UNIVERSITY OF CALIFORNIA, SAN DIEGO
Electrical & Computer Engineering Department
ECE 250 - Winter Quarter 2020
Random Processes

P.S. #8 with Solutions (self-study)

1 Stationary Processes (Strict-Sense and Wide-Sense)

1. *AM modulation.* Consider the AM modulated random process

$$X(t) = A(t)\cos(2\pi t + \Theta)\,,$$

where the amplitude $A(t)$ is a zero-mean WSS process with autocorrelation function $R_A(\tau) = e^{-\frac{1}{2}|\tau|}$, the phase Θ is a Unif$[0, 2\pi)$ random variable, and $A(t)$ and Θ are independent. Is $X(t)$ a WSS process? Justify your answer.

Solution: $X(t)$ is wide-sense stationary if $EX(t)$ is independent of t and if $R_X(t_1, t_2)$ depends only on $t_1 - t_2$. Consider

$$\begin{aligned}
E[X(t)] &= E[A(t)\cos(\omega t + \Theta)] \\
&= E[A(t)]E[\cos(\omega t + \Theta)] \quad \text{by independence} \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
R_X(t_1, t_2) &= E[X(t_1)X(t_2)] \\
&= E[A(t_1)\cos(\omega t_1 + \Theta)A(t_2)\cos(\omega t_2 + \Theta)] \\
&= E[A(t_1)A(t_2)\cos(\omega t_1 + \Theta)\cos(\omega t_2 + \Theta)] \\
&= E[A(t_1)A(t_2) \cdot E\cos(\omega t_1 + \Theta)\cos(\omega t_2 + \Theta)] \quad \text{by independence} \\
&= R_A(t_1 - t_2)E[\cos(\omega t_1 + \Theta)\cos(\omega t_2 + \Theta)] \\
&= R_A(t_1 - t_2)E\left[\frac{1}{2}\left(\cos(\omega(t_1 + t_2) + 2\Theta) + \cos(\omega(t_1 - t_2))\right)\right] \\
&= \frac{1}{2}R_A(t_1 - t_2)E\begin{pmatrix} \cos(\omega(t_1 + t_2))\cos(2\Theta) \\ -\sin(\omega(t_1 + t_2))\sin(2\Theta) \\ +\cos(\omega(t_1 - t_2)) \end{pmatrix} \\
&= \frac{1}{2}R_A(t_1 - t_2)\begin{pmatrix} E\cos(\omega(t_1 + t_2)) \cdot E\cos(2\Theta) \\ -E\sin(\omega(t_1 + t_2)) \cdot E\sin(2\Theta) \\ +E\cos(\omega(t_1 - t_2)) \end{pmatrix} \\
&= \frac{1}{2}R_A(t_1 - t_2)\cos(\omega(t_1 - t_2)),
\end{aligned}$$

which is a function of $t_1 - t_2$ only. Hence $X(t)$ is wide-sense stationary.

2. *Random binary waveform.* In a digital communication channel the symbol "1" is represented by the fixed duration rectangular pulse

$$g(t) = \begin{cases} 1 & \text{for } 0 \le t < 1 \\ 0 & \text{otherwise,} \end{cases}$$

and the symbol "0" is represented by $-g(t)$. The data transmitted over the channel is represented by the random process

$$X(t) = \sum_{k=0}^{\infty} A_k g(t-k), \quad \text{for } t \ge 0,$$

where $A_0, A_1, \ldots$ are i.i.d random variables with

$$A_i = \begin{cases} +1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2}. \end{cases}$$

(a) Find its first and second order pmfs.

(b) Find the mean and the autocorrelation function of the process $X(t)$.

