ECE 250: Stochastic Processes: Week #2

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Outline:

- Probability Space
- Properties of σ-algebras and probability measures
- Borel-Cantelli Lemma

Deterministic vs Random Processes (dynamics)

- Recall: A probability space is a tuple $(\Omega, \mathcal{F}, \Pr(\cdot))$, where $\mathcal{F}$ is a σ-algebra on Ω and $\Pr(\cdot)$ is a probability measure, i.e., it satisfies (i) $\Pr(\Omega) = 1$, and (ii) $\Pr(\cup_k A_k) = \sum_k \Pr(A_k)$ for countably many mutually disjoint sets A_k.
- Properties of σ-algebras:
  - For any family I of σ-algebras $\mathcal{F}_\alpha$ (countable or uncountable) on Ω, $\mathcal{F} = \cap_{\alpha \in I} F_\alpha$ is a σ-algebra.
  - We can define the smallest σ-algebra containing a given set of subsets $\mathcal{E}$, denoted by $\sigma(\mathcal{E})$ by:

$$\sigma(\mathcal{E}) := \bigcap_{\sigma-\text{algebra}\mathcal{F}:\mathcal{E}\subseteq\mathcal{F}} \mathcal{F}.$$

  - Example: if $\Omega = \{1, 2\}$, what is $\sigma(\{\{1\}\})$?
  - Definition: We refer to the σ-algebra generated by collection of open intervals $\{(a, b) \mid a < b\}$ in $\mathbb{R}$ to be the Borel σ-algebra in $\mathbb{R}$ and denote it by $\mathcal{B}(\mathbb{R})$.
  - What sets are in $\mathcal{B}(\mathbb{R})$? Do the sets of singletons $\{x\}$ belong to this σ-algebra? Answer: Yes.
    Does this mean that $\mathcal{B}(\mathbb{R})$ contains all the subsets of $\mathbb{R}$? Answer: No.
  - In general, we refer to the σ-algebra generated by $(a_1, b_1) \times \cdots \times (a_n, b_n)$ as the Borel σ-algebra in $\mathbb{R}^n$, denoted by $\mathcal{B}^n$.
  - For two σ-algebras $\mathcal{F}_1, \mathcal{F}_2$, is $\mathcal{F}_1 \cup \mathcal{F}_2$ a σ-algebra?
    Answer: No!

Properties of a Probability Measure

(Notation: We use indices k, i, j for countable index sets and $\alpha, \beta, ...$ for arbitrary ones)

1. Monotonicity: for $A, B \in \mathcal{F}$ with $A \subset B$, we have $\mathbf{Pr}(A) \leq \mathbf{Pr}(B)$.
2. Sub-additivity (Union Bound): $\mathbf{Pr}(\cup_k A_k) \leq \sum_k \mathbf{Pr}(A_k)$.
3. Continuity from Below: If $A_k \uparrow A$, i.e. $A_1 \subseteq A_2 \subseteq \cdots$ and $A = \cup_k A_k$, then $\mathbf{Pr}(A_k) \to \mathbf{Pr}(A)$ (implied from HW1).
4. Continuity from Above: If $A_k \downarrow A$, i.e. $\cdots \subseteq A_2 \subseteq A_1$ and $A = \cap_k A_k$, then $\mathbf{Pr}(A_k) \to \mathbf{Pr}(A)$ (implied from HW1).

Independent Events

(Recall: we refer to the members of the underlying σ-algebra of a probability space as events.)

- We say that two events $A, B \in \mathcal{F}$ are independent if $\Pr(A \cap B) = \Pr(A)\Pr(B)$.
- We say that a family of events $\{E_k\}$ are independent if for any finite $A_{i_1}, \ldots, A_{i_k} \in E$, $\Pr(A_{i_1} \cap \cdots \cap A_{i_k}) = \Pr(A_{i_1}) \cdots \Pr(A_{i_k})$, where $1 \leq i_1 < i_2 < \cdots < i_k$.
- For a countable sequence of events $\{A_k\}$ define its infinite often event $\{A_k, \text{ i.o.}\}$ as the set of $\omega \in \Omega$ such that ω belongs to infinitely many A_ks.
- Note that $\{A_k, \text{ i.o.}\} \in \mathcal{F}$ (why?).
- More precisely,

$$\{A_k, \text{ i.o.}\} = \bigcap_{k=1}^{\infty} \bigcup_{t=k}^{\infty} A_t.$$

Borel-Cantelli Lemma(s)

Theorem 1 (Borel-Cantelli)**.** *For a given sequence of events $\{A_k\}$, $\mathbf{Pr}(\{A_k,\ i.o.\}) > 0$ implies $\sum_{k=1}^{\infty} \mathbf{Pr}(A_k) = \infty$.*
Conversely, ***if*** $\{A_k\}$ ***is an independent sequence****,* $\sum_{k=1}^{\infty} \mathbf{Pr}(A_k) = \infty$ *implies*

$$\mathbf{Pr}(\{A_k,\ i.o.\}) = 1 > 0.$$

Proof. Let $E = \{A_k,\ \text{i.o.}\}$ and

$$E_k = \bigcup_{t=k}^{\infty} A_k.$$

Note that $E \subseteq E_k$ and because of sub-additivity:

$$\mathbf{Pr}(E_k) = \mathbf{Pr}(\bigcup_{t=k}^{\infty} A_t) \leq \sum_{t=k}^{\infty} \mathbf{Pr}(A_t).$$

