ECE 250: Stochastic Processes: Week #4

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Outline:

- Kolmogorov 0-1 Law
- Kolmogorov Maximal Inequality
- Kolmogorov Three-series Theorem

Sum of Independent Random Variables

- The study of independent increment processes is essentially the same as the study of (partial) sum process of an independent process:

$$S_n = \sum_{k=1}^{n} X_k.$$

- Main Question: When do independent increment processes converge?
- For example: suppose that we have a sequence of i.i.d. random variables $\{X_k\}$ in $\{-1, 1\}$ with probability 0.5 each. What can we say about the following sum?

$$S_n = \sum_{k=1}^{n} \frac{1}{k} X_k.$$

A General and Surprising Result

- Suppose that we are given a sequence of independent random variables $\{X_k\}$. We say that an event E is a tail event for $\{X_k\}$ if it only depends on the tail of these random variables, i.e., we can determine whether $\omega \in E$ or not it by knowing $X_k(\omega)$ for $k \geq t$ for all $t \geq 1$. In other words, $\omega \in E$ if the property holds even if we drop knowledge of $X_1, \ldots, X_t$ for all $t \geq 0$.
- Example: The event E where $\lim_{k\to\infty} X_k$ exists is a tail event.

  Theorem 1. *(Kolmogorov's 0-1 Law) A tail event of an independent process is a trivial event, i.e.,* $\Pr(E) = 0$ *or* $\Pr(E) = 1$*!*

- Note that the above result holds irrespective of the distributions of any of them!
- Implications:

  a. Probability of giant component on percolation process in $\mathbb{Z}^2$ is either 0 or 1.

  b. Probability of giant component on percolation process in $\mathbb{Z}^2$ that is edge dependent is either 0 or 1 irrespective of probability of each edge.

  c. $\Pr(\lim_{k\to\infty} S_k$ exists$)$ is 0 or 1 for partial sums of independent processes.

Kolmogorov's Maximal Inequality

- Recall Markov's Inequality: For a non-negative rv X

$$\Pr(X \geq \alpha) \leq \frac{\mathbb{E}[X]}{\alpha},$$

  for any $\alpha > 0$.

- Define $Var(X) = \mathbb{E}[(X - \bar{X})^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$. (if exists)

- **Chebyshev's inequality** (an extension of the Markov's inequality): For a random variable X we have

$$\Pr(|X - \bar{X}| \geq \alpha) \leq \frac{Var(X)}{\alpha^2}.$$

- Now suppose that we have an independent process $\{X_k\}$. What can we say about?

$$P(\max_{1 \leq k \leq n} |S_k| \geq \alpha)$$

  Theorem 2. *(Kolmogorov's Maximal Inequality) For a zero mean and independent process $\{X_k\}$ and any $n \geq 1$, we have*

$$P(\max_{1 \leq k \leq n} |S_k| \geq \alpha) \leq \frac{Var(S_n)}{\alpha^2}.$$

- Application: Estimating worst case scenario by day n of COVID

Two important intermediate results

- We say that a function $h : \mathbb{R}^n \to \mathbb{R}$ is measurable if $h^{-1}(B) \in \mathcal{B}$ for all Borel sets $B \in \mathcal{B}^n$.
- Important classes of measurable functions:
  - continuous functions,
  - piece-wise continuous functions, and
  - convex (concave) functions.
- Lemma 1: If X and Y are independent random variables, then $g_1(X)$ and $g_2(Y)$ are independent random variables for any measurable functions $g_1(\cdot), g_2(\cdot)$.
- More generally: for independent $X_1, \ldots, X_k, X_{k+1}, \ldots, X_n$, $g_1(X_1, \ldots, X_k)$ and $g_2(X_{k+1}, \ldots, X$ would be independent for any measurable functions g_1, g_2 of appropriate dimensions.
- Lemma 2: If X and Y are independent, then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

Kolmogorov's Maximal Inequality

Theorem 3. *For an independent sequence of random variables $\{X_k\}$ with zero mean, we have:*

$$\Pr(\max_{1\le k\le n} |S_k| \ge \alpha) \le \frac{Var(S_n)}{\alpha^2}.$$

Proof.

- Define $A_k = \{\omega \mid |S_k| \ge \alpha, |S_1|, \dots, |S_{k-1}| < \alpha\}$.
- $A_1, \dots, A_n$ are mutually exclusive and we have:

$$\begin{aligned}
\mathrm{Var}(S_n) &= \mathbb{E}[|S_n|^2] \ge \mathbb{E}[(\mathbf{1}_{A_1} + \mathbf{1}_{A_2} + \ldots + \mathbf{1}_{A_n})|S_n|^2] \\
&= \sum_{k=1}^{n} \mathbb{E}[\mathbf{1}_{A_k}|S_n|^2] \\
&= \sum_{k=1}^{n} \mathbb{E}[\mathbf{1}_{A_k}(S_n - S_k + S_k)^2] \\
&= \sum_{k=1}^{n} \mathbb{E}[\mathbf{1}_{A_k}((S_n - S_k)^2 + 2(S_n - S_k)S_k + S_k^2)] \\
&\ge \sum_{k=1}^{n} 2\mathbb{E}[\mathbf{1}_{A_k} S_k(S_n - S_k)] + \mathbb{E}[\mathbf{1}_{A_k}(S_k^2)] \\
&\ge \alpha^2 \sum_{k=1}^{n} \Pr(\mathbf{1}_{A_k}) \\
&= \alpha^2 \Pr(\max_{1\le k\le n} |S_k| \ge \alpha).
\end{aligned}$$

$\square$

Sums of Independent Random Variables is closely related to the Sums of Variances

Theorem 4. *For an independent sequence of random variables $\{X_k\}$ with zero mean, if*

$$\sum_{k=1}^{\infty} Var(X_k) < \infty,$$

then $\lim_{n\to\infty} \sum_{k=1}^{n} X_n$ *exists.*

Main idea of the proof: show that $\sum_{k=1}^{\infty} X_k$ is a Cauchy sequence almost surely by utilizing the Maximal inequality:

$$\Pr(\sup_{M\geq m} |S_M - S_m| \geq \epsilon) \leq \frac{1}{\epsilon^2} \sum_{k=m}^{\infty} \mathsf{Var}(X_k).$$

Kolmogorov's Three-Series Theorem

Theorem 5. *Sum of an independent random process $\{X_k\}$ converges almost surely if and only if for any $\alpha > 0$, if we let $Y_k = X_k \mathbf{1}_{|X_k| \leq \alpha}$, the following three (deterministic) series converges:*

- $\sum_{k=1}^{\infty} P(|X_k| \geq \alpha) < \infty$,
- $\sum_{k=1}^{\infty} \mathbb{E}[Y_k]$ *converges, and*
- $\sum_{k=1}^{\infty} Var(Y_k)$ *converges.*