UNIVERSITY OF CALIFORNIA, SAN DIEGO

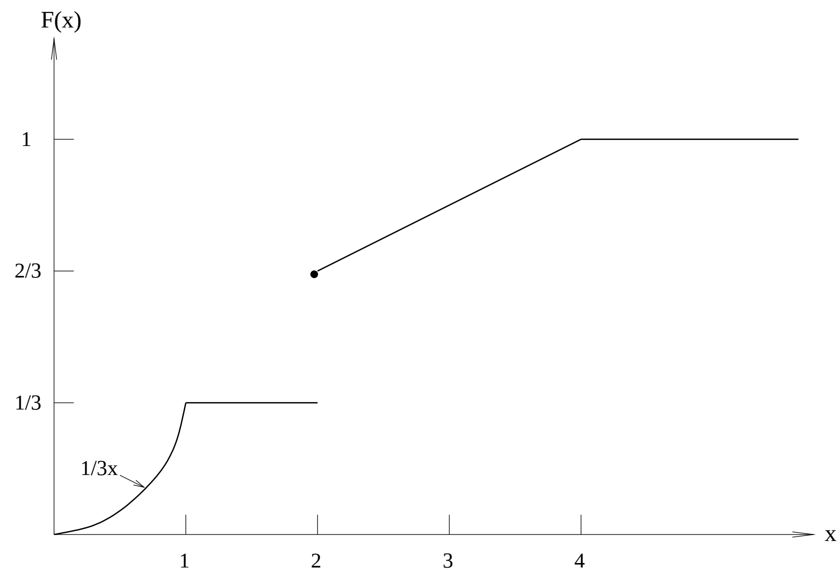
Electrical & Computer Engineering Department

ECE 250 - Winter Quarter 2020

Random Processes

Solutions to P.S. #2

1. *Probabilities from a cdf.* Let X be a random variable with the cdf shown below.



Find the probabilities of the following events.

(a) $\{X = 2\}$.

(b) $\{X < 2\}$.

(c) $\{X = 2\} \cup \{0.5 \leq X \leq 1.5\}$.

(d) $\{X = 2\} \cup \{0.5 \leq X \leq 3\}$.

Solution:

(a) There is a jump at $X = 2$, so we have

$$\begin{aligned}\mathsf{P}\{X = 2\} &= \mathsf{P}\{X \leq 2\} - \mathsf{P}\{X < 2\} \\ &= F(2) - F(2^-) \\ &= \frac{2}{3} - \frac{1}{3} \\ &= \frac{1}{3}.\end{aligned}$$

(b) $\mathsf{P}\{X < 2\} = F(2^-) = \frac{1}{3}$.

(c) since $\{X = 2\}$ and $\{0.5 \le X \le 1.5\}$ are two disjoint events,

$$\begin{aligned}\mathsf{P}(\{X = 2\} \cup \{0.5 \le X \le 1.5\}) &= \mathsf{P}\{X = 2\} + \mathsf{P}\{0.5 \le X \le 1.5\} \\ &= \frac{1}{3} + F(1.5) - F(0.5^-) \\ &= \frac{1}{3} + \frac{1}{3} - \frac{1}{3} \times 0.5^2 \\ &= \frac{7}{12}.\end{aligned}$$

(d) We have

$$\begin{aligned}\mathsf{P}(\{X = 2\} \cup \{0.5 \le X \le 3\}) &= \mathsf{P}\{0.5 \le X \le 3\} \\ &= F(3) - F(0.5^-) \\ &= \frac{5}{6} - \frac{1}{3} \times 0.5^2 \\ &= \frac{3}{4}.\end{aligned}$$

2. *Gaussian probabilities.* Let $X \sim N(1000, 400)$. Express the following in terms of the Q function.

(a) $\mathsf{P}\{0 < X < 1020\}$.

(b) $\mathsf{P}\{X < 1020 | X > 960\}$.

Solution: Using the fact that $\frac{X-\mu}{\sigma} \sim N(0,1)$, thus $F(x) = \Phi(\frac{x-\mu}{\sigma}) = 1 - Q(\frac{x-\mu}{\sigma})$.

(a) We have

$$\mathsf{P}\{0 < X < 1020\} = Q\left(\frac{0 - 1000}{20}\right) - Q\left(\frac{1020 - 1000}{20}\right) = Q(-50) - Q(1).$$

(b) We have

$$\begin{aligned}\mathsf{P}\{X < 1020 | X > 960\} &= \frac{\mathsf{P}\{960 < X < 1020\}}{\mathsf{P}\{X > 960\}} \\ &= \frac{Q(\frac{960-1000}{20}) - Q(\frac{1020-1000}{20})}{Q(\frac{960-1000}{20})} \\ &= \frac{Q(-2) - Q(1)}{Q(-2)}.\end{aligned}$$

3. *Laplacian.* Let $X \sim f(x) = \frac{1}{2}e^{-|x|}$.

