**ECE269: Linear Algebra and Applications**
**Fall 2021**

**Homework # 1 Solutions**

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1. (a) No it is not a subspace since the set is not closed under scalar multiplication. For example, suppose $\mathbf{x} \in S = \{\mathbf{x} \mid x_i \geq 0\}$ and $\alpha = -1$. Then

$$\alpha \mathbf{x} \notin S$$

(b) The set $S = \{\mathbf{x} \mid x_1 = 0\}$ is a subspace. Suppose $\mathbf{x}_1, \mathbf{x}_2 \in S$. Then

$$\mathbf{y} = \mathbf{x}_1 + \mathbf{x}_2; \quad y_1 = 0 \implies \mathbf{y} \in S$$

and let $\alpha \in \mathbb{R}$

$$\mathbf{z} = \alpha \mathbf{x}_1; \quad z_1 = 0 \implies \mathbf{z} \in S$$

Since the set is closed under addition and scalar multiplication, it is a subspace of $\mathbb{R}^n$.

(c) The set $S = \{\mathbf{x} \mid x_1 x_2 = 0\}$ is not a subspace since it is not closed under addition. To see this let $\mathbf{e}_1 = [1, 0, 0, \ldots, 0] \in S$ and $\mathbf{e}_2 = [0, 1, 0, \ldots, 0] \in S$. But

$$\mathbf{e}_1 + \mathbf{e}_2 \notin S$$

(d) The set $S = \{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b} \text{ where } \mathbf{b} \neq \mathbf{0}\}$ is not a subspace since it is not closed under addition. Suppose $\mathbf{x}_1, \mathbf{x}_2 \in S$. Then

$$\mathbf{A}(\mathbf{x_1} + \mathbf{x_2}) = \mathbf{A}\mathbf{x_1} + \mathbf{A}\mathbf{x_2} = 2\mathbf{b} \neq \mathbf{b}$$

Note that, this set is neither closed under scalar multiplication.

(e) Let set $S = \{[x_1, x_2, x_3, x_4] \in \mathbb{R}^4 \mid x_3 = x_1 + x_2,\ x_4 = x_1 - x_2\}$. Let $\mathbf{y}_1 = \begin{bmatrix} y_{11} \\ y_{12} \\ y_{13} \\ y_{14} \end{bmatrix} \in S$, $\mathbf{y}_2 = \begin{bmatrix} y_{21} \\ y_{22} \\ y_{23} \\ y_{24} \end{bmatrix} \in S$. First we check the addition $\mathbf{y}_1 + \mathbf{y}_2 = \begin{bmatrix} y_{11} + y_{21} \\ y_{12} + y_{22} \\ y_{13} + y_{23} \\ y_{14} + y_{24} \end{bmatrix}$, where $y_{13} + y_{23} = (y_{11} + y_{12}) + (y_{21} + y_{22}) = (y_{11} + y_{21}) + (y_{12} + y_{22})$, and $y_{14} + y_{24} = (y_{11} - y_{12}) + (y_{21} - y_{22}) = (y_{11} + y_{21}) - (y_{12} + y_{22})$. Thus $\mathbf{y}_1 + \mathbf{y}_2 \in S$. Let $\alpha \in \mathbb{R}$, $\alpha \mathbf{y}_1 = \begin{bmatrix} \alpha y_{11} \\ \alpha y_{12} \\ \alpha y_{13} \\ \alpha y_{14} \end{bmatrix}$, where $\alpha y_{13} = \alpha y_{11} + \alpha y_{12}$ and $\alpha y_{14} = \alpha y_{11} - \alpha y_{12}$. So clearly $\alpha \mathbf{y}_1 \in S$. Thus, we conclude set $S$ is closed under both addition and scalar multiplication.

(f) Let $S = \{[x_1, x_2, x_3] \in \mathbb{R}^3 \mid x_1 \leq x_2 \leq x_3\}$. Let $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in S$ and let $\alpha = -1 \in \mathbb{R}$, clearly $\alpha\mathbf{x} \notin S$. Thus, $S$ is not a subspace.

