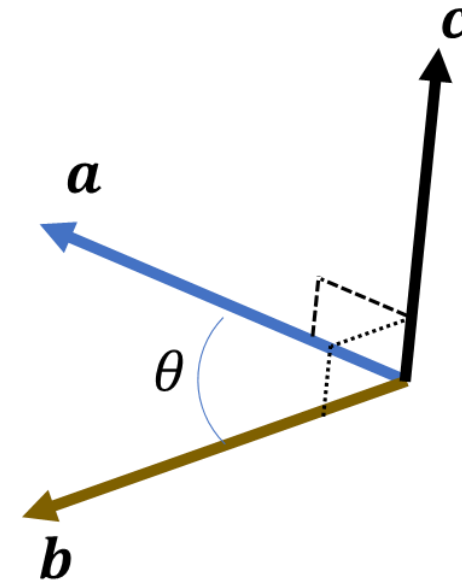
$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix} \quad \begin{array}{l} [x]: yz \\ [y]: zx \\ [z]: xy \end{array}$$



- Find a direction  $\mathbf{n}$  perpendicular to both  $\mathbf{a}$  and  $\mathbf{b}$

$$\mathbf{n} = \frac{\mathbf{a} \times \mathbf{b}}{\|\mathbf{a} \times \mathbf{b}\|}$$

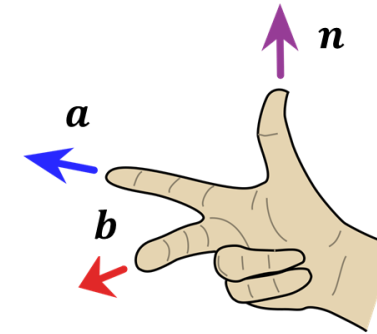
$$\begin{array}{l} \mathbf{a} \neq \mathbf{0}, \mathbf{b} \neq \mathbf{0} \\ \mathbf{a} \nparallel \mathbf{b} \end{array}$$





# Cross Product of 3D Vectors

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix} \quad \begin{array}{l} \textcolor{red}{[x]}: yz \\ \textcolor{blue}{[y]}: zx \\ \textcolor{green}{[z]}: xy \end{array}$$



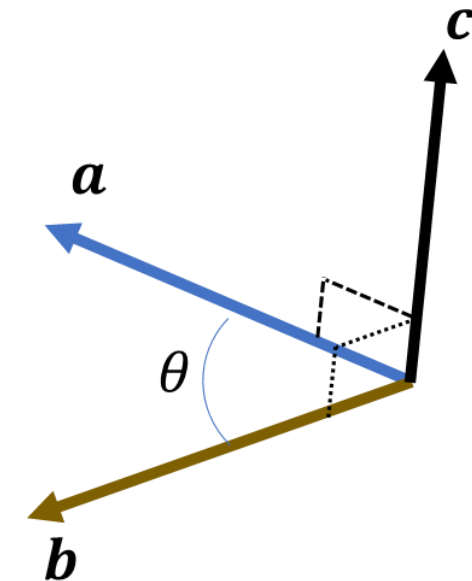
- Find a direction  $\mathbf{n}$  perpendicular to both  $\mathbf{a}$  and  $\mathbf{b}$

$$\mathbf{n} = \frac{\mathbf{a} \times \mathbf{b}}{\|\mathbf{a} \times \mathbf{b}\|}$$

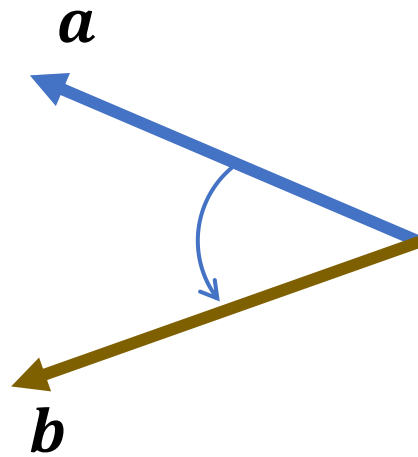
- Check if  $\mathbf{a}$  and  $\mathbf{b}$  are parallel

$$\mathbf{a} \times \mathbf{b} = \mathbf{0} \quad ?$$

$$\mathbf{a} \neq \mathbf{0}, \mathbf{b} \neq \mathbf{0}$$

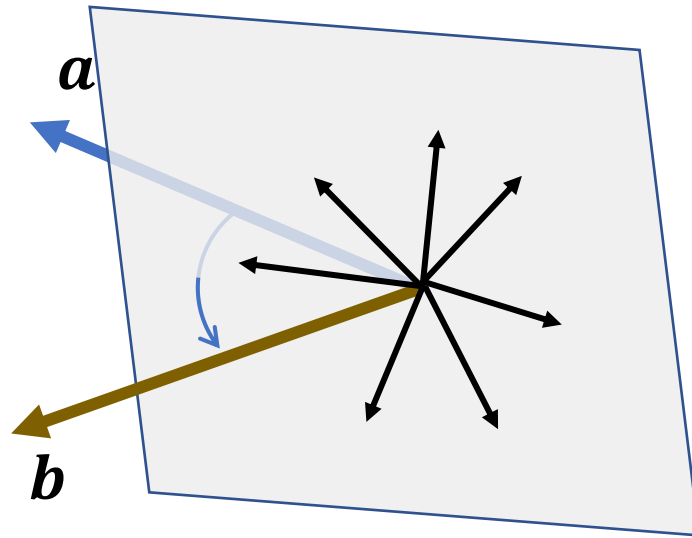


# How to find the rotation between vectors?



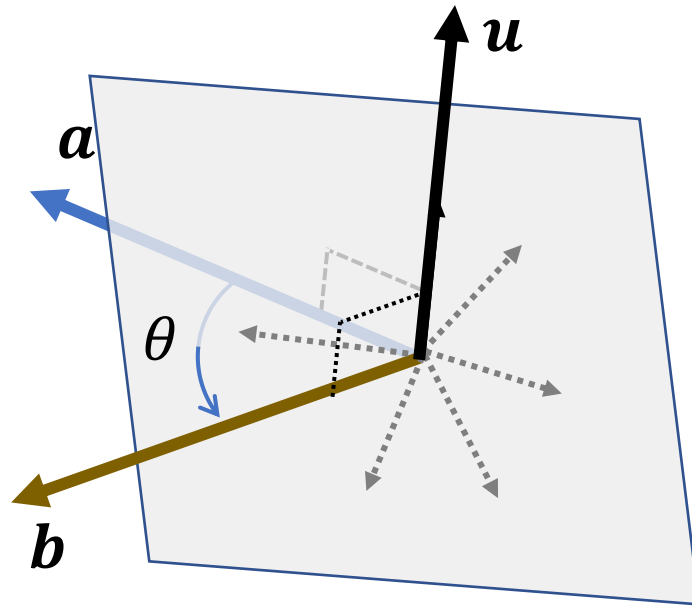
# How to find the rotation between vectors?

Any vector in the bisecting plane can be the axis

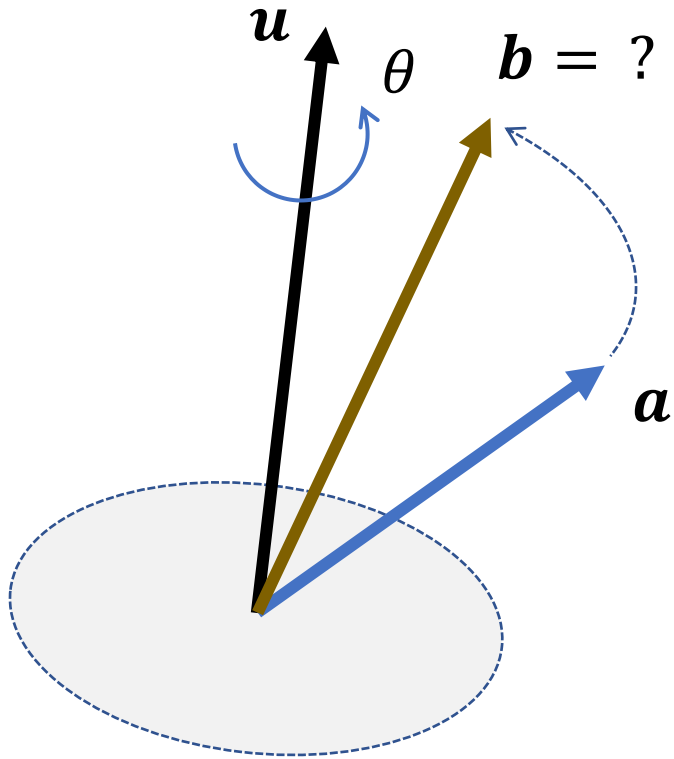


# How to find the rotation between vectors?

The minimum rotation:  $\mathbf{u} = \frac{\mathbf{a} \times \mathbf{b}}{\|\mathbf{a} \times \mathbf{b}\|}$       $\theta = \arg \cos \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|}$

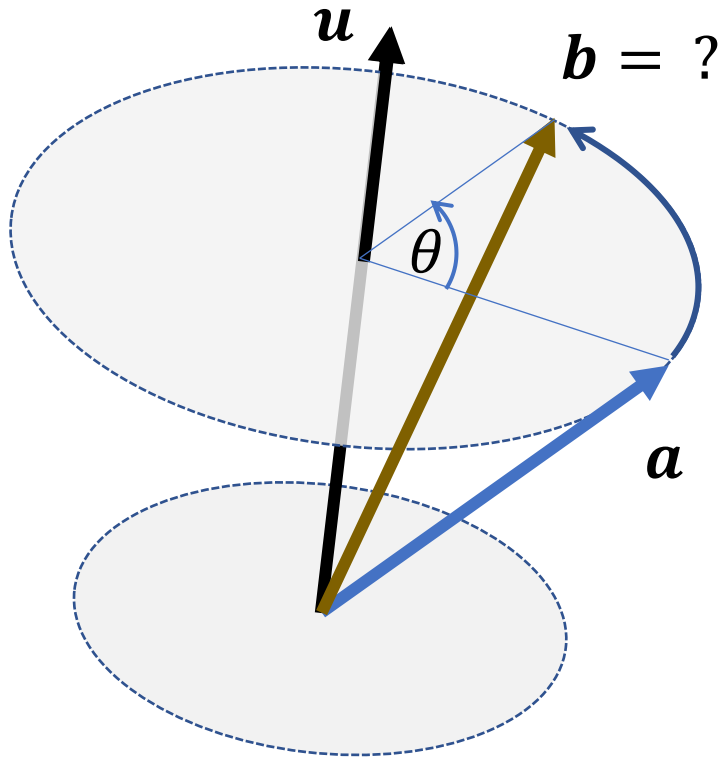


# How to rotate a vectors?



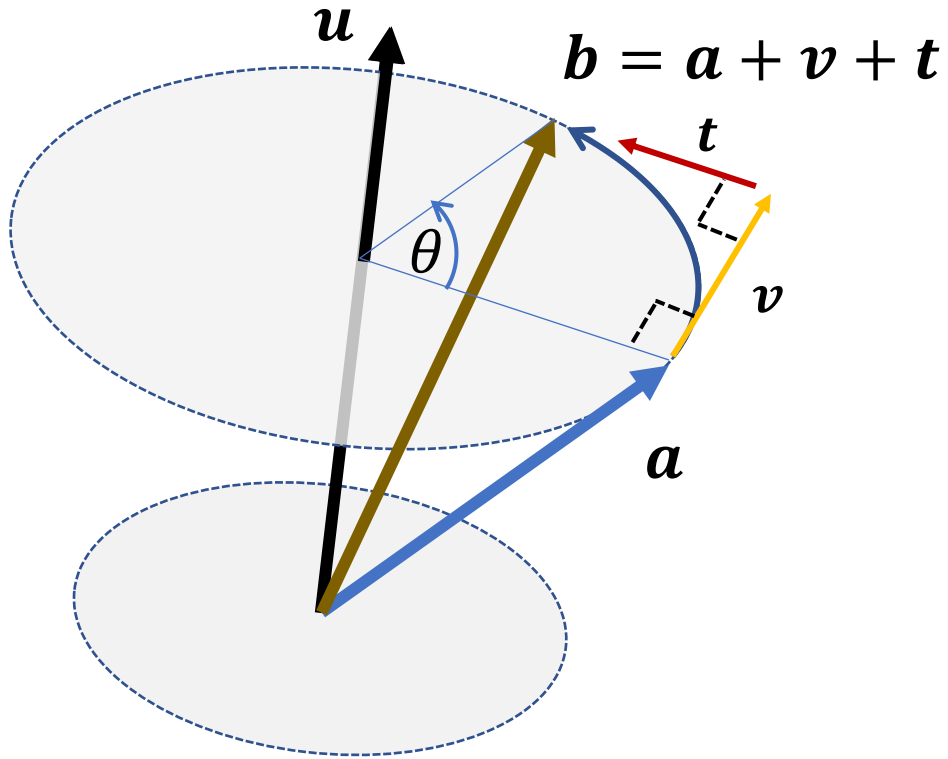
$$\|u\| = 1$$

# How to rotate a vectors?



$$\|u\| = 1$$

# How to rotate a vectors?

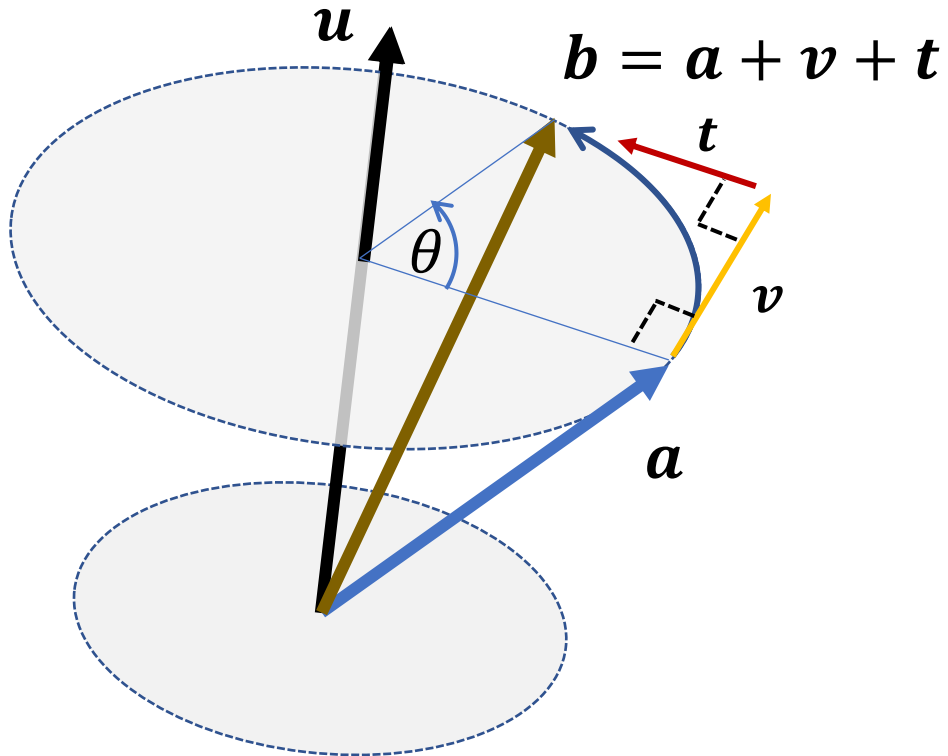


$$v \leftarrow u \times a$$

$$t \leftarrow u \times v = u \times (u \times a)$$

$$\|u\| = 1$$

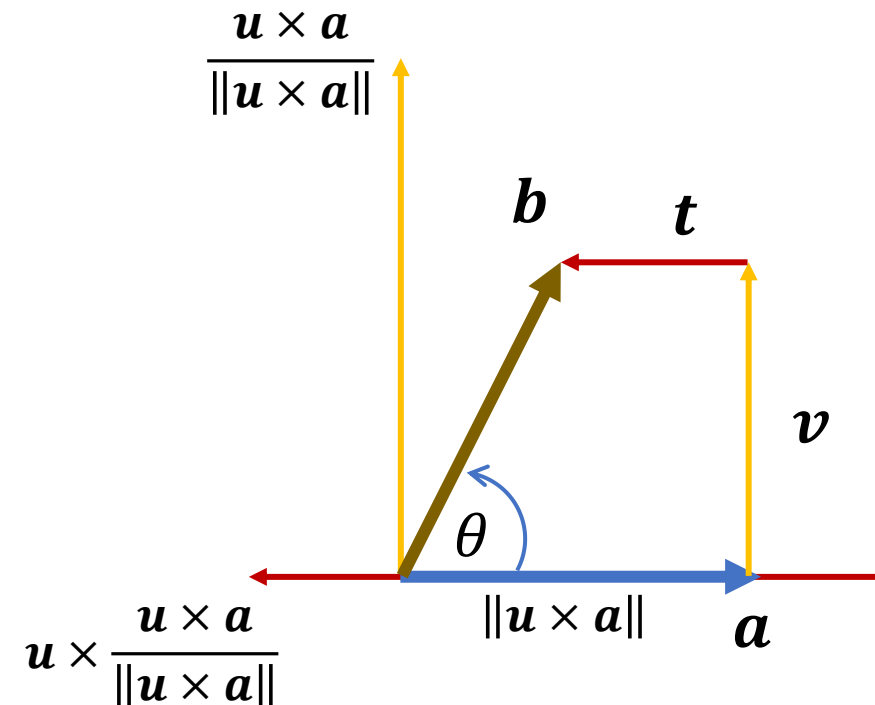
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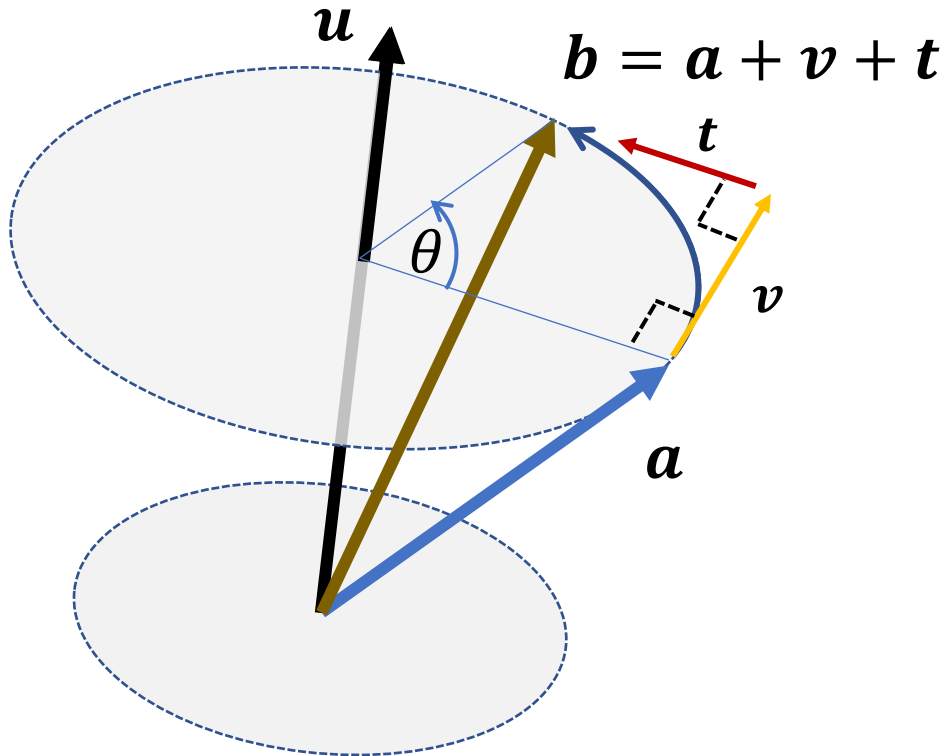
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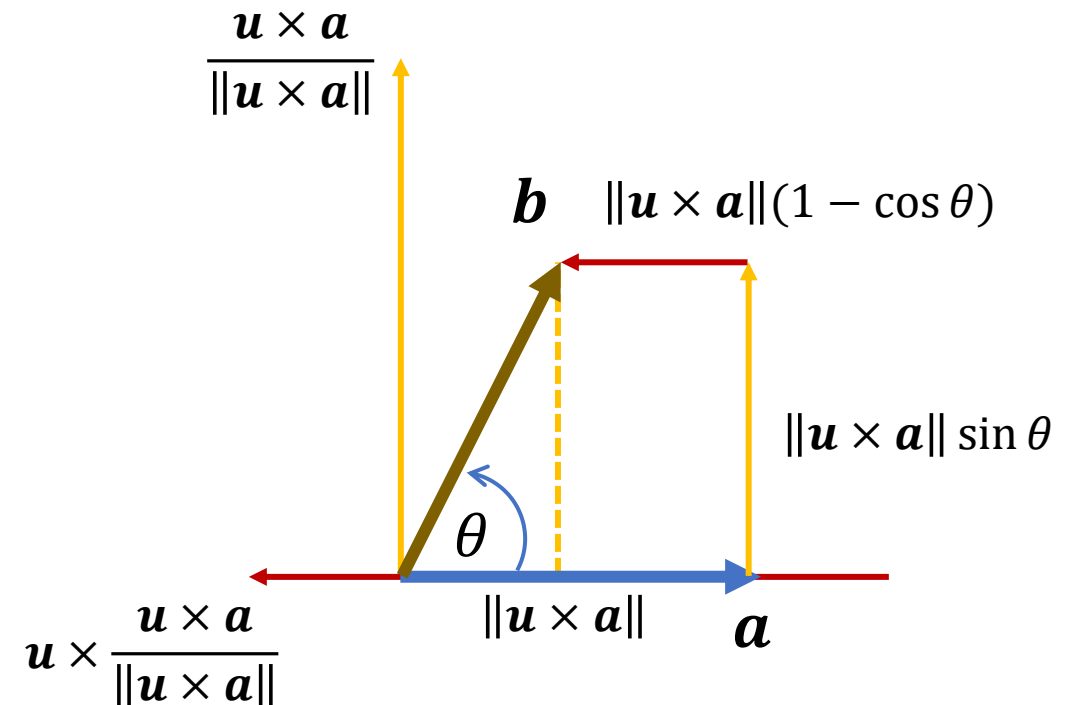
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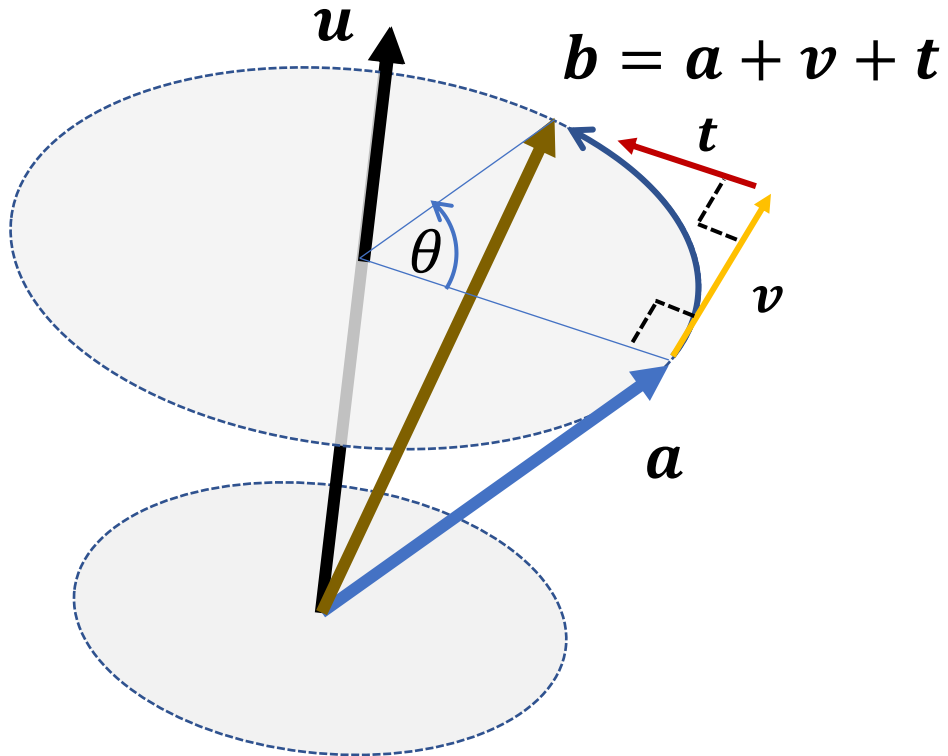
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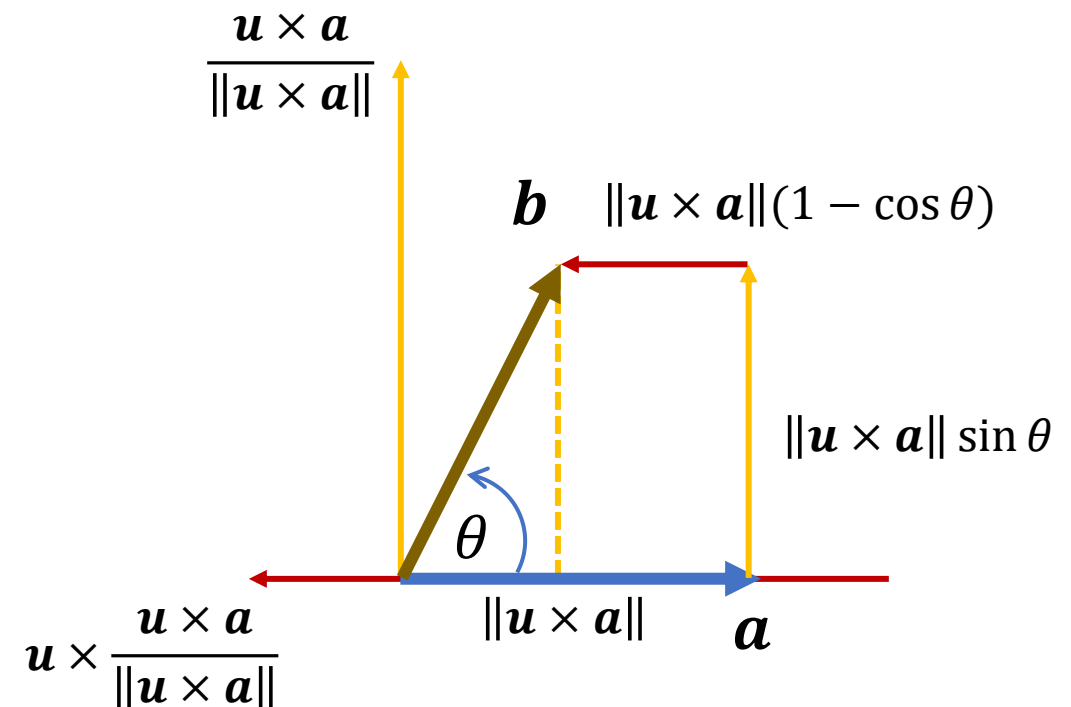
# How to rotate a vectors?



