Solutions to Homework 1

1. Let A, B, C be arbitrary sets. Prove or disprove the following statements. Note that to disprove a statement you need to provide an example that the statement fails.

   (a) $(A - (A - B)) = A \cap B$.

   (b) $A \cap (B \cup C) = (A \cap B) \cup C$.

   (c) if $A \subset C$, then $A \cup (C - A) = C$.

   Solution:

   (a) $A - B = A \cap B^c$. $A - (A - B) = A \cap (A - B)^c = A \cap (A \cap B^c)^c$. Using deMorgan's Law and distributive property of unions and intersections we get,

   $$A - (A - B) = A \cap (A^c \cup (B^c)^c) = A \cap (A^c \cup B) = (A \cap A^c) \cup (A \cap B) = \phi \cup (A \cap B) = A \cap B$$

   (b) False: Let $A = \{1, 2, 4\}, B = \{2, 3\}$ and, $C = \{3, 4\}$. Then $A \cap (B \cup C) = \{2, 4\}$, whereas $(A \cap B) \cup C = \{2, 3, 4\}$.

   (c) True: $A \cup (C - A) = A \cup (C \cap A^c) = (A \cup C) \cap (A \cup A^c)$. Since $A \subset C$, $A \cup C = C$ and $C \subseteq A \cup A^c$, therefore $A \cup (C - A) = C$.

2. Consider the sample space $\Omega = \{1, 2, 3, 4, 5\}$ and let $\mathcal{F} = \mathcal{P}(\Omega)$ be the set of all subsets of Ω. Consider a probability measure $P : \mathcal{F} \to [0, 1]$ satisfying $P(\{1, 2\}) = 0.2$ and $P(\{2, 3\}) = 0.3$ (and of course the axioms of probability measure). For each of the following events determine whether its probability can be uniquely determined (with this information) or not. If so, find the probability and if not, reason why you cannot find their probability.

   (a) $A = \{2\}$

   (b) $B = \{1, 3\}$

   (c) $C = \{5\}$

   Solution: Consider the following probability assignments

   1. $P(\{1\}) = 0.1, P(\{2\}) = 0.1, P(\{3\}) = 0.2, P(\{4\}) = 0.2, P(\{5\}) = 0.4,$ and
   2. $P(\{1\}) = 0.15, P(\{2\}) = 0.05, P(\{3\}) = 0.25, P(\{4\}) = 0.25, P(\{5\}) = 0.3.$

   (a) For assignment 1, $P(A) = 0.1$ while for assignment 2, $P(A) = 0.15$,

   (b) For assignment 1, $P(B) = 0.3$ while for assignment 2, $P(A) = 0.4$,

   (c) For assignment 1, $P(C) = 0.4$ while for assignment 2, $P(C) = 0.3$.

   In general, in order to have a probability measure on the σ-algebra provided it is sufficient to define the probability of the sets whose unions form all the other sets in the σ-algebra. In order to have a probability measure satisfying the mentioned constraints, defined through the probabilities of each element of Ω, we need the probabilities to satisfy the following set of equations (and inequalities)

   $$a + b = 0.2, \qquad b + c = 0.3, \qquad a + b + c + d + e = 1,$$
   $$0 \leq a, 0 \leq b, 0 \leq c, 0 \leq d, 0 \leq e,$$

   where $P(\{1\}) = a, P(\{2\}) = b, P(\{3\}) = c, P(\{4\}) = d, P(\{5\}) = e$. There are infinitely many solutions satisfying the given set of constraints.

