# Homework 2

1. Similar to Borel $\sigma$-algebra in $\mathbb{R}$, we define the Borel $\sigma$-algebra in $\Omega = \mathbb{R}^n$, to be the $\sigma$-algebra generated by the collection of open boxes, i.e.,

$$\mathcal{B}^n := \sigma(\{(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n) \mid a_i < b_i, a_i, b_i \in \mathbb{R} \text{ for all } i\}).$$

Using this, show that closed boxes are in the Borel $\sigma$-algebra in $\mathbb{R}^n$. Here, a closed box is a set of the form

$$[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n],$$

for some scalars $a_1 \leq b_2, a_2 \leq b_2, \ldots, a_n \leq b_n$.

**Solution:** We have

$$[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] = \bigcap_{k=1}^{\infty} (a_1 - \frac{1}{k}, b_1 + \frac{1}{k}) \times (a_2 - \frac{1}{k}, b_2 + \frac{1}{k}) \times \cdots \times (a_n - \frac{1}{k}, b_n + \frac{1}{k}).$$

Therefore the closed boxes $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ are also in $\sigma$-algebra since $(a_1 - \frac{1}{k}, b_1 + \frac{1}{k}) \times (a_2 - \frac{1}{k}, b_2 + \frac{1}{k}) \times \cdots \times (a_n - \frac{1}{k}, b_n + \frac{1}{k}) \in \mathcal{B}^n$ and countable intersections of sets in $\sigma$-algebra lies in the $\sigma$-algebra (similar to the one-dimensional equivalent $[a, b] = \bigcap_{k=1}^{\infty} (a - \frac{1}{k}, b + \frac{1}{k})$).

2. Show that if events $A, B$ are independent, then $A^c$ and $B$ are also independent.

**Solution:** We have

$$\begin{aligned} \mathrm{P}(B) &= \mathrm{P}(B \cap \Omega) = \mathrm{P}(B \cap (A \cup A^c)) = \mathrm{P}((B \cap A) \cup (B \cap A^c)) \\ &\overset{(a)}{=} \mathrm{P}(B \cap A) + \mathrm{P}(B \cap A^c) \\ &\overset{(b)}{=} \mathrm{P}(B)\mathrm{P}(A) + \mathrm{P}(B \cap A^c) \end{aligned}$$

where $(a)$ follows from $(B \cap A) \cap (B \cap A^c) = \emptyset$, and $(b)$ follow from the fact that $A, B$ are independent. Therefore, we have

$$\mathrm{P}(B \cap A^c) = \mathrm{P}(B) - \mathrm{P}(B)\mathrm{P}(A) = \mathrm{P}(B)(1 - \mathrm{P}(A)) = \mathrm{P}(B)\mathrm{P}(A^c).$$

3. Problem 2.10 of Prof. Kim's notes.

**Solution:**

a. We know $P(A_2) = P(A_1, A_2) + P(B_1, A_2) + P(C_1, A_2)$ and

$$\begin{aligned} P(A_2) &= P(A_1)P(A_2|A_1) + P(B_1)P(A_2|B_1) + P(C_1)P(A_2|C_1) \\ &= 0.5 \times 0.3 + 0.2 \times 0 + 0.3 \times 0.1 = 0.18. \end{aligned}$$

Similarly $P(B_2) = 0.5 \times 0.2 + 0.2 \times 0.2 + 0.3 \times 0.2 = 0.2$ and $P(C_2) = 0.5 \times 0.5 + 0.2 \times 0.8 + 0.3 \times 0.7 = 0.62$.

b. We know $P(A_1|A_2) = \frac{P(A_1, A_2)}{P(A_2)} = \frac{0.15}{0.18} = \frac{5}{6}$.
Similarly, $P(B_1|B_2) = \frac{P(B_1, B_2)}{P(B_2)} = \frac{0.04}{0.2} = \frac{1}{5}$ and $P(C_1|C_2) = \frac{P(C_1, C_2)}{P(C_2)} = \frac{0.21}{0.62} = \frac{21}{62}$.

c. Since $P(B_2|A_1) = 0.2$ and $P(B_2) = 0.2$, we know that $A_1, B_2$ are pairwise independent.

