

UNIVERSITY OF CALIFORNIA, SAN DIEGO
Electrical & Computer Engineering Department
ECE 250 - Winter Quarter 2022
Random Processes

Extended P.S. #8 with (Partial) Solutions
Problem numbers from original P.S. #8 marked as [*]

1 Stationary Processes (Strict-Sense and Wide-Sense)

1. [1] *QAM random process*. Consider the random process

$$X(t) = Z_1 \cos \omega t + Z_2 \sin \omega t, \quad -\infty < t < \infty,$$

where Z_1 and Z_2 are i.i.d. discrete random variables such that $p_{Z_i}(+1) = p_{Z_i}(-1) = \frac{1}{2}$.

- (a) Is $X(t)$ wide-sense stationary? Justify your answer.
(b) Is $X(t)$ strict-sense stationary? Justify your answer.
2. [2] *Mixture of two WSS processes*. Let $X(t)$ and $Y(t)$ be two zero-mean WSS processes with autocorrelation functions $R_X(\tau)$ and $R_Y(\tau)$, respectively. Define the process

$$Z(t) = \begin{cases} X(t), & \text{with probability } \frac{1}{2} \\ Y(t), & \text{with probability } \frac{1}{2}. \end{cases}$$

Find the mean and autocorrelation functions for $Z(t)$. Is $Z(t)$ a WSS process? Justify your answer.

Solution: To show that $Z(t)$ is WSS, we show that its mean and autocorrelation functions are time invariant. Consider

$$\begin{aligned} \mu_Z(t) &= E[Z(t)] \\ &= E(Z|Z=X)P\{Z=X\} + E(Z|Z=Y)P\{Z=Y\} \\ &= \frac{1}{2}(\mu_X + \mu_Y) \\ &= 0, \end{aligned}$$

and similarly

$$\begin{aligned} R_Z(t+\tau, t) &= E[Z(t+\tau)Z(t)] \\ &= \frac{1}{2}(R_X(\tau) + R_Y(\tau)). \end{aligned}$$

Since $\mu_Z(t)$ is independent of time and $R_Z(t+\tau, t)$ depends only on τ , $Z(t)$ is WSS.

3. [3] *Stationary Gauss-Markov process.* Let

$$X_0 \sim N(0, a)$$

$$X_n = \frac{1}{2}X_{n-1} + Z_n, \quad n \geq 1,$$

where Z_1, Z_2, Z_3, \dots are i.i.d. $N(0, 1)$ independent of X_0 .

- (a) Find a such that X_n is stationary. Find the mean and autocorrelation functions of X_n .
 - (b) (Difficult.) Consider the sample mean $S_n = \frac{1}{n} \sum_{i=1}^n X_i$, $n \geq 1$. Show that S_n converges to the process mean in probability even though the sequence X_n is not i.i.d. (A stationary process for which the sample mean converges to the process mean is called *mean ergodic*.)
4. [4] *AM modulation.* Consider the AM modulated random process

$$X(t) = A(t) \cos(\omega t + \Theta),$$

where the amplitude $A(t)$ is a zero-mean WSS process with autocorrelation function $R_A(\tau) = e^{-\frac{1}{2}|\tau|}$, the phase Θ is a $\text{Unif}[0, 2\pi)$ random variable, and $A(t)$ and Θ are independent. Is $X(t)$ a WSS process? Justify your answer.

5. *Random binary waveform.* In a digital communication channel the symbol “1” is represented by the fixed duration rectangular pulse

$$g(t) = \begin{cases} 1 & \text{for } 0 \leq t < 1 \\ 0 & \text{otherwise,} \end{cases}$$

and the symbol “0” is represented by $-g(t)$. The data transmitted over the channel is represented by the random process

$$X(t) = \sum_{k=0}^{\infty} A_k g(t - k), \quad \text{for } t \geq 0,$$

where A_0, A_1, \dots are i.i.d random variables with

$$A_i = \begin{cases} +1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2}. \end{cases}$$

- (a) Find its first and second order pmfs.
- (b) Find the mean and the autocorrelation function of the process $X(t)$.

