UNIVERSITY OF CALIFORNIA, SAN DIEGO
Electrical & Computer Engineering Department
ECE 250 - Winter Quarter 2022
Random Processes

Problem Set #1 Due Wednesday, January 12, 2022
Submit solutions to Problems 1, 4, 7, 10, 12 only

1. *σ-algebra.* Show that if $A_1, A_2, \ldots \in \mathcal{F}$, then $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$.

2. *Limits of probabilities.* Show

   (a) $\mathsf{P}(\bigcup_{i=1}^{\infty} A_i) = \lim_{n\to\infty} \mathsf{P}(\bigcup_{i=1}^{n} A_i)$.

   (b) $\mathsf{P}(\bigcap_{i=1}^{\infty} A_i) = \lim_{n\to\infty} \mathsf{P}(\bigcap_{i=1}^{n} A_i)$.

3. *Extension of a probability measure.* Consider a discrete probability space $(\Omega, 2^{\Omega}, \mathsf{P})$, where Ω is a subset of $\mathbb{R}$. Show that $\mathsf{P}(\cdot \cap \Omega)$ is a valid probability measure for the sample space $\mathbb{R}$ and the set of events $2^{\mathbb{R}}$, that is, it satisfies the axioms of probability.

4. *Independence.* Show that the events A and B are independent if $\mathsf{P}(A|B) = \mathsf{P}(A|B^c)$.

5. *Conditional probabilities.* Let $\mathsf{P}(A) = 0.8$, $\mathsf{P}(B^c) = 0.6$, and $\mathsf{P}(A \cup B) = 0.8$. Find

   (a) $\mathsf{P}(A^c|B^c)$.

   (b) $\mathsf{P}(B^c|A)$.

6. Let A, B be two events with $\mathsf{P}(A) \geq 0.5$ and $\mathsf{P}(B) \geq 0.75$. Show that $\mathsf{P}(A \cap B) \geq 0.25$.

7. *Monty Hall.* Gold is placed behind one of three curtains. A contestant chooses one of the curtains, Monty Hall (the game host) opens one of the unselected empty curtains. The contestant has a choice either to switch his selection to the third curtain or not.

   (a) What is the sample space for this random experiment? (Hint: An outcome consists of the curtain with gold, the curtain chosen by the contestant, and the curtain chosen by Monty.)

   (b) Assume that placement of the gold behind the three curtains is random, the contestant choice of curtains is random and independent of the gold placement, and that Monty Hall's choice of an empty curtain is random among the alternatives. Specify the probability measure for this random experiment and use it to compute the probability of winning the gold if the contestant decides to switch.

8. *Negative evidence.* Suppose that the evidence of an event B increases the probability of a criminal's guilt; that is, if A is the event that the criminal is guilty, then $\mathsf{P}(A|B) \geq \mathsf{P}(A)$. Does the absence of the event B decrease the criminal's probability of being guilty? In other words, is $\mathsf{P}(A|B^c) \leq \mathsf{P}(A)$? Prove or provide a counterexample.

9. *Random state transition.* Consider the *state diagram* in Figure 1. The sample space is

$$\Omega = \{(\alpha,\alpha),\ (\alpha,\beta),\ \ldots,\ (\gamma,\gamma)\},$$

where the first entry is the initial state and the second entry is the next state. Define the events

$$\begin{aligned} A_1 &= \{\text{the initial state is } \alpha\}, & A_2 &= \{\text{the next state is } \alpha\},\\ B_1 &= \{\text{the initial state is } \beta\}, & B_2 &= \{\text{the next state is } \beta\},\\ C_1 &= \{\text{the initial state is } \gamma\}, & C_2 &= \{\text{the next state is } \gamma\}. \end{aligned}$$

Assume that $\mathsf{P}(A_1) = 0.5$, $\mathsf{P}(B_1) = 0.2$, and $\mathsf{P}(C_1) = 0.3$.

(a) Find $\mathsf{P}(A_2)$, $\mathsf{P}(B_2)$, and $\mathsf{P}(C_2)$.

(b) Find $\mathsf{P}(A_1 \,|\, A_2)$, $\mathsf{P}(B_1 \,|\, B_2)$, and $\mathsf{P}(C_1 \,|\, C_2)$.

(c) Find two events among $A_1, A_2, B_1, B_2, C_1, C_2$ that are pairwise independent.

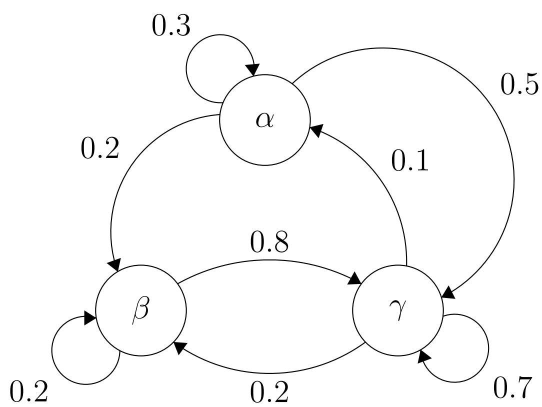

Figure 1: The state diagram for a three-state system. Here the label of each edge $i \to j$ denotes the transition probability from state i to state j, that is, the conditional probability that the next state is j given the initial state is i.

10. *Geometric pairs.* Consider a probability space consisting of the sample space

$$\Omega = \{1, 2, 3, \ldots\}^2 = \{(i,j) : i, j \in \mathbb{N}\},$$

i.e., all pairs of positive integers, the set of events 2^Ω, and the probability measure specified by

$$\mathsf{P}((i,j)) = p^2(1-p)^{i+j-2}, \quad 0 < p < 1.$$

(a) Find $\mathsf{P}(\{(i,j) : i \geq j\})$.

(b) Find $\mathsf{P}(\{(i,j) : i + j = k\})$.

(c) Find $\mathsf{P}(\{(i,j) : i \text{ is an odd number}\})$.

(d) Describe an experiment whose outcomes (i,j), $i, j \in \mathbb{N}$, have the probabilities $\mathsf{P}((i,j))$.

11. *Juror's fallacy.* Suppose that $\mathsf{P}(A|B) \geq \mathsf{P}(A)$ and $\mathsf{P}(A|C) \geq \mathsf{P}(A)$. Is it always true that $\mathsf{P}(A|B \cap C) \geq \mathsf{P}(A)$? Prove or provide a counterexample.

12. *Polya's urn.* Suppose we have an urn containing one red ball and one blue ball. We draw a ball at random from the urn. If it is red, we put the drawn ball plus another red ball into the urn. If it is blue, we put the drawn ball plus another blue ball into the urn. We then repeat this process. At the n-th stage, we draw a ball at random from the urn with $n+1$ balls, note its color, and put the drawn ball plus another ball of the same color into the urn.

    (a) Find the probability that the first ball is red.

    (b) Find the probability that the second ball is red.

    (c) Find the probability that the first three balls are all red.

    (d) Find the probability that two of the first three balls are red.