HW5

Monday, November 7, 2022 10:20 AM

1. $$f_{X,Y}(x,y) = \begin{cases} c, & \text{if } |x|+|y| \le \frac{1}{\sqrt{2}} \\ 0, & \text{otherwise} \end{cases}$$ $|y| \le \frac{1}{\sqrt{2}} - |x|$.

(a). $$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,dx$$

$$= \int_X \int_{-\frac{1}{\sqrt{2}}+|x|}^{\frac{1}{\sqrt{2}}-|x|} c\,dy\,dx$$

$$= \int_X c(\sqrt{2}-2|x|)\,dx$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} c(\sqrt{2}-2|x|)\,dx$$

$$= c\left[(\sqrt{2}x - x^2)\Big|_0^{\frac{1}{\sqrt{2}}} + (\sqrt{2}x + x^2)\Big|_{-\frac{1}{\sqrt{2}}}^{0}\right]$$

$$= c\left[(1-\tfrac{1}{2}) - (-1+\tfrac{1}{2})\right]$$

$$= c = 1.$$

(b) $$f_X(x) = \int f_{X,Y}(x,y)\,dy$$

$$= \int_{-\frac{1}{\sqrt{2}}+|x|}^{\frac{1}{\sqrt{2}}-|x|} 1\,dy$$

$$= \sqrt{2} - 2|x| \quad \left(|x| \le \tfrac{1}{\sqrt{2}}\right)$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(x)}$$

$$= \frac{1}{\sqrt{2}-2|y|}, \quad \begin{cases} |y| \le \frac{1}{\sqrt{2}} \\ |x|+|y| \le \frac{1}{\sqrt{2}} \end{cases}$$

(c) $$f_X(x) = \sqrt{2} - 2|x|$$

$$f_Y(y) = \sqrt{2} - 2|y|$$

$$f_{X,Y}(x,y) = 1 \ne f_X(x) f_Y(y) = 2 - 2\sqrt{2}(|x|+|y|) + 4|x||y|.$$

$\therefore$ X and Y are not independent.

(d) $\because Z = |X| + |Y|$.

$\therefore z \in [0, \frac{1}{\sqrt{2}}]$.

$\therefore P(Z \le z) = \int f_X(x) P(Z \le z \mid X = x) dx$.

$= \int f_X(x) P(|X| + |Y| \le z \mid X = x) dx$.

$= \int f_X(x) P(|Y| \le z - |x| \mid X = x) dx$.

$= \int_{-z}^{z} \int_{-z+|x|}^{z-|x|} f_{Y|X}(y|x) f_X(x) \, dy \, dx$.

$= \int_{-z}^{z} \int_{-z+|x|}^{z-|x|} \frac{1}{\sqrt{2} - 2|x|} (\sqrt{2} - 2|x|) \, dy \, dx$.

$= \int_{-z}^{z} (2z - 2|x|) \, dx$.

$= 4 \int_0^z (z - x) \, dx = 4 \left(zx - \frac{x^2}{2} \right) \Big|_0^z$.

$= 2z^2 = F_Z(z)$.

$\therefore f_Z(z) = \frac{d}{dz} F_Z(z)$.

$$= \begin{cases} 4z & , \text{if } z \in [0, \frac{1}{\sqrt{2}}] \\ 0 & , \text{otherwise} \end{cases}$$

2. $\{X\} = \{2, 3, 4\}$. $P(\text{girl}) = \frac{1}{2}$. indep.

(a) Let Y be the # of child.

$\therefore Y \le X$.

$\therefore$ the pmf $P_{Y,X}(y,x)$ is shown as below.

