1. (a) suppose $A^+$ is not unique. There exists $A_1^+$, $A_2^+$, such that: $A = AA_1^+A = AA_2^+A$

$$AA_1^+ = AA_2^+AA_1^+$$
$$= (A_2^+A)^T(A_1^+A)^T$$
$$= (A_2^+)^TA^T(A_1^+)^TA^T$$
$$= (A_2^+)^TA^T = (A_2^+A)^T$$
$$= AA_2^+$$

$$A_1^+A = A_1^+AA_2^+A$$
$$= A^T(A_1^+)^TA^T(A_2^+)^T$$
$$= (AA_1^+A)^T$$
$$= A^T(A_2^+)^T = (AA_2^+)^T$$
$$= A_2^+A$$

$$\because A_1^+ = A_1^+AA_1^+ = A_1^+AA_2^+ = A_2^+AA_2^+ = A_2^+$$

$\therefore$ $A^+$ is unique

(2) Suppose $(A^TA)^{-1}A^T = A^+$

$$A = AA^+A = A(A^TA)^{-1}A^TA = AI = A$$

$$A^+ = A^+AA^+$$
$$= (A^TA)^{-1}A^TA\,(A^TA)^{-1}A^T$$
$$= I\,(A^TA)^{-1}A^T = (A^TA)^{-1}A^T$$

$$A^+A = (A^TA)^{-1}A^TA = I$$

$\therefore$ $(A^TA)^{-1}A^T$ is the pseudoinverse and a left inverse of $A$.

(c) suppose $A^T(AA^T)^{-1} = A^+$

$AA^+ = AA^T(AA^T)^{-1} = I$

$AA^+A = IA = A$

$A^+AA^+ = A^+I = A^+$

$\therefore A^T(AA^T)^{-1}$ is the pseudoinverse and a right inverse of $A$.

(d) suppose $A^{-1} = A^+$

$AA^+A = AA^{-1}A = A$

$A^+AA^+ = A^{-1}AA^{-1} = A^{-1} = A^+$

$\therefore A^{-1}$ is the pesudoinverse of a full rank square matrix $A$.

(e) since $A$ is a projection matrix,

$A^2 = A$, $A^T = A$

suppose $A = A^+$,

$AA^+A = AAA = A^2A = AA = A^2 = A$

$A^+AA^+ = AAA = A = A^+$

$\therefore A$ is the pesudoinverse of a projection matrix $A$.

