UNIVERSITY OF CALIFORNIA, SAN DIEGO

Electrical & Computer Engineering Department

ECE 250 - Winter Quarter 2020

Random Processes

Solutions to P.S. #7

1. *Symmetric random walk.* Let X_n be a random walk defined by

$$\begin{aligned} X_0 &= 0, \\ X_n &= \sum_{i=1}^{n} Z_i, \end{aligned}$$

where $Z_1, Z_2, \ldots$ are i.i.d. with $\mathsf{P}\{Z_1 = -1\} = \mathsf{P}\{Z_1 = 1\} = \frac{1}{2}$.

(a) Find $\mathsf{P}\{X_{10} = 10\}$.

(b) Approximate $\mathsf{P}\{-10 \le X_{100} \le 10\}$ using the central limit theorem.

(c) Find $\mathsf{P}\{X_n = k\}$.

Solution:

(a) Since the event $\{X_{10} = 10\}$ is equivalent to $\{Z_1 = \cdots = Z_{10} = 1\}$, we have $\mathsf{P}\{X_{10} = 10\} = 2^{-10}$.

(b) Since $\mathsf{E}(Z_j) = 0$ and $\mathsf{E}(Z_j^2) = 1$, by the central limit theorem,

$$\begin{aligned} \mathsf{P}\{-10 \le X_{100} \le 10\} &= \mathsf{P}\left\{-1 \le \left(\frac{1}{\sqrt{100}} \sum_{i=1}^{100} Z_i\right) \le 1\right\} \\ &\approx 1 - 2Q(1) = 2\Phi(1) - 1 \\ &\approx 0.682. \end{aligned}$$

(c)

$$\begin{aligned} \mathsf{P}\{X_n = k\} &= \mathsf{P}\{(n+k)/2 \text{ heads in } n \text{ independent coin tosses}\} \\ &= \binom{n}{\frac{n+k}{2}} 2^{-n} \end{aligned}$$

for $-n \le k \le n$ with $n + k$ even.

2. *Absolute-value random walk.* Consider the symmetric random walk X_n in the previous problem. Define the absolute value random process $Y_n = |X_n|$.

(a) Find $\mathsf{P}\{Y_n = k\}$.

(b) Find $\mathsf{P}\{\max_{1 \le i < 20} Y_i = 10 \,|\, Y_{20} = 0\}$.

Solution:

(a) If $k \geq 0$ then
$$\mathsf{P}\{Y_n = k\} = \mathsf{P}\{X_n = +k \text{ or } X_n = -k\}\,.$$
If $k > 0$ then $\mathsf{P}\{Y_n = k\} = 2\mathsf{P}\{X_n = k\}$, while $\mathsf{P}\{Y_n = 0\} = \mathsf{P}\{X_n = 0\}$. Thus
$$\mathsf{P}\{Y_n = k\} = \begin{cases} \binom{n}{(n+k)/2}\left(\frac{1}{2}\right)^{n-1} & k > 0,\ n-k \text{ is even},\ n-k \geq 0 \\ \binom{n}{n/2}\left(\frac{1}{2}\right)^{n} & k = 0,\ n \text{ is even},\ n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(b) If $Y_{20} = |X_{20}| = 0$ then there are only two sample paths with $\max_{1\leq i<20}|X_i| = 10$ that is, $Z_1 = Z_2 = \cdots = Z_{10} = +1, Z_{11} = \cdots = Z_{20} = -1$ or $Z_1 = Z_2 = \cdots = Z_{10} = -1, Z_{11} = \cdots = Z_{20} = +1$. Since the total number of sample paths is $\binom{20}{10}$ and all paths are equally likely,
$$\mathsf{P}\Big\{\max_{1\leq i<20} Y_i = 10 | Y_{20} = 0\Big\} = \frac{2}{\binom{20}{10}} = \frac{2}{184756} = \frac{1}{92378}\,.$$

3. *Discrete-time Wiener process.* Let Z_n, $n \geq 0$ be a discrete time white Gaussian noise (WGN) process, i.e., $Z_1, Z_2, \ldots$ are i.i.d. $\sim \mathcal{N}(0,1)$. Define the process X_n, $n \geq 1$ as $X_0 = 0$, and $X_n = X_{n-1} + Z_n$ for $n \geq 1$.

(a) Is X_n an independent increment process? Justify your answer.

(b) Is X_n a Gaussian process? Justify your answer.