Solution:

- (a) The first order pmf is

$$\begin{aligned} p_{X(t)}(x) &= P(X(t) = x) \\ &= P\left(\sum_{k=0}^{\infty} A_k g(t - k) = x\right) \\ &= P(A_{\lfloor t \rfloor} = x) \\ &= P(A_0 = x) \quad \text{i.i.d} \\ &= \begin{cases} \frac{1}{2}, & x = \pm 1 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Now note that $X(t_1)$ and $X(t_2)$ are dependent only if t_1 and t_2 fall within the same time interval. Otherwise, they are independent. Thus, the second order pmf is

$$\begin{aligned}
p_{X(t_1)X(t_2)}(x, y) &= P(X(t_1) = x, X(t_2) = y) \\
&= P\left(\sum_{k=0}^{\infty} A_k g(t_1 - k) = x, \sum_{k=0}^{\infty} A_k g(t_2 - k) = y\right) \\
&= P(A_{\lfloor t_1 \rfloor} = x, A_{\lfloor t_2 \rfloor} = y) \\
&= \begin{cases} P(A_0 = x, A_0 = y), & \lfloor t_1 \rfloor = \lfloor t_2 \rfloor \\ P(A_0 = x, A_1 = y), & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{1}{2}, & \lfloor t_1 \rfloor = \lfloor t_2 \rfloor \text{ \& } (x, y) = (1, 1), (-1, -1) \\ \frac{1}{4}, & \lfloor t_1 \rfloor \neq \lfloor t_2 \rfloor \text{ \& } (x, y) = (1, 1), (1, -1), (-1, 1), (-1, -1) \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

(b) For $t \geq 0$,

$$\begin{aligned}
E[X(t)] &= E\left[\sum_{k=0}^{\infty} A_k g(t - k)\right] \\
&= \sum_{k=0}^{\infty} g(t - k) E[A_k] \\
&= 0.
\end{aligned}$$

For the autocorrelation $R_X(t_1, t_2)$, we note once again that only if t_1 and t_2 fall within the same interval, will $X(t_1)$ be dependent on $X(t_2)$; if they do not fall in the same interval then they are independent from one another. Then,

$$\begin{aligned}
R_X(t_1, t_2) &= E[X(t_1)X(t_2)] \\
&= \sum_{k=0}^{\infty} g(t_1 - k)g(t_2 - k)E[A_k^2] \\
&= \begin{cases} 1, & \lfloor t_1 \rfloor = \lfloor t_2 \rfloor \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

6. [6] *Finding time of flight.* Finding the distance to an object is often done by sending a signal and measuring the time of flight, the time it takes for the signal to return (assuming speed of signal, e.g., light, is known). Let $X(t)$ be the signal sent and $Y(t) = X(t - \delta) + Z(t)$ be the signal received, where δ is the unknown time of flight. Assume that $X(t)$ and $Z(t)$ (the sensor noise) are uncorrelated zero mean WSS processes. The estimated crosscorrelation function of $Y(t)$ and $X(t)$, $R_{YX}(t)$ is shown in Figure 1. Find the time of flight δ .

Solution: Consider

$$R_{YX}(\tau) = E(Y(t + \tau)X(t)) = E((X(t - \delta + \tau) + Z(t + \tau))X(t)) = R_X(\tau - \delta).$$

Now since the maximum of $|R_X(\alpha)|$ is achieved for $\alpha = 0$, by inspection of the given R_{YX} we get that $5 - \delta = 0$. Thus $\delta = 5$.