(g) Let $S = \{\mathbf{A} \in \mathbb{R}^{3\times 3} \mid [1,0,4]^T \in \mathcal{N}(\mathbf{A})\}$. Let $\mathbf{A}_1, \mathbf{A}_2 \in S$, such that $\mathbf{A}_1[1,0,4]^T = \mathbf{0}$ and $\mathbf{A}_2[1,0,4]^T = \mathbf{0}$. First check $(\mathbf{A}_1+\mathbf{A}_2)[1,0,4]^T = \mathbf{A}_1[1,0,4]^T + \mathbf{A}_2[1,0,4]^T = \mathbf{0}$, so the addition closure is satisfied. Then let $\alpha \in \mathbb{R}$, clearly $(\alpha\mathbf{A}_1)\mathbf{x} = \alpha(\mathbf{A}_1\mathbf{x}) = \mathbf{0}$, so that $[1,0,4]^T \in \mathcal{N}(\alpha\mathbf{A}_1)$. Thus, subset $S$ is a subspace.

(h) The set $S = \{\mathbf{B} \in \mathbb{R}^{n\times n} | \ \mathbf{AB} = \mathbf{BA}\}$ is a subspace. Let $\mathbf{B}_1, \mathbf{B}_2 \in S$,
Then,

$$\mathbf{A}(\mathbf{B_1}+\mathbf{B_2}) = \mathbf{AB_1}+\mathbf{AB_2} = \mathbf{B_1A}+\mathbf{B_2A} = (\mathbf{B_1}+\mathbf{B_2})\mathbf{A}$$

Thus, $\mathbf{B_1}+\mathbf{B_2} \in S$.
Let, $\alpha \in \mathbb{R}$

$$\mathbf{A}(\alpha\mathbf{B_1}) = \alpha(\mathbf{AB_1}) = \alpha(\mathbf{B_1A}) = (\alpha\mathbf{B_1})\mathbf{A}$$

Hence, $\alpha\mathbf{B_1} \in S$.
Since S is closed under addition and scalar multiplication, it is a subspace.

(i) No the set $S = \{\mathbf{X} \in \mathbb{R}^{n\times n} | \ \mathbf{X^2} = \mathbf{X}\}$ is not a subspace.
Let, $\alpha \in \mathbb{R}$ and $\mathbf{X} \in S$ where $\alpha \neq 0, 1$ and $\mathbf{X} \neq \mathbf{0}$

$$(\alpha\mathbf{X})^2 = \alpha^2\mathbf{X}^2 = \alpha^2\mathbf{X} \neq \alpha\mathbf{X}$$

Since the set is not closed under scalar multiplication, it is not a subspace. This set is also not closed under addition.

(j) The set $S = \{\mathbf{X} \in \mathbb{R}^{n\times n} | \ \mathbf{trace}(\mathbf{X}) = \mathbf{0}\}$ is a subspace. Let $\mathbf{X}_1, \mathbf{X}_2 \in S$,
Then,

$$trace(\mathbf{X_1}+\mathbf{X_2}) = trace(\mathbf{X_1}) + trace(\mathbf{X_2}) = 0 + 0 = 0$$

Thus, $\mathbf{X_1}+\mathbf{X_2} \in S$.
Let, $\alpha \in \mathbb{R}$

$$trace(\alpha\mathbf{X_1}) = \alpha\, trace(\mathbf{X_1}) = 0$$

Hence, $\alpha\mathbf{X_1} \in S$.
Since S is closed under addition and scalar multiplication, it is a subspace.

2. $\mathbb{P}_n(\mathbb{R}) = \{f(x) = \sum_{k=0}^{n} c_k x^k, c_k \in \mathbb{R}, k \in \{0, ..., n\}\}$.

(a) We show that the 10 vector space axioms hold. In the following, denote $f_1(x) = \sum_{k=0}^{n} a_k x^k$, $f_2(x) = \sum_{k=0}^{n} b_k x^k$, $f_3(x) = \sum_{k=0}^{n} c_k x^k$.$a_k, b_k, c_k, \alpha, \beta, \gamma \in \mathbb{R}$. $f_1(x), f_2(x), f_3(x) \in \mathbb{P}_n(\mathbb{R})$.

A1. For $f_1(x), f_2(x)$: $f_1(x)+f_2(x) = \sum_{k=0}^{n}(a_k+b_k)x^k, a_k+b_k \in \mathbb{R} \to f_1(x)+f_2(x) \in \mathbb{P}_n(\mathbb{R})$.

A2. For $f_1(x), f_2(x), f_3(x)$: $(f_1(x)+f_2(x))+f_3(x) = \sum_{k=0}^{n}(a_k+b_k+c_k)x^k = \sum_{k=0}^{n} a_k x^k + \sum_{k=0}^{n}(b_k+c_k)x^k = f_1(x)+(f_2(x)+f_3(x))$.