(c) Find the mean and autocorrelation functions of X_n.

(d) Specify the first order pdf of X_n.

(e) Specify the joint pdf of X_3, X_5, and X_8.

(f) Find $\mathsf{E}(X_{20}|X_1, X_2, \ldots, X_{10})$.

(g) Given $X_1 = 4$, $X_2 = 2$, and $0 \leq X_3 \leq 4$, find the minimum MSE estimate of X_4.

Solution:

(a) Yes. The increments X_{n_1}, $X_{n_2} - X_{n_1}, \ldots, X_{n_{k_1}} - X_{n_k}$ are sums of different Z_i random variables, and the Z_i are IID.

(b) Yes. Any set of samples of X_n, $n \geq 1$ are obtained by a linear transformation of IID $\mathcal{N}(0,1)$ random variables and therefore all nth order distributions of X_n are jointly Gaussian (it is not sufficient to show that the random variable X_n is Gaussian for each n).

(c) We have
$$\mathsf{E}[X_n] = \mathsf{E}\left[\sum_{i=1}^{n} Z_i\right] = \sum_{i=1}^{n} \mathsf{E}[Z_i] = \sum_{i=1}^{n} 0 = 0,$$

$$
\begin{aligned}
R_X(n_1, n_2) &= \mathsf{E}[X_{n_1} X_{n_2}] \\
&= \mathsf{E}\left[\sum_{i=1}^{n_1} Z_i \sum_{j=1}^{n_2} Z_j\right] \\
&= \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \mathsf{E}[Z_i Z_j] \\
&= \sum_{i=1}^{\min(n_1,n_2)} \mathsf{E}(Z_i^2) \quad \text{(IID)} \\
&= \min(n_1, n_2).
\end{aligned}
$$

(d) As shown above, X_n is Gaussian with mean zero and variance

$$
\begin{aligned}
\mathrm{Var}(X_n) &= \mathsf{E}[X_n^2] - \mathsf{E}^2[X_n] \\
&= R_X(n, n) - 0 \\
&= n.
\end{aligned}
$$

Thus, $X_n \sim \mathcal{N}(0, n)$.

$$
\mathrm{Cov}(X_{n_1}, X_{n_2}) = \mathsf{E}(X_{n_1} X_{n_2}) - \mathsf{E}(X_{n_1})\mathsf{E}(X_{n_2}) = \min(n_1, n_2).
$$

Therefore, X_{n_1} and X_{n_2} are jointly Gaussian random variables with mean $\mu = [0\ 0]^T$ and covariance matrix $\Sigma = \begin{pmatrix} n_1 & \min(n_1, n_2) \\ \min(n_1, n_2) & n_2 \end{pmatrix}$.

(e) X_n, $n \geq 1$ is a zero mean Gaussian random process. Thus

$$
\begin{bmatrix} X_3 \\ X_5 \\ X_8 \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} E[X_3] \\ E[X_5] \\ E[X_8] \end{bmatrix}, \begin{bmatrix} R_X(3,3) & R_X(3,5) & R_X(3,8) \\ R_X(5,3) & R_X(5,5) & R_X(5,8) \\ R_X(8,3) & R_X(8,5) & R_X(8,8) \end{bmatrix}\right)
$$

Substituting, we get

$$
\begin{bmatrix} X_3 \\ X_5 \\ X_8 \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 & 3 & 3 \\ 3 & 5 & 5 \\ 3 & 5 & 8 \end{bmatrix}\right).
$$

(f) Since X_n is an independent increment process,

$$
\begin{aligned}
\mathsf{E}(X_{20} | X_1, X_2, \ldots, X_{10}) &= \mathsf{E}(X_{20} - X_{10} + X_{10} | X_1, X_2, \ldots, X_{10}) \\
&= \mathsf{E}(X_{20} - X_{10} | X_1, X_2, \ldots, X_{10}) + \mathsf{E}(X_{10} | X_1, X_2, \ldots, X_{10}) \\
&= \mathsf{E}(X_{20} - X_{10}) + X_{10} \\
&= 0 + X_{10} \\
&= X_{10}.
\end{aligned}
$$

