UNIVERSITY OF CALIFORNIA, SAN DIEGO

Electrical & Computer Engineering Department

ECE 250 - Winter Quarter 2018

Random Processes

Solutions to P.S. #3

1. *Geometric with conditions.* Let X be a geometric random variable with pmf

$$p_X(k) = p(1-p)^{k-1}, \;\; k = 1, 2, \ldots .$$

Find and plot the conditional pmf $p_X(k|A) = P\{X = k | X \in A\}$ if:

(a) $A = \{X > m\}$ where m is a positive integer.

(b) $A = \{X < m\}$.

(c) $A = \{X \text{ is an even number}\}$.

Comment on the shape of the conditional pmf of part (a).

Solution:

(a) We have

$$\begin{aligned}
P(A) &= \sum_{n=m+1}^{\infty} p(1-p)^{n-1} \\
&= \sum_{n=0}^{\infty} p(1-p)^{n+m} \\
&= p(1-p)^m \sum_{n=0}^{\infty} (1-p)^n \\
&= (1-p)^m.
\end{aligned}$$

For $k \le m$, $p_X(k|A) = 0$. For $k > m$,

$$\begin{aligned}
p_X(k|A) &= P\{X = k | X > m\} \\
&= \frac{P\{X = k\}}{P\{X > m\}} \\
&= \frac{p(1-p)^{k-1}}{(1-p)^m} \\
&= p(1-p)^{k-m-1}.
\end{aligned}$$

(b) We have

$$\begin{aligned}
P(A) &= \sum_{n=0}^{m-2} p(1-p)^n \\
&= p\frac{1-(1-p)^{m-1}}{1-(1-p)} \\
&= 1-(1-p)^{m-1}.
\end{aligned}$$

For $k \geq m$ or $k \leq 0$, $p_X(k|A) = 0$. For $0 < jk < m$,

$$\begin{aligned}
p_X(k|A) &= P\{X = k | X < m\} \\
&= \frac{P\{X = k\}}{P\{X < m\}} \\
&= \frac{p(1-p)^{k-1}}{1-(1-p)^{m-1}}.
\end{aligned}$$

(c) We have

$$\begin{aligned}
P(A) &= \sum_{n \text{ even}} p(1-p)^{n-1} \\
&= \sum_{n'=0}^{\infty} p(1-p)((1-p)^2)^{n'} \\
&= \frac{p(1-p)}{1-(1-p)^2} \\
&= \frac{1-p}{2-p}.
\end{aligned}$$

For k odd, $\mathsf{P}_X(k|A) = 0$. For k even,

$$\begin{aligned}
p_X(k|A) &= P\{X = k | X \text{ is even}\} \\
&= \frac{P\{X = k\}}{P\{X \text{ is even}\}} \\
&= \frac{p(1-p)^{k-1}}{P(A)} \\
&= p(2-p)(1-p)^{k-2}.
\end{aligned}$$

Plots are shown in Figure 1. The shape of the conditional pmf in part (a) shows that the geometric random variable is memoryless:

$$p_X(x|X > k) = p_X(x-k), \quad \text{for } x \geq k.$$

Note that in all three parts $p_X(x)$ is defined for *all* x. This is required.