Solution:

(a) The first order pmf is

$$\begin{aligned} p_{X(t)}(x) &= \mathsf{P}\left(X\left(t\right) = x\right) \\ &= \mathsf{P}\left(\sum_{k=0}^{\infty} A_k g(t-k) = x\right) \\ &= \mathsf{P}\left(A_{\lfloor t \rfloor} = x\right) \\ &= \mathsf{P}\left(A_0 = x\right) \quad \mathsf{IID} \\ &= \begin{cases} \frac{1}{2}, & x = \pm 1 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Now note that $X(t_1)$ and $X(t_2)$ are dependent only if t_1 and t_2 fall within the same time interval. Otherwise, they are independent. Thus, the second order pmf is

$$\begin{aligned} p_{X(t_1)X(t_2)}(x,y) &= \mathsf{P}\left(X\left(t_1\right) = x, X\left(t_2\right) = y\right) \\ &= \mathsf{P}\left(\sum_{k=0}^{\infty} A_k g\left(t_1 - k\right) = x, \sum_{k=0}^{\infty} A_k g\left(t_2 - k\right) = y\right) \\ &= \mathsf{P}\left(A_{\lfloor t_1 \rfloor} = x, A_{\lfloor t_2 \rfloor} = y\right) \\ &= \begin{cases} \mathsf{P}(A_0 = x, A_0 = y), & \lfloor t_1 \rfloor = \lfloor t_2 \rfloor \\ \mathsf{P}(A_0 = x, A_1 = y), & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{1}{2}, & \lfloor t_1 \rfloor = \lfloor t_2 \rfloor \ \& \ (x,y) = (1,1), (-1,-1) \\ \frac{1}{4}, & \lfloor t_1 \rfloor \ne \lfloor t_2 \rfloor \ \& \ (x,y) = (1,1), (1,-1), (-1,1), (-1,-1) \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

(b) For $t \geq 0$,

$$\begin{aligned}
\mathsf{E}[X(t)] &= \mathsf{E}\left[\sum_{k=0}^{\infty} A_k g(t-k)\right] \\
&= \sum_{k=0}^{\infty} g(t-k) E[A_k] \\
&= 0.
\end{aligned}$$

For the autocorrelation $R_X(t_1, t_2)$, we note once again that only if t_1 and t_2 fall within the same interval, will $X(t_1)$ be dependent on $X(t_2)$; if they do not fall in the same interval then they are independent from one another. Then,

$$\begin{aligned}
R_X(t_1, t_2) &= E[X(t_1)X(t_2)] \\
&= \sum_{k=0}^{\infty} g(t_1 - k) g(t_2 - k) \mathsf{E}[A_k^2] \\
&= \begin{cases} 1, & \lfloor t_1 \rfloor = \lfloor t_2 \rfloor \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

3. *Mixture of two WSS processes.* Let $X(t)$ and $Y(t)$ be two zero-mean WSS processes with autocorrelation functions $R_X(\tau)$ and $R_Y(\tau)$, respectively. Define the process

$$Z(t) = \begin{cases} X(t), & \text{with probability } \frac{1}{2} \\ Y(t), & \text{with probability } \frac{1}{2}. \end{cases}$$

Find the mean and autocorrelation functions for $Z(t)$. Is $Z(t)$ a WSS process? Justify your answer.

Solution: To show that $Z(t)$ is WSS, we show that its mean and autocorrelation functions are time invariant. Consider

$$\begin{aligned}
\mu_Z(t) &= E[Z(t)] \\
&= E(Z|Z = X)\mathsf{P}\{Z = X\} + E(Z|Z = Y)\mathsf{P}\{Z = Y\} \\
&= \frac{1}{2}(\mu_X + \mu_Y) \\
&= 0,
\end{aligned}$$

and similarly

$$\begin{aligned}
R_Z(t+\tau, t) &= E[Z(t+\tau)Z(t)] \\
&= \frac{1}{2}(R_X(\tau) + R_Y(\tau)).
\end{aligned}$$

Since $\mu_Z(t)$ is independent of time and $R_Z(t+\tau, t)$ depends only on τ, $Z(t)$ is WSS.

4. *Stationary Gauss-Markov process.* Let

$$\begin{aligned}
X_0 &\sim \mathrm{N}(0, a) \\
X_n &= \tfrac{1}{2} X_{n-1} + Z_n, \quad n \geq 1,
\end{aligned}$$

where $Z_1, Z_2, Z_3, \ldots$ are i.i.d. $\mathrm{N}(0,1)$ independent of X_0.

(a) Find a such that X_n is stationary. Find the mean and autocorrelation functions of X_n.

