

# Homework 1

Due: 8:00pm (PT) Tuesday, Oct 4th, 2022

- Let  $A, B, C$  be arbitrary sets. Prove or disprove the following statements. Note that to disprove a statement you need to provide an example that the statement fails.
  - $(A - (A - B)) = A \cap B$ .
  - $A \cap (B \cup C) = (A \cap B) \cup C$ .
  - if  $A \subset C$ , then  $A \cup (C - A) = C$ .
- Consider the sample space  $\Omega = \{1, 2, 3, 4, 5\}$  and let  $\mathcal{F} = \mathcal{P}(\Omega)$  be the set of all subsets of  $\Omega$ . Consider a probability measure  $P : \mathcal{F} \rightarrow [0, 1]$  satisfying  $P(\{1, 2\}) = 0.2$  and  $P(\{2, 3\}) = 0.3$  (and of course the axioms of probability measure). For each of the following events determine whether its probability can be uniquely determined (with this information) or not. If so, find the probability and if not, reason why you cannot find their probability.
  - $A = \{2\}$
  - $B = \{1, 3\}$
  - $C = \{5\}$
- Solve Problem 2.1 from Prof. Kim notes (page 10).
- Solve Problem 2.2 from Prof. Kim notes (page 10).
- Solve Problem 2.3 from Prof. Kim notes (page 10).
- Let  $\Omega = \mathbb{R}$ , and  $\mathcal{F}$  be the collection of all subsets  $A$  of  $\mathbb{R}$  such that  $A$  or  $A^c$  is countable. Also, let

$$P(A) = \begin{cases} 0, & A \text{ is countable} \\ 1, & A^c \text{ is countable} \end{cases}.$$

Show that  $(\Omega, \mathcal{F}, P)$  is a probability space.

- Consider the random processes

$$\begin{aligned} x_t &= \frac{w_1 + w_2 + \dots + w_t}{t^{0.4}}, \\ y_t &= \frac{w_1 + w_2 + \dots + w_t}{\sqrt{t}}, \\ z_t &= e^{w_1} e^{w_2} \dots e^{w_t}. \end{aligned}$$

where  $\{w_t\}$  is an i.i.d random sequence that is Gaussian with zero mean and unit variance (you can use `normrnd` command in MATLAB).

- Find a function  $f_t(x, w)$  such that

$$x_t = f_t(x_{t-1}, w_t).$$

- Using MATLAB, plot 10 sample paths of  $x_t$ ,  $y_t$ , and  $z_t$  for  $t = 1, \dots, 1000$ . By investigating the plots and using your intuition, what do you think about the behavior of  $x_t$ ,  $y_t$ , and  $z_t$  as  $t \rightarrow \infty$ ? (Please include your code as well.)