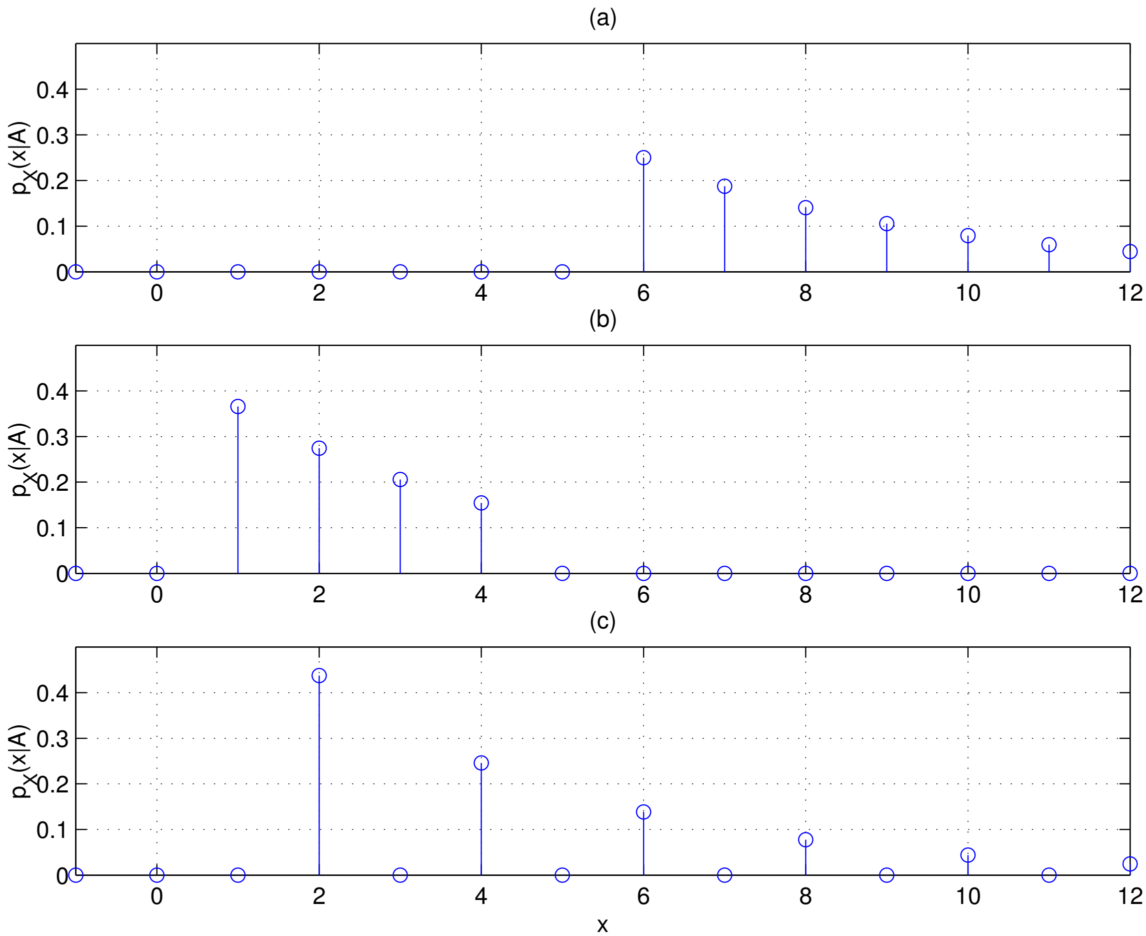


Figure 1: Plots of the conditional pmf's using $p = \frac{1}{4}$ and $m = 5$.

2. Let A be a nonzero probability event. Show that

   (a) $\mathsf{P}(A) = \mathsf{P}(A \,|\, X \le x) F_X(x) + \mathsf{P}(A \,|\, X > x)(1 - F_X(x))$.

   (b) $F_X(x \,|\, A) = \frac{\mathsf{P}(A|X \le x)}{\mathsf{P}(A)} F_X(x)$.

   Solution:

   (a) By law of total probability we have

$$\begin{aligned}\mathsf{P}(A) &= \mathsf{P}(A \,|\, X \le x)\mathsf{P}(X \le x) + \mathsf{P}(A \,|\, X > x)\mathsf{P}(X > x) \\ &= \mathsf{P}(A \,|\, X \le x) F_X(x) + \mathsf{P}(A \,|\, X > x)(1 - F_X(x)).\end{aligned}$$

   (b)

$$F_X(x \,|\, A) = \mathsf{P}(X \le x \,|\, A) = \frac{\mathsf{P}(A \,|\, X \le x)}{\mathsf{P}(A)} \mathsf{P}(X \le x) = \frac{\mathsf{P}(A \,|\, X \le x)}{\mathsf{P}(A)} F_X(x).$$

3. *Joint cdf or not.* Consider the function

$$G(x,y) = \begin{cases} 1 & \text{if } x+y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Can G be a joint cdf for a pair of random variables? Justify your answer.