(f) $A^T = [AA^+A]^T = A^T(A^+)^TA^T$

$(A^+)^T = [A^+AA^+]^T = (A^+)^TA^T(A^+)^T$

$[A^T(A^+)^T]^T = [A^T(A^+)^TA^T(A^+)^T]^T = A^+AA^+A = A^+A$

$[(A^+)^TA^T]^T = [(A^+)^TA^T(A^+)^TA^T]^T = AA^+AA^+ = AA^+$

$\therefore (A^T)^+ = (A^+)^T$

(g) $AA^T = AA^+AA^T = AA^+AA^+AA^T = A(A^+A)^TA^+AA^T$
$= AA^T(A^+)^TA^+AA^T$

$(A^+)^T(A^+) = (A^+)^TA^+AA^+AA^+ = (A^+)^TA^+A(A^+A)^TA^+$
$= (A^+)^TA^+(AA^T)(A^+)^TA^+$

$AA^T(A^+)^TA^+ = A(A^+A)^TA^+ = A(A^+A)A^+ = AA^+$

$[(A^+)^TA^+]AA^T = (A^+)^T(A^+A)^TA^T = (A^+AA^+)^TA^T = AA^+$

$\therefore (AA^T)^+ = (A^+)^TA^+$

$A^TA = A^TAA^+AA^+A = A^TAA^+(AA^+)^TA = A^TAA^+(A^+)^TA^TA$

$A^+(A^+)^T = A^+AA^+AA^+(A^+)^T = A^+(A^+)^TA^TAA^+(A^+)^T$

$A^+(A^+)^TA^TA = A^+AA^+A = A^+A$

$A^TAA^+(A^+)^T = (A^+AA^+A)^T = (A^+A)^T = AA^+$

$\therefore (A^TA)^+ = A^+(A^+)^T$

(f) Let $x \in R(A^+)$

$A^+y = x$

$A^+AA^+y = x$

$A^T(A^+)^TA^+y = x$

$\Rightarrow x \in R(A^T)$

Let $x \in R(A^T)$

$A^Ty = x$

$A^T(A^T)^+A^Ty = x$

$[(A^T)^+]^TAA^Ty = x$

$[(A^+)^T]^TAA^Ty = x$

$A^+AA^Ty = x$

$\Rightarrow x \in R(A^+)$

$\Rightarrow R(A^+) = R(A^T)$

(i) $\because$ both $AA^+$ and $A^+A$ are symmetric

$P^2 = AA^+AA^+ = AA^+$

$Q^2 = A^+AA^+A = A^+A$

$\therefore$ $P$ and $Q$ are projection matrix.

(j) Let $w \in R(A)$, $w = Ab$

$\langle x - Px, w \rangle$

$= w^T x - w^T P x$

$= w^T x - w^T AA^+ x$

$= b^T A^T x - b^T A^T AA^+ x$

$= b^T A^T x - b^T A^T (A^+)^T A^T x$

$= b^T A^T x - b^T A^T (A^T)^+ A^T x$

$= b^T A^T x - b^T A^T x = 0$

$\therefore$ $y = Px$ is the projection of $x$ onto $R(A)$.

Let $w \in R(A^T)$, $w = A^T b$

$\langle x - Qx, w \rangle$

$= w^T x - w^T Q x$

$= b^T A x - b^T A A^+ A x$

$= b^T A x - b^T A x = 0$

$\therefore$ $y = Qx$ is the projection of $x$ onto $R(A^T)$

(k) $\|Ax-b\|^2 = \|Ax - Ax^* + Ax^* - b\|^2$

$= \|Ax - Ax^*\|^2 + \|Ax^* - b\|^2$

$+ \langle A(x-x^*), Ax^*-b\rangle + \langle Ax^*-b, A(x-x^*)\rangle$

$\because \langle A(x-x^*), Ax^*-b\rangle + \langle Ax^*-b, A(x-x^*)\rangle$

$= 2(x-x^*)^T A^T (Ax^*-b)$

$= 2(x-x^*)^T (Ax^*-b)$

$= 2(x-x^*)^T (A^T(A^+)^T A^T b - A^T b)$

$= 0$

$\therefore \|Ax-b\|^2 = \|Ax - Ax^*\|^2 + \|Ax^*-b\|^2 \geq \|Ax^*-b\|^2$

(l) $\|x\| = \|x - x^* + x^*\| = \langle x-x^*, x^*\rangle + \langle x^*, x-x^*\rangle$

$+ \|x-x^*\| + \|x^*\|$

$\langle x-x^*, x^*\rangle + \langle x^*, x-x^*\rangle = 2(x^*)^T(x-x^*)$

$= 2b^T(A^+)^T(x-A^+b)$

$= 2b^T(A^T)^+(x-A^+b)$

$= 2b^T(A^T)^+A^T(A^T)^+(x-A^+b)$

$= 2b^T(A^T)^+[A^T(A^T)^+x - A^T(A^T)^+A^+b]$

$= 2b^T(A^T)^+(A^+Ax - A^+AA^+b)$

$= 2b^T(A^T)^+(A^+b - A^+b)$

$= 0$

$\therefore \|x\| = \|x-x^*\| + \|x^*\| \geq \|x^*\|$

Problem 2

(a) the characteristic polynomial of A:

$$p(\lambda) = \det(\lambda I - A) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$$

$$\because p(\lambda = 0) = \det(-A) = (-\lambda_1)(-\lambda_2)\cdots(-\lambda_n)$$

$$= (-1)^n \lambda_1 \lambda_2 \cdots \lambda_n$$

$$= (-1)^n \det(A)$$

$$\Rightarrow \det(A) = \lambda_1 \lambda_2 \cdots \lambda_n.$$

(b) $\because \lambda I - A^T = (\lambda I - A)^T$

$$\det((\lambda I - A)^T) = \det(\lambda I - A) = \det(A^T - \lambda I)$$

so $A^T$ and $A$ have the same set of $\lambda$.

