Homework 3-Solution

Reading assignment: Read Section 3.5 of Prof. Kim's notes on functions of a random variable before addressing Problem 1.

1. Problem 3.11 of Prof. Kim's notes.

   Solution: Note that if $-0.5 \leq X \leq 0.5$, then $Y = 0$. Otherwise, $Y \geq \frac{5}{4}$. Therefore, we have $F_Y(y) = 0$ for $y < 0$ and for $0 \leq y < \frac{5}{4}$, we have

$$F_Y(y) = \mathbf{P}[Y \leq y] = \mathbf{P}\left[|X| < \frac{1}{2}\right] = \frac{1}{2}.$$

   For $\frac{5}{4} \leq y < 2$, we can write

$$F_Y(y) = \mathbf{P}[Y \leq y] = \mathbf{P}[X^2 + 1 \leq y] = \mathbf{P}[X^2 \leq y - 1] = \mathbf{P}\left[-\sqrt{y-1} \leq X \leq \sqrt{y-1}\right] = \sqrt{y-1}.$$

   Since $Y \leq 2$, we have $F_Y(y) = 1$ for $y \geq 2$. Therefore, we have

$$F_Y(y) = \begin{cases} 0, & \text{if } y < 0 \\ \frac{1}{2}, & \text{if } 0 \leq y < \frac{5}{4} \\ \sqrt{y-1}, & \text{if } \frac{5}{4} \leq y < 2 \\ 1, & 2 \leq y \end{cases}.$$

2. Let $\{X_k\}$ to be a random process over an underlying probability space $(\Omega, \mathcal{F}, \mathbf{P}(\cdot))$.

   (a) For any $\alpha \in \mathbb{R}$, show that the event E_α where the limiting point of $X_k(\omega) = \alpha$ is an event in $\mathcal{F}$:

$$E_\alpha = \{\omega \in \Omega \mid \lim_{k\to\infty} X_k(\omega) = \alpha\}.$$

   (b) Show that the set E of sample points that X_k has limit is measurable (i.e., it is an event in $\mathcal{F}$):

$$E = \{\omega \in \Omega \mid \lim_{k\to\infty} X_k(\omega) \text{ exists}\}.$$

   Solution: First, we prove

$$\inf_k X_k \quad \sup_k X_k \quad \limsup_{k\to\infty} X_k \quad \liminf_{k\to\infty} X_k$$

   are random variables. ($\inf_k X_k \triangleq \inf\{X_1, X_2, \ldots\}$ and $\sup_k X_k \triangleq \sup\{X_1, X_2, \ldots\}$)
   Proof. We have to show for all a

$$(\inf_k X_k)^{-1}((-\infty, a)) \in \mathcal{F}.$$

   By definition, we have

$$(\inf_k X_k)^{-1}((-\infty, a)) = \left\{\omega \middle| \inf_k X_k(\omega) < a\right\} \triangleq \left\{\inf_k X_k < a\right\}.$$

   Since the infimum of a sequence is less than a if and only if some term is less than a (if all terms are greater or equal to a then so is the infimum), we have

$$\left\{\inf_k X_k < a\right\} = \bigcup_{k=1}^{\infty} \{X_k < a\} \in \mathcal{F},$$

where follows from the fact that X_ks are random variables. A similar argument shows $\{\sup_k X_k > a\} = \cup_k \{X_k > a\} \in \mathcal{F}$. For the last two, we observe

$$\liminf_{k\to\infty} X_k = \sup_k \left(\inf_{m\geq k} X_m\right)$$
$$\limsup_{k\to\infty} X_k = \inf_k \left(\sup_{m\geq k} X_m\right)$$

To complete the proof in the first case, note that $Y_k = \inf_{m\geq k} X_m$ is a random variable for each k, so $\sup_k Y_k$ is as well.