Solution: No. Note that for every x,

$$\lim_{y\to\infty} G(x,y) = 1.$$

But for any genuine marginal cdf,

$$\lim_{x\to-\infty} F_X(x) = 0 \neq 1\,.$$

Therefore $G(x,y)$ is *not* a cdf. Alternatively, assume that $G(x,y)$ is a joint cdf for X and Y, then

$$\begin{aligned} \mathsf{P}\{-1 < X \leq 2, -1 < Y \leq 2\} &= G(2,2) - G(-1,2) - G(2,-1) + G(-1,-1) \\ &= 1 - 1 - 1 + 0 = -1\,. \end{aligned}$$

But this violates the property that the probability of any event must be nonnegative.

4. *Time until the n-th arrival.* Let the random variable $N(t)$ be the number of packets arriving during time $(0,t]$. Suppose $N(t)$ is Poisson with pmf

$$p_N(n) = \frac{(\lambda t)^n}{n!}e^{-\lambda t} \qquad \text{for } n = 0,1,2,\ldots.$$

Let the random variable Y be the time to get the n-th packet. Find the pdf of Y.

Solution: To find the pdf $f_Y(t)$ of the random variable Y, note that the event $\{Y \leq t\}$ occurs iff the time of the nth packet is in $[0,t]$, that is, iff the number $N(t)$ of packets arriving in $[0,t]$ is at least n. Alternatively, $\{Y > t\}$ occurs iff $\{N(t) < n\}$. Hence, the cdf $F_Y(t)$ of Y is given by

$$F_Y(t) = \mathsf{P}\{Y \leq t\} = P\{N(t) \geq n\} = \sum_{k=n}^{\infty} \frac{(\lambda t)^k}{k!}e^{-\lambda t}.$$

Differentiating $F_Y(t)$ with respect to t, we get the pdf $f_Y(t)$ as

$$\begin{aligned} f_Y(t) &= \sum_{k=n}^{\infty}\left[-\lambda e^{-\lambda t}\frac{(\lambda t)^k}{k!} + \lambda e^{-\lambda t}\frac{(\lambda t)^{k-1}}{(k-1)!}\right] \\ &= \lambda e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!} - \sum_{k=n}^{\infty}\lambda e^{-\lambda t}\frac{(\lambda t)^k}{k!} + \sum_{k=n+1}^{\infty}\lambda e^{-\lambda t}\frac{(\lambda t)^{k-1}}{(k-1)!} \\ &= \lambda e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!} \end{aligned}$$

for $t > 0$.

Or we can use another way. Since we know that the time interval T between packet arrivals is an exponential random variable with pdf

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & \text{if } t \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Let T_i denote the i.i.d. exponential interarrival times, then $Y = T_1 + T_2 + \cdots + T_n$. By convolving $f_T(t)$ with itself $n-1$ times, which can be also computed by its Fourier transform (characteristic function), we can show that the pdf of Y is given by

$$f_Y(t) = \begin{cases} \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}, & \text{if } t \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

5. *Diamond distribution.* Consider the random variables X and Y with the joint pdf

$$f_{X,Y}(x,y) = \begin{cases} c & \text{if } |x| + |y| \leq 1/\sqrt{2} \\ 0 & \text{otherwise,} \end{cases}$$

where c is a constant.

(a) Find c.
(b) Find $f_X(x)$ and $f_{X|Y}(x|y)$.
(c) Are X and Y independent random variables? Justify your answer.
(d) Define the random variable $Z = (|X| + |Y|)$. Find the pdf $f_Z(z)$.