(b) (Difficult.) Consider the sample mean $S_n = \frac{1}{n}\sum_{i=1}^{n} X_i,\ n \geq 1$. Show that S_n converges to the process mean in probability even though the sequence X_n is not i.i.d. (A stationary process for which the sample mean converges to the process mean is called *mean ergodic.*)

Solution:

(a) We are asked to find a such that $\mathsf{E}(X_n)$ is independent of n and $R_X(n_1, n_2)$ depends only on $n_1 - n_2$. For X_n to be stationary, $\mathsf{E}(X_n^2)$ must be independent of n. Thus

$$\mathsf{E}(X_n^2) = \tfrac{1}{4}\mathsf{E}(X_{n-1}^2) + \mathsf{E}(Z_n^2) + \mathsf{E}(X_{n-1}Z_n) = \tfrac{1}{4}\mathsf{E}(X_n^2) + 1\,.$$

Therefore, $a = \mathsf{E}(X_0^2) = \mathsf{E}(X_n^2) = \frac{4}{3}$. We can easily verify that $\mathsf{E}(X_n) = 0$ for every n and that

$$R_X(n_1, n_2) = \mathsf{E}(X_{n_1}X_{n_2}) = \tfrac{4}{3}\, 2^{-|n_1 - n_2|}\,.$$

(b) To prove convergence in probability, we first prove convergence in mean square and then use the fact that mean square convergence implies convergence in probability.

$$\mathsf{E}(S_n) = \mathsf{E}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n}\mathsf{E}(X_i) = \frac{1}{n}\sum_{i=1}^{n} 0 = 0\,.$$

To show convergence in mean square we show that $\mathrm{Var}(S_n) \to 0$ as $n \to \infty$.

$$\begin{aligned}
\mathrm{Var}(S_n) &= \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \mathsf{E}\left(\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2\right) \qquad (\text{since } \mathsf{E}(X_i) = 0)\\
&= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} R_X(i,j) = \frac{4}{3n^2}\left(n + 2\sum_{i=1}^{n-1}(n-i)2^{-i}\right)\\
&\leq \frac{4}{3n}\left(1 + 2\sum_{i=1}^{n-1} 2^{-i}\right) \leq \frac{4}{3n}\left(1 + 2\sum_{i=1}^{\infty} 2^{-i}\right) = \frac{4}{n}\,.
\end{aligned}$$

Thus S_n converges to the process mean, even though the sequence is not i.i.d.

5. *Finding time of flight.* Finding the distance to an object is often done by sending a signal and measuring the time of flight, the time it takes for the signal to return (assuming speed of signal, e.g., light, is known). Let $X(t)$ be the signal sent and $Y(t) = X(t-\delta) + Z(t)$ be the signal received, where δ is the unknown time of flight. Assume that $X(t)$ and $Z(t)$ (the sensor noise) are uncorrelated zero mean WSS processes. The estimated crosscorrelation function of $Y(t)$ and $X(t)$, $R_{YX}(t)$ is shown in Figure 1. Find the time of flight δ.

Solution: Consider

$$R_{YX}(\tau) = E(Y(t+\tau)X(t)) = E((X(t-\delta+\tau) + Z(t+\tau))X(t)) = R_X(\tau - \delta).$$

Now since the maximum of $|R_X(\alpha)|$ is achieved for $\alpha = 0$, by inspection of the given R_{YX} we get that $5 - \delta = 0$. Thus $\delta = 5$.

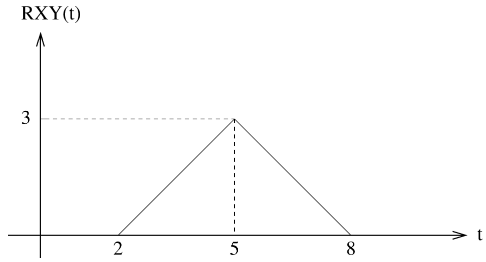

Figure 1: Crosscorrelation function.

2 Power Spectral Density

1. *Generating a random process with a prescribed power spectral density.* Let $S(f) \geq 0$, for $-\infty < f < \infty$, be a real and even function such that

$$\int_{-\infty}^{\infty} S(f)df = 1.$$

Define the random process

$$X(t) = \cos(2\pi F t + \Theta),$$

where $F \sim S(f)$ and $\Theta \sim \mathrm{U}[-\pi, \pi)$ are independent. Find the power spectral density of $X(t)$. Interpret the result.

