FALL 2021 Linear Algebra Practice Problems

We highly recommend you solving these problems by yourself first before checking the solutions, as this will be a good practice to prepare Quiz2.

1) Let $\mathcal{M}$ and $\mathcal{N}$ be subspaces of a vector space $\mathbb{C}^n$, and consider the associated orthogonal projectors $\mathbf{P}_{\mathcal{M}}$ and $\mathbf{P}_{\mathcal{N}}$.(Recall that in Homework 3, we define the orthogonal projection matrix as the symmetric matrix $\mathbf{P}^T = \mathbf{P} \in \mathbb{R}^{n\times n}$ satisfying $\mathbf{P}^2 = \mathbf{P}$)
   - a). Prove that $\mathbf{P}_{\mathcal{M}}\mathbf{P}_{\mathcal{N}} = \mathbf{0}$ if and only if $\mathcal{M} \perp \mathcal{N}$.
   - b). Is it true that $\mathbf{P}_{\mathcal{M}}\mathbf{P}_{\mathcal{N}} = \mathbf{0}$ if and only if $\mathbf{P}_{\mathcal{N}}\mathbf{P}_{\mathcal{M}} = \mathbf{0}$? Justify
   - c). Show $R(\mathbf{P}_{\mathcal{M}} + \mathbf{P}_{\mathcal{N}}) = R(\mathbf{P}_{\mathcal{M}}) + R(\mathbf{P}_{\mathcal{N}})$

   Solution 1: a).

$$\begin{aligned} & \mathbf{P}_{\mathcal{M}}\mathbf{P}_{\mathcal{N}} = 0 && (1)\\ & \Leftrightarrow \mathbf{P}_{\mathcal{M}}{}^T\mathbf{P}_{\mathcal{N}} = 0 \ (\mathbf{P}_{\mathcal{N}}{}^T\mathbf{P}_{\mathcal{M}} = 0) && (2)\\ & \Leftrightarrow \text{each column of } \mathbf{P}_{\mathcal{M}} \text{ is orthogonal to the columns of } \mathbf{P}_{\mathcal{N}} && (3)\\ & \Leftrightarrow R(\mathbf{P}_{\mathcal{N}}) \perp R(\mathbf{P}_{\mathcal{M}}) && (4)\\ & \Leftrightarrow \mathcal{M} \perp \mathcal{N} && (5) \end{aligned}$$

   b). Yes. This is a direct consequence of part (a). Alternatively, you could say

$$\mathbf{0} = \mathbf{P}_{\mathcal{M}}\mathbf{P}_{\mathcal{N}} \Leftrightarrow \mathbf{0} = (\mathbf{P}_{\mathcal{M}}\mathbf{P}_{\mathcal{N}})^T = \mathbf{P}_{\mathcal{N}}^T\mathbf{P}_{\mathcal{M}}^T = \mathbf{P}_{\mathcal{N}}\mathbf{P}_{\mathcal{M}} \quad (6)$$

   c).

$$\begin{aligned} [\mathbf{P}_{\mathcal{M}}\ \mathbf{P}_{\mathcal{N}}][\mathbf{P}_{\mathcal{M}}\ \mathbf{P}_{\mathcal{N}}]^T &= [\mathbf{P}_{\mathcal{M}}\ \mathbf{P}_{\mathcal{N}}]\begin{bmatrix}\mathbf{P}_{\mathcal{M}}^T\\ \mathbf{P}_{\mathcal{N}}^T\end{bmatrix} && (7)\\ &= [\mathbf{P}_{\mathcal{M}}\ \mathbf{P}_{\mathcal{N}}]\begin{bmatrix}\mathbf{P}_{\mathcal{M}}\\ \mathbf{P}_{\mathcal{N}}\end{bmatrix} && (8)\\ &= \mathbf{P}_{\mathcal{M}}^2 + \mathbf{P}_{\mathcal{N}}^2 = \mathbf{P}_{\mathcal{M}} + \mathbf{P}_{\mathcal{N}} && (9) \end{aligned}$$

   so that

$$\begin{aligned} R(\mathbf{P}_{\mathcal{M}} + \mathbf{P}_{\mathcal{N}}) &= R([\mathbf{P}_{\mathcal{M}}\ \mathbf{P}_{\mathcal{N}}][\mathbf{P}_{\mathcal{M}}\ \mathbf{P}_{\mathcal{N}}]^T) && (10)\\ &= R([\mathbf{P}_{\mathcal{M}}\ \mathbf{P}_{\mathcal{N}}]) = R(\mathbf{P}_{\mathcal{M}}) + R(\mathbf{P}_{\mathcal{N}}) && (11) \end{aligned}$$