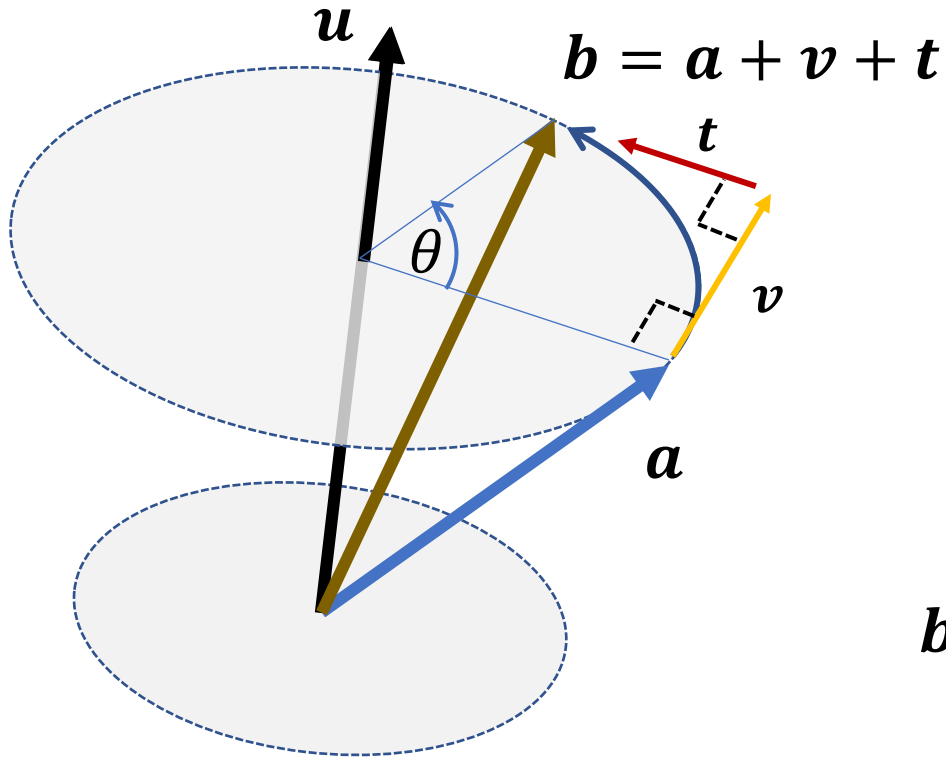
$$\|u\| = 1$$

$$v = (\sin \theta) u \times a$$

$$t = (1 - \cos \theta) u \times (u \times a)$$



# How to rotate a vectors?



$$\|u\| = 1$$

$$v = (\sin \theta) u \times a$$

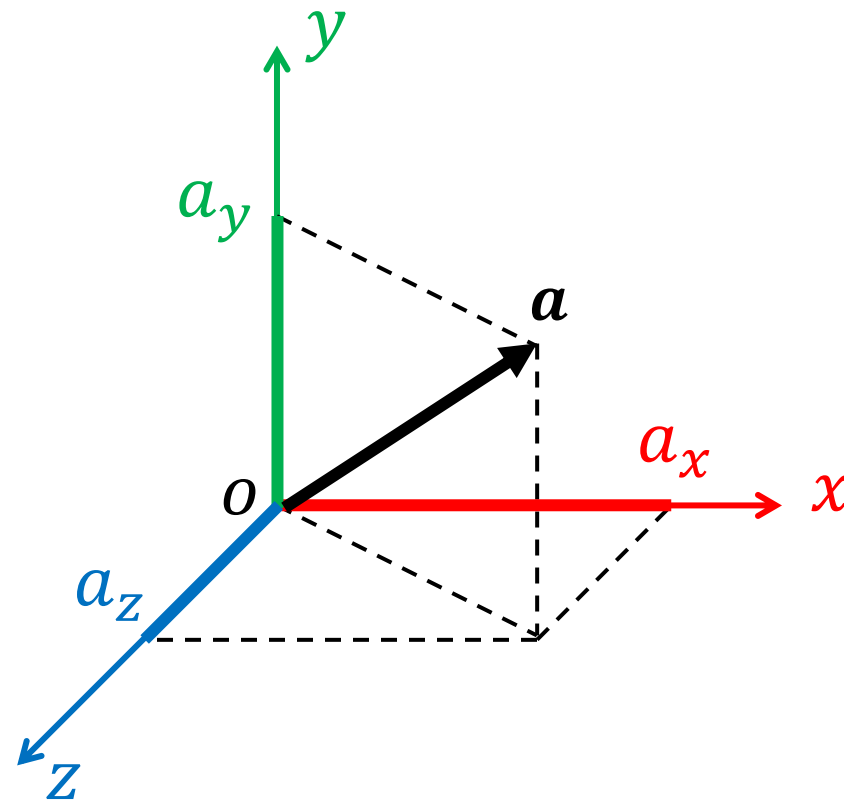
$$t = (1 - \cos \theta) u \times (u \times a)$$

Rodrigues' rotation formula

$$b = a + (\sin \theta) u \times a + (1 - \cos \theta) u \times (u \times a)$$

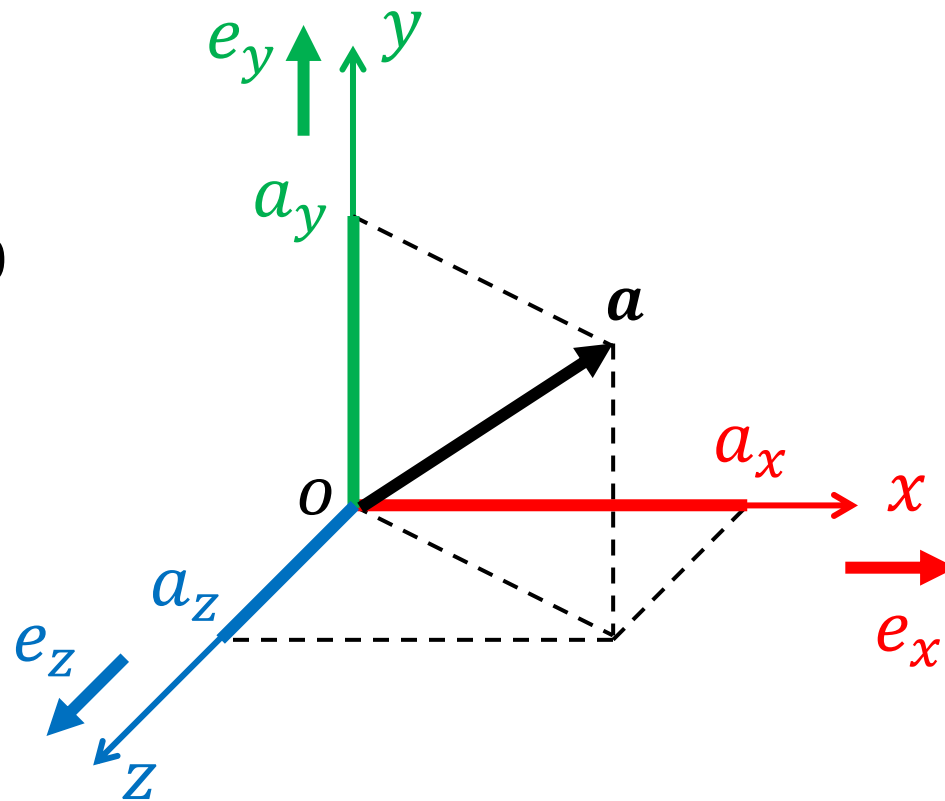
# Orthogonal Basis & Orthogonal Coordinates

$$\mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \in \mathbb{R}^3$$



# Orthogonal Basis & Orthogonal Coordinates

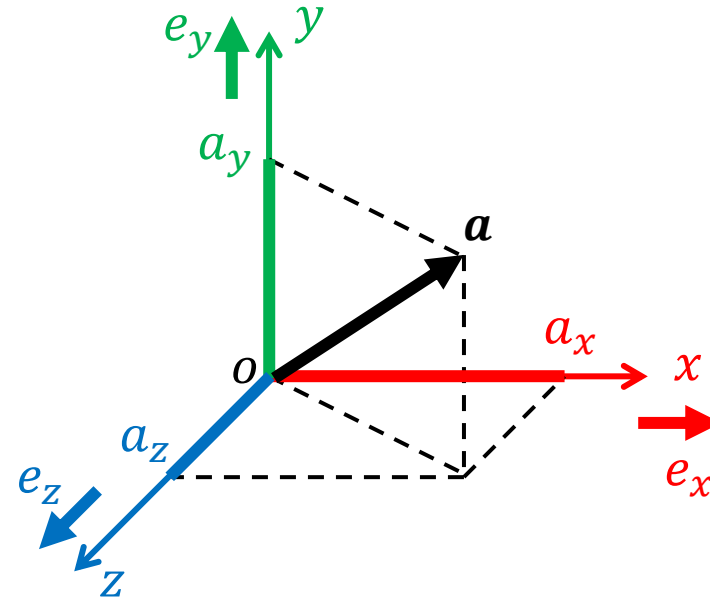
- $\|e_x\| = \|e_y\| = \|e_z\| = 1$
- $e_x \cdot e_y = e_y \cdot e_z = e_z \cdot e_x = 0$
- $e_x \times e_y = e_z$   
 $e_y \times e_z = e_x$   
 $e_z \times e_x = e_y$



# Orthogonal Basis & Orthogonal Coordinates

$$\mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \in \mathbb{R}^3$$

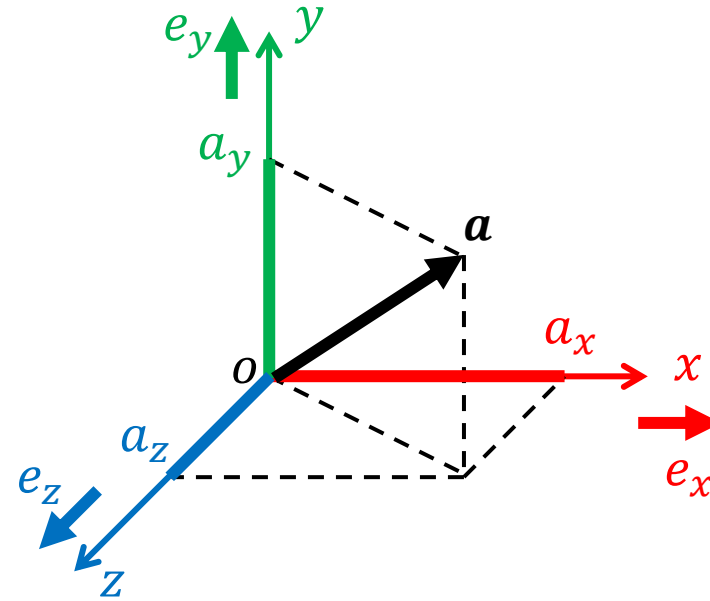
$$\mathbf{a} = a_x \mathbf{e}_x + a_y \mathbf{e}_y + a_z \mathbf{e}_z$$



# Orthogonal Basis & Orthogonal Coordinates

$$\mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \in \mathbb{R}^3$$

$$\mathbf{a} = a_x \mathbf{e}_x + a_y \mathbf{e}_y + a_z \mathbf{e}_z$$



$$\mathbf{a} \cdot \mathbf{b} = (a_x \mathbf{e}_x + a_y \mathbf{e}_y + a_z \mathbf{e}_z) \cdot (b_x \mathbf{e}_x + b_y \mathbf{e}_y + b_z \mathbf{e}_z)$$

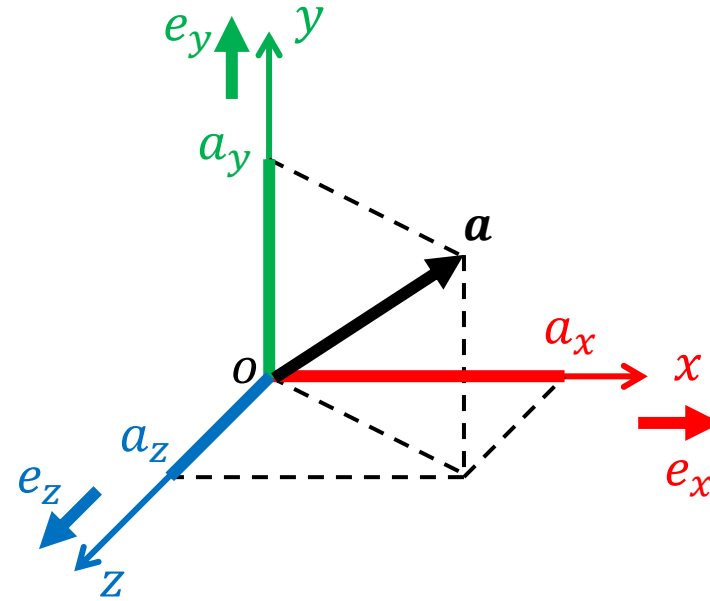
$$= a_x b_x \mathbf{e}_x \cdot \mathbf{e}_x + a_y b_y \mathbf{e}_y \cdot \mathbf{e}_y + a_z b_z \mathbf{e}_z \cdot \mathbf{e}_z$$

$$+ \sum_{i \neq j} a_i b_j \mathbf{e}_i \cdot \mathbf{e}_j$$

# Orthogonal Basis & Orthogonal Coordinates

$$\mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \in \mathbb{R}^3$$

$$\mathbf{a} = a_x \mathbf{e}_x + a_y \mathbf{e}_y + a_z \mathbf{e}_z$$



$$\mathbf{a} \cdot \mathbf{b} = (a_x \mathbf{e}_x + a_y \mathbf{e}_y + a_z \mathbf{e}_z) \cdot (b_x \mathbf{e}_x + b_y \mathbf{e}_y + b_z \mathbf{e}_z)$$

$$= a_x b_x \mathbf{e}_x \cdot \mathbf{e}_x + a_y b_y \mathbf{e}_y \cdot \mathbf{e}_y + a_z b_z \mathbf{e}_z \cdot \mathbf{e}_z$$

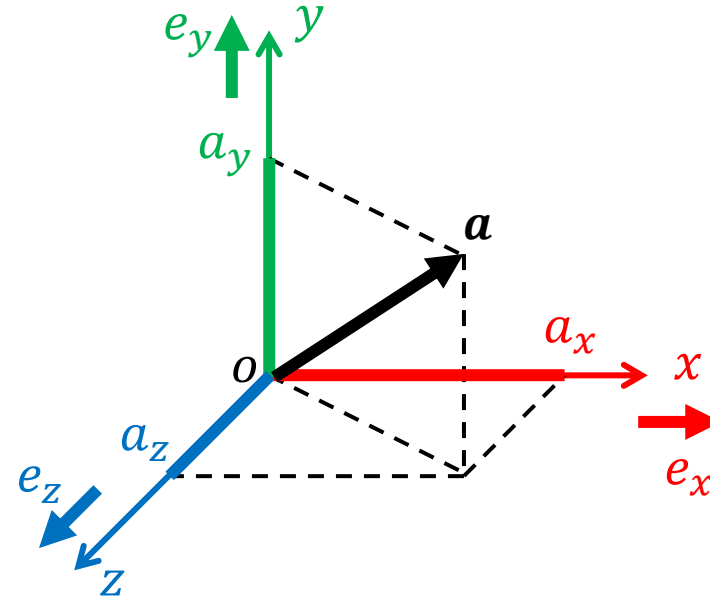
~~$$+ \sum_{i \neq j} a_i b_j \mathbf{e}_i \cdot \mathbf{e}_j$$~~



# Orthogonal Basis & Orthogonal Coordinates

$$\mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \in \mathbb{R}^3$$

$$\mathbf{a} = a_x \mathbf{e}_x + a_y \mathbf{e}_y + a_z \mathbf{e}_z$$

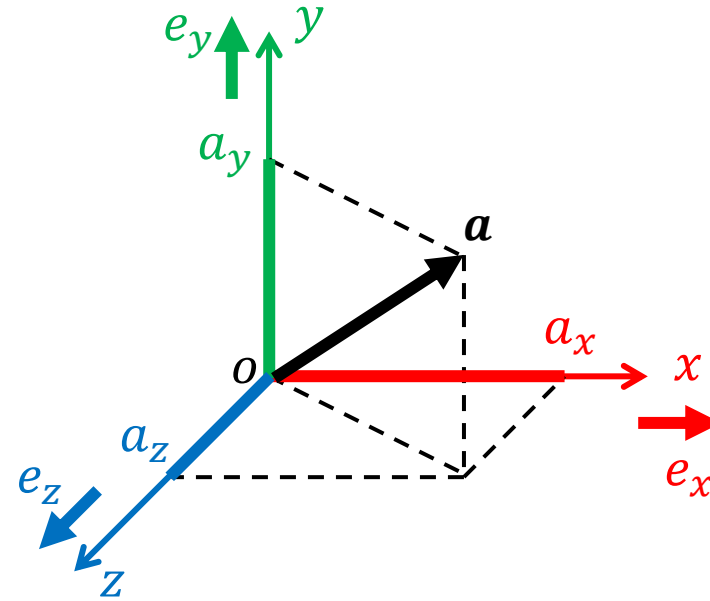


$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= (a_x \mathbf{e}_x + a_y \mathbf{e}_y + a_z \mathbf{e}_z) \times (b_x \mathbf{e}_x + b_y \mathbf{e}_y + b_z \mathbf{e}_z) \\ &= a_x b_x \mathbf{e}_x \times \mathbf{e}_x + a_x b_y \mathbf{e}_x \times \mathbf{e}_y + a_x b_z \mathbf{e}_x \times \mathbf{e}_z \\ &\quad + a_y b_x \mathbf{e}_y \times \mathbf{e}_x + a_y b_y \mathbf{e}_y \times \mathbf{e}_y + a_y b_z \mathbf{e}_y \times \mathbf{e}_z \\ &\quad + a_z b_x \mathbf{e}_z \times \mathbf{e}_x + a_z b_y \mathbf{e}_z \times \mathbf{e}_y + a_z b_z \mathbf{e}_z \times \mathbf{e}_z \end{aligned}$$

# Orthogonal Basis & Orthogonal Coordinates

$$\mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \in \mathbb{R}^3$$

$$\mathbf{a} = a_x \mathbf{e}_x + a_y \mathbf{e}_y + a_z \mathbf{e}_z$$

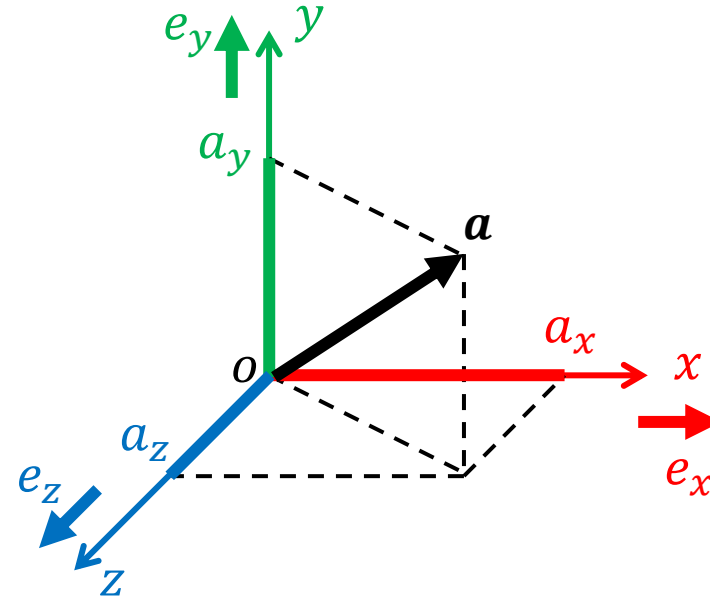


$$\begin{aligned} \mathbf{a} \times \mathbf{b} = & \cancel{a_x b_x \mathbf{e}_x \times \mathbf{e}_x} + a_x b_y \mathbf{e}_x \times \mathbf{e}_y + a_x b_z \mathbf{e}_x \times \mathbf{e}_z \\ & + a_y b_x \mathbf{e}_y \times \mathbf{e}_x + \cancel{a_y b_y \mathbf{e}_y \times \mathbf{e}_y} + a_y b_z \mathbf{e}_y \times \mathbf{e}_z \\ & + a_z b_x \mathbf{e}_z \times \mathbf{e}_x + a_z b_y \mathbf{e}_z \times \mathbf{e}_y + \cancel{a_z b_z \mathbf{e}_z \times \mathbf{e}_z} \end{aligned}$$

# Orthogonal Basis & Orthogonal Coordinates

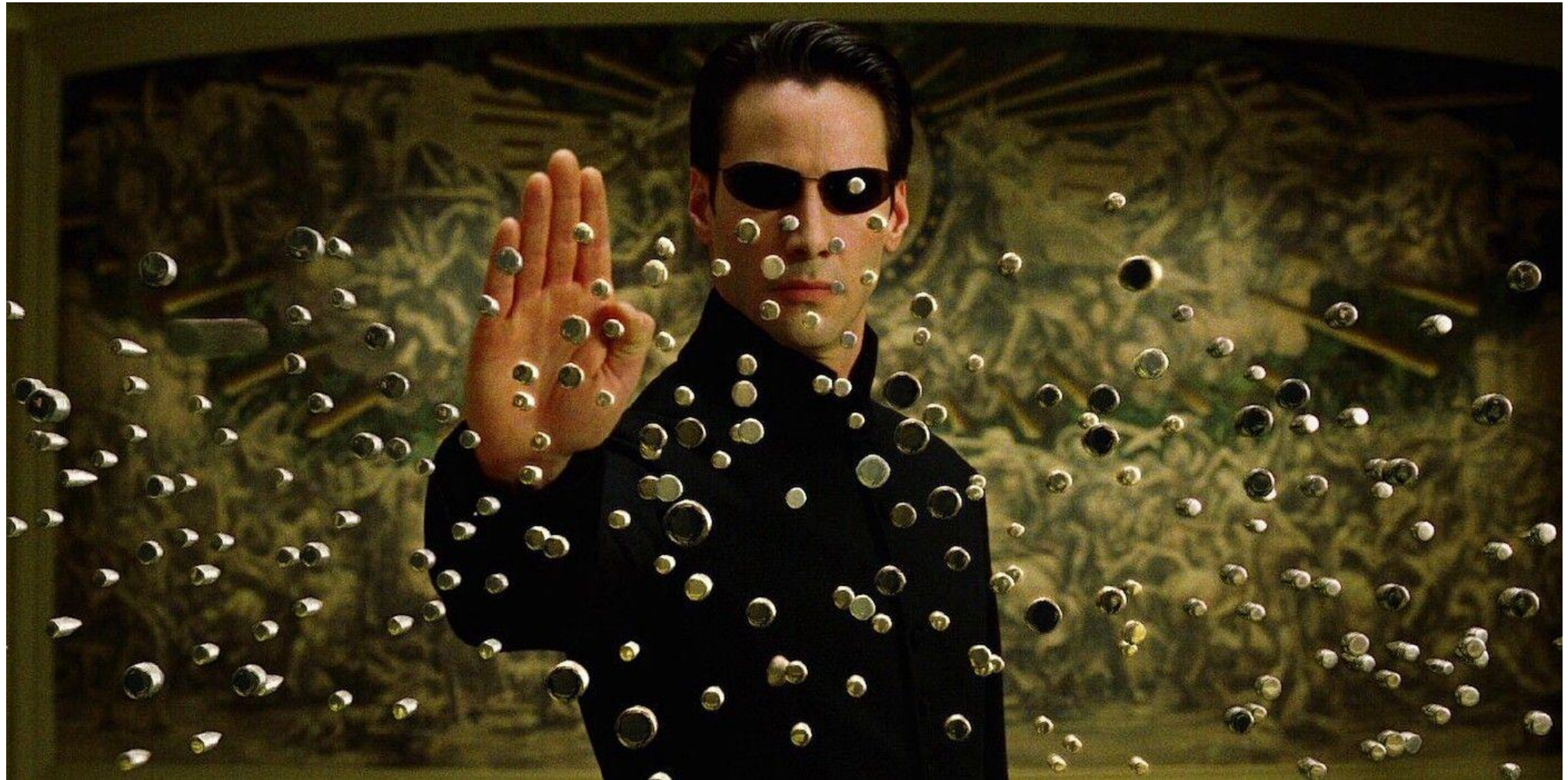
$$\mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \in \mathbb{R}^3$$

$$\mathbf{a} = a_x \mathbf{e}_x + a_y \mathbf{e}_y + a_z \mathbf{e}_z$$



$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= (a_y b_z - a_z b_y) \mathbf{e}_x \\ &\quad + (a_z b_x - a_x b_z) \mathbf{e}_y \\ &\quad + (a_x b_y - a_y b_x) \mathbf{e}_z \end{aligned}$$

# Matrix



*The Matrix, 1999*

# Matrix

- A 2D array of numbers

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in \mathbb{R}^{3 \times 3}$$

$$= [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3] = \begin{bmatrix} \mathbf{a}_{1*} \\ \mathbf{a}_{2*} \\ \mathbf{a}_{3*} \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^{3 \times 1}$$

# Matrix

- A 2D array of numbers

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in \mathbb{R}^{3 \times 3}$$

- Special matrices

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

identity

$$\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$$

diagonal

$$\begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}$$

symmetric

$$\begin{bmatrix} 0 & -a & b \\ a & 0 & -c \\ -b & c & 0 \end{bmatrix}$$

skew-symmetric

# Matrix Operation

- Transpose of a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in \mathbb{R}^{3 \times 3}$$

$$= [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3] = \begin{bmatrix} \mathbf{a}_{1*} \\ \mathbf{a}_{2*} \\ \mathbf{a}_{3*} \end{bmatrix}$$

Transpose

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \mathbf{a}_3^T \end{bmatrix} = [\mathbf{a}_{1*}^T \quad \mathbf{a}_{2*}^T \quad \mathbf{a}_{3*}^T]$$

# Matrix Operation

- Transpose of a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in \mathbb{R}^{3 \times 3}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

identity

$$A^T = A$$

$$\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$$

diagonal

$$A^T = A$$

$$\begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}$$

symmetric

$$A^T = A$$

$$\begin{bmatrix} 0 & -a & b \\ a & 0 & -c \\ -b & c & 0 \end{bmatrix}$$

skew-  
symmetric

$$A^T = -A$$



# Matrix Operation

- Scalar multiplication and matrix addition

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in \mathbb{R}^{3 \times 3}$$

$$sA = \begin{bmatrix} sa_{11} & sa_{12} & sa_{13} \\ sa_{21} & sa_{22} & sa_{23} \\ sa_{31} & sa_{32} & sa_{33} \end{bmatrix}$$

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \\ a_{31} + b_{31} & a_{32} + b_{32} & a_{33} + b_{33} \end{bmatrix}$$

# Matrix Operation

- Matrix multiplication

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in \mathbb{R}^{3 \times 3}$$

$$C = AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} * & ? & * \\ * & * & * \\ * & * & * \end{bmatrix} = [c_{ij} = \mathbf{a}_{i*} \cdot \mathbf{b}_j]$$

# Matrix Operation

- Matrix multiplication

$$AB \neq BA$$

$$ABC = (AB)C = A(BC)$$

$$A(B + C) = AB + AC$$

$$(AB)^T = B^T A^T \quad IA = A$$

- Inverse of a matrix

$$M = A^{-1} \Leftrightarrow AM = MA = I$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

# Matrix Form of Dot Product

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z$$

$$= \mathbf{a}^T \mathbf{b} = [a_x \quad a_y \quad a_z] \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

$$= \mathbf{b}^T \mathbf{a}$$

# Matrix Form of Cross Product

$$\begin{aligned}\mathbf{c} = \mathbf{a} \times \mathbf{b} &= \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix} \\ &= \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = [\mathbf{a}]_{\times} \mathbf{b}\end{aligned}$$

$$[\mathbf{a}]_{\times} + [\mathbf{a}]_{\times}^T = \mathbf{0} \quad \text{skew-symmetric}$$

# Matrix Form of Cross Product

$$\mathbf{a} \times \mathbf{b} = [\mathbf{a}]_{\times} \mathbf{b}$$

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= [\mathbf{a}]_{\times} ([\mathbf{b}]_{\times} \mathbf{c}) \\ &= [\mathbf{a}]_{\times} [\mathbf{b}]_{\times} \mathbf{c} \end{aligned}$$

$$\mathbf{a} \times (\mathbf{a} \times \mathbf{c}) = [\mathbf{a}]_{\times}^2 \mathbf{c}$$

$$[\mathbf{a}]_{\times} + [\mathbf{a}]_{\times}^T = \mathbf{0} \quad \begin{array}{l} \text{skew-} \\ \text{symmetric} \end{array}$$

