Homework 4

Due: 8:00pm (PDT) Thursday, October 27th, 2022
Please provide all the requested plots.

1. Let $\{Z_k\}$ be an i.i.d. random process with Z_k uniformly distributed over $[-1, 1]$. For each of the following random processes $\{X_k\}$, plot 100 sample paths for the corresponding partial sum sequence $\{S_k\}$ for $1 \leq k \leq 1000$. Conjecture, and theoretically prove whether the corresponding partial sum sequence converges or not. Explain your answer.

   (a) $X_k = \frac{Z_k}{k}$.

   (b) $X_k = \frac{Z_k}{k^{0.7}}$.

   (c) $X_k = \frac{Z_k}{k^{0.5}}$.

2. Consider the independent random process $\{X_k\}$ that takes values k^2 or 0 with the probabilities

$$P(X_k = k^2) = \frac{1}{k^2}, \text{ and}$$
$$P(X_k = 0) = 1 - \frac{1}{k^2}.$$

   Fix threshold $a = 1$. For each $k \geq 1$:

   (a) Determine $P(|X_k| \geq a)$.

   (b) Determine $\mathbb{E}[X_k 1_{\{|X_k| \leq a\}}]$.

   (c) Determine $\mathbf{Var}[X_k 1_{\{|X_k| \leq a\}}]$.

   (d) Using these series, determine whether $\sum_{k=1}^{\infty} X_k$ converges a.s. or not.

3. Problem 3.8 of Prof. Kim's notes.

4. Problem 5.7 of Prof. Kim's notes.

5. Consider the function

$$f(\alpha) = p \log(1 + \alpha) + (1 - p) \log(1 - \alpha),$$

   where p is a constant with $0.5 < p \leq 1$. Show that there exists an $\alpha^* \in [0, 1]$ such that $f(\alpha^*) > 0$,