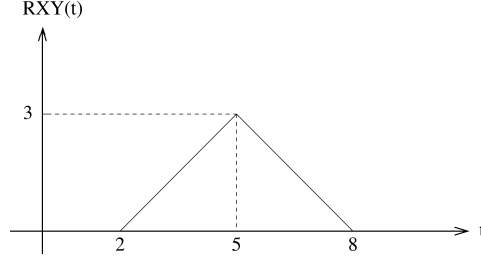


Figure 1: Crosscorrelation function.

2 Power Spectral Density

1. [8] *Generating a random process with a prescribed power spectral density.* Let $S(f) \geq 0$, for $-\infty < f < \infty$, be a real and even function such that

$$\int_{-\infty}^{\infty} S(f) df = 1.$$

Define the random process

$$X(t) = \cos(2\pi Ft + \Theta),$$

where $F \sim S(f)$ and $\Theta \sim U[-\pi, \pi)$ are independent. Find the power spectral density of $X(t)$. Interpret the result.

Solution: We have

$$\begin{aligned} E[X(t)] &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} \cos(2\pi ft + \theta) S(f) d\theta df \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} 0 \cdot S(f) df \\ &= 0. \end{aligned}$$

Also,

$$\begin{aligned} E[X(t)X(t+\tau)] &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} \cos(2\pi ft + \theta) \cos(2\pi f(t+\tau) + \theta) S(f) d\theta df \\ &= \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} \left(\cos(4\pi ft + 2\pi f\tau + 2\theta) + \cos(2\pi f\tau) \right) S(f) d\theta df \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \cos(2\pi f\tau) S(f) df, \end{aligned}$$

which is a function only of τ . Thus, $X(t)$ is WSS and has autocorrelation function

$$R_{XX}(\tau) = \frac{1}{2} \int_{-\infty}^{\infty} \cos(2\pi f\tau) S(f) df.$$

Now, $S(f)$ is even and hence, $S(f) \sin(2\pi f\tau)$ is odd. Thus, $\int_{-\infty}^{\infty} \sin(2\pi f\tau) S(f) df = 0$, and

hence,

$$\begin{aligned}\int_{-\infty}^{\infty} S(f) e^{i2\pi f\tau} df &= \int_{-\infty}^{\infty} \cos(2\pi f\tau) S(f) df + i \int_{-\infty}^{\infty} \sin(2\pi f\tau) S(f) df \\ &= \int_{-\infty}^{\infty} \cos(2\pi f\tau) S(f) df \\ &= 2R_{XX}(\tau).\end{aligned}$$

Thus, $R_{XX}(\tau) = \frac{1}{2} \mathcal{F}^{-1}(S(f))$.

Hence, the power spectral density of $X(t)$ is given by $S_X(f) = \frac{1}{2} S(f)$.

3 WSS Processes and LTI Systems

1. [5] *LTI system with WSS process input.* Let $Y(t) = h(t) * X(t)$ and $Z(t) = X(t) - Y(t)$ as shown in the Figure 2.

- (a) Find $S_Z(f)$.
- (b) Find $E(Z^2(t))$.

Your answers should be in terms of $S_X(f)$ and the transfer function $H(f) = \mathcal{F}[h(t)]$.

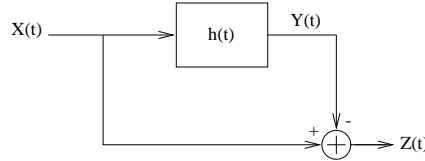


Figure 2: LTI system.

2. *Echo filtering.* A signal $X(t)$ and its echo arrive at the receiver as $Y(t) = X(t) + X(t - \Delta) + Z(t)$. Here the signal $X(t)$ is a zero-mean WSS process with power spectral density $S_X(f)$ and the noise $Z(t)$ is a zero-mean WSS with power spectral density $S_Z(f) = N_0/2$, uncorrelated with $X(t)$.

- (a) Find $S_Y(f)$ in terms of $S_X(f)$, Δ , and N_0 .
- (b) Find the best linear filter to estimate $X(t)$ from $\{Y(s)\}_{-\infty < s < \infty}$.