X \ Y	1	2	3	4
2	$\frac{1}{?}$	$\frac{1}{?}$	0	0

$X \backslash Y$	1	2	3	4
2	$\frac{1}{6}$	$\frac{1}{6}$	0	0
3	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{12}$	0
4	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{1}{24}$

$$\because P_X(x) = \begin{cases} \frac{1}{3}, & \text{if } X = 2, 3, 4. \\ 0, & \text{otherwise} \end{cases}$$

$$\therefore P_{Y|X}(y|x) = \frac{P_{Y,X}(y,x)}{P_X(x)}$$

$$\therefore ① \ P_{Y|X=2}(y|2) = \begin{cases} \frac{1}{2}, & y=1 \\ \frac{1}{2}, & y=2. \end{cases}$$

$$② \ P_{Y|X=3}(y|3) = \begin{cases} \frac{1}{2}, & y=1. \\ \frac{1}{4}, & y=2. \\ \frac{1}{4}, & y=3 \end{cases}$$

$$③ \ P_{Y|X=4}(y|4) = \begin{cases} \frac{1}{2}, & y=1 \\ \frac{1}{4}, & y=2 \\ \frac{1}{8}, & y=3 \\ \frac{1}{8}, & y=4. \end{cases}$$

(2) $$P_Y = \frac{P_{X,Y}(x,y)}{P_{Y|X}(y|x)}$$

$$= \begin{cases} \frac{1}{2}, & y=1. \\ \frac{1}{3}, & y=2 \\ \frac{1}{8}, & y=3 \\ \frac{1}{24}, & y=4 \end{cases}$$

3. (a) $p = 0.3$, $N = 1000$.

Let X_k be the event of flipping a coin.

Let X_k be the event of flipping a coin in k-th time.

$\therefore$ head: $P(X_k = 1) = p = 0.3$

number: $P(X_k = 0) = 1 - p = 0.7$.

$\therefore \mu_X = (1 \times 0.3) + (0 \times 0.7) = 0.3$.

$\sigma_X^2 = 0.21 = 0.3 - (0.3)^2$.

Let Z_k be the number of heads.

$\therefore Z_k = X_1 + \cdots + X_k$

$\therefore P(250 \leq Z_{1000} \leq 350)$

$= P\left(\frac{250}{\sqrt{1000}} \leq \frac{Z_{1000} + 1000(0.3)}{\sqrt{1000}} \leq \frac{350}{\sqrt{1000}}\right)$.

$= P\left(\frac{25}{\sqrt{10}} \leq \frac{Z_{1000} + 300}{10\sqrt{10}} \leq \frac{35}{\sqrt{10}}\right)$.

$\therefore$ By CTL: $\frac{Z_{1000} + 300}{10\sqrt{10}} \xrightarrow{\text{in law}} N(0.3, 0.21)$

$\therefore P\left(\frac{25}{\sqrt{10}} \leq \frac{Z_{1000}+300}{10\sqrt{10}} \leq \frac{35}{\sqrt{10}}\right)$

$= P\left(\frac{25}{\sqrt{10}} \leq Y \leq \frac{35}{\sqrt{10}}\right)$. $Y \sim N(0.3, 0.21)$

$= \int_{25/\sqrt{10}}^{35/\sqrt{10}} \frac{1}{\sqrt{0.42\pi}} e^{-\frac{(y-0.3)^2}{0.42}} dy$.

(b) $P(X_k = 1) = p = 0.5$.

$P(X_k = 0) = 1 - p = 0.5$.

$\mu_X = 0.5$, $\sigma_X^2 = 0.5 - 0.25 = 0.25$.

$Z_k = X_1 + \cdots + X_k$.

$P(250 \leq Z_{1000} \leq 350)$.

$P(250 \le Z_{1000} \le 350)$

$= P\left(\frac{25}{\sqrt{10}} \le \frac{Z_{1000}+500}{10\sqrt{10}} \le \frac{35}{\sqrt{10}}\right)$

By CTL: $\frac{Z_{1000}+500}{10\sqrt{10}} \xrightarrow{\text{in law}} N(0.5, 0.25)$.

$\therefore P(250 \le Z_{1000} \le 350)$

$= P\left(\frac{25}{\sqrt{10}} \le Y \le \frac{35}{\sqrt{10}}\right) \quad Y \sim N(0.5, 0.25)$

$= \int_{25/\sqrt{10}}^{35/\sqrt{10}} \frac{1}{\sqrt{0.5\pi}} e^{-\frac{(y-0.5)^2}{0.5}} dy$.