A3. For $f_1(x), f_2(x)$: $f_1(x)+f_2(x) = \sum_{k=0}^{n}(a_k+b_k)x^k = \sum_{k=0}^{n} b_k x^k + \sum_{k=0}^{n} a_k x^k = f_2(x)+f_1(x)$.

A4. Denote the zero polynomial as $\mathbf{0} = \sum_{k=0}^{n} 0 \cdot x^k$, so: $f_1(x)+\mathbf{0} = \sum_{k=0}^{n}(a_k+0)x^k = \sum_{k=0}^{n} a_k x^k = f_1(x)$.

A5. For any $f_1(x) = \sum_{k=0}^{n}(a_k)x^k \in \mathbb{P}_n(\mathbb{R})$, let $f_I(x) = \sum_{k=0}^{n}(-a_k)x^k \in \mathbb{P}_n(\mathbb{R}) \implies f_1(x)+f_I(x) = \mathbf{0}$

M1. $\alpha \cdot f_1(x) = \sum_{k=0}^{n}(\alpha a_k)x^k \in \mathbb{P}_n(\mathbb{R}), \alpha a_k \in \mathbb{R}$.

M2. $(\alpha\beta)f_1(x) = \sum_{k=0}^{n}(\alpha\beta a_k)x^k = \alpha(\beta\sum_{k=0}^{n} a_k x^k) = \alpha(\beta f_1(x))$.

M3. $\alpha(f_1(x)+f_2(x)) = \sum_{k=0}^{n} \alpha(a_k+b_k)x^k = \sum_{k=0}^{n}(\alpha a_k + \alpha b_k)x^k = \alpha\sum_{k=0}^{n} a_k x^k + \alpha\sum_{k=0}^{n} b_k x^k = \alpha f_1(x) + \alpha f_2(x)$.

M4. $(\alpha+\beta)f_1(x) = \sum_{k=0}^{n}(\alpha+\beta)a_k x^k = \alpha\sum_{k=0}^{n} a_k x^k + \beta\sum_{k=0}^{n} a_k x^k = \alpha f_1(x) + \beta f_1(x)$.

M5. $1 \cdot f_1(x) = \sum_{k=0}^{n} 1 \cdot a_k x^k = \sum_{k=0}^{n} a_k x^k = f_1(x)$.

Thus $\mathbb{P}_n(\mathbb{R})$ is a vector space.
It is easy to see that the set $\{1, x, ..., x^n\}$ is a basis for $\mathbb{P}_n(\mathbb{R})$. By definition of spanning set, $\mathbb{P}_n(\mathbb{R})$ is spanned by this set. Also, by definition of linear independence, $\sum_{k=0}^{n} c_k x^k = 0, c_i \in \mathbb{R}, i \in \{0, ..., n\}\}$ if and only if $c_k = 0, \forall k \in \{0, ..., n\}$ by equating polynomial coefficients. Thus the set $\{1, x, ..., x^n\}$ is also linearly independent, therefore it is a basis for $\mathbb{P}_n(\mathbb{R})$. Clearly there are $n+1$ elements in this set, so the dimension of $\mathbb{P}_n(\mathbb{R})$ is $n+1$.

(b) The union $\bigcup_{n=1}^{m} \mathbb{P}_n$ is a vector space. Consider $f(x) = \sum_{i=0}^{n} \alpha_i x^i \in \mathbb{P}_n$, so for $n \leq m$, $f(x) = \sum_{i=0}^{n} \alpha_i x^i + \sum_{i=n+1}^{m} 0 \cdot x^i \in \mathbb{P}_m$ thus $\mathbb{P}_n \subseteq \mathbb{P}_m$ so the union is a vector space.
In class we learned that a union of subspaces is a subspace, only when one of the subspaces is contained in another. So, our result settles with the claim made in class.

(c) The standard basis for $\mathbb{P}_4$ is given by $\{1, x, x^2, x^3, x^4\}$. Notice that $x^2 = \frac{1}{2}((x^2+1)+(x^2-1)), 1 = \frac{1}{2}((x^2+1)-(x^2-1))$. It is clear that the rest of the standard basis is independent of the set elements as they are polynomials of different degrees, thus a possible basis would be $\{x, x^2+1, x^2-1, x^3, x^4\}$.

