ECE 250: Stochastic Processes: Week #1

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Outline:

- Curious Random Processes
- Probability Space
- Random Variables

Deterministic vs Random Processes (dynamics)

- Deterministic: starting from $x_0 \in \mathbb{R}^n$ for all $t \geq 0$

  $$x_{t+1} = f(t, x_t).$$

  More generally: $x_{t+1} = f(t, x_t, \ldots, x_{t-m})$ where m is the memory of the system.

  Example: $n = 1$, starting at $x_0 > 0$, a simple (deterministic) population growth model:

  $$x_{t+1} = r_0 x_t.$$

  Note that $x_t = r_0^t x_0$.

- Random process: starting from $x_0 \in \mathbb{R}^n$ for all $t \geq 0$

  $$x_{t+1} = f(t, x_t, w_t).$$

  More generally: $x_{t+1} = f(t, x_t, \ldots, x_{t-m}, w_t)$ where m is the memory of the system and w_t is a random variable/vector.

  Example: Beginning phase of a pandemics: for some initially infected population $x_0 > 0$, the population of infected people at the beginning phase of a pandemics can be modeled by:

  $$x_{t+1} = r_t x_t,$$

  where r_t is a non-negative random variable independent of r_k for $k < t$ with some $\mathbb{E}[r_t] = r_0$.

Some Curious Examples of Random Processes

- Averaging: suppose that $\{w_t\}$ is an independently and identically distributed random process with $\mathbb{E}[w_k] = \mu$.

  How does the running average $x_t = \frac{w_1 + \ldots + w_t}{t}$ behave as $t \to \infty$?

  In this case:

$$
\begin{aligned}
tx_t &= (t-1)x_{t-1} + w_t \\
x_t &= (1 - \frac{1}{t})x_{t-1} + \frac{1}{t}w_t \\
x_t &= f_t(x_{t-1}, w_t) \\
f_t(x, w) &= (1 - \frac{1}{t})x + \frac{1}{t}w.
\end{aligned}
$$

- What happens if we use other weights such as: $x_t = \frac{w_1 + \ldots + w_t}{\sqrt{t}}$?
- What if we don't have any weights at all, i.e., $x_t = w_1 + \ldots + w_t$? What happens?
- What can we say about asymptotic behavior of such processes in general?

More Complicated Random Processes

- **Polar Codes**: Suppose that x_t is a random process whose evolution is given by

$$x_{t+1} = \begin{cases} 2x_t - x_t^2 & \text{with probability } 1/2 \\ x_t^2 & \text{with probability } 1/2 \end{cases}.$$

  More precisely, let $\{w_t\}$ be a sequence of i.i.d. binary random process with

$$P(w_t = 1) = P(w_t = 0) = \frac{1}{2}$$

  and

$$x_{t+1} = f(x_t, w_t) = w_t(2x_t - x_t^2) + (1 - w_t)x_t^2.$$

  Note that both x^2 and $x(2 - x)$ map $[0, 1]$ to $[0, 1]$ (why?). What happens if we start with $x_1 \in (0, 1)$? Does this process converge or not?

- **Pandemics**: Suppose that $\{w_t\}$ are i.i.d. with $w_t > 0$ with probability one. If $x_0 > 0$, what happens with the process

$$x_{t+1} = w_t x_t$$

  as $t \to \infty$? Do we have some sort of exponential behavior in this case as well?

- Notations: $\mathbb{R}$: set of real numbers, $\mathbb{Z}$: set of integers, $\mathbb{Q}$: set of rational numbers, $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$ and $\mathbb{Z}^+ = \{a \in \mathbb{Z} \mid a \geq 0\}$. For a set X, we denote the set of all its subsets by $\mathcal{P}(X)$.

Probability Theory: Axiomatic and Mathematical Approach

- This course: a theory course on probability and stochastic processes that provides **systematic/axiomatic** approach to probability
- Probability spaces are triplets of $(\Omega, \mathcal{F}, \Pr(\cdot))$ consisting of a *sample space* Ω, set of *events* $\mathcal{F}$, and *a probability measure* $\Pr(\cdot)$
  - *Sample space*: any set Ω
  - *Events* $\mathcal{F}$: This is a set consisting of **subsets** of Ω satisfying:
    a. $\Omega \in \mathcal{F}$
    b. **Closed under complement**: $E \in \mathcal{F}$ implies $E^c \in \mathcal{F}$, and
    c. **Closed under countable union**: for any countably many subsets $E_1, \ldots, E_k, \ldots \in \mathcal{F}$, we have $\cup_{k=1}^{\infty} E_k \in \mathcal{F}$
  - *Probability measure* $\Pr(\cdot)$: is a function from $\mathcal{F}$ to $\mathbb{R}^+ := \{x \in \mathbb{R} \mid x \geq 0\}$ that satisfies:
    i. $\Pr(\Omega) = 1$, and
    ii. For a countably many subsets $\{E_k\}$ in $\mathcal{F}$ that is mutually disjoint (i.e., $E_i \cap E_j = \emptyset$ for all $i \neq j$), we have
    $$\Pr(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \Pr(E_k).$$
- Note
  - Any $\mathcal{F}$ that satisfies the properties a,b, and c is called a σ-algebra over Ω.
  - Any $(\Omega, \mathcal{F})$ is called a *measurable space*.
  - Any function $\mu : \mathcal{F} \to \mathbb{R}^+$ that only satisfies $\Pr(\emptyset) = 0$ and (ii) of probability measure is called a *measure*. Probability measures are measures with unit measure for the entire space.
  - Volume is a measure!

Examples of measurable spaces

- Example: $\Omega = \{0, 1\}$, and $\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.
- Example: In general, $\mathcal{P}(\Omega)$ is a σ-algebra for any Ω.
- Example: $(\mathbb{Z}^+, \mathcal{P}(\mathbb{Z}^+))$. Can we make this a probability space by adding a proper probability measure?
- Example: $\{\emptyset, \Omega\}$ is a σ-algebra for any Ω.
- When Ω has countably many elements, then we can throw (meaningful) probability measures on any $(\Omega, \mathcal{F})$ for any σ-algebra $\mathcal{F} \subseteq \mathcal{P}(\Omega)$.
- What if Ω is uncountable such as $\mathbb{R}^n$?
- Related Important Historic Question: Can we find the volume of all the subsets of $\mathbb{R}^n$? Answer: No!

  Theorem 1. *Banach-Tarski Theorem (Paradox) For any two balls E, F in $\mathbb{R}^n$ ($n \geq 3$), there exist* finite partitions *$\{E_1, \ldots, E_d\}$ of E and $\{F_1, \ldots, F_d\}$ of F such that E_i can be obtained by a combination of translations, rotations, or reflections of F_i for $i = 1, \ldots, d$.*

- Fortunately, for $\mathbb{R}^n$ there exists a σ-algebra that we can define meaningful measures in $\mathbb{R}^n$, namely the *Borel* σ-algebra.