UNIVERSITY OF CALIFORNIA, SAN DIEGO
Electrical & Computer Engineering Department
ECE 250 - Winter Quarter 2022
Random Processes

Problem Set #6 **Due Friday, February 25, 2022 at 11:59pm**
Submit solutions to Problems 1, 3, 8, 12 only

1. *Covariance matrices.* Which of the following matrices can be a covariance matrix? Justify your answer either by constructing a random vector $\mathbf{X}$, as a function of the i.i.d zero mean unit variance random variables Z_1, Z_2, and Z_3, with the given covariance matrix, or by establishing a contradiction.

$$\text{(a)} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \text{(c)} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \qquad \text{(d)} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 3 \end{bmatrix}$$

2. *Spagetti.* We have a bowl with n spaghetti strands. You randomly pick two strand ends and join them. The process is continued until there are no ends left. Let L be the number of spaghetti loops formed. Find $\mathsf{E}[L]$.

3. *Gaussian random vector.* Given a Gaussian random vector $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$, where $\boldsymbol{\mu} = (1\,5\,2)^T$ and

$$\Sigma = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}.$$

   (a) Find the pdfs of
      i. X_1,
      ii. $X_2 + X_3$,
      iii. $2X_1 + X_2 + X_3$,
      iv. X_3 given (X_1, X_2), and
      v. (X_2, X_3) given X_1.

   (b) What is $\mathsf{P}\{2X_1 + X_2 - X_3 < 0\}$? Express your answer using the Q function.

   (c) Find the joint pdf on $\mathbf{Y} = A\mathbf{X}$, where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$

4. *Gaussian Markov chain.* Let X, Y, and Z be jointly Gaussian random variables with zero mean and unit variance, i.e., $\mathsf{E}(X) = \mathsf{E}(Y) = \mathsf{E}(Z) = 0$ and $\mathsf{E}(X^2) = \mathsf{E}(Y^2) = \mathsf{E}(Z^2) = 1$. Let $\rho_{X,Y}$ denote the correlation coefficient between X and Y, and let $\rho_{Y,Z}$ denote the correlation coefficient between Y and Z. Suppose that X and Z are conditionally independent given Y.

(a) Find $\rho_{X,Z}$ in terms of $\rho_{X,Y}$ and $\rho_{Y,Z}$.

(b) Find the MMSE estimate of Z given (X,Y) and the corresponding MSE.

5. *Sufficient statistic.* The bias of a coin is a random variable $P \sim \mathrm{U}[0,1]$. Let $Z_1, Z_2, \ldots, Z_{10}$ be the outcomes of 10 coin flips. Thus $Z_i \sim \mathrm{Bern}(P)$ and $Z_1, Z_2, \ldots, Z_{10}$ are conditionally independent given P. If X is the total number of heads, then $X|\{P = p\} \sim \mathrm{Binom}(10, p)$. Assuming that the total number of heads is 9, show that

$$f_{P|Z_1,Z_2,\ldots,Z_{10}}(p|z_1, z_2, \ldots, z_{10}) = f_{P|X}(p|9)$$

is independent of the order of the outcomes.

6. *Noise cancellation.* A classical problem in statistical signal processing involves estimating a weak signal (e.g., the heart beat of a fetus) in the presence of a strong interference (the heart beat of its mother) by making two observations; one with the weak signal present and one without (by placing one microphone on the mother's belly and another close to her heart). The observations can then be combined to estimate the weak signal by "canceling out" the interference. The following is a simple version of this application.

   Let the weak signal X be a random variable with mean μ and variance P, and the observations be $Y_1 = X + Z_1$ (Z_1 being the strong interference), and $Y_2 = Z_1 + Z_2$ (Z_2 is a measurement noise), where Z_1 and Z_2 are zero mean with variances N_1 and N_2, respectively. Assume that X, Z_1 and Z_2 are uncorrelated. Find the MMSE linear estimate of X given Y_1 and Y_2 and its MSE. Interpret the results.

7. *Nonlinear estimator.* Consider a channel with the observation $Y = XZ$, where the signal X and the noise Z are uncorrelated Gaussian random variables. Let $\mathsf{E}[X] = 1$, $\mathsf{E}[Z] = 2$, $\sigma_X^2 = 5$, and $\sigma_Z^2 = 8$.

