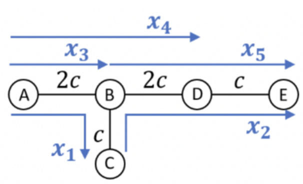
ECE158a Assignment 4

solution

1 Questions

1. Consider the network graph below. There are five flows: flow 1 from A to C, flow 2 from C to E, flow 3 from A to B, flow 4 from A to D, flow 5 from B to E. Denote the rates of the flows as x_1, x_2, x_3, x_4, x_5. Link AB and link BD have capacity 2c, link BC and link DE have capacity c. Assume the utility of each flow is evaluated as $U(x_i) = ln(x_i)$, determine the flow rates which maximizes the total utility under the link capacity constraints.

   Figure 1: Traffic flows

   A: Each link poses a constraint on the flow rates:

   $$
   \begin{aligned}
   AB &: x_1 + x_3 + x_4 = 2C \\
   BC &: x_1 + x_2 = C \\
   BD &: x_2 + x_4 + x_5 = 2C \\
   DE &: x_2 + x_5 = C
   \end{aligned}
   $$

   Note that we have 4 equations and 5 variables, leaving exactly 1 degree of freedom. We can eliminate all variables but one using the equations. Here we write x_2, x_3, x_4, x_5 as functions of x_1:

   $$
   \begin{aligned}
   x_1 &= x_1 \\
   x_2 &= C - x_1 \\
   x_3 &= C - x_1 \\
   x_4 &= C \\
   x_5 &= x_1
   \end{aligned}
   $$

   Then the total utility can be written as a univariate function of x_1: $\sum_{i=1}^{5} U(x_i) = 2ln(x_1) + 2ln(C - x_1) + ln(C)$. To maximize the total utility, take derivative in respect to x1, and let the derivative equal to 0. $\frac{2}{x_1} - \frac{2}{C-x_1} = 0$. Solving the equation above gives $x_1 = C/2$.

2. Figure 2 below shows the change of congestion window over time for an example operation of TCP protocol. The x-axis indicates the transmission round, while y-axis represents the window size. Answer the following questions based on the figure:

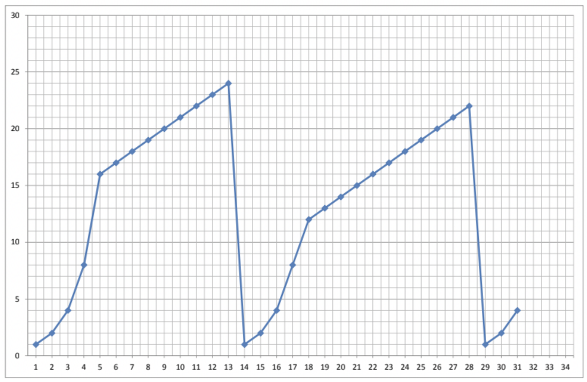
   a) Determines the time instances where timeout events occur.

   b) Determine intervals of time that TCP is operating in slow start and congestion avoidance (AIMD) phases.

c) Suppose that a triple duplicate ACK event occurs at the 24th transmission round, what would the congestion window size at the 25th round become?

d) How many packets have the sender sent by the end of the 8th transmission round? (Assuming no packets have been sent before the 1st transmission round)

Figure 2: TCP congestion window size



A:

a) Whenever timeout occurs, the window size is reset to 1. So from the figure, we can see that timeouts occur at t=13 and t=28.

b) Slow start phases are from t = [1, 5], t = [14, 18] and t = [29, 31]. Congestion avoidance intervals are [5, 13] and [18, 28].

c) Whenever a triple duplicate ACK event occurs, the slow-start threshold ssthresh is set to W, and the window size is halved. So the window size at t = 25 becomes 9.

d) The number of packets sent by the eighth round is (1+2+4+8+16+17+18+19) = 85.

3. Consider the network shown in Figure 3 with three flows. Flow 1 goes from A to C with rate r_1, flow 2 goes from A to B with rate r_2, and flow 3 goes from B to C with rate r_3. Suppose the capacity of the two links are $C_1 = 10$ and $C_2 = 12$. Let the utility function of the rate vector $r = (r_1, r_2, r_3)$ be $U(r) = \sum_{i=1}^{3} \sqrt{r_i}$, and the penalty function on each link be $B_l(y) = (max\{0, y - C_l\})^2$ for link $l = 1, 2$.

