HW3

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$$Y = \lfloor X \rfloor$$

2. $X \sim Exp(\lambda)$, $Y = k$ such that $k \le X < k+1$, $k = 0, 1, 2, \cdots$

(a) pmf of Y:

$$\begin{aligned} p_Y(k) &= P(Y=k) \\ &= P(k \le X < k+1) \\ &= \int_k^{k+1} f_X(x)\,dx \\ &= \int_k^{k+1} \lambda e^{-\lambda x}\,dx \\ &= e^{-\lambda k} - e^{-\lambda(k+1)} \\ &= e^{-\lambda k}(1-e^{-\lambda}), \quad \text{for } k = 0, 1, 2, \cdots \end{aligned}$$

(b) $Z = X - Y = X - \lfloor X \rfloor$

$\therefore$ Z is the fractional part of X.

$\therefore$ $0 \le Z < 1$.

for $0 \le r < 1$.

$$\begin{aligned} F_Z(r) &= P(Z \le r) \\ &= P(X - \lfloor X \rfloor \le r) \\ &= \sum_{k=0}^{\infty} P(k \le X \le k+r) \\ &= \sum_{k=0}^{\infty} e^{-\lambda k}(1-e^{-\lambda r}) \\ &= (1-e^{-\lambda r}) \sum_{k=0}^{\infty} e^{-\lambda k} \\ &= \frac{1-e^{-\lambda r}}{1-e^{-\lambda}} \end{aligned}$$

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$
$$= \frac{\lambda e^{-\lambda z}}{1-e^{-\lambda}}, \quad 0 \le z < 1$$

6. $\because P(X \in M) = 1$

$M = \{m_k \mid k \geq 1\}$

$\therefore P(X \notin M) = 1 - P(X \in M) = 0$

$\therefore X = 1_M$

$\therefore P(X \in M) = \sum_{k=1}^{\infty} P_X(m_k) = 1$

$\therefore E[X] = \sum x P_X(x) = \sum_{x \in M} x P_X(x) + \sum_{x \notin M} x P_X(x)$

$= \sum_{x \in M} x P_X(x)$

$= \sum_{k=1}^{\infty} m_k P_X(m_k)$

7. $P(X = \infty) = P(X = -\infty) = 0$

(a) Let $a < b$, $a, b \in \mathbb{R}$

$\therefore F_X(a) = P(X \le a) = P(X \in (-\infty, a]\}$

$F_X(b) = P(X \le b) = P(X \in (-\infty, b]\}$

$\because (-\infty, a] \subseteq (-\infty, b]$

$\therefore P(X \le a) \le P(X \le b)$

$\therefore F_X(x)$ is non-decreasing

(b) $\because F_X(x) = P(X \le x) = P(X \in (-\infty, x])$

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Let $\{x_n\}_{n\ge1}$ be a sequence of decreasing number.

such that $x_n \to -\infty$ as $n \to \infty$.

Let $A_{x_n} = \{\omega \in \Omega : X(\omega) \le x_n\}$.

$\therefore x_n \ge x_{n+1}$, $A_{x_n} \supseteq A_{x_{n+1}}$.

$$\therefore \lim_{x\to-\infty} F_X(x) = \lim_{x\to\infty} P(X \le x) = \lim_{n\to\infty} P(A_{x_n})$$

$$= P\left(\bigcap_{n=1}^{\infty} A_{x_n}\right) = P(\emptyset) = 0.$$

Similar to above, let $\{x_n\}_{n\ge1}$ be a sequence of increasing real numbers, such that $n \to \infty$, $x \to \infty$.

Let $A_{x_n} = \{\omega \in \Omega : X(\omega) \le x_n\}$

$\therefore x_{n+1} \ge x_n$, $A_{x_{n+1}} \supseteq A_{x_n}$.

$$\therefore \lim_{x\to\infty} F_X(x) = \lim_{x\to\infty} P(X \le x) = \lim_{n\to\infty} P(A_{x_n}) = P\left(\bigcup_{n=1}^{\infty} A_{x_n}\right)$$

$$= P(\Omega) = 1.$$

(c) Let $A_n = \{\omega : X(\omega) \le x + \frac{1}{n}\}$, $x \in \mathbb{R}$, $n = 1, 2, \cdots$

$\therefore A_1 \supset A_2 \supset \cdots \supset A_n$

$\therefore \{A_n\}_{n\ge1}$ is a decreasing sequence.