(c).

4. $X_{k+1} = W_k X_k = W_k W_{k-1} X_{k-1}$

$= W_k W_{k-1} \cdots W_1 X_1$.

(a) $\{W_k\}$ is i.i.d $\quad W_k \sim U[0.4, 3]$.

Let $Y = \log W_k$. $\quad \therefore Y \in [\log(0.4), \log(3)]$.

$F_Y(y) = P(Y \le y)$

$= P(\log W_k \le y)$

$= P(W_k \le e^y)$.

$= F_{W_k}(e^y)$.

$\because f_{W_k}(w) = \int \frac{1}{2.6}, \quad W_k \in [0.4, 3]$.

$$\because f_{W_k}(w) = \begin{cases} \frac{1}{2.6}, & W_k \in [0.4, 3]. \\ 0, & \text{otherwise}. \end{cases}$$

$$\therefore F_{W_k}(e^y) = \int_{0.4}^{e^y} \frac{1}{2.6} \, dw.$$

$$= \frac{e^y - 0.4}{2.6}$$

$$\therefore f_Y(y) = \frac{dF_W(e^y)}{dy}.$$

$$= \frac{e^y}{2.6}$$

$$\therefore f_Y(y) = \begin{cases} \frac{e^y}{2.6}, & y \in [\log(0.4), \log(3)]. \\ 0, & \text{otherwise} \end{cases}$$

$$F_Y(y) = \begin{cases} 0, & y < \log(0.4) \\ \frac{e^y - 0.4}{2.6}, & y \in [\log(0.4), \log(3)]. \\ 1, & y \geq \log(3) \end{cases}$$

(b) $$E[\log W_k] = E[Y] = \int_{\log(0.4)}^{\log(3)} y \frac{(e^y)}{2.6} dy.$$

$$= \frac{1}{2.6} \int_{\log(0.4)}^{\log(3)} y(e^y) dy.$$

$$= \frac{1}{2.6} \left[e^y (y-1) \right] \Big|_{\log(0.4)}^{\log(3)}.$$

$$= \frac{1}{2.6} [3(\log 3 - 1) - 0.4(\log(0.4) - 1)].$$

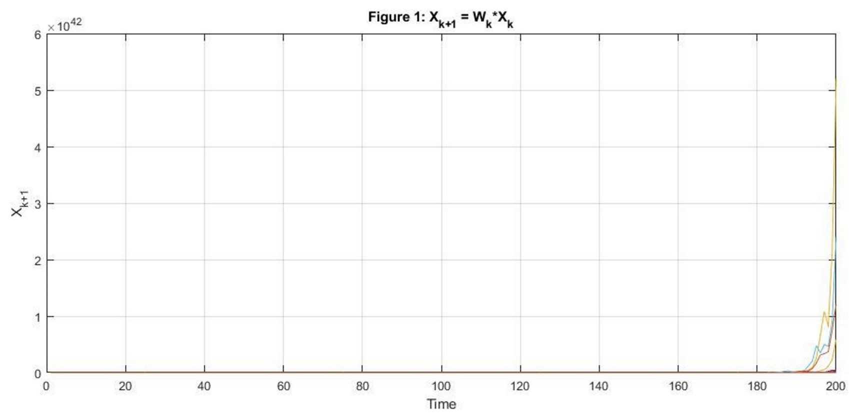
$$= \frac{1}{2.6} (3\log 3 - 0.4\log(0.4) - 2.6).$$