Solution:

- (a) We can write $Y(t) = g(t) * X(t) + Z(t)$ where $g(t) = \delta(t) + \delta(t - \Delta)$.
Thus, $S_Y(f) = |G(f)|^2 S_X(f) + S_Z(f) = |1 + e^{-j2\pi\Delta f}|^2 S_X(f) + \frac{N_0}{2}$.

(b) Since $S_{YX}(f) = (1 + e^{-j2\pi\Delta f})S_X(f)$,

$$\hat{X}(t) = h(t) * Y(t),$$

where the linear filter $h(t)$ has the transfer function

$$H(f) = \frac{S_{XY}(f)}{S_Y(f)} = \frac{S_{YX}(-f)}{S_Y(f)} = \frac{(1 + e^{j2\pi\Delta f})S_X(f)}{|1 + e^{-j2\pi\Delta f}|^2 S_X(f) + \frac{N_0}{2}}.$$

3. *Discrete-time LTI system with white noise input.* Let $\{X_n : -\infty < n < \infty\}$ be a discrete-time white noise process, i.e., $E(X_n) = 0, -\infty < n < \infty$, and

$$R_X(n) = \begin{cases} 1 & n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The process is filtered using a linear time invariant system with impulse response

$$h(n) = \begin{cases} \alpha & n = 0, \\ \beta & n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find α and β such that the output process Y_n has

$$R_Y(n) = \begin{cases} 2 & n = 0, \\ 1 & |n| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Solution: We are given that $R_X(n)$ is a discrete-time unit impulse. Therefore

$$R_Y(n) = h(n) * R_X(n) * h(-n) = h(n) * h(-n).$$

The impulse response $h(n)$ is the sequence $(\alpha, \beta, 0, 0, \dots)$. The convolution with $h(-n)$ has only finitely many nonzero terms.

$$\begin{aligned} R_Y(0) &= 2 = (h(n) * h(-n)) \Big|_{n=0} = \alpha^2 + \beta^2 \\ R_Y(+1) &= 1 = (h(n) * h(-n)) \Big|_{n=1} = \alpha\beta \\ R_Y(-1) &= 1 = R_Y(1) \end{aligned}$$

This pair of equations has two solutions: $\alpha = +1$ and $\beta = +1$ or $\alpha = -1$ and $\beta = -1$.

4. [7] *Finding impulse response of LTI system.* To find the impulse response $h(t)$ of an LTI system (e.g., a concert hall), i.e., to *identify* the system, white noise $X(t)$, $-\infty < t < \infty$, is applied to its input and the output $Y(t)$ is measured. Given the input and output sample functions, the crosscorrelation $R_{YX}(\tau)$ is estimated. Show how $R_{YX}(\tau)$ can be used to find $h(t)$.

Solution: Since white noise has a flat psd, the crosspower spectral density of the input $X(t)$ and the output $Y(t)$ is just the transfer function of the system scaled by the psd of the white noise.

$$S_{YX}(f) = H(f)S_X(f) = H(f)\frac{N_0}{2}$$

$$R_{YX}(\tau) = \mathcal{F}^{-1}(S_{YX}(f)) = \frac{N_0}{2}h(\tau)$$

Thus to estimate the impulse response of a linear time invariant system, we apply white noise to its input, estimate the crosscorrelation function of its input and output, and scale it by $2/N_0$.

5. [9] *Integrators.* Let $Y(t)$ be a short-term integration of a WSS process $X(t)$:

$$Y(t) = \frac{1}{T} \int_{t-T}^t X(u) du.$$

Find $S_Y(f)$ in terms of $S_X(f)$.