Solution: We have

$$\begin{aligned} \mathsf{E}[X(t)] &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\pi}^{\pi} \cos(2\pi f t + \theta) S(f) d\theta df \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 0 \cdot S(f) df \\ &= 0. \end{aligned}$$

Also,

$$\begin{aligned} \mathsf{E}[X(t)X(t+\tau)] &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\pi}^{\pi} \cos(2\pi f t + \theta)\cos(2\pi f t + 2\pi f \tau + \theta) S(f) d\theta df \\ &= \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\pi}^{\pi} \Big(\cos(4\pi f t + 2\pi f \tau + 2\theta)\cos(2\pi f \tau)\Big) S(f) d\theta df \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \cos(2\pi f \tau) S(f) df, \end{aligned}$$

which is a function only of τ. Thus, $X(t)$ is WSS and has autocorrelation function

$$R_{XX}(\tau) = \frac{1}{2}\int_{-\infty}^{\infty} \cos(2\pi f \tau) S(f) df.$$

Now, $S(f)$ is even and hence, $S(f)\sin(2\pi f \tau)$ is odd. Thus, $\int_{-\infty}^{\infty} \sin(2\pi f \tau) S(f) df = 0$, and hence,

$$\begin{aligned}\int_{-\infty}^{\infty} S(f)e^{i2\pi f\tau}df &= \int_{-\infty}^{\infty}\cos(2\pi f\tau)S(f)df + i\int_{-\infty}^{\infty}\sin(2\pi f\tau)S(f)df\\ &= \int_{-\infty}^{\infty}\cos(2\pi f\tau)S(f)df\\ &= 2R_{XX}(\tau).\end{aligned}$$

Thus, $R_{XX}(\tau) = \frac{1}{2}\mathcal{F}^{-1}\Big(S(f)\Big)$.

Hence, the power spectral density of $X(t)$ is given by $S_X(f) = \frac{1}{2}S(f)$.

3 WSS Processes and LTI Systems

1. *LTI system with WSS process input.* Let $Y(t) = h(t) * X(t)$ and $Z(t) = X(t) - Y(t)$ as shown in the Figure 2.

   (a) Find $S_Z(f)$.

   (b) Find $E(Z^2(t))$.

   Your answers should be in terms of $S_X(f)$ and the transfer function $H(f) = \mathcal{F}[h(t)]$.

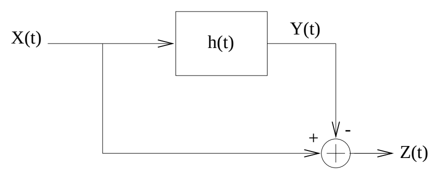

Figure 2: LTI system.

Solution:

(a) To find $S_Z(f)$, we first find the autocorrelation function

$$\begin{aligned}R_Z(\tau) &= E(Z(t)Z(t+\tau))\\ &= E((X(t) - Y(t))(X(t+\tau) - Y(t+\tau)))\\ &= R_X(\tau) + R_Y(\tau) - R_{YX}(-\tau) - R_{XY}(-\tau)\\ &= R_X(\tau) + R_Y(\tau) - R_{XY}(\tau) - R_{XY}(-\tau).\end{aligned}$$

Now, taking the Fourier Transform, we get

$$\begin{aligned}S_Z(f) &= S_X(f) + S_Y(f) - S_{XY}(f) - S_{XY}(-f)\\ &= S_X(f) + |H(f)|^2 S_X(f) - H(-f)S_X(f) - H(f)S_X(f)\\ &= S_X(f)\left(1 + |H(f)|^2 - 2\mathrm{Re}[H(f)]\right)\\ &= S_X(f)|1 - H(f)|^2.\end{aligned}$$

(b) To find the average power of $Z(t)$, we find the area under $S_Z(f)$

$$E(Z^2(t)) = \int_{-\infty}^{\infty} |1 - H(f)|^2 S_X(f)\, df.$$

2. *Echo filtering.* A signal $X(t)$ and its echo arrive at the receiver as $Y(t) = X(t) + X(t-\Delta) + Z(t)$. Here the signal $X(t)$ is a zero-mean WSS process with power spectral density $S_X(f)$ and the noise $Z(t)$ is a zero-mean WSS with power spectral density $S_Z(f) = N_0/2$, uncorrelated with $X(t)$.