2) Suppose we are given a set of orthonormal basis vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N\}$ in $\mathbb{C}^N$
   - a). Let $\mathbf{x} \in \mathcal{U}$, we can find a unique representation of $\mathbf{x} = \sum_{i=1}^N \alpha_i \mathbf{u}_i$. Prove

$$||\mathbf{x}||_2^2 = \sum_{i=1}^N |\alpha_i|^2 \quad (12)$$

     (Note: this is known as Parseval's identity)
   - b). Suppose you have a subset of orthonormal vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_s\}$ (where $s < N$) from the given basis. Show that any vector $\mathbf{v} \in \mathcal{U}$ satisfies

$$||\mathbf{v}||_2^2 \geq \sum_{i=1}^s |\langle\, \mathbf{v}, \mathbf{u}_i\rangle|^2 \quad (13)$$

     (Hint: Similar to Problem 1 in Homework 3, we can define an orthogonal complement of the subspace spanned by $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_s\}$)

Solution 2: a). Because $\{\mathbf{u}_1, \dots, \mathbf{u}_N\}$ is a set of orthonormal vectors, we have $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 1$ when $i = j$, otherwise the inner product is 0.

$$
\begin{aligned}
||\mathbf{x}||_2^2 &= \langle \sum_{i=1}^{N} \alpha_i \mathbf{u}_i, \sum_{i=1}^{N} \alpha_i \mathbf{u}_i \rangle && (14)\\
&= \langle \alpha_1 \mathbf{u}_1, \sum_{i=1}^{N} \alpha_i \mathbf{u}_i \rangle + \dots + \langle \alpha_N \mathbf{u}_N, \sum_{i=1}^{N} \alpha_i \mathbf{u}_i \rangle && (15)\\
&= \alpha_1 \langle \mathbf{u}_1, \sum_{i=1}^{N} \alpha_i \mathbf{u}_i \rangle + \dots + \alpha_N \langle \mathbf{u}_N, \sum_{i=1}^{N} \alpha_i \mathbf{u}_i \rangle && (16)\\
&= \alpha_1 \overline{\langle \sum_{i=1}^{N} \alpha_i \mathbf{u}_i, \mathbf{u}_1 \rangle} + \dots + \alpha_N \overline{\langle \sum_{i=1}^{N} \alpha_i \mathbf{u}_i, \mathbf{u}_N \rangle} && (17)\\
&= \alpha_1 \Big(\overline{\langle \alpha_1 \mathbf{u}_1, \mathbf{u}_1 \rangle} + \dots + \overline{\langle \alpha_N \mathbf{u}_N, \mathbf{u}_1 \rangle} \Big) + \dots + \alpha_n \Big(\overline{\langle \alpha_1 \mathbf{u}_1, \mathbf{u}_N \rangle} + \dots + \overline{\langle \alpha_N \mathbf{u}_N, \mathbf{u}_N \rangle} \Big) && (18)\\
&= \alpha_1 \Big(\bar{\alpha_1} \langle \mathbf{u}_1, \mathbf{u}_1 \rangle + \dots \bar{\alpha_N} \langle \mathbf{u}_1, \mathbf{u}_N \rangle \Big) + \dots + \alpha_N \Big(\bar{\alpha_1} \langle \mathbf{u}_N, \mathbf{u}_1 \rangle + \dots \bar{\alpha_N} \langle \mathbf{u}_N, \mathbf{u}_N \rangle \Big) && (19)\\
&= \sum_{i=1}^{N} |\alpha_i|^2 && (20)
\end{aligned}
$$

b). From a), we know that $||\mathbf{v}||_2^2 = \sum_{i=1}^{N} |\alpha_i|^2$. Then we have

$$||\mathbf{v}||_2^2 = \sum_{i=1}^{N} |\alpha_i|^2 \geq \sum_{i=1}^{s} |\alpha_i|^2 \tag{21}$$

