Stochastic Processes
UCSD ECE 250 Midterm-Solution
Instructor: Behrouz Touri

1. Determine whether each of the following statements is True or False. If True, prove it, if False, provide a counterexample.

   (a) Consider the probability space $(\Omega, \mathcal{F}, \mathbf{P}(\cdot))$. Let $X, Y : \Omega \to \mathbb{R}$, be such that $X + Y$ is a random variable. Then X, Y are random variables on this probability space.

   Solution: : False. For any any mapping X, let $Y = -X$, then $X + Y$ is random variable but we can choose the mapping X such that X, Y are not random variables.

   (b) If $A_1, A_2, \ldots$ are events such that $\mathbf{P}(A_n \text{ i.o.}) = 0$, then $\sum_{n=1}^{\infty} \mathbf{P}(A_n) < \infty$.

   Solution: False. Let X be a random variable with uniform distribution over $[0, 1]$. Let $A_n = \{0 < X < \frac{1}{n}\}$. Then, we have $\sum_{n=1}^{\infty} \mathbf{P}(A_n) = \infty$ and $\mathbf{P}(A_n \text{ i.o.}) = 0$.

   (c) If $\mathcal{F}_1$ and $\mathcal{F}_2$ are both σ-algebras on a sample space Ω, then $\mathcal{F}_1 \cup \mathcal{F}_2$ is a σ-algebra on Ω.

   Solution: False. $\Omega = \{1, 2, 3\}, \mathcal{F}_1 = \{\emptyset, \{1\}, \{2, 3\}, \Omega\}, \mathcal{F}_2 = \{\emptyset, \{2\}, \{1, 3\}, \Omega\}$. $\mathcal{F}_1 \bigcup \mathcal{F}_2$ is not a σ-algebra.

2. Let Θ be a uniformly distributed random variable over $[0, 2\pi]$. Let

$$R = \begin{cases} \cos(\Theta) & \text{if } \Theta \geq \frac{\pi}{3} \\ 0 & \text{otherwise} \end{cases}.$$

(a) Find the Cumulative Distribution Function (CDF) of R.
Solution: For $0.5 \leq r \leq 1$, we have

$$P(R > r) = P(\cos\Theta > r) = P(\Theta \in (2\pi - \arccos r, 2\pi]) = \frac{\arccos r}{2\pi}.$$

For $-1 \leq r < 0$, we have

$$\begin{aligned} P(R \leq r) &= P(\cos\Theta \leq r) \\ &= P(\cos\Theta \leq r, \Theta \in [0, \pi]) + P(\cos\Theta \leq r, \Theta \in [\pi, 2\pi]) \\ &= P(\Theta \in [\arccos r, \pi]) + P(\Theta \in [\pi, 2\pi - \arccos r]) \\ &= \frac{\pi - \arccos r}{\pi}. \end{aligned}$$

For $0 \leq r \leq 0.5$, we have

$$\begin{aligned} P(R \leq r) &= P(\cos\Theta \leq r) + P(\Theta \in [0, \pi/3)) \\ &= P(\cos\Theta \leq r, \Theta \in [0, \pi]) + P(\cos\Theta \leq r, \Theta \in [\pi, 2\pi]) + P(\Theta \in [0, \frac{\pi}{3})) \\ &= P(\Theta \in [\arccos r, \pi]) + P(\Theta \in [\pi, 2\pi - \arccos r]) + \frac{1}{6} \\ &= \frac{\pi - \arccos r}{\pi} + \frac{1}{6}. \end{aligned}$$

Therefore, we have

$$P(R \leq r) = \begin{cases} 0 & r < -1 \\ 1 - \frac{\arccos r}{\pi} & -1 \leq r < 0 \\ 1 - \frac{\arccos r}{\pi} + \frac{1}{6} & 0 \leq r < 0.5 \\ 1 - \frac{\arccos r}{2\pi} & 0.5 \leq r < 1 \end{cases}$$

(b) Is R a continuous or a discrete random variable?
Solution: R is not continuous as the CDF is discontinuous at 0. It is not discrete as $P(R = 0) = \frac{1}{6}$ and $P(Y = \alpha) = 0$ for any other $\alpha \neq 0$.

(c) Find $\mathbb{E}[\cdot]$. **Solution:**

$$\begin{aligned} \mathbb{E}[R] &= \int_{\frac{\pi}{3}}^{2\pi} \frac{1}{2\pi} \cos\theta d\theta \\ &= \frac{1}{2\pi} [\sin\theta]_{\frac{\pi}{3}}^{2\pi} = -\frac{\sqrt{3}}{4\pi} \end{aligned}$$

3. For each of the following processes, determine whether the **partial sum sequence** $S_k = \sum_{i=1}^{k} X_i$ converges almost surely or not, i.e., $\lim_{k\to\infty} S_k$ exists almost surely or not.