   (a) Find $S_Y(f)$ in terms of $S_X(f)$, Δ, and N_0.

   (b) Find the best linear filter to estimate $X(t)$ from $\{Y(s)\}_{-\infty<s<\infty}$.

   Solution:

   (a) We can write $Y(t) = g(t) * X(t) + Z(t)$ where $g(t) = \delta(t) + \delta(t-\Delta)$. Thus, $S_Y(f) = |G(f)|^2 S_X(f) + S_Z(f) = |1 + e^{-j2\pi\Delta f}|^2 S_X(f) + \frac{N_0}{2}$.

   (b) Since $S_{YX}(f) = (1 + e^{-j2\pi\Delta f}) S_X(f)$,

$$\hat{X}(t) = h(t) * Y(t),$$

   where the linear filter $h(t)$ has the transfer function

$$H(f) = \frac{S_{XY}(f)}{S_Y(f)} = \frac{S_{YX}(-f)}{S_Y(f)} = \frac{(1 + e^{j2\pi\Delta f}) S_X(f)}{|1 + e^{-j2\pi\Delta f}|^2 S_X(f) + \frac{N_0}{2}}.$$

3. *Discrete-time LTI system with white noise input.* Let $\{X_n : -\infty < n < \infty\}$ be a discrete-time white noise process, i.e., $\mathsf{E}(X_n) = 0, -\infty < n < \infty$, and

$$R_X(n) = \begin{cases} 1 & n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

   The process is filtered using a linear time invariant system with impulse response

$$h(n) = \begin{cases} \alpha & n = 0, \\ \beta & n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

   Find α and β such that the output process Y_n has

$$R_Y(n) = \begin{cases} 2 & n = 0, \\ 1 & |n| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

   Solution: We are given that $R_X(n)$ is a discrete-time unit impulse. Therefore

$$R_Y(n) = h(n) * R_X(n) * h(-n) = h(n) * h(-n)\,.$$

The impulse response $h(n)$ is the sequence $(\alpha, \beta, 0, 0, \ldots)$. The convolution with $h(-n)$ has only finitely many nonzero terms.

$$
\begin{aligned}
R_Y(0) &= 2 = h(0) * h(0) = \alpha^2 + \beta^2 \\
R_Y(+1) &= 1 = h(1) * h(-1) = \alpha\beta \\
R_Y(-1) &= 1 = R_Y(1)
\end{aligned}
$$

This pair of equations has two solutions: $\alpha = +1$ and $\beta = +1$ or $\alpha = -1$ and $\beta = -1$.

4. *Finding impulse response of LTI system.* To find the impulse response $h(t)$ of an LTI system (e.g., a concert hall), i.e., to *identify* the system, white noise $X(t)$, $-\infty < t < \infty$, is applied to its input and the output $Y(t)$ is measured. Given the input and output sample functions, the crosscorrelation $R_{YX}(\tau)$ is estimated. Show how $R_{YX}(\tau)$ can be used to find $h(t)$.

   Solution: Since white noise has a flat psd, the crosspower spectral density of the input $X(t)$ and the output $Y(t)$ is just the transfer function of the system scaled by the psd of the white noise.

$$
\begin{aligned}
S_{YX}(f) &= H(f) S_X(f) = H(f) \frac{N_0}{2} \\
R_{YX}(\tau) &= \mathcal{F}^{-1}(S_{YX}(f)) = \frac{N_0}{2} h(\tau)
\end{aligned}
$$

   Thus to estimate the impulse response of a linear time invariant system, we apply white noise to its input, estimate the crosscorrelation function of its input and output, and scale it by $2/N_0$.

5. *Integrators.* Let $Y(t)$ be a short-term integration of a WSS process $X(t)$:

$$
Y(t) = \frac{1}{T} \int_{t-T}^{t} X(u)\,du.
$$

   Find $S_Y(f)$ in terms of $S_X(f)$.