We learned in Lecture 7 that the coefficients of orthonormal basis $\mathbf{u}_i$ can be obtained by the following formula

$$\alpha_i = \frac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{||\mathbf{u}_i||_2^2} = \langle \mathbf{v}, \mathbf{u}_i \rangle \tag{22}$$

so that

$$||\mathbf{v}||_2^2 \geq \sum_{i=1}^{s} |\alpha_i|^2 \geq \sum_{i=1}^{s} |\langle \mathbf{v}, \mathbf{u}_i \rangle|^2 \tag{23}$$

3) Let $\mathbf{A} \in \mathbb{C}^{m \times n}$. Suppose $\mathbf{A}^H \mathbf{A} = \mathbf{A}\mathbf{A}^H$. Show that $R(\mathbf{A}) \perp N(\mathbf{A})$, i.e for all $\mathbf{x} \in R(\mathbf{A})$, $\mathbf{y} \in N(\mathbf{A})$, $\mathbf{x}^H \mathbf{y} = 0$
Solution: In Discussion 5, we showed that

$$N(\mathbf{A}) = N(\mathbf{A^H A}) \tag{24}$$

Now we will show $R(\mathbf{A}) = R(\mathbf{A A^H})$. We have

$$R(\mathbf{A A^H}) \subset R(\mathbf{A}) \tag{25}$$

because

$$\mathbf{y} \in R(\mathbf{A A^H}) \implies \mathbf{y} = \mathbf{A A^H x} = \mathbf{A}(\mathbf{A^H x}) \implies \mathbf{y} \in R(\mathbf{A})$$

The Eq (24) implies $N(\mathbf{A}^H) = N(\mathbf{A A^H})$. Then from rank-nullity theorem

$$dim(R(\mathbf{A})) = dim(R(\mathbf{A}^H)) = m - N(\mathbf{A^H}) = m - dim(N(\mathbf{A A^H})) = dim(R(\mathbf{A A^H}))$$

Together with Eq (25), $dim(R(\mathbf{A})) = dim(R(\mathbf{AA^H}))$ shows $R(\mathbf{A}) = R(\mathbf{AA^H})$. Now let $\mathbf{x} \in R(\mathbf{A})$, $\mathbf{y} \in N(\mathbf{A})$. We know $\mathbf{x} \in R(\mathbf{AA^H})$ and there exists $\mathbf{z}$ such that $\mathbf{x} = \mathbf{AA^H z}$. Now

$$\mathbf{x}^H\mathbf{y} = (\mathbf{AA^H z})^H\mathbf{y} = \mathbf{z}^H\mathbf{A^H A y} = \mathbf{z}^H\mathbf{A^H A y} = 0$$

The last step is due to the fact that $\mathbf{y} \in N(\mathbf{A}^H\mathbf{A})$. Hence we are done.

4) **Householder Reflections:** A Householder matrix is defined as

$$\mathbf{Q} = \mathbf{I} - \mathbf{2uu^T}$$

for a unit vector $\mathbf{u} \in \mathbb{R}^n$

a) Show that $\mathbf{Q}$ is orthogonal.
b) Show that $\mathbf{Qu} = -\mathbf{u}$ and that $\mathbf{Qv} = \mathbf{v}$ for every $\mathbf{v} \perp \mathbf{u}$. Thus, the linear transformation $\mathbf{y} = \mathbf{Qx}$ reflects $\mathbf{x}$ through the hyperplane with normal vector $\mathbf{u}$.
c) Given $\mathbf{y}$, find $\mathbf{x}$ such that $\mathbf{y} = \mathbf{Qx}$.
d) Given nonzero vectors $\mathbf{x}$ and $\mathbf{y}$, find a unit vector u such that $(\mathbf{I} - \mathbf{2uu^T})\mathbf{x} \in \text{span}(\mathbf{y})$, in terms of $\mathbf{x}$ and $\mathbf{y}$.

