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JACOBS SCHOOL OF ENGINEERING

ECE/SIOC 228 Machine Learning for Physical Applications
Lecture 13: Markov Decision Process II

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Reference & Acknowledgement: David Silver's Course on RL https://www.davidsilver.uk/teaching/
Richard Sutton and Andrew Barto, Reinforcement Learning: An Introduction http://incompleteideas.net/book/the-book-2nd.html

Outline

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- **Markov Decision Process**
- Value Iteration
- Policy Iteration

Markov Decision Process

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- A Markov Decision Process in a tuple $< S, A, P, R, \gamma >$
  - S is a finite set of states
  - A is a finite set of actions
  - P is a state transition probability matrix, $P^a_{ss'} = P[S_{t+1} = s' | S_t = s, A_t = a]$
  - R is a reward function, $R^a_s = R(R_{t+1} | S_t = s, A_t = a)$
  - γ is a discount factor $\gamma \in [0, 1]$

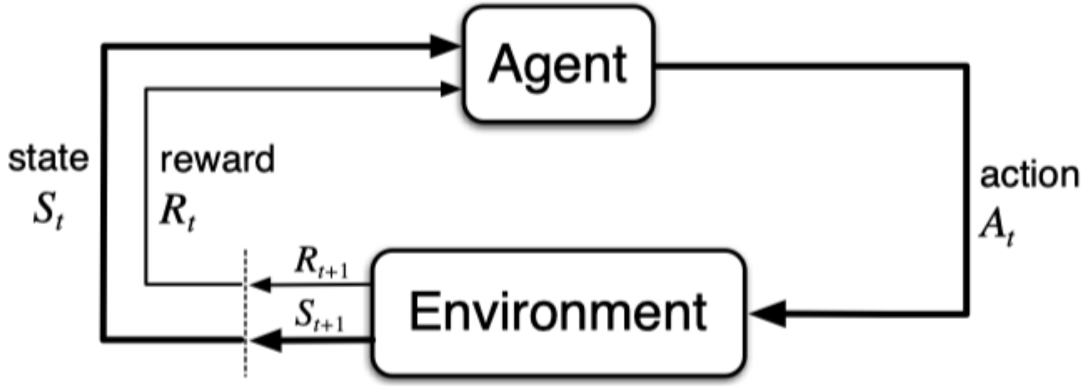

Figure 3.1: The agent–environment interaction in a Markov decision process.

Markov Decision Process Example

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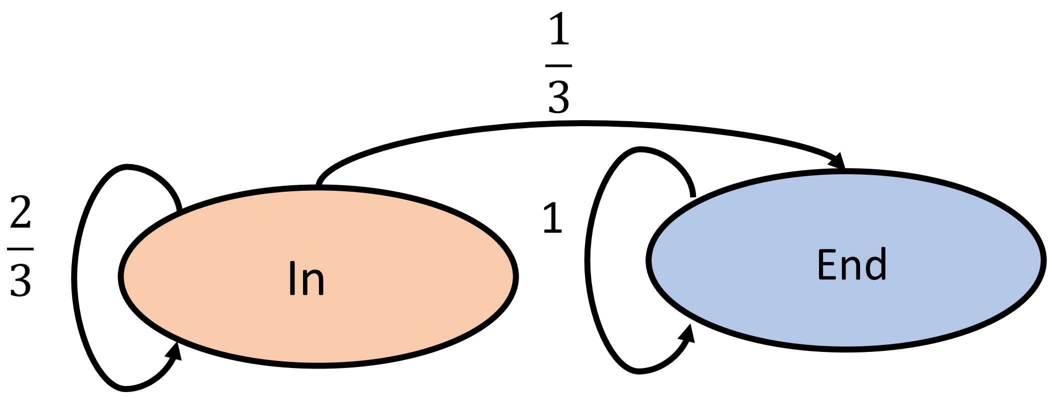

Dice Game Markov Chain

For each round $r = 1,2, \ldots$

- You choose stay or quit
- If quit, you get $\$10$ and the game ends
- If stay, you get $\$4$ and then we roll a 6-sided dice
  - If the dice results in 1 or 2, game ends
  - Otherwise, game continues to the next round