   Solution: It is easy to see that the system that generates $Y(t)$ from $X(t)$ is linear and time-invariant. Writing $\delta(t)$ in place of $X(t)$, the impulse response of the system can then be obtained as

$$
\begin{aligned}
h(t) &= \frac{1}{T} \int_{t-T}^{t} \delta(u)\,du \\
&= \begin{cases} \frac{1}{T}, & 0 \le t \le T \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

   Alternatively, we can find $h(t)$ by attempting to write $Y(t)$ as $Y(t) = \displaystyle\int_{\infty}^{\infty} h(\tau) X(t-\tau)\,d\tau$. We have

$$
\begin{aligned}
Y(t) &= \frac{1}{T} \int_{t-T}^{t} X(u)\,du \\
&= \frac{1}{T} \int_{0}^{T} X(t-\tau)\,d\tau.
\end{aligned}
$$

This shows, as before, that $h(\tau) = \begin{cases} \frac{1}{T}, & 0 \le \tau \le T \\ 0, & \text{otherwise.} \end{cases}$

Thus, the frequency response is given by

$$\begin{aligned} H(f) &= \frac{1}{T}\int_0^T e^{-i2\pi f t}dt \\ &= \frac{1}{T}\Big(\frac{e^{-i2\pi f T}-1}{-i2\pi f}\Big) \\ &= \frac{e^{-i\pi f T}}{\pi f T}\Big(\frac{e^{i\pi f T}-e^{-i\pi f T}}{2i}\Big) \\ &= e^{-i\pi f T}\frac{\sin(\pi f T)}{\pi f T}. \end{aligned}$$

Thus,

$$\begin{aligned} S_Y(f) &= S_X(f)|H(f)|^2 \\ &= S_X(f)\frac{\sin^2(\pi f T)}{\pi^2 f^2 T^2}. \end{aligned}$$

Alternative Method:

We have

$$\begin{aligned} \mathsf{E}[Y(t)Y(t+\tau)] &= \frac{1}{T^2}\mathsf{E}\Big[\int_{t-T}^{t}\int_{t+\tau-T}^{t+\tau} X(u)X(v)dvdu\Big] \\ &= \frac{1}{T^2}\mathsf{E}\Big[\int_{t-T}^{t}\int_{t-T}^{t} X(u)X(v+\tau)dvdu\Big] \\ &= \frac{1}{T^2}\int_{t-T}^{t}\int_{t-T}^{t} R_{XX}(v+\tau-u)dvdu. \end{aligned}$$

Writing $w = u - v$ in the integral, we see that $-T \le w \le T$, and for each fixed w, $t-T+w \le u \le t+w$. Thus,

$$\begin{aligned} \mathsf{E}[Y(t)Y(t+\tau)] &= \frac{1}{T^2}\int_{-T}^{T}\int_{\max(t-T,t-T+w)}^{\min(t,t+w)} R_{XX}(\tau-w)dudw \\ &= \frac{1}{T^2}\int_{-T}^{T} R_{XX}(\tau-w)\Big(\min(t,t+w)-\max(t-T,t-T+w)\Big)dw \\ &= \int_{-T}^{T} R_{XX}(\tau-w)g(w)dw, \end{aligned}$$

where

$$\begin{aligned} g(w) &= \frac{1}{T^2}\Big(\min(t,t+w)-\max(t-T,t-T+w)\Big) \\ &= \begin{cases} \frac{T+w}{T^2}, & -T \le w \le 0 \\ \frac{T-w}{T^2}, & 0 \le w \le T \end{cases} \\ &= \frac{T-|w|}{T^2}. \end{aligned}$$