(g) The MMSE estimate of X_4 given $X_1 = x_1$, $X_2 = x_2$ and $a \le X_3 \le b$ equals $\mathsf{E}[X_4|X_1 = x_1, X_2 = x_2, a \le X_3 \le b]$. Thus, the MMSE estimate is given by

$$\begin{aligned}
&\mathsf{E}\Big[X_4\Big|\{X_1 = x_1, X_2 = x_2, a \le X_3 \le b\}\Big] \\
&= \mathsf{E}\Big[X_2 + Z_3 + Z_4\Big|\{X_1 = x_1, X_2 = x_2, a \le X_2 + Z_3 \le b\}\Big] \\
&= x_2 + \mathsf{E}\Big[Z_3\Big|\{a - x_2 \le Z_3 \le b - x_2\}\Big] + \mathsf{E}[Z_4] \\
&\text{(since } Z_3 \text{ is independent of } (X_1, X_2)\text{, and } Z_4 \text{ is independent of } (X_1, X_2, Z_3) \text{)} \\
&= x_2 + \mathsf{E}\Big[Z_3\Big|\{a - x_2 \le Z_3 \le b - x_2\}\Big].
\end{aligned}$$

Plugging in the values, the required MMSE estimate is $\hat{X}_4 = 2 + \mathsf{E}\Big[Z_3\Big|\{Z_3 \in [-2, 2]\}\Big]$. We have

$$f_{Z_3|Z_3\in[-2,2]}(z_3) = \begin{cases} \dfrac{f_{Z_3}(z_3)}{\mathsf{P}(Z_3 \in [-2,2])}, & z_3 \in [-2,2] \\ 0, & \text{otherwise} \end{cases},$$

which is symmetric about $z_3 = 0$. Thus, $\mathsf{E}\Big[Z_3\Big|\{Z_3 \in [-2, 2]\}\Big] = 0$ and we have $\hat{X}_4 = 2$.

4. *Wiener process.* Recall the following definition of the (standard) Wiener process:

- $W(0) = 0$,
- $\{W(t)\}$ is independent increment with $W(t) - W(s) \sim \mathrm{N}(0, t - s)$ for all $t > s$,
- $\mathsf{P}\{\omega : W(\omega, t) \text{ is continuous in } t\} = 1$.

Let $W_1(t)$ and $W_2(t)$ be independent Wiener processes.

(a) Find the mean and the variance of

$$X(t) = \frac{1}{\sqrt{2}}\big(W_1(t) + W_2(t)\big).$$

Is $\{X(t)\}$ a Wiener process? Justify your answer.

(b) Find the mean and the variance of

$$Y(t) = \frac{1}{\sqrt{2}}\big(W_1(t) - W_2(t)\big).$$

Is $\{Y(t)\}$ a Wiener process? Justify your answer.

(c) Find $\mathsf{E}[X(t)Y(s)]$.

Solution:

(a) Since $W_1(t)$ and $W_2(t)$ are independent, Gaussian random variables with mean zero and variance t, the sum $S(t) = W_1(t) + W_2(t)$ is Gaussian with mean zero and variance $2t$. The random variable $X(t) = \frac{1}{\sqrt{2}}(W_1(t) + W_2(t))$ is therefore Gaussian with mean zero and variance t. Noting that $W_i(t) - W_i(s) \sim \mathrm{N}(0, t - s),\ t > s$ for $i = 1, 2$, a similar argument shows that $X(t) - X(s) \ \sim \mathrm{N}(0, t-s),\ t > s$. It remains to show that $X(t)$ is independent increment. 'For this, we will use transform methods and exploit the fact that for independent random variables X, Y,

$$\Phi_{X,Y}(\omega_1, \omega_2) = \Phi_X(\omega_1)\Phi_Y(\omega_2).$$

Specifically, we will show that, for $t_0 < t_1$, the random variables $W(t_0)$ and $W(t_1) - W(t_0)$ are independent. A similar argument can then be applied to other pairs of increments. We have