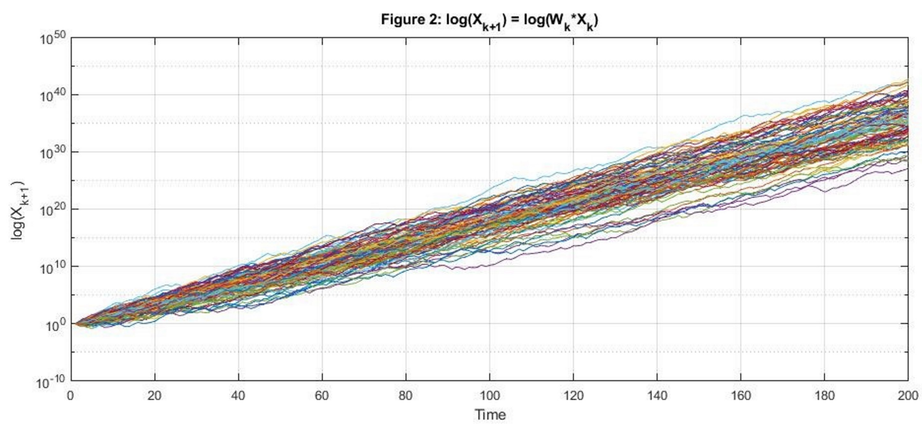
$$= \frac{1}{2.6} \left(\log\left[\frac{27}{(0.4)^{0.4}}\right] \right) - 1 \approx 0.41 > 0.$$

$\because E[\log W_k] > 0$

$\therefore \lim_{k\to\infty} X_k = \infty$ a.s.

(c)





(d) $\because X_{k+1} = W_k X_k = W_k W_{k-1} X_{k-1} = W_k \cdots W_1 X_1$.

$\therefore \log X_{k+1} = \log(W_k) + \cdots + \log(W_1) + \log(X_1)$.

Let $Y_k = \log X_{k+1}$, $Z_k = \log(W_k)$.

$\therefore Y_k - Y_1 = \sum_{i=1}^{k} Z_i$.

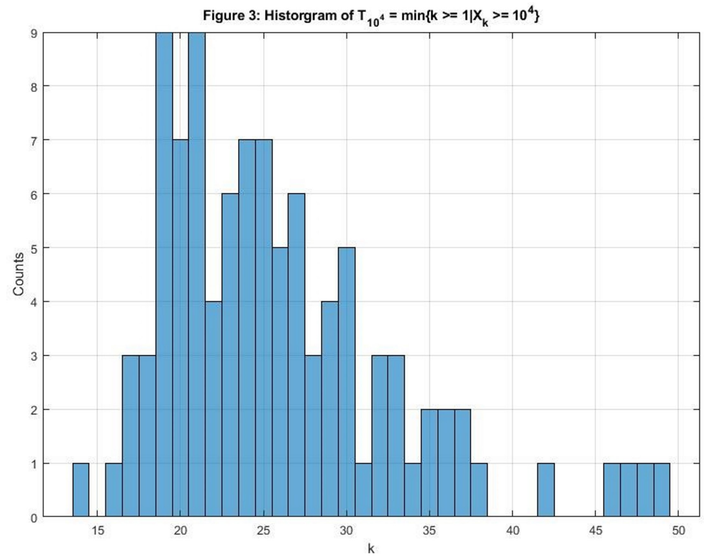
$\because E[Z_k] > 0$.

by SLLN:

$$\lim_{k\to\infty} \frac{Y_k - Y_1}{k} \xrightarrow{a.s} \mu_{Z_k} > 0.$$

$$\therefore \lim_{k\to\infty} X_k \xrightarrow{a.s} \lim_{k\to\infty} e^{Y_1 \cdot k\mu_{Z_k}} = \infty$$

(f)



(g)

6. $Y_i = X + Z_i$ for $i = 1, 2, \cdots, k$, where $X, Z_1, \cdots, Z_k \sim N(0,1)$ are all indep.