   In other words, we transform the constrained optimization problem

$$\max_r U(r) = \sum_{i=1}^{3} \sqrt{r_i}$$

$$s.t. \ \sum_{i=1}^{3} a_{li} r_i \leq C_l, \ \ l = 1, 2.$$

   into an unconstrained problem

$$\max_r F(r) = \sum_{i=1}^{3} \sqrt{r_i} - \sum_{l=1}^{2} B_l(\sum_{i=1}^{3} a_{li} r_i)$$

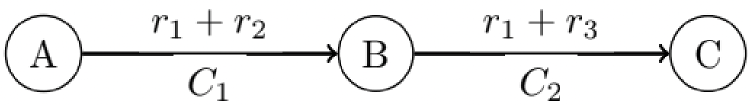
   where the routing matrix $A = [a_{ij}]$ is determined by the topology as shown in Figure 3.

   Using the gradient descent (ascent) method with the above mentioned utility and penalty functions and the step size of k = 0.1, obtain improved rate allocations with the following starting values:

   a) $\mathbf{r} = (1, 2, 1)$

   b) $\mathbf{r} = (6, 6, 4)$

Figure 3: Traffic flows for this problem



c) $\mathbf{r} = (8,\ 4,\ 6)$.

A: Let us define $g(\hat{r}) = \sum_{i=1}^{3} \sqrt{r_i} - B_1(y_1) - B_2(y_2)$; where

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A. \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Then, to obtain the next point according to the gradient descent method, we need to compute the gradient of the function $g(\hat{r})$.

$$\nabla g(\underline{r}) = \begin{pmatrix} \frac{\partial g(\underline{r})}{\partial r_1} \\ \frac{\partial g(\underline{r})}{\partial r_2} \\ \frac{\partial g(\underline{r})}{\partial r_3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2\sqrt{r_1}} - \frac{\partial B_1}{\partial r_1} - \frac{\partial B_2}{\partial r_1} \\ \frac{1}{2\sqrt{r_2}} - \frac{\partial B_1}{\partial r_2} - \frac{\partial B_2}{\partial r_2} \\ \frac{1}{2\sqrt{r_1}} - \frac{\partial B_1}{\partial r_3} - \frac{\partial B_2}{\partial r_3} \end{pmatrix}$$

and we have for $j = 1, 2$

$$\frac{\partial B_j(y_j)}{\partial y_j} = \begin{cases} 0, & \text{if } y_j < C_j. \\ 2(y_j - C_j), & \text{otherwise.} \end{cases}$$

Then we use chain rule to obtain

$$\frac{\partial B_j(y_j)}{\partial r_i} = \frac{\partial B_j(y_j)}{\partial y_j} \frac{\partial y_j}{\partial r_i} = \frac{\partial B_j(y_j)}{\partial y_j} a_{ji}$$

for $j = 1, 2$ and $i = 1, 2, 3$ to get the final expression of the gradient (Here aji is the (j, i) element of the matrix A). Once we have the gradient, we can write the update of the gradient descent algorithm as :

$$\hat{r}^{new} = \hat{r}^{old} + \nabla g(\hat{r}^{old})$$

where k = 0.1 is the step size. Now we can write the updated rate allocations for the three cases:

(a) In this case, none of the constraints are violated. So the updated value of $\underline{r}$ is

$$\underline{r}^{new} = \underline{r}^{old} + 0.1 \begin{pmatrix} 1/2 \\ 1/2\sqrt{2} \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1.05 \\ 2.0354 \\ 1.05 \end{pmatrix}$$

(b) In this case, the first constraint is violated as $r_1 + r_2$ is larger than the capacity of the first link. So we have $\frac{\partial B_1}{\partial r_1} = \frac{\partial B_1}{\partial r_1} = 2 * (6 + 6 - 10) = 4$ which gives us:

$$\underline{r}^{new} = \underline{r}^{old} + 0.1 \begin{pmatrix} \frac{1}{2\sqrt{6}} - 4 \\ \frac{1}{2\sqrt{6}} - 4 \\ \frac{1}{2\sqrt{4}} \end{pmatrix} = \begin{pmatrix} 5.6204 \\ 5.6204 \\ 4.0250 \end{pmatrix}$$

(c) In this case, both the constraints are violated, and we have $\frac{\partial B_1}{\partial y_1} = 2(8 + 4 - 10) = 4$ and $\frac{\partial B_2}{\partial y_2} = 2(8 + 6 - 12) = 4$. So we have the following:

$$\underline{r}^{new} = \underline{r}^{old} + 0.1 \begin{pmatrix} \frac{1}{2\sqrt{8}} - 4 - 4 \\ \frac{1}{2\sqrt{4}} - 4 - 0 \\ \frac{1}{2\sqrt{6}} - 0 - 4 \end{pmatrix} = \begin{pmatrix} 7.2177 \\ 3.6250 \\ 5.6204 \end{pmatrix}$$