Markov Decision Process Example

Policy

- **A policy $\boldsymbol{\pi}$** is a mapping from each state $s \in S$ to an action $a \in A$

$$\pi[a|s] = P[A_t = a|S_t = s]$$

- A policy full defines the behavior of an agent
- In MDP, policies only depend on the current state S_t, not the history
- Policy is stationary (time independent), $A_t \sim \pi(\cdot\,|S_t), \forall t > 0$
- Policy can be both stochastic and deterministic

Example: policy "stay" // policy "quit" // policy "50% stay, 50% quit"

Episode

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- **An episode** is a sequence of states, actions and rewards

$$S_t, A_t, R_{t+1}, S_{t+1}, A_{t+1}, R_{t+2}, S_{t+2}, A_{t+2}, R_{t+3}, S_{t+4}, \ldots\ldots$$

Sample **episodes** starting from $\mathrm{S}_1 = \mathrm{In}$

- [In (stay), End] $4
- [In (stay), In (stay), In (stay), End] $12
- [In (stay), In (stay), End] $8
- [In (stay), In (stay), In (stay), In (stay), End] $16
- ……

- Episode with finite length (episodic) v.s. infinite length (continuing tasks)
- Episodes are independent from each other

Return

- **Each episode is associated with a return $\boldsymbol{G_t}$**, that is the total discounted reward from time step t

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$$

- The discount $\gamma \in [0, 1]$ is the present value of future rewards
- The value of receiving reward R after $k + 1$ time-steps is $\gamma^k R$
- This values immediate reward above delayed reward
  - γ close to 0 leads to "greedy" evaluation (only care about current step)
  - γ close to 1 leads to "far-sighted" evaluation

Return

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- **The return G_t** is the total discounted reward from time step t

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$$

For each round r = 1,2, ...

- You choose stay or quit
- If quit, you get \$10 and the game ends
- If stay, you get \$4 and then we roll a 6-sided dice
  - If the dice results in 1 or 2, game ends
  - Otherwise, game continues to the next round

Discount $\gamma = 1$: [In (stay), In (stay), In (stay), In (stay), End] $4 + 4 + 4 + 4 = \$16$

Discount $\gamma = 0$: [In (stay), In (stay), In (stay), In (stay), End] $4 + 0 \cdot 4 + 0 \cdot 4 + 0 \cdot 4 = \$4$

Discount $\gamma = 1/2$: [In (stay), In (stay), In (stay), In (stay), End] $4 + 0.5 \cdot 4 + 0.5^2 \cdot 4 + 0.5^3 \cdot 4 = \$7.5$

Why Discount?

- Mathematically convenient to use discount rewards
- Animal/human behavior shows preference for immediate reward
- If the reward is financial, immediate rewards may earn more interest than delayed rewards
- Uncertainty about the future may not be fully represented
- Sometimes, it is possible to use undiscounted reward if all possible episodes are finite
- ……

Value function

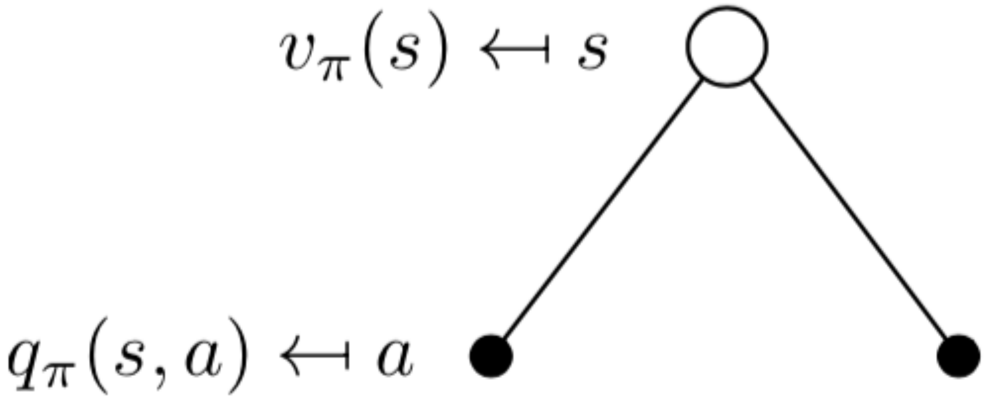
The **state-value function $v_\pi(s)$ of a policy π** in an MDP is the expected return starting from state s, and then following policy π

