UCSD ECE 250: Stochastic Processes: Week #3

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Outline:

- Random Variables, Random Vector, and Random Processes
- Almost Sure Limit of Random Processes
- Distribution of Random Variables
- Independence, Independent Processes, and Independent Increment Processes

How to characterize a Random Variable

- Question: How to show that a function $X : \Omega \to \mathbb{R}$ is a random variable?
- There are several ways:
  a. By definition, showing that for all $B \in \mathcal{B}$, it suffices to show that $X^{-1}((-\infty, a]) \in \mathcal{F}$ for all $a \in \mathbb{R}$.
  b. Practical way: Let $g : \mathbb{R}^n \to \mathbb{R}$ be a continuous mapping and let $X_1, \ldots, X_n$ be r.v's. Then $X = g(X_1, X_2, \ldots, X_n)$ is a r.v. This allows us to construct new random variables from the old ones: for example if X, Y are random variables, $X+Y$, $X-Y$, $X \times Y$, X^Y, etc. are all random variables.

Example 1. Let X, Y be r.v's. Then $E = \{\omega | X(\omega) = Y(\omega)\}$ is an event. Why?

- Let $Z = X - Y$.
- By property (c), Z is a random variable.
- $\{0\}$ is a Borel set (as $\{0\} = \cap_{i=1}^{\infty}(-\frac{1}{i}, \frac{1}{i})$).
- Therefore
$$Z^{-1}(\{0\}) = \{\omega \mid Z(\omega) = X(\omega) - Y(\omega) = 0\} \in \mathcal{F}$$
and hence, $E = Z^{-1}(\{0\})$ is an Event!

Limits of Stochastic Processes

- **Motivation**: Early stage of an epidemics dynamics: We have an initial infected population X_0, which, at each iteration (day) $t \geq 1$, is getting multiplied by a positive random variable w_t, i.e., $X_{t+1} = w_t X_t$, where w_t is an **independently** and **identically distributed** random variables/process. Then, if $\mathbb{E}[\log(w_t)] > 0$, we have $\lim_{t\to\infty} X_t = \infty$ almost surely.
- We say that b is an upper bound for a sequence $\{\alpha_k\}$, if $\alpha_k \leq b$ for all k. Smallest such b is called is the supremum of $\{\alpha_k\}$ and is denoted by $\sup_{k\geq 1} \alpha_k$. We always assume that $+\infty$ is an upper bound for a sequence and hence, supremum always exists.
- Similarly, we say that b is a lower bound for a sequence $\{\alpha_k\}$, if $\alpha_k \geq b$ for all k. Largest such b is called is the infimum of $\{\alpha_k\}$ and is denoted by $\inf_{k\geq 1} \alpha_k$.
- We define:

$$\limsup_{k\to\infty} \alpha_k = \inf_{t\geq 1} \sup_{k\geq t} \alpha_k$$
$$\liminf_{k\to\infty} \alpha_k = \sup_{t\geq 1} \inf_{k\geq t} \alpha_k.$$

- Note that for a sequence of r.v.s $\{X_k\}$ and for an $\omega \in \Omega$, $\{X_i(\omega)\}$ is a sequence in $\mathbb{R}$. Then

  I. $X(\omega) = \sup_{k\geq 1} X_k(\omega)$ is a random variable,

  II. $X(\omega) = \inf_{k\geq 1} X_k(\omega)$ is a random variable,

  III. $\overline{X}(\omega) := \limsup_{k\to\infty} X_k(\omega)$ is a random variable,

  IV. $\underline{X}(\omega) := \liminf_{k\to\infty} X_k(\omega)$is a random variable,

  V. if $\overline{X}(\omega) = \underline{X}(\omega)$ for almost all $\omega \in \Omega$, X defined by $X = \lim_{k\to\infty} X_k(\omega)$ is a random variable (HW 3).