Solution: It is easy to see that the system that generates $Y(t)$ from $X(t)$ is linear and time-invariant. Writing $\delta(t)$ in place of $X(t)$, the impulse response of the system can then be obtained as

$$h(t) = \frac{1}{T} \int_{t-T}^t \delta(u) du$$

$$= \begin{cases} \frac{1}{T}, & 0 \leq t \leq T \\ 0, & \text{otherwise.} \end{cases}$$

Alternatively, we can find $h(t)$ by attempting to write $Y(t)$ as $Y(t) = \int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau$.

We have

$$Y(t) = \frac{1}{T} \int_{t-T}^t X(u) du$$

$$= \frac{1}{T} \int_0^T X(t-\tau) d\tau.$$

This shows, as before, that $h(\tau) = \begin{cases} \frac{1}{T}, & 0 \leq \tau \leq T \\ 0, & \text{otherwise.} \end{cases}$

Thus, the frequency response is given by

$$H(f) = \frac{1}{T} \int_0^T e^{-i2\pi f t} dt$$

$$= \frac{1}{T} \left(\frac{e^{-i2\pi f T} - 1}{-i2\pi f} \right)$$

$$= \frac{e^{-i\pi f T}}{\pi f T} \left(\frac{e^{i\pi f T} - e^{-i\pi f T}}{2i} \right)$$

$$= e^{-i\pi f T} \frac{\sin(\pi f T)}{\pi f T}.$$

Thus,

$$\begin{aligned} S_Y(f) &= S_X(f)|H(f)|^2 \\ &= S_X(f) \frac{\sin^2(\pi f T)}{\pi^2 f^2 T^2}. \end{aligned}$$

Alternative Method:

We have

$$\begin{aligned} \mathbb{E}[Y(t)Y(t+\tau)] &= \frac{1}{T^2} \mathbb{E} \left[\int_{t-T}^t \int_{t+\tau-T}^{t+\tau} X(u)X(v)dvdu \right] \\ &= \frac{1}{T^2} \mathbb{E} \left[\int_{t-T}^t \int_{t-T}^t X(u)X(v+\tau)dvdu \right] \\ &= \frac{1}{T^2} \int_{t-T}^t \int_{t-T}^t R_{XX}(v+\tau-u)dvdu. \end{aligned}$$

Writing $w = u - v$ in the integral, we see that $-T \leq w \leq T$, and for each fixed w , $t - T + w \leq u \leq t + w$. Thus,

$$\begin{aligned} \mathbb{E}[Y(t)Y(t+\tau)] &= \frac{1}{T^2} \int_{-T}^T \int_{\max(t-T, t-T+w)}^{\min(t, t+w)} R_{XX}(\tau-w)du dw \\ &= \frac{1}{T^2} \int_{-T}^T R_{XX}(\tau-w) \left(\min(t, t+w) - \max(t-T, t-T+w) \right) dw \\ &= \int_{-T}^T R_{XX}(\tau-w)g(w)dw, \end{aligned}$$

where

$$\begin{aligned} g(w) &= \frac{1}{T^2} \left(\min(t, t+w) - \max(t-T, t-T+w) \right) \\ &= \begin{cases} \frac{T+w}{T^2}, & -T \leq w \leq 0 \\ \frac{T-w}{T^2}, & 0 \leq w \leq T \end{cases} \\ &= \frac{T-|w|}{T^2}. \end{aligned}$$