$$v_\pi(s) = \mathbb{E}_\pi[G_t | S_t = s]$$

The **state-action value function $q_\pi(s, a)$ of a policy π** (sometimes also refer to as action value function) is the expected return starting from state s, taking action a, and then following policy π

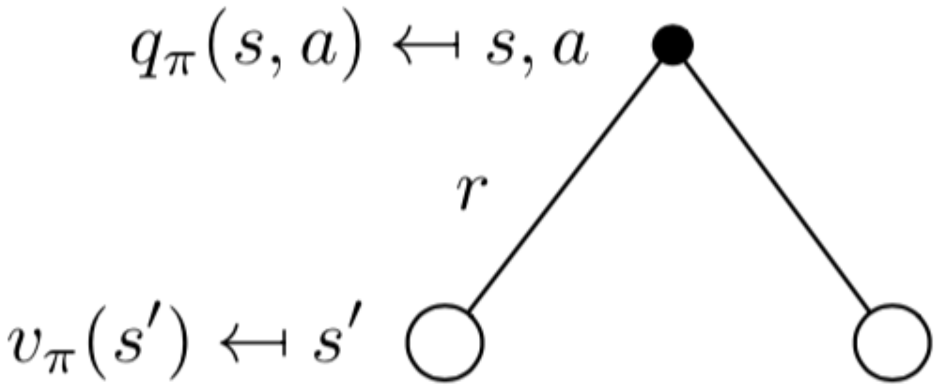
$$q_\pi(s, a) = \mathbb{E}_\pi[G_t | S_t = s, A_t = a]$$

Value function



$$v_\pi(s) = \sum_{a \in \mathcal{A}} \pi(a|s) q_\pi(s,a)$$

Value function



$$q_\pi(s, a) = \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_\pi(s')$$

Value function

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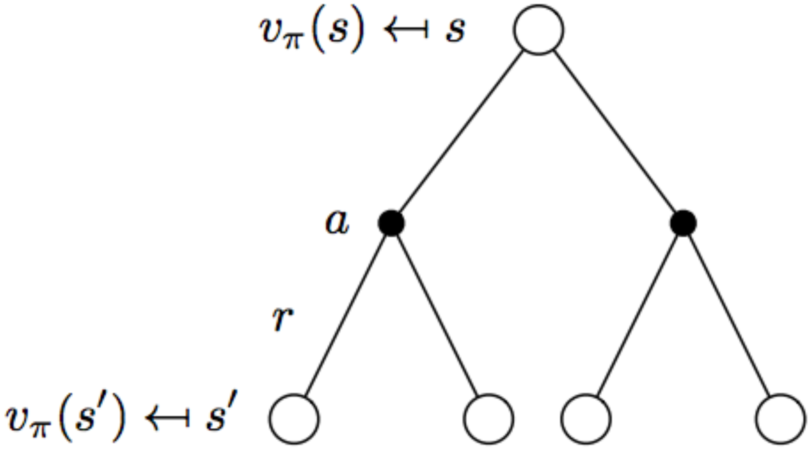

$$v_\pi(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_\pi(s') \right)$$

Policy Evaluation

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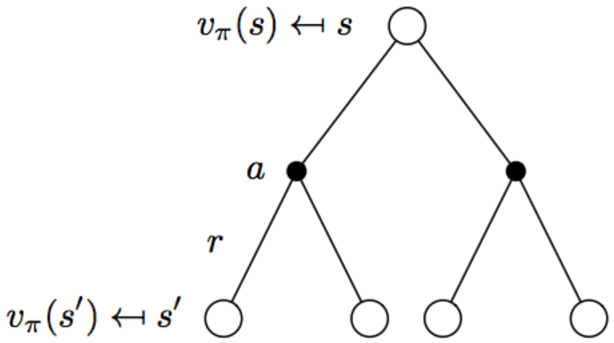
- **Given a policy π, how can we evaluate it, i.e., compute $v_\pi(s)$ and $q_\pi(s, a)$?**
- Solve a linear system



