HW2

Sunday, October 9, 2022 8:50 PM

2. Show if A, B are independent, then A^c and B are also independent.

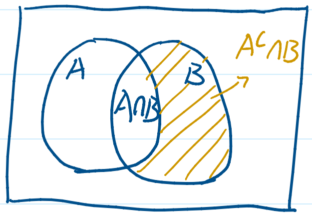
$\because$ A, B are independent

$\therefore P(A\cap B) = P(A)P(B)$

$\because P(A) = 1 - P(A^c)$

$\therefore P(A\cap B) = P(A)P(B)$

$= (1-P(A^c))P(B)$



$\therefore P(A)P(B) = P(B) - P(A^c)P(B)$

$\therefore P(A^c)P(B) = P(B) - P(A)P(B)$

$= P(B) - P(A\cap B)$

$= P(A^c \cap B)$

$\therefore$ A^c and B are independent

3. Prof. Kim 2.10

$P(A_1) = 0.5$, $P(B_1) = 0.2$, $P(C_1) = 0.3$

(a) $P(A_2) = P(\{\alpha,\alpha\}) + P(\{\gamma,\alpha\})$

$= 0.5(0.3) + 0.3(0.1)$

$= 0.15 + 0.03$

$= 0.18$

$P(B_2) = P(\{\alpha,\beta\}) + P(\{\beta,\beta\}) + P(\{\gamma,\beta\})$

$= 0.5(0.2) + 0.2(0.2) + 0.3(0.2)$

$= 0.1 + 0.04 + 0.06$

$= 0.2$

$P(C_2) = P(\{\alpha,\gamma\}) + P(\{\beta,\gamma\}) + P(\{\gamma,\gamma\})$

$= 0.5(0.5) + 0.2(0.8) + 0.3(0.7)$

$= 0.25 + 0.16 + 0.21$

$$= 0.25 + 0.16 + 0.21$$
$$= 0.62.$$

(b) $P(A_1)P(A_2|A_1) = P(A_2)P(A_1|A_2)$.

$$P(A_1|A_2) = P(A_1)P(A_2|A_1)/P(A_2)$$
$$= 0.15/0.18$$
$$= \frac{5}{6} = 0.8333$$

$$P(B_1|B_2) = P(B_1)P(B_2|B_1)/P(B_2)$$
$$= 0.04/0.2$$
$$= 0.2$$

$$P(C_1|C_2) = P(C_1)P(C_2|C_1)/P(C_2)$$
$$= 0.21/0.62$$
$$= 0.3387$$

(c) A_1, B_1, C_1 are not pairwise independent.

$$P(A_1 \cap B_1) = P(A_1 \cap C_1) = P(B_1 \cap C_1) = P(\emptyset) = 0.$$

A_2, B_2, C_2 are not pairwise independent either for similar reason.

$$P(A_1)P(A_2) = 0.5(0.18) = 0.09 \neq P(A_1 \cap A_2) = P(\{\alpha, \alpha\}) = 0.15.$$

$$P(A_1)P(B_2) = 0.5(0.2) = 0.1 = P(A_1 \cap B_2) = P(\{\alpha, \beta\}) = 0.1.$$

$$P(A_1)P(C_2) = 0.5(0.62) = 0.31 \neq P(A_1 \cap C_2) = P(\{\alpha, \gamma\}) = 0.25.$$

$$P(B_1)P(A_2) = 0.2(0.18) = 0.036 \neq P(B_1 \cap A_2) = P(\emptyset) = 0.$$

$P(B_1)P(A_2) = 0.2(0.18) = 0.036 \neq P(B_1 \cap A_2) = P(\phi) = 0$

$P(B_1)P(C_2) = 0.2(0.62) = 0.124 \neq P(B_1 \cap C_2) = P(\{\beta, \gamma\}) = 0.16$

$P(C_1)P(A_2) = 0.3(0.18) = 0.054 \neq P(\{\gamma, \alpha\}) = 0.3(0.1) = 0.03$

$P(C_1)P(B_2) = 0.3(0.2) = 0.06 = P(\{\gamma, \beta\}) = P(C_1 \cap B_2)$

From above, A_1 and B_2, C_1 and B_2 are pairwise independent.

4. Prof. Kim 2.11

$\Omega = \{1, 2, 3, \cdots\}^2 = \{(i,j) : i, j \in \mathbb{N}\}$

$P((i,j)) = p^2(1-p)^{i+j-2}, \quad 0<p<1$

(a) $P(\{(i,j) : i \geq j\}) = \sum_{j=1}^{\infty} \sum_{i=j}^{\infty} p^2(1-p)^{i+j-2}$

$= \frac{p^2}{(1-p)^2} \sum_{j=1}^{\infty} \sum_{i=j}^{\infty} (1-p)^{i+j}$

$\because i \geq j \quad \therefore i-j \geq 0$

set $m = i-j$. $\quad i = j+m$

$\therefore P(\{(i,j) : i \geq j\}) = \frac{p^2}{(1-p)^2} \sum_{j=1}^{\infty} \sum_{m=0}^{\infty} (1-p)^{2j+m}$