Thus, $R_{YY}(\tau) = R_{XX}(\tau) * g(\tau)$, and hence

$$\begin{aligned}
S_Y(f) &= S_X(f)\int_{-\infty}^{\infty} g(t)e^{-i2\pi ft}dt \\
&= S_X(f)\frac{1}{T^2}\int_{-T}^{T}(T-|t|)e^{-i2\pi ft}dt \\
&= S_X(f)\frac{1}{T^2}\int_{-T}^{T}(T-|t|)\cos(2\pi ft)dt \\
&= S_X(f)\Big(\frac{\sin(2\pi fT)}{\pi fT} - \frac{1}{4\pi^2 f^2T^2}\int_{-2\pi fT}^{2\pi fT}|u|\cos u du\Big) \\
&= S_X(f)\left(\frac{\sin(2\pi fT)}{\pi fT} - \frac{1}{4\pi^2 f^2T^2}\cdot 2\Big(2\pi fT\sin(2\pi fT) + \cos(2\pi fT) - \cos 0\Big)\right) \\
&= \frac{S_X(f)}{2\pi^2 f^2T^2}\Big(1-\cos(2\pi fT)\Big) \\
&= \frac{S_X(f)}{2\pi^2 f^2T^2}\Big(2\sin^2(\pi fT)\Big) \\
&= S_X(f)\frac{\sin^2(\pi fT)}{\pi^2 f^2T^2}.
\end{aligned}$$

4 Linear Estimation of Random Processes

1. *Prediction.* Let $\mathbf{X}$ be a random process with zero mean and covariance matrix

$$\Sigma_{\mathbf{X}} = \begin{bmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{n-1} \\ \alpha & 1 & \alpha & & \\ \alpha^2 & \alpha & 1 & & \\ \vdots & & & \ddots & \\ \alpha^{n-1} & & & \cdots & 1 \end{bmatrix}$$

for $|\alpha| < 1$. $X_1, X_2, \ldots, X_{n-1}$ are observed, find the best linear MSE estimate (predictor) of X_n. Compute its MSE.

Solution: Let $\mathbf{Y} = (X_1, X_2, \ldots X_{n-1})^T$. Since $\mathsf{E}[X_n] = 0$ and $\mathsf{E}[\mathbf{Y}] = \mathbf{0}$, the best linear MSE estimate of X_n given $\mathbf{Y}$ is given by $\hat{X}_n = h^T\mathbf{Y}$, where the vector h satisfies the equation

$$\mathsf{E}[\mathbf{Y}X_n] = \mathsf{E}[\mathbf{Y}\mathbf{Y}^T]h. \tag{1}$$

We see that $\mathsf{E}[\mathbf{Y}\mathbf{Y}^T]$ is simply $\Sigma_{\mathbf{X}}$ with the last row and last column removed, and $\mathsf{E}[\mathbf{Y}X_n]$ is simply the last column of $\Sigma_{\mathbf{X}}$, with the last element removed.

Thus, $\mathsf{E}[\mathbf{Y}X_n] = \begin{bmatrix} \alpha^{n-1} \\ \alpha^{n-2} \\ \vdots \\ \alpha \end{bmatrix}$, and the last column of $\mathsf{E}[\mathbf{Y}\mathbf{Y}^T]$ is $\begin{bmatrix} \alpha^{n-2} \\ \alpha^{n-3} \\ \vdots \\ 1 \end{bmatrix}$.

Thus, $\mathsf{E}[\mathbf{Y}X_n]$ is simply a constant multiple of the last column of $\mathsf{E}[\mathbf{Y}\mathbf{Y}^T]$, and thus, (2) is solved by

$$h = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \alpha \end{bmatrix}.$$

Hence, the best linear MSE estimate of X_n given $\mathbf{Y}$ is given by $\hat{X}_n = \alpha X_{n-1}$.

The MSE of this estimate is

$$\begin{aligned} \mathsf{E}[(X_n - \alpha X_{n-1})^2] &= \mathsf{E}[X_n^2] + \alpha^2 \mathsf{E}[X_{n-1}^2] - 2\alpha \mathsf{E}[X_n X_{n-1}] \\ &= 1 - \alpha^2. \end{aligned}$$

2. *Arrow of time.* Let X_0 be a Gaussian random variable with zero mean and unit variance, and $X_n = \alpha X_{n-1} + Z_n$ for $n \geq 1$, where α is a fixed constant with $|\alpha| < 1$ and $Z_1, Z_2, \ldots$ are i.i.d. $\sim N(0, 1-\alpha^2)$, independent of X_0.

   (a) Is the process $\{X_n\}$ Gaussian?

   (b) Is $\{X_n\}$ Markov?