$$\therefore \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_k \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} X + \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_k \end{bmatrix}$$

$$\therefore \hat{X} = \text{Cov}[X, Y]\,\text{Cov}^{-1}[Y, Y](Y - \bar{Y}) + \bar{X}$$

$$= \text{Cov}[X, Y]\,\text{Cov}^{-1}[Y, Y]\,Y$$

$$= \text{Cov}[X, X]\left(\text{Cov}[X, X] + \text{Cov}[Z, Z]\right)^{-1} Y$$

$$= 1^T \cdot (1 \cdot 1^T + I)^{-1} Y$$

$$- \quad 1^T Y$$

$$= 1 \cdot (1 \cdot 1^{T} + 1) \; Y$$

$$= \frac{1^{T} Y}{(k+1)}.$$

7. $S_n = \sum_{k=1}^{n} X_k.$

$$\therefore \frac{S_n}{n} = \frac{\sum_{k=1}^{n} X_k}{n} = \frac{n\mu}{n} = \mu.$$

$$\therefore Var\left(\frac{S_n}{n}\right) = Var\left(\frac{\sum_{k=1}^{n} X_k}{n}\right) \qquad E\left[\left(\frac{S_n}{n}\right)^2\right] - E\left[\frac{S_n}{n}\right]^2$$

$$= Var\left(\frac{X_1 + X_2 + \cdots + X_n}{n}\right)$$

$$= \frac{1}{n^2} \cdot Var(X_1 + X_2 + \cdots + X_n).$$

$\because \{X_k\}$ i.i.d.

$$\therefore Var\left(\frac{S_n}{n}\right) = \frac{1}{n^2}\left[Var(X_1) + Var(X_2) + \cdots + Var(X_n)\right].$$

$$= \frac{n}{n^2} \sigma^2.$$

$$= \frac{\sigma^2}{n}.$$

by Chebyshev's inequality:

$$P\left[\| Y - \bar{Y} \|^2 \geq \alpha\right] \leq \frac{Var(Y)}{\alpha^2},$$

where $\alpha > 0$, $Y = \frac{S_n}{n}$.

$$\therefore \bar{Y} = \frac{n\mu}{n} = \mu.$$

$$\therefore P\left[\left\| \frac{S_n}{n} - \mu \right\|^2 \geq \alpha\right] \leq \frac{Var\left(\frac{S_n}{n}\right)}{\alpha^2} = \frac{\sigma^2}{n\alpha^2}.$$

$$\because \sigma^2 < \infty$$

$$\therefore \lim_{n \to \infty} P\left[\left\| \frac{S_n}{n} - \mu \right\|^2 \geq \alpha\right] = 0.$$

$\therefore \lim_{n\to\infty} \|\frac{S_n}{n} - \mu\|^2 = 0 = \lim_{n\to\infty} Var(\frac{S_n}{n}) = \lim_{n\to\infty} \frac{\sigma^2}{n}$

5. $\vec{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$ which a $n\times 1$ random vector.

its $n\times n$ covariance matrix : $C = E[(X-\bar{X})(X-\bar{X})^T] \geq 0$.

Let $\vec{u} \in \mathbb{R}^n$ be non-zero $n\times 1$ random vector.

$\therefore \quad u^T C u = E[u^T(X-\bar{X})(X-\bar{X})^T u] \geq 0$

$= E[(Z-\bar{Z})(Z-\bar{Z})^T]$, where $Z = u^T X$

$\therefore$ the covariance matrix of X is always positive semi-definite.

the covariance matrix is not positive semi-definite in the case like:

$C = E[(X-\bar{X})(Y-\bar{Y})^T]$, where $Y = -X$.

```
K = 200;
N = 100;

w_k = 0.4+2.6*rand(N,K);
x_k = zeros(N,K);
x_k(:,1) = 1;

for i = 1:N
    for j = 1:K-1
        x_k(i,j+1)=x_k(i,j)*w_k(i,j);
    end
end

figure(1)
plot(x_k'); grid on;
title("Figure 1: X_{k+1} = W_k*X_k");
ylabel("X_{k+1}");
xlabel("Time");

figure(2)
semilogy(x_k'); grid on;
title("Figure 2: log(X_{k+1}) = log(W_k*X_k)");
ylabel("log(X_{k+1}) ");
xlabel("Time");

%%
a = 1e4;
A = x_k;
A(A<a) = inf;
[~,I] = min(A,[],2);

%%
figure(3)
histogram(I); grid on;
title("Figure 3: Historgram of T_{10^4} = min\{k >= 1|X_k >= 10^4\}");
ylabel("Counts");
xlabel("k");
```