HW1

Sunday, October 2, 2022 5:11 PM

1. (a) $(A-(A-B))=A\cap B$

$\because A-B=A\cap B^c$

$\therefore (A-(A-B))=A\cap(A\cap B^c)^c.$

$=A\cap(A^c\cup B)$

$=(A\cap A^c)\cup(A\cap B)$

$=\emptyset\cup(A\cap B)$

$=A\cap B.$

(b) $A\cap(B\cup C)$

by Distribute Property

$=(A\cap B)\cup(A\cap C)$

$\neq(A\cap B)\cup C$ for any arbitary sets A,B,C.

Let $A=\{1,2\}.$

$B=\{1,3,4\}.$

$C=\{3,4,5\}.$

$\therefore B\cup C=\{1,3,4,5\}$

$A\cap(B\cup C)=\{1\}.$

$A\cap B=\{1\}.$

$(A\cap B)\cup C=\{1,3,4,5\}\neq\{1\}=A\cap(B\cup C).$

(c) $A\cup(C-A)$

$=A\cup(C\cap A^c)$

$=(A\cup C)\cap(A\cup A^c)$

$=(A\cup C)\cap U$

$= (A \cup C) \cap U$

$= A \cup C$

$\because A \subseteq C$

$\therefore A \cup C = C = A \cup (C - A)$

2. $\Omega = \{1, 2, 3, 4, 5\}$.

$\mathcal{F} = \mathcal{P}(\Omega) = 2^{\Omega}$

$P: \mathcal{F} \to [0, 1]$

$P(\{1,2\}) = 0.2$, $P(\{2,3\}) = 0.3$

(a) $A = \{2\}$.

$P(A) = P(\{2\}) = P(\{1,2\} \cap \{2,3\})$.

$\therefore 0 \le P(A) \le \min(P(\{1,2\}), P(\{2,3\})) = P(\{1,2\}) = 0.2$.

$\therefore 0 \le P(A) \le 0.2$.

(b) $B = \{1, 3\}$

$P(B) = P(\{1,3\}) = P(\{1,2\} \cup \{2,3\} - \{2\})$.

$= P(\{1,2\} \cup \{2,3\}) - P(\{2\})$.

$= P(\{1,2\}) + P(\{2,3\}) - P(\{1,2\} \cap \{2,3\}) - P(\{2\})$.

$= 0.5 - 2P(\{2\})$.

$\therefore 0.1 \le P(B) \le 0.5$.

(c) $C = \{5\}$.

Let $D = \{4, 5\}$. $D^c = \{1, 2, 3\} = \{1,2\} \cup \{2,3\}$.

$\therefore P(D) = 1 - P(D^c) = 1 - (P(\{1,2\}) + P(\{2,3\}) - P(\{2\}))$.

$= 0.5 + P(A)$.

$\therefore 0.5 \le P(D) \le 0.7$.

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$\because C \subseteq D$.

$\therefore P(C) \le P(D) \Rightarrow 0.5 \le P(C) \le 0.7$.

Based on the provided information, the probability of A, B, and C can only be a range of values rather than specific number.

3. $A_1, A_2, \cdots \in \mathcal{F}$, proof $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$

$\because \forall A \in \mathcal{F}$, $A^c \in \mathcal{F}$ (closed on complement).

$\therefore \bigcup_{i=1}^{\infty} A_i^c \in \mathcal{F}$ (closed on countable unions).

$\therefore \left(\bigcup_{i=1}^{\infty} A_i^c\right)^c \in \mathcal{F}$ (closed on complement)

$= \bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$

4. (1) show $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty}\left(\bigcup_{i=1}^{n} A_i\right)$.

Let $B_n = \bigcup_{i=1}^{n} A_i$.

$\therefore B_1 \subseteq B_2 \subseteq \cdots \subseteq B_n$

i.e. $P(B_1) \le P(B_2) \le \cdots \le P(B_n)$

$\therefore n \to \infty$, $P(B_n) = \lim_{n\to\infty}\left(\bigcup_{i=1}^{n} A_i\right) = P\left(\bigcup_{i=1}^{\infty} A_i\right)$.