$$\begin{aligned}
\Phi_{W(t_0),W(t_1)-W(t_0)}(\omega_1,\omega_2) &= \mathsf{E}[e^{j\omega_1 W(t_0) + j\omega_2(W(t_1)-W(t_0))}] \\
&= \mathsf{E}[e^{j\omega_1(W_1(t_0)+W_2(t_0))/\sqrt{2} + j\omega_2((W_1(t_1)+W_2(t_1))-(W_1(t_0)+W_2(t_0)))/\sqrt{2}}] \\
&= \mathsf{E}[e^{\frac{1}{\sqrt{2}}j(\omega_1 W_1(t_0) + j\omega_2(W_1(t_1)-W_1(t_0)))} e^{\frac{1}{\sqrt{2}}(j\omega_1 W_2(t_0) + j\omega_2(W_2(t_1)-W_2(t_0)))}] \\
(W_1(t), W_2(t))\text{independent} &= \mathsf{E}[e^{\frac{1}{\sqrt{2}}j(\omega_1 W_1(t_0) + j\omega_2(W_1(t_1)-W_1(t_0)))}]\mathsf{E}[e^{\frac{1}{\sqrt{2}}(j\omega_1 W_2(t_0) + j\omega_2(W_2(t_1)-W_2(t_0)))}] \\
\text{ind.increment} &= \mathsf{E}[e^{\frac{1}{\sqrt{2}}j(\omega_1 W_1(t_0)}]\mathsf{E}[e^{\frac{1}{\sqrt{2}}j\omega_2(W_1(t_1)-W_1(t_0)))}] \\
&\quad \cdot \mathsf{E}[e^{\frac{1}{\sqrt{2}}j\omega_1 W_2(t_0)}]\mathsf{E}[e^{\frac{1}{\sqrt{2}}j\omega_2(W_2(t_1)-W_2(t_0)))}] \\
&= \mathsf{E}[e^{\frac{1}{\sqrt{2}}j(\omega_1 W_1(t_0)}]\mathsf{E}[e^{\frac{1}{\sqrt{2}}j\omega_1 W_2(t_0)}] \\
&\quad \cdot \mathsf{E}[e^{\frac{1}{\sqrt{2}}j\omega_2(W_1(t_1)-W_1(t_0)))}]\mathsf{E}[e^{\frac{1}{\sqrt{2}}j\omega_2(W_2(t_1)-W_2(t_0)))}] \\
(W_1(t), W_2(t))\text{independent} &= \mathsf{E}[e^{\frac{1}{\sqrt{2}}j\omega_1(W_1(t_0)+W_2(t_0))}]\mathsf{E}[e^{\frac{1}{\sqrt{2}}j\omega_2((W_1(t_1)+W_2(t_1))-(W_1(t_0)+W_2(t_0))))}] \\
\\
&= \mathsf{E}[e^{j\omega_1 W(t_0)}]\mathsf{E}[e^{j\omega_2(W(t_1)-W(t_0))}] \\
&= \Phi_{W(t_0)}(\omega_1)\Phi_{W(t_1)-W(t_0)}(\omega_2).
\end{aligned}$$

which proves independence of $W(t_0)$ and $W(t_1) - W(t_0)$

(b) The arguments and answers from part (a) apply to the difference $Y(t)$.

(c)

$$\begin{aligned}
E[X(t)Y(s)] &= \frac{1}{2}\mathsf{E}[(W_1(t) + W_2(t))(W_1(s) - W_2(s))] \\
&= \frac{1}{2}\mathsf{E}[W_1(t)W_1(s) + W_2(t)W_1(s) - W_1(t)W_2(s) - W_2(t)W_2(s)] \\
&= \frac{1}{2}(\mathsf{E}[W_1(t)W_1(s)] + \mathsf{E}[W_2(t)W_1(s)] - \mathsf{E}[W_1(t)W_2(s)] - \mathsf{E}[W_2(t)W_2(s)]) \\
&= \frac{1}{2}(R_{W_1}(t,s) + 0 + 0 - R_{W_2}(t,s)) \\
&= 0
\end{aligned}$$

where we have use the independence of the processes $\{W_1(t)\}$ and $\{W_2(t)\}$, along with the fact that $R_{W_1}(t,s) = R_{W_2}(t,s)$.

5. *Autoregressive process.* Let $X_0 = 0$ and $X_n = \frac{1}{2}X_{n-1} + Z_n$ for $n \geq 1$, where $Z_1, Z_2, \ldots$ are i.i.d. $\sim \mathrm{N}(0,1)$. Find the mean and autocorrelation function of X_n.

Solution: This autoregressive process is an example of a Gauss-Markov process, as described in Section 8.2.5, with parameter $\alpha = \frac{1}{2}$ and noise process variance $N = 1$. The mean function is

$$\mu_X(n) = 0,\ n \geq 0.$$

The autocorrelation function is

$$\begin{aligned} R_X(n_1, n_2) &= \alpha^{|n_2-n_1|}\frac{1-\alpha^{2\min(n_1,n_2)}}{1-\alpha^2}N \\ &= 2^{-|n_1-n_2|}\frac{4}{3}[1-4^{-\min(n_1,n_2)}] \end{aligned}$$

6. *Moving average process.* Let $Z_0, Z_1, Z_2, \ldots$ be i.i.d. $\sim \mathcal{N}(0,1)$.