Therefore, $\mathbf{Pr}(E) \leq \mathbf{Pr}(E_k) \leq \sum_{t=k}^{\infty} \mathbf{Pr}(A_t)$ and hence, if $\sum_{t=1}^{\infty} \mathbf{Pr}(A_k) < \infty$, $\mathbf{Pr}(E) = 0$ (why?).
For the converse part, if $\{A_k\}$ are independent, we have:

$$\begin{aligned}
\mathbf{Pr}(E_k) = \mathbf{Pr}(\bigcup_{t=k}^{\infty} A_t) &= 1 - \mathbf{Pr}((\bigcup_{t=k}^{\infty} A_t)^c) \\
&= 1 - \mathbf{Pr}(\bigcap_{t=k}^{\infty} A_t^c) \\
&= 1 - \prod_{t=k}^{\infty} (1 - \mathbf{Pr}(A_t)),
\end{aligned}$$

where the last equality follows from the independence of A_k^cs (see HW2).

Note that $1 - x \leq e^{-x}$. Therefore,

$$\begin{aligned}
\mathbf{Pr}(E_k) &= 1 - \prod_{t=k}^{\infty}(1 - \mathbf{Pr}(A_t)) \\
&\geq 1 - \prod_{t=k}^{\infty} e^{-\mathbf{Pr}(A_t)} \\
&= 1 - e^{-\sum_{t=k}^{\infty} \mathbf{Pr}(A_t)} \\
&= 1 - e^{-\infty} = 1.
\end{aligned}$$

Since, $E_k \downarrow E$, $\lim_{k\to\infty} \mathbf{Pr}(E_k) = \mathbf{Pr}(E)$. Therefore, $\mathbf{Pr}(E) = \lim_{k\to\infty} \mathbf{Pr}(E_k) = 1$. $\square$

Borel-Cantelli Lemma(s)-Application

- Suppose that we have an independent sequence of $0, 1$ with equal probability.
- What is the probability of infinitely many of them becoming 1? Why?
- What if we have an independent sequence $0, 1$ but now the kth random variable is 1 with probability $1/k$. What is the probability of infinitely many of them being 1?
  Answer: 1! Because of the Borel-Cantelli lemma as $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$.

Random Variables

The most important objects in probability theory are *random variables*.

Definition 1. *Let* $(\Omega, \mathcal{F}, \mathbf{Pr}(\cdot))$ *be a probability space. The mapping* $X : \Omega \to \mathbb{R}$ *is called a random variable if the pre-image of any interval* $(-\infty, a]$ *belongs to* $\mathcal{F}$*, i.e.*

$$X^{-1}((-\infty, a]) \in \mathcal{F} \qquad \textit{for all } a \in \mathbb{R}, \tag{1}$$

where

$$X^{-1}(B) := \{\omega \in \Omega \mid X(\omega) \in B\}.$$

Important Result: If we have a random variable X, then (1) implies that

$$X^{-1}(B) \in \mathcal{F} \qquad \text{for all } B \in \mathcal{B}.$$

Notation: We use three notations interchangeably $\{\omega \in \Omega \mid X(\omega) \in B\} := \{X \in B\} := X^{-1}(B)$.

(Important) Example of Random Variable

Example 1. *(Indicator Function)* For a set $E \subseteq \Omega$, define the indicator function of E as

$$\mathbf{1}_E(\omega) = \begin{cases} 1 & \text{if } \omega \in E \\ 0 & \text{if } \omega \notin E \end{cases}.$$

Indeed, $\mathbf{1}_E(\omega)$ is a random-variable iff $E \in \mathcal{F}$. To show this, let $a \in \mathbb{R}$. Then

1. if $a < 0$, $\mathbf{1}_E^{-1}((-\infty, a]) = \emptyset$,
2. if $0 \leq a < 1$, then $\mathbf{1}_E^{-1}((-\infty, a]) = E^c$, and
3. if $1 \leq a$, then $\mathbf{1}_E^{-1}((-\infty, a]) = \Omega$.

Therefore, Since $\mathcal{F}$ is a σ-algebra and $E \in \mathcal{F}$, therefore, it follows that $E^c \in \mathcal{F}$ and hence, $\mathbf{1}_E^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{B}$.

Random Vectors and Random Processes

- *Random Vectors*: Any mapping $X : \Omega \to \mathbb{R}^n$ with $X(\omega) = (X_1(\omega), X_2(\omega), \ldots, X_n(\omega))$ is called a random vector if X_i is a random variable for all $i = 1, \ldots, n$.
- *Random Process*: An infinitely indexed collection $\{X_\alpha\}_{\alpha \in I}$ of random variables on $(\Omega, \mathcal{F}, \Pr)$ is called a random process.
- If the index set I is a discrete set (usually $I = \mathbb{Z}^+$), the random process is called a discrete-time random process. When $I = \mathbb{R}$ or $I = \mathbb{R}^+$, the random process is called a continuous-time random process.

Some Comments on Random Processes

- It is extremely important that the underlying probability space is shared between all these random variables.
- Note that for an $\omega \in \Omega$, $\{X_t(\omega)\}$ would be a sequence of real-numbers (i.e., a usual sequence). Such a sequence is called a **sample-path** for the process.
- Example: HW 1-Problem 5

This course: Mostly focuses on discrete-time random processes, i.e., when $I = \mathbb{Z}, \mathbb{Z}^+$. Therefore, we drop the index set and unless otherwise explicitly stated, the index variable $k, t, n, \ldots$ are discrete.