$$v_\pi(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_\pi(s') \right)$$

$$v_\pi = \mathcal{R}^\pi + \gamma \mathcal{P}^\pi v_\pi$$

$$\textit{where } R^\pi(s) = \textstyle\sum_{a \in \mathcal{A}} \pi(a|s)\, R_s^a,\ \mathcal{P}^\pi(s'|s) = \textstyle\sum_{a \in \mathcal{A}} \pi(a|s)\, P(s'|s, a)$$

$$\begin{bmatrix} v_\pi(1) \\ v_\pi(2) \\ \vdots \\ v_\pi(n) \end{bmatrix} = \begin{bmatrix} R^\pi(1) \\ R^\pi(2) \\ \vdots \\ R^\pi(n) \end{bmatrix} + \gamma \begin{bmatrix} \mathcal{P}_{11}^\pi & \mathcal{P}_{12}^\pi & \cdots & \mathcal{P}_{1n}^\pi \\ \mathcal{P}_{21}^\pi & \mathcal{P}_{22}^\pi & \cdots & \mathcal{P}_{2n}^\pi \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{P}_{n1}^\pi & \mathcal{P}_{n2}^\pi & \cdots & \mathcal{P}_{nn}^\pi \end{bmatrix} \begin{bmatrix} v_\pi(1) \\ v_\pi(2) \\ \vdots \\ v_\pi(n) \end{bmatrix}$$

Policy Evaluation

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- The Bellman Equation is a linear equation:

$$v_\pi = \mathcal{R}^\pi + \gamma \mathcal{P}^\pi v_\pi$$

- We can solve it directly by,

$$(I - \gamma \mathcal{P}^\pi) v = \mathcal{R}^\pi$$
$$v = (I - \gamma \mathcal{P}^\pi)^{-1} \mathcal{R}^\pi$$

- Direct solution only possible for small MDPs

- **Method 2: Iterative methods for large MDPs,**

$$v_\pi^{(k+1)}(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \left(\mathcal{R}_s^a + \gamma \sum_{s' \in S} P_{ss'}^a v_\pi^{(k)}(s')\right)$$

$v_\pi^{(k+1)} = \mathcal{R}^\pi + \gamma \mathcal{P}^\pi v_\pi^{(\mathrm{k})}, \quad v_\pi^{(1)} \to v_\pi^{(2)} \to \cdots \to v_\pi$ (contraction mapping → convergence)

Summary

- Markov Decision Process $< S, A, P, R, \gamma >$
  - State
  - Action
  - Transition Probability
  - Reward (and discount factor)
- Policy
- Return
- Value function: state value function $v_\pi(s)$, action value function $q_\pi(s, a)$

Outline

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- Markov Decision Process
- **Value Iteration**
- Policy Iteration

Optimal Value Function

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- The optimal state-value function $v_*(s)$ is the maximum value function over all policies

$$v_*(s) = \max_{\pi} v_{\pi}(s)$$

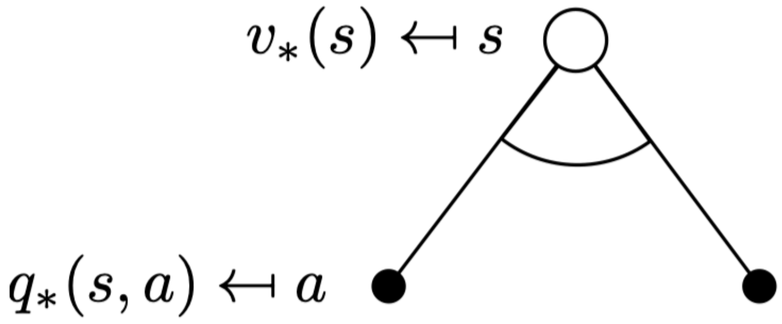
- The optimal state-action value function $q_*(s, a)$ is the maximum state-action value function over all policies

$$q_*(s, a) = \max_{\pi} q_{\pi}(s, a)$$

- An MDP is "solved" when we know the optimal value function
- Notice if we have $q_*(s, a)$, we can obtain the optimal policy π_*: $\pi_*(s) = \arg\max_{a \in \mathcal{A}} q_*(s, a), \forall s \in S$
- **Value iteration and policy iteration are two ways to "solve" the MDP, a.k.a. find the optimal policy (or equivalently, find the optimal value function)**