(c) $Av = \lambda_i v$. $i = 1, 2, \cdots, n$

$$A\,Av = A\lambda_i v = \lambda_i Av = \lambda_i^2 v$$

$$A^2 Av = A^2 \lambda_i v = \lambda_i A\,Av = \lambda A \lambda_i v = \lambda_i^2 Av = \lambda_i^3 v$$

$$\vdots$$

$$A^k v = \lambda_i^k v$$

$\therefore \lambda_i^k$, $i = 1, 2, \cdots, n$ are eigenvalues of matrix $A^k$

(d) suppose A has a zero eigenvalue $\lambda$

$\therefore Av = \lambda v = 0$ , $v \neq 0$

$\because v \in N(A)$ , A is invertible

$\therefore \dim(N(A)) = 0 \Rightarrow N(A) = 0 \Rightarrow v = 0$

$\therefore$ if A is invertible, it does not have a zero eigenvalue

(d) If $A$ does not have a zero eigenvalue, there is no $v \neq 0$, s.t

$$Av = \lambda v = 0$$

$\therefore$ $v = 0$ is the only solution

$\Rightarrow \dim(N(A)) = 0$

$\therefore$ $A$ is invertible iff $A$ does not have a zero eigenvalue

(e) $Av = \lambda_i v \Rightarrow A^{-1}Av = \lambda_i A^{-1} v$

$\Rightarrow v = \lambda_i A^{-1} v$

$\Rightarrow \lambda_i^{-1} v = A^{-1} v$

$\therefore$ $\lambda_i^{-1}$, $i = 1, 2, \cdots, n$ are the eigenvalues of $A^{-1}$.

## Problem 3

a) Let $\lambda$ be an eigenvalue of $A$. and $v$ is the eigen vector, $v \neq 0$

$$Av = \lambda v$$

$$A^2 v = \lambda (Av) = \lambda^2 v$$

$\because$ $A$ is nilpotent matrix

$\therefore$ $A^k = 0$ for som $k$

$\Rightarrow \quad A^k v = \lambda^k v = 0$

$\Rightarrow \quad \lambda^k = 0$

$\Rightarrow \lambda = 0$ for all $\lambda$

(a) $\because$ all eigenvalues of a nilpotent matrix must be zero

(b) suppose $A^k = 0,\ k > n$

$\therefore A^{n+1} = A^{n+2} = \cdots = 0$

the characteristic polynomial of $A$

$$P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

By Cayley Hamilton

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 = 0$$

$$A(A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0) = 0$$

$$a_{n-1}A^n + \cdots + a_1 A^2 + a_0 A = 0$$

$$A(a_{n-1}A^n + \cdots + a_1 A^2 + a_0 A) = 0$$

$$a_{n-2}A^n + \cdots + a_1 A^3 + a_0 A^2 = 0$$

$$\vdots$$

$$\therefore a_0 A^n = 0$$

$$\Rightarrow A^n = 0$$

$$\therefore A^k = 0,\ k \leq n$$

Problem 4

a) $\|A^H A\| = \|A^H\| \cdot \|A\| = \|A\| \cdot \|A\| = \|A\|^2$

b) since $U$ and $V$ are unitarily matrice,

$\|UA\| = \|A\| = \|AU\|$

Let $B = AV$

$\therefore \|UAV\| = \|UB\|$

$= \|B\|$

$= \|AV\|$

$= \|A\|$

P5. (a) $A \geq 0$, $B \geq 0$

Let $B = \sum_{i=1}^{n} \lambda_i v_i v_i^T$

$\therefore$ $\mathrm{Trace}(AB) = \mathrm{Trace}(A \sum_{i=1}^{n} \lambda_i v_i v_i^T)$

$$= \sum_{i=1}^{n} \lambda_i \, \mathrm{Trace}(A v_i v_i^T)$$

$$= \sum_{i=1}^{n} \lambda_i v_i^T A v_i \quad \geqslant 0$$

(b)