

Solutions to Exercise Set #1

1. Using the definition of σ -algebra, show that $A_1, A_2, \dots \in \mathcal{F}$ implies that $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$.

Solution: Suppose that $A_1, A_2, \dots \in \mathcal{F}$. Then, by the property that \mathcal{F} is closed under complement, that is $A \in \mathcal{F}$ implies $A^c \in \mathcal{F}$,

$$A_1^c, A_2^c, \dots \in \mathcal{F}.$$

Also, by the property of closure under countable union,

$$\bigcup_{i=1}^{\infty} A_i^c \in \mathcal{F}.$$

Finally, by the closure under complement again,

$$\bigcap_{i=1}^{\infty} A_i = \left(\bigcup_{i=1}^{\infty} A_i^c \right)^c \in \mathcal{F}.$$

2. Using the axioms of probability, show that $A \subset B$ implies that $P(A) \leq P(B)$.

Solution: Since $(B \cap A^c)$ and $(B \cap A) = A$ are disjoint, by using the axioms of probability we have

$$P(B) = P((B \cap A^c) \cup A) = P(A) + P(B \cap A^c).$$

Also note that $P(B \cap A^c) \geq 0$ by the axioms of probability, therefore

$$P(B) \geq P(A).$$

3. *Independence.* Show that the events A and B are independent if $P(A|B) = P(A|B^c)$.

Solution: It is given that

$$P(A|B) = P(A|B^c).$$

Now

$$\frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B^c)}{P(B^c)}.$$

Therefore

$$\begin{aligned} P(A \cap B)P(B^c) &= P(A \cap B^c)P(B) \\ &= (P(A) - P(A \cap B))P(B), \end{aligned}$$

which implies that

$$P(A \cap B)(P(B^c) + P(B)) = P(A \cap B) = P(A)P(B).$$

4. *Conditional probabilities.* Let $P(A) = 0.8$, $P(B^c) = 0.6$, and $P(A \cup B) = 0.8$. Find

(a) $P(A^c|B^c)$.

(b) $P(B^c|A)$.

Solution:

(a) Consider

$$P(A^c|B^c) = \frac{P(A^c \cap B^c)}{P(B^c)} = \frac{P((A \cup B)^c)}{P(B^c)} = \frac{1 - P(A \cup B)}{P(B^c)} = \frac{0.2}{0.6} = \frac{1}{3}.$$

(b) Consider

$$P(B^c|A) = 1 - P(B|A) = 1 - \frac{P(A \cap B)}{P(A)} = 1 - \frac{P(A) + P(B) - P(A \cup B)}{P(A)} = \frac{1}{2}.$$

5. Let A, B be two events with $P(A) \geq 0.5$ and $P(B) \geq 0.75$. Show that $P(A \cap B) \geq 0.25$.

Solution: We have

$$\begin{aligned} P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ &\geq P(A) + P(B) - 1 \\ &\geq 0.5 + 0.75 - 1 \\ &= 0.25. \end{aligned}$$

6. Two distinguishable dice are tossed. The number of dots facing up on each die is counted and noted, recording the number of dots on them.

(a) Find the sample space.

(b) Find the set A corresponding to the event that the total number of dots showing is even.

(c) Find the set B corresponding to the event that both dice are even.

(d) Does A imply B or does B imply A ? Find $A \cap B^c$ and describe this event in words.

(e) Let C be the event that the number of dots on the two dice differ by one. Find $A \cap C$.

Solution:

(a) The sample space for the toss of the two dice is

$$\Omega = \{1, 2, \dots, 6\} \times \{1, 2, \dots, 6\} = \left\{ \begin{array}{cccccc} (1, 1) & (1, 2) & (1, 3) & (1, 4) & (1, 5) & (1, 6) \\ (2, 1) & (2, 2) & (2, 3) & (2, 4) & (2, 5) & (2, 6) \\ (3, 1) & (3, 2) & (3, 3) & (3, 4) & (3, 5) & (3, 6) \\ (4, 1) & (4, 2) & (4, 3) & (4, 4) & (4, 5) & (4, 6) \\ (5, 1) & (5, 2) & (5, 3) & (5, 4) & (5, 5) & (5, 6) \\ (6, 1) & (6, 2) & (6, 3) & (6, 4) & (6, 5) & (6, 6) \end{array} \right\}$$

