HW4

Saturday, October 22, 2022 8:55 PM

2. $P(X_k = k^2) = \frac{1}{k^2}$.

$P(X_k = 0) = 1 - \frac{1}{k^2}$.

$a = 1$.

(a) $P(|X_k| \geq a) = P(|X_k| \geq 1) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} < \infty$

$\therefore P(|X_k| \geq 1)$ converges.

(b) $E[X_k \mathbb{1}_{\{|X_k| \leq a\}}] = E[X_k \mathbb{1}_{\{|X_k| \leq 1\}}]$.

$\because$ if $|X_k| \leq 1$, $X_k = 0$.

$\therefore E[X_k \mathbb{1}_{\{|X_k| \leq 1\}}] = 0 < \infty$

$\therefore E[X_k \mathbb{1}_{\{|X_k| \leq 1\}}]$ converges.

(c) $Var[X_k \cdot \mathbb{1}_{\{|X_k| \leq a\}}] = Var[X_k \cdot \mathbb{1}_{\{|X_k| \leq 1\}}]$.

$= Var(0) = 0 < \infty$.

$\therefore Var[X_k \cdot \mathbb{1}_{\{|X_k| \leq a\}}]$ converges.

(d) $Var(X_k) = 0 + \sum_{k=1}^{\infty} 1 = \infty$.

as (a), (b), (c) shown above, by Kolmogorov's three series theorem:

$\lim_{k \to \infty} \sum_{i=1}^{k} X_i$ exists, i.e., $\sum_{k=1}^{\infty} X_k$ converges.

3. $\because \Theta \sim U[-\pi, \pi]$.

$\therefore f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & \theta \in [-\pi, \pi] \\ 0, & \text{otherwise} \end{cases}$

$F_\Theta(\theta) = \begin{cases} 0, & \theta < -\pi \\ \frac{\theta+\pi}{2\pi}, & \theta \in [-\pi, \pi] \\ 1, & \theta > \pi \end{cases}$

$Y(t) = \sin(\omega t + \theta) \quad \in [-1, 1]$.

$\therefore P(Y \le y) = P(\sin(\omega t + \theta) \le y)$

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$= P(\theta \le \sin^{-1}(y) - \omega t)$

$= F_\Theta(\sin^{-1}(y) - \omega t)$

$= \dfrac{\sin^{-1}(y) + \pi - \omega t}{2\pi}$

$\therefore f_Y(y) = \dfrac{d}{dy} \dfrac{\sin^{-1}(y) + \pi - \omega t}{2\pi}$

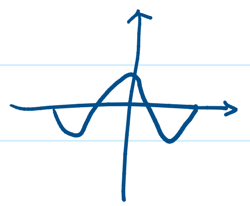
$= \dfrac{1}{2\pi} \cdot \dfrac{1}{\sqrt{1-y^2}} = \dfrac{1}{2\pi\sqrt{1-y^2}}, \quad y \in [-1, 1]$.

$\therefore$ the pdf of $Y(t)$ is independent on t.

4. $Y(t) = \sin(\omega t + \theta)$

$\Theta \sim \text{Unif}[-\pi, \pi]$.

$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & \theta \in [-\pi, \pi] \\ 0, & \text{otherwise} \end{cases}$



$\therefore E[Y] = \int_{-\pi}^{\pi} \sin(\omega t + \theta) \frac{1}{2\pi} \, d\theta$

$$= -\frac{\cos(wt+\theta)}{2\pi}\Big|_{-\pi}^{\pi}$$

$$= 0.$$

$$\therefore Var(Y) = E[Y^2] - (E[Y])^2$$

$$= E[Y^2].$$

$$= \int_{-\pi}^{\pi} \frac{1}{2\pi} \sin^2(wt+\theta)\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-\cos(2\theta+2wt)}{2}\, d\theta$$

$$= \frac{1}{2\pi}\left[\frac{\theta}{2} - \frac{\sin(2\theta+2wt)}{4}\right]\Big|_{-\pi}^{\pi}.$$

$$= \frac{\pi}{2\pi} = \frac{1}{2}.$$

1. $Z_k \sim Unif[-1,1]$, $\{Z_k\}$ is i.i.d R.P.

