ECE 250: Stochastic Processes: Week #10

Instructor: Behrouz Touri

Outline:

- Mean, Autocorrelation, and Autocovariance Functions
- Strict and Wide Sense Stationary Processes
- Power spectral density

Mean, Autocorrelation, and Autocovariance Functions

- For a discrete or cont's time random process $\{X_t\}_{t \in I}$, we define the **deterministic** functions:

  (a) **Mean**: function $\mu_X : I \to \mathbb{R}$ defined by:

  $$\mu_X(t) = \mathbb{E}[X_t].$$

  (b) **Autocorrelation function**: function $R_X : I \times I \to \mathbb{R}$ defined by

  $$R_X(t_1, t_2) = \mathbb{E}[X_{t_1} X_{t_2}].$$

  (c) **Autocovariance function**: function $R_X : I \times I \to \mathbb{R}$ defined by

  $$C_X(t_1, t_2) = \mathbb{E}[(X_{t_1} - \bar{X}_{t_1})(X_{t_2} - \bar{X}_{t_2})] = R_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2).$$

WSS and SSS

Example: i.i.d. process $\{X_t\}$:

(a) For the mean function:

$$\mu_X(t) = \mathbb{E}[X_t] = \mathbb{E}[X_0].$$

Therefore, we have a constant function.

(b) For the **Autocorrelation function**:

$$R_X(t_1, t_2) = \mathbb{E}[X_{t_1} X_{t_2}] = \begin{cases} \mathbb{E}[X_{t_1}^2] = \mathbb{E}[X_1^2] & t_1 = t_2 \\ \mathbb{E}[X_{t_1}]\mathbb{E}[X_{t_2}] = \mu_X(t_1)\mu_X(t_2) = \mu_X(0)^2 & t_1 \neq t_2 \end{cases}.$$

(c) **Autocovariance function**:

$$C_X(t_1, t_2) = R_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2) = \begin{cases} \mathsf{Var}(X_1) & t_1 = t_2 \\ 0 & t_1 \neq t_2 \end{cases}.$$

- Definition: We say that a random process is **Wide Sense Stationary** (WSS) if (i) the mean function does not depend on time t, and (ii) the $R_X(t_1, t_2) = f(t_1 - t_2)$, i.e., autocorrelation function is just a function of $t_1 - t_2$. Example: i.i.d. processes

- Definition: Definition: We say that a random process is **Strict Sense Stationary** (SSS) if the (finite) joint probability distributions are invariant under shift, i.e., for all $t_1 < t_2 < \cdots < t_k$ and all $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$:

  $$F_{X_{t_1}, \ldots, X_{t_k}}(\alpha_1, \ldots, \alpha_k) = F_{X_{t_1+s}, \ldots, X_{t_k+s}}(\alpha_1, \ldots, \alpha_k)$$

  for all $-t_1 \leq s$.
  Example: i.i.d. processes as

  $$F_{X_{t_1}, \ldots, X_{t_k}}(\alpha_1, \ldots, \alpha_k) = F_{X_{t_1}}(\alpha_1) \cdots F_{X_{t_1}}(\alpha_k) = F_X(\alpha_1) \cdots F_X(\alpha_k).$$

Example: Random Walk

Let $\{X_k\}$ be a random walk, given by $X_{k+1} = X_k + Z_k$ where $\{Z_k\}$ is i.i.d. with zero mean and variance σ^2 and $X_0 = 0$ a.s.

(a) For the mean function:

$$\mu_X(k) = \mathbb{E}[X_{k-1} + Z_{k-1}] = \mathbb{E}[X_{k-1}].$$

Therefore, $\mu_X(k) = \mu_X(k-1) = \ldots = \mu_X(0) = 0$.

(b) For the **Autocorrelation function**: Let $k_1 \leq k_2$:

$$\begin{aligned} R_X(k_1, k_2) &= \mathbb{E}[X_{k_1} X_{k_2}] = \mathbb{E}[X_{k_1}(X_{k_2} - X_{k_1} + X_{k_1})] \\ &= \mathbb{E}[X_{k_1}(X_{k_2} - X_{k_1})] + \mathbb{E}[X_{k_1}^2] \\ &= \mathbb{E}[X_{k_1}^2] = k_1 \sigma^2. \end{aligned}$$

Therefore, $R_X(k_1, k_2) = \min(k_1, k_2)\sigma^2$. Thus, such a process is not WSS and hence, not an SSS.