Thus, $R_{YY}(\tau) = R_{XX}(\tau) * g(\tau)$, and hence

$$\begin{aligned}
S_Y(f) &= S_X(f) \int_{-\infty}^{\infty} g(t) e^{-i2\pi ft} dt \\
&= S_X(f) \frac{1}{T^2} \int_{-T}^T (T - |t|) e^{-i2\pi ft} dt \\
&= S_X(f) \frac{1}{T^2} \int_{-T}^T (T - |t|) \cos(2\pi ft) dt \\
&= S_X(f) \left(\frac{\sin(2\pi fT)}{\pi fT} - \frac{1}{4\pi^2 f^2 T^2} \int_{-2\pi fT}^{2\pi fT} |u| \cos u du \right) \\
&= S_X(f) \left(\frac{\sin(2\pi fT)}{\pi fT} - \frac{1}{4\pi^2 f^2 T^2} \cdot 2 \left(2\pi fT \sin(2\pi fT) + \cos(2\pi fT) - \cos 0 \right) \right) \\
&= \frac{S_X(f)}{2\pi^2 f^2 T^2} (1 - \cos(2\pi fT)) \\
&= \frac{S_X(f)}{2\pi^2 f^2 T^2} (2 \sin^2(\pi fT)) \\
&= S_X(f) \frac{\sin^2(\pi fT)}{\pi^2 f^2 T^2}.
\end{aligned}$$

4 Linear Estimation of Random Processes

1. *Prediction.* Let \mathbf{X} be a random process with zero mean and covariance matrix

$$\Sigma_{\mathbf{X}} = \begin{bmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{n-1} \\ \alpha & 1 & \alpha & & \\ \alpha^2 & \alpha & 1 & & \\ \vdots & & & \ddots & \\ \alpha^{n-1} & & & \cdots & 1 \end{bmatrix}$$

for $|\alpha| < 1$. X_1, X_2, \dots, X_{n-1} are observed, find the best linear MSE estimate (predictor) of X_n . Compute its MSE.

Solution: Let $\mathbf{Y} = (X_1, X_2, \dots, X_{n-1})^T$. Since $E[X_n] = 0$ and $E[\mathbf{Y}] = \mathbf{0}$, the best linear MSE estimate of X_n given \mathbf{Y} is given by $\hat{X}_n = h^T \mathbf{Y}$, where the vector h satisfies the equation

$$E[\mathbf{Y}X_n] = E[\mathbf{Y}\mathbf{Y}^T]h. \quad (1)$$

We see that $E[\mathbf{Y}\mathbf{Y}^T]$ is simply $\Sigma_{\mathbf{X}}$ with the last row and last column removed, and $E[\mathbf{Y}X_n]$ is simply the last column of $\Sigma_{\mathbf{X}}$, with the last element removed.

$$\text{Thus, } E[\mathbf{Y}X_n] = \begin{bmatrix} \alpha^{n-1} \\ \alpha^{n-2} \\ \vdots \\ \alpha \end{bmatrix}, \text{ and the last column of } E[\mathbf{Y}\mathbf{Y}^T] \text{ is } \begin{bmatrix} \alpha^{n-2} \\ \alpha^{n-3} \\ \vdots \\ 1 \end{bmatrix}.$$

Thus, $\mathbf{E}[\mathbf{Y}X_n]$ is simply a constant multiple of the last column of $\mathbf{E}[\mathbf{Y}\mathbf{Y}^T]$, and thus, (2) is solved by

$$h = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \alpha \end{bmatrix}.$$

Hence, the best linear MSE estimate of X_n given \mathbf{Y} is given by $\hat{X}_n = \alpha X_{n-1}$.

The MSE of this estimate is

$$\begin{aligned} \mathbf{E}[(X_n - \alpha X_{n-1})^2] &= \mathbf{E}[X_n^2] + \alpha^2 \mathbf{E}[X_{n-1}^2] - 2\alpha \mathbf{E}[X_n X_{n-1}] \\ &= 1 - \alpha^2. \end{aligned}$$

2. *Arrow of time.* Let X_0 be a Gaussian random variable with zero mean and unit variance, and $X_n = \alpha X_{n-1} + Z_n$ for $n \geq 1$, where α is a fixed constant with $|\alpha| < 1$ and Z_1, Z_2, \dots are i.i.d. $\sim N(0, 1 - \alpha^2)$, independent of X_0 .