(2) show $P\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty}\left(\bigcap_{i=1}^{n} A_i\right)$

Let $C_n = \bigcap_{i=1}^{n} A_i$.

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$\therefore C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots \supseteq C_n$.

i.e. $P(C_1) \geqslant P(C_2) \geqslant \cdots \geqslant P(C_n)$

$\therefore n \to \infty$, $P(C_n) = \lim_{n\to\infty} (\bigcap_{i=1}^{n} A_i) = P(\bigcap_{i=1}^{\infty} A_i)$.

5. $\because \Omega \in \mathbb{R}$

$\therefore \Omega^c \in \mathbb{R}$.

$\therefore \forall A \in 2^{\mathbb{R}}$,

$A \in 2^{\Omega}$ or $A \in 2^{\Omega^c}$.

for Prob. measure $P(\cdot \cap \Omega)$.

$$P(A \cap \Omega) = \begin{cases} P(A), & A \in 2^{\Omega} \\ P(\emptyset), & A \in 2^{\Omega^c} \end{cases}$$

$\therefore P(A \cap \Omega) \geqslant 0$, $\forall A \in 2^{\mathbb{R}}$ ... ①

$P(R \cap \Omega) = P(\Omega) = 1$ ... ②

Let $A_1, A_2, \cdots \in 2^{\mathbb{R}}$ be disjoint, i.e. $A_i \cap A_j = \emptyset$, $i \neq j$

$\therefore \bigcup_{i=1}^{\infty} A_i = (\bigcup_{i=1}^{\infty} B_i) \cup (\bigcup_{i=1}^{\infty} C_i)$, where $B_i \in 2^{\Omega}$, $C_i \in 2^{\Omega^c}$.

$$\therefore P(\bigcup_{i=1}^{\infty} A_i \cap \Omega) = P([(\bigcup_{i=1}^{\infty} B_i) \cup (\bigcup_{i=1}^{\infty} C_i)] \cap \Omega)$$

$$= P(\bigcup^{\infty} (B_i \cap \Omega)) + P(\bigcup^{\infty} (C_i \cap \Omega))$$

$$= P\left(\bigcup_{i=1}^{\infty}(B_i \cap \Omega)\right) + P\left(\bigcup_{i=1}^{\infty}(C_i \cap \Omega)\right)$$

$$= P\left(\bigcup_{i=1}^{\infty} B_i\right) + P(\emptyset).$$

$$= P\left(\bigcup_{i=1}^{\infty} B_i\right). = \sum_{i=1}^{\infty} P(B_i)$$

Based on above, $P(\cdot \cap \Omega)$ is a valid probability measure on sample space $\mathbb{R}$, and set on events 2^{Ω}.

6. $\Omega = \mathbb{R}$;

$\mathcal{F}$ is the collection of all subsets A of $\mathbb{R}$, such that A or A^c is countable;

$$P(A) = \begin{cases} 0, & A \text{ is countable} \\ 1, & A^c \text{ is countable.} \end{cases}$$

show $(\Omega, \mathcal{F}, P)$ is a probability space

① $\emptyset \in \mathcal{F}$;

② $\forall A \in \mathcal{F}$, $A^c \in \mathcal{F}$;

③ if $\forall i$, A_i is countable,

then $\bigcup_{i=1}^{\infty} A_i$ is countable

$\therefore \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

④ if $\forall i$, at least one A_i is uncountable,

then $\bigcup_{i=1}^{\infty} A_i$ is uncountable.

Let A_{n_0} be the uncountable $A_i \in \mathcal{F}$,

$\because A_{n_0} \in \mathcal{F}$ $\therefore A_{n_0}^c$ is countable

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$\because A_{n_0} \in \mathcal{F}$ $\therefore A_{n_0}^c$ is countable

$\therefore (\bigcup_{i=1}^{\infty} A_i)^c = \bigcap_{i=1}^{\infty} A_i^c \subseteq A_{n_0}^c$

$\therefore (\bigcup_{i=1}^{\infty} A_i)^c$ is countable.

