Homework 3-Solution

Reading assignment: Read Section 3.5 of Prof. Kim's notes on functions of a random variable before addressing Problem 1.

1. Let $\{X_k\}_{k\geq 1}$ be a random process over an underlying probability space $(\Omega, \mathcal{F}, \mathbf{P}(\cdot))$. Show that $X = \inf_{k\geq 1} X_k$ is a random variable.

   Solution: First note that in order to show that a function $Z : \Omega \to \mathbb{R}$ is a random variable, it is enough to show that $Z^{-1}([a, \infty)) \in \mathcal{F}$ for all $a \in \mathbb{R}$. This is true as this condition holds if and only if $Y^{-1}((-\infty, -a]) \in \mathcal{F}$ for $Y = -Z$ and all $a \in \mathbb{R}$, which holds iff Y is a random variable. But if Y is a random variable, $Y = -Z$ would be a random variable.

   So, for $X = \inf k \geq 1 X_k$, it suffice to show that for all a

   $$X^{-1}([a, \infty)) \in \mathcal{F}.$$

   By definition, we have

   $$(\inf_k X_k)^{-1}([a, \infty)) = \left\{\omega \middle| \inf_k X_k(\omega) \geq a\right\} \triangleq \left\{\inf_k X_k \geq a\right\}.$$

   Since the infimum of a sequence is greater than or equal to a if and only if every term is greater than or equal to a, we have

   $$\left\{\inf_k X_k \geq a\right\} = \bigcap_{k=1}^{\infty} \{X_k \geq a\} \in \mathcal{F},$$

   where follows from the fact that X_ks are random variables.

2. Problem 3.9 of Prof. Kim's notes.

   Solution:

   (a) For $x \geq 0$, we have $P(X < x) = 1 - e^{-\lambda x}$ and $P(X < x) = 0$ for $x < 0$.
   Let k be a non-negative integer. Then we have,

   $$\begin{aligned} P(Y = k) &= P(k \leq X < k + 1) \\ &= P(X < k + 1) - P(X < k) = e^{-\lambda k} - e^{-\lambda(k+1)} = e^{-\lambda k}(1 - e^{-1}). \end{aligned} \tag{1}$$

   (b) Note that for $z \in [0, 1)$, we have $\{Z < z\} = \cup_{k=0}^{\infty}\{k \leq X < k + z\}$. Therefore,

   $$\begin{aligned} P(Z \leq z) &= \sum_{k=0}^{\infty} P(k \leq X < k + z) \\ &\sum_{k=0}^{\infty} e^{-\lambda k} - e^{-\lambda(k+z)} = \sum_{k=0}^{\infty} e^{-\lambda k}(1 - e^{-\lambda z}) \\ &= \frac{1 - e^{-\lambda z}}{1 - e^{-\lambda}}. \end{aligned}$$

   Then, the pdf of Z is given by $\frac{dP(Z<z)}{dz} = \frac{\lambda e^{-\lambda z}}{1-e^{-\lambda}}$ for $z \in [0, 1)$ and 0 otherwise.

3. Let $\{X_k\}$ to be a random process over an underlying probability space $(\Omega, \mathcal{F}, \mathbf{P}(\cdot))$.

   (a) For any $\alpha \in \mathbb{R}$, show that the event E_α where the limiting point of $X_k(\omega) = \alpha$ is an event in $\mathcal{F}$:

   $$E_\alpha = \{\omega \in \Omega \mid \lim_{k\to\infty} X_k(\omega) = \alpha\}.$$

(b) Show that the set E of sample points that X_k has limit is measurable (i.e., it is an event in $\mathcal{F}$):

$$E = \{\omega \in \Omega \mid \lim_{k\to\infty} X_k(\omega) \text{ exists}\}.$$

Solution: First, we prove

$$\inf_k X_k \quad \sup_k X_k \quad \limsup_{k\to\infty} X_k \quad \liminf_{k\to\infty} X_k$$

are random variables. ($\inf_k X_k \triangleq \inf\{X_1, X_2, \ldots\}$ and $\sup_k X_k \triangleq \sup\{X_1, X_2, \ldots\}$)
Proof. We have to show for all a

$$(\inf_k X_k)^{-1}((-\infty, a)) \in \mathcal{F}.$$

By definition, we have

$$(\inf_k X_k)^{-1}((-\infty, a)) = \left\{\omega \middle| \inf_k X_k(\omega) < a\right\} \triangleq \left\{\inf_k X_k < a\right\}.$$

Since the infimum of a sequence is less than a if and only if some term is less than a (if all terms are greater or equal to a then so is the infimum), we have

$$\left\{\inf_k X_k < a\right\} = \bigcup_{k=1}^{\infty} \{X_k < a\} \in \mathcal{F},$$

where follows from the fact that X_ks are random variables. A similar argument shows $\{\sup_k X_k > a\} = \cup_k \{X_k > a\} \in \mathcal{F}$. For the last two, we observe

$$\liminf_{k\to\infty} X_k = \sup_k \left(\inf_{m\geq k} X_m\right)$$
$$\limsup_{k\to\infty} X_k = \inf_k \left(\sup_{m\geq k} X_m\right)$$

To complete the proof in the first case, note that $Y_k = \inf_{m\geq k} X_m$ is a random variable for each k, so $\sup_k Y_k$ is as well.