Bellman Optimality Equations

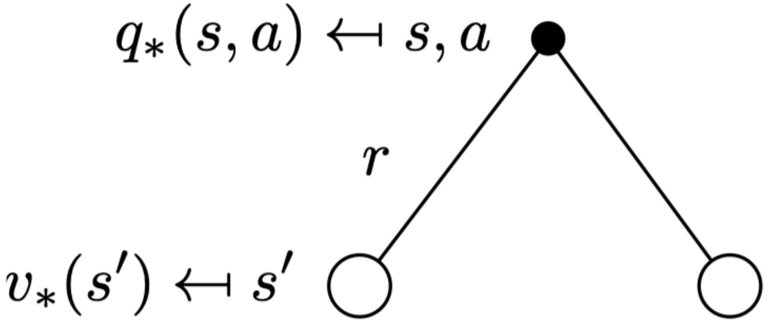
- The optimal value functions are recursively related by the Bellman optimality equations:



$$v_*(s) = \max_a q_*(s,a)$$

Bellman Optimality Equations

- The optimal value functions are recursively related by the Bellman optimality equations:



$$q_*(s, a) = \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_*(s')$$

Bellman Optimality Equations

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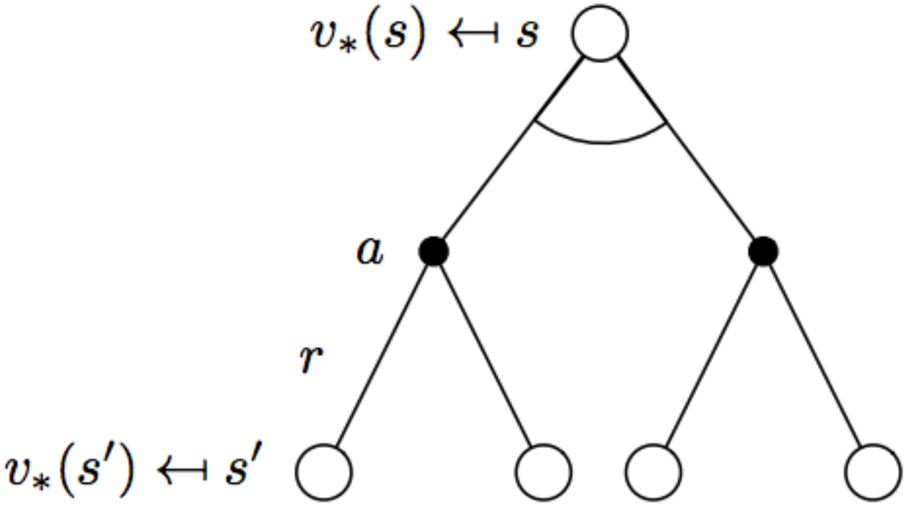
- The optimal value functions are recursively related by the Bellman optimality equations:



Value Iteration

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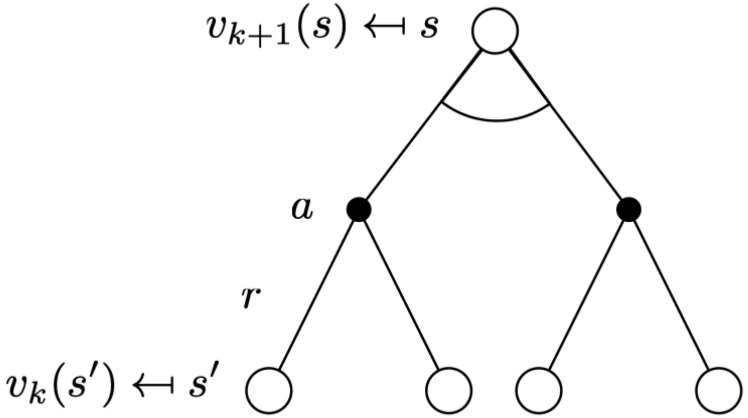
- Goal: find the optimal value function
- Solution: iterative application of the Bellman optimality equation
- $v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_*$



$$v_{k+1}(s) = \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_k(s') \right)$$

