Homework 4-Solution

1. For each of the following random processes $\{X_k\}$, plot 100 sample paths for the corresponding partial sum sequence $\{S_k\}$ for $1 \leq k \leq 1000$. Conjecture, and theoretically prove whether the corresponding partial sum sequence converges or not. Explain your answer.

   (a) $X_k = \frac{Z_k}{k}$ where Z_k is i.i.d. and uniformly distributed over $[-1, 1]$.

   (b) $X_k = \frac{Z_k}{k^{0.7}}$ where Z_k is i.i.d. and uniformly distributed over $[-1, 1]$.

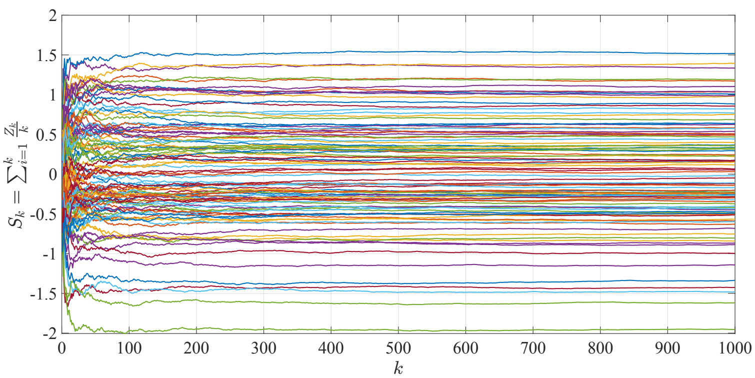
   (c) $X_k = \frac{Z_k}{k^{0.5}}$ where Z_k is i.i.d. and uniformly distributed over $[-1, 1]$.

   Solution:

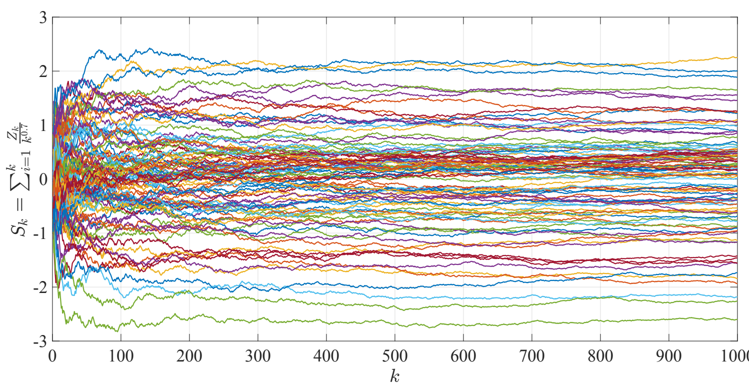
   (a) We have $\mathbb{E}[X_k] = 0$ and $\mathbf{Var}[X_k] = \frac{1}{3k^2}$. Therefore, since $\sum_{k=1}^{\infty} \mathbf{Var}(X_k) < \infty$ and $\mathbb{E}[X_k] = 0$ for all k, $\lim_{n\to\infty} \sum_{k=1}^{n} X_n$ exists.

   (b) We have $\mathbb{E}[X_k] = 0$ and $\mathbf{Var}[X_k] = \frac{1}{3k^{1.4}}$. Therefore, since $\sum_{k=1}^{\infty} \mathbf{Var}(X_k) < \infty$ and $\mathbb{E}[X_k] = 0$ for all k, $\lim_{n\to\infty} \sum_{k=1}^{n} X_n$ exists.

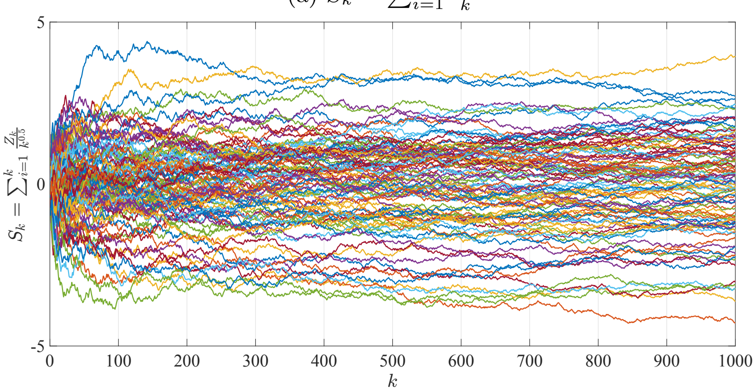
   (c) We have $\mathbb{E}[X_k] = 0$ and $\mathbf{Var}[X_k \mathbf{1}_{|X_k|\leq 1}] = \frac{1}{3k}$. Therefore, since $\sum_{k=1}^{\infty} \mathbf{Var}(X_k) = \infty$, $\lim_{n\to\infty} \sum_{k=1}^{n} X_k$ does not exist.