(a) Let $X_n = \frac{1}{2}Z_{n-1} + Z_n$ for $n \geq 1$. Find the mean and autocorrelation function of X_n.
(b) Is $\{X_n\}$ wide-sense stationary? Justify your answer.
(c) Is $\{X_n\}$ Gaussian? Justify your answer.
(d) Is $\{X_n\}$ strict-sense stationary? Justify your answer.
(e) Find $\mathsf{E}(X_3|X_1, X_2)$.
(f) Find $\mathsf{E}(X_3|X_2)$.
(g) Is $\{X_n\}$ Markov? Justify your answer.
(h) Is $\{X_n\}$ independent increment? Justify your answer.

Solution:

(a)

$$\mathsf{E}(X_n) = \frac{1}{2}\mathsf{E}(Z_{n-1}) + \mathsf{E}(Z_n) = 0.$$

$$\begin{aligned} R_X(m,n) &= \mathsf{E}(X_m X_n) \\ &= \mathsf{E}\left[\left(\frac{1}{2}Z_{n-1} + Z_n\right)\left(\frac{1}{2}Z_{m-1} + Z_m\right)\right] \\ &= \begin{cases} \frac{1}{2}\mathsf{E}[Z_{n-1}^2], & n-m=1 \\ \frac{1}{4}\mathsf{E}[Z_{n-1}^2] + \mathsf{E}[Z_n^2], & n=m \\ \frac{1}{2}\mathsf{E}[Z_n^2], & m-n=1 \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{5}{4}, & n=m \\ \frac{1}{2}, & |n-m|=1 \\ 0\,, & \text{otherwise.} \end{cases} \end{aligned}$$

(b) Since the mean and autocorrelation functions are time-invariant, the process is WSS.

(c) Since $(X_1, \ldots, X_n)$ is a linear transform of a GRV $(Z_0, Z_1, \ldots, Z_n)$, the process is Gaussian.

(d) Since the process is WSS and Gaussian, it is SSS.

(e) Since the process is Gaussian, the conditional expectation (MMSE estimate) is linear. Hence,

$$\mathsf{E}(X_3|X_1, X_2) = \begin{bmatrix} 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{5}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{4} \end{bmatrix}^{-1} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \frac{1}{21}(10X_2 - 4X_1).$$

(f) Similarly, $\mathsf{E}(X_3|X_2) = (2/5)X_2$.

(g) Since $\mathsf{E}(X_3|X_1, X_2) \neq \mathsf{E}(X_3|X_2)$, the process is not Markov.

(h) Since the process is not Markov, it is not independent increment.

7. *QAM random process.* Consider the random process

$$X(t) = Z_1 \cos \omega t + Z_2 \sin \omega t\,, \quad -\infty < t < \infty\,,$$

where Z_1 and Z_2 are i.i.d. discrete random variables such that $p_{Z_i}(+1) = p_{Z_i}(-1) = \frac{1}{2}$.

(a) Is $X(t)$ wide-sense stationary? Justify your answer.

(b) Is $X(t)$ strict-sense stationary? Justify your answer.

Solution:

Note that $\mathsf{E}[Z_1] = \mathsf{E}[Z_2] = 0$ and $\mathsf{E}[Z_1^2] = \mathsf{E}[Z_2^2] = 1$.

(a) We first check the mean.

$$\mathsf{E}(X(t)) = \mathsf{E}(Z_1)\cos \omega t + \mathsf{E}(Z_2)\sin \omega t = 0 \cdot \cos(\omega t) \;+\; 0 \cdot \sin(\omega t) = 0\,.$$

The mean is independent of t. Next we consider the autocorrelation function.

$$\begin{aligned}
\mathsf{E}(X(t+\tau)X(t)) &= \mathsf{E}((Z_1 \cos(\omega(t+\tau)) + Z_2 \sin(\omega(t+\tau)))(Z_1 \cos(\omega t) + Z_2 \sin(\omega t))) \\
&= \mathsf{E}(Z_1^2)\cos(\omega(t+\tau))\cos(\omega t) + \mathsf{E}(Z_2^2)\sin(\omega(t+\tau))\sin(\omega t) \\
&= \cos(\omega(t+\tau))\cos(\omega t) + \sin(\omega(t+\tau))\sin(\omega t) \\
&= \cos(\omega(t+\tau) - \omega t)) = \cos \omega\tau\,.
\end{aligned}$$

The autocorrelation function is also time invariant. Therefore $X(t)$ is WSS.