Convergence of Value Iteration

Theorem. Value iteration converges to v_*. At convergence,

$$\forall s \in S,\ v_*(s) = \max_{a\in\mathcal{A}}\left(R_s^a + \gamma \sum_{s'\in\mathcal{S}} \mathcal{P}_{ss'}^a v_*(s')\right)$$

- Proof is left as Homework (HW2 Q3)
- By first showing the Bellman optimality operator $Bell(v) = \max_{a\in\mathcal{A}}\left(R_s^a + \gamma \sum_{s'\in\mathcal{S}} \mathcal{P}_{ss'}^a v(s')\right)$ is a contraction mapping.
- Show the fixed point is unique.
- We can also derive the optimal policy from the optimal value function v_*

Illustration of Value Iteration: Deterministic Environment

- Simple grid world with a "goal state" with reward 10 and a "bad state" with reward -10. Move 1 step incurs reward -1.
- Both the "goal state" and "bad state" are terminal state, value of terminal state equal to 0 (future expected return equal to 0).
- Actions at each state: *up, down, left, right*. Taking an action that would bump into a wall leaves agent where it is.
- Discount factor $\gamma = 0.9$

0	0	10
0	0	-10
0	0	0

Original reward function

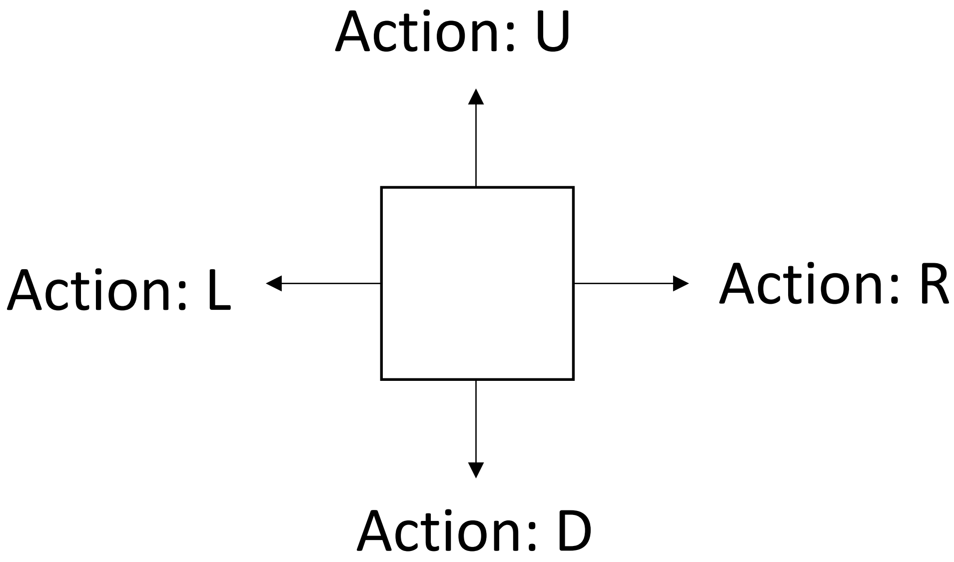


Illustration of Value Iteration

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0	0	0
0	0	0
0	0	0

Initialize value function: $v_0(s)$

-1	9	0
-1	-1	0
-1	-1	-1

Value function after 1 iteration: $v_1(s)$

$$v_{k+1}(s) = \max_{a \in \mathcal{A}} \left(R_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_k(s') \right)$$

Illustration of Value Iteration

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7.1	9	0
-1.9	7.1	0
-1.9	-1.9	-1.9