(a) $S_k = \sum_{i=1}^{k} \frac{Z_k}{k}$



(b) $S_k = \sum_{i=1}^{k} \frac{Z_k}{k^{0.7}}$



(c) $S_k = \sum_{i=1}^{k} \frac{Z_k}{k^{0.5}}$

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T=1000;
N=100;
z=2*rand(N,T)-1;
for i=1:N
    x(i,:)=cumsum(z(i,:)./([1:T]));
    y(i,:)=cumsum(z(i,:)./([1:T].^.7));
    v(i,:)=cumsum(z(i,:)./([1:T].^.5));
end
plot(x')
xlabel('$k$','Interpreter','latex')
ylabel('$S_k=\sum_{i=1}^k \frac{Z_k}{k}$','Interpreter','latex')
figure
plot(y')
xlabel('$k$','Interpreter','latex')
ylabel('$S_k=\sum_{i=1}^k \frac{Z_k}{k^{0.7}}$','Interpreter','latex')
figure
plot(v')
xlabel('$k$','Interpreter','latex')
ylabel('$S_k=\sum_{i=1}^k \frac{Z_k}{k^{0.5}}$','Interpreter','latex')
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2. Consider the independent random process $\{X_k\}$ that takes values k^2 or 0 with the probabilities

$$P(X_k = k^2) = \frac{1}{k^2}, \text{ and}$$
$$P(X_k = 0) = 1 - \frac{1}{k^2}.$$

Fix threshold $a = 1$. For each $k \geq 1$:

(a) Determine $P(|X_k| \geq a)$.
Solution: Since, X_k takes value k^2 or 0 and $k^2 \geq 1$ for all k we have

$$P(X_k \geq 1) = P(X_k = k^2) = \frac{1}{k^2}.$$

(b) Determine $\mathbb{E}[X_k 1_{\{|X_k| \leq a\}}]$.
Solution: X_k takes value k^2 or 0 and $k^2 > 1$ for all $k \geq 2$. Therefore for $k \geq 2$, $Y_k = X_k 1_{\{|X_k| \leq a\}}$ takes value 0 ($k^2 \times 0$) or (0×1) in both cases. In other words $Y_k = 0$ for all $k \geq 2$. Note that $Y_1 = X_1$. Therefore $\mathbb{E}[Y_1] = 1$ and $\mathbb{E}[Y_k] = 0$ for all $k \geq 2$.

(c) Determine $\mathbf{Var}[X_k 1_{\{|X_k| \leq a\}}]$. **Solution:** Since $Y_k = 0$ for all $k \geq 2$, $\mathbf{Var}(Y_k) = 0$ for all $k \geq 2$. Since $Y_1 = X_1$, $\mathbf{Var}(Y_1) = \mathbf{Var}(X_1) = 0$.

(d) Using these series, determine whether $\sum_{k=1}^{\infty} X_k$ converges a.s. or not. **Solution:** From the previous parts we know

i. $\sum_{k=1}^{\infty} P(|X_k| \geq a) = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$
ii. $\sum_{k=1}^{\infty} \mathbb{E}[X_k 1_{\{|X_k| \leq a\}}] = 1 < \infty$
iii. $\sum_{k=1}^{\infty} \mathbf{Var}[X_k 1_{\{|X_k| \leq a\}}] = 0 < \infty$

Therefore by the Three-Series theorem X_k converges a.s.

3. Problem 3.8 of Prof. Kim's notes.
Solution:
We are given that $\Theta \sim U[-\pi, \pi]$ and we need to find the pdf of $Y = \sin(\omega t + \Theta)$.
We know that $\sin^{-1} y = \{2k\pi + \arcsin y, (2k+1)\pi - \arcsin y, \forall k \in \mathbb{Z}\}$.
Since $\Theta \sim U[-\pi, \pi]$, $P(\sin(\omega t + \Theta) \leq y) = \frac{\pi + 2 \arcsin y}{2\pi}$ for $y \in [-1, 1]$.
Therefore the pdf of Y is given by $f_Y(y) = \frac{1}{\pi\sqrt{1-y^2}}$ for $y \in [-1, 1]$.

The pdf of Y is independent of ω and t.

4. Problem 5.7 of Prof. Kim's notes.
Solution: Mean of $Y(t)$:

$$\mathbb{E}[Y(t)] = \int_{-\pi}^{\pi} \frac{1}{2\pi} \sin(\omega t + \theta) d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin(\omega t + \theta) d\theta = 0.$$

Variance of $Y(t)$:

$$\mathbb{E}[Y(t)] = \int_{-\pi}^{\pi} \frac{1}{2\pi} \sin^2(\omega t + \theta) d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - \cos(2\omega t + 2\theta)}{2} d\theta = \frac{\pi}{2\pi} = \frac{1}{2}.$$

The mean and variance of Y are independent of ω and t.

5. Consider the function

$$f(\alpha) = p \log(1 + \alpha) + (1 - p) \log(1 - \alpha),$$

where p is a constant with $0.5 < p \leq 1$. Show that there exists an $\alpha^* \in [0, 1]$ such that $f(\alpha^*) > 0$.
Solution: The statement is clearly true when $p = 1$ since $f(\alpha) = \log(1 + \alpha) > 0$ for all $\alpha \in (0, 1]$.

Let $0.5 < p < 1$. $f(\alpha)$ is a differentiable function for $\alpha \in (0, 1)$. We have

$$f'(\alpha) = \frac{p}{1 + \alpha} - \frac{1 - p}{1 - \alpha} = \frac{2p - 1 - \alpha}{1 - \alpha^2}.$$

Therefore $f'(\alpha) = 0$ when $\alpha = 2p - 1 \in (0, 1)$.

Define $\alpha^* = 2p - 1$. We know that $f''(\alpha) = -\frac{p}{(1+\alpha)^2} - \frac{1-p}{(1-\alpha)^2} < 0$ for all $\alpha \in (0, 1)$. Therefore, $f(\alpha^*)$ is the maximum value of $f(\alpha)$. Since f is continuous over $[0, 1]$ and $f(0) = 0$ we have $f(\alpha^*) \geq f(0)$.

Note that if $f(2p - 1) = 0$ then by the mean value theorem there exists $c \in (0, 2p - 1)$ such that $f'(c) = 0$ which is impossible since $\alpha = 2p - 1$ is the only stationary point for $f(\alpha)$.