$\therefore \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

Based on ①.②,③,④, $\mathcal{F}$ is a σ-algebra on Ω.

For mutuall disjoint $A_1, A_2, \cdots \in \mathcal{F}$,

⑤ if A_i is countable $\forall i$,

$P(\bigcup_{i=1}^{\infty} A_i) = 0 = \sum_{i=1}^{\infty} P(A_i)$.

⑥ if at least one A_i is uncountable,

$\bigcup_{i=1}^{\infty} A_i$ is uncountable.

Let A_{n_0} be uncountable,

$A_{n_0}^c$ is countable.

$A_i \subseteq A_{n_0}^c \quad \forall i \neq n_0$,

$P(\bigcup_{i=1}^{\infty} A_i) = P(\bigcup_{i=1, i \neq n_0}^{\infty} A_i) + P(A_{n_0}) = 0 + 1 = 1$.

⑦ $\because \phi \in \mathcal{F}$, $\Omega^c = \phi$

$P(\Omega) = 1$.

Based on ①.②,$\cdots$,⑦, $(\Omega, \mathcal{F}, P)$ is a probability space.

space.

7. (a) find a $f_t(x.w)$: $x_t = f_t(x_{t-1}, w_t)$.

$$\because x_t = \frac{w_1 + w_2 + \cdots + w_t}{t^{0.4}}.$$

$$x_{t-1} = \frac{w_1 + w_2 + \cdots + w_{t-1}}{(t-1)^{0.4}}.$$

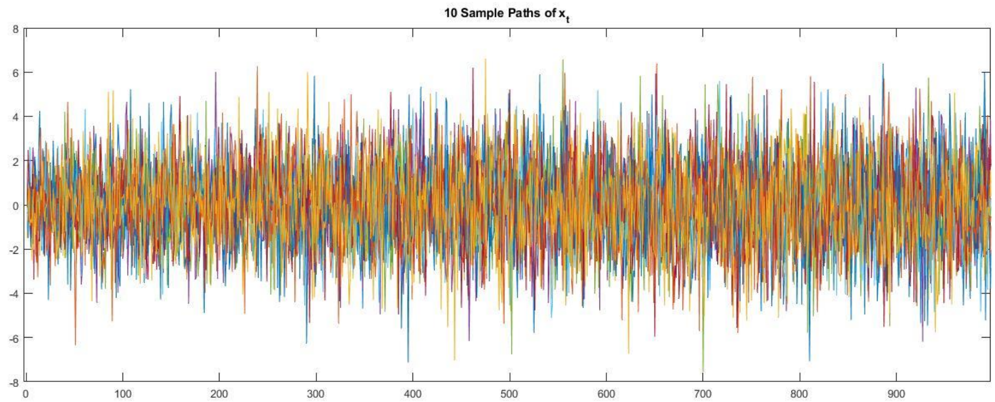
$$\therefore t^{0.4} x_t = (t-1)^{0.4} x_{t-1} + w_t.$$

