# UNIVERSITY OF CALIFORNIA, SAN DIEGO
## Electrical & Computer Engineering Department
## ECE 250 - Winter Quarter 2022
*Random Processes*

**Problem Set #3 Due Friday, January 28, 2022 at 11:59pm**
**Submit solutions to Problems 5, 6, 7, 8 only**

1. *Geometric with conditions.* Let $X$ be a geometric random variable with pmf
$$p_X(k) = p(1-p)^{k-1}, \;\; k = 1, 2, \ldots .$$
Find the conditional pmf $p_X(k|A) = \mathsf{P}\{X = k | X \in A\}$ if:
   (a) $A = \{X > m\}$ where $m$ is a positive integer. Show that the geometric random variable is memoryless.
   (b) $A = \{X < m\}$.
   (c) $A = \{X \text{ is an even number}\}$.

2. Let $A$ be a nonzero probability event $A$. Show that
   (a) $\mathsf{P}(A) = \mathsf{P}(A|X \le x)F_X(x) \; + \; \mathsf{P}(A|X > x)(1 - F_X(x))$.
   (b) $F_X(x|A) = \dfrac{\mathsf{P}(A|X \le x)}{\mathsf{P}(A)} F_X(x)$.

3. *Joint cdf or not.* Consider the function
$$G(x, y) = \begin{cases} 1 & \text{if } x + y \ge 0 \\ 0 & \text{otherwise.} \end{cases}$$
Can $G$ be a joint cdf for a pair of random variables? Justify your answer.

4. *Time until the $n$-th arrival.* Let the random variable $N(t)$ be the number of packets arriving during time $(0, t]$. Suppose $N(t)$ is Poisson with pmf
$$p_N(n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \quad \text{for } n = 0, 1, 2, \ldots .$$
Let the random variable $Y$ be the time to get the $n$-th packet. Find the pdf of $Y$.

5. *Diamond distribution.* Consider the random variables $X$ and $Y$ with the joint pdf
$$f_{X,Y}(x, y) = \begin{cases} c, & \text{if } |x| + |y| \le 1/\sqrt{2}, \\ 0, & \text{otherwise,} \end{cases}$$
where $c$ is a constant.

(a) Find $c$.

(b) Find $f_X(x)$ and $f_{X|Y}(x|y)$.

(c) Are $X$ and $Y$ independent random variables? Justify your answer.

(d) Define the random variable $Z = (|X| + |Y|)$. Find the pdf $f_Z(z)$.

6. *Coin with random bias.* You are given a coin but are not told what its bias (probability of heads) is. You are told instead that the bias is the outcome of a random variable $P \sim \mathrm{U}[0,1]$. To get more infromation about the coin bias, you flip it independently 10 times. Let $X$ be the number of heads you get. Thus $X \sim \mathrm{Binom}(10, P)$. Assuming that $X = 9$, find and sketch the *a posteriori* probability of $P$, i.e., $f_{P|X}(p|9)$.

7. *First available teller.* Consider a bank with two tellers. The service times for the tellers are independent exponentially distributed random variables $X_1 \sim \mathrm{Exp}(\lambda_1)$ and $X_2 \sim \mathrm{Exp}(\lambda_2)$, respectively. You arrive at the bank and find that both tellers are busy but that nobody else is waiting to be served. You are served by the first available teller once he/she is free.

   (a) What is the probability that you are served by the first teller? (Hint: Recall that the exponential distribution has the memoryless property.)

   (b) Let the random variable $Y$ denote your waiting time. Find the pdf of $Y$.

8. *Two independent uniform random variables.* Let $X$ and $Y$ be independently and uniformly drawn from the inverval $[0, 1]$.

   (a) Find the pdf of $U = \max(X, Y)$.

   (b) Find the pdf of $V = \min(X, Y)$.

   (c) Find the pdf of $W = U - V$.

   (d) Find the probability $\mathsf{P}\{|X - Y| \geq 1/2\}$.

9. *Maximal correlation.*

   (a) For any pair of random variables $(X, Y)$, show that

$$F_{X,Y}(x, y) \leq \min\{F_X(x), F_Y(y)\}.$$

   Now let $F$ and $G$ be continuous and invertible cdf's and let $X \sim F$.

   (b) Find the distribution of

$$Y = G^{-1}(F(X)).$$

   (c) Show that

$$F_{X,Y}(x, y) = \min\{F(x), G(y)\}.$$