Solution: $\mathbf{Q} = \mathbf{I} - 2\mathbf{uu}^T$
For unit vector $\mathbf{u} \in \mathbb{R}^n$. Notice that $\mathbf{Q} \in \mathbb{R}^{n\times n}$.
Also, notice that if a matrix $\mathbf{Q} \in \mathbb{R}^{n\times n}$ is orthogonal then by definition $\mathbf{Q}^T\mathbf{Q} = \mathbf{QQ}^T = \mathbf{I}$.

(a) $\mathbf{QQ}^T = (\mathbf{I} - 2\mathbf{uu}^T)(\mathbf{I} - 2\mathbf{uu}^T)^T = (\mathbf{I} - 2\mathbf{uu}^T)(\mathbf{I} - 2\mathbf{uu}^T) = \mathbf{I} - 2\mathbf{uu}^T - 2\mathbf{uu}^T + 4\mathbf{uu}^T\mathbf{uu}^T = \mathbf{I} - 4\mathbf{uu}^T + 4\mathbf{uu}^T = \mathbf{I}$.
Similarly $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$

(b) For unit vector $\mathbf{u} \in \mathbb{R}^n$:
$\mathbf{Qu} = (\mathbf{I} - 2\mathbf{uu}^T)\mathbf{u} = \mathbf{u} - 2\mathbf{uu}^T\mathbf{u} = \mathbf{u} - 2\mathbf{u} = -\mathbf{u}$.
For any $\mathbf{u} \perp \mathbf{v}$:
$\mathbf{Qv} = (\mathbf{I} - 2\mathbf{uu}^T)\mathbf{v} = \mathbf{v} - 2\mathbf{uu}^T\mathbf{v} = \mathbf{v} - 0 = \mathbf{v}$.
Any $\mathbf{x} \in \mathbb{R}^n$ can be written as $\mathbf{x} = (\mathbf{u}^T\mathbf{x})\mathbf{u}+\mathbf{v}$ for some $\mathbf{v} \in \mathbb{R}^n$, where $\mathbf{u}^T\mathbf{v} = \mathbf{u}^T(\mathbf{x}-(\mathbf{u}^T\mathbf{x})\mathbf{u}) = 0$, meaning $\mathbf{v} \perp \mathbf{u}$. Thus, $\mathbf{Qx} = -(\mathbf{u}^T\mathbf{x})\mathbf{u} + \mathbf{v}$ which can be interpreted as the reflection of $\mathbf{x}$ through a hyperplane with the normal vector $\mathbf{u}$.

(c) Since $\mathbf{Q}$ is symmetric and orthogonal, $\mathbf{Q}^{-1} = \mathbf{Q}^T = \mathbf{Q}$. Hence, $\mathbf{x} = \mathbf{Q}^{-1}\mathbf{y} = \mathbf{Qy}$ which is the reflection back from $\mathbf{y}$.

