UNIVERSITY OF CALIFORNIA, SAN DIEGO
Electrical & Computer Engineering Department
ECE 250 - Winter Quarter 2022
Random Processes

Problem Set #8 Due Friday, March 11, 2022 (Submit 1, 3, 4, 5)

1. *QAM random process.* Consider the random process

$$X(t) = Z_1 \cos \omega t + Z_2 \sin \omega t\,, \quad -\infty < t < \infty\,,$$

where Z_1 and Z_2 are i.i.d. discrete random variables such that $p_{Z_i}(+1) = p_{Z_i}(-1) = \frac{1}{2}$.

   (a) Is $X(t)$ wide-sense stationary? Justify your answer.

   (b) Is $X(t)$ strict-sense stationary? Justify your answer.

2. *Mixture of two WSS processes.* Let $X(t)$ and $Y(t)$ be two zero-mean WSS processes with autocorrelation functions $R_X(\tau)$ and $R_Y(\tau)$, respectively. Define the process

$$Z(t) = \begin{cases} X(t), & \text{with probability } \frac{1}{2} \\ Y(t), & \text{with probability } \frac{1}{2}. \end{cases}$$

Find the mean and autocorrelation functions for $Z(t)$. Is $Z(t)$ a WSS process? Justify your answer.

3. *Stationary Gauss–Markov process.* Let

$$X_0 \sim \mathrm{N}(0, \sigma^2)$$
$$X_n = \tfrac{1}{2} X_{n-1} + Z_n\,, \quad n \geq 1\,,$$

where $Z_1, Z_2, Z_3, \ldots$ are i.i.d. $\mathrm{N}(0,1)$ independent of X_0.

   (a) Find σ^2 such that X_n is stationary. Find the mean and autocorrelation functions of X_n.

   (b) (Difficult.) Consider the sample mean $S_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, $n \geq 1$. Show that S_n converges to the process mean in probability even though the sequence X_n is not i.i.d. (A stationary process for which the sample mean converges to the process mean is called *mean ergodic*.)

4. *AM modulation.* Consider the AM modulated random process

$$X(t) = A(t) \cos(2\pi t + \Theta)\,,$$

where the amplitude $A(t)$ is a zero-mean WSS process with autocorrelation function $R_A(\tau) = e^{-\frac{1}{2}|\tau|}$, the phase Θ is a $\mathrm{Unif}[0, 2\pi)$ random variable, and $A(t)$ and Θ are independent. Is $X(t)$ a WSS process? Justify your answer.

5. *LTI system with WSS process input.* Let $Y(t) = h(t) * X(t)$ and $Z(t) = X(t) - Y(t)$ as shown in the Figure 1.

   (a) Find $S_Z(f)$.

   (b) Find $E(Z^2(t))$.

   Your answers should be in terms of $S_X(f)$ and the transfer function $H(f) = \mathcal{F}[h(t)]$.

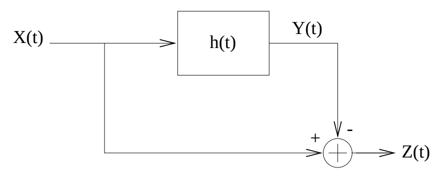

Figure 1: LTI system.

6. *Finding time of flight.* Finding the distance to an object is often done by sending a signal and measuring the time of flight, the time it takes for the signal to return (assuming speed of signal, e.g., light, is known). Let $X(t)$ be the signal sent and $Y(t) = X(t - \delta) + Z(t)$ be the signal received, where δ is the unknown time of flight. Assume that $X(t)$ and $Z(t)$ (the sensor noise) are uncorrelated zero mean WSS processes. The estimated crosscorrelation function of $Y(t)$ and $X(t)$, $R_{YX}(t)$ is shown in Figure 2. Find the time of flight δ.

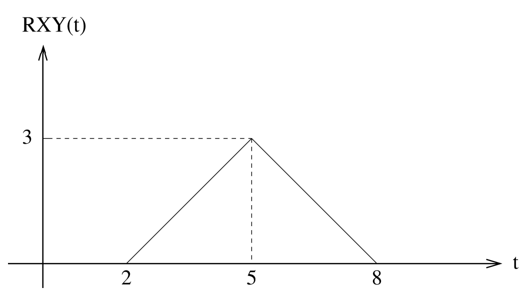

Figure 2: Crosscorrelation function.

7. *Finding impulse response of LTI system.* To find the impulse response $h(t)$ of an LTI system (e.g., a concert hall), i.e., to *identify* the system, white noise $X(t)$, $-\infty < t < \infty$, is applied to its input and the output $Y(t)$ is measured. Given the input and output sample functions, the crosscorrelation $R_{YX}(\tau)$ is estimated. Show how $R_{YX}(\tau)$ can be used to find $h(t)$.

8. *Generating a random process with a prescribed power spectral density.* In the lecture we showed that the psd $S_X(f)$ for a WSS process is real, even, and nonnegative. In this problem you will show that if a function $S(f)$ is real, even, and nonnegative (with $\int_{-\infty}^{\infty} S(f)df < \infty$) then it can be a psd for a WSS random process. Thus a function $S(f)$ is psd for some WSS iff it is even and nonnegative (with finite integral). Let $S(f) \geq 0$, for $-\infty < f < \infty$, be a real and even function such that

$$\int_{-\infty}^{\infty} S(f)df = 1.$$

Define the random process

$$X(t) = \cos(2\pi F t + \Theta),$$

where $F \sim S(f)$ and $\Theta \sim \mathrm{U}[-\pi, \pi)$ are independent. Find the power spectral density of $X(t)$. Interpret the result.

9. *Integrators.* Let $Y(t)$ be a short-term integration of a WSS process $X(t)$:

$$Y(t) = \frac{1}{T}\int_{t-T}^{t} X(u)du.$$

Find $S_Y(f)$ in terms of $S_X(f)$.