ECE 269: Linear Algebra and Applications
Fall 2021

Homework # 1
Due: October 12, 11:59pm, via Gradescope

Collaboration Policy: This homework set allows limited collaboration. You are expected to try to solve the problems on your own. You may discuss a problem with other students to clarify any doubts, but you must fully understand the solution that you turn in and write it up entirely on your own. Blindly copying another student's result will be considered a violation of academic integrity. *

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1. **Suggested Reading**. Sections 4.1 − 4.4 of Carl D. Meyer's book "Matrix Analysis and Applied Linear Algebra".

2. **Problem 1: Subspaces of $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$.**
   Determine which of the following subsets of $\mathbb{R}^n$, and $\mathbb{R}^{n\times n}$ are subspaces $(n > 2)$.

   (a) $\{\mathbf{x} \mid x_i \geq 0\}$
   (b) $\{\mathbf{x} \mid x_1 = 0\}$
   (c) $\{\mathbf{x} \mid x_1 x_2 = 0\}$
   (d) $\{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b} \text{ where } \mathbf{b} \neq \mathbf{0}\}$
   (e) $\{[x_1, x_2, x_3, x_4] \in \mathbb{R}^4 \mid x_3 = x_1 + x_2,\ x_4 = x_1 - x_2\}$
   (f) $\{[x_1, x_2, x_3] \in \mathbb{R}^3 \mid x_1 \leq x_2 \leq x_3\}$
   (g) $\{\mathbf{A} \in \mathbb{R}^{3\times 3} \mid [1, 0, 4]^T \in N(\mathbf{A})\}$
   (h) All matrices that commute with a given matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$
   (i) $\{\mathbf{X} \in \mathbb{R}^{n\times n} \mid \mathbf{X}^2 = \mathbf{X}\}$
   (j) $\{\mathbf{X} \in \mathbb{R}^{n\times n} \mid \text{trace}(\mathbf{X}) = 0\}$

3. **Problem 2: Vector Spaces of Polynomials.**
   Consider the set $\mathbb{P}_n(\mathbb{R})$ of all real valued polynomials of degree $\leq n$ with real coefficients:

$$\mathbb{P}_n(\mathbb{R}) = \{f(x) = \sum_{k=0}^{n} c_k x^k, c_0, c_1, \cdots, c_n \in \mathbb{R}\} \tag{1}$$

   (a) Show that $\mathbb{P}_n(\mathbb{R})$ is a vector space. What is the dimension of this vector space?
   (b) Is the union $\cup_{n=1}^{m} \mathbb{P}_n$ a vector space? Does this contradict or comply with something you learned in class?

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(c) Find a basis for $\mathbb{P}_4$ containing $\{x^2+1, x^2-1\}$

(d) Find a basis for $\mathbb{P}_2$ from the set $\{1+x, x+x^2, x+2x^2, 2x+3x^2, 1+2x+x^2\}$

4. **Problem 3: Linear Independence.**

(a) Consider the stacked vectors

$$\mathbf{z}_1 = \begin{bmatrix} \mathbf{x}_1 \\ \\ \mathbf{y}_1 \end{bmatrix}, \cdots, \mathbf{z}_n = \begin{bmatrix} \mathbf{x}_n \\ \\ \mathbf{y}_n \end{bmatrix},$$

i. Suppose $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent (no assumption is made on $\mathbf{y}_1, \ldots, \mathbf{y}_n$). Can we conclude that the vectors $\mathbf{z}_1, \ldots, \mathbf{z}_n$ are linearly independent? If yes, provide a proof. If no, give a counterexample.

ii. Suppose $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly dependent (no assumption is made on $\mathbf{y}_1, \ldots, \mathbf{y}_n$). Can we conclude that the vectors $\mathbf{z}_1, \ldots, \mathbf{z}_n$ are linearly dependent? If yes, prove the result. If no, give a counterexample.

(b) Let $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ be a basis for a vectorspace $\mathcal{V}$ over some **arbitrary field** $\mathbb{F}$. Is $\{\mathbf{x}+\mathbf{y}, \mathbf{y}+\mathbf{z}, \mathbf{z}+\mathbf{x}\}$ also a basis for $\mathcal{V}$? Either prove the statement or provide a counterexample.

5. **Problem 4: Matrix Products.** Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$. Prove or provide a counterexample to each of the following statements.

(a) If $\mathbf{AB} = 0$, then either $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$.

(b) If $\mathbf{A}^2 = \mathbf{0}$, then $\mathbf{A} = \mathbf{0}$.

(c) If $\mathbf{A}^T\mathbf{A} = \mathbf{0}$, then $\mathbf{A} = \mathbf{0}$.

Programming Assignment

In this assignment, we will apply our knowledge of linear algebra to channel coding, an important building block in the design of communication systems. Don't worry, no prior knowledge is assumed!