(a) We have

$$\begin{aligned} E_\alpha &= \{\omega \in \Omega \mid \lim_{k\to\infty} X_k(\omega) = \alpha\} \\ &= \{\omega \in \Omega \mid \liminf_{k\to\infty} X_k(\omega) = \alpha\} \cap \{\omega \in \Omega \mid \limsup_{k\to\infty} X_k(\omega) = \alpha\} \in \mathcal{F}, \end{aligned}$$

where follow from the fact $\liminf_{k\to\infty} X_k$ and $\limsup_{k\to\infty} X_k$ are random variables, and so

$$(\liminf_{k\to\infty} X_k)^{-1}(\{\alpha\}), (\limsup_{k\to\infty} X_k)^{-1}(\{\alpha\}) \in \mathcal{F}.$$

(b) We have

$$\begin{aligned} E &= \{\omega \in \Omega \mid \lim_{k\to\infty} X_k(\omega) \text{ exists}\} \\ &= \{\omega \in \Omega \mid \liminf_{k\to\infty} X_k(\omega) = \limsup_{k\to\infty} X_k(\omega)\}, \\ &= \{\omega \in \Omega \mid \liminf_{k\to\infty} X_k(\omega) - \limsup_{k\to\infty} X_k(\omega) = 0\}, \\ &= (\liminf_{k\to\infty} X_k - \limsup_{k\to\infty} X_k)^{-1}(\{0\}) \in \mathcal{F}. \end{aligned}$$

4. Find a sequence of random variables (i.e., a random process) $\{X_k\}$ such that its limit exists and $\mathbb{E}[\lim_{k\to\infty} X_k] \neq \lim_{k\to\infty} \mathbb{E}[X_k]$. (if you cannot do it by yourself, do research on finding such random variables)

   Solution: Let
$$X_k = \begin{cases} 0, & \text{with probability } 1-\frac{1}{k^2} \\ k^2, & \text{with probaility } \frac{1}{k^2} \end{cases}.$$
   Therefore, $\mathbb{E}[X_k] = 1$ for all k, and hence $\lim_{k\to\infty} \mathbb{E}[X_k] = 1$. Now, we want to show that $\lim_{k\to\infty} X_k = 0$, and hence $\mathbb{E}[\lim_{k\to\infty} X_k] = 0$, which is not equal to $\lim_{k\to\infty} \mathbb{E}[X_k] = 1$.

   To prove, consider the sequence of events
$$E_k = \{X_k > 0\}$$
   which happens with probability $\mathbf{P}(E_k) = \frac{1}{k^2}$. Since $\sum_{k=1}^{\infty} \mathbf{P}(E_k) < \infty$ and these events are independent, the Borel-Cantelli lemma implies that $\mathbf{P}(\{E_k. \text{ i.o.}\}) = 0$. This implies that for almost all $\omega \in \Omega$, there exists some $T(\omega)$ such that $X_k(\omega) = 0$ for $k \geq T(\omega)$, i.e., $\lim_{k\to\infty} X_k(\omega) = 0$ almost surely.

5. In the spirit of HW1-Problem 7, let $\{w_k\}$ be an i.i.d. random process that is uniformly distributed over $[-1, 1]$, i.e., they admit the PDF
$$f(x) = \begin{cases} \frac{1}{2} & x \in [-1,1] \\ 0 & \text{otherwise} \end{cases}.$$
   (a) Show that $P(\{|w_k| > \frac{1}{4} \text{ i.o.}\}) = 1$.

   (b) Using this and the definition of convergent series, show that the process $\{w_k\}$ is almost surely not summable, i.e., $\sum_{k=1}^{\infty} w_k$ does not exists with probability one.

   Solution:

   (a) Consider the sequence of events
$$E_k = \left\{ w_k \geq \frac{1}{4} \right\}.$$
   Hence, $\mathbf{P}(E_k) = \frac{3}{8}$, and so $\sum_{k=1}^{\infty} \mathbf{P}(E_k) = \infty$. Since, w_ks are independent, Borel-Cantelli lemma implies that $\mathbf{P}(\{E_k. \text{ i.o.}\}) = 1$.