# Matrix Form of Cross Product

$$\mathbf{a} \times \mathbf{b} = [\mathbf{a}]_{\times} \mathbf{b}$$

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= [\mathbf{a}]_{\times} ([\mathbf{b}]_{\times} \mathbf{c}) \\ &= [\mathbf{a}]_{\times} [\mathbf{b}]_{\times} \mathbf{c} \end{aligned}$$

$$\mathbf{a} \times (\mathbf{a} \times \mathbf{c}) = [\mathbf{a}]_{\times}^2 \mathbf{c}$$

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = [\mathbf{a}]_{\times} [\mathbf{b}]_{\times} \mathbf{c} \quad ???$$

$$[\mathbf{a}]_{\times} + [\mathbf{a}]_{\times}^T = \mathbf{0} \quad \begin{array}{l} \text{skew-} \\ \text{symmetric} \end{array}$$

# Matrix Form of Cross Product

$$\mathbf{a} \times \mathbf{b} = [\mathbf{a}]_{\times} \mathbf{b}$$

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= [\mathbf{a}]_{\times} ([\mathbf{b}]_{\times} \mathbf{c}) \\ &= [\mathbf{a}]_{\times} [\mathbf{b}]_{\times} \mathbf{c} \end{aligned}$$

$$\mathbf{a} \times (\mathbf{a} \times \mathbf{c}) = [\mathbf{a}]_{\times}^2 \mathbf{c}$$

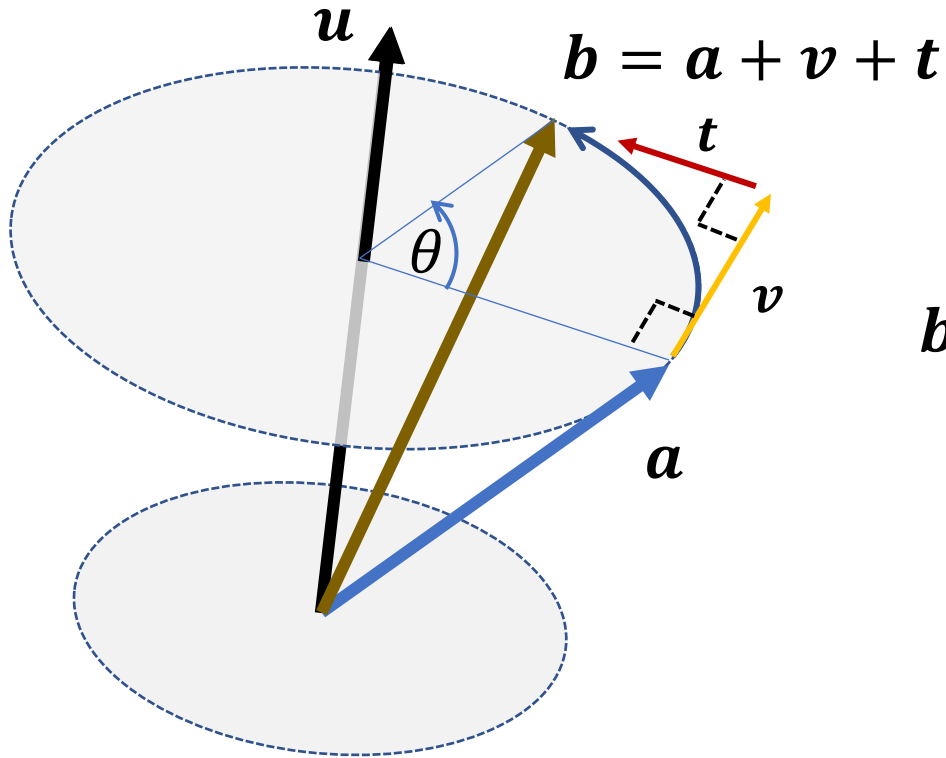
$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq [\mathbf{a}]_{\times} [\mathbf{b}]_{\times} \mathbf{c} \quad ???$$

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = [\mathbf{a} \times \mathbf{b}]_{\times} \mathbf{c}$$

$$[\mathbf{a}]_{\times} + [\mathbf{a}]_{\times}^T = \mathbf{0} \quad \text{skew-symmetric}$$



# How to rotate a vectors?



$$\|u\| = 1$$

$$v = (\sin \theta) u \times a$$

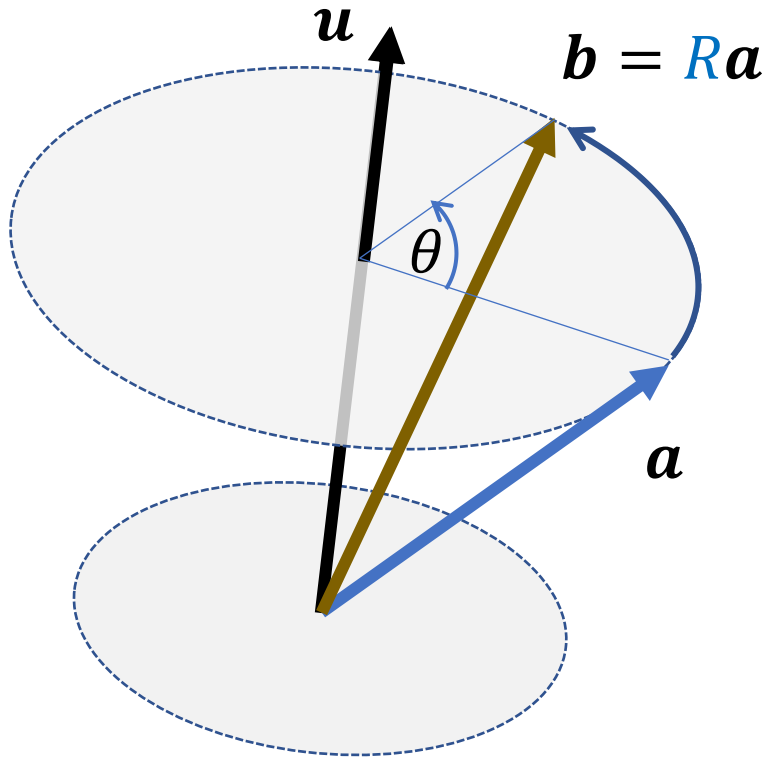
$$t = (1 - \cos \theta) u \times (u \times a)$$

$$b = a + (\sin \theta) u \times a + (1 - \cos \theta) u \times (u \times a)$$

$$b = (I + (\sin \theta) [u]_{\times} + (1 - \cos \theta) [u]_{\times}^2) a$$

$$= Ra$$

# How to rotate a vectors?



$$\|u\| = 1$$

Rodrigues' rotation formula

$$R = I + (\sin \theta) [u]_{\times} + (1 - \cos \theta) [u]_{\times}^2$$

# Orthogonal Matrix

- A matrix whose columns (& rows) are orthogonal vectors

$$A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3] \quad \mathbf{a}_i^T \mathbf{a}_j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

$$A^T A = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \mathbf{a}_3^T \end{bmatrix} [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3] = \begin{bmatrix} \mathbf{a}_1^T \mathbf{a}_1 & \mathbf{a}_1^T \mathbf{a}_2 & \mathbf{a}_1^T \mathbf{a}_3 \\ \mathbf{a}_2^T \mathbf{a}_1 & \mathbf{a}_2^T \mathbf{a}_2 & \mathbf{a}_2^T \mathbf{a}_3 \\ \mathbf{a}_3^T \mathbf{a}_1 & \mathbf{a}_3^T \mathbf{a}_2 & \mathbf{a}_3^T \mathbf{a}_3 \end{bmatrix} = \mathbf{I}$$

$$A^T = A^{-1}$$

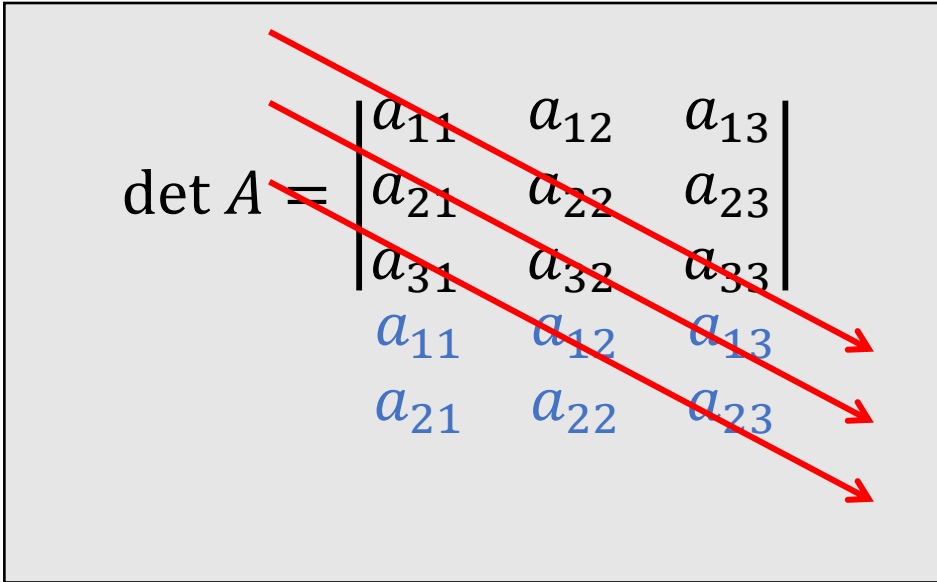
# Determinant of a Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in \mathbb{R}^{3 \times 3}$$

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

# Determinant of a Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in \mathbb{R}^{3 \times 3}$$



$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$a_{11} \quad a_{12} \quad a_{13}$

$a_{21} \quad a_{22} \quad a_{23}$

# Determinant of a Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in \mathbb{R}^{3 \times 3}$$

The diagram illustrates the calculation of the determinant of a 3x3 matrix  $A$  using the rule of Sarrus. It shows the matrix  $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$  with blue arrows indicating the positive terms (downward diagonal) and red arrows indicating the negative terms (upward diagonal).

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Blue arrows (positive terms):

- $a_{11}a_{22}a_{33}$
- $a_{21}a_{32}a_{13}$
- $a_{31}a_{12}a_{23}$

Red arrows (negative terms):

- $a_{13}a_{22}a_{31}$
- $a_{12}a_{33}a_{21}$
- $a_{11}a_{23}a_{32}$

# Determinant of a Matrix

- $\det I = 1$
- $\det AB = \det A * \det B$
- $\det A^T = \det A$
- If  $A$  is invertible,  $\det A^{-1} = (\det A)^{-1}$
- If  $U$  is orthogonal,  $\det U = \pm 1$

# Cross Product as a Determinant

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}$$

$$= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix}$$



# Eigenvalues and Eigenvectors

For a matrix  $A$ , if a **nonzero** vector  $\mathbf{x}$  satisfies

$$A\mathbf{x} = \lambda\mathbf{x}$$

Then:

$\lambda$ : an eigenvalue of  $A$

$\mathbf{x}$ : an eigenvector of  $A$

# Eigenvalues and Eigenvectors

For a matrix  $A$ , if a **nonzero** vector  $\mathbf{x}$  satisfies

$$A\mathbf{x} = \lambda\mathbf{x}$$

Then:

$\lambda$ : an eigenvalue of  $A$

$\mathbf{x}$ : an eigenvector of  $A$

Especially, a **3 × 3 orthogonal** matrix  $U$

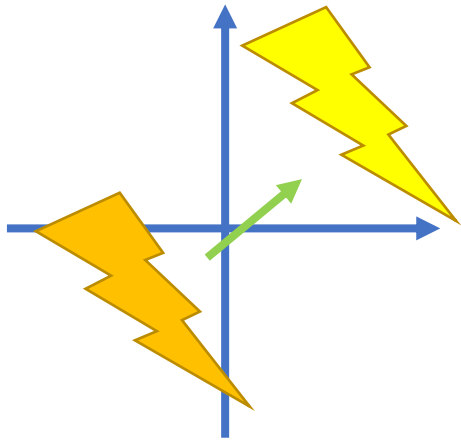
has at least one real eigenvalue:  $\lambda = \det U = \pm 1$



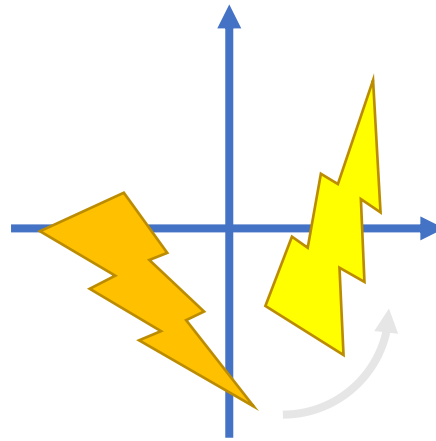
# Rigid Transformation

Translation, rotation, and coordinate transformation

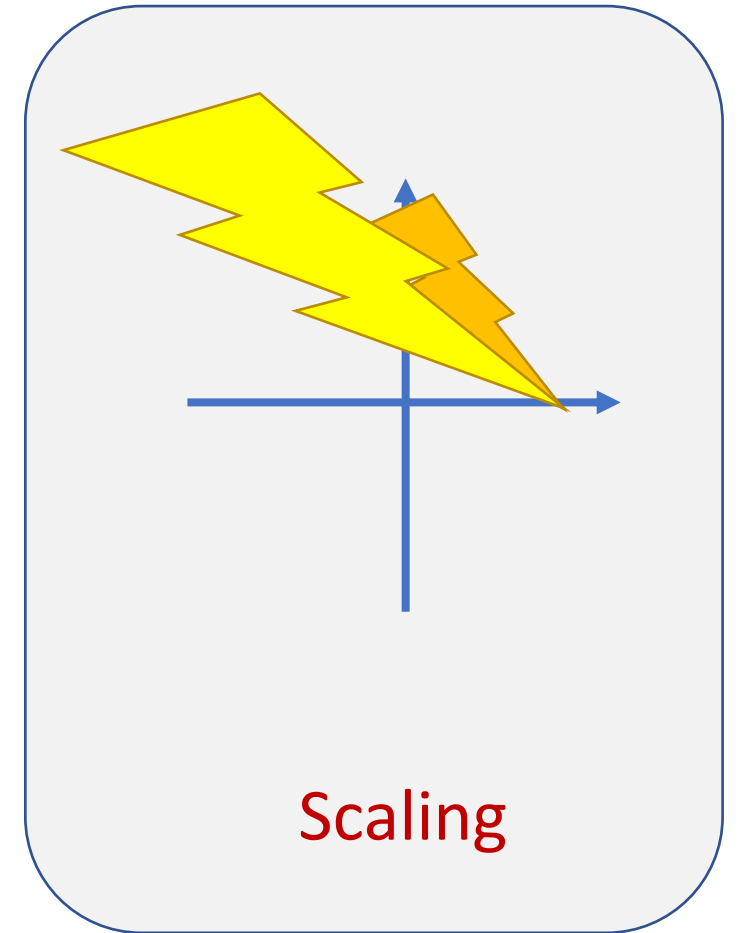
# Rigid Transformation: Translation + Rotation



Translation

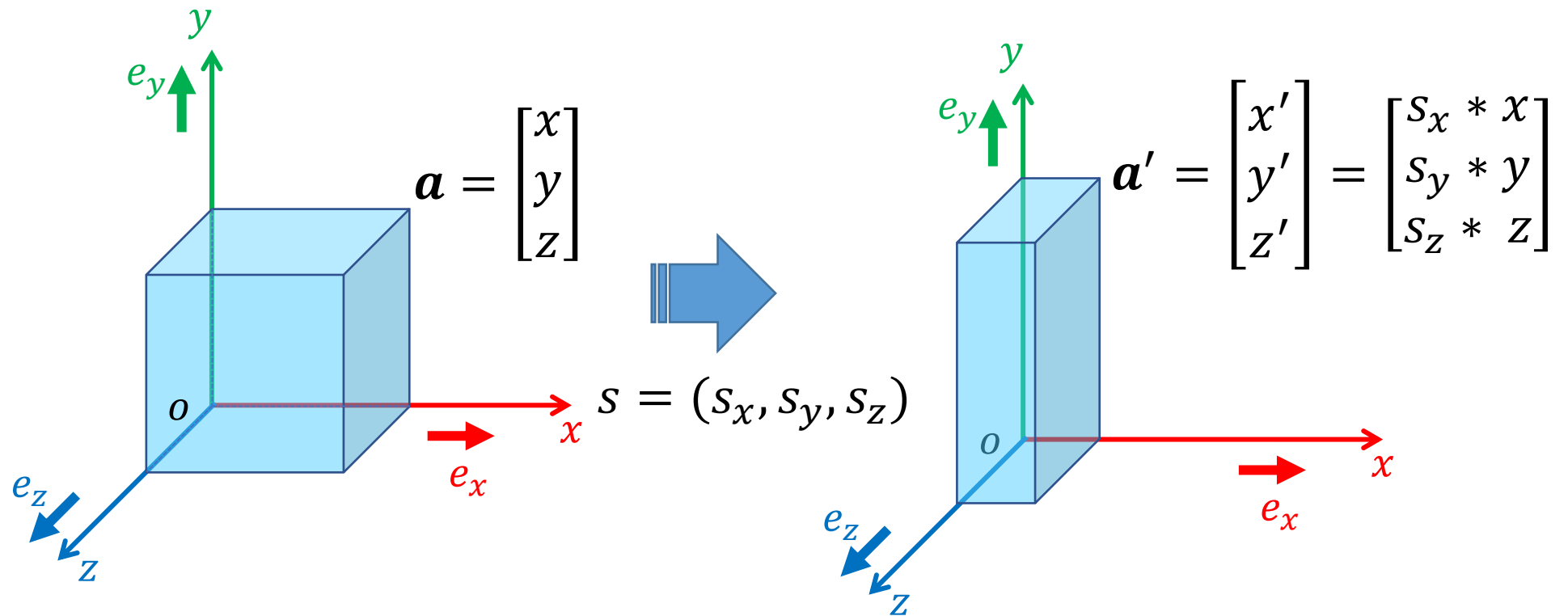


Rotation



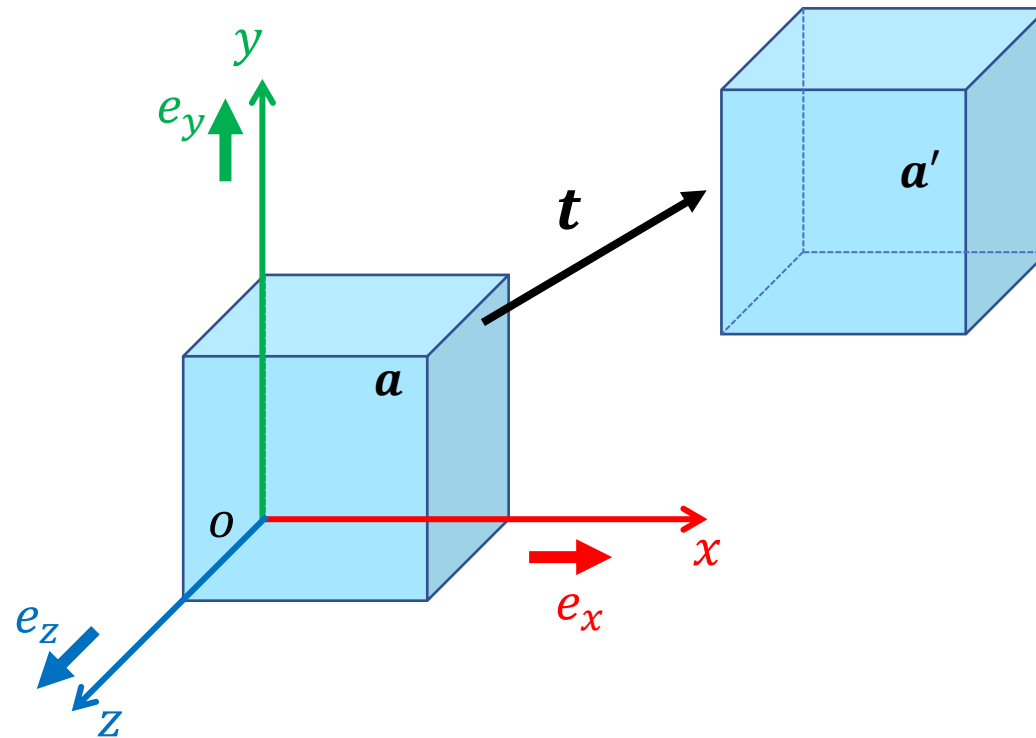
Scaling

# Scaling



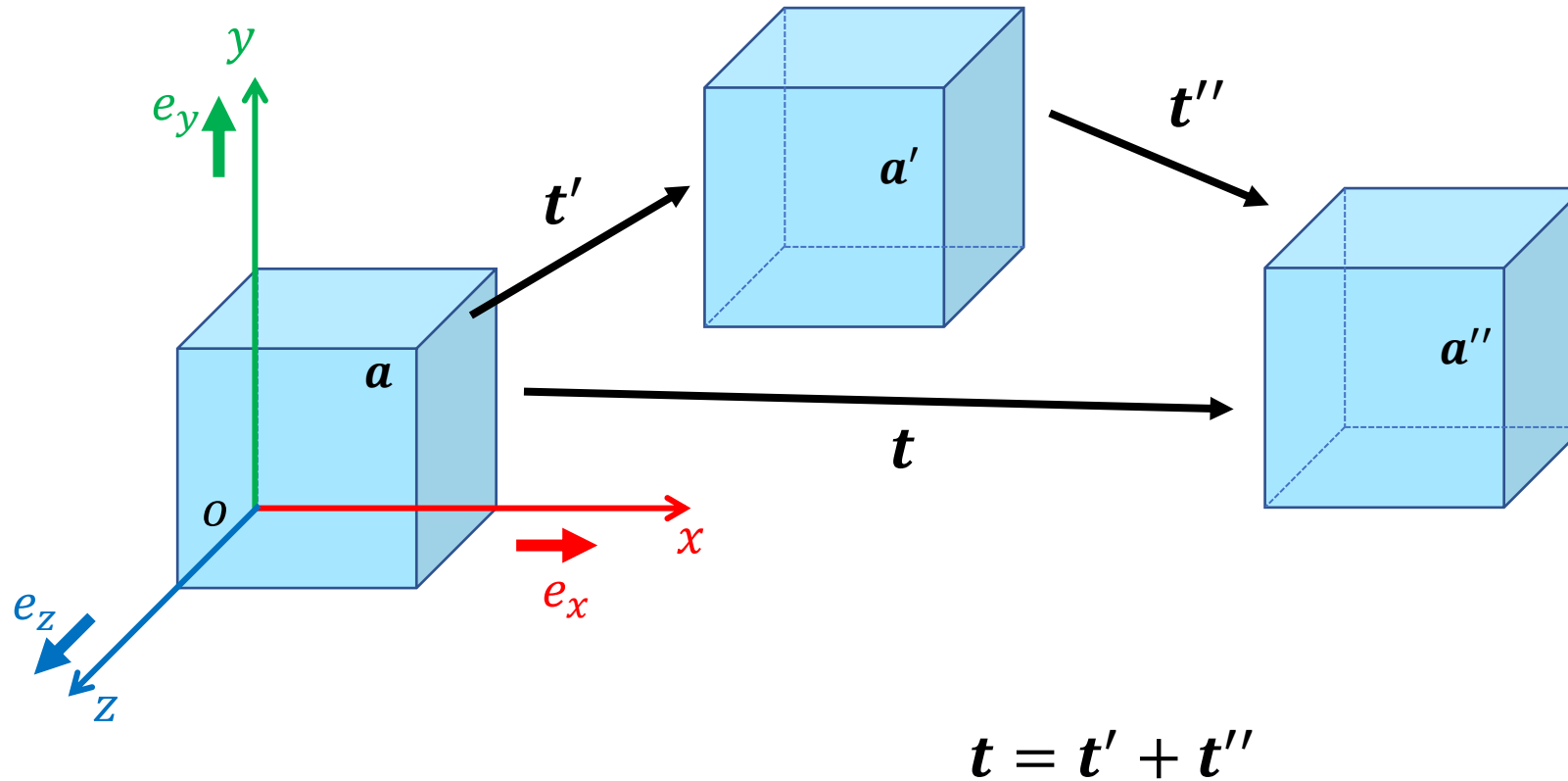
$$a' = \begin{bmatrix} s_x & & \\ & s_y & \\ & & s_z \end{bmatrix} a$$

