Homework 2

Due: 8:00pm (PDT) Tuesday, Oct 11, 2022

Reading assignment: read Section 1.2 of Bruce Hajek book and 2.5 of Young-Han Kim's notes on conditional probability.

1. Similar to Borel σ-algebra in $\mathbb{R}$, we define the Borel σ-algebra in $\Omega = \mathbb{R}^n$, to be the σ-algebra generated by the collection of open boxes, i.e.,

$$\mathcal{B}^n := \sigma(\{(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n) \mid a_i < b_i, a_i, b_i \in \mathbb{R} \text{ for all } i\}).$$

   Using this, show that closed boxes are in the Borel σ-algebra in $\mathbb{R}^n$. Here, a closed box is a set of the form

$$[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n],$$

   for some scalars $a_1 \leq b_2, a_2 \leq b_2, \ldots, a_n \leq b_n$.

2. Show that if events A, B are independent, then A^c and B are also independent.

3. Problem 2.10 of Prof. Kim's notes.

4. Problem 2.11 of Prof. Kim's notes.

5. Suppose that an ordinary deck of 52 cards is shuffled so that any order of the 52 cards are equally likely to appear. The cards are then turned over one at a time until the first ace appears. Given that the first ace is the 20th card to appear, what is the conditional probability that the card following it is the
   (a) ace of spades?
   (b) two of clubs?

6. Let $\Omega = \mathbb{R}$ and $\mathcal{F}$ be the σ-algebra generated by $\mathcal{A} = \{\mathbb{Q}\}$ (i.e., $\mathcal{F} = \sigma(\{\mathbb{Q}\})$ where $\mathbb{Q}$ is the set of rational numbers).
   (a) Write explicitly all the sets belonging to $\mathcal{F}$.
   (b) Characterize all the mappings $X : \Omega \to \mathbb{R}$ that are random variables with respect to $(\Omega, \mathcal{F})$.