Solution:

(a) The integral of the pdf $f_{X,Y}(x,y)$ over $-\infty < x < \infty, -\infty < y < \infty$ is c, and therefore by the definition of joint density

$$c = 1.$$

(b) The marginal pdf is obtained by integrating the joint pdf with respect to y. For $0 \leq x \leq \frac{1}{\sqrt{2}}$,

$$f_X(x) = \int_{-\frac{1}{\sqrt{2}}+x}^{\frac{1}{\sqrt{2}}-x} c\, dy = 2\left(\frac{1}{\sqrt{2}} - x\right),$$

and for $-\frac{1}{\sqrt{2}} \leq x \leq 0$,

$$f_X(x) = \int_{-\frac{1}{\sqrt{2}}-x}^{\frac{1}{\sqrt{2}}+x} c\, dy = 2\left(\frac{1}{\sqrt{2}} + x\right).$$

So the marginal pdf may be written as

$$f_X(x) = \begin{cases} \sqrt{2} - 2|x| & |x| \leq \frac{1}{\sqrt{2}} \\ 0 & \text{otherwise.} \end{cases}$$

Now since $f_{XY}(x,y)$ is symmetrical, $f_Y(y) = f_X(y)$. Thus,

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ &= \begin{cases} \frac{1}{\sqrt{2}-2|y|} & |x|+|y| \le \frac{1}{\sqrt{2}},\ |y| \le \frac{1}{\sqrt{2}} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(c) X and Y are not independent since

$$f_{X,Y}(x,y) \neq f_X(x) f_Y(y).$$

Alternatively, X and Y are not independent since $f_{X|Y}(x|y)$ depends on the value of y.

(d) We have, for $0 \le z < 1/\sqrt{2}$,

$$\begin{aligned} F_Z(z) &= \mathsf{P}(Z \le z) \\ &= \mathsf{P}(|X|+|Y| \le z) \\ &= \int_{-z}^{z} \mathsf{P}\Big(|X|+|Y| \le z \Big| Y = y\Big) f_Y(y) dy \\ &= \int_{-z}^{z} \mathsf{P}\Big(|X| \le z - |y| \Big| Y = y\Big) f_Y(y) dy \\ &= \int_{-z}^{z} \int_{-(z-|y|)}^{(z-|y|)} f_{X|Y}(x|y) f_Y(y) dx dy \\ &= \int_{-z}^{z} \frac{2(z-|y|)}{\sqrt{2}-2|y|} (\sqrt{2}-2|y|) dy \\ &= 4 \int_{0}^{z} (z-y) dy \\ &= 2z^2. \end{aligned}$$

Thus, $f_Z(z) = \begin{cases} 4z, & z \in (0, 1/\sqrt{2}) \\ 0, & \text{otherwise.} \end{cases}$

6. *Coin with random bias.* You are given a coin but are not told what its bias (probability of heads) is. You are told instead that the bias is the outcome of a random variable $P \sim \text{Unif}[0,1]$. To get more infromation about the coin bias, you flip it independently 10 times. Let X be the number of heads you get. Thus $X \sim \text{B}(10, P)$. Assuming that $X = 9$, find and sketch the *a posteriori* probability of P, i.e., $f_{P|X}(p|9)$.

Solution: In order to find the conditional pdf of P, apply Bayes' rule for mixed random variables to get

$$f_{P|X}(p|x) = \frac{p_{X|P}(x|p)}{p_X(x)} f_P(p) = \frac{p_{X|P}(x|p)}{\int_0^1 p_{X|P}(x|p) f_P(p)\, dp} f_P(p).$$

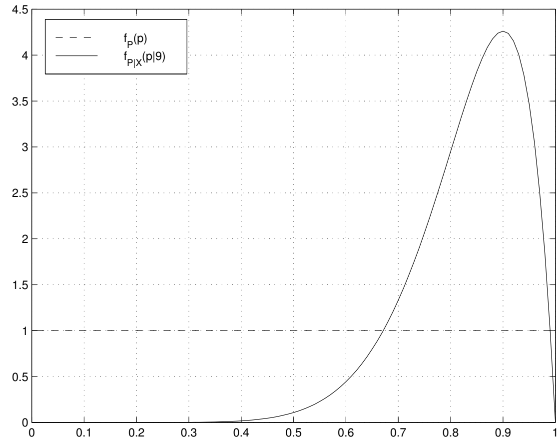


Figure 2: Comparison of *a priori* and *a posteriori* pdfs of P

Now it is given that $X = 9$, thus for $0 \le p \le 1$

$$\begin{aligned} f_{P|X}(p|9) &= \frac{p^9(1-p)}{\int_0^1 p^9(1-p)\,dp} \\ &= \frac{p^9(1-p)}{\frac{1}{110}} \\ &= 110p^9(1-p). \end{aligned}$$