$v_2(s)$

7.1	9	0
5.39	7.1	0
-2.71	5.39	-2.71

$v_3(s)$

7.1	9	0
5.39	7.1	0
3.85	5.39	3.85

$v_4(s)$

7.1	9	0
5.39	7.1	0
3.85	5.39	3.85

$v_5(s)$

$$v_{k+1}(s) = \max_{a \in \mathcal{A}} \left(R_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_k(s') \right)$$

Illustration of Value Iteration

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7.1	9	0
5.39	7.1	0
3.85	5.39	3.85

$v_5(s)$

→	→	
↑	↑	
↑	↑	←

Best policy based on $v_5(s)$

An optimal policy can be found by maximizing over $q_*(s,a) = \mathcal{R}_s^a + \gamma \sum_{s' \in S} P_{ss'}^a v_*(s')$

$$\pi_*(a|s) = \begin{cases} 1 & \text{if } a = \operatorname*{argmax}_{a \in \mathcal{A}} q_*(s,a) \\ 0 & \text{otherwise} \end{cases}$$

Outline

- Markov Decision Process
- Value Iteration
- **Policy Iteration**

Policy Iteration

- Given a policy π
  - Evaluate the policy π → $v_\pi(s)$
  - Improve the policy by acting greedily with respect to v_π:

$$\pi'(s) \leftarrow \underset{a \in \mathcal{A}}{\operatorname{argmax}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in S} P(s'|s,a) v_\pi(s') \right) = \underset{a \in \mathcal{A}}{\operatorname{argmax}}\, q_\pi(s,a)$$

Policy Iteration Algorithm:

1. Initialize policy π (e.g., randomly)
2. Perform policy evaluation
3. Perform policy improvement , obtain policy π'
4. If policy changed in last iteration, return to Step 2

Policy Iteration

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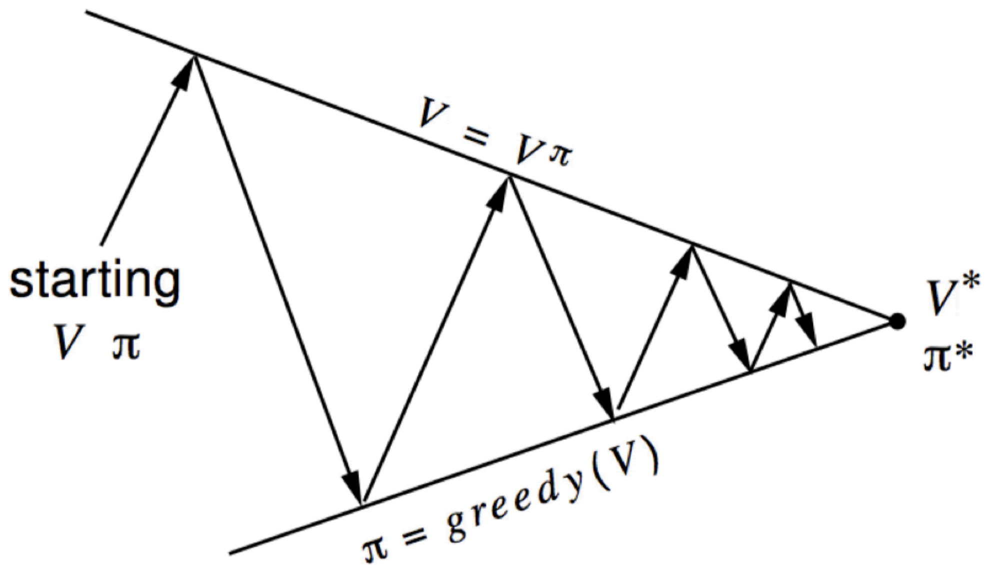
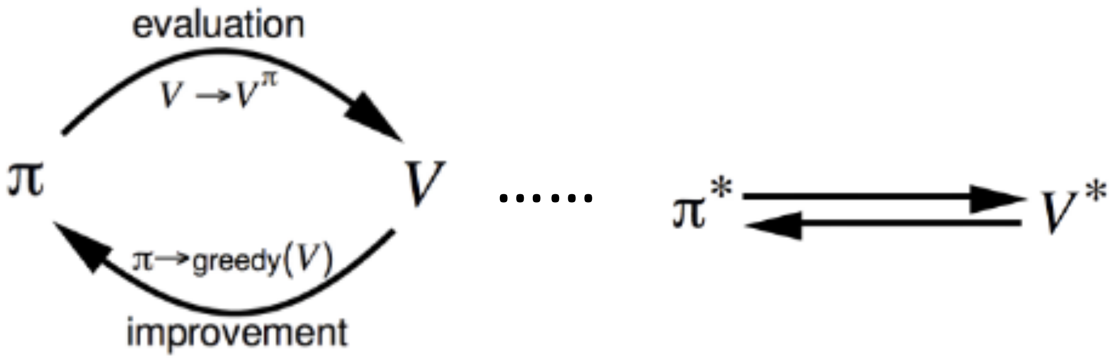