(c) **Autocovariance function**: since the process is zero mean $C_X = R_X$.

Continuous Time Random Processes

- Example: for a deterministic $\alpha > 0$ and frequency ω, let $X_t = \alpha \cos(\omega t + \theta)$ where $\theta \sim U([0, 2\pi])$.
  - You can verify that we have the marginal PDF $f_{X_t}(x) = \frac{sgn(\alpha)}{\pi\sqrt{\alpha^2 - x^2}}$ for $x \in (-\alpha, \alpha)$.
  - The mean function:

$$\mu_X(t) = \mathbb{E}[\alpha \cos(\omega t + \theta)] = \frac{1}{2\pi} \int_0^{2\pi} \alpha \cos(\omega t + \theta) d\theta = 0.$$

  - The correlation function:

$$\begin{aligned}
R_X(t_1, t_2) &= \mathbb{E}[\alpha \cos(\omega t_1 + \theta) \alpha \cos(\omega t_2 + \theta)] \\
&= \frac{1}{2\pi} \int_0^{2\pi} \alpha^2 \cos(\omega t_1 + \theta) \cos(\omega t_2 + \theta) d\theta \\
&= \frac{\alpha^2}{2\pi} \int_0^{2\pi} \cos(\omega t_1 + \theta) \cos(\omega t_2 + \theta) d\theta \\
&= \frac{\alpha^2}{4\pi} \int_0^{2\pi} \cos(\omega(t_1 + t_2) + 2\theta) d\theta + \frac{\alpha^2}{4\pi} \int_0^{2\pi} \cos(\omega(t_1 - t_2)) d\theta \\
&= \frac{\alpha^2}{4\pi} \int_0^{2\pi} \cos(\omega(t_1 - t_2)) d\theta \\
&= \frac{\alpha^2}{2} \cos(\omega(t_1 - t_2)).
\end{aligned}$$

Independent Increment Continuous Time Random Processes

- Recall $\{X_t\}$ is an *Independent Increment* process if its increments over non-overlapping intervals are independent, i.e., for any $0 \leq a_1 < b_1 \leq a_2 < b_2 \leq \ldots \leq a_k < b_k$, the random variables $X_{b_1} - X_{a_1}, X_{b_2} - X_{a_2}, \ldots, X_{b_k} - X_{a_k}$ are independent. Examples:
  - A **Wiener** Process is an independent increment CT random process $\{X_t\}_{t\geq 0}$ that satisfies (i) $X_0 = 0$, (ii) *Gaussian Increments*: $X_t - X_s \sim \mathcal{N}(0, t-s)$ for any $t > s$, and (iii) *a.s. Continuous*: for almost all ω, $X_t(\omega)$ is a continuous function.
  - Recall: A discrete random variable X has the Poisson distribution with parameter $\lambda > 0$, if X takes non-negative integers with $P(X = k) = \frac{\lambda^k}{k!}e^{-\lambda}$. We denote this by $X \sim$ Pois(λ).
  - A **Poison Point Process** $\{X_t\}_{t\geq 0}$ with parameter $\lambda > 0$ is a CT independent increment process with (i) $X_0 = 0$, and (ii) *Poisson Increments*: $X_t - X_s \sim$ Pois$(\lambda(t-s))$.

Properties of WSS Processes

- Some properties of a WSS process $\{X_t\}$:
  1. $R_X(\tau) = \mathbb{E}[X(t)X(t+\tau)]$ is an even, i.e., $R_X(\tau) = R_X(-\tau)$.
  2. $R_X(0) \geq R_X(\tau)$ for all τ.
  3. For independent processes $\{X(t)\}$ and $\{Y(t)\}$ with zero mean, $R_{X+Y}(\tau) = R_X(\tau) + R_Y(\tau)$.
  4. Necessary and Sufficient Condition: A function $R(\tau) = R(t_1 - t_2)$ is a autocorrelation function of a WSS process iff (i) $R(\tau)$ is even, and (ii) it is non-negative definite: for all $t_1, \ldots, t_n$ and $a_1, \ldots, a_n$, $\sum_{i,j} a_i a_j R(t_i - t_j) \geq 0$.