3. Solve Problem 2.1 from Prof. Kim notes (page 10).

   Solution: Suppose that $A_1, A_2, \ldots \in \mathcal{F}$. Then, by the property that $\mathcal{F}$ is closed under complement, that is $A \in \mathcal{F}$ implies $A^c \in \mathcal{F}$,

   $$A_1^c, A_2^c, \ldots \in \mathcal{F}.$$

   Also, by the property of closure under countable union,

   $$\bigcup_{i=1}^{\infty} A_i^c \in \mathcal{F}.$$

   Finally, by the closure under complement again,

   $$\bigcap_{i=1}^{\infty} A_i = \left(\bigcup_{i=1}^{\infty} A_i^c\right)^c \in \mathcal{F}.$$

4. Solve Problem 2.2 from Prof. Kim notes (page 10).

   Solution:

   (a) Let $B_n = A_n \cap \left(\bigcup_{i=1}^{n-1} A_i\right)^c$, for $n \geq 1$. Clearly $B_i \bigcap B_j = \emptyset$, for $i \neq j$. Note that B_n may be empty. Then,

   $$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i$$

   and

   $$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i.$$

   By the countable additivity property

   $$\mathrm{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathrm{P}\left(B_i\right)$$

   and

   $$\mathrm{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathrm{P}\left(B_i\right)$$

   Then,

   $$\lim_{n\to\infty} \mathrm{P}\left(\bigcup_{i=1}^{n} A_i\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \mathrm{P}\left(B_i\right) = \sum_{i=1}^{\infty} \mathrm{P}\left(B_i\right) = \mathrm{P}\left(\bigcup_{i=1}^{\infty} A_i\right).$$

   (b) Let $Z_n = A_n^c$, for $n \geq 1$. Then by De Morgan's Law,

   $$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} Z_i^c = \left(\bigcup_{i=1}^{n} Z_i\right)^c$$

   and

   $$\bigcap_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} Z_i^c = \left(\bigcup_{i=1}^{\infty} Z_i\right)^c$$

   It follows that

   $$\mathrm{P}\left(\bigcap_{i=1}^{\infty} A_i\right) = \mathrm{P}\left(\left(\bigcup_{i=1}^{\infty} Z_i\right)^c\right) = 1 - \mathrm{P}\left(\bigcup_{i=1}^{\infty} Z_i\right)$$

From part (a),

$$
\begin{aligned}
1-\mathrm{P}\left(\bigcup_{i=1}^{\infty} Z_i\right) &= 1-\lim_{n\to\infty}\mathrm{P}\left(\bigcup_{i=1}^{n} Z_i\right)\\
&= 1-\lim_{n\to\infty}\mathrm{P}\left(\left(\bigcap_{i=1}^{n} A_i\right)^c\right)\\
&= 1-\lim_{n\to\infty}\left(1-\mathrm{P}\left(\bigcap_{i=1}^{n} A_i\right)\right)\\
&= \lim_{n\to\infty}\mathrm{P}\left(\bigcap_{i=1}^{n} A_i\right)
\end{aligned}
$$

5. Solve Problem 2.3 from Prof. Kim notes (page 10).

   Solution: For $A \in 2^{\mathbb{R}}$, i.e., for $A \subseteq \mathbb{R}$, note that $A \cap \Omega \subseteq \Omega$, so $A \cap \Omega \in 2^{\Omega}$. Therefore, $\mathrm{P}(A\cap\Omega)$ is well-defined as a function on $2^{\mathbb{R}}$ Now, we verify the axioms of probability.

   (a) $\mathrm{P}(A\cap\Omega) \geq 0$ for every $A \in 2^{\mathbb{R}}$, because $A\cap\Omega \in 2^{\Omega}$, and $\mathrm{P}(\cdot)$ is a probability measure satisfying $P(B) \geq 0$ for every $B \in 2^{\Omega}$.

   (b) $\mathrm{P}(\mathbb{R}\cap\Omega) = \mathrm{P}(\Omega) = 1$.

   (c) If $A_1, A_2, \ldots$ are disjoint subsets in $\mathbb{R}$, then $A_1\cap\Omega, A_2\cap\Omega, \ldots$ are disjoint in Ω. Therefore,

$$
\mathrm{P}\left(\left(\bigcup_{i=1}^{\infty} A_i\right)\cap\Omega\right) = \mathrm{P}\left(\bigcup_{i=1}^{\infty}(A_i\cap\Omega)\right) = \sum_{i=1}^{\infty}\mathrm{P}\left(A_i\cap\Omega\right).
$$

6. Let $\Omega = \mathbb{R}$, and $\mathcal{F}$ be all subsets so that A or A^c is countable. Also, let

$$
P(A) = \begin{cases} 0, & A \text{ is countable} \\ 1, & A^c \text{ is countable} \end{cases}.
$$

   Show that $(\Omega, \mathcal{F}, P)$ is a probability space.