(a) Sketch the cdf of X.

(b) Find $\mathsf{P}\{|X| \le 2 \text{ or } X \ge 0\}$.

(c) Find $\mathsf{P}\{|X| + |X-3| \le 3\}$.

(d) Find $\mathsf{P}\{X \ge 0 \,|\, X \le 1\}$.

Solution:

(a) We have

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{2} e^{-|u|} \mathrm{d}u = \begin{cases} \frac{1}{2}e^{x}, & \text{if } x < 0 \\ 1 - \frac{1}{2}e^{-x}, & \text{if } x \ge 0. \end{cases}$$

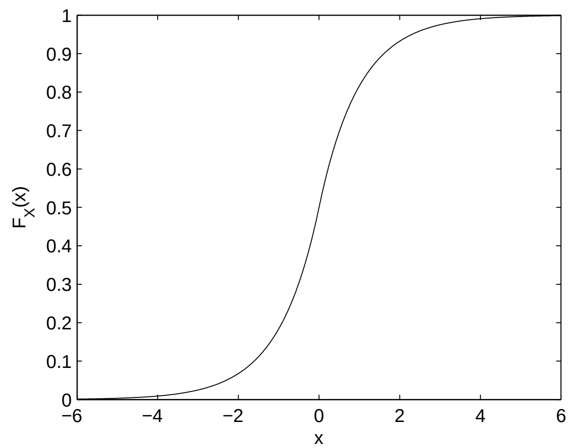


Figure 1: cdf of X

(b) We have

$$\begin{aligned} \mathsf{P}\{|X| \le 2 \text{ or } X \ge 0\} &= \mathsf{P}\{X \ge -2\} \\ &= 1 - \mathsf{P}\{X < -2\} \\ &= 1 - \int_{-\infty}^{-2} \frac{1}{2} e^{-|x|} \mathrm{d}x \\ &= 1 - \frac{1}{2}e^{-2}. \end{aligned}$$

(c) We have

$$\begin{aligned} \mathsf{P}\{|X| + |X-3| \le 3\} &= P\{0 \le X \le 3\} \\ &= \int_{0}^{3} \frac{1}{2} e^{-|x|} \mathrm{d}x \\ &= \frac{1}{2} - \frac{1}{2}e^{-3}. \end{aligned}$$

(d) We have

$$\mathsf{P}\{X \geq 0 \,|\, X \leq 1\} = \frac{\mathsf{P}\{0 \leq X \leq 1\}}{\mathsf{P}\{X \leq 1\}} = \frac{F_X(1) - F_X(0^-)}{F_X(1)} = \frac{1/2 - 1/2e^{-1}}{1 - 1/2e^{-1}} = \frac{1 - e^{-1}}{2 - e^{-1}}.$$

4. *Distance to the nearest star.* Let the random variable N be the number of stars in a region of space of volume V. Assume that N is a Poisson r.v. with pmf

$$p_N(n) = \frac{e^{-\rho V}(\rho V)^n}{n!}, \quad \text{for } n = 0, 1, 2, \ldots,$$

where ρ is the "density" of stars in space. We choose an arbitrary point in space and define the random variable X to be the distance from the chosen point to the nearest star. Find the pdf of X (in terms of ρ).

Solution: The trick in this problem, as in many others, is to find a way to connect events regarding X with events regarding N. In our case, for $x \geq 0$:

$$\begin{aligned} F_X(x) &= \mathsf{P}\{X \leq x\} \\ &= 1 - \mathsf{P}\{X > x\} \\ &= 1 - \mathsf{P}\{\text{No stars within distance } x\} \\ &= 1 - \mathsf{P}\{N = 0 \text{ in sphere centered at origin of radius } x\} \\ &= 1 - e^{-\rho \frac{4}{3}\pi x^3}. \end{aligned}$$

Now differentiating, we get

$$f_X(x) = 4\pi\rho x^2 e^{-\rho\frac{4}{3}\pi x^3}.$$

For $x < 0$, both the cdf and the pdf are zero everywhere.

5. *Uniform arrival.* The arrival time of a professor to his office is uniformly distributed in the interval between 8 and 9 am. Find the probability that the professor will arrive during the next minute given that he has not arrived by 8:30. Repeat for 8:50.

Solution: For convenience, let us denote the length of one hour by a.