# Translation

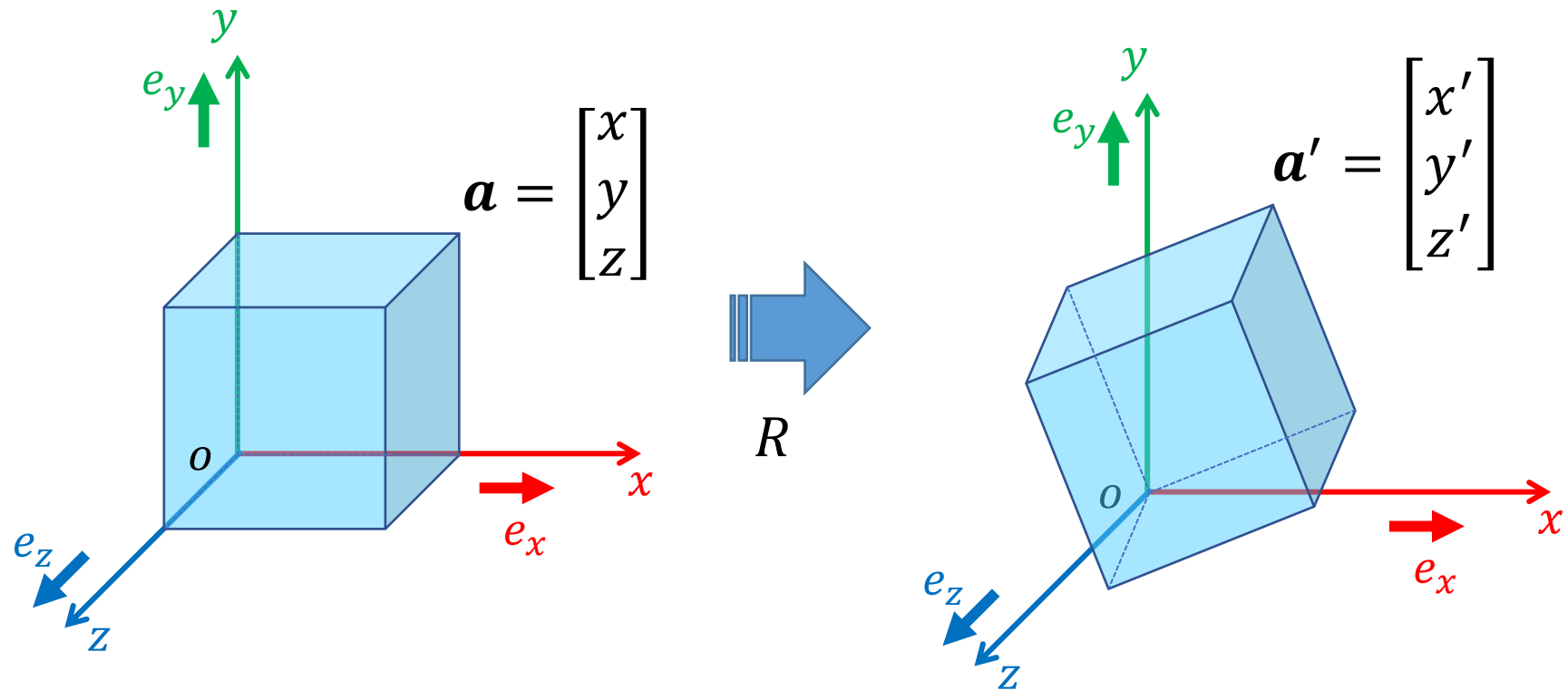


$$a' = a + t$$

# Combination of Translations



# Rotation



$$\mathbf{a}' = R\mathbf{a}$$

$R$ : Rotation Matrix



# Rotation Matrix

- Rotation matrix is orthogonal:

$$R^{-1} = R^T \quad R^T R = R R^T = I$$

- Determinant of  $R$

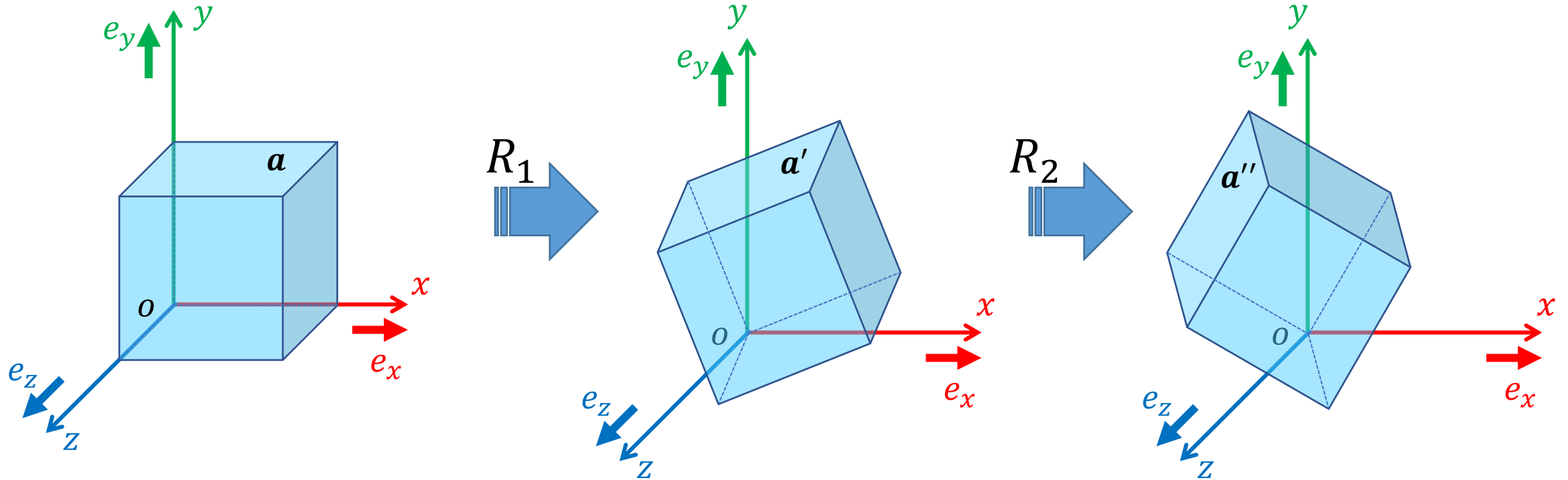
$$\det R = +1$$

- Rotation maintains length of vectors

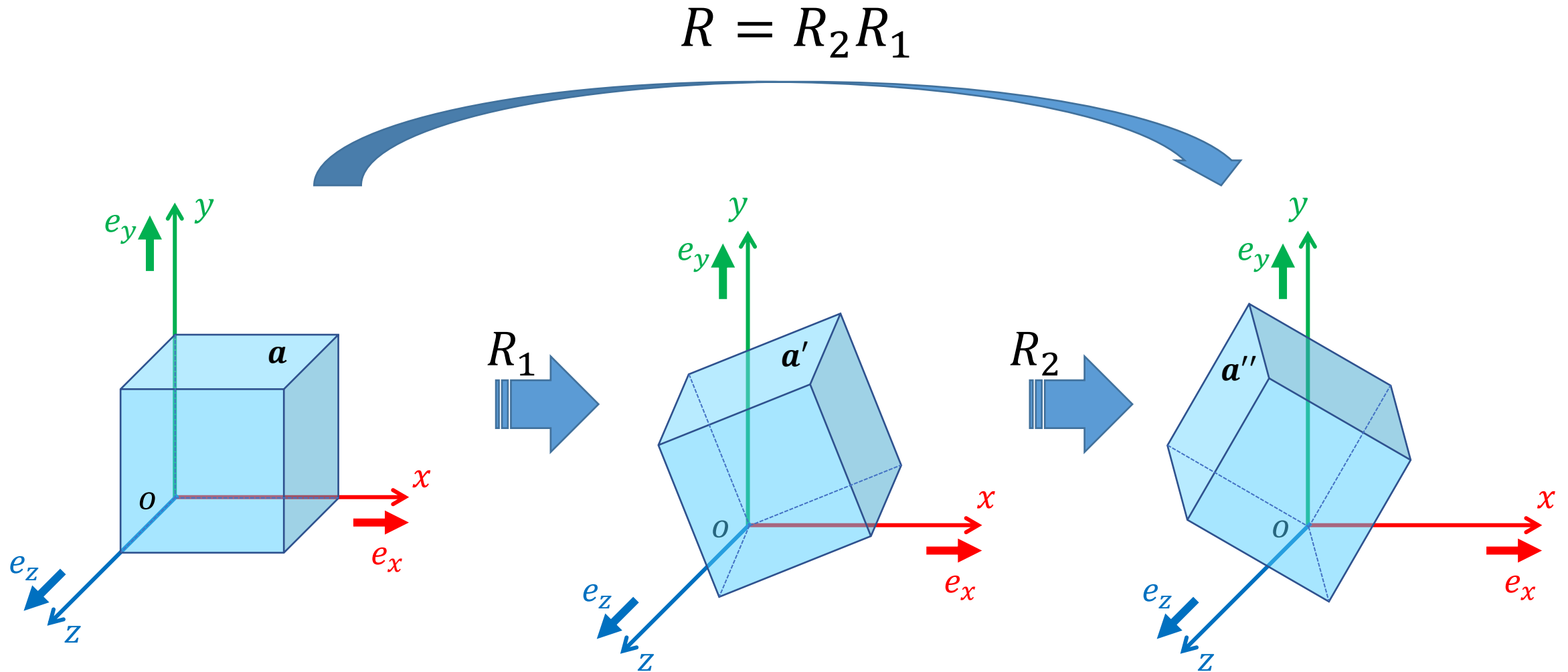
$$\|R\mathbf{x}\| = \|\mathbf{x}\|$$

# Combination of Rotations

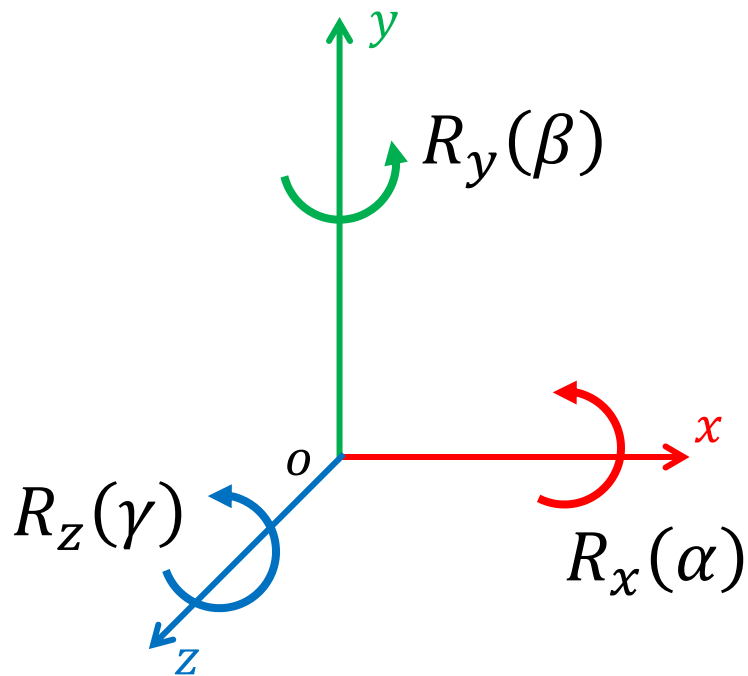
$$R = R_1 R_2 \text{ ???}$$



# Combination of Rotations



# Rotation around Coordinate Axes



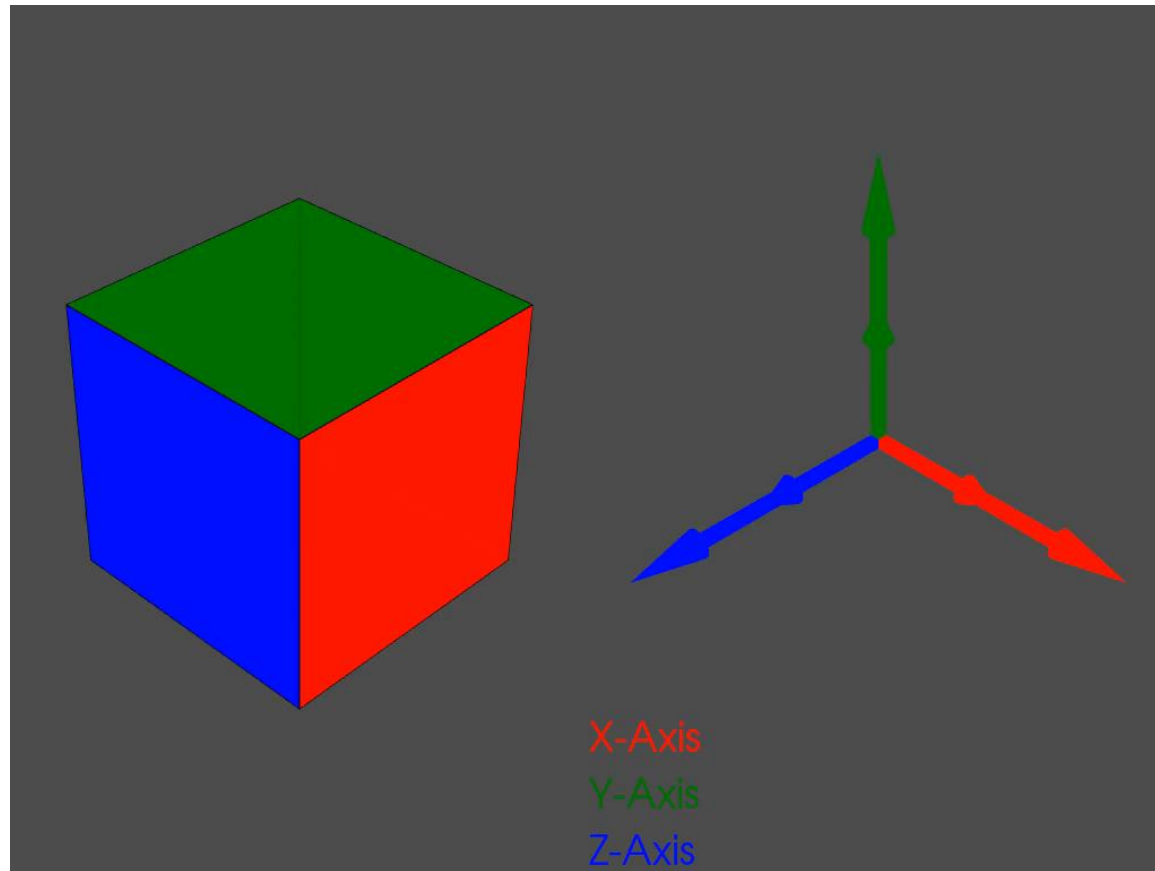
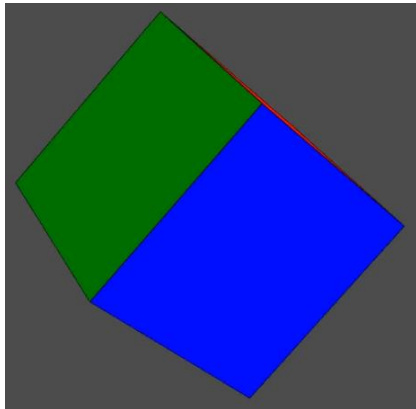
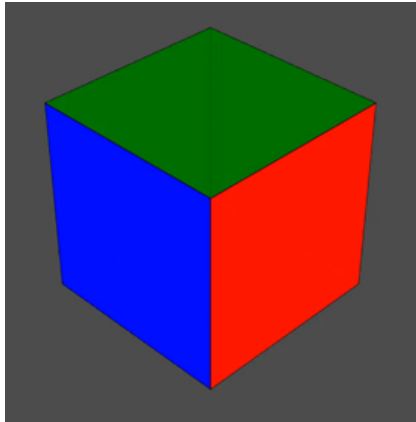
$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{pmatrix}$$

$$R_y(\beta) = \begin{pmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{pmatrix}$$

$$R_z(\gamma) = \begin{pmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

# Rotation around Coordinate Axes

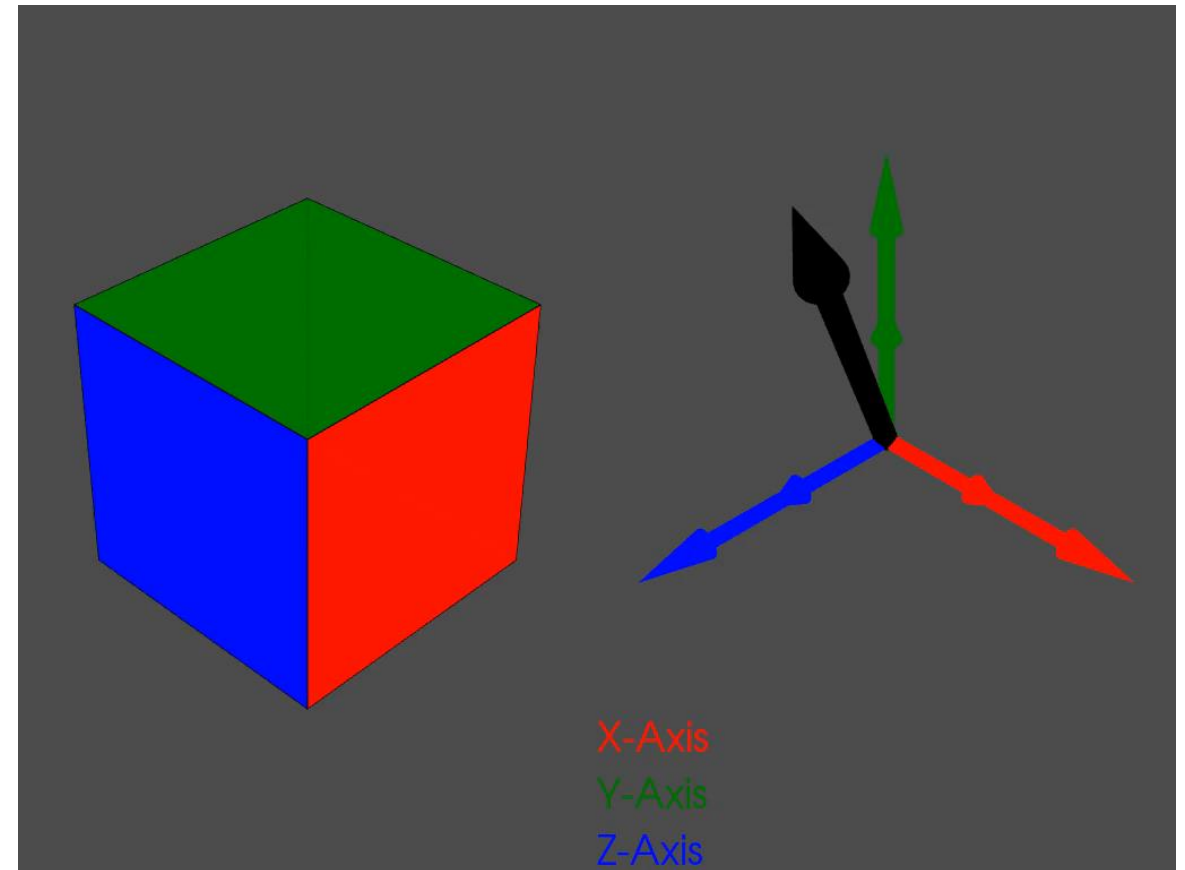
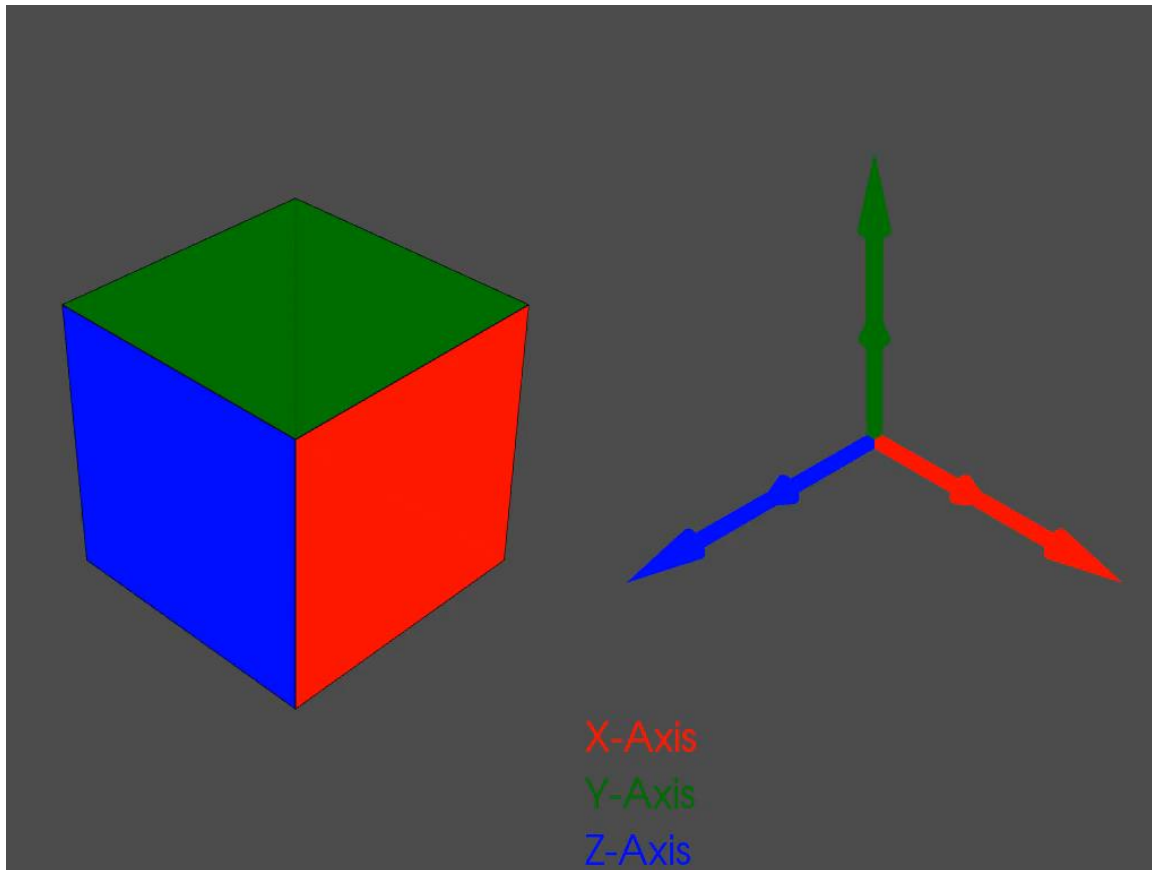
$$R_z(72^\circ)R_y(45^\circ)R_x(60^\circ)$$



# Rotation around Coordinate Axes

$$R_z(72^\circ)R_y(45^\circ)R_x(60^\circ)$$

$$u = (0.28, 0.83, 0.48) \quad \theta = 81.1^\circ$$



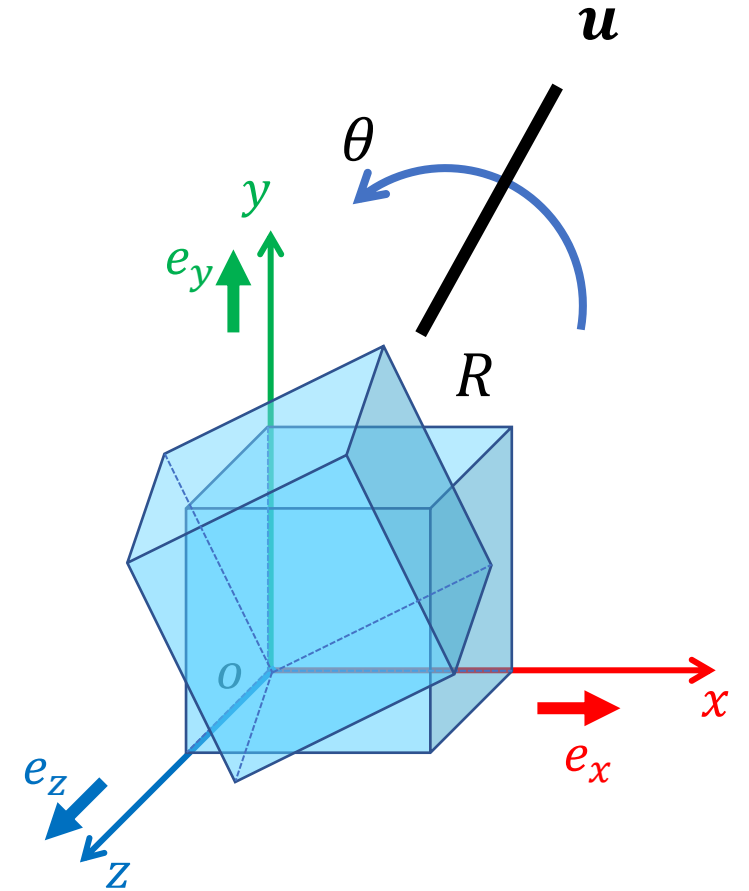
# Rotation Axis and Angle

Rotation matrix  $R$  has a real eigenvalue:  $+1$

$$R\mathbf{u} = \mathbf{u}$$

In other words,  $R$  can be considered as a rotation around **axis  $\mathbf{u}$**  by some **angle  $\theta$**

How to find **axis  $\mathbf{u}$**  and **angle  $\theta$** ?



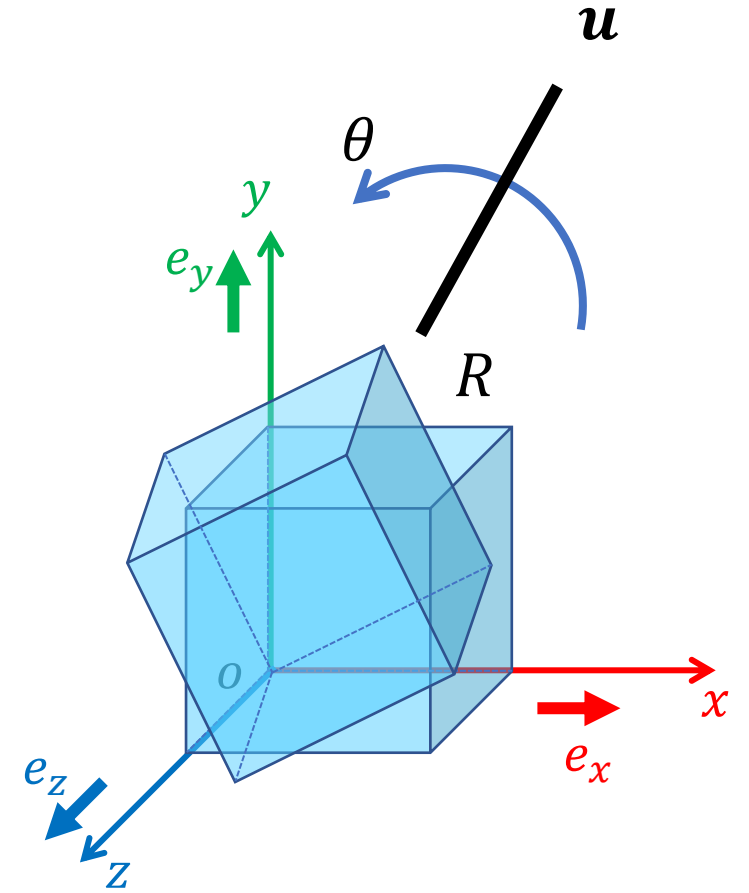
# Rotation Axis and Angle

$$R\mathbf{u} = \mathbf{u} \quad \Rightarrow \quad \mathbf{u} = R^T\mathbf{u}$$

$$(R - R^T)\mathbf{u} = 0$$

$$\begin{bmatrix} 0 & -(r_{21} - r_{12}) & r_{13} - r_{31} \\ r_{21} - r_{12} & 0 & -(r_{32} - r_{23}) \\ -(r_{13} - r_{31}) & r_{32} - r_{23} & 0 \end{bmatrix} \mathbf{u} = 0$$

Skew-symmetric





# Rotation Axis and Angle

$$R\mathbf{u} = \mathbf{u} \quad \Rightarrow \quad \mathbf{u} = R^T\mathbf{u}$$

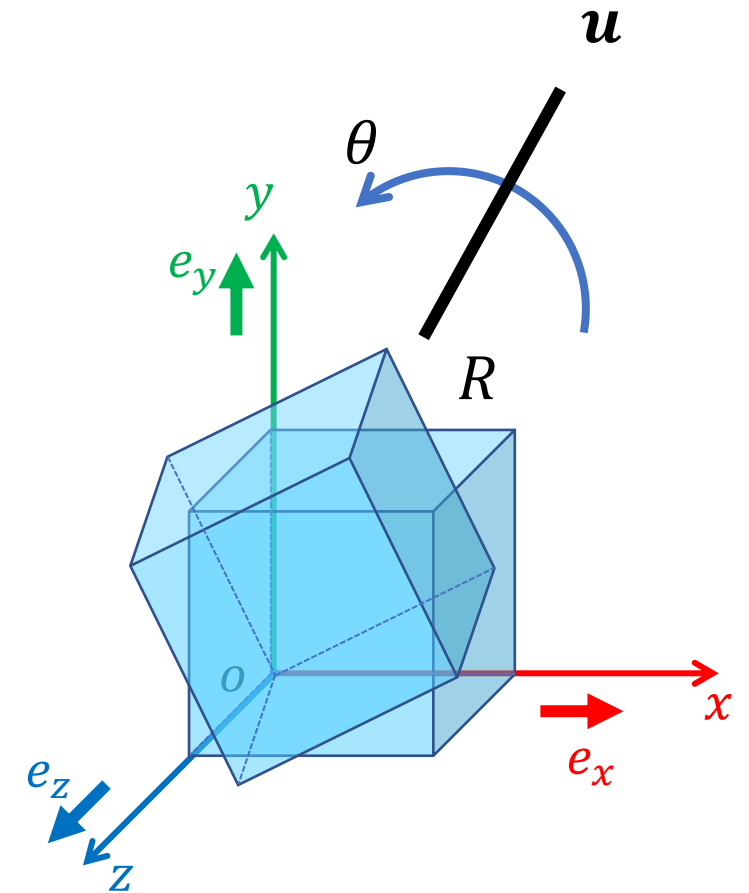
$$(R - R^T)\mathbf{u} = 0$$

$$\begin{bmatrix} 0 & -(r_{21} - r_{12}) & r_{13} - r_{31} \\ r_{21} - r_{12} & \mathbf{u}' \times \mathbf{u} = 0 & (r_{32} - r_{23}) \\ -(r_{13} - r_{31}) & r_{32} - r_{23} & 0 \end{bmatrix} \mathbf{u} = 0$$