Figure 2 compares the unconditional and the conditional pdfs for P. It may be seen that given the information that 10 independent tosses resulted in 9 heads, the pdf is shifted towards the value $\frac{9}{10}$.

7. *First available teller.* Consider a bank with two tellers. The service times for the tellers are independent exponentially distributed random variables $X_1 \sim \text{Exp}(\lambda_1)$ and $X_2 \sim \text{Exp}(\lambda_2)$ respectively. You arrive at the bank and find that both tellers are busy but that nobody else is waiting to be served. You are served by the first available teller once he/she is free.

   (a) What is the probability that you are served by the first teller?

   (b) Let the random variable Y denote your waiting time. Find the pdf of Y.

   Solution:

   (a) From the memoryless property of the exponential distribution, the remaining services for the tellers are also independent exponentially distributed random variables with parameters λ_1 and λ_2, respectively. The probability that you will be served by the first teller is the probability that the first teller finishes the service before the second teller does. Thus,

$$\begin{aligned}
\mathsf{P}\{X_1 < X_2\} &= \int_{x_2 > x_1} f_{X_1,X_2}(x_1,x_2)\,dx_2\,dx_1 \\
&= \int_{x_1=0}^{\infty}\int_{x_2=x_1}^{\infty} \lambda_1 e^{-\lambda_1 x_1}\lambda_2 e^{-\lambda_2 x_2}\,dx_2\,dx_1 \\
&= \int_{x_1=0}^{\infty} \lambda_1 e^{-(\lambda_1+\lambda_2)x_1}\,dx_1 \\
&= \frac{\lambda_1}{\lambda_1+\lambda_2}.
\end{aligned}$$

(b) First observe that $Y = \min(X_1, X_2)$. Since

$$\begin{aligned}
\mathsf{P}\{Y > y\} &= \mathsf{P}\{X_1 > y, X_2 > y\} \\
&= \mathsf{P}\{X_1 > y\}\mathsf{P}\{X_2 > y\} \\
&= e^{-\lambda_1 y} \times e^{-\lambda_2 y} \\
&= e^{-(\lambda_1+\lambda_2)y}
\end{aligned}$$

for $y \ge 0$, Y is an exponential random variable with pdf

$$f_Y(y) = \begin{cases} (\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)y}, & y \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

8. *Two independent uniform random variables.*
Let X and Y be independently and uniformly drawn from the interval $[0,1]$.

(a) Find the pdf of $U = \max(X,Y)$.
(b) Find the pdf of $V = \min(X,Y)$.
(c) Find the pdf of $W = U - V$.
(d) Find the probability $\mathsf{P}\{|X-Y| \ge 1/2\}$.

Solution:

(a) We have

$$\begin{aligned}
F_U(u) &= \mathsf{P}\{U \le u\} \\
&= \mathsf{P}\{\max(X,Y) \le u\} \\
&= \mathsf{P}\{X \le u, Y \le u\} \\
&= \mathsf{P}\{X \le u\}\mathsf{P}\{Y \le u\} \\
&= u^2
\end{aligned}$$

for $0 \le u \le 1$. Hence,

$$f_U(u) = \begin{cases} 2u, & 0 \le u \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Similarly,

$$\begin{aligned}1 - F_V(v) &= \mathsf{P}\{V > v\}\\ &= \mathsf{P}\{\min(X,Y) > v\}\\ &= \mathsf{P}\{X > v, Y > v\}\\ &= \mathsf{P}\{X > v\}\mathsf{P}\{Y > v\}\\ &= (1-v)^2,\end{aligned}$$