Then, without loss of generality, we can consider the random variable T, denoting the arrival time of the professor in his office, to be distributed uniformly in $[0, a]$.

Let $0 < \eta < 1$.

Then we have to calculate the probability that T will lie between ηa and $\eta a + a/60$, given that T does not lie in $[0, \eta a]$, for $\eta = 1/2$ and $\eta = 5/6$.

We have

$$
\begin{aligned}
P\Big(T \le \eta a + a/60 \Big| T > \eta a\Big) &= \frac{P\Big(\eta a < T \le \eta a + a/60\Big)}{P(T > \eta a)} \\
&= \frac{(1/a)(a/60)}{(1/a)(a - \eta a)} \\
&= \frac{1}{60(1-\eta)}.
\end{aligned}
$$

For 8:30, the probability is found by substituting $\eta = 1/2$, and comes out as $1/30$.

For 8:50, the probability is found by substituting $\eta = 5/6$, and comes out as $1/10$.

Note that conditioned on the professor not having arrived till time t, the arrival time distribution becomes uniform on the remaining time. Hence, as we move closer to 9 am without the professor having yet arrived, the probability of him arriving during the next minute increases.

6. *Lognormal distribution.* Let $X \sim \mathrm{N}(0, \sigma^2)$. Find the pdf of $Y = e^X$ (known as the *lognormal* pdf).

   Solution: $Y = e^X > 0$ implies $f_Y(y) = 0$ if $y \le 0$. For $y > 0$

   $$\mathsf{P}(Y \le y) = \mathsf{P}(e^X \le y) = \mathsf{P}(X \le \ln(y)) = F_X(\ln(y))$$

   taking derivative with respect to y,

   $$f_Y(y) = \frac{1}{y} f_X(\ln(y)) = \frac{1}{y}\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\ln(y))^2}{2\sigma^2}} \qquad \text{for } y > 0.$$

7. *Random phase signal.* Let $Y(t) = \sin(\omega t + \Theta)$ be a sinusoidal signal with random phase $\Theta \sim U[-\pi, \pi]$. Find the pdf of the random variable $Y(t)$ (assume here that both t and the radial frequency ω are constant). Comment on the dependence of the pdf of $Y(t)$ on time t.

   Solution: We can easily see (by plotting y vs. θ) that for $y \in (-1, 1)$

   $$
   \begin{aligned}
   P(Y \le y) &= P(\sin(\omega t + \Theta) \le y) \\
   &= P(\sin(\Theta) \le y) \\
   &= \frac{2\left(\sin^{-1}(y) + \frac{\pi}{2}\right)}{2\pi} \\
   &= \frac{\sin^{-1}(y)}{\pi} + \frac{1}{2}.
   \end{aligned}
   $$

   By differentiating with respect to y, we get

   $$f_Y(y) = \frac{1}{\pi\sqrt{1-y^2}}.$$

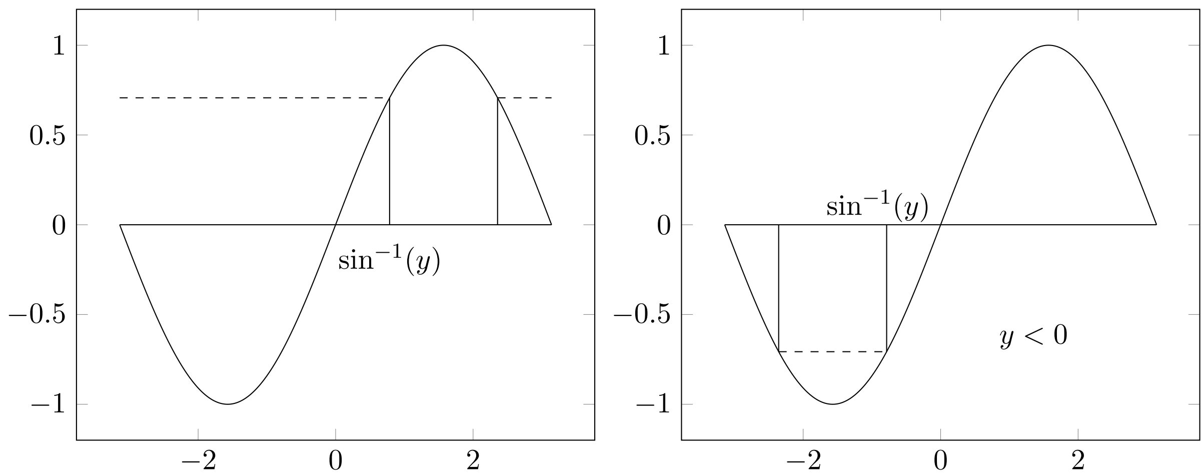
Referring to the figure, for $y > 0$, the probability is

$$(2\pi - 2(\frac{\pi}{2} - \sin^{-1}(y)))/2\pi = (\pi + 2\sin^{-1}(y))/2\pi = \frac{\sin^{-1}(y)}{\pi} + \frac{1}{2}$$

and for $y < 0$,the probability is

$$(2(\frac{\pi}{2} + \sin^{-1}(y)))/2\pi = \frac{\sin^{-1}(y)}{\pi} + \frac{1}{2}.$$