Skew-symmetric  
Matrix



Cross Product



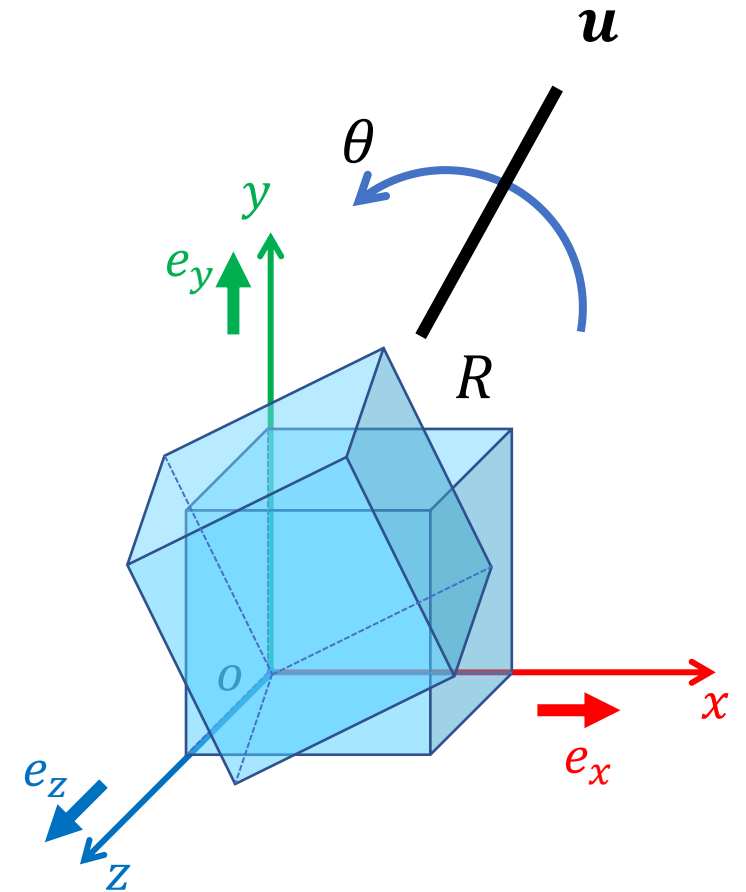
# Rotation Axis and Angle

$$R\mathbf{u} = \mathbf{u} \quad \Rightarrow \quad \mathbf{u} = R^T\mathbf{u}$$

$$(R - R^T)\mathbf{u} = 0$$

$$\mathbf{u} \leftarrow \mathbf{u}' = \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

When  $R \neq R^T \Leftrightarrow \sin \theta \neq 0 \Leftrightarrow \theta \neq 0^\circ \text{ or } 180^\circ$



# Rotation Axis and Angle

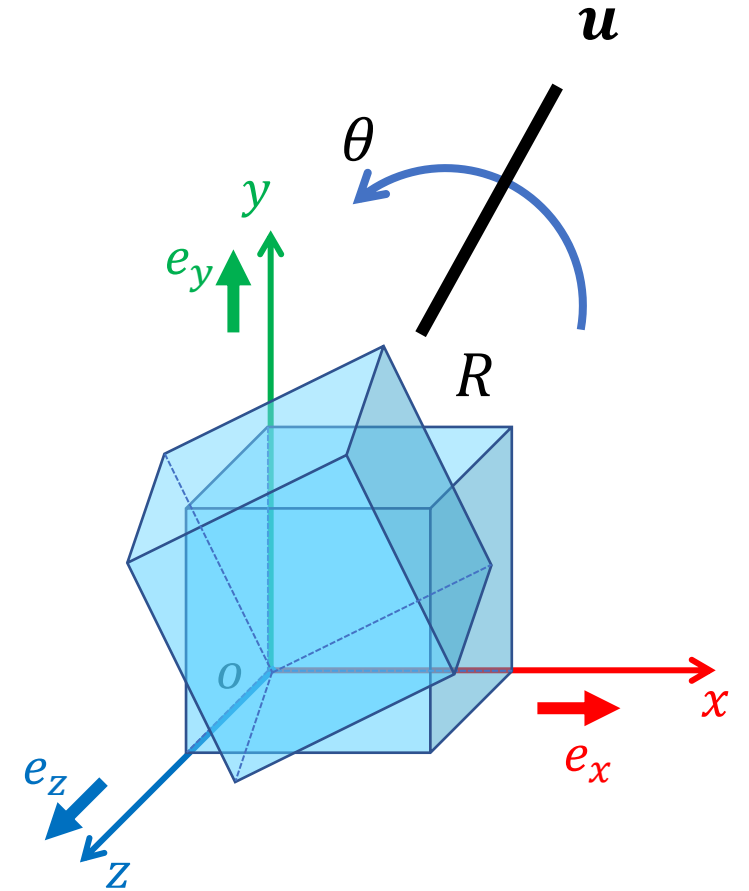
$$R = I + (\sin \theta) [\mathbf{u}]_{\times} + (1 - \cos \theta) [\mathbf{u}]_{\times}^2$$



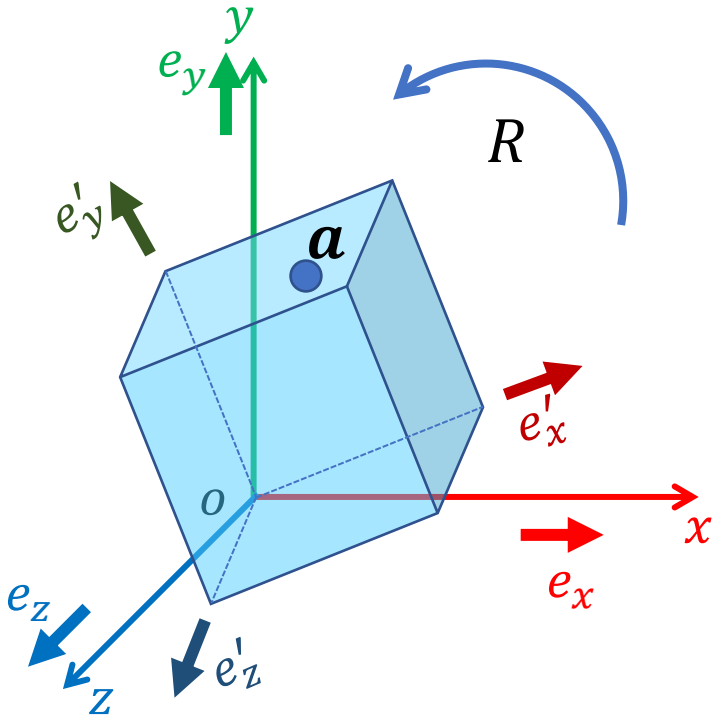
$$\mathbf{u} \leftarrow \mathbf{u}' = \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} \leftarrow R - R^T$$

$$\|\mathbf{u}'\| = 2 \sin \theta$$

When  $R \neq R^T \Leftrightarrow \sin \theta \neq 0 \Leftrightarrow \theta \neq 0^\circ \text{ or } 180^\circ$



# Coordinate Transformation

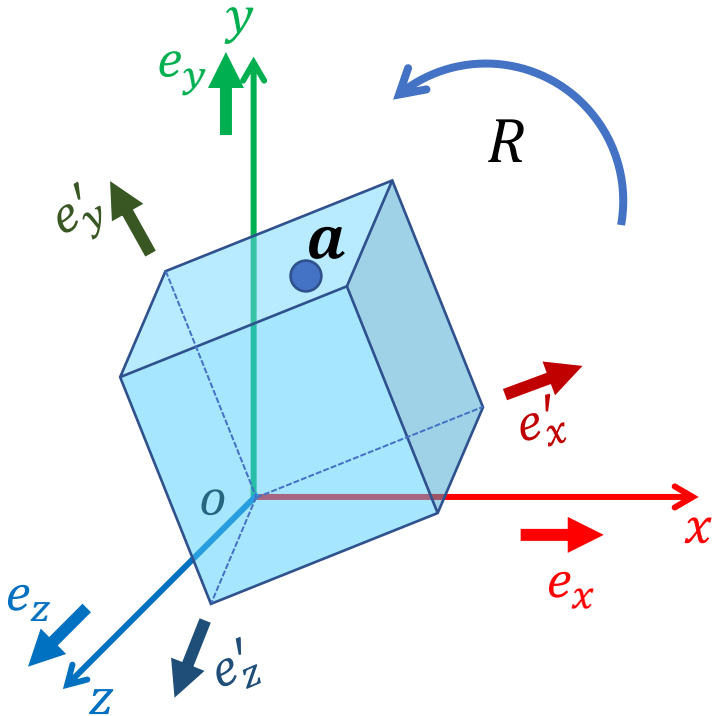


$(x', y', z')^T$ :  $a$  in *object* system

$(x, y, z)^T$ :  $a$  in *global* system

$$\begin{aligned} \mathbf{a} &= \begin{bmatrix} | & | & | \\ \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ | & | & | \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ &= \begin{bmatrix} | & | & | \\ \mathbf{e}'_x & \mathbf{e}'_y & \mathbf{e}'_z \\ | & | & | \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \end{aligned}$$

# Coordinate Transformation



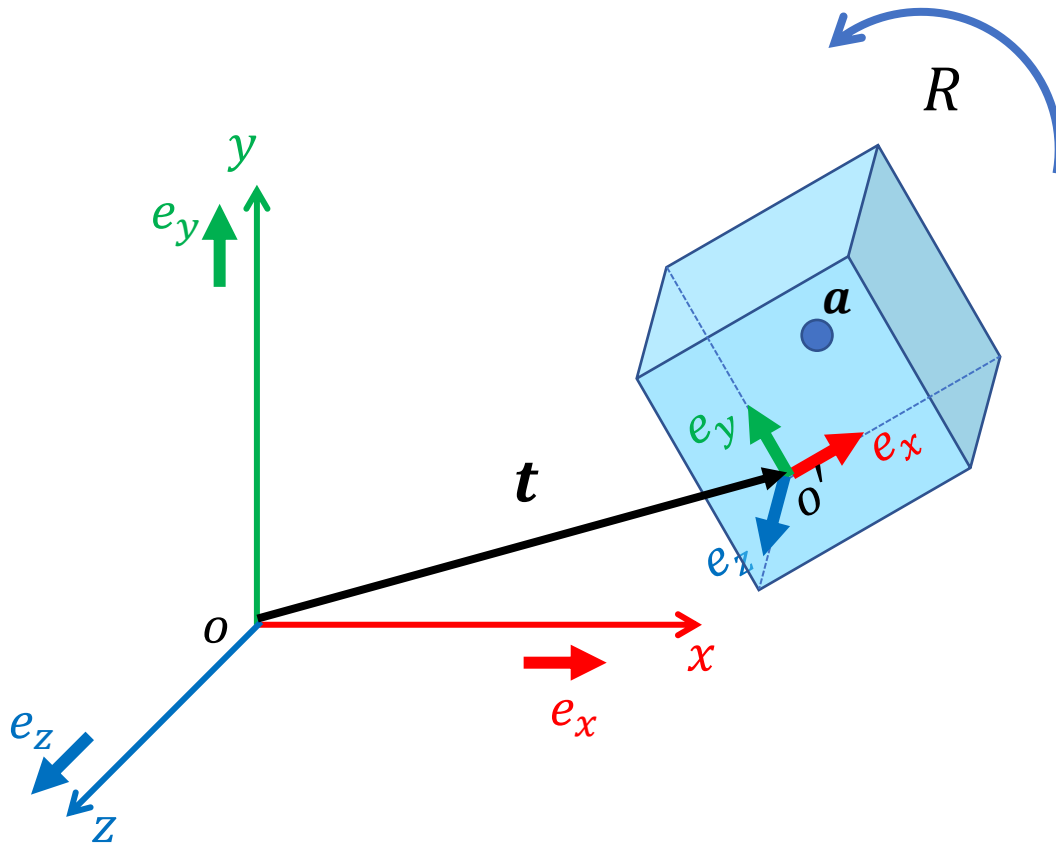
$(x', y', z')^T$ :  $\mathbf{a}$  in *object* system

$(x, y, z)^T$ :  $\mathbf{a}$  in *global* system

$$R = \begin{bmatrix} | & | & | \\ \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ | & | & | \end{bmatrix}^{-1} \begin{bmatrix} | & | & | \\ \mathbf{e}'_x & \mathbf{e}'_y & \mathbf{e}'_z \\ | & | & | \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = R \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

# Coordinate Transformation



$(x', y', z')^T$ :  $a$  in *object* system

$(x, y, z)^T$ :  $a$  in *global* system

*object*  $\rightarrow$  *global*

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = R \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} + t$$

*global*  $\rightarrow$  *object*

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R^T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} - t \right)$$



# Representations of 3D Rotation

回回回回

# Parameterization of Rotation

- A rotation matrix, 9 parameters:  $a_{ij}$

$$R = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$



# Parameterization of Rotation

- A rotation matrix, 9 parameters:  $a_{ij}$

$$R = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$R^T R = I$$

$$\begin{cases} a_{11}^2 + a_{21}^2 + a_{31}^2 = 1 \\ a_{12}^2 + a_{22}^2 + a_{32}^2 = 1 \\ a_{13}^2 + a_{23}^2 + a_{33}^2 = 1 \end{cases} \quad \begin{cases} a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} = 0 \\ a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} = 0 \\ a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} = 0 \end{cases}$$

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degrees of freedom (DoF) = 3

# Parameterization of Rotation

- A rotation matrix, 9 parameters:  $a_{ij}$

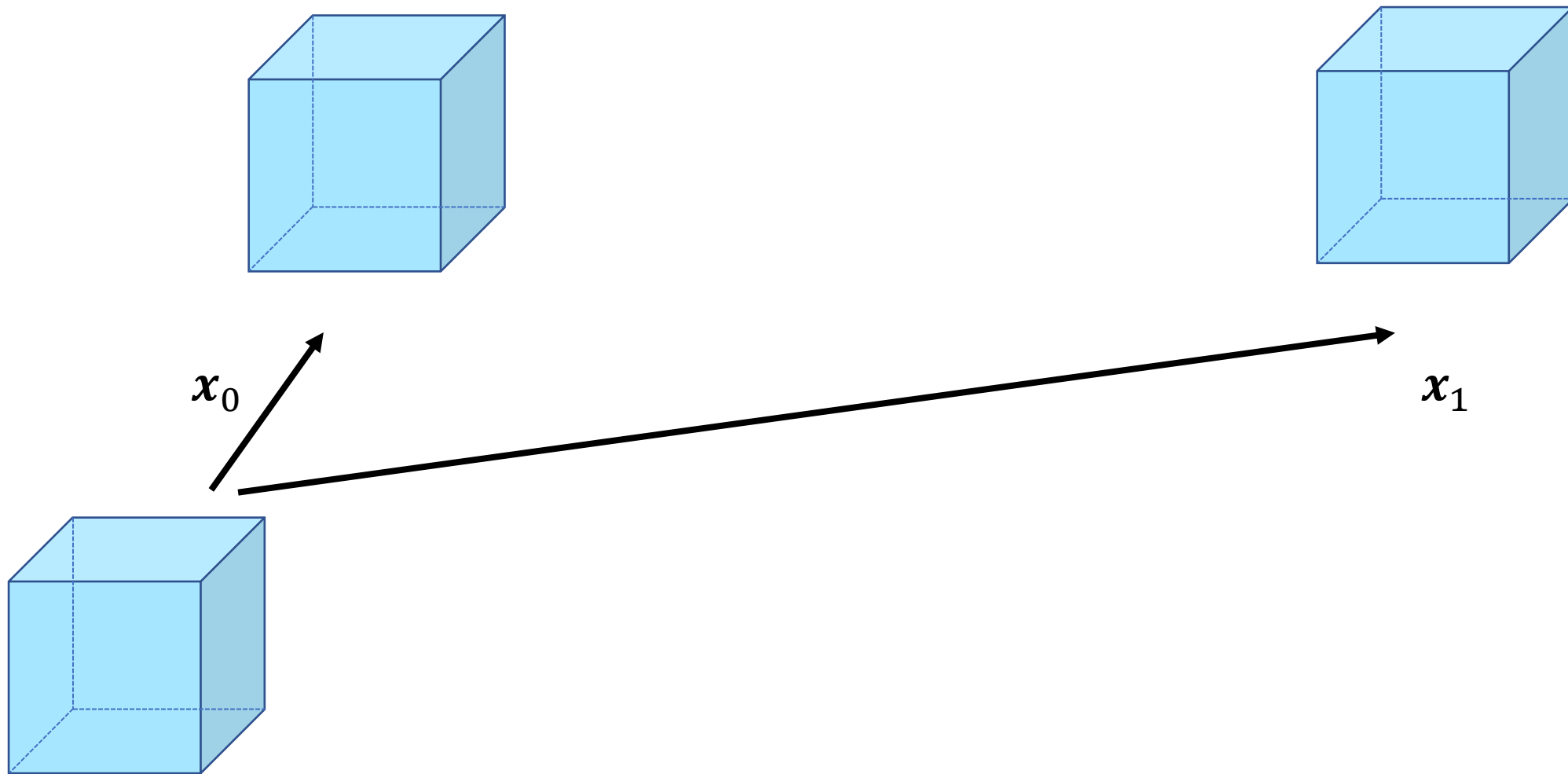
$$R = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$R^T R = I \quad \det R = 1$$

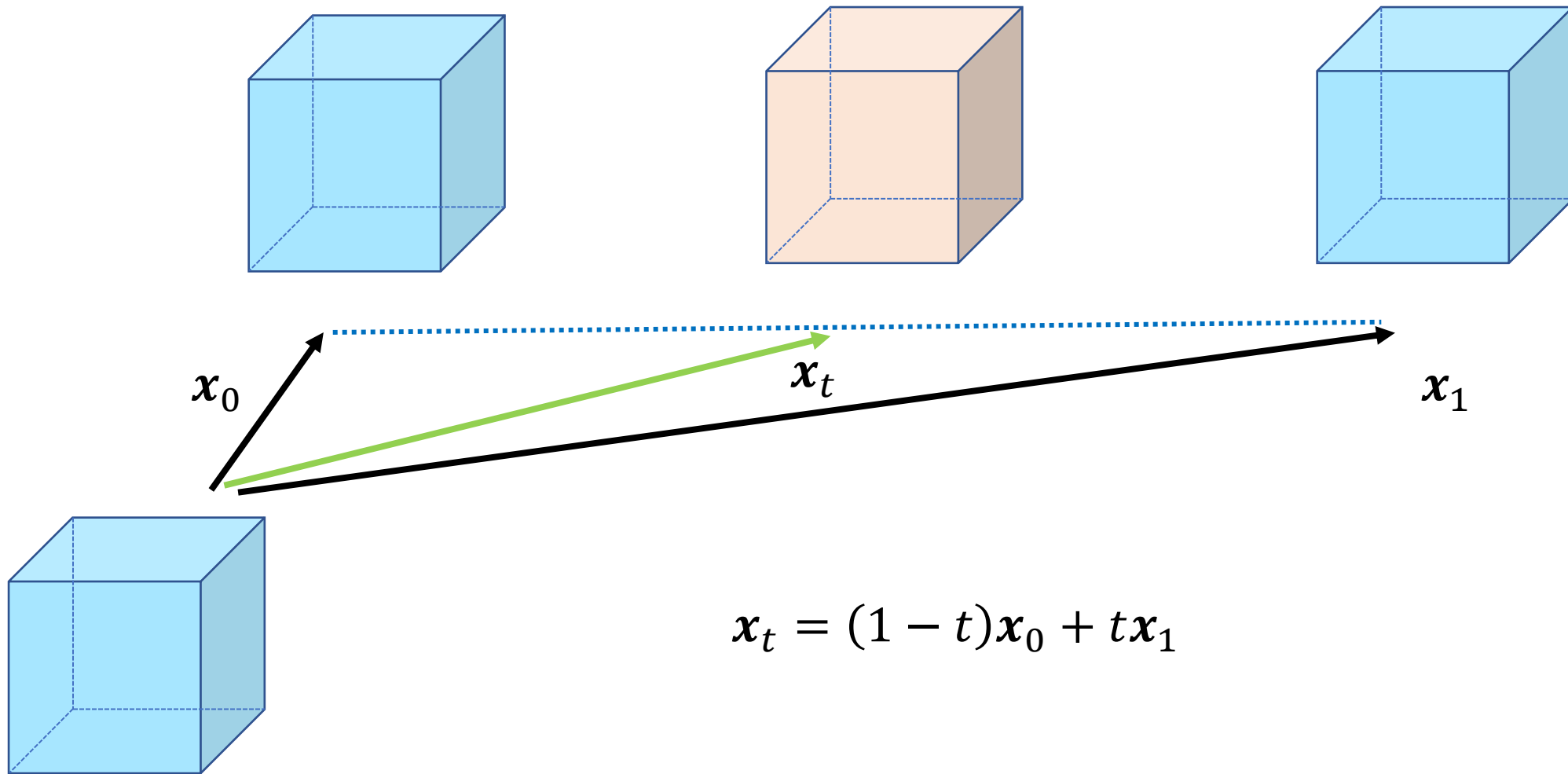
$$\begin{cases} a_{11}^2 + a_{21}^2 + a_{31}^2 = 1 \\ a_{12}^2 + a_{22}^2 + a_{32}^2 = 1 \\ a_{13}^2 + a_{23}^2 + a_{33}^2 = 1 \end{cases} \quad \begin{cases} a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} = 0 \\ a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} = 0 \\ a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} = 0 \end{cases}$$