(a) We have

$$\begin{aligned}E_\alpha &= \{\omega \in \Omega \mid \lim_{k\to\infty} X_k(\omega) = \alpha\}\\ &= \{\omega \in \Omega \mid \liminf_{k\to\infty} X_k(\omega) = \alpha\} \cap \{\omega \in \Omega \mid \limsup_{k\to\infty} X_k(\omega) = \alpha\} \in \mathcal{F},\end{aligned}$$

where follow from the fact $\liminf_{k\to\infty} X_k$ and $\limsup_{k\to\infty} X_k$ are random variables, and so

$$(\liminf_{k\to\infty} X_k)^{-1}(\{\alpha\}), (\limsup_{k\to\infty} X_k)^{-1}(\{\alpha\}) \in \mathcal{F}.$$

(b) We have

$$\begin{aligned}E &= \{\omega \in \Omega \mid \lim_{k\to\infty} X_k(\omega) \text{ exists}\}\\ &= \{\omega \in \Omega \mid \liminf_{k\to\infty} X_k(\omega) = \limsup_{k\to\infty} X_k(\omega)\},\\ &= \{\omega \in \Omega \mid \liminf_{k\to\infty} X_k(\omega) - \limsup_{k\to\infty} X_k(\omega) = 0\},\\ &= (\liminf_{k\to\infty} X_k - \limsup_{k\to\infty} X_k)^{-1}(\{0\}) \in \mathcal{F}.\end{aligned}$$

3. Find a sequence of random variables (i.e., a random process) $\{X_k\}$ such that its limit exists and $\mathbf{E}[\lim_{k\to\infty} X_k] \neq \lim_{k\to\infty} \mathbf{E}[X_k]$. (if you cannot do it by yourself, do research on finding such random variables)

Solution: Let

$$X_k = \begin{cases} 0, & \text{with probaility } 1 - \frac{1}{k^2} \\ k^2, & \text{with probaility } \frac{1}{k^2} \end{cases}.$$

Therefore, $\mathbf{E}[X_k] = 1$ for all k, and hence $\lim_{k\to\infty} \mathbf{E}[X_k] = 1$. Now, we want to show that $\lim_{k\to\infty} X_k = 0$, and hence $\mathbf{E}[\lim_{k\to\infty} X_k] = 0$, which is not equal to $\lim_{k\to\infty} \mathbf{E}[X_k] = 1$.

To prove, consider the sequence of events

$$E_k = \{X_k > 0\}$$

which happens with probability $\mathbf{P}(E_k) = \frac{1}{k^2}$. Since $\sum_{k=1}^{\infty} \mathbf{P}(E_k) < \infty$ and these events are independent, the Borel-Cantelli lemma implies that $\mathbf{P}(\{E_k.\text{ i.o.}\}) = 0$. This implies that for almost all $\omega \in \Omega$, there exists some $T(\omega)$ such that $X_k(\omega) = 0$ for $k \geq T(\omega)$, i.e., $\lim_{k\to\infty} X_k(\omega) = 0$ almost surely.

4. Using the general definition of $\mathbf{E}[X]$ that we discussed in the class, show that for a non-negative discrete random variable X

$$\mathbf{E}[X] = \sum_{k=1}^{\infty} m_k p_X(m_k),$$

where $\mathbf{P}(X \in M) = 1$ and $M = \{m_k \mid k \geq 1\}$.

Solution: Let us define the simple function $X_i = \sum_{k=1}^{i} m_k \mathbf{1}_{\{X=m_k\}}$ for all $i \geq 1$. Therefore, from the definition of expectation of simple functions, $\mathbf{E}[X_i] = \sum_{k=1}^{i} m_k p_X(m_k)$. Since, $X_1 \leq X_2 \leq \cdots \leq X = \lim_{i\to\infty} X_i$, from Monotone Convergence Theorem, we have

$$\mathbf{E}[X] = \mathbf{E}[\lim_{k\to\infty} X_k] = \lim_{k\to\infty} \mathbf{E}[X_k] = \sum_{k=1}^{\infty} m_k p_X(m_k).$$

5. Let X be a finite random variable on $(\Omega, \mathcal{F}, \mathbf{P})$ (i.e., $P(X = \infty) = P(X = -\infty) = 0$. Show that its distribution function F_X satisfies the following properties:

   (a) F_X is non-decreasing.

   (b) $\lim_{x\to-\infty} F_X(x) = 0$, and $\lim_{x\to\infty} F_X(x) = 1$.

