HW6

Monday, November 21, 2022 2:49 PM

1. $X \sim N(0, \sigma^2)$, $Y \sim N(0, v^2)$ are indep. $Z = X + Y$.

$\therefore Z \sim N(0, \sigma^2 + v^2)$

(a) $$\hat{X} = a^* Z + b^*$$
$$= \frac{Cov(X,Z)}{Var(Z)}(Z - E[Z]) + E[X].$$
$$= \frac{Cov(X,Z)}{Var(Z)} \cdot Z$$
$$= \frac{\sigma^2}{\sigma^2 + v^2} \cdot Z$$

(b) $$f_{Z|X}(z|x) = f_Y(z - x) = \frac{1}{\sqrt{2\pi v^2}} e^{-\frac{(z-x)^2}{v^2}}$$

(c) $$f_{X|Z}(z|x) = \frac{f_{Z|X}(z|x) f_X(x)}{f_Z(z)}.$$
$$= \frac{\left(\frac{1}{\sqrt{2\pi v^2}} e^{-\frac{(z-x)^2}{v^2}}\right) \cdot \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{\sigma^2}}\right)}{\frac{1}{\sqrt{2\pi(v^2+\sigma^2)}} e^{-\frac{z^2}{v^2+\sigma^2}}}.$$
$$= \sqrt{\frac{v^2+\sigma^2}{2\pi v^2 \sigma^2}} \cdot e^{\frac{z^2}{v^2+\sigma^2} - \frac{(z-x)^2}{v^2} - \frac{x^2}{\sigma^2}}.$$
$$= \sqrt{\frac{v^2+\sigma^2}{2\pi v^2 \sigma^2}} e^{-\frac{[(v^2+\sigma^2)x - \sigma^2 z]^2}{v^2\sigma^2(v^2+\sigma^2)}}.$$
$$= \sqrt{\frac{v^2+\sigma^2}{2\pi v^2 \sigma^2}} e^{-\frac{\left(x - \frac{\sigma^2}{v^2+\sigma^2} z\right)^2}{\frac{v^2\sigma^2}{v^2+\sigma^2}}}.$$

(d) MMSE $= E[X|Z]$.

from (c), we know $X \mid \{Z = z\} \sim N\left(\frac{\sigma^2}{v^2+\sigma^2} z, \frac{v^2\sigma^2}{v^2+\sigma^2}\right)$.

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$\therefore$ $MMSE = E[X \mid z]$

$$= \frac{\sigma^2}{V^2+\sigma^2} z = LMMSE$$

2. $X \sim U(0,1)$, $Y \sim U(0,1)$ are indep., $Z = X + \frac{1}{2}Y$.

(a) $MMSE = E[X \mid z]$.

$$= \int_{-\infty}^{\infty} x \, f_{X|Z}(x|z) \, dx.$$

$$f_{X|Z}(x|z) = \frac{f_{Z|X}(z|x) f_X(x)}{f_Z(z)}$$

$f_X(x) = 1$.

$f_{Z|X}(z|x) = f_Y(2(z-x)) = 1$.

$\because Z = X + \frac{1}{2}Y$ $\therefore Z \in [0, \frac{3}{2}]$.

① $z \in [0,1] \Rightarrow z - x \geq 0 \Rightarrow x \leq z$.

$$\therefore f_Z = \int_{-\infty}^{\infty} f_Y(2(z-x)) f_X(x) dx = \int_0^z f_Y(2(z-x)) dx + \int_z^1 f_Y(2\cancel{(z-x)}^{0}) dx$$

$$= \int_0^z 1 \, dx = z.$$

$$f_{X|Z}(x|z) = \frac{1}{z}$$

$$\underline{E[X|Z] = \int_0^z \frac{x}{z} dx = \frac{z}{2}}$$

② $z \in (1, \frac{3}{2}] \Rightarrow z - x \leq \frac{1}{2} \Rightarrow x \geq z - \frac{1}{2}$

$$\therefore f_Z = \int_0^{z-\frac{1}{2}} f_Y(2\cancel{(z-x)}^{0}) dx + \int_{z-\frac{1}{2}}^1 f_Y(2(z-x)) dx.$$

