ECE 250: Stochastic Processes: Week #5

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Outline:

- Strong Law of Large Numbers
- Applications of SLLN: Kelly Gambling, Pandemics Model
- Central Limit Theorem

Sums of Independent Random Variables and their Variances

Theorem 1. *For an independent sequence of random variables $\{X_k\}$ with zero mean, if*

$$\sum_{k=1}^{\infty} Var(X_k) < \infty,$$

then $\lim_{n\to\infty} \sum_{k=1}^{n} X_n$ *exists.*

Main idea of the proof: show that $\sum_{k=1}^{\infty} X_k$ is a Cauchy sequence almost surely by utilizing the Maximal inequality:

$$\Pr(\sup_{M\geq m} |S_M - s_m| \geq \epsilon) \leq \sum_{k=m}^{\infty} \mathsf{Var}(X_k).$$

Kolmogorov's Three-Series Theorem

Theorem 2. *Sum of an independent random process* $\{X_k\}$ *converges almost surely if and only if for any* $\alpha > 0$*, if we let* $Y_k = X_k \mathbf{1}_{|X_k| \le \alpha}$*, the following three (deterministic) series converges:*

- $\sum_{k=1}^{\infty} P(|X_k| \ge \alpha) < \infty$,
- $\sum_{k=1}^{\infty} \mathbb{E}[Y_k]$ *converges, and*
- $\sum_{k=1}^{\infty} Var(Y_k)$ *converges.*

Strong Law of Large Numbers: main idea of the proof

Theorem 3. *Let $\{X_k\}$ be a sequence of i.i.d. random variables with finite mean $\mathbb{E}[X_k] = \mu$. Then, the running average sequence:*

$$Y_k = \frac{X_1 + \ldots + X_n}{n}, \tag{1}$$

converges almost surely to μ.

Lemma 1. *(Kronecker's Lemma) Suppose that $\sum_{k=1}^{\infty} \frac{x_k}{a_k}$ converges for two deterministic sequences $\{x_k\}$ and $\{a_k\}$ with $a_k \to \infty$. Then*

$$\frac{x_1 + \ldots + x_n}{a_n} \to 0.$$

Proof. (Proof of SLLN for the case of $\mathrm{Var}(X_k) = \sigma^2 < \infty$)

- By Theorem 1, $\sum_{k=1}^{\infty} \frac{X_k - \mu}{k}$ converges almost surely as $\mathrm{Var}(\frac{X_k-\mu}{k}) = \frac{\sigma^2}{k^2}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$.
- Therefore, using the Kronecker's Lemma,

$$\frac{\sum_{k=1}^{n}(X_k - \mu)}{n} = \frac{S_n}{n} - \mu$$

  converges to 0 almost surely as $k \to \infty$.

□

Strong Law of Large Numbers: Applications

- Kelly Gambling:
  - Suppose that we have a slot machine
  - At time k, we put in $Z(k)$ dollars in we win with probability $p = 0.51$ and loose with probability $1 - p = 0.49$
  - If win, the machine returns the money and matches the investment, otherwise, lose the money
  - Question: If we initially have $X_0 =$\$10, can we become a millionaire with probability one?

Strong Law of Large Numbers: Applications cont.

- Pandemics Modeling: Suppose that $X_{k+1} = W_k X_k$ for some $X_0 > 0$ and an i.i.d. process $\{W_k\}$ that is positive a.s.
- What happens if $\mathbb{E}[\log(W_k)] = \gamma > 0$?
- Answer: For any $0 < \lambda < \gamma$, for almost all ω, there exists a $T(\omega)$ such that, we have

  $$X_k(\omega) \geq e^{\lambda k},$$

  for all $k \geq T(\omega)$.
- What happens if $\mathbb{E}[\log(W_k)] = \gamma < 0$?
- Answer: For any $0 < \lambda < -\gamma$, for almost all ω, there exists a $T(\omega)$ such that, we have

  $$X_k(\omega) \leq e^{-\lambda k},$$

  for all $k \geq T(\omega)$.

Convergence in Distribution

- **Definition**: We say that a random process $\{X_k\}$ converges in distribution to a random variable X if $\lim_{k\to\infty} \Pr(X_k \leq \alpha) = \Pr(X \leq \alpha)$ for all $\alpha \in \mathbb{R}$.
- In other words, we say that $\{X_k\}$ converges in distribution if the distribution of $\{X_k\}$ converges (point-wise) to the of distribution X.
- We denote this by $X_k \stackrel{\text{law}}{\to} \mu$.
- Sometimes this mode of convergence is defined as: We say that a random process $\{X_k\}$ converges in distribution to a **distribution** η if $\lim_{k\to\infty} \Pr(X_k \leq \alpha) = \eta(\alpha)$ for all $\alpha \in \mathbb{R}$.
- This convergence is much weaker than almost sure convergence.
- Recall we say that a continuous r.v. X is normally distributed with mean μ and variance σ^2 if
$$f_X(x) = \frac{d}{dx} F_X(x) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$
We denote this by $X \sim \mathcal{N}(\mu, \sigma^2)$.

Central Limit Theorem

Theorem 4. *(Central Limit Theorem) Suppose that $\{X_k\}$ is an i.i.d. sequence with finite mean μ and* ***finite variance*** *σ^2. Then,*

$$\begin{aligned}
\frac{(X_1-\mu)+(X_2-\mu)+\ldots+(X_n-\mu)}{\sqrt{n}} &= \frac{X_1+X_2+\ldots+X_n}{\sqrt{n}} - \sqrt{n}\mu \\
&= \frac{S_n - n\mu}{\sqrt{n}} \stackrel{law}{\to} X,
\end{aligned}$$

where $X \sim \mathcal{N}(0, \sigma^2)$.

- Application: Suppose that we have a fair coin and we flip it $N = 1000$ times. How likely is it to have between 450 and 550 of heads?
- Solution: Let $X_k = 1$ if the coin comes head and $X_k = 0$ if the coin comes tail. Then, $\mathbb{E}[X_k] = \frac{1}{2}$ and $\mathsf{Var}(X_k) = \frac{1}{4}$.
- Then,

$$\begin{aligned}
\Pr(450 \le X_1 + \cdots + X_{1000} \le 550) &= \Pr(-50 \le S_{1000} - 1000\mu \le 50) \\
&= \Pr(-\frac{50}{10\sqrt{10}} \le \frac{S_{1000}-1000\mu}{10\sqrt{10}} \le \frac{50}{10\sqrt{10}}) \\
&= \Pr(-\frac{\sqrt{10}}{2} \le \frac{S_{1000}-1000\mu}{10\sqrt{10}} \le \frac{\sqrt{10}}{2}) \\
&\approx \Pr(-1.5811 \le X \le 1.5811) \qquad (2)
\end{aligned}$$

  for a random variable $X \sim \mathcal{N}(0, \frac{1}{4})$.

- Using Gaussian integral tables, $\Pr(-1.5811 \le X \le 1.5811) \approx 0.9984$.