Power Spectral Density (PSD) and Some Geometry

- In the space of complex-valued signals, inner product can be defined by $\langle x(t), y(t)\rangle = \int_{-\infty}^{\infty} x(t)y^*(t)dt$.
- Interpretation: $\langle x(t), y(t)\rangle$ is the component of $x(t)$ along $y(t)$.
- For a CT WSS process $X(t)$ (that is integrable), we can find the "power density" at frequency f (Hertz):

$$\begin{aligned} S(f) &:= \mathbb{E}[|\langle X(t), e^{j2\pi f t}\rangle|^2] \\ &= FT[R_X(\tau)] = \int_{-\infty}^{\infty} R_X(\tau)e^{-j2\pi f\tau}d\tau \end{aligned}$$

- $S(f)$ is called the Power Spectral Density (PSD) of $X(t)$.
- If using the rad/s unit for frequency (denoted by ω),

$$S(\omega) = FT[R_X(\tau)] = \int_{-\infty}^{\infty} R_X(\tau)e^{-j\omega\tau}d\tau$$

- Using the inverse Fourier Transform for f Hz or ω rad/sec, we get:

$$\begin{aligned} R_x(\tau) &= \int_{-\infty}^{\infty} S(f)e^{j2\pi f\tau}df \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\omega)e^{j\omega\tau}df \end{aligned}$$

PSD of DT WSS Processes

- Similarly, for a DT WSS process $\{X_k\}$, the PSD is given (in Hertz) by

$$S(f) = FT(R_X(k)) = \sum_{k=-\infty}^{\infty} R_X(k) e^{-j2\pi fk}.$$

- In rad/sec (notation: ω), we get

$$S(\omega) = FT(R_X(k)) = \sum_{k=-\infty}^{\infty} R_X(k) e^{-j\omega k}.$$

- Note that $S_X(f)$ is periodic with period 1, and $S_X(\omega)$ is periodic with period 2π for DT WSS processes.
- Inverse Relations:

$$\begin{aligned} R_X(k) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} S_X(f) e^{j2\pi fk} df \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} S_X(\omega) e^{j\omega k} d\omega. \end{aligned}$$

Properties of PSD

- PSD is even and real: this follows from the fact that $R_X(\tau)$ is even and real
- By the definiton $S_X(f) \geq 0$
- Average power in $[f_1, f_2]$ would be:

$$\int_{-f_2}^{-f_1} S_X(f)df + \int_{f_1}^{f_2} S_X(f)df = 2\int_{f_1}^{f_2} S_X(f)df.$$

- Average power: $\int_{-\infty}^{\infty} S_X(f)df = R_X(0) = \mathbb{E}[X(0)^2]$.

PSD: Examples

- CT Sinusoidal with Random Phase: $X(t) = \alpha \cos(w_0 t + \theta)$ with $\theta\ U(0, 2\pi)$, we got $R_X(\tau) = \frac{\alpha^2}{2}\cos(\omega_0\tau)$. Therefore, $S_X(f) = \frac{\alpha^2}{4}(\delta(2\pi f - w_0) + \delta(2\pi f + \omega_0))$.
- DT White Noise:
  - Definition: We say that a DT random process $\{X_k\}$ is a white noise if it is WSS with $\mu_X(k) = 0$ and $R_X(k) = \sigma^2\delta[k]$.
  - Therefore, $S_X(f) = FT(R_X(\tau)) = 1$ for all f.
- CT White Noise:
  - **Bandlimited White Noise**: A CT WSS process $\{X(t)\}$ is said to be a band-limited white noise if $\mu_X(t) = 0$ and $R_X(\tau) = \sigma^2 B \frac{\sin(2\pi B\tau)}{2\pi B\tau}$. $2B$ is called the bandwidth of the noise.
  - For a band-limited white noise $\{X_t\}$, $S_X(t) = \sigma^2 \text{rect}_B(f)$.
  - Letting $B \to \infty$, we have a white noise $\{X_t\}$: A a CT WSS process is a **White Noise** if it has zero mean and autocorrelation function $R_X(\tau) = \sigma^2\delta(\tau)$.
  - In this case, $S_X(f) = \sigma^2$.
  - CT White noise is equivalent of CT dirac's delta function: physically meaningless, practically very useful!
  - A practical note: when we talk about a CT White noise in a context, we are really talking about a band-limited white noise (which is realizable) with a bandwith that far exceeds the bandwidth of the underlying systems/signals involved.