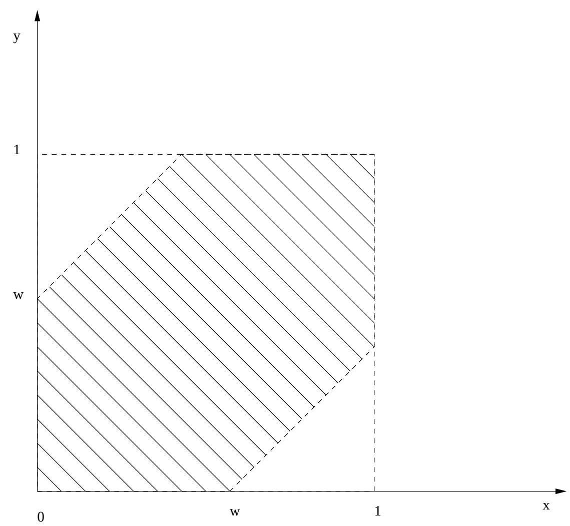
or equivalently, $F_V(v) = 1 - (1-v)^2$, for $0 \le v \le 1$. Hence,

$$f_V(v) = \begin{cases} 2(1-v), & 0 \le v \le 1,\\ 0, & \text{otherwise.}\end{cases}$$

(c) First note that $W = U - V = |X - Y|$. (Why?) Hence,

$$\begin{aligned}\mathsf{P}\{W \le w\} &= \mathsf{P}\{|X - Y| \le w\}\\ &= P(-w \le X - Y \le w).\end{aligned}$$

Since X and Y are uniformly distributed over $[0,1]$, the above integral is equal to the area of the shaded region in the following figure:



The area can be easily calculated as $1 - (1-w)^2$ for $0 \le w \le 1$. Hence $F_W(w) = 1 - (1-w)^2$ and

$$f_W(w) = \begin{cases} 2(1-w), & 0 \le w \le 1,\\ 0, & \text{otherwise.}\end{cases}$$

(d) From the figure above,

$$\mathsf{P}\{|X - Y| \ge 1/2\} = \mathsf{P}\{W \ge 1/2\} = 1/4.$$

9. *Maximal correlation.*

(a) For any pair of random variables (X, Y), show that

$$F_{X,Y}(x, y) \leq \min\{F_X(x), F_Y(y)\}.$$

Now let F and G be continuous and invertible cdf's and let $X \sim F$.

(b) Find the distribution of

$$Y = G^{-1}(F(X)).$$

(c) Show that

$$F_{X,Y}(x, y) = \min\{F(x), G(y)\}.$$

Solution:

(a) We have

$$F_{X,Y}(x, y) = P\{X \leq x, Y \leq y\} \leq P\{X \leq x\} = F_X(x),$$

and similarly, $F_{X,Y} \leq F_Y(y)$. Thus,

$$F_{X,Y} \leq \min\{F_X(x), F_Y(y)\}.$$

(b) We have

$$\begin{aligned}
F_Y(y) &= P\{Y \leq y\} \\
&= P\{G^{-1}(F(X)) \leq y\} \\
&= P\{F(X) \leq G(y)\} \\
&= P\{X \leq F^{-1}(G(y))\} \\
&= F(F^{-1}(G(y))) \\
&= G(y).
\end{aligned}$$

(c) We have

$$\begin{aligned}
F_{X,Y}(x, y) &= P(X \leq x, Y \leq y) \\
&= P(X \leq x, X \leq F^{-1}(G(y))) \\
&= P(X \leq \min\{x, F^{-1}(G(y))) \\
&= \min\{F(x), F(F^{-1}(G(y)))\} \\
&= \min\{F(x), G(y)\}.
\end{aligned}$$

From part (a), this is the maximal joint cdf for any (X, Y) with the given marginal cdf's $F(x)$ and $G(y)$.