(b) The set A corresponding to the event “total number of dots showing is even” is

$$A = \left\{ \begin{pmatrix} (1, 1) & (1, 3) & (1, 5) \\ (2, 2) & (2, 4) & (2, 6) \\ (3, 1) & (3, 3) & (3, 5) \\ (4, 2) & (4, 4) & (4, 6) \\ (5, 1) & (5, 3) & (5, 5) \\ (6, 2) & (6, 4) & (6, 6) \end{pmatrix} \right\}$$

(c) The set B corresponding to the event “both dice are even” is

$$B = \left\{ \begin{pmatrix} (2, 2) & (2, 4) & (2, 6) \\ (4, 2) & (4, 4) & (4, 6) \\ (6, 2) & (6, 4) & (6, 6) \end{pmatrix} \right\}$$

(d) B implies A since $B \subseteq A$.

$$A \cap B^c = \left\{ \begin{pmatrix} (1, 1) & (1, 3) & (1, 5) \\ (3, 1) & (3, 3) & (3, 5) \\ (5, 1) & (5, 3) & (5, 5) \end{pmatrix} \right\}$$

$A \cap B^c$ is the set corresponding to the event “both dice are odd.”

(e) The set C corresponding to the event “number of dots differs by one” is

$$C = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (2, 1), (3, 2), (4, 3), (5, 4), (6, 5)\}.$$

The intersection $A \cap C = \emptyset$ is the empty set.

7. *Monty Hall.* Gold is placed behind one of three curtains. A contestant chooses one of the curtains, Monty Hall (the game host) opens one of the unselected empty curtains. The contestant has a choice either to switch his selection to the third curtain or not.

- What is the sample space for this random experiment? (Hint: An outcome consists of the curtain with gold, the curtain chosen by the contestant, and the curtain chosen by Monty.)
- Assume that placement of the gold behind the three curtains is random, the contestant choice of curtains is random and independent of the gold placement, and that Monty Hall's choice of an empty curtain is random among the alternatives. Specify the probability measure for this random experiment and use it to compute the probability of winning the gold if the contestant decides to switch.

Solution:

- The sample space consists of triplets of the form (Curtain with Gold behind it, Curtain chosen by the Player, Curtain that Monty opens). We denote the curtains by A, B , and C . So we can write our sample space as:

$$\Omega = \left\{ \begin{pmatrix} (A, A, B), (A, A, C), (A, B, C), (A, C, B), (B, B, A), (B, B, C), \\ (B, A, C), (B, C, A), (C, C, A), (C, C, B), (C, B, A), (C, A, B) \end{pmatrix} \right\}.$$

- (b) As discussed in class, for a discrete sample space the probability measure can be completely specified by probabilities of the single outcome events. For this problem we can specify the probabilities as follows:

$$\begin{aligned}
 P\{(A, A, B)\} &= P\{\text{Monty opens } B \mid \text{Gold behind } A, \text{ player's first choice is } A\} \times \\
 &\quad P\{\text{Gold behind } A, \text{ player's first choice is } A\} \\
 &= \frac{1}{2} P\{\text{Gold is behind } A\} \times P\{\text{player's first choice is } A\} \\
 &= \frac{1}{2} \times \frac{1}{3} \times \frac{1}{3} \\
 &= \frac{1}{18}.
 \end{aligned}$$

Note that the last action of Monty Hall is not independent of where the gold is placed and what the player's initial choice was. That's why we used conditional probability in the above derivation. Using similar arguments we get: $P\{(A, A, C)\} = P\{(B, B, A)\} = P\{(B, B, C)\} = P\{(C, C, A)\} = P\{(C, C, B)\} = \frac{1}{18}$. This covers all the cases where the gold placement and the initial choice of contestant coincide. For the other cases

$$\begin{aligned}
 P\{(A, B, C)\} &= P\{\text{Monty opens } C \mid \text{Gold behind } A, \text{ player's first choice is } B\} \times \\
 &\quad P\{\text{Gold is behind } A, \text{ player's first choice is } B\} \\
 &= 1 \times P\{\text{Gold is behind } A\} \times P\{\text{player's first choice is } B\} \\
 &= 1 \times \frac{1}{3} \times \frac{1}{3} \\
 &= \frac{1}{9}.
 \end{aligned}$$

Using similar arguments, $P\{(A, C, B)\} = P\{(B, A, C)\} = P\{(B, C, A)\} = P\{(C, A, B)\} = P\{(C, B, A)\} = \frac{1}{9}$ and we now have a complete description of the probability space for this random experiment.

