UNIVERSITY OF CALIFORNIA, SAN DIEGO

Electrical & Computer Engineering Department

ECE 250 - Winter Quarter 2022

Random Processes

Problem Set #5 **Due Wednesday, February 16, 2022. (Submit 2, 3, 10, 11).**

1. *Neural net.* Let $Y = X + Z$, where the signal $X \sim \mathrm{U}[-1, 1]$ and noise $Z \sim \mathrm{N}(0, 1)$ are independent.

   (a) Find the function $g(y)$ that minimizes

   $$\mathrm{MSE} = \mathsf{E}\left[(\mathrm{sgn}(X) - g(Y))^2\right],$$

   where

   $$\mathrm{sgn}(x) = \begin{cases} -1 & x \le 0 \\ +1 & x > 0. \end{cases}$$

   (b) Plot $g(y)$ vs. y.

2. *Additive shot noise channel.* Consider an additive noise channel $Y = X + Z$, where the signal $X \sim \mathcal{N}(0, 1)$, and the noise $Z|\{X = x\} \sim \mathcal{N}(0, x^2)$, i.e., the noise power increases linearly with the signal squared.

   (a) Find $E(Z^2)$.

   (b) Find the best linear MSE estimate of X given Y.
   [Hint: To compute $\mathrm{Var}(Y)$, use the law of conditional variance.]

3. *Additive uniform noise channel.* Let the signal

   $$X = \begin{cases} +1, & \text{with probability } \frac{1}{2} \\ -1, & \text{with probability } \frac{1}{2}, \end{cases}$$

   and the noise $Z \sim \mathrm{Unif}[-2, 2]$ be independent random variables. Their sum $Y = X + Z$ is observed.

   (a) Find the minimum MSE estimate of X given Y and its MSE.

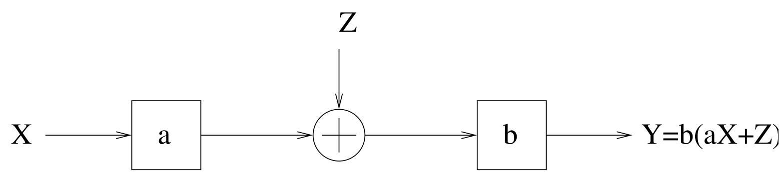
   (b) Now suppose we use a decoder to decide whether $X = +1$ or $X = -1$ so that the probability of error is minimized. Find the optimal decoder and its probability of error. Compare the optimal decoder's MSE to the minimum MSE of part (a).

4. *Linear estimator.* Consider a channel with the observation $Y = XZ$, where the signal X and the noise Z are uncorrelated Gaussian random variables. Let $E[X] = 1$, $E[Z] = 2$, $\sigma_X^2 = 5$, and $\sigma_Z^2 = 8$.

   (a) Find the best MSE linear estimate of X given Y.

   (b) Suppose your friend from Caltech tells you that he was able to derive an estimator with a lower MSE. Your friend from UCLA disagrees, saying that this is not possible because the signal and the noise are Gaussian, and hence the best linear MSE estimator will also be the best MSE estimator. Could your UCLA friend be wrong?

5. *Additive-noise channel with path gain.* Consider the additive noise channel shown in the figure below, where X and Z are zero mean and uncorrelated, and a and b are constants.



   Find the MMSE linear estimate of X given Y and its MSE in terms only of σ_X, σ_Z, a, and b.

6. *Worst noise distribution.* Consider an additive noise channel $Y = X+Z$, where the signal $X \sim \mathcal{N}(0, P)$ and the noise Z has zero mean and variance N. Assume X and Z are independent. Find a distribution of Z that maximizes the minimum MSE of estimating X given Y, i.e., the distribution of the worst noise Z that has the given mean and variance. You need to justify your answer.

7. *Image processing.* A pixel signal $X \sim \mathrm{U}[-k, k]$ is digitized to obtain

$$\tilde{X} = i + \frac{1}{2}, \text{ if } i < X \leq i+1, \; i = -k, \; -k+1, \ldots, \; k-2, \; k-1.$$

   To improve the the visual appearance, the digitized value $\tilde{X}$ is dithered by adding an independent noise Z with mean $\mathsf{E}(Z) = 0$ and variance $\mathrm{Var}(Z) = N$ to obtain $Y = \tilde{X} + Z$.

   (a) Find the correlation of X and Y.

   (b) Find the best linear MSE estimate of X given Y. Your answer should be in terms only of k, N, and Y.

8. *Orthogonality.* Let $\hat{X}$ be the minimum MSE estimate of X given Y.

   (a) Show that for any function $g(y)$, $E((X - \hat{X})g(Y)) = 0$, i.e., the error $(X - \hat{X})$ and $g(Y)$ are orthogonal.

   (b) Show that

$$\mathrm{Var}(X) = E(\mathrm{Var}(X|Y)) + \mathrm{Var}(\hat{X}).$$

   Provide a geometric interpretation for this result.

9. *Difference from sum.* Let X and Y be two random variables. Let $Z = X + Y$ and let $W = X - Y$. Find the best linear estimate of W given Z as a function of $E(X), E(Y), \sigma_X, \sigma_Y, \rho_{XY}$ and Z.

10. *Sum of exponentials via transforms.* Let $X \sim \text{Exp}(\lambda)$ and $Y \sim \text{Exp}(\mu)$ be independent exponential random variables, where λ and μ are positive constants. Using transform methods, evaluate the probability density function of $Z = X + Y$ if $\mu \neq \lambda$ and if $\mu = \lambda$.

11. *Moment theorem application.* The discrete random variable X has probability mass function given by

$$p_X(n) = \left(\frac{1}{2}\right)^n, \quad n = 1, 2, \ldots.$$

Evaluate the mean and variance of X using the moment theorem.