$$= \int_{z-\frac{1}{2}}^1 1 \, dx = \frac{3}{2} - z$$

$\therefore f_Z = \int_0 f_Y(z-x)\,dx + \int_{z-\frac{1}{2}} f_Y(\ldots)$

$$= \int_{z-\frac{1}{2}}^{1} 1\,dx = \frac{3}{2} - z$$

$$\therefore f_{X|Z}(x|z) = \frac{1}{\frac{3}{2} - z}$$

$$\therefore E[X|Z] = \int_{z-\frac{1}{2}}^{1} \frac{x}{\frac{3}{2}-z}\,dx = \frac{\frac{1}{2}\left(1 - (z-\frac{1}{2})^2\right)}{\frac{3}{2} - z}$$

$$= \frac{(1 - z^2 + z - \frac{1}{4})}{\frac{3}{4} - \frac{1}{2}z}$$

$$= \frac{-z^2 + z + \frac{3}{4}}{\frac{3}{4} - \frac{1}{2}z}$$

$$= \frac{-4z^2 + 4z + 3}{3 - 2z}$$

(b) $\hat{X} = a^* Z + b^*$

$$= \frac{Cov(X, Z)}{Var(Z)}(Z - E[Z]) + E[X]$$

$$= \frac{E[X^2]}{Var(Z)}(Z - E[Z]) + E[X]$$

$$= \frac{E[X^2]}{E[Z^2] - (E[Z])^2}(Z - E[Z]) + E[X]$$

$=$

5. $P = \begin{pmatrix} 1/4 & 3/4 & 0 \\ 1/4 & 0 & 3/4 \\ 0 & 1/2 & 1/2 \end{pmatrix}$

(a) $\pi^T P = \pi^T$.

$\Rightarrow [\pi_0 \quad \pi_1 \quad \pi_2] \begin{bmatrix} 1/4 & 3/4 & 0 \\ 1/4 & 0 & 3/4 \\ 0 & 1/2 & 1/2 \end{bmatrix} = [\pi_0 \quad \pi_1 \quad \pi_2]$.

$\Rightarrow \begin{cases} \frac{1}{4}\pi_0 + \frac{1}{4}\pi_1 = \pi_0 \\ \frac{3}{4}\pi_0 + \frac{1}{2}\pi_2 = \pi_1 \\ \frac{3}{4}\pi_1 + \frac{1}{2}\pi_2 = \pi_2 \end{cases}$

$\Rightarrow \quad \pi_1 = 3\pi_0$

$\pi_2 = \frac{3}{2}\pi_1 = \frac{9}{2}\pi_0$

$\because \pi_0 + \pi_1 + \pi_2 = 1$

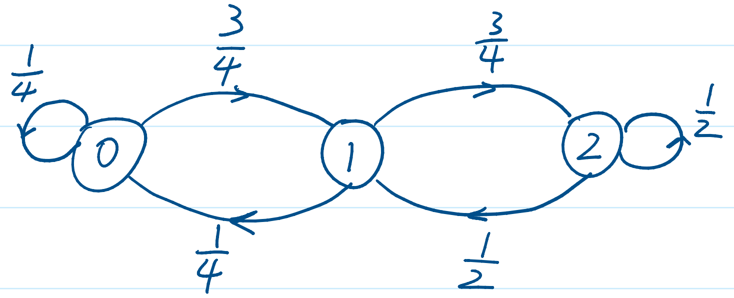
$\therefore \pi_0 + 3\pi_0 + \frac{9}{2}\pi_0 = 1$

$\frac{17}{2}\pi_0 = 1$

$\pi_0 = \frac{2}{17}, \quad \pi_1 = \frac{6}{17}, \quad \pi_2 = \frac{9}{17}$

$$\therefore \pi = \begin{bmatrix} \frac{2}{17} \\ \frac{6}{17} \\ \frac{9}{17} \end{bmatrix}$$

(b)



$\therefore$ this Markov chain is irreducible

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$\therefore$ it is aperiodic.

$d(i) = 1, \ \forall i.$

(c)

$$P^4 = \left(\begin{bmatrix} 1/4 & 3/4 & 0 \\ 1/4 & 0 & 3/4 \\ 0 & 1/2 & 1/2 \end{bmatrix} \right)^4$$

$$= \begin{bmatrix} 37/256 & 75/256 & 9/16 \\ 25/256 & 27/64 & 123/256 \\ 1/8 & 41/128 & 71/128 \end{bmatrix}$$

$$X_4^T = [1/3 \quad 1/3 \quad 1/3] \cdot P^4$$

$$= \begin{bmatrix} 47/384 & 265/768 & 409/768 \end{bmatrix}.$$

$$\therefore P(X_4 = S_3) = \frac{409}{768}$$

(d) $P(X_4 = S_2) + P(X_4 = S_3) = \frac{265+409}{768} = \frac{674}{768} = \frac{337}{384}.$