$= \frac{p^2}{(1-p)^2} \sum_{j=1}^{\infty} (1-p)^{2j} \sum_{m=0}^{\infty} (1-p)^m$

$= \frac{p^2}{(1-p)^2} \sum_{j=1}^{\infty} (1-p)^{2j} \cdot \frac{1}{1-(1-p)}$

$= \frac{p}{(1-p)^2} \sum_{j=1}^{\infty} (1-p)^{2j} = \frac{p}{(1-p)^2} \cdot \frac{(1-p)^2}{1-(1-p)^2}$

$= \frac{p}{1-(1-p)^2} = \frac{p}{2p-p^2} = \frac{1}{2-p}$

Let $a = 1-p$

$S = 1+a+a^2+\cdots$

$aS = a+a^2+\cdots$

$S - aS = 1$

$S = \frac{1}{1-a}$

$S = a^2+a^4+\cdots$

$a^2S = a^4+a^6+\cdots$

$S - a^2S = a^2$

$S = \frac{a^2}{1-a^2}$

(b) $P(\{(i,j): i+j=k\}) = \sum_{j=1}^{k-1} p^2(1-p)^{k-2}$

$= p^2(1-p)^{k-2} \sum_{j=1}^{k-1} 1$

$= p^2(1-p)^{k-2}(k-1)$.

(c) $P(\{(i,j): i \text{ is an odd number}\})$

$= \sum_{j=1}^{\infty} \sum_{i \text{ is odd}} p^2(1-p)^{i+j-2}$

$= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} p^2(1-p)^{(2i-1)+j-2}$

$= \frac{p^2}{(1-p)^3} \sum_{j=1}^{\infty} (1-p)^j \cdot \sum_{i=1}^{\infty} (1-p)^{2i}$

$= \frac{p^2}{(1-p)^3} \cdot \frac{1-p}{1-(1-p)} \cdot \frac{(1-p)^2}{1-(1-p)^2}$

$= \frac{p}{2p-p} = \frac{1}{2-p}$

$s = a + a^2 + \dots$

$as = \dots$

$s = \frac{a}{1-a}$

(d) $P((i,j)) = p^2(1-p)^{i+j-2}$

$= [p(1-p)^{i-1}] \cdot [p(1-p)^{j-1}]$

The experiment can be:
flip a coin repeatly,
first stops at the occurance of the first head, then flip it again until we get the second head appeared.

5. A = "the first Ace is at 20th draw"

a) $P(A) = \binom{48}{19}\binom{4}{1}$.

B = "the following card is the Ace of spades".

$$P(B|A) = P(B \cap A)/P(A)$$
$$= \binom{48}{19}\binom{3}{1}\cdot\frac{1}{32} \bigg/ \left[\binom{48}{19}\binom{4}{1}\right]$$
$$= \frac{3}{32}\cdot\frac{1}{4} = \frac{3}{128}$$

b) C = "the following card is two of clubs.

$$P(C|A) = P(C \cap A)/P(A)$$
$$= \binom{47}{19}\binom{4}{1}\cdot\left(\frac{1}{32}\right) \bigg/ \left[\binom{48}{19}\binom{4}{1}\right]$$
$$= \frac{47!}{19!\,28!}\cdot\frac{1}{32}\cdot\frac{19!\,29!}{48!}$$
$$= \frac{29}{48}\cdot\frac{1}{32} = \frac{29}{1536}$$

1. $\mathbb{B}^n := \sigma(\{(a_1, b_1)\times(a_2, b_2)\times\cdots\times(a_n, b_n) \mid a_i < b_i,\ a_i, b_i \in \mathbb{R}\ \forall i\})$.

show $[a_1, b_1]\times[a_2, b_2]\times\cdots\times[a_n, b_n] \in \mathbb{B}^n$.

for one dimension $\mathbb{R}$: $[a, b]$, $a < b$, $a, b \in \mathbb{R}$.

$$[a, b] = \bigcap_{k=1}^{\infty}\left(a - \frac{1}{k}, b - \frac{1}{k}\right)$$

which is an intersection of countable many open intervals $(a-\frac{1}{k}, b-\frac{1}{k}) \in \mathbb{B}$.

$\therefore [a,b] \in \mathbb{B}$.

for $\mathbb{R}^n$; $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$.

$$= \bigcap_{k=1}^{\infty} (a_1-\tfrac{1}{k}, b_1-\tfrac{1}{k}) \times \bigcap_{k=1}^{\infty} (a_2-\tfrac{1}{k}, b_2-\tfrac{1}{k}) \times \cdots \times \bigcap_{k=1}^{\infty} (a_n-\tfrac{1}{k}, b_n-\tfrac{1}{k})$$

$\therefore [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \in \mathbb{B}^n$.

6. $\Omega = \mathbb{R}$, $\mathcal{F} = \sigma(\{Q\})$.

(a) $\mathcal{F} = \{\emptyset, \{Q\}, \{Q^c\}, \Omega\}$.

(b) $Q \subseteq \Omega$, $Q^c \subseteq \Omega$

Set $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$,

$$\therefore X(\omega) = \begin{cases} 0, & \emptyset \\ \alpha, & \omega \in Q \\ \beta, & \omega \in Q^c \end{cases}$$