$$\therefore f_{Z_k}(z) = \begin{cases} \frac{1}{2}, & z \in [-1,1] \\ 0, & \text{otherwise}. \end{cases}$$

(a) $X_k = \frac{Z_k}{K}$.

$$\therefore E[X_k] = \int X_k\, f_{X_k}(x)\, dx.$$

$$= \int_{-1}^{1} \frac{Z_k}{K}\cdot\frac{1}{2}\, dz_k.$$

$$= \frac{1}{4K}\left[Z_k^2\right]\Big|_{-1}^{1} = 0.$$

$$\therefore Var[X_k] = Var(Z_k/k) = E\left[\frac{Z_k^2}{K^2}\right] = \frac{1}{2K^2}\int_{-1}^{1} Z_k^2\, dZ_k.$$

$$= \frac{1}{6K^2}\left[Z_k^3\Big|_{-1}^{1}\right] = \frac{1}{3K^2}.$$

$$\because \sum_{K=1}^{\infty} Var[X_k] = \sum_{K=1}^{\infty} \frac{1}{3K^2} < \infty$$

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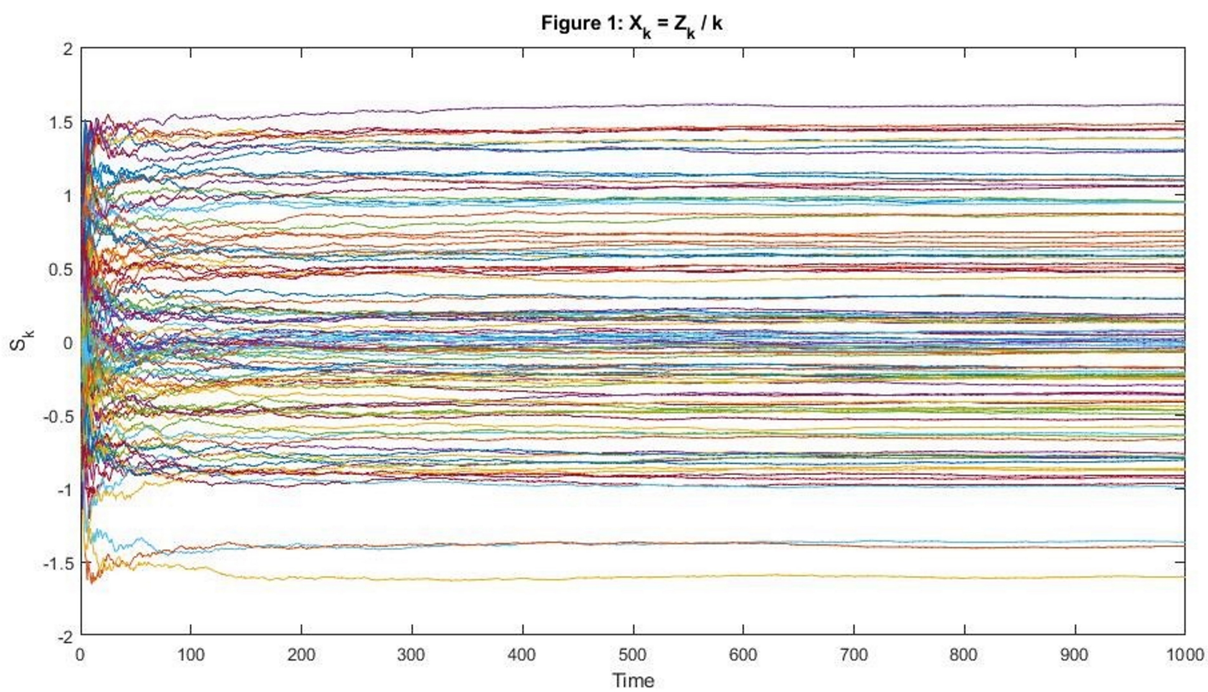
by "Noname Theorem":