   Solution: Consider these two lemma:

   Lemma 1 *Union of countably many countable set is countable.*

   Proof: The proof is similar to the proof of countability of $\mathbb{Q}$. ■

   Lemma 2 *If A, B are countable, then $A^c \cap B^c \neq \emptyset$.*

   Proof: Since A, B are countable, $A\cup B$ are countable, and hence, $A \cup B \neq \mathbb{R}$, and so, $A^c\cap B^c = (A\cup B)^c \neq \emptyset$ ■

   To solve the problem, first, we need to show that $\mathcal{F}$ is a σ-algebra.

   (a) $\emptyset \in \mathcal{F}$.

   (b) If $A \in \mathcal{F}$, then A or A^c is countable. Therefore, A^c or A is countable, and hence, $A^c \in \mathcal{F}$.

   (c) If A_i are countable for all i, then from Lemma 1, $\cup_{i=1}^{\infty} A_i$ is countable, and hence $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$. If one of them, say k, is not countable, then A_k^c is countable. Also, we have

$$
\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c \subset A_k^c.
$$

   Therefore, $(\cup_{i=1}^{\infty} A_i)^c$ is countable and in $\mathcal{F}$.

Now, we need to show that $(\Omega, \mathcal{F}, P)$ is a probability space.

(a) $1 \geq P(\cdot) \geq 0$.

(b) $\mathbb{R}^c = \emptyset$ is countable, so $P(\Omega) = 1$.

(c) If A_i are countable for all i, then $\cup_{i=1}^{\infty} A_i$ is countable, and hence

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = 0 = \sum_{i=1}^{\infty} P(A_i).$$

If one of them, say k, is not countable, then A_k^c is countable, and hence $P(A_k) = 1$. Also, $\left(\cup_{i=1}^{\infty} A_i\right)^c$ is countable. Therefore, we have

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = 1 = \sum_{i=1}^{\infty} P(A_i).$$

We cannot have two disjoint sets $A_k, A'_k \in \mathcal{F}$ that are not countable. Since, if A_k, A'_k are not countable, then $A_k^c, A_k'^c$ are countable, and from Lemma 2, we know that $(A_k^c)^c \cap (A_k'^c)^c \neq \emptyset$.

7. Consider the random processes

$$\begin{aligned} x_t &= \frac{w_1 + w_2 + \cdots + w_t}{t^{0.4}}, \\ y_t &= \frac{w_1 + w_2 + \cdots + w_t}{\sqrt{t}}, \\ z_t &= e^{w_1} e^{w_2} \cdots e^{w_t}. \end{aligned}$$

where $\{w_t\}$ is an i.i.d random sequence that is Gaussian with zero mean and unit variance (you can use `normrnd` comman in MATLAB).

(a) Find a function $f_t(x, w)$ such that
$$x_t = f_t(x_{t-1}, w_t).$$

(b) Using MATLAB, plot 10 sample paths of x_t, y_t, and z_t for $t = 1, \ldots, 1000$. By investigating the plots and using your intuition, what do you think about the behavior of x_t, y_t, and z_t as $t \to \infty$? (Please include your code as well.)

Solution:

(a) We have

$$x_t \cdot t^{0.4} = (t-1)^{0.4} \cdot x_{t-1} + w_t.$$

Therefore,

$$x_t = \left(1 - \frac{1}{t}\right)^{0.4} x_{t-1} + \frac{1}{t^{0.4}} w_t,$$

and hence $f_t(x, w) = x\left(1 - \frac{1}{t}\right)^{0.4} + \frac{1}{t^{0.4}} w$.

