Stochastic Processes
UCSD ECE 250 Midterm
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1. Determine whether each of the following statements is True or False. If True, prove it, if False, provide a counterexample.
   (a) Consider the probability space $(\Omega, \mathcal{F}, \mathbf{P}(\cdot))$. Let $X, Y : \Omega \to \mathbb{R}$, be such that $X + Y$ is a random variable. Then X, Y are random variables on this probability space.
   (b) If $A_1, A_2, \ldots$ are events such that $\mathbf{P}(A_n \text{ i.o.}) = 0$, then $\sum_{n=1}^{\infty} \mathbf{P}(A_n) < \infty$.
   (c) If $\mathcal{F}_1$ and $\mathcal{F}_2$ are both σ-algebras on a sample space Ω, then $\mathcal{F}_1 \cup \mathcal{F}_2$ is a σ-algebra on Ω.
2. Let Θ be a uniformly distributed random variable over $[0, 2\pi]$. Let
$$R = \begin{cases} \cos(\Theta) & \text{if } \Theta \geq \frac{\pi}{3} \\ 0 & \text{otherwise} \end{cases}.$$
   (a) Find the Cumulative Distribution Function (CDF) of R.
   (b) Is R a continuous or a discrete random variable?
   (c) Find $\mathbb{E}[\cdot]$.
3. For each of the following processes, determine whether the **partial sum sequence** $S_k = \sum_{i=1}^{k} X_i$ converges almost surely or not, i.e., $\lim_{k\to\infty} S_k$ exists almost surely or not.
   (a) $\{X_k\}$ is an independent process such that X_k takes three values $-k, 0, k^k$ with
$$\mathbf{P}(X_k = k^k) = \mathbf{P}(X_k = -k) = \frac{1}{2k^2}$$
   and $\mathbf{P}(X_k = 0) = 1 - \frac{1}{k^2}$.
   (b) $\{X_k\}$ is given by $X_k = \frac{Y_k - 1}{k}$ where $\{Y_k\}$ is an i.i.d. Gaussian random process with $Y_k \sim \mathcal{N}(0, 1)$ (i.e., a Gaussian with zero mean and unit variance).
4. Three players P1, P2, and P3 taking turns on flipping a fair coin (starting from P1, then P2, then P3, and then P1 again, etc.). The person who gets the first head is the winner.
   (a) Let Ω be the set of all the binary sequences of the form $1, 01, 001, 0001, \ldots$. Provide the set of events $\mathcal{F}$ and a probability measure $\mathbf{P}(\cdot)$, such that the probability space $(\Omega, \mathcal{F}, \mathbf{P}(\cdot))$ models this experiment.
   (b) Determine the event B where P2 is winning and find $\mathbf{P}(B)$.
   (c) Let A be the event that P1 is winning. Is A and B independent?