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} \sum_{i=1}^{k} X_i \text{ exists, i.e. } \{S_k\} \text{ converges.}$$



(b). $X_k = Z_k / k^{0.7}$

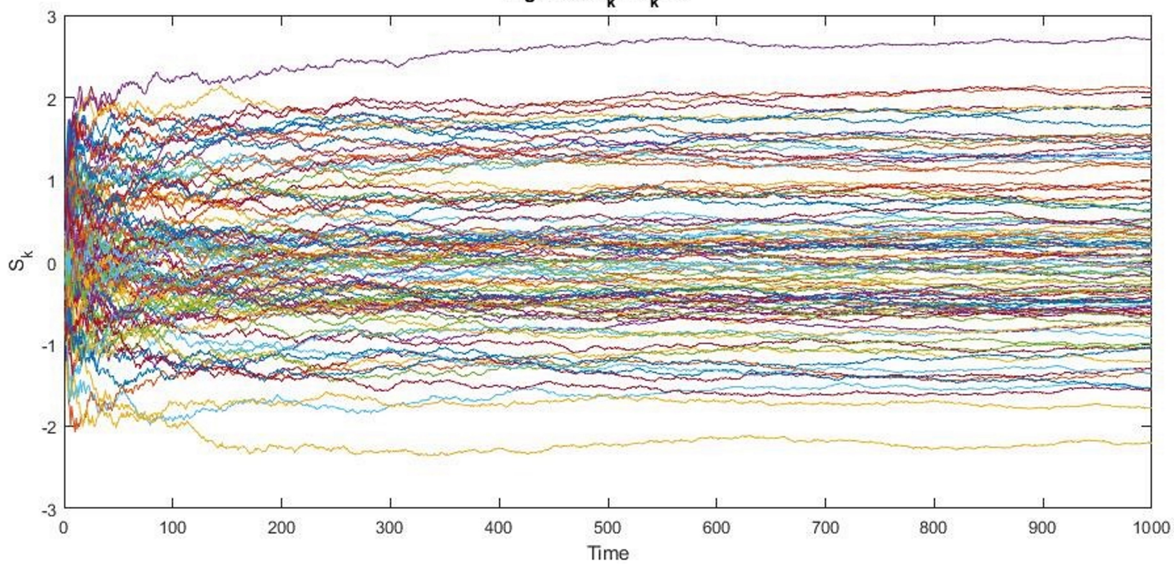
$$E[X_k] = E\left[Z_k / k^{0.7}\right] = 0.$$

$$Var[X_k] = Var\left(Z_k / k^{0.7}\right) = E\left[Z_k^2 / k^{1.4}\right] = \frac{1}{3k^{1.4}}$$

$$\because \sum_{k=1}^{\infty} Var(X_k) = \sum_{k=1}^{\infty} \frac{1}{3k^{1.4}} < \infty$$

$$\therefore \lim_{k\to\infty} S_k = \lim_{k\to\infty} \sum_{i=1}^{k} X_i \text{ exists, i.e. } \{S_k\} \text{ converges.}$$

Figure 2: $X_k = Z_k / k^{0.7}$



(c) $X_k = \frac{Z_k}{k^{0.5}}$.