degrees of freedom (DoF) = 3

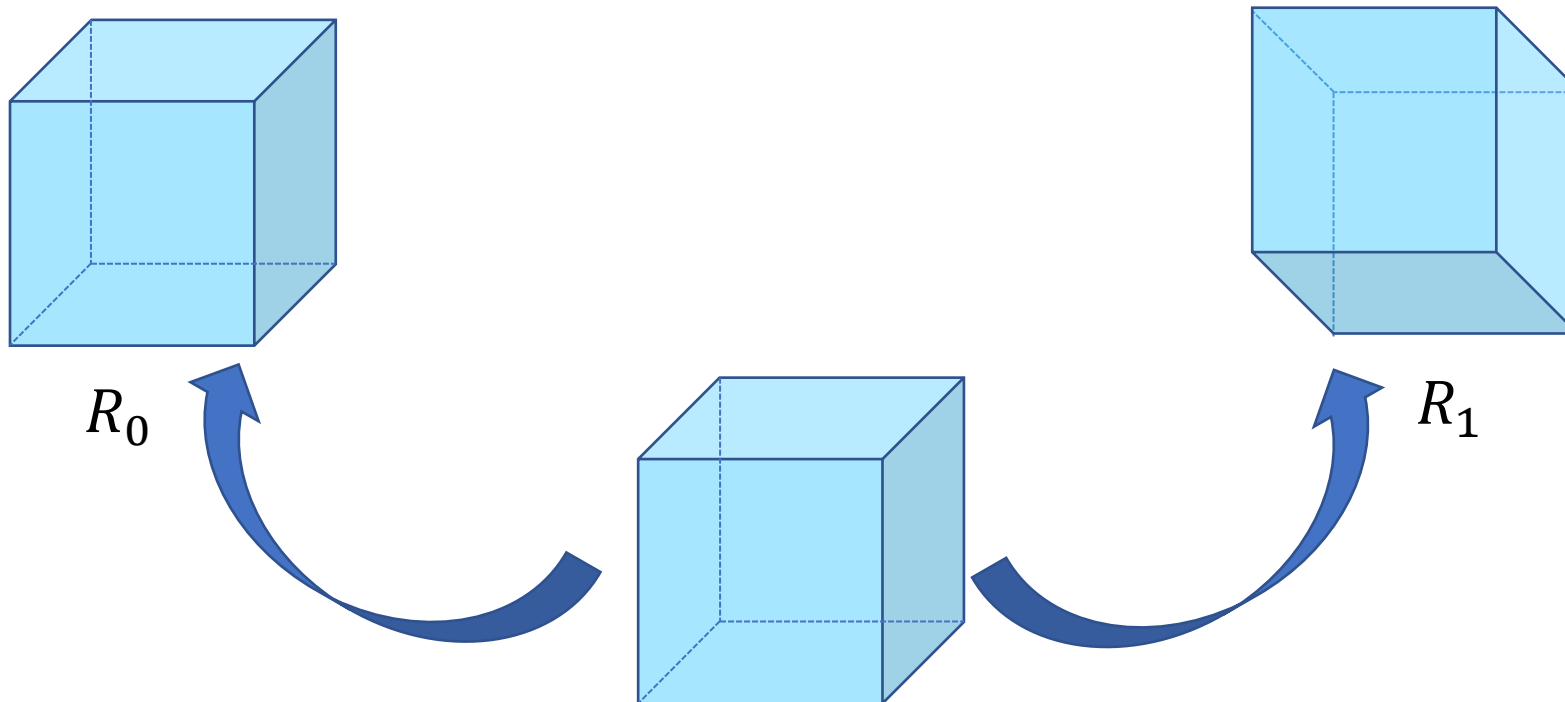
# Interpolation of Translations



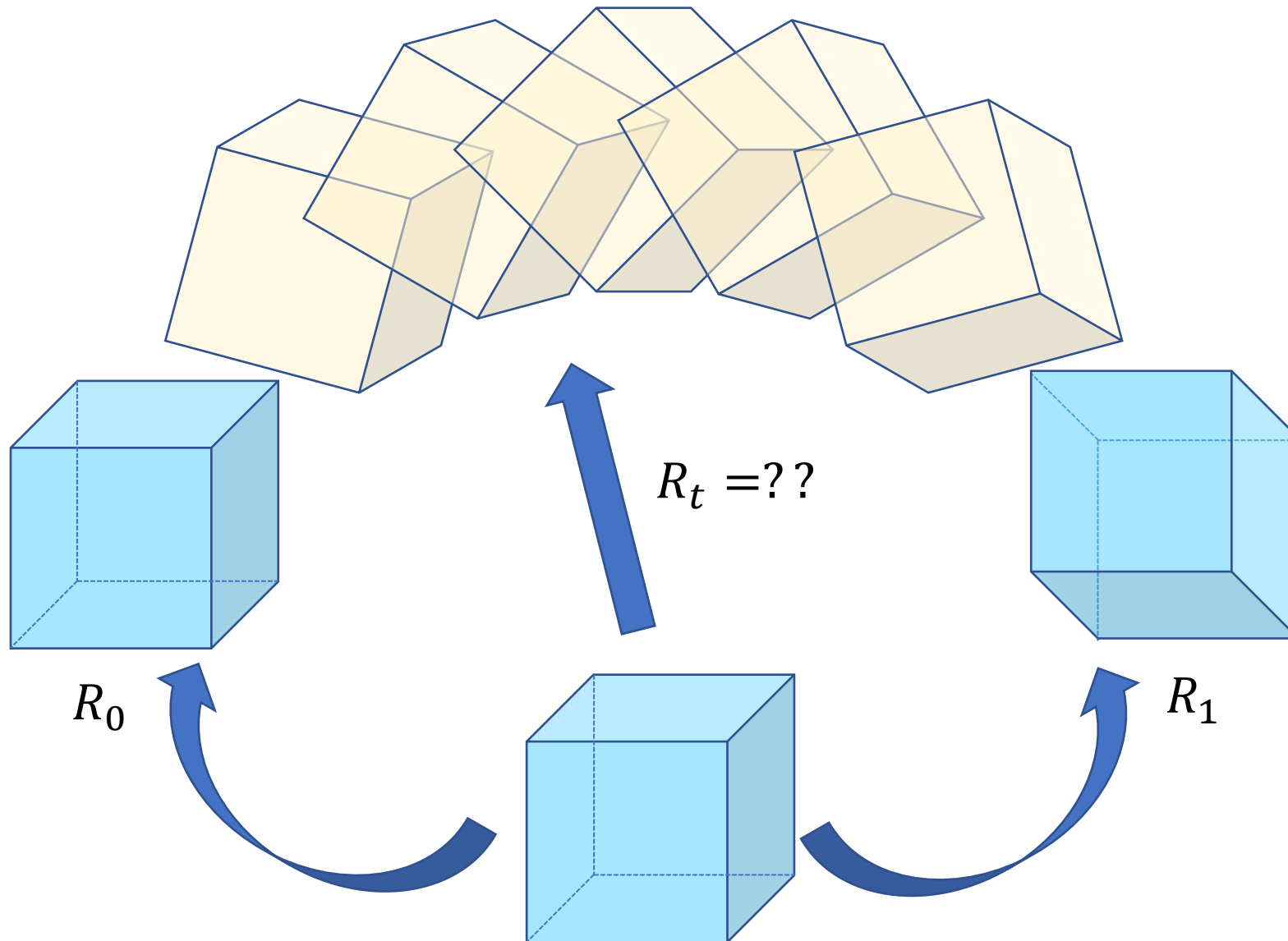
# Interpolation of Translations



# Interpolation of Rotations

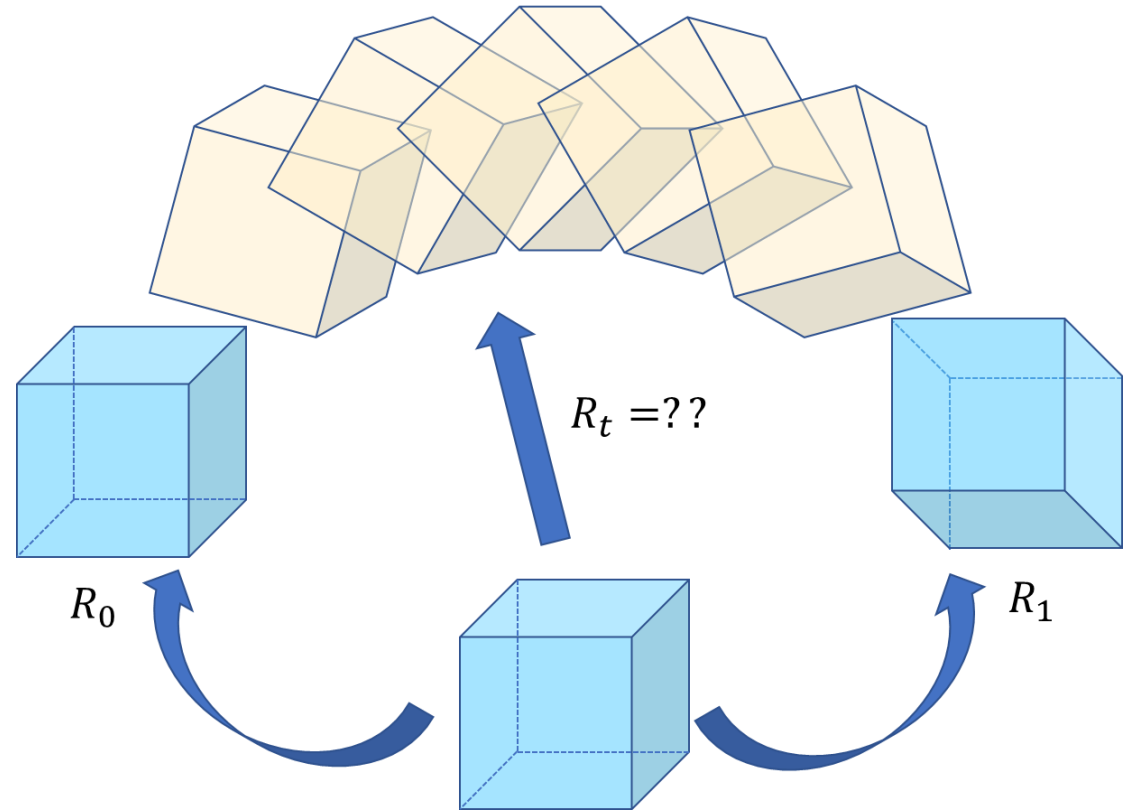


# Interpolation of Rotations



# Interpolation of Rotations

$$R_t = (1 - t)R_0 + tR_1 ??$$





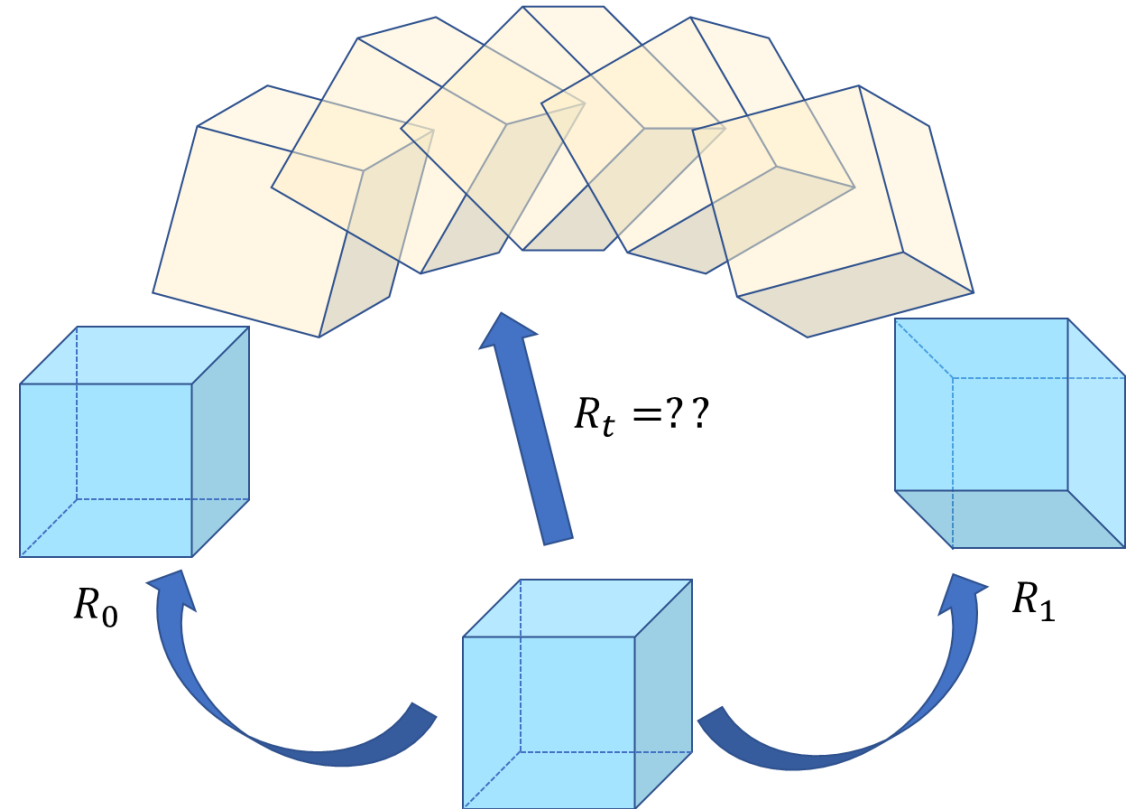
# Interpolation of Rotations

$$R_t = (1 - t)R_0 + tR_1 \quad ??$$

$$R_0 = R_y(-90^\circ) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

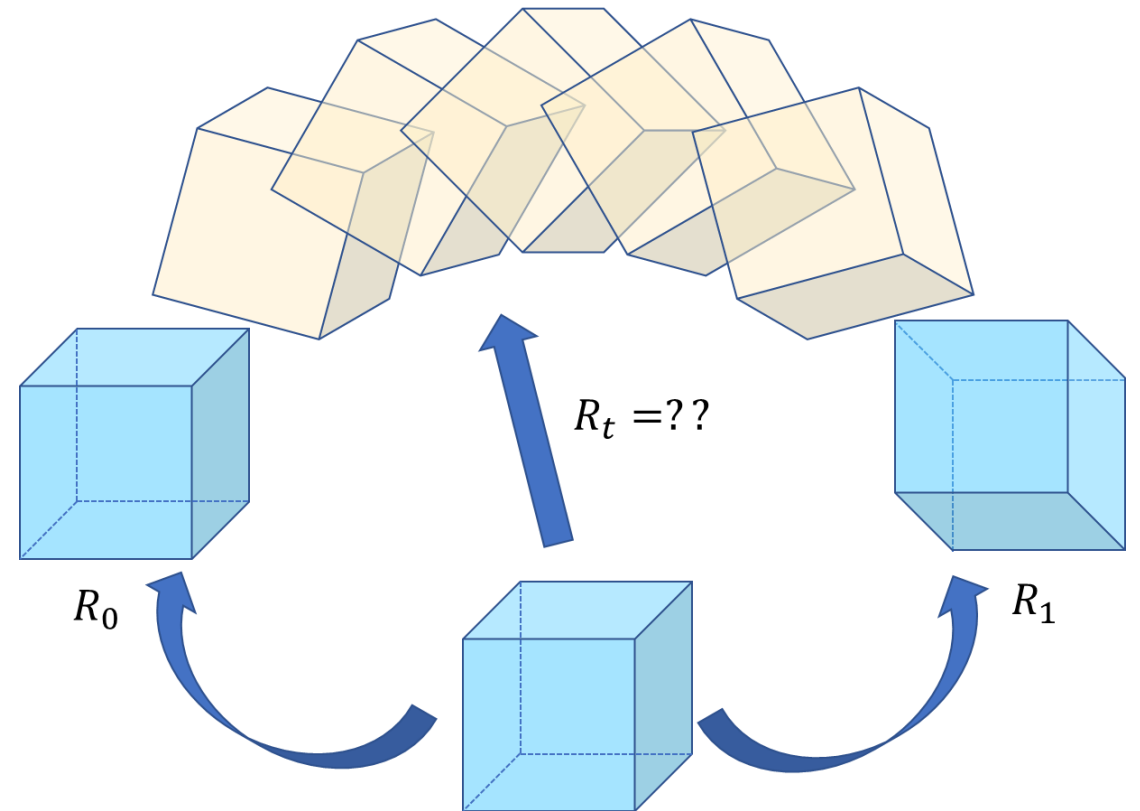
$$R_1 = R_y(+90^\circ) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$R_{0.5} = 0.5(R_0 + R_1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$






# Interpolation of Rotations

- What is good interpolation?
  - Rotation is valid at any time  $t$
  - Constant rotational speed is preferred



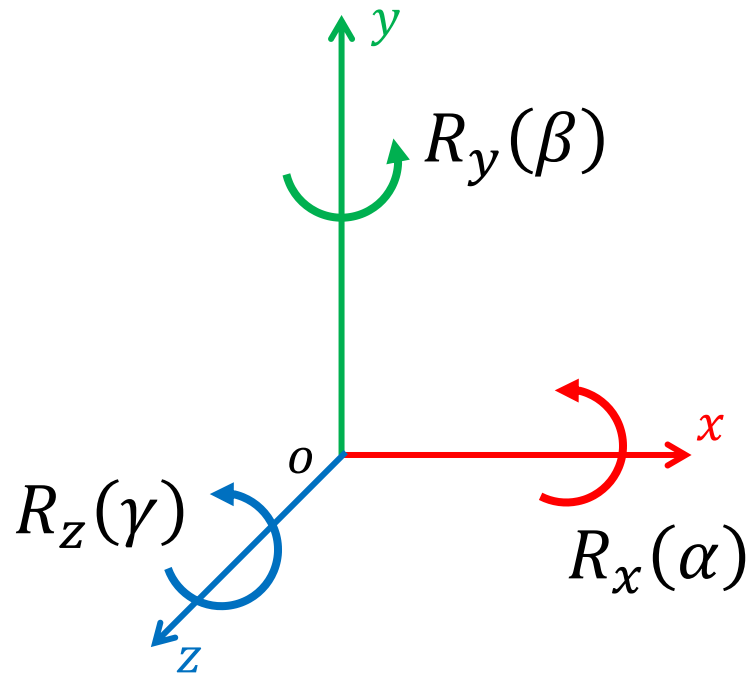
# [R] Rotation Matrix

$$R = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad R^T R = I$$

- Easy to compose? 
- Easy to apply? 
- Easy to interpolate? 

# [ 回 ] Euler angles

- Basic rotations



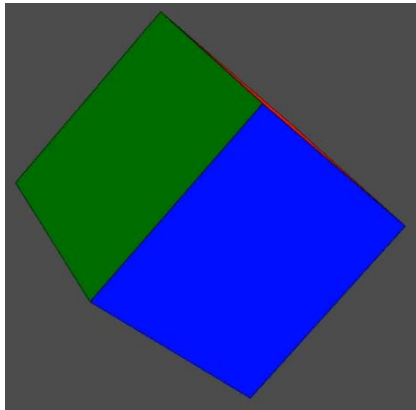
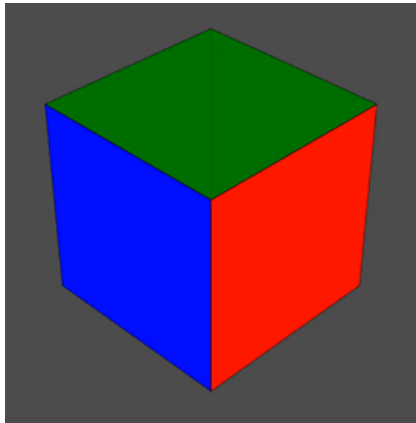
$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{pmatrix}$$

$$R_y(\beta) = \begin{pmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{pmatrix}$$

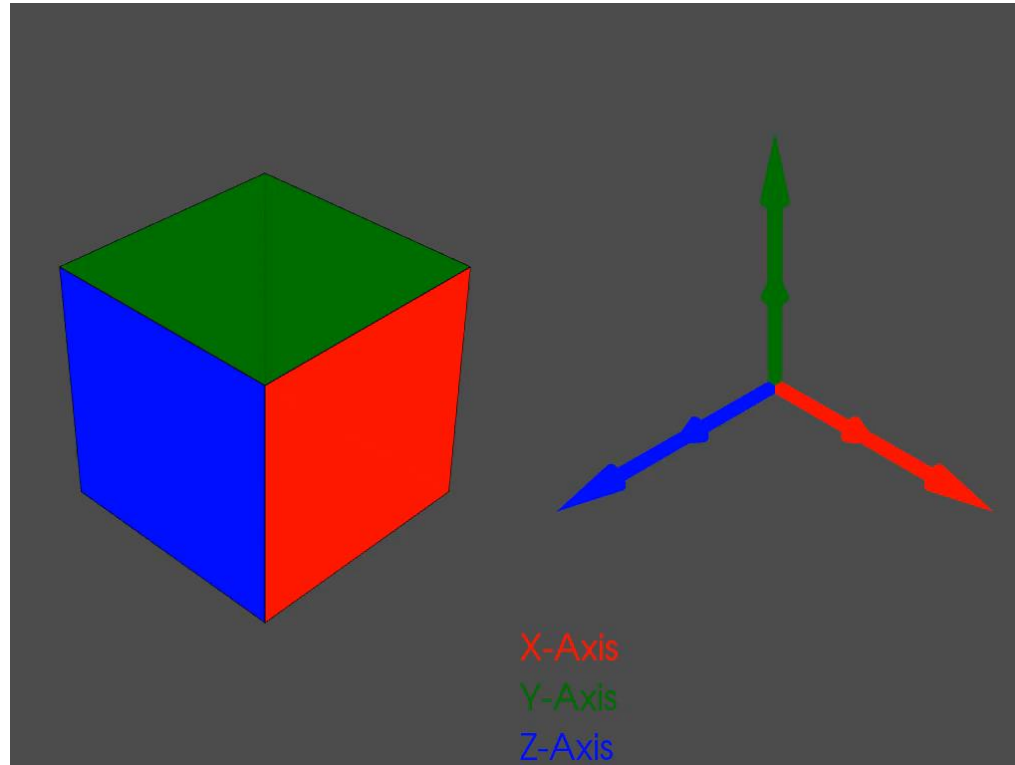
$$R_z(\gamma) = \begin{pmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

# [E] Euler Angles

- Any rotation can be represented as a combination of three basic rotations

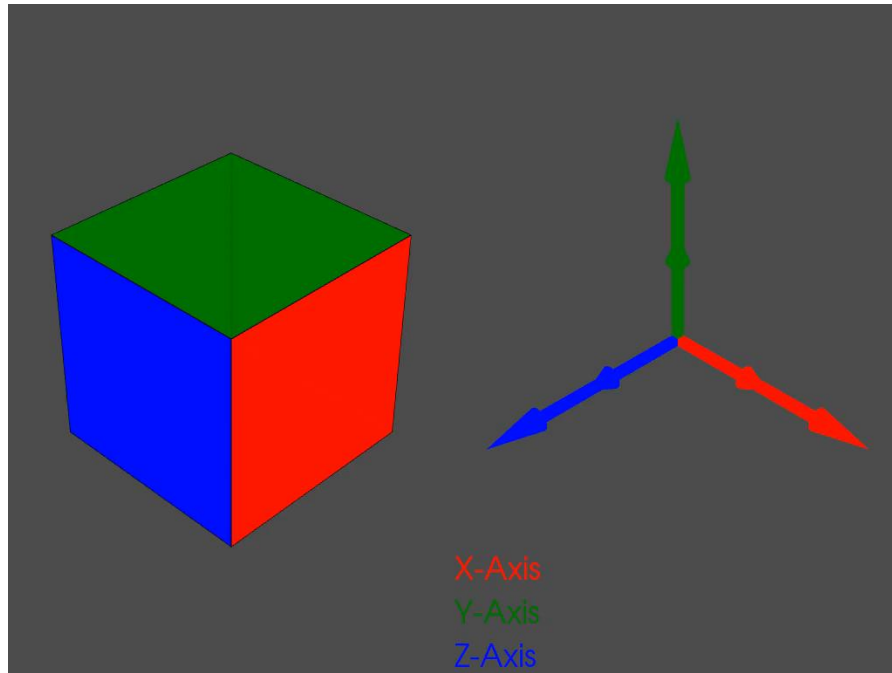


$$R_z(72^\circ)R_y(45^\circ)R_x(60^\circ)$$

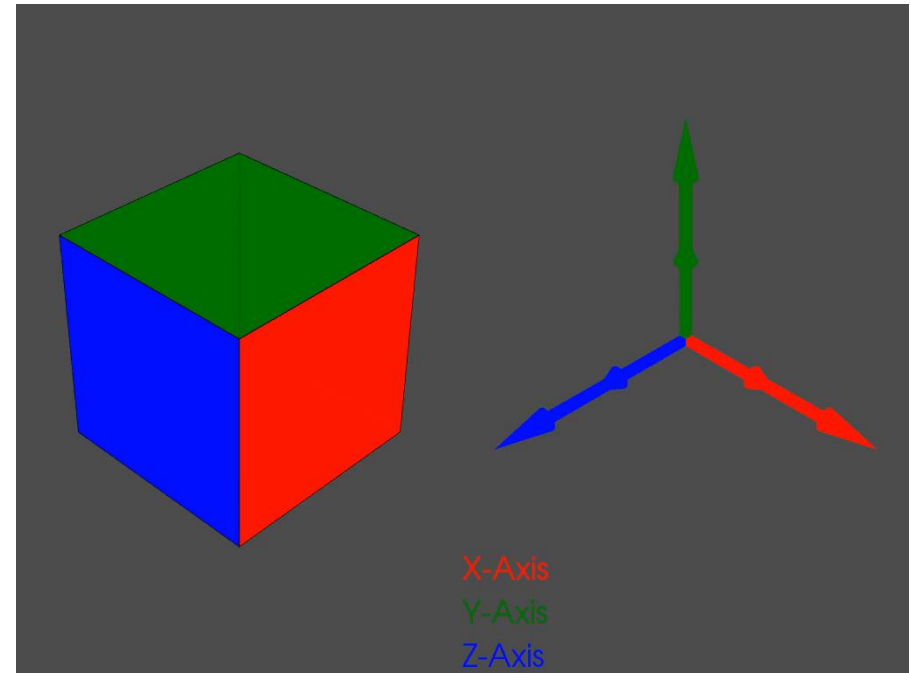


# [同] Euler Axes

- Any combination of three basic rotations are allowed
  - Excluding those rotate twice around the same axis
  - XYZ, XZY, YZX, YXZ, ZYX, ZXY, XYX, XZX, YXY, YZY, ZXZ, ZYZ



$$R_z(72^\circ)R_y(45^\circ)R_x(60^\circ)$$

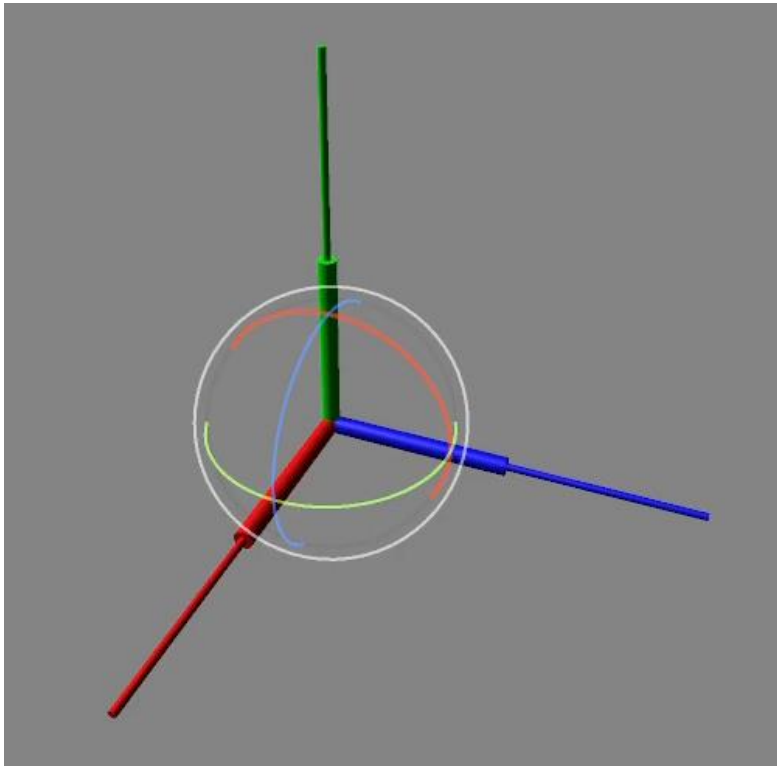


$$R_x(69.2^\circ)R_y(4.0^\circ)R_z(42.4^\circ)$$

# [同] Conventions of Euler Angles

intrinsic rotations:

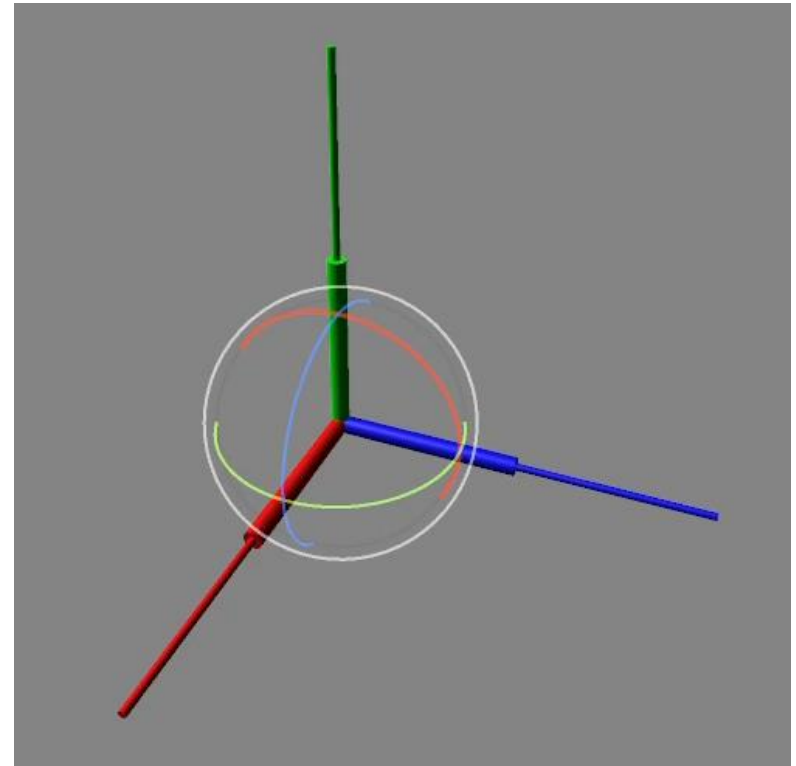
axes attached to the object



$$R_x(\alpha)R_y(\beta)R_z(\gamma)$$

extrinsic rotations:

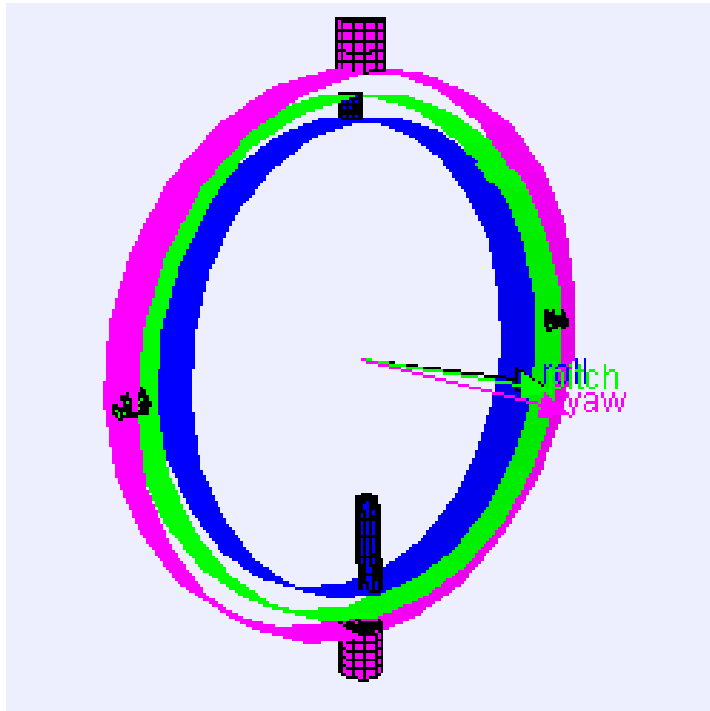
axes fixed to the world



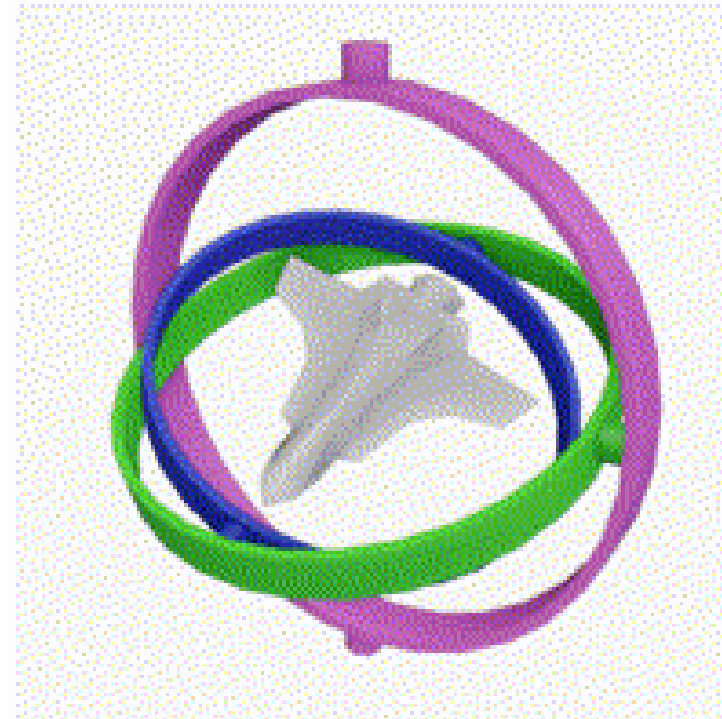
$$R_z(\gamma)R_y(\beta)R_x(\alpha)$$

# [回] Gimbal Lock

- When two local axes are driven into a parallel configuration, one degree of freedom is “locked”