(d) The question is asking to find the vector $\mathbf{u}$ such that $\mathbf{x}$ and $\alpha\mathbf{y}$ for some α are reflections of each other through the hyperplane $\{\mathbf{z} : \mathbf{u}^T\mathbf{z} = 0\}$. Since $\mathbf{Qx}$ is given to be in the span of $\mathbf{y}$ we can write $\mathbf{Qx} = \alpha\mathbf{y}$ for some constant α, or $\|\mathbf{x}\| = \|\mathbf{Qx}\| = \|\alpha\mathbf{y}\|$, which implies that $\alpha = \pm\frac{\|\mathbf{x}\|}{\|\mathbf{y}\|}$.
We consider the case $\mathbf{Qx} = (\frac{\|\mathbf{x}\|}{\|\mathbf{y}\|})\mathbf{y} = \bar{\mathbf{x}}$, but the other case works equally well and provides an alternative answer. Now note that if the hyperplane $\{\mathbf{z} : \mathbf{u}^T\mathbf{z} = 0\}$ reflects $\mathbf{x}$ to $\bar{\mathbf{x}}$ and vice versa, then $\mathbf{x} - \bar{\mathbf{x}}$ is normal to the hyperplane. To see this $\mathbf{x} - \bar{\mathbf{x}} = (\mathbf{I} - \mathbf{Q})\mathbf{x} = 2\mathbf{u^T x u}$.
Suppose $\mathbf{x} \notin \textbf{span}(\mathbf{y})$ so $\mathbf{x} - \bar{\mathbf{x}} \neq 0$(If that is not the case, then we can not represent $\mathbf{u}$ in term of $\mathbf{x}$ and $\mathbf{y}$). So:
$\mathbf{u} = \pm\frac{\mathbf{x}-\bar{\mathbf{x}}}{\|\mathbf{x}-\bar{\mathbf{x}}\|} = \pm\frac{\mathbf{x}-(\frac{\|\mathbf{x}\|}{\|\mathbf{y}\|})\mathbf{y}}{\|\mathbf{x}-(\frac{\|\mathbf{x}\|}{\|\mathbf{y}\|})\mathbf{y}\|} = \pm\frac{\|\mathbf{y}\|\mathbf{x}-\|\mathbf{x}\|\mathbf{y}}{\|\|\mathbf{y}\|\mathbf{x}-\|\mathbf{x}\|\mathbf{y}\|}$
which is the desired unit normal vector.
Alternatively, if we take $-\ \mathbf{x}$ as the reflection of $\mathbf{x}$, then $\mathbf{u} = \pm\frac{\mathbf{x}+\bar{\mathbf{x}}}{\|\mathbf{x}+\bar{\mathbf{x}}\|} = \pm\frac{\|\mathbf{y}\|\mathbf{x}+\|\mathbf{x}\|\mathbf{y}}{\|\|\mathbf{y}\|\mathbf{x}+\|\mathbf{x}\|\mathbf{y}\|}$.

Challenge: We provided some hints to help you understand the following two problems. Please make sure to complete the remaining steps by yourself.

5) Consider a system whose input $x(n)$ and output $y(n)$ are related by:

$$y(n) = \sum_{k=0}^{L-1} h(k)x(n-k), \text{ where } n = 0, 1, 2, \dots \tag{26}$$

Here $h(n)$ is called the impulse response of the system. Suppose you are given an input signal $\bar{x}(n)$ (non-zero for all n) and are able to observe the output $\bar{y}(n)$ (for all n) contaminated with noise $w(n)$, i.e., you observe

$$\bar{y}(n) = \sum_{k=0}^{L-1} h(k)\bar{x}(n-k) + w(n) \text{ where } n = 0, 1, 2, \dots \tag{27}$$

Using the idea of orthogonal projection, describe a method to estimate the impulse response $h(n)$ using $\bar{y}(n)$ and $\bar{x}(n)$. In the absense of noise, under what conditions can you exactly identify $h(n)$? Justify your answer.

Hints: (i). **When there exists noise in the system**: Given $\bar{y}(n)$ and $\bar{x}(n)$, your goal is to find $h(0), \dots, h(L-1)$. Suppose we observe output sequence y from time index n_1 to n_2, this problem can be cast as finding $\mathbf{h} \in \mathbb{R}^L$ using

$$\min_{\mathbf{h}\in\mathbb{R}^L} \sum_{n=n_1}^{n_2} \Big(\bar{y}(n) - \sum_{k=0}^{L-1} h(k)\bar{x}(n-k)\Big)^2 \tag{28}$$

$$= \min_{\mathbf{h}\in\mathbb{R}^L} \Bigg\| \begin{bmatrix} \bar{y}(n_1) \\ \vdots \\ \bar{y}(n_2) \end{bmatrix} - \begin{bmatrix} \bar{x}(n_1) & \dots & \bar{x}(n_1 - L + 1) \\ \vdots & \ddots & \vdots \\ \bar{x}(n_2) & \dots & \bar{x}(n_2 - L + 1) \end{bmatrix} \begin{bmatrix} \bar{h}(0) \\ \vdots \\ \bar{h}(L-1) \end{bmatrix} \Bigg\|_2^2 \tag{29}$$

$$= \min_{\mathbf{h}\in\mathbb{R}^L} ||\bar{\mathbf{y}} - \bar{\mathbf{X}}\mathbf{h}||_2^2 \tag{30}$$

such that the impulse response $h(n)$ can be estimated from the normal equation

$$\bar{\mathbf{X}}^{\mathbf{T}}\mathbf{y} = (\bar{\mathbf{X}}^{\mathbf{T}}\bar{\mathbf{X}})\mathbf{h} \tag{31}$$