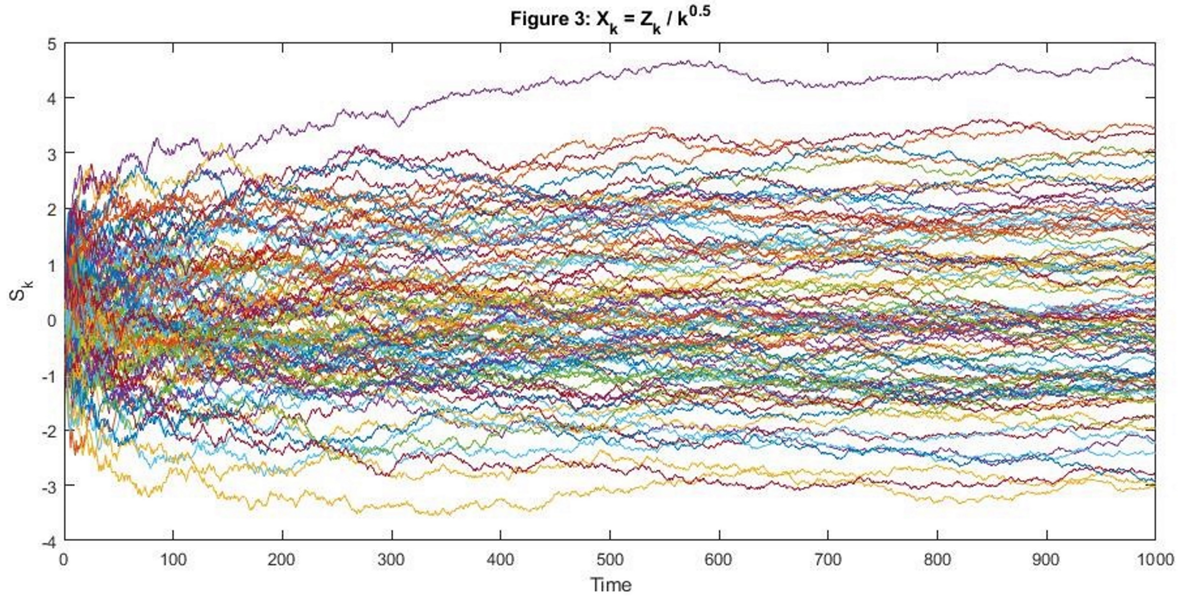
$E[X_k] = E\left[\frac{Z_k}{k^{0.5}}\right] = 0$

$Var[X_k] = E\left[\frac{Z_k^2}{k}\right] = \frac{1}{3k}$.

$\sum_{k=1}^{\infty} Var(X_k) = \sum_{k=1}^{\infty} \frac{1}{3k} = \infty$

$\therefore \lim_{k \to \infty} \sum_{i=1}^{k} X_i$ does not exist.

Figure 3: $X_k = Z_k / k^{0.5}$



5. $f(\alpha) = p\log(1+\alpha) + (1-p)\log(1-\alpha)$, $0.5 < p \leq 1$.

show $\alpha^* \in [0,1]$ such that $f(\alpha^*) > 0$ exists

$$\frac{d}{d\alpha} f(\alpha) = \frac{p}{1+\alpha} - \frac{(1-p)}{1-\alpha} = 0$$

$\therefore$ $p(1-\alpha) = (1-p)(1+\alpha)$.

$p - p\alpha = 1 - p + \alpha - p\alpha$.

$\therefore$ $\alpha = 2p - 1$.

$\because$ $0.5 < p \leq 1$.

$\therefore$ $0 < \alpha \leq 1$.

$f(0) = p\log(1) + (1-p)\log(1) = 0$

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T = 1000;
N = 100;
Z = -1+2*rand(N,T);

for i = 1:N
    X1(i,:)= cumsum(Z(i,:)./[1:T]);
    X2(i,:)= cumsum(Z(i,:)./([1:T].^0.7));
    X3(i,:)= cumsum(Z(i,:)./([1:T].^0.5));
end

figure(1)
plot(X1');
title("Figure 1: X_k = Z_k / k");
ylabel("S_k");
xlabel("Time");

figure(2)
plot(X2');
title("Figure 2: X_k = Z_k / k^{0.7}");
ylabel("S_k");
xlabel("Time");

figure(3)
plot(X3');
title("Figure 3: X_k = Z_k / k^{0.5}");
ylabel("S_k");
xlabel("Time");
```