   (b) To show that the probability of $z_k := w_1 + \cdots + w_k$ not having a limit is equal to one it is sufficient to show that the probability of $\limsup_{k\to\infty} w_k > 0$ is one.
   So, to solve the problem, it is enough to show that $\mathbf{P}(\{\limsup_{k\to\infty} w_k > 0\}) = 1$.
   We know $P(\{|w_k| > \frac{1}{4} \text{ i.o.}\}) = 1$. This implies that $\limsup_{k\to\infty} w_k \geq \frac{1}{4}$ with probability 1 which proves the result.

6. Using the general definition of $\mathbb{E}[X]$ that we discussed in the class, show that for a non-negative discrete random variable X
$$\mathbb{E}[X] = \sum_{k=1}^{\infty} m_k p_X(m_k),$$
   where $\mathbf{P}(X \in M) = 1$ and $M = \{m_k \mid k \geq 1\}$.

   Solution: Let us define the simple function $X_i = \sum_{k=1}^{i} m_k \mathbf{1}_{\{X=m_k\}}$ for all $i \geq 1$. Therefore, from the definition of expectation of simple functions, $\mathbb{E}[X_i] = \sum_{k=1}^{i} m_k p_X(m_k)$. Since, $X_1 \leq X_2 \leq \cdots \leq X = \lim_{i\to\infty} X_i$, from Monotone Convergence Theorem, we have
$$\mathbb{E}[X] = \mathbb{E}[\lim_{k\to\infty} X_k] = \lim_{k\to\infty} \mathbb{E}[X_k] = \sum_{k=1}^{\infty} m_k p_X(m_k).$$

7. Let X be a finite random variable on $(\Omega, \mathcal{F}, \mathbf{P})$ (i.e., $P(X=\infty)=P(X=-\infty)=0$. Show that its distribution function F_X satisfies the following properties:

   (a) F_X is non-decreasing.

   (b) $\lim_{x\to-\infty} F_X(x)=0$, and $\lim_{x\to\infty} F_X(x)=1$.

   (c) $F_X(\cdot)$ **is right-continuous**, i.e., for any $x\in\mathbb{R}$, $\lim_{y\to x^+} F_X(y)=F_X(x)$.

   (d) Define $F_X(x^-) := \lim_{y\uparrow x} F_X(y)$, then

   $$F_X(x^-)=\mathbf{P}[X<x]=\mathbf{P}[\{\omega\in\Omega \mid X(\omega)<x\}].$$

   (e) For any $x\in\mathbb{R}$, we have $\mathbf{P}[X=x]=F_X(x)-F_X(x^-)$.

   Solution:

   (a) If $x<y$, then $\{X\le x\}\subset\{X\le y\}$, which implies that $F_X(x)\le F_Y(y)$.

   (b) If $x_n, n\ge 1$ is an increasing sequence such that $x_n\to\infty$, then the events $E_n=\{X\le x_n\}$ form an increasing sequence with

   $$\{X<\infty\}=\bigcup_{n=1}^{\infty} E_n.$$

   It follows from the continuity properties of probability measures that

   $$\lim_{n\to\infty} F_X(x_n)=\lim_{n\to\infty}\mathbf{P}(E_n)=\mathbf{P}(X<\infty)=1.$$

   Likewise, if $x_n, n\ge 1$ is a decreasing sequence such that $x_n\to-\infty$, then the events $E_n=\{X\le x_n\}$ form a decreasing sequence with

   $$\emptyset=\bigcap_{n=1}^{\infty} E_n.$$

   In this case, the continuity properties of measures imply that

   $$\lim_{n\to\infty} F_X(x_n)=\lim_{n\to\infty}\mathbf{P}(E_n)=\mathbf{P}(\emptyset)=0.$$

   (c) If $x_n, n\ge 1$ is a decreasing sequence converging to x, then the sets $E_n=\{X\le x_n\}$ also form a decreasing sequence with

   $$\{X\le x\}=\bigcap_{n=1}^{\infty} E_n.$$

   Consequently,

   $$\lim_{n\to\infty} F_X(x_n)=\lim_{n\to\infty}\mathbf{P}(E_n)=\mathbf{P}\{X\le x\}=F_X(x)$$

   (d) If $x_n, n\ge 1$ is a increasing sequence converging to x, then the sets $E_n=\{X\le x_n\}$ also form a increasing sequence with

   $$\{X<x\}=\bigcup_{n=1}^{\infty} E_n.$$

   Consequently,

   $$\lim_{n\to\infty} F_X(x_n)=\lim_{n\to\infty}\mathbf{P}(E_n)=\mathbf{P}\{X<x\}=F_X(x^-).$$

   (e) We have

   $$\mathbf{P}[X=x]=\mathbf{P}[X\le x]-\mathbf{P}[X<x]=F_X(x)-F_X(x^-).$$