(ii). When the system is noise-free: In this case, the design of your input $\bar{x}(n)$ is crucial. Under what condition will the system become identifiable? Try to come up with a necessary condition based on matrix $\bar{\mathbf{X}}$.

6) a). Given $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{c} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$, consider the following problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} ||\mathbf{x} + \mathbf{c}||_2 \tag{32}$$

$$s.t.\ \mathbf{A}\mathbf{x} = \mathbf{b} \tag{33}$$

Cast it as an orthogonal projection problem. Identify the subspace you are projecting on? What is the point being projected?

Hints: i). Case 1: If $\mathbf{b} \notin R(\mathbf{A})$, this optimization problem cannot be solved.

ii). Case 2: If $\mathbf{b} \in R(\mathbf{A})$, $rank(\mathbf{A}) = n$, only one point in the set $\{\mathbf{x}|\mathbf{A}\mathbf{x} = \mathbf{b}\}$ can be the solution, that is, $\mathbf{x} = \mathbf{B}\mathbf{b}$, where $\mathbf{B}$ is the left inverse of matrix $\mathbf{A}$. Such that the orthogonal projection problem can be cast as

$$\min_{\mathbf{x}\in\mathbb{R}^n} ||\mathbf{x} + \mathbf{c}||_2 = ||\mathbf{B}\mathbf{b} + \mathbf{c}||_\mathbf{2} \tag{34}$$

iii). Case 3: If $\mathbf{b} \in R(\mathbf{A})$, $rank(\mathbf{A}) < n$, there are infinite solutions, that is

$$\mathbf{A}\mathbf{x} = \mathbf{b} \Leftrightarrow \mathbf{x} = \mathbf{x_0} + \mathbf{y}, \text{where } \mathbf{y} \in N(\mathbf{A}) \tag{35}$$

such that the optimization problem can be cast as

$$\min_{\mathbf{y}} ||\mathbf{y} + \mathbf{x_0} + \mathbf{c}||_2 \tag{36}$$

$$s.t.\ \mathbf{y} \in N(\mathbf{A}) \tag{37}$$

which is the least square projection of $-(\mathbf{x_0}+\mathbf{c})$ onto $N(\mathbf{A})$.
b). Given $\mathbf{x}_0 \in \mathbb{R}^n$, $\mathbf{a} \in \mathbb{R}^n, b \in \mathbb{R}$, solve the following problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} ||\mathbf{x}_0 - \mathbf{x}||_2 \tag{38}$$

$$s.t\ \mathbf{a}^T\mathbf{x} = b \tag{39}$$

Derive a closed form of the solution.
Hints: We need to solve the orthonormal projection for $N(\mathbf{a}^T)$ first. As already shown in Homework 3, $N(\mathbf{A}^T) \perp R(\mathbf{A})$ holds for any matrix $\mathbf{A} \in \mathbb{R}^{m\times n} \implies N(\mathbf{a}^T) \perp R(\mathbf{a})$.
Clearly, orthonormal basis for $R(\mathbf{a})$ can be represented as follows along with its corresponding orthogonal projection matrix

$$\mathbf{u_a} = \frac{\mathbf{a}}{||\mathbf{a}||_2},\ \text{with corresponding projection matrix}\ \mathbf{u_a}\mathbf{u_a^T} = \frac{\mathbf{aa^T}}{||\mathbf{a}||_2^2} \tag{40}$$

Thus,

$$\text{the orthonormal projector onto}\ N(\mathbf{a}^T) = \mathbf{I} - \frac{\mathbf{aa^T}}{||\mathbf{a}||_\mathbf{2}^\mathbf{2}} \tag{41}$$

Please make sure to complete the remaining steps by yourself.