Note that $f_Y(y)$ does not depend on time t, i.e., is time invariant (or stationary) (more on this later in the course).



8. *Quantizer.* Let $X \sim \text{Exp}(\lambda)$, i.e., an exponential random variable with parameter λ and $Y = \lfloor X \rfloor$, i.e., $Y = k$ for $k \leq X < k + 1$, $k = 0, 1, 2, \ldots$. Find the pmf of Y. Define the quantization error $Z = X - Y$. Find the pdf of Z.

Solution: For $k < 0$, $p_Y(k) = 0$. Elsewhere

$$\begin{aligned} p_Y(k) &= P\{Y = k\} \\ &= P\{k \leq X < k + 1\} \\ &= F_X(k + 1) - F_X(k) \\ &= \left(1 - e^{-\lambda(k+1)}\right) - \left(1 - e^{-\lambda k}\right) \\ &= e^{-\lambda k} - e^{-\lambda(k+1)} \\ &= e^{-\lambda k}\left(1 - e^{-\lambda}\right). \end{aligned}$$

Since $Z = X - Y = X - \lfloor X \rfloor$ is the fractional part of X, $f_Z(z) = 0$ for $z < 0$ or $z \geq 1$. For

$0 \leq z < 1$, we have

$$\begin{aligned} F_Z(z) &= P(Z \leq z) \\ &= \sum_{k=0}^{\infty} P(k \leq X \leq k + z) \\ &= \sum_{k=0}^{\infty} e^{-\lambda k} - e^{-\lambda(k+z)} \\ &= \frac{1 - e^{-\lambda z}}{1 - e^{-\lambda}}. \end{aligned}$$

By differentiating with respect to z, we get

$$f_Z(z) = \frac{\lambda e^{-\lambda z}}{1 - e^{-\lambda}}$$

for $0 \leq z < 1$.

Refer to Figure 2 for a graphical explanation of the above.