To find the probability of winning if the player decides to switch, we need to find the subset W of the sample space corresponding to this event, which is

$$W = \{(A, B, C), (A, C, B), (B, A, C), (B, C, A), (C, A, B), (C, B, A)\}.$$

But the probability of the set W is the sum of the probabilities of its members, and, therefore, the probability of winning is

$$\begin{aligned}
 P(W) &= \frac{1}{9} \times 6 \\
 &= \frac{2}{3}.
 \end{aligned}$$

8. *Negative evidence.* Suppose that the evidence of an event B increases the probability of a criminal's guilt; that is, if A is the event that the criminal is guilty, then $P(A|B) \geq P(A)$. Does the absence of the event B decrease the criminal's probability of being guilty? In other words, is $P(A|B^c) \leq P(A)$? Prove or provide a counterexample.

Solution: We know that $P(A|B) \geq P(A)$. From the law of total probability, we have

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c).$$

Thus

$$\begin{aligned}P(A|B^c)P(B^c) &= P(A) - P(A|B)P(B) \\&\leq P(A)(1 - P(B)) \\&= P(A)P(B^c).\end{aligned}$$

Finally, dividing both sides by $P(B^c)$, we can conclude that $P(A|B^c) \leq P(A)$; that is, the absence of a positive evidence is a negative evidence.

9. *Binary communication channel.* Consider the binary communication channel discussed in Lecture Notes 1 with $p(1|0) = 0.1$ and $p(0|1) = 0.2$. Assume that the inputs are equiprobable.

- (a) Find the probability that the output is 0.
- (b) Find the probability that the input was 0 given that the output is 1.

Solution:

- (a) Denote the output by Y and the input by X . Using the law of total probability and conditional probability we get

$$\begin{aligned}P(Y = 0) &= P(X = 1)P(0|1) + P(X = 0)P(0|0) \\&= 0.5 \times 0.2 + 0.5 \times 0.9 \\&= 0.55.\end{aligned}$$

- (b) The conditional probability is

$$\begin{aligned}P(X = 0|Y = 1) &= \frac{P(X = 0, Y = 1)}{P(Y = 1)} \\&= \frac{P(X = 0)P(1|0)}{1 - P(Y = 0)} \\&= \frac{1}{9}.\end{aligned}$$

10. *Ternary communication channel.* Consider a communication channel with three inputs and three outputs, depicted in the Figure 1. Suppose that the input symbols 0, 1, and 2 occur with probability $\frac{1}{2}, \frac{1}{4}$ and $\frac{1}{4}$ respectively.

- (a) Find the probabilities of the output symbols.
- (b) Suppose that a 1 is observed as an output. What is the probability that the input was 0? 1? 2?

Your answers should be in terms of the conditional error probability ϵ .

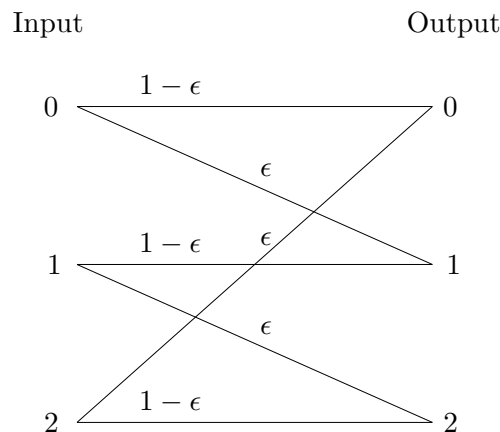


Figure 1: Ternary communication channel

Solution:

- (a) Denote the output by Y and the input by X . Using the law of total probability and conditional probability we get

$$\begin{aligned} P\{Y = 0\} &= P\{Y = 0|X = 0\}P\{X = 0\} + P\{Y = 0|X = 1\}P\{X = 1\} \\ &= \frac{1}{2}(1 - \epsilon) + \frac{1}{4}\epsilon = \frac{1}{2} - \frac{1}{4}\epsilon, \\ P\{Y = 1\} &= \frac{1}{2}\epsilon + \frac{1}{4}(1 - \epsilon) = \frac{1}{4} + \frac{1}{4}\epsilon, \\ P\{Y = 2\} &= \frac{1}{4}\epsilon + \frac{1}{4}(1 - \epsilon) = \frac{1}{4}. \end{aligned}$$

Note that the sum of the probabilities is 1.