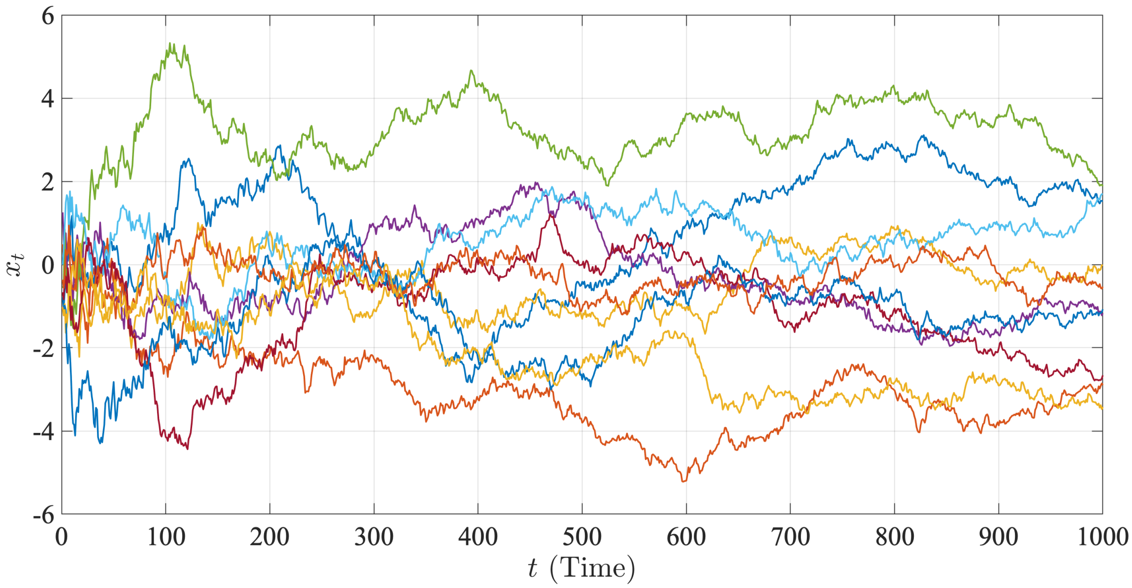
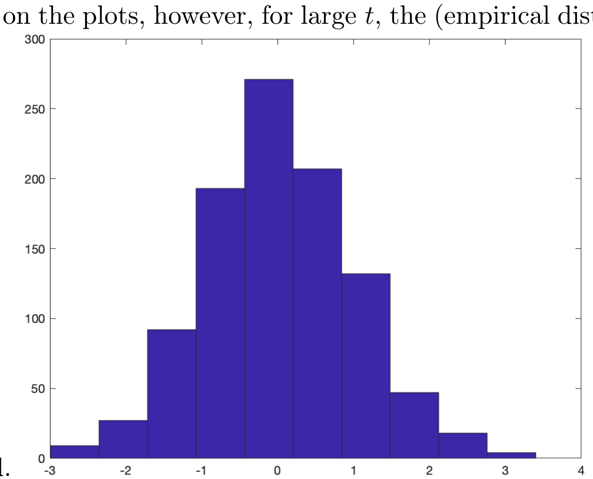


Figure 1: x_t

(b) As can be observed here z_t converges to 0 or diverges to ∞. In the case of x_t and y_t, there is no convergence claim to be made based on the plots, however, for large t, the (empirical distribution) of y_t is roughly normally distributed.




```
close all;
clear all;
T=1000;
N=10;
w=randn(N,T);
for i=1:N
    x(i,:)=cumsum(w(i,:))./([1:T].^0.4);
    y(i,:)=cumsum(w(i,:))./(sqrt([1:T]));
    z(i,:)=exp(cumsum(w(i,:)));
end
plot(x')
ylabel('$x_t$','Interpreter','latex');
xlabel('$t$ (Time)','Interpreter','latex');
figure
plot(y')
ylabel('$y_t$','Interpreter','latex');
```

