Homework 6

Due: 8:00pm (PT) **Tuesday**, Nov 22nd, 2022

1. Let $X \sim \mathcal{N}(0, \sigma^2)$ and $Y \sim \mathcal{N}(0, \nu^2)$ be independent random variables and $Z = X + Y$.
   (a) Find the LMMSE estimator of X given Z.
   (b) Find the conditional distribution $f_{Z|X}(z \mid x)$.
   (c) Using the previous part, find the conditional distribution $f_{X|Z}(x|z)$.
   (d) Find the MMSE estimator of X given Z and show that it is the same as the LMMSE estimator in Part (a).

2. Let $X, Y \sim U(0, 1)$ be independent random variables uniformly distributed over $(0, 1)$ and $Z = X + \frac{1}{2}Y$.
   (a) Find the LMMSE estimator of X given Z.
   (b) Find the MMSE estimator of X given Z. Is LMMSE and MMSE estimator the same?

3. Consider an additive noise channel $Y = X + Z$, where the signal $X \sim \mathcal{N}(0, \sigma^2)$ and the noise Z has zero mean and variance ν^2. Assume X and Z are independent. Find a distribution of Z that maximizes the minimum MSE of estimating X given Y, i.e., the distribution of the worst noise Z that has the given mean and variance. Justify your answer.

4. Using the definition of conditional expectation using the projection, show that for any variables $Y_1, \ldots, Y_k, Z \in L_2(\Omega, \mathcal{F}, \mathbf{P}(\cdot))$ and any (measurable) function $h : \mathbb{R}^k \to \mathbb{R}$,

$$\mathbb{E}[Zh(Y_1, \ldots, Y_k) \mid Y_1, \ldots, Y_k] = \mathbb{E}[Z \mid Y_1, \ldots, Y_k]h(Y_1, \ldots, Y_k).$$

   This is called the product rule for conditional expectation.

5. Consider a discrete time finite state Markov chain $\{X_k\}$ on three states S_1, S_2, S_3 with the transition matrix

$$P = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

   (a) Find a stationary distribution π for P. Is it unique?
   (b) Is this Markov chain aperiodic? Is it irreducible?
   (c) Suppose that $x_0 = (\mathbf{P}(X_0 = S_1), \mathbf{P}(X_0 = S_2), \mathbf{P}(X_0 = S_3))^T = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^T$ is the probability mass function at time 0. What is the probability that by time $T = 4$ we visit state S_3?
   (d) What is the probability that we visit state S_2 or S_3 by time $T = 4$?