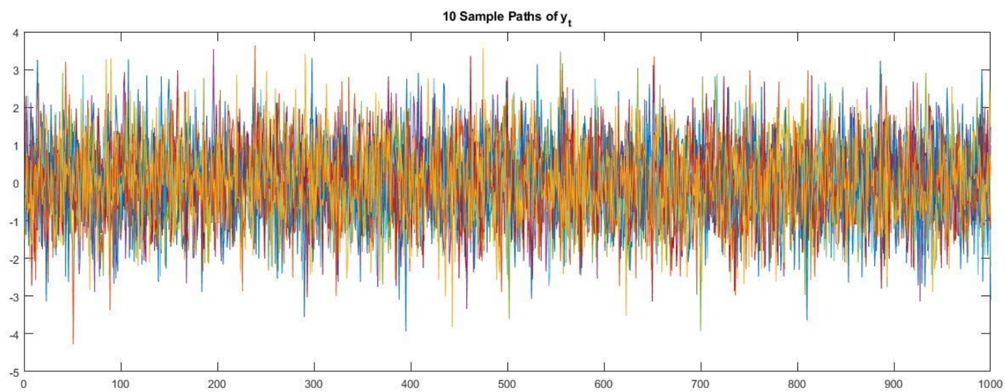
$$\begin{aligned} x_t &= \left(\frac{t-1}{t}\right)^{0.4} x_{t-1} + \left(\frac{1}{t}\right)^{0.4} w_t \\ &= \left(1-\frac{1}{t}\right)^{0.4} x_t + \left(\frac{1}{t}\right)^{0.4} w_t. \\ &= f_t(x_{t-1}, w_t). \end{aligned}$$

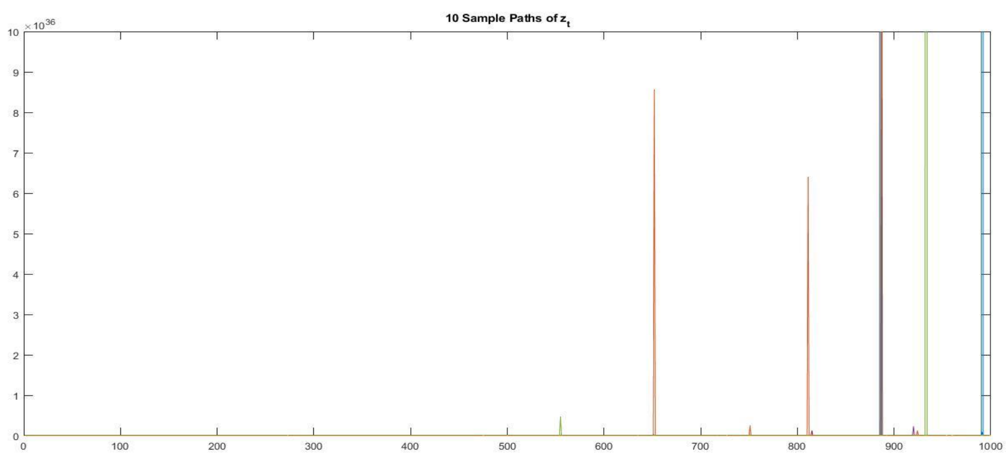
$$\therefore f_t(x, w) = \left(1-\frac{1}{t}\right)^{0.4} x + \left(\frac{1}{t}\right)^{0.4} w.$$

(b)

7. b) From the 10 sample paths of x_t, y_t, z_t shown below, as $t \to \infty$, both x_t and y_t will continuously oscillates around the horizontal 0-axis while their amplitudes become stronger, i.e., the absolute value of their maximum and minimum is increasing. As for z_t, its amplitude also becomes stronger, its maximum will approach to ∞, and its minimum will approach to 0.







```
x_t = zeros(10,1000);
y_t = x_t;
z_t = x_t;


for n = 1: 10
    for t = 1: 1000
        omega = normrnd(0,1,[1,t]);

        x_t(n,t) = sum(omega)/t^(0.4);

        y_t(n,t) = sum(omega)/sqrt(t);

        z_t(n,t) = exp(sum(omega));
    end
end
%%
figure (1)
for n = 1:10
    plot(1:1000,x_t(n,:)); hold on;
end
hold off;
title("10 Sample Paths of x_{t}");

figure (2)
for n = 1:10
    plot(1:1000,y_t(n,:)); hold on;
end
hold off;
title("10 Sample Paths of y_{t}");

figure (3)
for n = 1:10
    plot(1:1000,z_t(n,:)); hold on;
end
hold off;
title("10 Sample Paths of z_{t}");
ylim([0,1e37]);
```