   (a) $\{X_k\}$ is an independent process such that X_k takes three values $-k, 0, k^k$ with

   $$\mathbf{P}(X_k = k^k) = \mathbf{P}(X_k = -k) = \frac{1}{2k^2}$$

   and $\mathbf{P}(X_k = 0) = 1 - \frac{1}{k^2}$.

   (b) $\{X_k\}$ is given by $X_k = \frac{Y_k - 1}{k}$ where $\{Y_k\}$ is an i.i.d. Gaussian random process with $Y_k \sim \mathcal{N}(0,1)$ (i.e., a Gaussian with zero mean and unit variance).

   Solution:

   (a) It is convergence almost surely. To show this, we invoke Kolmogorov's three series theorem with $\alpha = 0.5$. Note that $Y_k = 1_{|X_k|<\alpha} X_k = 0$ surely. Therefore, we have

      i. $\sum_{k=1}^{\infty} P(|X_k| > \alpha) = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$,

      ii. $\sum_{k=1}^{\infty} \mathbb{E}[Y_k] = 0$, and

      iii. $\sum_{k=1}^{\infty} Var[Y_k] = 0$.

   Therefore, by Kolmogorov's Three Series Theorem, the sequence is convergent almost surely.

   Alternatively, you could argue that $P(\{X_k \neq 0 \text{ i.o.}\}) = 0$ and hence, $\lim_{k\to\infty} X_k = 0$ almost surely.

   (b) This series is not convergent. Let $Z_k = \frac{Y_k}{k}$. Therefore, $X_k = Z_k - \frac{1}{k}$. Note that $\mathbb{E}[Z_k] = 0$ and $Var(Z_k) = \frac{1}{k^2}$. Therefore, since the process is independent, the series $\sum_{k=1}^{\infty} Z_k$ is convergent almost surely. But the sequence $-\frac{1}{k}$ is not convergent and hence, $\sum_{k=1}^{\infty} \frac{Y_k - 1}{k} = \sum_{k=1}^{\infty} Z_k - \frac{1}{k}$ is not convergent.

4. Three players P1, P2, and P3 taking turns on flipping a fair coin (starting from P1, then P2, then P3, and then P1 again, etc.). The person who gets the first head is the winner.

   (a) Let Ω be the set of all the binary sequences of the form $1, 01, 001, 0001, \ldots$. Provide the set of events $\mathcal{F}$ and a probability measure $\mathbf{P}(\cdot)$, such that the probability space $(\Omega, \mathcal{F}, \mathbf{P}(\cdot))$ models this experiment.

   (b) Determine the event B where P2 is winning and find $\mathbf{P}(B)$.

   (c) Let A be the event that P1 is winning. Is A and B independent?

   Solution:

   (a) Let's denote a sequence of coin flip with a binary sequence where tail is represented with 0 and head with 1. It is clear that our experiment ends when we see a head. Therefore, a binary sequence $\omega_k = \underbrace{0 \ldots 0}_{k-1 \text{ times}} 1$ represent the experiment that we have $k-1$ tails and then a head at the kth flip. With the set of such sequences being Ω, then we can simply set $\mathcal{F} = \mathcal{P}(\Omega)$. Note that the probability of the event of a single outcome ω_k is exactly $P(\{\omega_k\}) = \frac{1}{2^k}$. Let us denote this with p_{ω_k}. Once we know the probability of these singleton sets, we can set $P(E) = \sum_{\omega \in E} p_\omega$ for any $E \in \mathcal{F}$.

   (b) Note that the winner is P2 if $k \bmod 3 = 2$, i.e.,

$$B = \{\omega_2, \omega_5, \ldots\} = \{\omega_{3k+2} \mid k = 0, 1, 2, \ldots\}.$$

   Therefore,

$$\begin{aligned} P(B) &= \sum_{k=0}^{\infty} p_{\omega_{3k+2}} = \sum_{k=0}^{\infty} \frac{1}{2^{3k+2}} \\ &= \frac{1}{4} \sum_{k=0}^{\infty} \frac{1}{2^{3k}} \\ &= \frac{1}{4} \sum_{k=0}^{\infty} \frac{1}{8^k} \\ &= \frac{1}{4} \times \frac{1}{1 - \frac{1}{8}} = \frac{2}{7}. \end{aligned}$$

   (c) Using the same argument as above, you can show that $P(A) = 2P(B) = \frac{4}{7} \neq 0$. But $A \cap B = \emptyset$ and hence, $0 = P(A \cap B) \neq P(A)P(B)$. Therefore, the two events are not independent.