UNIVERSITY OF CALIFORNIA, SAN DIEGO

Electrical & Computer Engineering Department

ECE 250 - Winter Quarter 2022

Random Processes

Problem Set #4 Due Monday, February 7, 2022 at 11:59pm
Submit solutions to Problems 1, 4, 6, 9 only

1. *Optical communication channel.* Let the signal input to an optical channel be given by

$$X = \begin{cases} 1 & \text{with probability } \frac{1}{2} \\ 10 & \text{with probability } \frac{1}{2}. \end{cases}$$

The conditional pmf of the output of the channel $Y|\{X = 1\} \sim \text{Poisson}(1)$, i.e., Poisson with intensity $\lambda = 1$, and $Y|\{X = 10\} \sim \text{Poisson}(10)$.

(a) Show that the MAP rule reduces to

$$d^*(y) = \begin{cases} 1, & y < y^* \\ 10, & \text{otherwise.} \end{cases}$$

(b) Find y^* and the corresponding probability of error.

2. *Iocane or Sennari.* An absent-minded chemistry professor forgets to label two identically looking bottles. One bottle contains a chemical named "Iocane" and the other bottle contains a chemical named "Sennari". It is well known that the radioactivity level of "Iocane" has the $\mathrm{U}[0, 1]$ distribution, while the radioactivity level of "Sennari" has the $\mathrm{Exp}(1)$ distribution.

(a) Let X be the radioactivity level measured from one of the bottles. What is the optimal decision rule (based on the measurement X) that maximizes the chance of correctly identifying the content of the bottle?

(b) What is the associated probability of error?

3. *Radar signal detection.* The signal for a radar channel $S = 0$ if there is no target and a random variable $S \sim N(0, P)$ if there is a target. Both occur with equal probability. Thus

$$S = \begin{cases} 0, & \text{with probability } \frac{1}{2} \\ X \sim N(0, P), & \text{with probability } \frac{1}{2}. \end{cases}$$

The radar receiver observes $Y = S + Z$, where the noise $Z \sim N(0, N)$ is independent of S. Find the optimal decoder for deciding whether $S = 0$ or $S = X$ and its probability of error? Provide your answer in terms of intervals of y and provide the boundary points of the intervals in terms of P and N.

4. *Mean and variance.* Let X and Y be random variables with joint pdf

$$f_{X,Y}(x,y) = \begin{cases} 1 & \text{if } |x| + |y| \le 1/\sqrt{2} \\ 0 & \text{otherwise} \end{cases}$$

Define the random variable $Z = |X| + |Y|$. Find the mean and variance of Z without first finding the pdf of Z.

5. *Random phase signal.* Let $Y(t) = \sin(\omega t + \Theta)$ be a sinusoidal signal with random phase $\Theta \sim \text{Unif}[-\pi, \pi]$. Assume here that ω and t are constants. Find the mean and variance of $Y(t)$. Do they depend on t?

6. *Iterated expectation.* Let Λ and X be two random variables with

$$\Lambda \sim f_\Lambda(\lambda) = \begin{cases} \frac{5}{3}\lambda^{\frac{2}{3}}, & 0 \le \lambda \le 1 \\ 0, & \text{otherwise,} \end{cases}$$

and $X|\{\Lambda = \lambda\} \sim \text{Exp}(\lambda)$. Find $E(X)$.

7. *Sum of packet arrivals.* Consider a network router with two types of incoming packets, wireline and wireless. Let the random variable $N_1(t)$ denote the number of *wireline* packets arriving during time $(0, t]$ and let the random variable $N_2(t)$ denote the number of *wireless* packets arriving during time $(0, t]$. Suppose $N_1(t)$ and $N_2(t)$ are independent Poisson with pmfs

$$\mathsf{P}\{N_1(t) = n\} = \frac{(\lambda_1 t)^n}{n!} e^{-\lambda_1 t} \qquad \text{for } n = 0, 1, 2, \ldots$$
$$\mathsf{P}\{N_2(t) = k\} = \frac{(\lambda_2 t)^k}{k!} e^{-\lambda_2 t} \qquad \text{for } k = 0, 1, 2, \ldots.$$

Let $N(t) = N_1(t) + N_2(t)$ be the total number of packets arriving at the router during time $(0, t]$.

   (a) Find the mean $E(N(t))$ and variance $\text{Var}(N(t))$ of the total number of packet arrivals.

   (b) Find the pmf of $N(t)$.

   (c) Let the random variable Y be the time to receive the first packet of either type. Find the pdf of Y.

   (d) What is the probability that the first received packet is wireless?

8. *Conditioning on an event.* Let X be a r.v. with pdf

$$f_X(x) = \begin{cases} 2(1-x) & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and let the event $A = \{X \ge 1/3\}$. Find $f_{X|A}(x)$, $\mathsf{E}(X|A)$, and $\text{Var}(X|A)$.

9. *Inequalities.* Label each of the following statements with $=$, $\leq$, or $\geq$. Justify each answer.

   (a) $\frac{1}{E(X^2)}$ vs. $E\left(\frac{1}{X^2}\right)$.

   (b) $(E(X))^2$ vs. $E(X^2)$.

   (c) $\mathrm{Var}(X)$ vs. $\mathrm{Var}(E(X|Y))$.

   (d) $E(X^2)$ vs. $E((E(X|Y))^2)$.

10. *Correlation coefficient.*Let X and Y have correlation coefficient $\rho_{X,Y}$.

   (a) What is the correlation coefficient between X and $3Y$?

   (b) What is the correlation coefficient between $2X$ and $-5Y$?

11. *Cauchy–Schwartz inequality.*

   (a) Prove the following inequality: $(E(XY))^2 \leq E(X^2)E(Y^2)$. (Hint: Use the fact that for any real t, $E((X+tY)^2) \geq 0$.)

   (b) Prove that equality holds if and only if $X = cY$ for some constant c. Find c in terms of the second moments of X and Y.

   (c) Use the Cauchy–Schwartz inequality to show the correlation coefficient $|\rho_{X,Y}| \leq 1$.

   (d) Prove the *triangle inequality*: $\sqrt{E((X+Y)^2)} \leq \sqrt{E(X^2)} + \sqrt{E(Y^2)}$.