   (a) Using the fact that $\mathsf{E}(W^3) = \mu^3 + 3\mu\sigma^2$ and $\mathsf{E}(W^4) = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$ for $W \sim \mathcal{N}(\mu, \sigma^2)$, find the mean and covariance matrix of $[X \;\; Y \;\; Y^2]^T$.

   (b) Find the MMSE linear estimate of X given Y and the corresponding MSE.

   (c) Find the MMSE linear estimate of X given Y^2 and the corresponding MSE.

   (d) Find the MMSE linear estimate of X given Y and Y^2 and the corresponding MSE.

   (e) Compare your answers in parts (b) through (d). Is the MMSE estimate of X given Y (namely, $\mathsf{E}(X|Y)$) linear?

8. *Sample mean convergence.* Consider the sequence of i.i.d. random variables $X_1, X_2, \ldots$ with

$$X_i = \begin{cases} 0 & \text{w.p. } \frac{1}{2}, \\ 2 & \text{w.p. } \frac{1}{2}, \end{cases}$$

   for all $i \geq 1$.

   Define the sequence

$$Y_n = \begin{cases} X_n, & \text{for all } n \quad \text{w.p. } \frac{1}{3}, \\ \frac{1}{2}X_n, & \text{for all } n \quad \text{w.p. } \frac{1}{3}, \\ 0, & \text{for all } n \quad \text{w.p. } \frac{1}{3}. \end{cases}$$

Let

$$M_n = \frac{1}{n}\sum_{i=1}^{n} Y_i.$$

(a) Determine the probability mass function (pmf) of Y_n.

(b) Determine the random variable (or constant) that M_n converges to (in probability) as n approaches infinity. Justify your answer.

(c) Use the central limit theorem to estimate the probability that the random variable M_{84} exceeds $\frac{2}{3}$.

9. *Minimum waiting time.* Let $X_1, X_2, \ldots$ be i.i.d. exponentially distributed random variables with parameter λ, i.e., $f_{X_i}(x) = \lambda e^{-\lambda x}$, for $x \geq 0$.

   (a) Show that $Y_n = \min\{X_1, X_2, \ldots, X_n\}$ converges in probability as n approaches infinity. What is the limit?

   (b) Does $Z_n = nY_n$ converge to the same limit in probability?

10. *Roundoff errors.* The sum of a list of 200 real numbers is to be computed. Suppose that these numbers are rounded off to the nearest integer so that each number has an error that is uniformly distributed in the interval $(-0.5, 0.5)$. Use the central limit theorem to estimate the probability that the total error in the sum of the 200 numbers exceeds 10.

11. The signal received over a wireless communication channel can be represented by two sums

$$\begin{aligned} X_{1n} &= \frac{1}{\sqrt{n}}\sum_{j=1}^{n} Z_j \cos\Theta_j, \text{ and} \\ X_{2n} &= \frac{1}{\sqrt{n}}\sum_{j=1}^{n} Z_j \sin\Theta_j, \end{aligned}$$

where $Z_1, Z_2, \ldots$ are i.i.d. with mean μ and variance σ^2 and $\Theta_1, \Theta_2, \ldots$ are i.i.d. $\mathrm{U}[0, 2\pi]$ independent of $Z_1, Z_2, \ldots$. Find the distribution of $\begin{bmatrix} X_{1n} \\ X_{2n} \end{bmatrix}$ as n approaches ∞.

12. *Convergence.* Consider the following sequences of random variables defined on the probability space $(\Omega, \mathcal{F}, \mathsf{P})$, where $\Omega = \{0, 1, 2, \ldots, m-1\}$, $\mathcal{F}$ is the collection of all subsets of Ω, and P is the uniform distribution over Ω.

$$X_n(w) = \begin{cases} \frac{1}{n}, & \omega = n \bmod m \\ 0, & \text{otherwise} \end{cases}$$

$$Y_n(w) = \begin{cases} 2^n, & \omega = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$Z_n(w) = \begin{cases} 1, & \omega = 1 \\ 0, & \text{otherwise} \end{cases}$$

Which of these sequences converges to zero

(a) with probability one?

(b) in mean square?

(c) in probability?