HW3

Saturday, November 5, 2022 8:22 PM

1. ~~① time slots $t \in \{0, 1, \dots\}$.~~

② $P(ON) = p$; $P(OFF) = 1-p$

③ $P(\text{arrive}) = \lambda$; $P(\text{not arrive}) = 1-\lambda$

④ queue : FIFO

(a) T : the time required for packet at the head-of-line of the queue to be transmitted successfully.

$$P(T=t) = (1-p)^{t-1}$$

pdf of T:
$$F_T(t) = \sum_{k=0}^{t} (1-p)^{k-1} p = p \, \frac{1-(1-p)^t}{p} = 1-(1-p)^t$$

$$E[T] = \sum_{t=1}^{\infty} t (1-p)^{t-1} p = p\left(-\frac{d}{dp} \sum_{t=1}^{\infty} (1-p)^t\right) = p\left(-\frac{d}{dp}\frac{1}{p}\right) = p\left(\frac{1}{p^2}\right) = \frac{1}{p}$$

$\therefore$ the service rate $\mu = \frac{1}{E[T]} = p$.

To make the queue stable, $\mu > \lambda$

$\therefore \lambda < p$

(b) $\because$ ① the packet memorylessly arrives at rate λ.

② a packet takes a random transmission time with probability p.

③ arrival and departure are i.i.d.

$\therefore$ we can treat it as a M/G/1 queue.

$\therefore$ mean queue length $L = \dfrac{\lambda E[T]^2}{2(1-\lambda/\mu)} = \dfrac{\lambda E[T^2]}{2(1-\lambda/p)}$.

$$E[T^2] = \sum_{t=1}^{\infty} t^2 (1-p)^{t-1} p$$

$$= \frac{p^2 - 3p + 2}{(1-p)p^2} = \frac{2-p}{p^2}.$$

$\therefore$ mean wait time $\bar{T}_{wait} = \dfrac{L}{\lambda}$

$$= \frac{E[T^2]}{2(1-\lambda/p)}$$

$$= \frac{2-p}{2p^2(1-\lambda/p)}$$

$$= \frac{2-p}{2(p^2-\lambda p)}$$

(c) $\because$ the time out period is two time slot.

$\therefore$ ① successful transmission at $T=1$: $P(T=1) = p$.

② ... at $T=3$. $\quad P(T=3) = p(1-p)$.

③ ... at $T=5$ $\quad P(T=5) = p(1-p)^2$

$\vdots$

... at $T=2k-1$ $\quad P(T=2k-1) = p(1-p)^{k-1}$. $k \in \mathbb{Z}^+$

$$\therefore E[T] = \sum_{k=1}^{\infty} (2k-1)(1-p)^{k-1} p$$

$$= p \sum_{k=1}^{\infty} \left[2k(1-p)^{k-1} - (1-p)^{k-1}\right].$$

$$= p\left[-2\frac{d}{dp}\sum_{k=1}^{\infty}(1-p)^k - \sum_{k=1}^{\infty}(1-p)^{k-1}\right].$$

$$= p\left(\frac{2}{p^2} - \frac{1}{p}\right)$$

$$= \frac{2-p}{p}.$$

$\therefore$ similar to (a), $\lambda < \dfrac{1}{E[T]} = \dfrac{p}{2-p}$.

2. (a) Assume $N=3$.

suppose queues matrix

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix},$$

consider one of the matching matrice

$$S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

$\because$ only X_{11}, X_{32} are selected

$\therefore$ the size of $S_1 = 2 < N=3$.

$W_{S_1} = \sum_{ij} X_{ij} S_{ij} = 1+2 = 3$.

Consider another one of the matching matrice

$$S_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\because$ X_{11}, X_{22}, X_{33} are selected

$\therefore$ the size of $S_2 = 3 = N$

$W_{S_2} = 1+1+1 = 3$.

from above, both S_1 and S_2 have the maximum Weight $=3$. but the size of S_1 < maximum size $=3$.

$\therefore$ this algorithm can result in a matching which does not have the maximum size.

(b) Prob. of picking any matching matrix : $P(S = S_k) = \frac{1}{N!}$

By this algorithm, we at most have d unique matching matrice each time. Therefore, we can at most cover $d \cdot N$ VOQs.

Therefore, at least $N^2 - d \cdot N$ VOQs are not served.

$$P_{min}(VOQ_{ij} \text{ is not served}) = \binom{N^2}{1}\left(\frac{N^2 - dN}{N^2}\right) \cdot \left(\frac{d}{N!}\right)$$

$$= (N^2 dN) \cdot \frac{d}{N!}.$$

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