Distributions of Random Variables

- For a r.v. X, we define the distribution function (or cumulative distribution function (CDF)) of X, to be the mapping $F_X : \mathbb{R} \to [0,1]$ defined by $F(x) = \Pr(X^{-1}\big((-\infty, x]\big))$.
- Properties of Distribution Functions (see HW 3):

   a. F_X is non-decreasing.

   b. $\lim_{x\to-\infty} F_X(x) = 0$, and $\lim_{x\to\infty} F_X(x) = 1$.

   c. $F_X(\cdot)$ **is right-continuous**, i.e., for any $x \in \mathbb{R}$, $\lim_{y\to x^+} F_X(y) = F_X(x)$.

   d. Define $F_X(x^-) := \lim_{y\uparrow x} F_X(y)$, then

   $$F_X(x^-) = \Pr(X < x) = \Pr(\{\omega \in \Omega \mid X(\omega) < x\}).$$

   e. For any $x \in \mathbb{R}$, we have $\Pr(X = x) = F_X(x) - F_X(x^-)$.

- If we are given a distribution F, the first question one may ask is *Does there exist a probability space* $(\Omega, \mathcal{F}, \mathbf{Pr})$ *and a function* $X : \Omega \to \mathbb{R}$ *such that* X *has the given distribution* F*?* And the following theorem is the answer to this question.

   Theorem 1. *Suppose that a function* $F : \mathbb{R} \to [0,1]$ *satisfies the above properties (a), (b) and (c), then there exists a probability space* $(\Omega, \mathcal{F}, \mathbf{Pr})$ *and a r.v.* X *such that* F *is the distribution function of* X*.*

Expected Value of a Random Variable

- How to define expected value?
- Many of the constructions in probability theory starts from simple functions (r.v.s): we say that a random variable X is a simple r.v. if $X = \sum_{i=1}^{m} \alpha_i \mathbf{1}_{A_i}$ for some finite $m \geq 1$, where $A_1, \ldots, A_m \in \mathcal{F}$ and $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$.
- We define the *expected value* of a *random variable* using the following steps:
  - Expected value of simple r.v.s: For $X = \sum_{i=1}^{m} \alpha_i \mathbf{1}_{A_i}$, we define

$$\mathbb{E}[X] := \sum_{i=1}^{m} \alpha_i \Pr(A_i).$$

  - For a positive random variable X (i.e., $X \geq 0$ almost surely), we define:

$$\mathbb{E}[X] := \sup\{\mathbb{E}[Y] \mid Y \leq X \text{ and } Y \text{ is a simple function}\}.$$

  - Define positive and negative side of a random variable as: $X^+ = \mathbf{1}_{X \geq 0} X$ and $X^- = -\mathbf{1}_{X \leq 0} X$. Note that they are both non-negative r.v.s.
  - We say that the expected value of X exists if either $\mathbb{E}[X^+] < \infty$ or $\mathbb{E}[X^-] < \infty$ and we let it be

$$\mathbb{E}[X] := \mathbb{E}[X^+] - \mathbb{E}[X^-].$$

Properties of Expected Value

- If $X \geq 0$, then $\mathbb{E}[X] \geq 0$.
- *monotonicity*: if $X \leq Y$, then $\mathbb{E}[X] \leq \mathbb{E}[Y]$.
- $\mathbb{E}[|X|] = 0$ if and only if $X = 0$ almost surely.
- Markov Inequality: For a non-negative random variable X,

  $$\Pr(X \geq \alpha) \leq \frac{\mathbb{E}[X]}{\alpha}$$

  for any $\alpha > 0$.
- Jensen's Inequality: For a convex function $\Phi : \mathbb{R} \to \mathbb{R}$,

  $$\Phi(\mathbb{E}[X]) \leq \mathbb{E}[\Phi(X)].$$

  Since $-\Phi$ is a concave function, the reverse inequality holds for concave functions.
- Very^n important result: Monotone Convergence Theorem (MCT): Suppose that $X_1 \leq X_2 \leq \cdots \leq X = \lim_{k\to\infty} X_k$. Then,

  $$\lim_{k\to\infty} \mathbb{E}[X_k] = \mathbb{E}[\lim_{k\to\infty} X_k] = \mathbb{E}[X].$$