Policy evaluation Estimate v_π
Iterative policy evaluation

Policy improvement Generate $\pi' \geq \pi$
Greedy policy improvement

- Convergence property of policy iteration: $\pi \rightarrow \pi^*$
- Proof involves showing that each policy evaluation is a contraction and policy must improve each step, or be optimal policy (policy improvement theorem)



Policy Improvement Theorem

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- Consider a deterministic policy, $a = \pi(s)$
- We can improve the policy by acting greedily

$$\pi'(s) = \underset{a \in \mathcal{A}}{\operatorname{argmax}}\, q_\pi(s, a)$$

- This improves the value from any state s over one step,

$$q_\pi\big(s, \pi'(s)\big) = \max_{a \in \mathcal{A}} q_\pi(s, a) \geq q_\pi\big(s, \pi(s)\big) = v_\pi(s)$$

- It therefore improves the value function,

Policy Improvement Theorem

- If improvements stop,

$$q_\pi(s, \pi'(s)) = \max_{a \in \mathcal{A}} q_\pi(s, a) = q_\pi(s, \pi(s)) = v_\pi(s)$$

- The Bellman optimality equation has been satisfied,

$$v_\pi(s) = \operatorname*{argmax}_{a \in \mathcal{A}} q_\pi(s, a)$$

- Therefore, $v_\pi(s) = v_*(s)$ for all $s \in S$

- So π is am optimal policy

Illustration of Policy Iteration

0	0	10
0	0	-10
0	0	0

Original reward function

- Initialize with a policy $\pi(s) = [\mathrm{U}, \mathrm{D}, \mathrm{L}, \mathrm{R}]$ with prob 0.25

Illustration of Policy Iteration

-5.78 →	-1.97 →	0
-7.7 ↑	-7.69 ↑	0
-8.62 ↑	-8.93 ↑	-10.02 ←

v_π of the initial random policy

- Initialize with a policy $\pi(s) = [\mathrm{U}, \mathrm{D}, \mathrm{L}, \mathrm{R}]$ with prob 0.25
- Perform policy evaluation of policy π
- Perform policy improvement

$$\pi'(s) \leftarrow \underset{a \in \mathcal{A}}{\operatorname{argmax}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in S} P(s'|s,a) v_\pi(s') \right)$$

- **Find the optimal policy in 1 iteration!**
- Can also compute optimal value function by running policy evaluation of π_*

7.1	9	0
5.39	7.1	0
3.85	5.39	3.85

v_{π_*}

Policy Iteration or Value Iteration?

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- Policy iteration requires fewer iterations that value iteration, but each iteration requires solving a linear system instead of just applying Bellman operator

- For small MDPs, policy iteration is often faster

- For MDPs with large state spaces, solving for v_π explicitly would involve solving a large system of linear equations, and could be difficult. In these problems, value iteration may be preferred.

What is next?

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- **Planning** in Markov Decision Process (offline)
  - Have the environment model (reward, transition)
  - Find the optimal policy via value iteration and policy iteration

Today's lecture

- **Reinforcement Learning** in Markov Decision Process (online)
  - Don't know how the environment works
  - Find the optimal policy by interacting with the environment (taking actions and collect reward)

Next lectures