Normal Situation



Gimbal Lock



# [Euler] Euler Angles

$$R_x(\alpha)R_y(\beta)R_z(\gamma)$$

3 parameters:  $\alpha, \beta, \gamma$

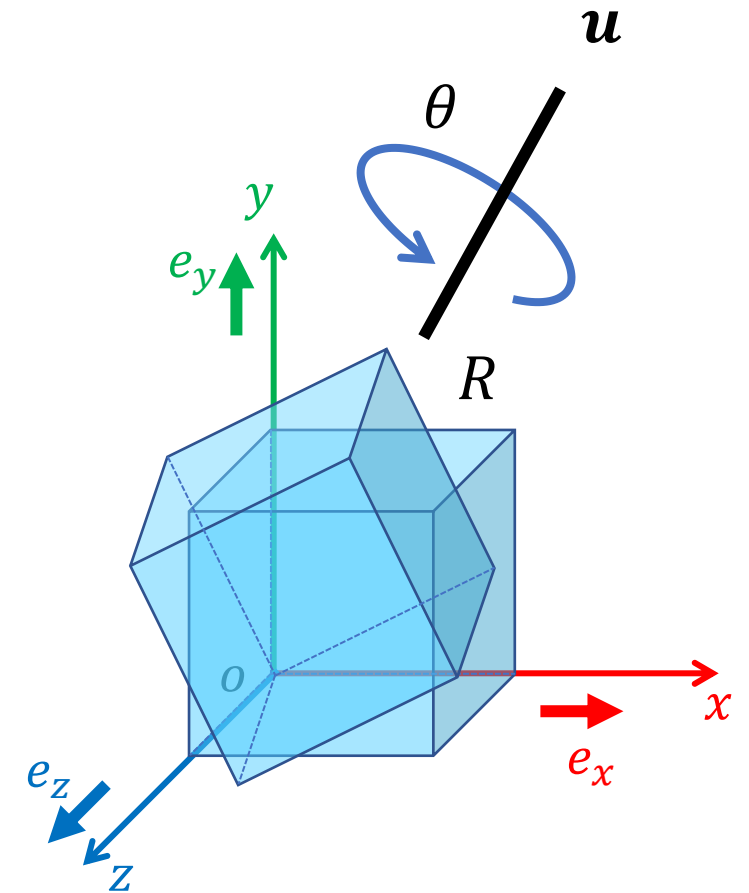
12 variations: XYZ, XZY, YZX, YXZ, ZYX, ZXY,  
XYX, XZX, YXY, YZY, ZXZ, ZYZ

Intrinsic/Extrinsic rotations

- Easy to compose? ✓ But hard to create specific rotations
- Easy to apply? ✓ Need three matrix multiplications
- Easy to interpolate? ✓ Need to deal with singularities
- Gimbal lock ✗ rotational speed is not constant

# [H] Rotation Vectors / Axis Angles

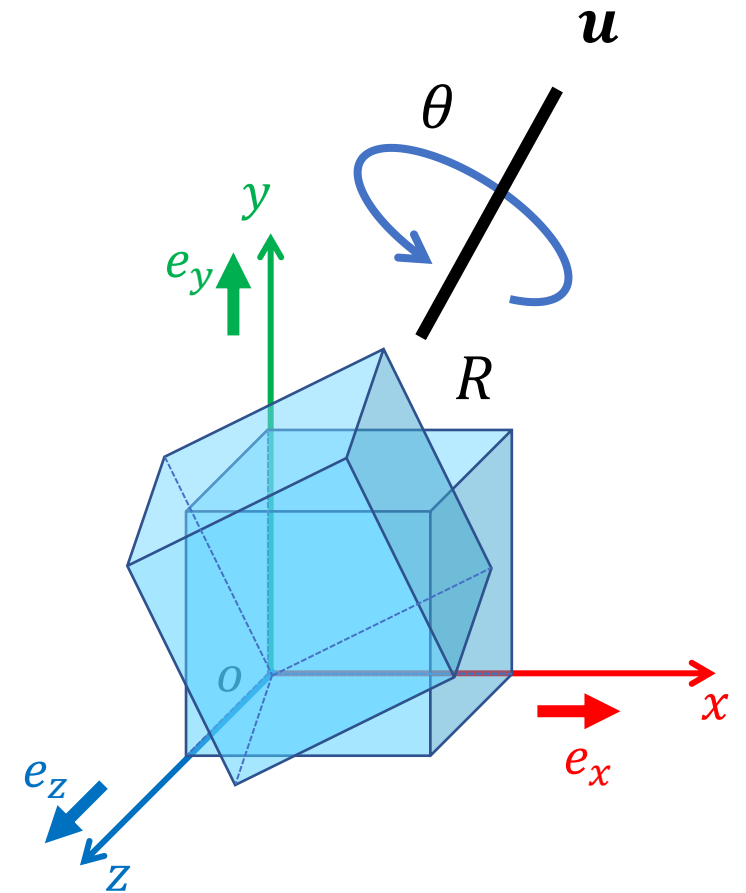
- Axis angle  $(\mathbf{u}, \theta)$ : represent a rotation using
  - A vector  $\mathbf{u}$ : rotation axis
  - A scalar  $\theta$ : rotation angle



# [H] Rotation Vectors / Axis Angles

- Axis angle  $(\mathbf{u}, \theta)$ : represent a rotation using
  - A vector  $\mathbf{u}$ : rotation axis
  - A scalar  $\theta$ : rotation angle
- Rotation vector: represent a rotation as
  - $\boldsymbol{\theta} = \theta \mathbf{u}$
  - Obviously:

$$\theta = \|\boldsymbol{\theta}\| \quad \mathbf{u} = \frac{\boldsymbol{\theta}}{\|\boldsymbol{\theta}\|}$$



# [H] Applying Rotation Vectors / Axis Angles

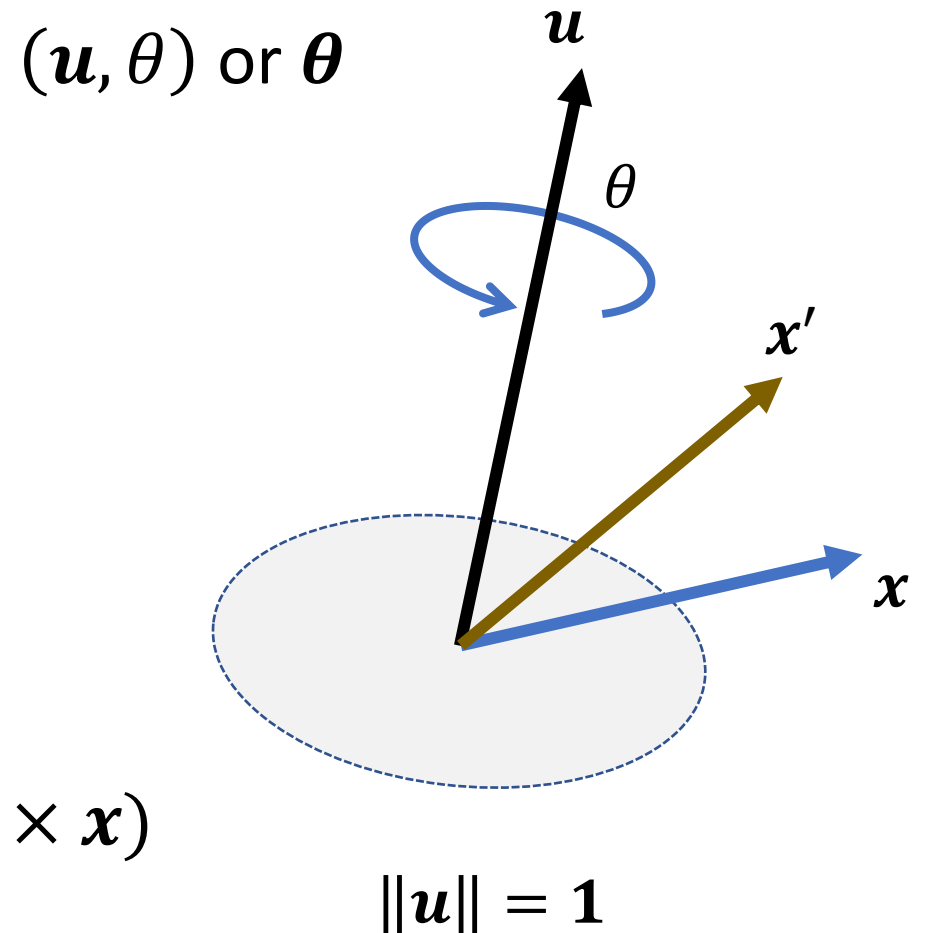
Rodrigues' rotation formula

$$\mathbf{x}' = R\mathbf{x}$$

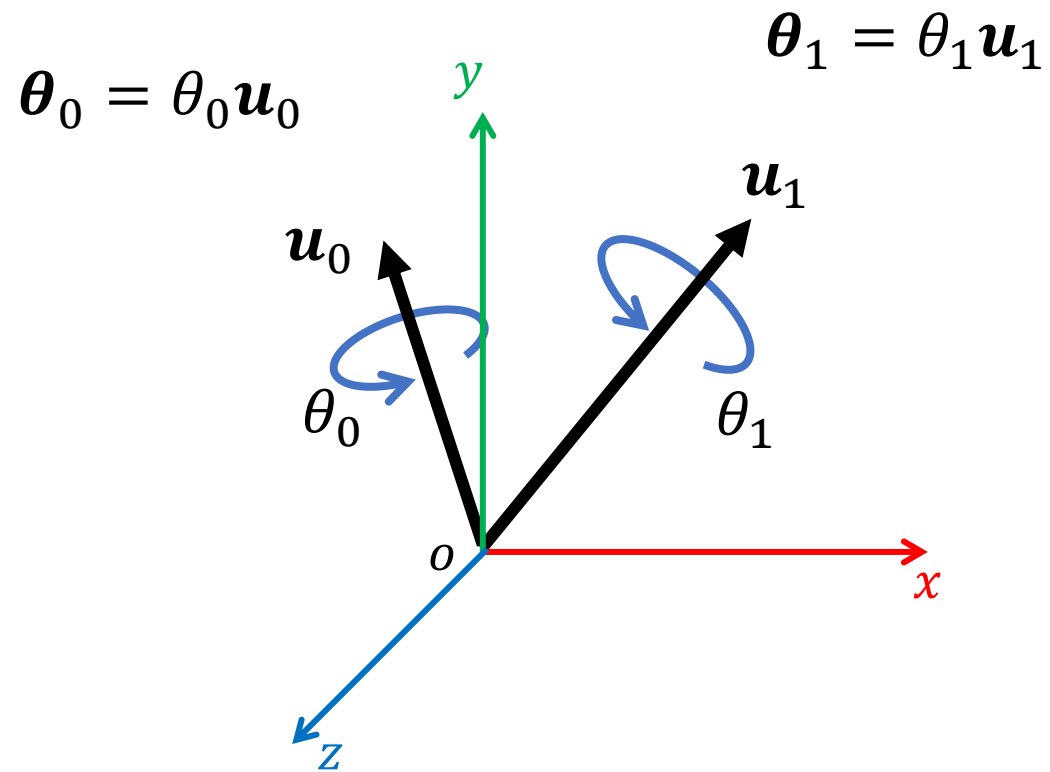
$$R = I + (\sin \theta) [\mathbf{u}]_{\times} + (1 - \cos \theta) [\mathbf{u}]_{\times}^2$$

or

$$\mathbf{x}' = \mathbf{x} + (\sin \theta) \mathbf{u} \times \mathbf{x} + (1 - \cos \theta) \mathbf{u} \times (\mathbf{u} \times \mathbf{x})$$



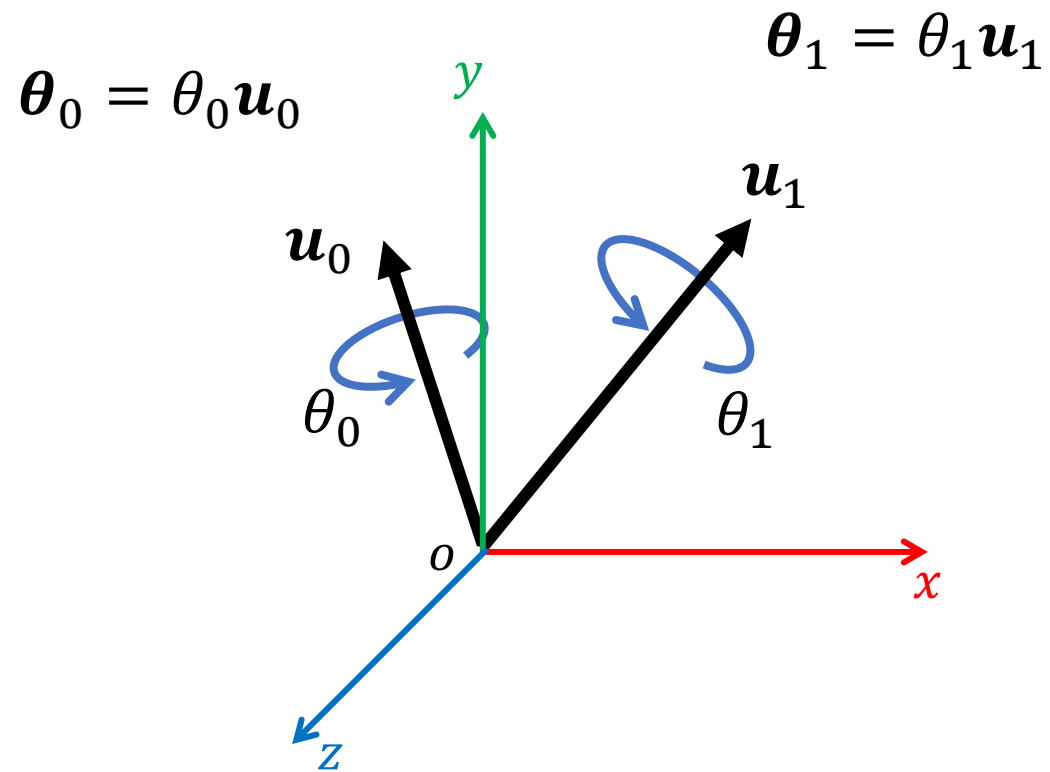
# [ 田 ] Interpolating Rotation Vectors / Axis Angles



Linear interpolation

$$\theta_t = (1 - t)\theta_0 + t\theta_1$$

# [H] Interpolating Rotation Vectors / Axis Angles

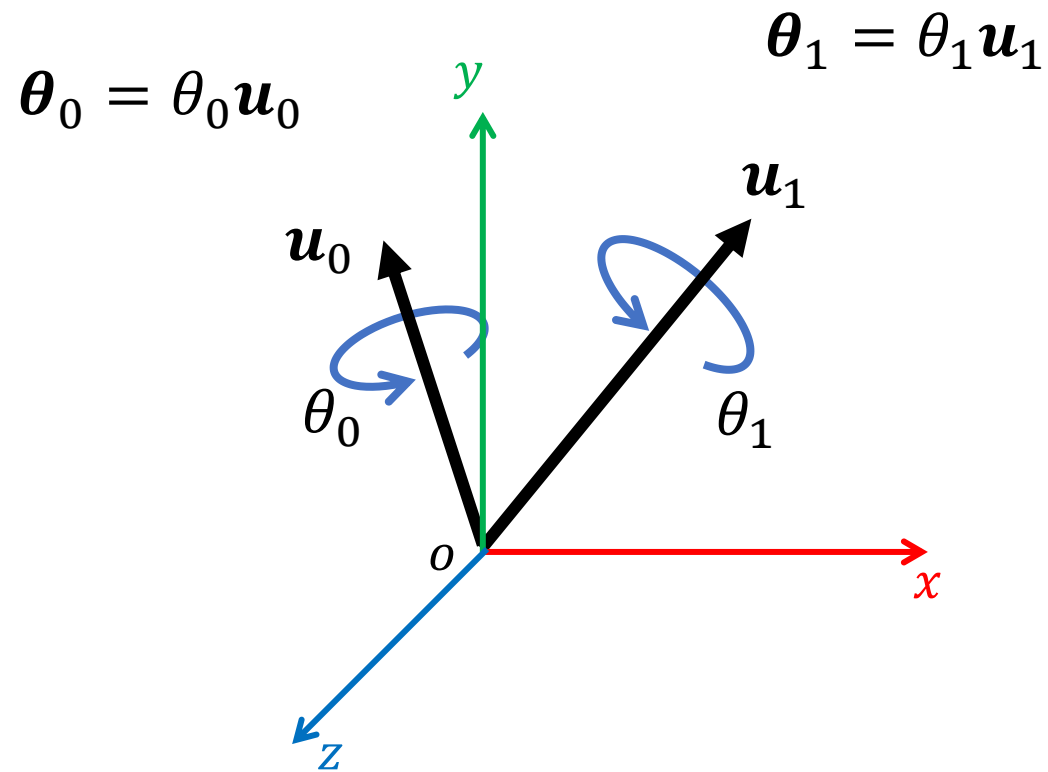


Linear interpolation

$$\theta_t = (1 - t)\theta_0 + t\theta_1$$

- $\theta_t$  is valid ✓
- Constant speed? Not quite

# [H] Interpolating Rotation Vectors / Axis Angles



Compute offset rotation

$$R(\delta\theta) = R^T(\theta_0)R(\theta_1)$$

$$\delta\theta_t = (1 - t)\mathbf{0} + t\delta\theta$$

$$R(\theta_t) = R(\theta_0)R(\delta\theta_t)$$

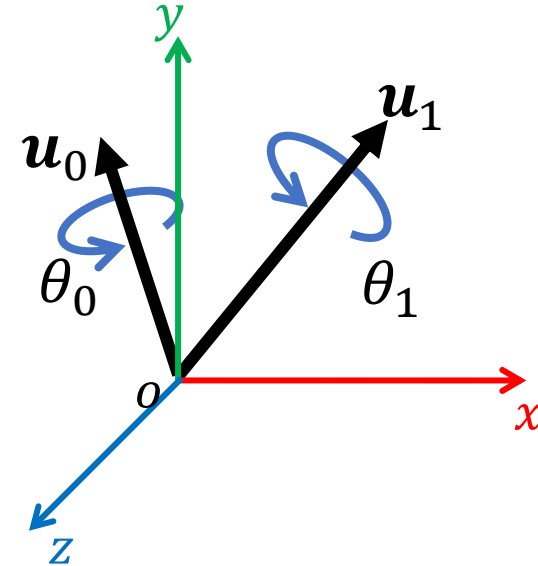
- $\theta_t$  is valid ✓
- Constant speed ✓

# [H] Rotation Vectors / Axis Angles

$$(\mathbf{u}, \theta) \text{ or } \boldsymbol{\theta} = \theta \mathbf{u}$$

Representation is not unique

$$(\mathbf{u}, \theta), \quad (-\mathbf{u}, -\theta), \quad (\mathbf{u}, \theta + 2n\pi)$$



- Easy to compose? ✓ But hard to manipulate
- Easy to apply? ✗ Need to convert to matrix
- Easy to interpolate? ✓ Linear interpolation works, but not perfect  
need to deal with singularities
- No Gimbal lock ✓





# Quaternions

[目]

# [圖] Quaternions

- Recall: a 2D rotation can be represented as a **complex**

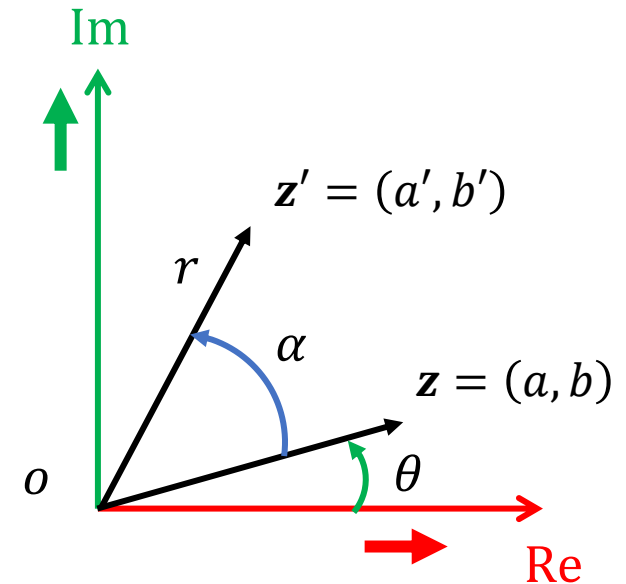
$$z = a + bi = re^{i\theta} \in \mathbb{C}, \quad i^2 = -1$$

$$z' = re^{i(\theta+\alpha)}$$

$$= e^{i\alpha} \times re^{i\theta}$$

$$= e^{i\alpha} z$$

- How to deal with 3D rotation?



# [圖] Quaternions

- Extending complex numbers

$$z = a + bi + \textcolor{red}{c}\textcolor{red}{j} + \textcolor{blue}{d}\textcolor{blue}{k} + \textcolor{yellow}{????}$$

$$i^2 = -1$$

$$\textcolor{red}{j}^2 = -1, \textcolor{red}{j} \neq \textcolor{red}{i}$$

$$\textcolor{blue}{k}^2 = -1, \textcolor{blue}{k} \neq \textcolor{red}{i}, \textcolor{red}{j}$$

# [H] Quaternions

- Extending complex numbers

$$q = a + bi + cj + dk \in \mathbb{H}, a, b, c, d \in \mathbb{R}$$

- $i^2 = j^2 = k^2 = ijk = -1$
- $ij = k, ji = -k$  (\*cross product)
- $jk = i, kj = -i$
- $ki = j, ik = -j$



William Rowan Hamilton

# [ $\mathbb{H}$ ] Quaternion Arithmetic

$$\mathbf{q} = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H}, a, b, c, d \in \mathbb{R}$$

Conjugation:  $\mathbf{q}^* = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$

Scalar product:  $t\mathbf{q} = ta + tb\mathbf{i} + tc\mathbf{j} + td\mathbf{k}$

Addition:  $\mathbf{q}_1 + \mathbf{q}_2 = (a_1 + a_2) + (b_1 + b_2)\mathbf{i} + (c_1 + c_2)\mathbf{j} + (d_1 + d_2)\mathbf{k}$

Dot product:  $\mathbf{q}_1 \cdot \mathbf{q}_2 = a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2$

Norm:  $\|\mathbf{q}\| = \sqrt{a^2 + b^2 + c^2 + d^2} = \sqrt{\mathbf{q} \cdot \mathbf{q}}$

# [圖] Quaternion Multiplication

$$\mathbf{q}_1 \mathbf{q}_2 = (a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}) * (a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k})$$

$$\begin{aligned}\mathbf{q}_1\mathbf{q}_2 = & a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2 \\ & + (b_1a_2 + a_1b_2 - d_1c_2 + c_1d_2)\mathbf{i} \\ & + (c_1a_2 + d_1b_2 + a_1c_2 - b_1d_2)\mathbf{j} \\ & + (d_1a_2 - c_1b_2 + b_1c_2 + a_1d_2)\mathbf{k}\end{aligned}$$

note:

- $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$
- $\mathbf{ij} = \mathbf{k}, \mathbf{ji} = -\mathbf{k}$  (\*cross product)
- $\mathbf{jk} = \mathbf{i}, \mathbf{kj} = -\mathbf{i}$
- $\mathbf{ki} = \mathbf{j}, \mathbf{ik} = -\mathbf{j}$

# [四] Quaternions

$$q = w + xi + yj + zk \quad \rightarrow \quad q = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} w \\ \mathbf{v} \end{bmatrix}$$

$$q = [w, \mathbf{v}]^T \in \mathbb{H}, \quad w \in \mathbb{R}, \quad \mathbf{v} \in \mathbb{R}^3$$

$$w = [w, \mathbf{0}]^T : \text{scalar quaternion}$$

$$\mathbf{v} = [0, \mathbf{v}]^T : \text{pure quaternion}$$

# [目] Quaternion Arithmetic

$$\mathbf{q} = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad \rightarrow \quad \mathbf{q} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} w \\ \mathbf{v} \end{bmatrix}$$

Conjugation:  $\mathbf{q}^* = [w, -\mathbf{v}]^T$

Scalar product:  $t\mathbf{q} = [tw, t\mathbf{v}]^T$

Addition:  $\mathbf{q}_1 + \mathbf{q}_2 = [w_1 + w_2, \mathbf{v}_1 + \mathbf{v}_2]^T$

Dot product:  $\mathbf{q}_1 \cdot \mathbf{q}_2 = w_1 w_2 + \mathbf{v}_1 \cdot \mathbf{v}_2$

Norm:  $\|\mathbf{q}\| = \sqrt{w_1 w_2 + \mathbf{v}_1 \cdot \mathbf{v}_2} = \sqrt{\mathbf{q} \cdot \mathbf{q}}$



# [目] Quaternion Multiplication

$$\mathbf{q}_1 \mathbf{q}_2 = \begin{bmatrix} w_1 \\ \mathbf{v}_1 \end{bmatrix} \begin{bmatrix} w_2 \\ \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} w_1 w_2 - \mathbf{v}_1 \cdot \mathbf{v}_2 \\ w_1 \mathbf{v}_2 + w_2 \mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2 \end{bmatrix}$$

# [圖] Quaternion Multiplication

$$\mathbf{q}_1 \mathbf{q}_2 = \begin{bmatrix} w_1 \\ \mathbf{v}_1 \end{bmatrix} \begin{bmatrix} w_2 \\ \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} w_1 w_2 - \mathbf{v}_1 \cdot \mathbf{v}_2 \\ w_1 \mathbf{v}_2 + w_2 \mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2 \end{bmatrix}$$

Non-Commutativity:

$$\mathbf{q}_1 \mathbf{q}_2 \neq \mathbf{q}_2 \mathbf{q}_1$$

Associativity:

$$\mathbf{q}_1 \mathbf{q}_2 \mathbf{q}_3 = (\mathbf{q}_1 \mathbf{q}_2) \mathbf{q}_3 = \mathbf{q}_1 (\mathbf{q}_2 \mathbf{q}_3)$$

# [圖] Quaternion Multiplication

$$\mathbf{q}_1 \mathbf{q}_2 = \begin{bmatrix} w_1 \\ \mathbf{v}_1 \end{bmatrix} \begin{bmatrix} w_2 \\ \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} w_1 w_2 - \mathbf{v}_1 \cdot \mathbf{v}_2 \\ w_1 \mathbf{v}_2 + w_2 \mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2 \end{bmatrix}$$