   (c) $F_X(\cdot)$ **is right-continuous**, i.e., for any $x \in \mathbb{R}$, $\lim_{y\to x^+} F_X(y) = F_X(x)$.

   (d) Define $F_X(x^-) := \lim_{y\uparrow x} F_X(y)$, then

   $$F_X(x^-) = \mathbf{P}[X < x] = \mathbf{P}[\{\omega \in \Omega \mid X(\omega) < x\}].$$

   (e) For any $x \in \mathbb{R}$, we have $\mathbf{P}[X = x] = F_X(x) - F_X(x^-)$.

   Solution:

   (a) If $x < y$, then $\{X \leq x\} \subset \{X \leq y\}$, which implies that $F_X(x) \leq F_Y(y)$.

   (b) If $x_n, n \geq 1$ is an increasing sequence such that $x_n \to \infty$, then the events $E_n = \{X \leq x_n\}$ form an increasing sequence with

   $$\{X < \infty\} = \bigcup_{n=1}^{\infty} E_n.$$

   It follows from the continuity properties of probability measures that

   $$\lim_{n\to\infty} F_X(x_n) = \lim_{n\to\infty} \mathbf{P}(E_n) = \mathbf{P}(X < \infty) = 1.$$

   Likewise, if $x_n, n \geq 1$ is a decreasing sequence such that $x_n \to -\infty$, then the events $E_n = \{X \leq x_n\}$ form a decreasing sequence with

   $$\emptyset = \bigcap_{n=1}^{\infty} E_n.$$

   In this case, the continuity properties of measures imply that

   $$\lim_{n\to\infty} F_X(x_n) = \lim_{n\to\infty} \mathbf{P}(E_n) = \mathbf{P}(\emptyset) = 0.$$

   (c) If $x_n, n \geq 1$ is a decreasing sequence converging to x, then the sets $E_n = \{X \leq x_n\}$ also form a decreasing sequence with

   $$\{X \leq x\} = \bigcap_{n=1}^{\infty} E_n.$$

   Consequently,

   $$\lim_{n\to\infty} F_X(x_n) = \lim_{n\to\infty} \mathbf{P}(E_n) = \mathbf{P}\{X \leq x\} = F_X(x)$$

   (d) If $x_n, n \geq 1$ is a increasing sequence converging to x, then the sets $E_n = \{X \leq x_n\}$ also form a increasing sequence with

   $$\{X < x\} = \bigcup_{n=1}^{\infty} E_n.$$

   Consequently,

   $$\lim_{n\to\infty} F_X(x_n) = \lim_{n\to\infty} \mathbf{P}(E_n) = \mathbf{P}\{X < x\} = F_X(x^-).$$

(e) We have

$$\mathbf{P}[X = x] = \mathbf{P}[X \leq x] - \mathbf{P}[X < x] = F_X(x) - F_X(x^-).$$

6. For a random process $\{X_k\}$ define $\pi_k = \prod_{i=1}^{k} X_i$. Is the event $\lim_{k\to\infty} \pi_k = \infty$ a tail event?

Solution: It is NOT a tail event. Assume that

$$X_1 = \begin{cases} 0, & \text{with probability } \frac{1}{2} \\ 1, & \text{with probability } \frac{1}{2} \end{cases},$$

and $X_k = 2$ for $k > 1$. Therefore, $\{\lim_{k\to\infty} \pi_k = \infty\} = \{X_1 = 1\}$, and hence, it is not a tail event. You can view this by inspecting that the $\mathbf{P}(\pi_k \to \infty) = \frac{1}{2}$ that is neither 0 nor 1.

7. For each of the following random processes $\{X_k\}$, plot 100 sample paths for the corresponding partial sum sequence $\{S_k\}$ for $1 \leq k \leq 1000$. Conjecture, and theoretically prove whether the corresponding partial sum sequence converges or not. Explain your answer.

(a) $X_k = \frac{Z_k}{k}$ where Z_k is i.i.d. and uniformly distributed over $[-1, 1]$.

(b) $X_k = \frac{Z_k}{k^{0.75}}$ where Z_k is i.i.d. and uniformly distributed over $[-1, 1]$.