(b) Note that $X(0) = Z_1 \cos 0 + Z_2 \sin 0 = Z_1$, so $X(0)$ has the same pmf as Z_1. On the other hand,

$$\begin{aligned}
X(\frac{\pi}{4\omega}) &= Z_1 \cos(\pi/4) + Z_2(\sin \pi/4) \\
&= \frac{1}{\sqrt{2}}(Z_1 + Z_2) \\
&= \begin{cases} \frac{2}{\sqrt{2}} = \sqrt{2} & \text{w.p. } \frac{1}{4} \\ 0 & \text{w.p. } \frac{1}{2} \\ \frac{-2}{\sqrt{2}} = -\sqrt{2} & \text{w.p. } \frac{1}{4} \end{cases}
\end{aligned}$$

This shows that $X(\pi/4\omega)$ does not have the same pdf or even same range as $X(0)$. Therefore $X(t)$ is not first-order stationary and as a result is not SSS.

8. *Convergence of random processes.* Let $\{N(t)\}_{t=0}^{\infty}$ be a Poisson process with rate λ. Recall that the process is independent increment and $N(t) - N(s)$, $0 \le s < t$, has the pmf

$$p_{N(t)-N(s)}(n) = \frac{e^{-\lambda(t-s)}(\lambda(t-s))^n}{n!}, \quad n = 0, 1, \ldots.$$

Define

$$M(t) = \frac{N(t)}{t}, \quad t > 0.$$

(a) Find the mean and autocorrelation function of $\{M(t)\}_{t>0}$.
$N(t) \sim \text{Poisson}(\lambda t)$, so $\mathsf{E}[M(t)] = \lambda t/t = \lambda$.
In the discussion session, we found the autocorrelation function of the Poisson process,

$$R_N(t_1, t_2) = \lambda \min(t_1, t_2) + \lambda^2 t_1 t_2.$$

Now,

$$\begin{aligned}
R_M(t_1, t_2) &= \mathsf{E}[M(t_1)M(t_2)] \\
&= \frac{1}{t_1 t_2}\mathsf{E}[N(t_1)N(t_2)] \\
&= \frac{1}{t_1 t_2} R_N(t_1, t_2) \\
&= \frac{1}{t_1 t_2}(\lambda \min(t_1, t_2) + \lambda^2 t_1 t_2) \\
&= \frac{\lambda}{\max(t_1, t_2)} + \lambda^2
\end{aligned}$$

Note also that $\text{Var}(M(t)) = R_M(t, t) - \mathsf{E}[M]^2 = \frac{\lambda}{t}$.

(b) Does $\{M(t)\}_{t>0}$ converge in mean square as $t \to \infty$, that is,

$$\lim_{t\to\infty} \mathsf{E}\big[(M(t) - M)^2\big] = 0$$

for some random variable (or constant) M? If so, what is the limit?
The Chebyschev Inequality states that

$$\mathsf{P}(|M(t) - E[M(t)]| \ge \epsilon) \le \frac{\text{Var}(M(t))}{\epsilon^2}$$

so

$$\mathsf{P}(|M(t) - \lambda| \ge \epsilon) \le \frac{\lambda}{t\epsilon^2}.$$

Now,

$$\lim_{t\to\infty} \frac{\lambda}{t\epsilon^2} = 0$$

so

$$\lim_{t \to \infty} \mathsf{P}(|M(t) - \lambda| \geq \epsilon) = 0$$

or

$$\lim_{t \to \infty} \mathsf{P}(|M(t) - \lambda| \leq \epsilon) = 1.$$

Thus, $M(t)$ converges in probability to λ, so we conjecture that it converges in mean square as well. We find that

$$\mathsf{E}[(M(t) - \lambda)^2] = \mathsf{E}[M(t)^2] - 2\lambda\mathsf{E}[M(t)] + \lambda^2 = \frac{\lambda}{t}$$

so

$$\lim_{t \to \infty} \mathsf{E}[(M(t) - \lambda)^2] = \lim_{t \to \infty} \frac{\lambda}{t} = 0,$$

confirming the conjecture.