- **Preliminaries:** Our vector space in this example is $\mathbb{F}_2^n$ over the field $\mathbb{F}_2$. Recall definitions of the binary field $\mathbb{F}_2$ and the vector space $\mathbb{F}_2^n$ defined over the field $\mathbb{F}_2$.

- **Communicating over a noisy channel and Decoding:** Let $\mathbf{H} \in \mathbb{F}_2^{(n-k)\times n}$ be a full-rank matrix ($k < n$). A binary linear code $\mathcal{C}$ is defined as follows:

$$\mathcal{C} = \{\mathbf{x} \in \mathbb{F}_2^n \,|\, \mathbf{x} \in \mathcal{N}(\mathbf{H})\},$$

where $\mathcal{N}(\mathbf{H})$ is the nullspace of $\mathbf{H}$. $\mathcal{C}$ is referred to as the set of codewords or the *codebook*.

1. How many codewords are there in $\mathcal{C}$? Given $\mathcal{C}$, is $\mathbf{H}$ unique?

2. Find the cardinality of $\mathcal{R}(\mathbf{H})$, the range space of $\mathbf{H}$.

3. Given a vector $\mathbf{x} \in \mathbb{F}_2^n$ and matrix $\mathbf{H} \in \mathbb{F}_2^{(n-k)\times n}$, write a function `checkCodeword` that checks if $\mathbf{x}$ belongs to the codebook.

   For what follows, assume that

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \tag{2}$$

   Test your function with $\mathbf{H}$ defined in Eq. (2) and following codewords and print your result in your report:

$$\mathbf{x}_1^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\mathbf{x}_2^T = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\mathbf{x}_3^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Communicating codewords over a noisy channel and Decoding: Assume that $\mathbf{x} \in \mathcal{C}$ is transmitted over a communication channel, and the received vector is $\mathbf{r} = \mathbf{x} + \mathbf{e}$, where $\mathbf{e} \in \mathbb{F}_2^n$.

Given $\mathbf{r}$, we would like to recover an estimate of $\mathbf{x}$. It is clear that this is equivalent to recovering $\mathbf{e}$ since knowing $\mathbf{e}$, $\mathbf{x}$ can be easily computed as $\mathbf{x} = \mathbf{r} - \mathbf{e}$. There can be many ways to estimate $\mathbf{e}$, one of which is by using a "Maximum Likelihood Decoder". A maximum likelihood decoder aims to find a vector $\mathbf{e}$ with the smallest ℓ_1-norm (the ℓ_1 norm of a vector $\mathbf{e} \in \mathbb{F}_2^n$ is the number of ones in $\mathbf{e}$). Also, notice the following:

$$\mathbf{s} := \mathbf{Hr} = \mathbf{H}(\mathbf{x} + \mathbf{e}) = \mathbf{Hx} + \mathbf{He} = \mathbf{He},$$

i.e. $\mathbf{H}$ maps both $\mathbf{r}$ and $\mathbf{e}$ to the same vector $\mathbf{s} \in \mathbb{F}_2^{n-k}$. The vector $\mathbf{s}$ is called the syndrome of $\mathbf{e}$.

4. Write a function `buildTable` that takes as input a matrix $\mathbf{H} \in \mathbb{F}_2^{(n-k)\times n}$ and generates a matrix $\mathbf{E} \in \mathbb{F}_2^{n\times(n+1)}$ of vectors $\mathbf{e}$ of ℓ_1-norm at most one, along with the matrix $\mathbf{S} \in \mathbb{F}_2^{(n-k)\times(n+1)}$ of their corresponding syndrome vectors.

5. Show that any $\mathbf{s} \in \mathbb{F}_2^{n-k}$ is the syndrome vector of some vector $\mathbf{e}$, that has an ℓ_1-norm of at most one.

6. (a) Using the function you wrote in part 4 and for $\mathbf{H}$ given in (2), write a function `channelDecode` that takes as input a received vector $\mathbf{r} \in \mathbb{F}_2^n$ and generates an estimate of $\mathbf{e}$ that has the smallest ℓ_1-norm, along with the corresponding estimated vector $\mathbf{x}$.

   (b) Suppose you want to transmit the following vector

$$\mathbf{x}^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3}$$

   Suppose your channel is such that it only introduces error in a single bit. Write a script that generates all possible received signals corresponding to the above $\mathbf{x}$, and decodes them using `channelDecode`. Count how many times you successfully decode the transmitted signal. What happens if you repeat this exercise for a different $\mathbf{x}$?

   (c) You again transmit the same vector $\mathbf{x}$ as above, but this time, the channel introduces errors in exactly two bits. Write a script that generates all possible received signals, and decode them using `channelDecode`. Count how many times you successfully recover the transmitted signal. Why does this happen? Based on these observations, what can you say about the case when there are errors in exactly 3 bits?