Signal Processing of Random Processes

- Main Question: What can we say about the output process $\{Y_t\}$ for the random process $\{X_t\}$ fed into an LTI system with impulse response $h(t)$?
- Definition: For (CT or DT) random processes $\{X_t\}$ and $\{Y_t\}$, we define the cross-correlation function $R_{XY}(t_1, t_2) = \mathbb{E}[X_{t_1} Y_{t_2}]$.
- Definition: We say that $\{X_t\}$ and $\{Y_t\}$ are jointly WSS, if (i) each is WSS, and (ii) the cross-correlation function is time-invariant, i.e., $R_{XY}(t_1, t_2) = R_{XY}(t_1 + \tau, t_2 + \tau)$ for all s. (Abused) Notation: $R_{XY}(\tau) := R_{XY}(0, \tau) = R_{XY}(t_1, t_1 + \tau)$.
- Note that $R_{XY}(\tau) = R_{YX}(-\tau)$.
- Definition: Cross-Spectral Density: $S_{XY}(f) = FT(R_{XY}(\tau))$.

LTI Processing of WSS Processes: A Key Result

Theorem: Let $\{Y_t\}$ be the output of an LTI system with impulse response $h(\cdot)$ to a WSS random process $\{X_t\}$, i.e., $Y = X * h$. Then,

a. $\{X_t\}$ and $\{Y_t\}$ are jointly WSS and hence, $\{Y_t\}$ is WSS.

b. $\mu_Y = H(0)\mu_X$.

c. $R_{YX}(\tau) = h(\tau) * R_X(\tau)$.

d. $R_Y(\tau) = R_{YX}(\tau) * h(-\tau) = h(\tau) * R_X(\tau) * h(\tau)$.

- Taking FT of c. and d. yields:
  - $S_{YX}(f) = H(f)S_X(f)$.
  - $S_Y(f) = S_{YX}(f)H^*(f) = S_X(f)|H(f)|^2$.
- Example: Let $\{X_t\}$ be a CT zero mean WSS signal and let H be an ideal band-pass filter with cut-off frequencies $f_1 < f_2$. For the output $\{Y_t\}$ for $\{X_t\}$, what is Var(Y_t)?
- Since X_t is zero mean, Y_t would be zero mean and hence, $R_Y(0) = \mathbb{E}[Y_0^2] =$ Var(Y_t). But
$$R_Y(0) = \int_{-\infty}^{\infty} S_Y(f)df = \int_{-\infty}^{\infty} S_X(f)|H(f)|^2 df = 2\int_{f_1}^{f_2} S_X(f)df.$$
- Application (System ID using white noise):
  - Suppose that we have an LTI system with unknown impulse response $h(t)$. How to find $h(t)$? Various approaches: one approach using white noises as the input.
  - If $\{Y_t\}$ is the output for the white noise input $\{X_t\}$, then we know:
$$S_{YX}(f) = X(f)H(f) = \sigma^2 H(f).$$
  - Therefore, $R_{YX}(\tau) = \sigma^2 h(\tau)$.
  - We can estimate $R_{YX}(\tau)$ by empirical averaging in time domain, i.e.,
$$R_{YX}(\tau) \approx \lim_{T\to\infty} \frac{1}{T}\int_0^T Y(t)X(t+\tau)dt \qquad \text{(for CT)}$$
$$R_{YX}(\tau) \approx \lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} Y(t)X(t+\tau) \qquad \text{(for DT)}.$$