UCSD ECE250 — Handout #2
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Solutions to ECE250 Aptitude Test

1. $\displaystyle\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$.

   Solution: Let
   $$S = 1 + r + r^2 + \cdots .$$
   (The series converges since $0 < r < 1$.) Then
   $$rS = r + r^2 + r^3 + \cdots .$$
   Taking the difference between two equations, we have $S - rS = 1$, or equivalently,
   $$S = \frac{1}{1-r}.$$

2. $\displaystyle\sum_{n=0}^{\infty} nr^n = \frac{r}{(1-r)^2}$.

   Solution: There are two ways to evaluate this. First, we can write
   $$S = r + 2r^2 + 3r^3 + \cdots .$$
   Then
   $$rS = r^2 + 2r^3 + 3r^4 + \cdots .$$
   By taking the difference between S and rS, we can easily see that
   $$(1-r)S = r + r^2 + r^3 + \cdots = \frac{r}{1-r}.$$
   Alternatively, we have
   $$\begin{aligned}\sum_{n=0}^{\infty} nr^n = r\sum_{n=0}^{\infty} nr^{n-1} &= r\sum_{n=0}^{\infty} \frac{d}{dr}\left(r^n\right) \\ &= r\frac{d}{dr}\left(\sum_{n=0}^{\infty} r^n\right) = r\frac{d}{dr}\left(\frac{1}{1-r}\right) = \frac{r}{(1-r)^2}.\end{aligned}$$
   (Note: The interchange of the differentiation and the summation can be justified. We already learned a similar and more general method in ECE109 with moment generating functions; the given identity is nothing but the mean of a geometric random variable.)

3. $\displaystyle\sum_{n=0}^{\infty} \frac{r^n}{n!} = e^r.$

Solution: This is one of famous Maclaurin series (a special form of Taylor series evaluated at 0). The Maclaurin series generated by a function $f(r)$ is

$$f(r) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} r^n$$

where $f^{(n)}(0)$ denotes the nth derivative of f evaluated at 0.

Let $f(r) = e^r$. Since $f^{(n)}(r) = e^r$ for all nonnegative integer n, $f^{(n)}(0) = e^0 = 1$ for all n. Therefore,

$$e^r = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} r^n = \sum_{n=0}^{\infty} \frac{r^n}{n!}.$$

Alternatively, the probability mass function of a Poisson random variable X with mean r is given by

$$\mathrm{P}\{X = n\} = e^{-r} \frac{r^n}{n!}$$

which sums to 1, as

$$e^{-r} \sum_{n=0}^{\infty} \frac{r^n}{n!} = 1.$$

By multiplying e^r to both sides, we get

$$\sum_{n=0}^{\infty} \frac{r^n}{n!} = e^r.$$

4. $\displaystyle\sum_{k=0}^{n} \binom{n}{k} r^k (1-r)^{n-k} = 1.$

Solution: This is just 1 since each term of the summation is the probability of k successes out of n independent trials, each with success probability r. In other words, each term is the probability mass function of a binomial random variable with parameter r which should sum to 1.

Alternatively, from the binomial expansion of $(p+q)^n$ which is

$$(p+q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}.$$

Taking $p = r$ and $q = 1 - r$, we get

$$1 = (r + 1 - r)^n = \sum_{k=0}^{n} \binom{n}{k} r^k (1-r)^{n-k}.$$

5. $\displaystyle\sum_{k=0}^{n} k\binom{n}{k} r^k(1-r)^{n-k} = rn.$

Solution: Since

$$k\binom{n}{k} = k\frac{n!}{k!(n-k)!} = \frac{n!}{(k-1)!(n-k)!} = n\binom{n-1}{k-1}$$

for $k \neq 0$, we have from the previous problem

$$\begin{aligned}\sum_{k=0}^{n} k\binom{n}{k} r^k(1-r)^{n-k} &= \sum_{k=1}^{n} n\binom{n-1}{k-1} r^k(1-r)^{n-k} \\ &= rn\sum_{j=0}^{n-1}\binom{n-1}{j} r^j(1-r)^{n-1-j} \\ &= rn.\end{aligned}$$

Alternatively, we can note from ECE109 that the given summation is the mean of a binomial random variable with parameter r.

6. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n} = \infty.$

Solution: This is a well-known divergent series. To see this, consider

$$\begin{gathered}\frac{1}{1} = \frac{1}{1} \\ \frac{1}{2} = \frac{1}{2} \\ \frac{1}{3}+\frac{1}{4} \geq \frac{1}{4}+\frac{1}{4} = \frac{1}{2} \\ \frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} \geq \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} = \frac{1}{2} \\ \vdots \\ \sum_{n=2^k+1}^{2^{k+1}}\frac{1}{n} \geq 2^k\cdot\frac{1}{2^{k+1}} = \frac{1}{2}.\end{gathered}$$

Clearly, the infinite sum of the RHS is unbounded, and so is the LHS.

7. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$

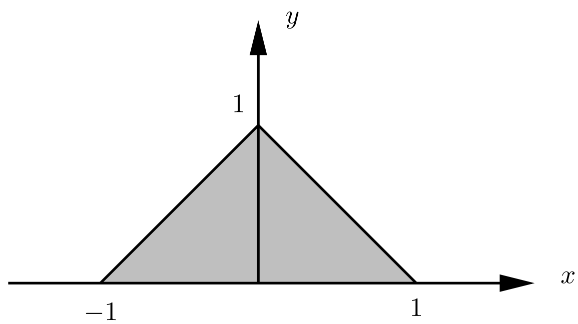
Solution: This is a famous identity that goes back to Euler. There are at least 15 different methods to derive it, and 14 of them are listed in an article by Robin Chapman.

8. $\displaystyle\int_{-1}^{1} 1 - |x|\,dx = 1.$

Solution: We have

$$\begin{aligned}\int_{-1}^{1} 1 - |x|\,dx &= \int_{-1}^{0} 1 + x\,dx + \int_{0}^{1} 1 - x\,dx \\ &= x + \frac{x^2}{2}\bigg|_{-1}^{0} + x - \frac{x^2}{2}\bigg|_{0}^{1} \\ &= (1 - 1/2) + (1 - 1/2) = 1.\end{aligned}$$

Alternatively, we can easily see that the given integral is nothing but the area of the following triangle.



9. $\displaystyle\int_{0}^{\infty} e^{-x}dx = 1.$

Solution: We have

$$\int_{0}^{\infty} e^{-x}dx = -e^{-x}|_0^\infty = 0 - (-1) = 1.$$

10. $\displaystyle\int_{0}^{\infty} xe^{-x}dx = 1.$

Solution: Integration by parts. Since

$$(xe^{-x})' = e^{-x} - xe^{-x},$$

we have

$$0 = xe^{-x}|_0^\infty = \int e^{-x} - \int xe^{-x} = 1 - \int xe^{-x}.$$

11. $\displaystyle\int_{0}^{\infty}\int_{x}^{\infty} e^{-y}dydx = 1.$

Solution: There are two ways to evaluate this. Consider

$$\int_{0}^{\infty}\int_{x}^{\infty} e^{-y}dydx = \int_{0}^{\infty}\left(-e^{-y}|_x^\infty\right)dx = \int_{0}^{\infty} e^{-x}dx = 1.$$

Here we used the answer of Problem 9.

Alternatively,

$$\begin{aligned}
\int_0^\infty \int_x^\infty e^{-y} dy dx &= \int_0^\infty \int_0^y e^{-y} dx dy \\
&= \int_0^\infty e^{-y} \int_0^y dx dy \\
&= \int_0^\infty y e^{-y} dy = 1.
\end{aligned}$$

Here we used the answer of Problem 10. (Or we could solve Problem 10 from Problem 11 without intergration by parts.)

12. $\displaystyle\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}.$

Solution: There are two ways to see this. The classical approach is to observe that

$$\begin{aligned}
\left(\int_{-\infty}^\infty e^{-x^2} dx\right)^2 &= \int_{-\infty}^\infty e^{-x^2} dx \int_{-\infty}^\infty e^{-y^2} dy \\
&= \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(x^2+y^2)} dx dy \\
&= \int_0^{2\pi} \int_0^\infty r e^{-r^2} dr d\theta \\
&= \int_0^{2\pi} \frac{1}{2} d\theta = \pi.
\end{aligned}$$

Here we used change of cordinate from the Cartesian (x, y) to the polar $(r\cos\theta, r\sin\theta)$ with $dxdy = rdrd\theta$.

Alternatively, we know from ECE109 that a Gaussian random variable with zero mean and variance σ^2 has the density

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}},$$

which integrates to 1. Thus by taking $\sigma^2 = 1/2$, we have

$$\int_{-\infty}^\infty \frac{1}{\sqrt{\pi}} e^{-x^2} dx = 1.$$

13. $\displaystyle\int_{-\infty}^\infty x e^{-x^2} dx = 0.$

Solution: Since the integrand $f(x) = xe^{-x^2}$ is odd, i.e., $f(x) = -f(-x)$, the integral is zero. (Or consider the mean of the corresponding Gaussian distribution.)

14. $\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}^{-1} = \dfrac{1}{1-r^2} \begin{pmatrix} 1 & -r \\ -r & 1 \end{pmatrix}.$

Solution: This comes from the well-known 2×2 matrix inversion formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

(Note: There are many ways to find the inverse of the general $n \times n$ matrices. One of common methods is performing Gaussian elimination to the original matrix and the identity matrix.)

15. $\lim_{x \to 0} x \ln x = 0.$

Solution:

$$\begin{aligned} \lim_{x \to 0} x \ln x &= \lim_{x \to 0} \frac{\ln x}{\frac{1}{x}} \\ &\overset{(a)}{=} \lim_{x \to 0} \frac{(\ln x)'}{\left(\frac{1}{x}\right)'} \\ &= \lim_{x \to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \\ &= \lim_{x \to 0} -x \\ &= 0. \end{aligned}$$

Here we applied the L'hopital's rule in (a).