$\because \{A_n\}_{n\geq 1}$ is a decreasing sequence.

$\therefore \lim_{n\to\infty} P(A_n) = P(\bigcap_{n=1}^{\infty} A_k)$.

$= P(\{w : X(w) \leq x\})$.

$\therefore \lim_{n\to\infty} P(X \leq x+\frac{1}{n}) = P(X \leq x)$.

$\lim_{n\to\infty} F_X(x+\frac{1}{n}) = F_X(x)$.

Let $y = x+\frac{1}{n}$.

as $n\to\infty$. $y \downarrow x$.

$\therefore \lim_{y\to x^+} F_X(y) = F_X(x) \quad \forall x \in \mathbb{R}$.

$\therefore F_X(\cdot)$ is right-continuous.

(d) define: $F_X(x^-) := \lim_{y\uparrow x} F_X(y)$.

$\because F_X(y) = P(X \leq y)$.

$\therefore$ as $y \uparrow x$, $P(X\leq y) \uparrow P(X<x)$.

$\therefore \lim_{y\uparrow x} F_X(y) = P(X<x) = P(\{x\in\Omega \mid X(w)<x\})$.

(e) $\because F_X(x) = P(X\leq x)$, $F_X(x^-) = P(X<x)$.

$\therefore P(X=x) = F_X(x) - F_X(x^-)$.

f $\quad \because \lim \quad \int \frac{1}{x} \quad x\in[-1,1]$

5. $\because f(x) = \begin{cases} \frac{1}{2}, & x \in [-1, 1]. \\ 0, & \text{otherwise}. \end{cases}$

$\therefore F(x) = \int_{-\infty}^{x} f(x)dx = \begin{cases} 1 & , x > 1 \\ \frac{1}{2}x + 1 & , x \in [-1, 1]. \\ 0 & , x < -1 \end{cases}$

(a) $\because \sum_{k=1}^{\infty} P(\{|W_k| > \frac{1}{4}\}) = \sum_{k=1}^{\infty} (1 - P(\{-\frac{1}{4} \le W_k \le \frac{1}{4}\}))$

$= \sum_{k=1}^{\infty} \frac{3}{4} = \infty$

$\{W_k\}$ is i.i.d random process.

$\therefore$ by Borel-Cantelli Lemma:

$P(\{|W_k| > \frac{1}{4} \text{ i.o}\}) = 1.$

(b)

1. $\because$ infimum of a sequence is the maximum lower bound of this sequence.

$\therefore$ for all $\{X_k < a\}_{k \ge 1} \in \mathcal{F}$, $a \in \mathbb{R}$.

$\{\inf_{k \ge 1} X_k < a\} = \bigcup_{k \ge 1} \{X_k < a\} \in \mathcal{F}.$

$\because P(\{\inf_{k\geq 1} X_k < a\}) \geq 0$.

$\therefore \inf_{k\geq 1} X_k$ is also a random variable.

3. (b) $\because \inf_{k\geq 1} X_k$ and $\sup_{k\geq 1} X_k$ are random variables for a random process $\{X_k\}$

$\therefore E = \{\omega \in \Omega \mid \lim_{k\to\infty} X_k(\omega) \text{ exists}\}$.

$= \{\omega \in \Omega \mid \lim_{k\to\infty} \sup X_k = \lim_{k\to\infty} \inf X_k\}$.

which is measurable.

(a) Similarity.

$E_\alpha = \{\omega \in \Omega \mid \lim_{k\to\infty} X_k(\omega) = \alpha\}$.

$= \{\omega \in \Omega \mid \lim_{k\to\infty} \sup X_k = \alpha\} \cap \{\omega \in \Omega \mid \lim_{k\to\infty} \inf X_k = \alpha\}$.

4. Let $X_k = \begin{cases} k, & p = \frac{1}{k}. \\ 0, & p = 1 - \frac{1}{k}. \end{cases}$

$E[X_k] = 1$.

$\lim_{k\to\infty} E[X_k] = 1$, $E[\lim_{k\to\infty} X_k] = 0$.

$$\therefore \lim_{k \to \infty} E[X_k] \neq E[\lim_{k \to \infty} X_k]$$