(d) Consider the set $1+x, x+x^2, 2x+3x^2$.
Consider $a(1+x)+b(x+x^2)+c(2x+3x^2) = 0$
Rewriting,$(a)1+(a+b+2c)x+(b+3c)x^2 = 0$
Hence $a = 0, (a+b+2c) = 0$, and $(b+3c) = 0$
Solving the equations, $a = b = c = 0$. Hence the 3 elements from the set are linearly independent. Now we know that a basis for $\mathbb{P}_2$ will contain 3 elements, hence a set of 3 linearly independent elements from $\mathbb{P}_2$ will make a basis. Hence the set $1+x, x+x^2, 2x+3x^2$ is a basis.

3. (a) i. Yes. We will prove it by contradiction. It is given that $\mathbf{x}_1, ..., \mathbf{x}_n$ are linearly independent, so by definition

$$\sum_{i=1}^{n} \alpha_i \mathbf{x}_i = 0 \implies \alpha_i = 0, \ i = 1, \ldots, n.$$

Now let us assume that $\mathbf{z}_1, ..., \mathbf{z}_n$ are linearly dependent, meaning there exists some constants, $\beta_i$, not all zero, such that $\sum_{i=1}^n \beta_i \mathbf{z}_i = 0$. So:

$$\mathbf{0} = \sum_{i=1}^n \beta_i \mathbf{z}_i = \sum_{i=1}^n \beta_i (\mathbf{x}_i, \mathbf{y}_i)^T = (\sum_{i=1}^n \beta_i \mathbf{x}_i, \sum_{i=1}^n \beta_i \mathbf{y}_i)^T \implies \sum_{i=1}^n \beta_i \mathbf{x}_i = 0.$$

This contradicts the fact that linear independence assumption.

ii. No. Consider the counterexample $n = 2, x_1 = x_2 = 1, y_1 = 1, y_2 = 0$.
Clearly $x_1, x_2$ are linearly dependent as they are the identical, but $z_1 = (1,1)^T, z_2 = (1,0)^T$ are linearly independent.

(b) No, $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ forming a basis for $\mathcal{V}$ over an arbitrary field does not always imply $\{\mathbf{x}+\mathbf{y}, \mathbf{y}+\mathbf{z}, \mathbf{z}+\mathbf{x}\}$is also a basis for $\mathcal{V}$ over that field.
Consider $\mathcal{V} = \mathbb{F}_2^3$, where $\mathbb{F}_2$ is the binary field $\{0, 1\}$. Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be a basis of $\mathcal{V}$ over field $\mathbb{F}_2$,

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Then,

$$\mathbf{x}+\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{y}+\mathbf{z} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{z}+\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

We observe that $\{\mathbf{z}+\mathbf{x}\}$ can be written as a linear combination of $\{\mathbf{x}+\mathbf{y}\}$ and $\{\mathbf{y}+\mathbf{z}\}$, i.e.,

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Hence, $\{\mathbf{x}+\mathbf{y}, \mathbf{y}+\mathbf{z}, \mathbf{z}+\mathbf{x}\}$ are not linearly independent and therefore not a basis of $\mathcal{V}$.

4. (a) $\mathbf{AB} = \mathbf{0} \not\Rightarrow \mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$
Consider

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{then} \quad \mathbf{AB} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}$$

(b) $\mathbf{A}^2 = \mathbf{0} \not\Rightarrow \mathbf{A} = \mathbf{0}$
Consider

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{then} \quad \mathbf{A}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}$$

(c) $\mathbf{A}^T\mathbf{A} = \mathbf{0} \implies \mathbf{A} = \mathbf{0}$
Note that $\mathbb{F} = \mathbb{R}$. Consider the $j$th diagonal element of $\mathbf{A}$.

$$(\mathbf{A}^T\mathbf{A})_{jj} = \sum_{i=1}^n \mathbf{a}_{ij}\mathbf{a}_{ij} = \sum_{i=1}^n \mathbf{a}_{ij}^2 = 0$$

Since $\mathbf{a}_{ij} \in \mathbb{R}$, $\sum_{i=1}^{n} \mathbf{a}_{ij}^2 = 0 \implies \mathbf{a}_{ij} = 0,\ i = 1, \ldots, n$.
Since $(\mathbf{A}^T\mathbf{A})_{jj} = 0$ for $j = 1, \ldots, n$; we conclude that $\mathbf{a}_{ij} = 0,\ i, j = 1, \ldots, n$. In other words, $\mathbf{A} = \mathbf{0}$.