Conjugation:

$$(\mathbf{q}_1 \mathbf{q}_2)^* = \mathbf{q}_2^* \mathbf{q}_1^*$$

Norm:

$$\|\mathbf{q}\|^2 = \mathbf{q}^* \mathbf{q} = \mathbf{q} \mathbf{q}^*$$

Reciprocal:

$$\mathbf{q} \mathbf{q}^{-1} = \mathbf{1} \quad \Rightarrow \quad \mathbf{q}^{-1} = \frac{\mathbf{q}^*}{\|\mathbf{q}\|^2}$$
$$\mathbf{q}^{-1} \mathbf{q} = \mathbf{1}$$

# [目] Unit Quaternions

$$\mathbf{q} = \begin{bmatrix} w \\ \mathbf{v} \end{bmatrix} \quad \|\mathbf{q}\| = 1$$

For any non-zero quaternion  $\tilde{\mathbf{q}}$ :

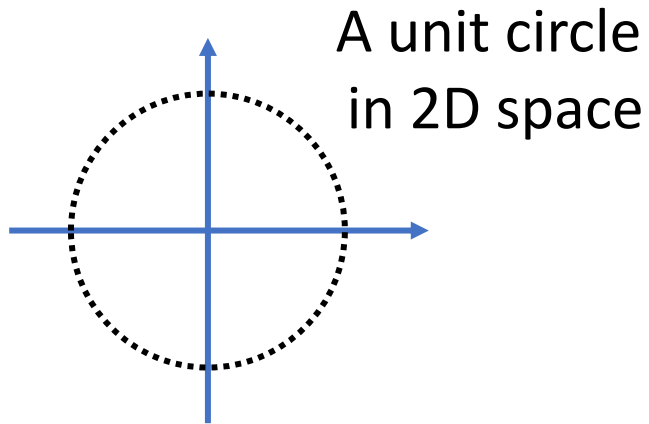
$$\mathbf{q} = \frac{\tilde{\mathbf{q}}}{\|\tilde{\mathbf{q}}\|}$$

Reciprocal:

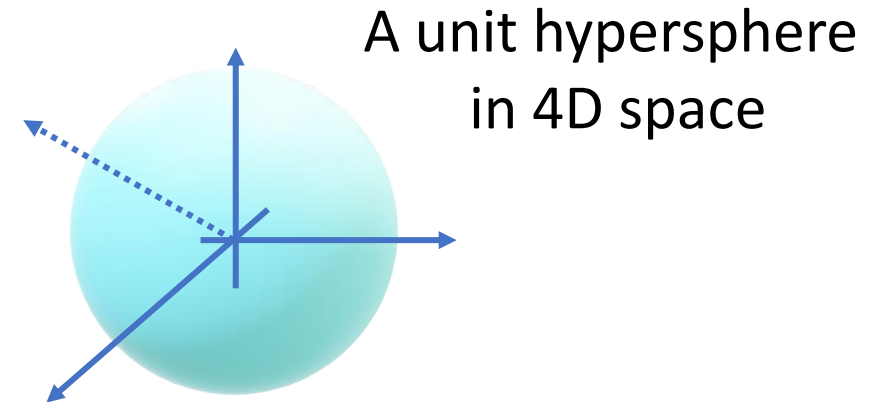
$$\mathbf{q}^{-1} = \mathbf{q}^* = \begin{bmatrix} w \\ -\mathbf{v} \end{bmatrix} \quad \Leftrightarrow \quad R^{-1} = R^T$$

# [圖] Unit Quaternions

$$\mathbf{q} = \begin{bmatrix} w \\ \mathbf{v} \end{bmatrix} \quad \|\mathbf{q}\| = 1$$



unit complex number  
 $z = \cos \theta + i \sin \theta$

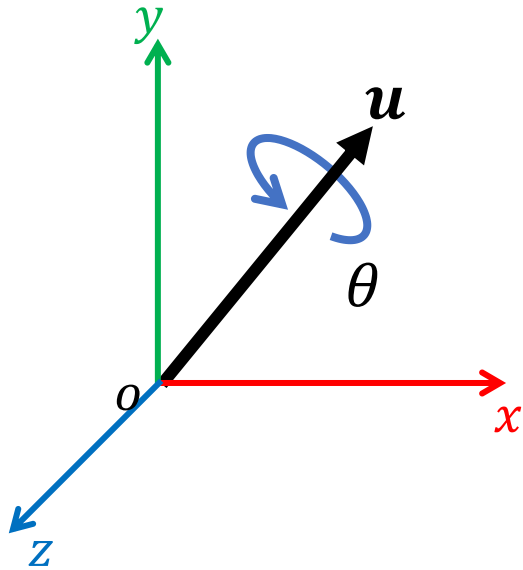


unit quaternion

$$\mathbf{q} = \left[ \cos \frac{\theta}{2}, \mathbf{u} \sin \frac{\theta}{2} \right] \quad \|\mathbf{u}\| = 1$$

# [圖] Unit Quaternions

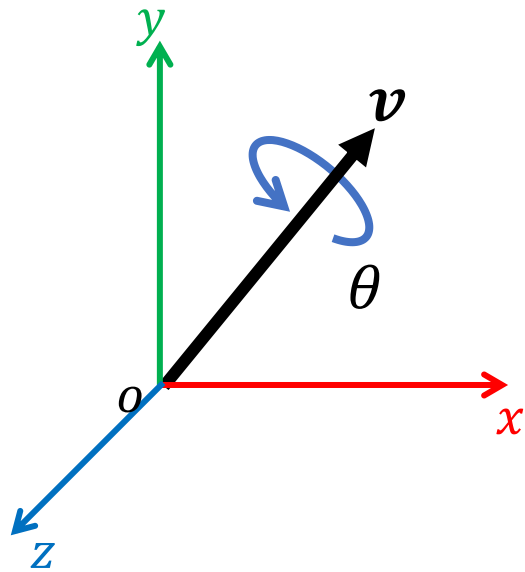
$$\mathbf{q} = \begin{bmatrix} w \\ \mathbf{v} \end{bmatrix} = \left[ \cos \frac{\theta}{2}, \mathbf{u} \sin \frac{\theta}{2} \right] \quad \|\mathbf{u}\| = 1$$



same information as axis angles  $(\mathbf{u}, \theta)$   
But in a different form

# [圖] Unit Quaternions as 3D Rotations

Any 3D rotation  $(\mathbf{v}, \theta)$  can be represented as a **unit quaternion**



$$\mathbf{q} = \begin{bmatrix} w \\ \mathbf{v} \end{bmatrix} = \left[ \cos \frac{\theta}{2}, \mathbf{u} \sin \frac{\theta}{2} \right]$$

Angle:  $\theta = 2 \arg \cos w$

Axis:  $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$

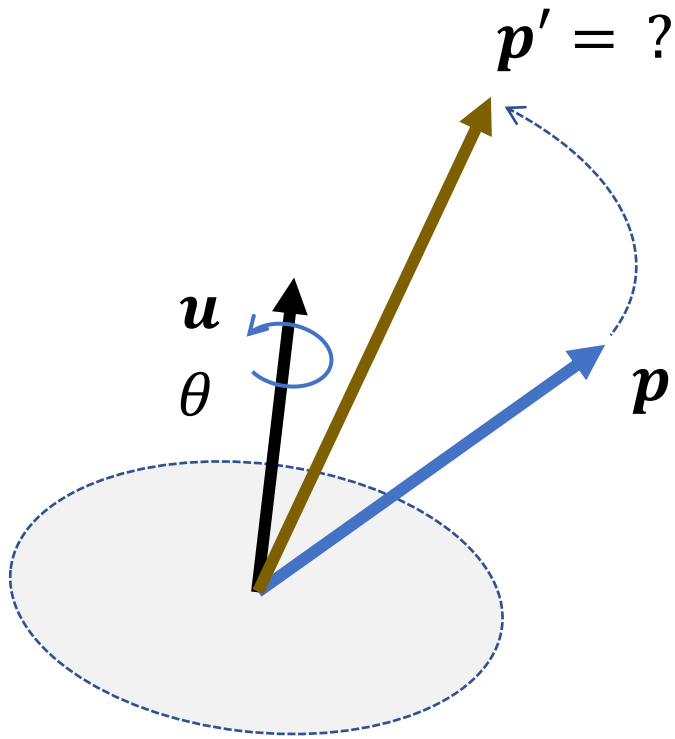
# [圖] Rotation a Vector Using Unit Quaternions

$$\text{Unit quaternion: } \mathbf{q} = \begin{bmatrix} w \\ \mathbf{v} \end{bmatrix} = \left[ \cos \frac{\theta}{2}, \mathbf{u} \sin \frac{\theta}{2} \right]$$

3D vector:  $\mathbf{p}$       Rotation result:  $\mathbf{p}'$

Then the rotation can be applied by  
quaternion multiplication:

$$\begin{bmatrix} 0 \\ \mathbf{p}' \end{bmatrix} = \mathbf{q} \begin{bmatrix} 0 \\ \mathbf{p} \end{bmatrix} \mathbf{q}^*$$





# [圖] Rotation a Vector Using Unit Quaternions

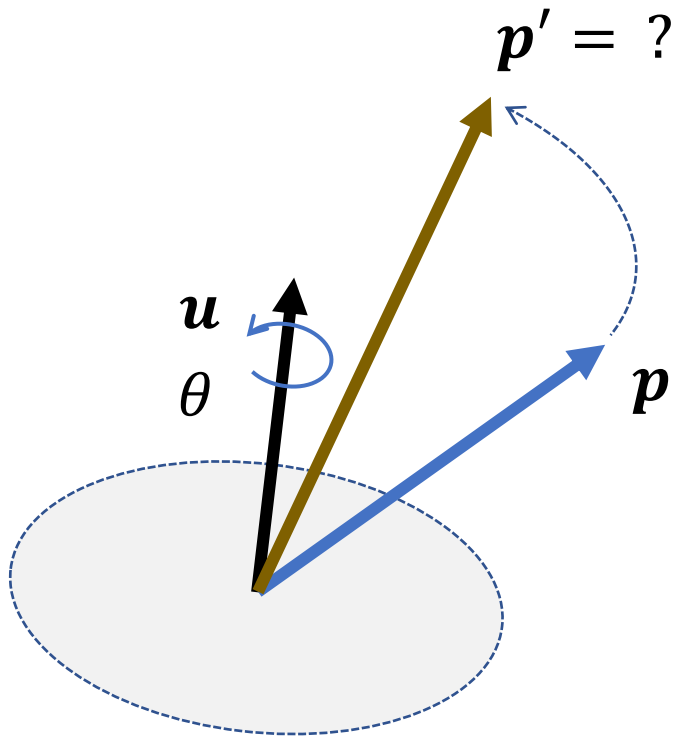
$$\text{Unit quaternion: } \mathbf{q} = \begin{bmatrix} w \\ \mathbf{v} \end{bmatrix} = \left[ \cos \frac{\theta}{2}, \mathbf{u} \sin \frac{\theta}{2} \right]$$

3D vector:  $\mathbf{p}$       Rotation result:  $\mathbf{p}'$

Then the rotation can be applied by  
quaternion multiplication:

$$\begin{bmatrix} 0 \\ \mathbf{p}' \end{bmatrix} = \mathbf{q} \begin{bmatrix} 0 \\ \mathbf{p} \end{bmatrix} \mathbf{q}^* = (-\mathbf{q}) \begin{bmatrix} 0 \\ \mathbf{p} \end{bmatrix} (-\mathbf{q})^*$$

$\mathbf{q}$  and  $-\mathbf{q}$  represent the same rotation



# [目] Combination of Rotations

Unit quaternion:  $\mathbf{q}_1, \mathbf{q}_2$

3D vector:  $\mathbf{p}$

$$\begin{bmatrix} 0 \\ \mathbf{p}' \end{bmatrix} = \mathbf{q}_1 \begin{bmatrix} 0 \\ \mathbf{p} \end{bmatrix} \mathbf{q}_1^*$$

$$\begin{aligned} \begin{bmatrix} 0 \\ \mathbf{p}'' \end{bmatrix} &= \mathbf{q}_2 \begin{bmatrix} 0 \\ \mathbf{p}' \end{bmatrix} \mathbf{q}_2^* = \mathbf{q}_2 \left( \mathbf{q}_1 \begin{bmatrix} 0 \\ \mathbf{p} \end{bmatrix} \mathbf{q}_1^* \right) \mathbf{q}_2^* = (\mathbf{q}_2 \mathbf{q}_1) \begin{bmatrix} 0 \\ \mathbf{p} \end{bmatrix} (\mathbf{q}_2 \mathbf{q}_1)^* \\ &= \mathbf{q} \begin{bmatrix} 0 \\ \mathbf{p} \end{bmatrix} \mathbf{q}^* \end{aligned}$$

# [目] Combination of Rotations

Unit quaternion:  $q_1, q_2$



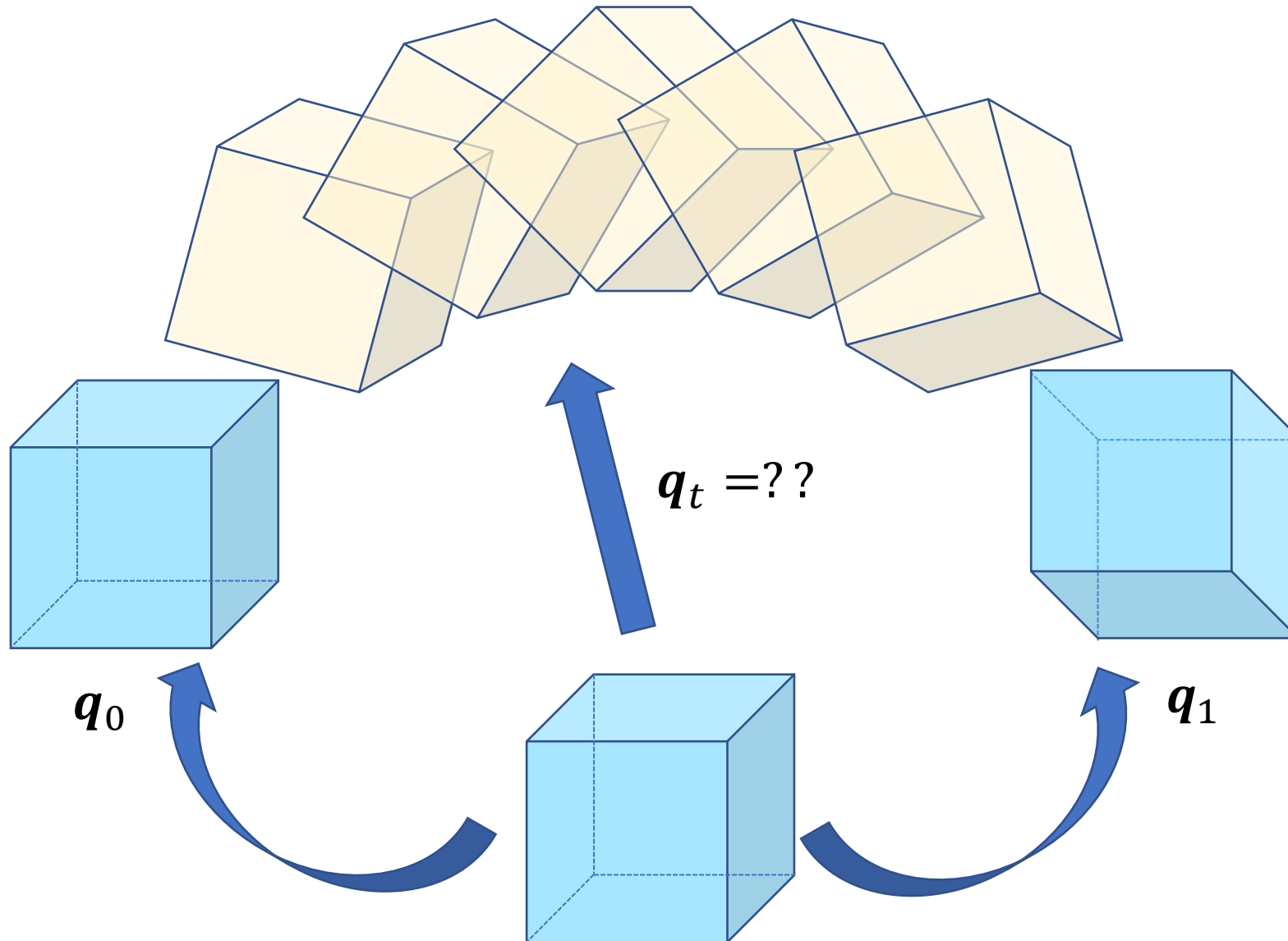
Combined rotation:  $q = q_2 q_1$

3D vector:  $p$

$$\begin{bmatrix} 0 \\ p' \end{bmatrix} = q_1 \begin{bmatrix} 0 \\ p \end{bmatrix} q_1^*$$

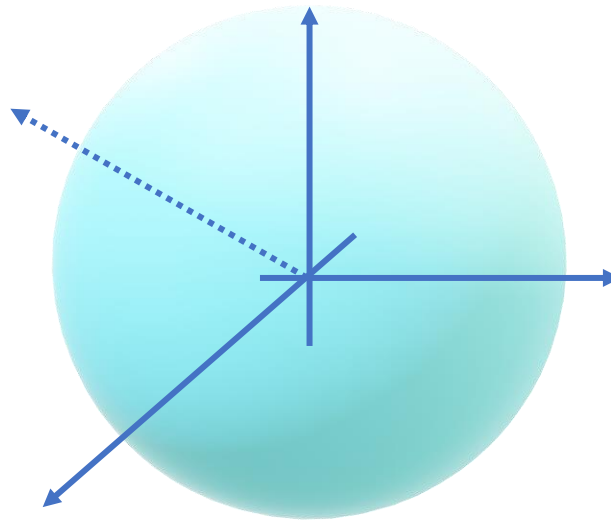
$$\begin{aligned} \begin{bmatrix} 0 \\ p'' \end{bmatrix} &= q_2 \begin{bmatrix} 0 \\ p' \end{bmatrix} q_2^* = q_2 \left( q_1 \begin{bmatrix} 0 \\ p \end{bmatrix} q_1^* \right) q_2^* = (q_2 q_1) \begin{bmatrix} 0 \\ p \end{bmatrix} (q_2 q_1)^* \\ &= q \begin{bmatrix} 0 \\ p \end{bmatrix} q^* \end{aligned}$$

# [圖] Quaternion Interpolation



# [圖] Quaternion Interpolation

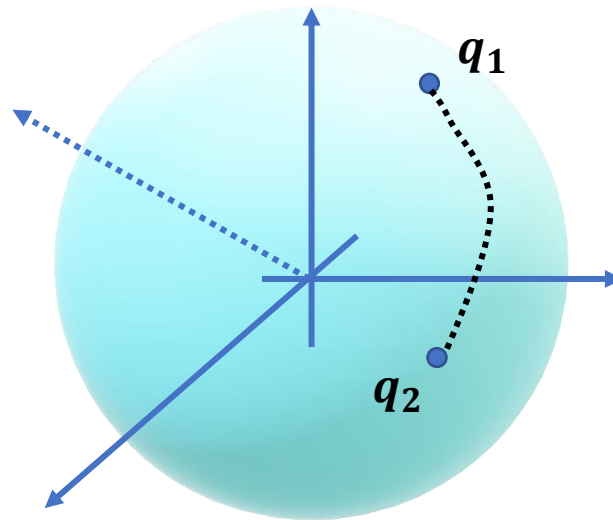
$$\mathbf{q} = \begin{bmatrix} w \\ \mathbf{v} \end{bmatrix} \quad \|\mathbf{q}\| = 1$$



A unit hypersphere  
in 4D space

# [圖] Quaternion Interpolation

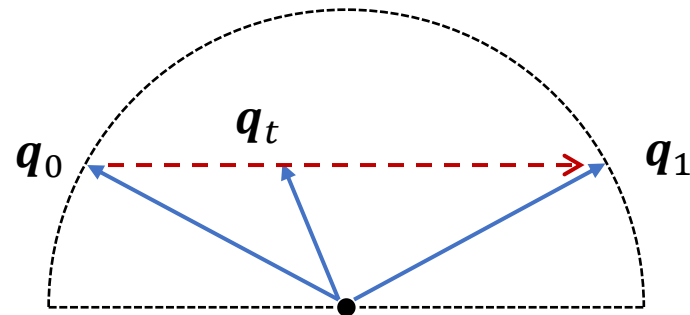
$$q = \begin{bmatrix} w \\ v \end{bmatrix} \quad \|q\| = 1$$



A unit hypersphere  
in 4D space

# [圖] Linear Interpolation

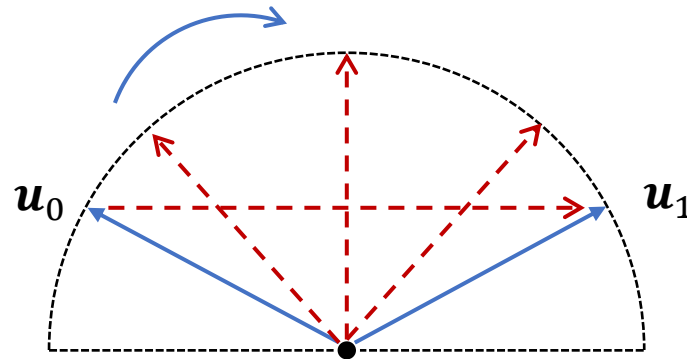
$$\mathbf{q}_t = (1 - t)\mathbf{q}_0 + t\mathbf{q}_1$$



$\mathbf{q}_t$  is not a unit quaternion

# [圖] Linear Interpolation + Projection

$$\tilde{q}_t = (1 - t)q_0 + tq_1 \qquad q_t = \frac{\tilde{q}_t}{\|\tilde{q}_t\|}$$



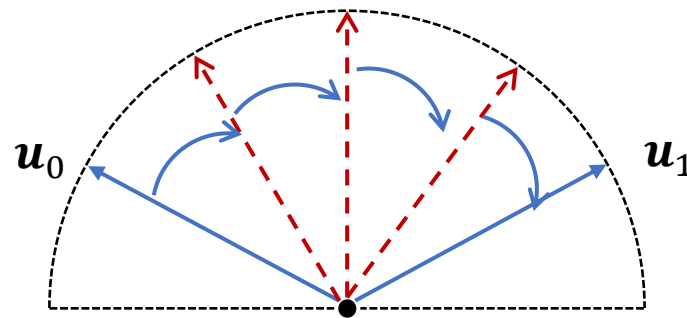
$q_t$  is a unit quaternion

Rotational speed is not constant



# [圖] SLERP: Spherical Linear Interpolation

$$\mathbf{q}_t = a(t)\mathbf{q}_0 + b(t)\mathbf{q}_1$$



# [圖] SLERP: Spherical Linear Interpolation

$$r = a(t)p + b(t)q$$

Consider the angle  $\theta$  between  $p, q$ :  $\cos \theta = p \cdot q$

We have:

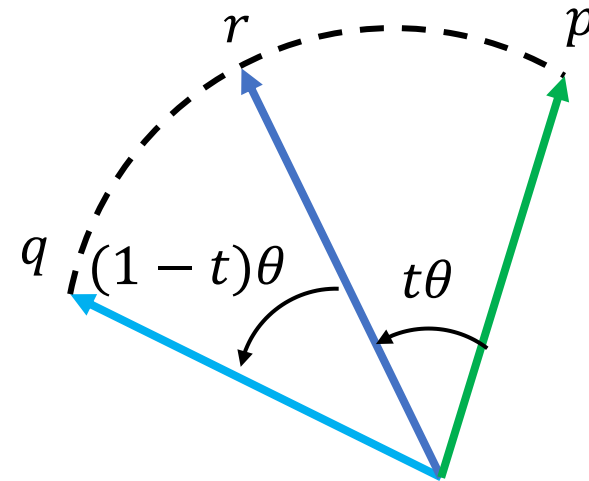
$$\begin{aligned} p \cdot r &= a(t)p \cdot p + b(t)q \cdot p \\ \Rightarrow \cos t\theta &= a(t) + b(t) \cos \theta \end{aligned}$$

similarly

$$\begin{aligned} q \cdot r &= a(t)q \cdot p + b(t) \\ \Rightarrow \cos(1-t)\theta &= a(t) \cos \theta + b(t) \end{aligned}$$

then we have:

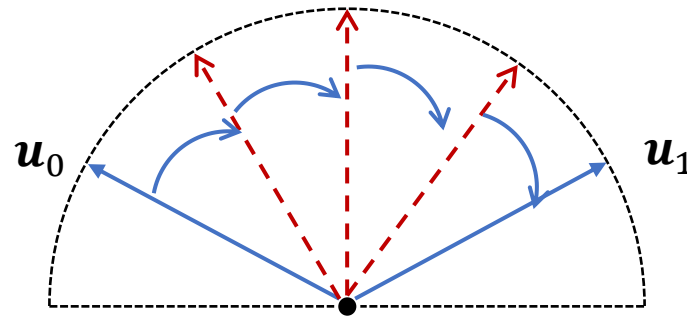
$$a(t) = \frac{\sin[(1-t)\theta]}{\sin \theta}, \quad b(t) = \frac{\sin t\theta}{\sin \theta}$$



# [圖] SLERP: Spherical Linear Interpolation

$$\mathbf{q}_t = \frac{\sin[(1-t)\theta]}{\sin \theta} \mathbf{q}_0 + \frac{\sin t\theta}{\sin \theta} \mathbf{q}_1$$

$$\cos \theta = \mathbf{q}_0 \cdot \mathbf{q}_1$$



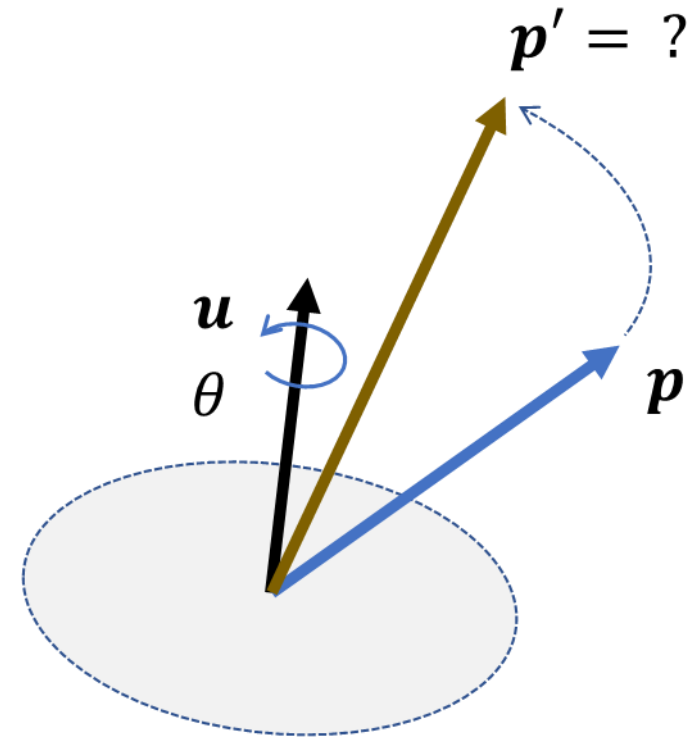
# [目] Quaternions

Rotations can be represented by **unit quaternions**

$$\mathbf{q} = \begin{bmatrix} w \\ \mathbf{v} \end{bmatrix} = \left[ \cos \frac{\theta}{2}, \mathbf{u} \sin \frac{\theta}{2} \right]$$

Representation is not unique

$\mathbf{q}, -\mathbf{q}$  represent the same rotation



- Easy to compose? ✓ Need normalization, hard to manipulate,
- Easy to apply? ✓ Quaternion multiplication
- Easy to interpolate? ✓ SLERP, need to deal with singularities
- No Gimbal lock ✓

# Questions?