4. Problem 2.11 of Prof. Kim's notes.

   **Solution:** (a) We have

$$\begin{aligned}
\mathrm{P}(\{(i,j): i \geq j\}) &= \sum \sum_{\{(i,j):i\geq j\}} p(i,j) = \sum_{j=1}^{\infty}\sum_{i=j}^{\infty} p^2(1-p)^{i+j-2} \\
&= \sum_{j=1}^{\infty}\sum_{i=0}^{\infty} p^2(1-p)^{i+2j-2} \\
&= \frac{p^2}{(1-p)^2}\sum_{j=1}^{\infty}(1-p)^{2j}\sum_{i=0}^{\infty}(1-p)^i \\
&= p^2\sum_{j=1}^{\infty}(1-p)^{2j}\frac{1}{1-(1-p)} = p\frac{1}{1-(1-p)^2} = \frac{1}{2-p}.
\end{aligned}$$

   (b) We have

$$\begin{aligned}
\mathrm{P}(\{(i,j): i+j=k\}) &= \sum \sum_{\{(i,j):i+j=k\}} p(i,j) \\
&= \sum_{j=1}^{k-1} p^2(1-p)^{j+(k-j)-2} = \sum_{j=1}^{k-1} p^2(1-p)^{k-2} \\
&= p^2(1-p)^{k-2}\sum_{j=1}^{k-1} 1 = p^2(1-p)^{k-2}(k-1).
\end{aligned}$$

   (c) We have

$$\begin{aligned}
\mathrm{P}(\{(i,j): i\text{odd}\}) &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} p^2(1-p)^{(2i-1)+j-2} \\
&= \sum_{i=0}^{\infty} p(1-p)^{2i}\sum_{j=0}^{\infty} p(1-p)^j = \frac{p}{1-(1-p)^2} = \frac{1}{2-p}.
\end{aligned}$$

   (d) The probability mass function can be factored as

$$\mathrm{P}((i,j)) = p^2(1-p)^{i+j-2} = p(1-p)^{i-1}p(1-p)^{j-1}.$$

   which is the product of two geometric pmfs. Consider a coin whose probability that a head occurs is $p$. Then $\mathrm{P}((i,j))$ is the probability that when the coin is tossed repeatedly, the first head occurs on the $i$ -th toss and the second head occurs on $j$ -th toss after the first one ($i+j$ -th toss ).

5. Suppose that an ordinary deck of 52 cards is shuffled so that any order of the 52 cards are equally likely to appear. The cards are then turned over one at a time until the first ace appears. Given that the first ace is the 20th card to appear, what is the conditional probability that the card following it is the

   (a) ace of spades?

   (b) two of clubs?

   **Solution:** Since the probability of any order appearing is same, we need to count ratio of the number of orders satisfying the conditions to the total number of possible outcomes for the same number of draws.

   Let $A$ be the event that first ace is the 20th card to appear, $B$ be the event that the following card (21st card) is ace of spades, and $C$ be the event that the following card (21st card) is two of clubs.

The total number of orderings for 21 draws $52 \cdot 51 \cdot 50 \cdots 33 \cdot 32$ since first card being drawn has 52 possibilities, second card has 51 excluding the card drawn in first draw, so on until 20-th card being drawn has 33 possibilities and 19-th card has 32. In general the total number of ordering for $k$ draws is $52 \cdot 51 \cdots (52-(k-1))$.

The total number of possible orderings for 20th card being drawn being the first ace is $48 \cdot 47 \cdots 30 \cdot 4$ since the first card drawn has 48 possibilities (excluding 4 aces), second card having 47, so on until the 19th card having 30 possibilities and 20th card has to be one of the 4 aces.

Similarly for 20th card being drawn being the first ace and the 21st card being ace of spades the total number of possible orderings is $48 \cdot 47 \cdots 30 \cdot 3 \cdot 1$, with the 20th card being any ace except the ace of spades and 21st card being ace of spades.