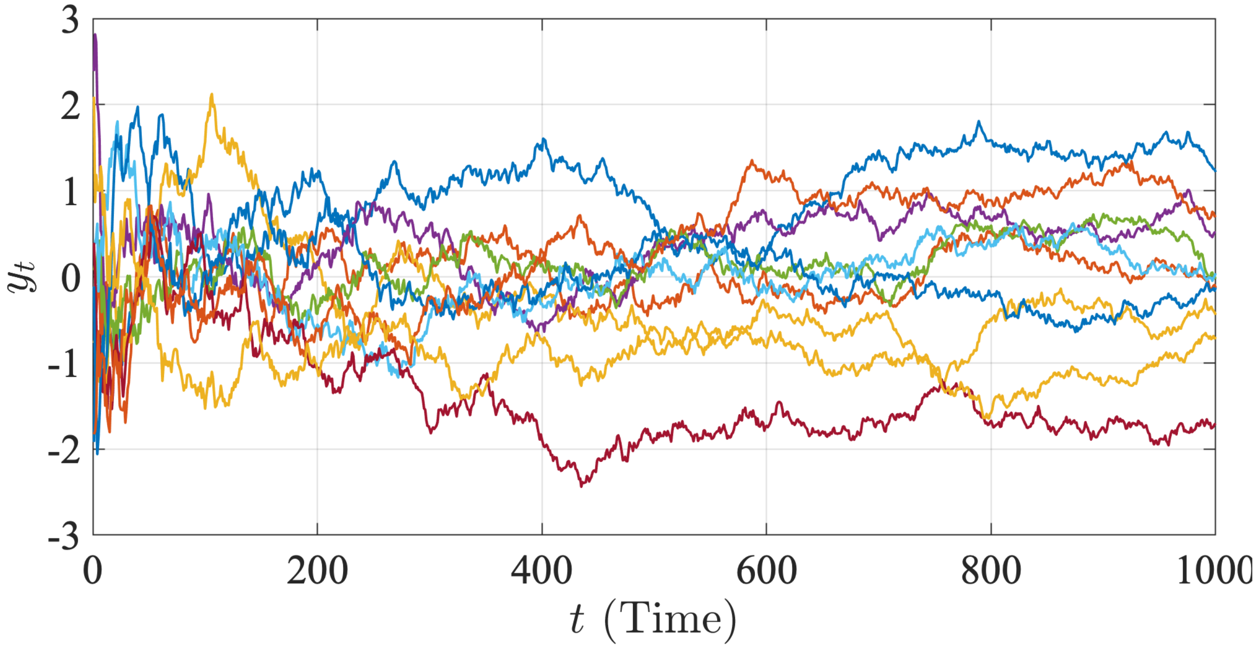


Figure 2: y_t

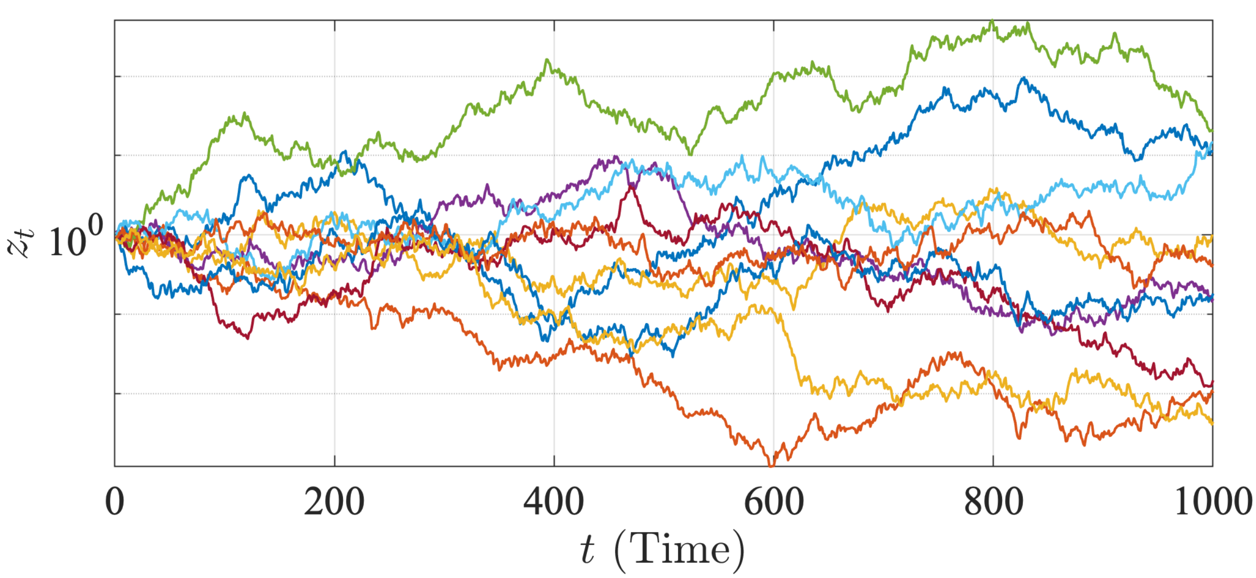


Figure 3: log scale plot z_t

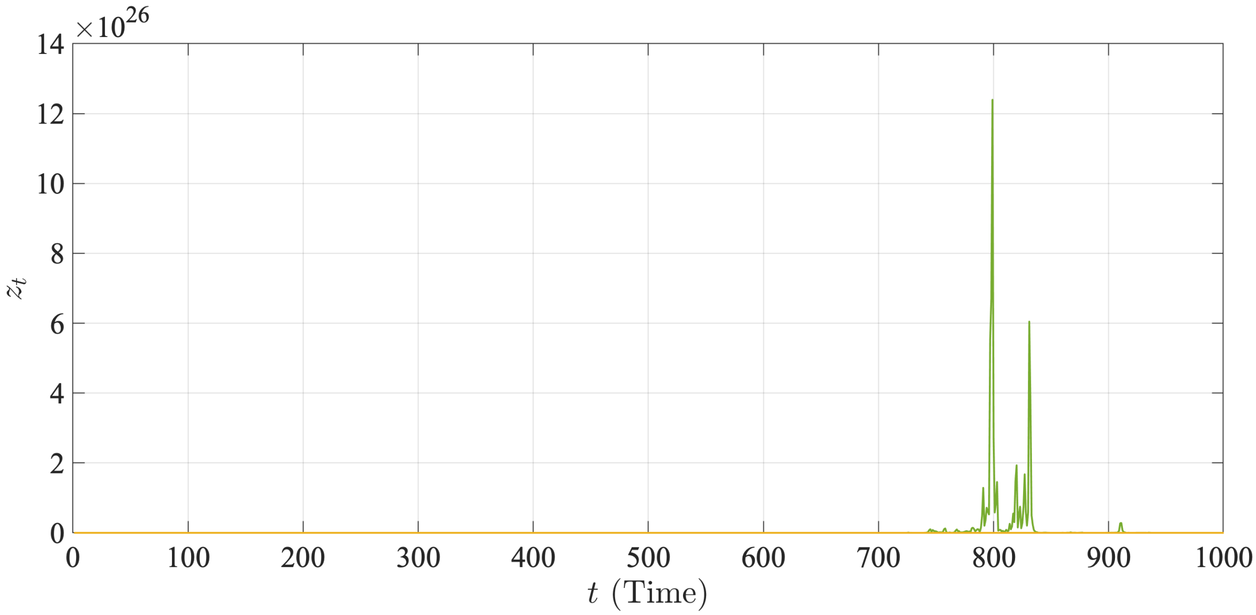


Figure 4: linear scale plot z_t

```
xlabel('$t$ (Time)','Interpreter','latex');
figure
plot(z')
ylabel('$z_t$','Interpreter','latex');
xlabel('$t$ (Time)','Interpreter','latex');
figure
semilogy(z')
ylabel('$z_t$','Interpreter','latex');
xlabel('$t$ (Time)','Interpreter','latex');
```