PDF, PMF, Continuous, and Discrete Random Variables

- We say that X is a continuous r.v., if $F_X(x)$ is continuous. If further, $F_X(x) = \int_{-\infty}^{x} f_X(u)du$ for a non-negative function $f_X : \mathbb{R} \to \mathbb{R}^+$, we say that X has a probability density function (PDF) $f_X(x)$.

- **Very important**: For a (continuous) r.v. X with the pdf $f_X(x)$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)dx.$$

- More generally (and important result), for any integrable function $g(\cdot)$, for the random variable $Z = g(X)$, we have

$$\mathbb{E}[Z] = \mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)dx.$$

- We say that a random variable is a discrete random variable if $\Pr(X \in B) = 1$ for a (finite or) countable set $B = \{b_k \mid k \geq 1\}$.

- We define the *probability mass function* $p : \mathbb{R} \to [0,1]$ of a discrete random variable X to be defined by:

$$p_X(x) = \begin{cases} \Pr(X = b_k) & \text{if } x = b_k \text{ for some } k \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

- **Very important**: For a discrete r.v. X (see HW3)

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} b_k p_X(b_k).$$

Independent Random Variables and Processes

- Motivation: We have an initial infected population X_0 and at each iteration (day) $t \geq 1$, its getting multiplied by a positive random w_t variable, i.e., $X_{t+1} = w_t X_t$, where w_t is an **independently** and **identically distributed** random variables. Then, if $\mathbb{E}[\log(w_t)] > 0$, we have $\lim_{t\to\infty} X_t = \infty$ almost surely.

- We say X, Y are two random variables, if $X^{-1}(B_1)$ and $Y^{-1}(B_2)$ are independent for any Borel sets $B_1, B_2 \in \mathcal{B}$, i.e.,

$$\Pr(X \in B \text{ and } Y \in B) = \Pr(X \in B)\Pr(Y \in B).$$

- **Important Fact (lemma)**: X, Y are independent if $X^{-1}((-\infty, \alpha])$ and $Y^{-1}((-\infty, \beta])$ are independent for all $\alpha, \beta \in \mathbb{R}$, i.e., it suffices to hold the above for sets of the form $(-\infty, \alpha]$. In other words, X, Y are independent if and only if

$$\Pr(X \leq \alpha, Y \leq \beta) = F_X(\alpha)F_Y(\beta).$$

- Similarly, we say that $X_1, \ldots, X_n$ are independent if for any collection of Borel-sets $B_1, \ldots, B_n$, the events $X_1^{-1}(B_1), \ldots, X_n^{-1}(B_n)$ are independent.

- Again it follows from a result[1] that $X_1, \ldots, X_n$ are independent iff for any selection of real numbers $\alpha_1, \ldots, \alpha_n$:

$$\Pr(X_1 \leq \alpha_1, X_2 \leq \alpha_2, \ldots, X_n \leq \alpha_n) = F_{X_1}(\alpha_1) \cdots F_{X_n}(\alpha_n).$$

[1] If interested, look for Dynkin's π-λ Theorem.

Independent and Independent Increment Processes

- We say that a DT or a CT random process $\{X_t\}$ is
  1. An independent process: if any finite collection $X_{t_1}, \ldots, X_{t_n}$ are independent for any $n \geq 2$ and $t_1 < t_2 < \ldots < t_n$.
  2. An independent increment process: if for any $n \geq 2$, and $a_1 < b_1 \leq a_2 < b_2 \leq \ldots \leq a_n < b_n$, the increments $X_{b_1} - X_{a_1}$, $X_{b_2} - X_{a_2}$, ..., $X_{b_n} - X_{a_n}$ are independent.