On the other hand for 20th card being drawn being the first ace and the 21st card being two of clubs the total number of orderings is $47 \cdot 46 \cdots 29 \cdot 4 \cdot 1$ with first card having 47 possibilities (excluding 4 aces and the two of clubs), second card having 46 possibilities and so on until the 19th card having 29 options, 20th card being one of the four aces and 21st card being the two of clubs. $P(A) = \frac{48 \cdot 47 \cdots 30 \cdot 4}{52 \cdot 51 \cdots 34 \cdot 33}$, $P(A \cap B) = \frac{48 \cdot 47 \cdots 30 \cdot 3 \cdot 1}{52 \cdot 51 \cdots 34 \cdot 33 \cdot 32}$. Therefore, $P(B \mid A) = \frac{3}{4}\frac{1}{32}$.
$P(A \cap C) = \frac{47 \cdot 46 \cdots 29 \cdot 4 \cdot 1}{52 \cdot 51 \cdots 34 \cdot 33 \cdot 32} = \frac{47}{52}\frac{46}{51} \cdots \frac{29}{34}\frac{4}{33}\frac{1}{32}$. Therefore $P(C \mid A) = \frac{1}{32}\frac{29}{48}$.

6. Let $\Omega = \mathbb{R}$ and $\mathcal{F}$ be the $\sigma$-algebra generated by $A = \mathbb{Q}$ (i.e., $\mathcal{F} = \sigma(\{\mathbb{Q}\})$ where $\mathbb{Q}$ is the set of integers).

   (a) Write explicitly all the sets belonging to $\mathcal{F}$.

   (b) Characterize all the mappings $X : \Omega \to \mathbb{R}$ that are random variables with respect to $(\Omega, \mathcal{F})$.

   **Solution:**

   (a) For the $\mathcal{F}$, since we need to have $\mathbb{Q} \in \mathcal{F}$, we also need to have $\mathbb{Q}^c \in \mathcal{F}$ and we always have $\emptyset, \Omega \in \mathcal{F}$. But $\mathcal{F} = \{\mathbb{Q}, \mathbb{Q}^c, \Omega, \emptyset\}$ is already a $\sigma$-algebra and hence, it is the smallest one containing $\mathbb{Q}$.

   (b) In this case a function $X : \Omega \to \mathbb{R}$ is a random variable if and only if $X(\omega) = \alpha$ for some $\alpha \in \mathbb{R}$ and all $\omega \in \mathbb{Q}$ and $X(\omega) = \beta$ for some $\beta \in \mathbb{R}$ and all $\omega \in \mathbb{Q}^c$. In other words, $X = \alpha \mathbf{1}_{\mathbb{Q}} + \beta \mathbf{1}_{\mathbb{Q}^c}$.
   Note that, since $\mathbf{1}_{\mathbb{Q}}$ and $\mathbf{1}_{\mathbb{Q}^c}$ are both random variables, any of their linear combinations is a random variable and hence, any function of the form $X = \alpha \mathbf{1}_{\mathbb{Q}} + \beta \mathbf{1}_{\mathbb{Q}^c}$ is a random variable.
   Conversely, suppose that we have a random variable that either $X(\omega) = \alpha$ for $\omega \in \mathbb{Q}$ or $X(\omega) = \beta$ for all $\omega \in \mathbb{Q}^c$ does not hold. Without loss of generality assume that $X$ is not consistent across $\mathbb{Q}$, and $X(\omega_1) \neq X(\omega_2)$ for $\omega_1, \omega_2 \in \mathbb{Q}$. Let $a = X(\omega_1)$. Then, the set $\{a\}$ is a Borel set in $\mathbb{R}$, and $E = X^{-1}(\{a\})$ is a set that contains $\omega_1$ but not $\omega_2$. Therefore,

   - $E \neq \emptyset$ as $\omega_1 \in E$,
   - $E \neq \Omega$ as $\omega_2 \notin E$,
   - $E \neq \mathbb{Q}$ as $\omega_1 \in E$ and $\omega_2 \notin E$,
   - $E \neq \mathbb{Q}^c$ as $\omega_1 \in E$.

   Therefore, $E$ is not in $\mathcal{F}$ and hence, $X$ is not a random variable.