(c) $X_k = \frac{Z_k}{k^{0.5}}$ where Z_k is i.i.d. and uniformly distributed over $[-1, 1]$.

(d) $X_k = \frac{Z_k - k^{-2}}{k}$ where Z_k is i.i.d. and Normally distributed with zero mean and unit variance.

Solution:

(a) We have $\mathbf{E}[X_k] = 0$ and $\mathbf{Var}[X_k] = \frac{1}{3k^2}$. Therefore, since $\sum_{k=1}^{\infty} \mathbf{Var}(X_k) < \infty$ and $\mathbf{E}[X_k] = 0$ for all k, $\lim_{n\to\infty} \sum_{k=1}^{n} X_n$ exists.

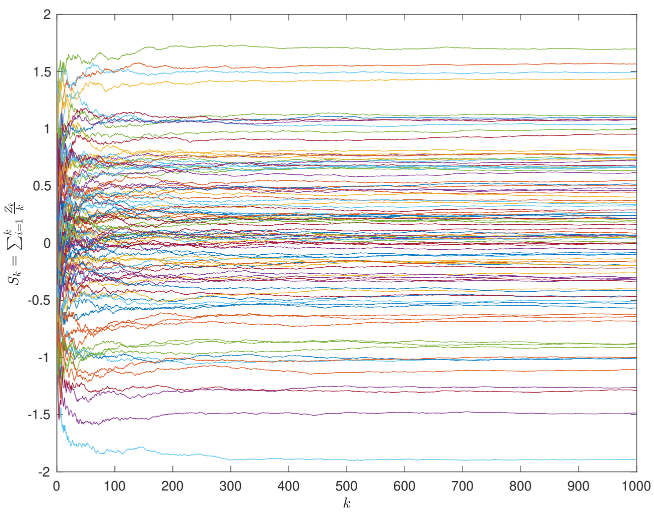
(b) We have $\mathbf{E}[X_k] = 0$ and $\mathbf{Var}[X_k] = \frac{1}{3k^{1.5}}$. Therefore, since $\sum_{k=1}^{\infty} \mathbf{Var}(X_k) < \infty$ and $\mathbf{E}[X_k] = 0$ for all k, $\lim_{n\to\infty} \sum_{k=1}^{n} X_n$ exists.

(c) We have $\mathbf{E}[X_k] = 0$ and $\mathbf{Var}[X_k \mathbf{1}_{|X_k| \leq 1}] = \frac{1}{3k}$. Therefore, since $\sum_{k=1}^{\infty} \mathbf{Var}(X_k) = \infty$, $\lim_{n\to\infty} \sum_{k=1}^{n} X_k$ does not exist.

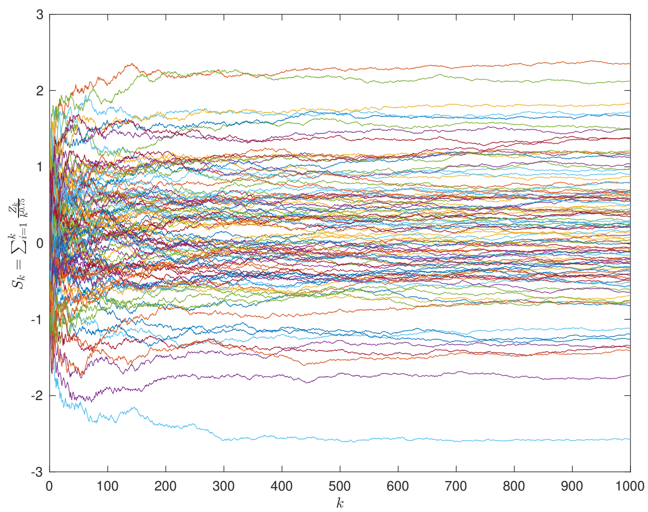
(d) Let $Y_k = \frac{Z_k}{k}$. Then, we have $X_k = Y_k - \frac{1}{k^3}$. We have $\mathbf{E}[Y_k] = 0$ and $\mathbf{Var}[Y_k] = \frac{1}{k^2}$. Therefore, since $\sum_{k=1}^{\infty} \mathbf{Var}(Y_k) < \infty$ and $\mathbf{E}[Y_k] = 0$ for all k, $\lim_{n\to\infty} \sum_{k=1}^{n} Y_n$ exists. Hence,

$$\lim_{n\to\infty} \sum_{k=1}^{n} X_k = \lim_{n\to\infty} \sum_{k=1}^{n} Y_k - \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^3},$$