- (a) Is the process $\{X_n\}$ Gaussian?
- (b) Is $\{X_n\}$ Markov?
- (c) Find $R_X(n, m)$.
- (d) Find the (nonlinear) MMSE estimate of X_{100} given $(X_1, X_2, \dots, X_{99})$.
- (e) Find the MMSE estimate of X_{100} given $(X_{101}, X_{102}, \dots, X_{199})$.
- (f) Find the MMSE estimate of X_{100} given $(X_1, \dots, X_{99}, X_{101}, \dots, X_{199})$.

Solution:

- (a) Yes, the process is Gaussian, since it is the linear transform of white Gaussian process $\{Z_n\}$.
- (b) Yes, the process is Markov since $X_n | \{X_1 = x_1, \dots, X_{n-1} = x_{n-1}\} \sim N(\alpha x_{n-1}, 1 - \alpha^2)$, which depends only on x_{n-1} .
- (c) First note that $R_X(n, n) = \alpha^2 R_X(n-1, n-1) + (1 - \alpha^2) = 1$ for all n . Since we can express

$$X_n = \alpha^k X_{n-k} + \alpha^{k-1} Z_{n-k+1} + \dots + Z_n,$$

and X_{n-k} is independent of (Z_{n-k+1}, \dots, Z_n) , we have $R_X(n, n-k) = \mathbf{E}(X_n X_{n-k}) = \alpha^k$. Thus, $R_X(n, m) = \alpha^{|n-m|}$.

- (d) Because the process is Gaussian, the MMSE estimator is linear. From Markovity, $\hat{X}_{100} = E(X_{100} | X_1, \dots, X_{99}) = E(X_{100} | X_{99}) = \frac{R_X(100, 99)}{R_X(99, 99)} X_{99} = \alpha X_{99}$.
- (e) Again from the Markovity and the symmetry, (X_1, \dots, X_n) has the same distribution as (X_n, \dots, X_1) hence $\hat{X}_{100} = \mathbf{E}(X_{100} | X_{101}, \dots, X_{199}) = \mathbf{E}(X_{100} | X_{101}) = \alpha X_{101}$.

- (f) First note that X_{100} is conditionally independent of $(X_1, \dots, X_{98}, X_{102}, \dots, X_{199})$ given (X_{99}, X_{101}) . To see this, consider

$$\begin{aligned}
& f(x_{100}|x_1, \dots, x_{99}, x_{101}, \dots, x_{199}) \\
&= \frac{f(x_1, \dots, x_{199})}{f(x_1, \dots, x_{99}, x_{101}, \dots, x_{199})} \\
&= \frac{f(x_1)f(x_2|x_1) \cdots f(x_{99}|x_{98})f(x_{100}|x_{99})f(x_{101}|x_{100})f(x_{102}|x_{101}) \cdots f(x_{199}|x_{198})}{f(x_1)f(x_2|x_1) \cdots f(x_{99}|x_{98})f(x_{101}|x_{99})f(x_{102}|x_{101}) \cdots f(x_{199}|x_{198})} \\
&= \frac{f(x_{100}|x_{99})f(x_{101}|x_{100})}{f(x_{101}|x_{99})} \\
&= \frac{f(x_{100}, x_{101}|x_{99})}{f(x_{101}|x_{99})} \\
&= f(x_{100}|x_{99}, x_{101}).
\end{aligned}$$

Thus,

$$\begin{aligned}
\hat{X}_{100} &= \mathbf{E}(X_{100}|X_{99}, X_{101}) \\
&= [R_X(100, 99) \quad R_X(100, 101)] \begin{bmatrix} R_X(99, 99) & R_X(99, 101) \\ R_X(101, 99) & R_X(101, 101) \end{bmatrix}^{-1} \begin{bmatrix} X_{99} \\ X_{101} \end{bmatrix} \\
&= [\alpha \quad \alpha] \begin{bmatrix} 1 & \alpha^2 \\ \alpha^2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} X_{99} \\ X_{101} \end{bmatrix} \\
&= \frac{\alpha}{1 + \alpha^2} (X_{99} + X_{101}).
\end{aligned}$$