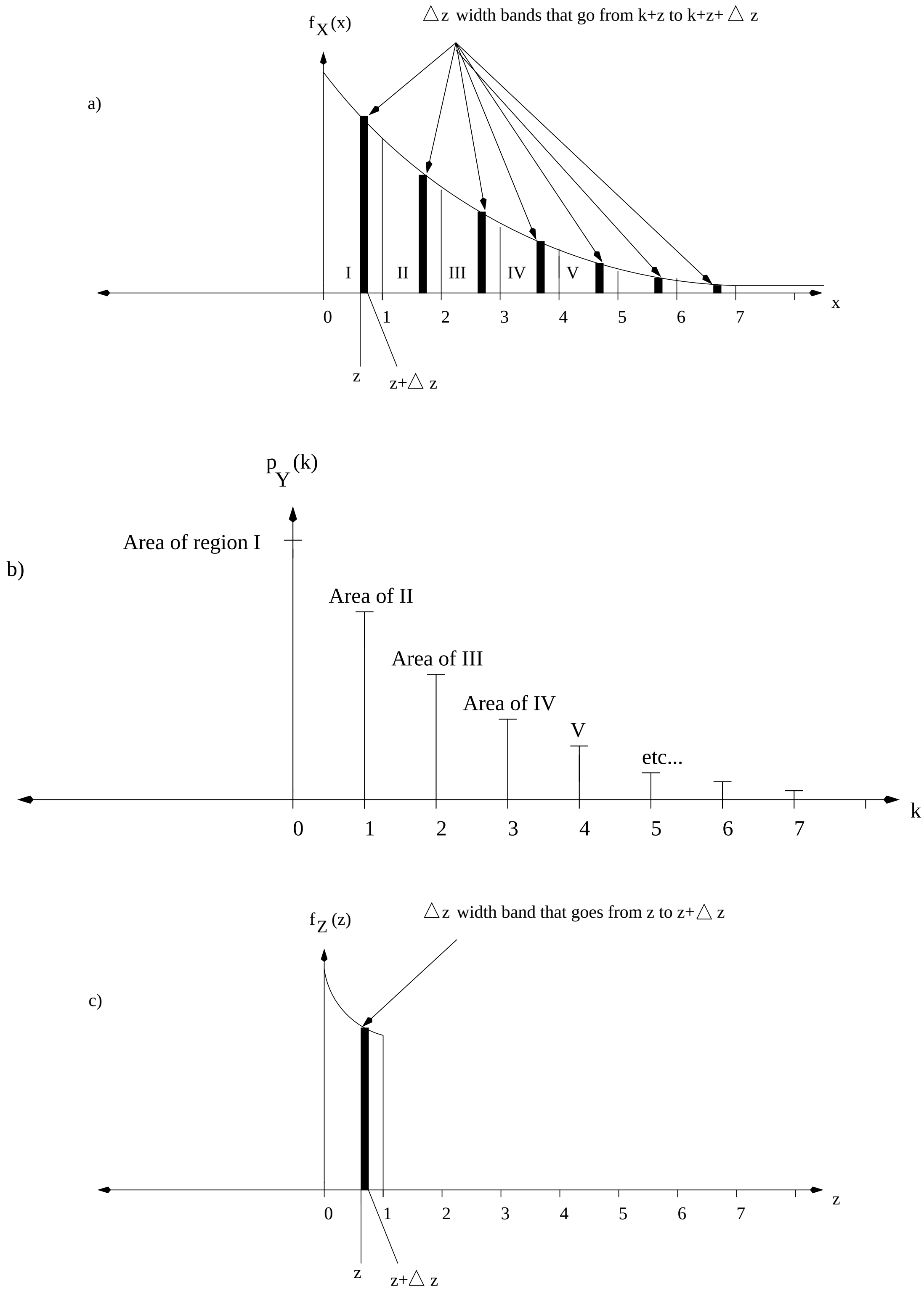


Figure 2: a) pdf of X, b) pmf of Y, c) pdf of Z

9. *Gambling.* Alice enters a casino with one unit of capital. She looks at her watch to generate a uniform random variable $U \sim \text{unif}\,[0,1]$, then bets the amount U on a fair coin flip. Her wealth is thus given by the r.v.

$$X = \begin{cases} 1+U, & \text{with probability } 1/2, \\ 1-U, & \text{with probability } 1/2. \end{cases}$$

Find the cdf of X.

Solution: First note that $U \in [0,1]$ with probability one, so $X \in [0,2]$ with probability one.

Hence, $F_X(x) = 0$ for $x < 0$, and $F_X(x) = 1$ for $x \geq 2$.

We note that $1 - U$ also follows the uniform distribution on $[0,1]$, while $1 + U$, which is simply a shifted version of U, follows the uniform distribution on $[1,2]$. Thus, it is intuitively clear that $X \sim$ unif$[0,2]$. In order to formally show this, we proceed as follows.

For $0 \leq x < 1$, we have

$$\begin{aligned}
F_X(x) &= P(X \leq x) \\
&= P(X \leq x | \text{ Alice wins })P(\text{ Alice wins }) + P(X \leq x | \text{ Alice loses })P(\text{ Alice loses }) \\
&= \frac{1}{2}\Big[P(1+U \leq x) + P(1-U \leq x)\Big] \\
&= \frac{1}{2}\Big[P(U \leq x-1) + P(U \geq 1-x)\Big] \\
&= \frac{1}{2}\Big[0 + (1-(1-x))\Big] \\
&\text{(since } x < 1, \text{ we have } x-1 < 0 \text{ and so the first probability is zero)} \\
&= \frac{x}{2}.
\end{aligned}$$

For $1 \leq x < 2$, we have

$$\begin{aligned}
F_X(x) &= P(X \leq x) \\
&= P(X \leq x | \text{ Alice wins })P(\text{ Alice wins }) + P(X \leq x | \text{ Alice loses })P(\text{ Alice loses }) \\
&= \frac{1}{2}\Big[P(1+U \leq x) + P(1-U \leq x)\Big] \\
&= \frac{1}{2}\Big[P(U \leq x-1) + P(U \geq 1-x)\Big] \\
&= \frac{1}{2}\Big[(x-1) + 1\Big] \\
&\text{(since } x \geq 1, \text{ we have } 1-x \leq 0 \text{ and so the second probability is one)} \\
&= \frac{x}{2}.
\end{aligned}$$

Thus, $F_X(x) = \begin{cases} 0, & x < 0 \\ x/2, & 0 \leq x < 2 \\ 1, & x \geq 2. \end{cases}$

Thus $X \sim$ unif$[0,2]$.