# Homework 3

Due: 8:00pm (PDT) Thursday, October 20, 2022

**Reading assignment:** Read Section 3.5 of Prof. Kim's notes on functions of a random variable before solving Problem 1.

1. Let $\{X_k\}_{k\geq 1}$ be a random process over an underlying probability space $(\Omega, \mathcal{F}, \mathbf{P}(\cdot))$. Show that $X = \inf_{k\geq 1} X_k$ is a random variable.

2. Problem 3.9 of Prof. Kim's notes.

3. Let $\{X_k\}$ to be a random process over an underlying probability space $(\Omega, \mathcal{F}, \mathbf{P}(\cdot))$.

   (a) For any $\alpha \in \mathbb{R}$, show that the event $E_\alpha$ where the limiting point of $X_k(\omega) = \alpha$ is an event in $\mathcal{F}$

   $$E_\alpha = \{\omega \in \Omega \mid \lim_{k\to\infty} X_k(\omega) = \alpha\}.$$

   (b) Show that the set $E$ of sample points that $X_k$ has limit is measurable (i.e., it is an event in $\mathcal{F}$):

   $$E = \{\omega \in \Omega \mid \lim_{k\to\infty} X_k(\omega) \text{ exists}\}.$$

4. Find a sequence of random variables (i.e., a random process) $\{X_k\}$ such that its almost sure limit exists and $\mathbb{E}[\lim_{k\to\infty} X_k] \neq \lim_{k\to\infty} \mathbb{E}[X_k]$. (if you cannot do it by yourself, do research on finding such random variables)

5. In the spirit of HW1-Problem 7, let $\{w_k\}$ be an i.i.d. random process that is uniformly distributed over $[-1, 1]$, i.e., they admit the PDF

   $$f(x) = \begin{cases} \frac{1}{2} & x \in [-1, 1] \\ 0 & \text{otherwise} \end{cases}. \tag{1}$$

   (a) Show that $P(\{|w_k| > \frac{1}{4} \text{ i.o.}\}) = 1$.

   (b) Using this and the definition of convergent series, show that the process $\{w_k\}$ is almost surely not summable, i.e., $\sum_{k=1}^{\infty} w_k$ does not exists with probability one.

6. Using the general definition and the properties of $\mathbb{E}[X]$ that we discussed in the class, show that for a non-negative discrete random variable $X$

   $$\mathbb{E}[X] = \sum_{k=1}^{\infty} m_k p_X(m_k),$$

   where $\mathbf{P}(X \in M) = 1$ and $M = \{m_k \mid k \geq 1\}$.

7. Let $X$ be a finite random variable on $(\Omega, \mathcal{F}, \mathbf{P})$ (i.e., $P(X = \infty) = P(X = -\infty) = 0$). Show that its distribution function $F_X$ satisfies the following properties:

   (a) $F_X$ is non-decreasing.

   (b) $\lim_{x\to-\infty} F_X(x) = 0$, and $\lim_{x\to\infty} F_X(x) = 1$.

   (c) $F_X(\cdot)$ **is right-continuous**, i.e., for any $x \in \mathbb{R}$, $\lim_{y\to x^+} F_X(y) = F_X(x)$.

   (d) Define $F_X(x^-) := \lim_{y\uparrow x} F_X(y)$, then

   $$F_X(x^-) = \mathbf{P}[X < x] = \mathbf{P}[\{\omega \in \Omega \mid X(\omega) < x\}].$$

   (e) For any $x \in \mathbb{R}$, we have $\mathbf{P}[X = x] = F_X(x) - F_X(x^-)$.