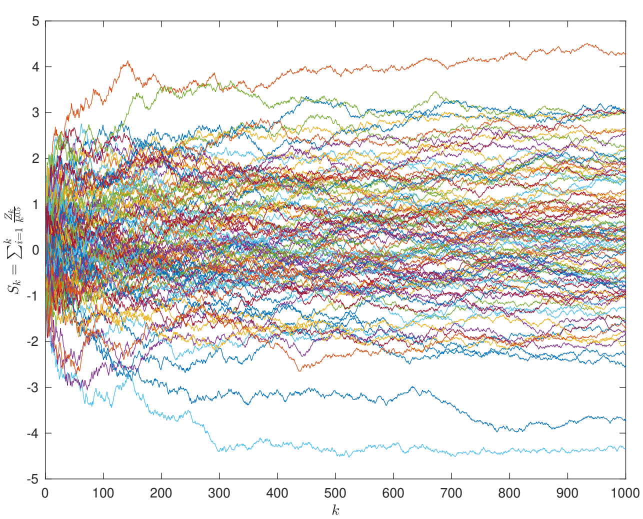
exists.



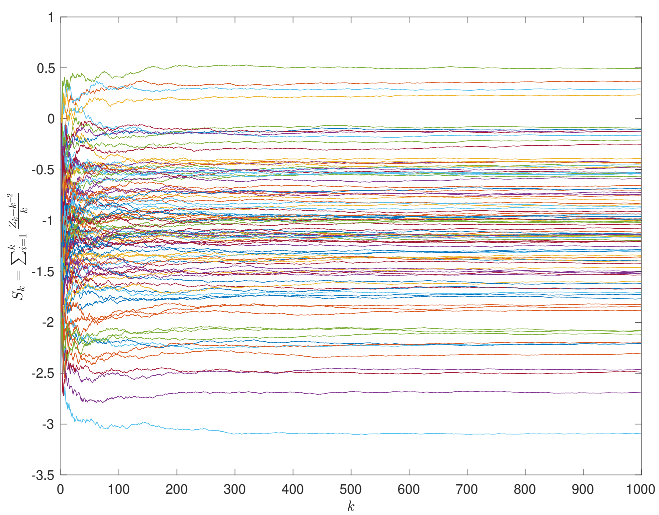
(a) $S_k = \sum_{i=1}^{k} \frac{Z_k}{k}$



(b) $S_k = \sum_{i=1}^{k} \frac{Z_k}{k^{0.75}}$



(c) $S_k = \sum_{i=1}^{k} \frac{Z_k}{k^{0.5}}$



(d) $S_k = \sum_{i=1}^{k} \frac{Z_k - k^{-2}}{k}$

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T=1000;
N=100;
z=2*rand(N,T)-1;
for i=1:N
    x(i,:)=cumsum(z(i,:)./([1:T]));
    y(i,:)=cumsum(z(i,:)./([1:T].^.75));
    v(i,:)=cumsum(z(i,:)./([1:T].^.5));
    w(i,:)=cumsum((z(i,:)-[1:T].^-2)./([1:T]));
end
plot(x')
xlabel('$k$','Interpreter','latex')
ylabel('$S_k=\sum_{i=1}^k \frac{Z_k}{k}$','Interpreter','latex')
figure
plot(y')
xlabel('$k$','Interpreter','latex')
ylabel('$S_k=\sum_{i=1}^k \frac{Z_k}{k^{0.75}}$','Interpreter','latex')
figure
plot(v')
xlabel('$k$','Interpreter','latex')
ylabel('$S_k=\sum_{i=1}^k \frac{Z_k}{k^{0.5}}$','Interpreter','latex')
figure
plot(w')
xlabel('$k$','Interpreter','latex')
ylabel('$S_k=\sum_{i=1}^k \frac{Z_k-k^{-2}}{k}$','Interpreter','latex')
```