- (b) The conditional probabilities are

$$\begin{aligned} P\{X = 0|Y = 1\} &= \frac{P\{X = 0, Y = 1\}}{P\{Y = 1\}} = \frac{\frac{1}{2}\epsilon}{\frac{1}{2}\epsilon + \frac{1}{4}(1 - \epsilon)} = \frac{2\epsilon}{1 + \epsilon}, \\ P\{X = 1|Y = 1\} &= \frac{\frac{1}{4}(1 - \epsilon)}{\frac{1}{2}\epsilon + \frac{1}{4}(1 - \epsilon)} = \frac{1 - \epsilon}{1 + \epsilon}, \\ P\{X = 2|Y = 1\} &= 0. \end{aligned}$$

Again, note that the sum of the conditional probabilities is 1.

11. *Geometric pairs.* Consider a probability space consisting of the sample space

$$\Omega = \{1, 2, 3, \dots\}^2 = \{(i, j) : i, j \in \mathbb{Z}^+\},$$

i.e., all pairs of positive integers, where the set of events is the power set of Ω and the probability measure on points in the sample space is

$$P((i, j)) = p^2(1 - p)^{i+j-2}, \quad 0 < p < 1.$$

- (a) Find $P(\{(i, j) : i \geq j\})$.
 (b) Find $P(\{(i, j) : i + j = k\})$.
 (c) Find $P(\{(i, j) : i \text{ is an odd number}\})$.
 (d) Describe an experiment whose outcomes (i, j) have the probabilities $P((i, j))$.

Solution:

- (a) Consider

$$\begin{aligned}
 P(\{(i, j) : i \geq j\}) &= \sum \sum_{\{(i, j) : i \geq j\}} p(i, j) = \sum_{j=1}^{\infty} \sum_{i=j}^{\infty} p^2(1-p)^{i+j-2} \\
 &= \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} p^2(1-p)^{i+2j-2} \\
 &= \frac{p^2}{(1-p)^2} \sum_{j=1}^{\infty} (1-p)^{2j} \sum_{i=0}^{\infty} (1-p)^i \\
 &= p^2 \sum_{j=1}^{\infty} (1-p)^{2j} \frac{1}{1-(1-p)} = p \frac{1}{1-(1-p)^2} = \frac{1}{2-p}.
 \end{aligned}$$

- (b) Consider

$$\begin{aligned}
 P(\{(i, j) : i + j = k\}) &= \sum \sum_{\{(i, j) : i+j=k\}} p(i, j) \\
 &= \sum_{j=1}^{k-1} p^2(1-p)^{j+(k-j)-2} = \sum_{j=1}^{k-1} p^2(1-p)^{k-2} \\
 &= p^2(1-p)^{k-2} \sum_{j=1}^{k-1} 1 = p^2(1-p)^{k-2}(k-1).
 \end{aligned}$$

- (c) Consider

$$\begin{aligned}
 P(\{(i, j) : i \text{ odd}\}) &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} p^2(1-p)^{(2i-1)+j-2} \\
 &= \sum_{i=0}^{\infty} p(1-p)^{2i} \sum_{j=0}^{\infty} p(1-p)^j = \frac{p}{1-(1-p)^2} = \frac{1}{2-p}.
 \end{aligned}$$

- (d) The probability mass function can be factored as

$$P((i, j)) = p^2(1-p)^{i+j-2} = p(1-p)^{i-1}p(1-p)^{j-1},$$

which is the product of two geometric pmfs. Consider a coin whose probability that a head occurs is p . Then $P((i, j))$ is the probability that when the coin is tossed repeatedly, the first head occurs on the i -th toss and the second head occurs on j -th toss after the first one ($i + j$ -th toss).