UNIVERSITY OF CALIFORNIA, SAN DIEGO
Electrical & Computer Engineering Department
ECE 250 - Winter Quarter 2019
Random Processes

Problem Set #7 **Due Sunday, March 8, 2020 at 2pm**
Submit solutions to Problems 5, 7 only

1. *Symmetric random walk.* Let X_n be a random walk defined by

$$\begin{aligned} X_0 &= 0, \\ X_n &= \sum_{i=1}^{n} Z_i, \end{aligned}$$

where $Z_1, Z_2, \ldots$ are i.i.d. with $\mathsf{P}\{Z_1 = -1\} = \mathsf{P}\{Z_1 = 1\} = \frac{1}{2}$.

   (a) Find $\mathsf{P}\{X_{10} = 10\}$.

   (b) Approximate $\mathsf{P}\{-10 \le X_{100} \le 10\}$ using the central limit theorem.

   (c) Find $\mathsf{P}\{X_n = k\}$.

2. *Absolute-value random walk.* Consider the symmetric random walk X_n in the previous problem. Define the absolute value random process $Y_n = |X_n|$.

   (a) Find $\mathsf{P}\{Y_n = k\}$.

   (b) Find $\mathsf{P}\{\max_{1\le i<20} Y_i = 10 \,|\, Y_{20} = 0\}$.

3. *Discrete-time Wiener process.* Let Z_n, $n \ge 0$, be a discrete time white Gaussian noise process, i.e., $Z_1, Z_2, \ldots$ are i.i.d $N(0,1)$. Define the process X_n, $n \ge 1$, such that $X_0 = 0$, and $X_n = X_{n-1} + Z_n$, for $n \ge 1$.

   (a) Is X_n an independent increment process? Justify your answer.

   (b) Is X_n a Gaussian process? Justify your answer.

   (c) Find the mean and autocorrelation functions of X_n.

   (d) Specify the first-order pdf of X_n.

   (e) Specify the joint pdf of X_3, X_5, and X_8.

   (f) Find $\mathsf{E}(X_{20}|X_1, X_2, \ldots, X_{10})$.

   (g) Given $X_1 = 4$, $X_2 = 2$, and $0 \le X_3 \le 4$, find the minimum MSE estimate of X_4.

4. *Wiener process.* Recall the following definition of the (standard) Wiener process:

   - $W(0) = 0$,
   - $\{W(t)\}$ is independent increment with $W(t) - W(s) \sim \mathrm{N}(0, t-s)$ for all $t > s$,

- $\mathsf{P}\{\omega : W(\omega, t) \text{ is continuous in } t\} = 1$.

Let $W_1(t)$ and $W_2(t)$ be independent Wiener processes.

(a) Find the mean and the variance of

$$X(t) = \frac{1}{\sqrt{2}}\big(W_1(t) + W_2(t)\big).$$

Is $\{X(t)\}$ a Wiener process? Justify your answer.

(b) Find the mean and the variance of

$$Y(t) = \frac{1}{\sqrt{2}}\big(W_1(t) - W_2(t)\big).$$

Is $\{Y(t)\}$ a Wiener process? Justify your answer.

(c) Find $\mathsf{E}[X(t)Y(s)]$.

5. *Autoregressive process.* Let $X_0 = 0$ and $X_n = \frac{1}{2}X_{n-1} + Z_n$ for $n \geq 1$, where $Z_1, Z_2, \ldots$ are i.i.d. $\sim \mathrm{N}(0,1)$. Find the mean and autocorrelation function of X_n.

6. *Moving average process.* Let $Z_0, Z_1, Z_2, \ldots$ be i.i.d. $\sim \mathcal{N}(0,1)$. Let $X_n = \frac{1}{2}Z_{n-1} + Z_n$ for $n \geq 1$.

(a) Find the mean and autocorrelation function of X_n.

(b) Find $\mathsf{E}(X_3|X_1, X_2)$.

(c) Find $\mathsf{E}(X_3|X_2)$.

(d) Is $\{X_n\}$ Markov? Justify your answer.

(e) Is $\{X_n\}$ independent increment? Justify your answer.

7. *QAM random process.* Consider the random process

$$X(t) = Z_1 \cos \omega t + Z_2 \sin \omega t\,, \quad -\infty < t < \infty\,,$$

where Z_1 and Z_2 are i.i.d. discrete random variables such that $p_{Z_i}(+1) = p_{Z_i}(-1) = \frac{1}{2}$.

(a) Is $X(t)$ wide-sense stationary? Justify your answer.

(b) Is $X(t)$ strict-sense stationary? Justify your answer.

8. *Convergence of random processes.* Let $\{N(t)\}_{t=0}^{\infty}$ be a Poisson process with rate λ. Recall that the process is independent increment and $N(t) - N(s)$, $0 \leq s < t$, has the pmf

$$p_{N(t)-N(s)}(n) = \frac{e^{-\lambda(t-s)}(\lambda(t-s))^n}{n!}, \quad n = 0, 1, \ldots.$$

Define

$$M(t) = \frac{N(t)}{t}, \quad t > 0.$$

(a) Find the mean and autocorrelation function of $\{M(t)\}_{t>0}$.

(b) Does $\{M(t)\}_{t>0}$ converge in mean square as $t \to \infty$, that is,

$$\lim_{t\to\infty} \mathsf{E}\big[(M(t) - M)^2\big] = 0$$

for some random variable (or constant) M? If so, what is the limit?

Now consider

$$L(t) = \frac{1}{t}\int_0^t \frac{N(s)}{s}\, ds, \quad t > 0.$$

(c) Does $\{L(t)\}_{t>0}$ converge in mean square as $t \to \infty$? If so, what is the limit?