   (c) Find $R_X(n, m)$.

   (d) Find the (nonlinear) MMSE estimate of X_{100} given $(X_1, X_2, \ldots, X_{99})$.

   (e) Find the MMSE estimate of X_{100} given $(X_{101}, X_{102}, \ldots, X_{199})$.

   (f) Find the MMSE estimate of X_{100} given $(X_1, \ldots, X_{99}, X_{101}, \ldots, X_{199})$.

   Solution:

   (a) Yes, the process is Gaussian, since it is the linear transform of white Gaussian process $\{Z_n\}$.

   (b) Yes, the process is Markov since $X_n | \{X_1 = x_1, \ldots, X_{n-1} = x_{n-1}\} \sim N(\alpha x_{n-1}, 1 - \alpha^2)$, which depends only on x_{n-1}.

   (c) First note that $R_X(n, n) = \alpha^2 R_X(n-1, n-1) + (1 - \alpha^2) = 1$ for all n. Since we can express

   $$X_n = \alpha^k X_{n-k} + \alpha^{k-1} Z_{n-k+1} + \cdots + Z_n,$$

   and X_{n-k} is independent of $(Z_{n-k+1}, \ldots, Z_n)$, we have $R_X(n, n-k) = \mathsf{E}(X_n X_{n-k}) = \alpha^k$. Thus, $R_X(n, m) = \alpha^{|n-m|}$.

   (d) Because the process is Gaussian, the MMSE estimator is linear. From Markovity, $\hat{X}_{100} = E(X_{100} | X_1, \ldots, X_{99}) = E(X_{100} | X_{99}) = \frac{R_X(100,99)}{R_X(99,99)} X_{99} = \alpha X_{99}$.

   (e) Again from the Markovity and the symmetry, $(X_1, \ldots, X_n)$ has the same distribution as $(X_n, \ldots, X_1)$ hence $\hat{X}_{100} = \mathsf{E}(X_{100} | X_{101}, \ldots, X_{199}) = \mathsf{E}(X_{100} | X_{101}) = \alpha X_{101}$.

(f) First note that X_{100} is conditionally independent of $(X_1, \dots, X_{98}, X_{102}, \dots, X_{199})$ given (X_{99}, X_{101}). To see this, consider

$$\begin{aligned}
&f(x_{100}|x_1, \dots, x_{99}, x_{101}, \dots, x_{199}) \\
&\quad = \frac{f(x_1, \dots, x_{199})}{f(x_1, \dots, x_{99}, x_{101}, \dots, x_{199})} \\
&\quad = \frac{f(x_1)f(x_2|x_1)\cdots f_(x_{99}|x_{98})f(x_{100}|x_{99})f(x_{101}|x_{100})f(x_{102}|x_{101})\cdots f(x_{199}|x_{198})}{f(x_1)f(x_2|x_1)\cdots f(x_{99}|x_{98})f(x_{101}|x_{99})f(x_{102}|x_{101})\cdots f(x_{199}|x_{198})} \\
&\quad = \frac{f(x_{100}|x_{99})f(x_{101}|x_{100})}{f(x_{101}|x_{99})} \\
&\quad = \frac{f(x_{100}, x_{101}|x_{99})}{f(x_{101}|x_{99})} \\
&\quad = f(x_{100}|x_{99}, x_{101}).
\end{aligned}$$

Thus,

$$\begin{aligned}
\hat{X}_{100} &= \mathsf{E}(X_{100}|X_{99}, X_{101}) \\
&= \begin{bmatrix} R_X(100, 99) & R_X(100, 101) \end{bmatrix} \begin{bmatrix} R_X(99, 99) & R_X(99, 101) \\ R_X(101, 99) & R_X(101, 101) \end{bmatrix}^{-1} \begin{bmatrix} X_{99} \\ X_{101} \end{bmatrix} \\
&= \begin{bmatrix} \alpha & \alpha \end{bmatrix} \begin{bmatrix} 1 & \alpha^2 \\ \alpha^2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} X_{99} \\ X_{101} \end{bmatrix} \\
&= \frac{\alpha}{1 + \alpha^2}(X_{99} + X_{101}).
\end{aligned}$$