Question: 5. Problem 5: Properties of PSD matrices. Let A = AT e Rnxn and B = BT e Rnxn. Prove the followin...

5. **Problem 5: Properties of PSD matrices.** Let $A = A^T \in \mathbb{R}^{n\times n}$ and $B = B^T \in \mathbb{R}^{n\times n}$. Prove the following statements.

   (a) If $A \succeq 0$ and $B \succeq 0$, then $\text{Trace}(AB) \geq 0$.

   (b) If $A \succeq 0$, then $A + B \succeq B$.

   (c) If $A \succeq B$, then $-B \succeq -A$.

   (d) If $A \succeq I$, then $I \succeq A^{-1}$.

   (e) If $A \succeq B \succ 0$, then $B^{-1} \succeq A^{-1} \succ 0$.

Expert Answer

5 $A = A^T \in \mathbb{R}^{n\times n}$

$B = B^T \in \mathbb{R}^{n\times n}$

(a) $A \succcurlyeq 0$ and $B \succcurlyeq 0$,

$\Rightarrow$ A is positive semi definite and B is positive semidefinite

$\Rightarrow$ eigen values of $A \geq 0$ and eigen values of B are ≥ 0

$\Rightarrow$ eigen values of (AB) are greater than or equal to zero

and trace = sum of eigen values

$\Rightarrow tr(AB) \geq 0$

(b) $A \succcurlyeq 0$

$\Rightarrow A + B - B \succcurlyeq 0$ (adding and subtracting B)

We know that, $A \preccurlyeq B$ if and only if $A - B \preccurlyeq 0$

$\Rightarrow A + B \succcurlyeq B$

(c) $A \geq B$

$\Rightarrow A - A \geq B - A$

$\Rightarrow 0 \geq B - A$

$\Rightarrow 0 - B \geq B - A - B$

$\Rightarrow -B \geq -A$

(d) $A \geq I$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigen values of A

then $\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_n - 1$ are eigen values of $A - I$

$\because A \geq I \Rightarrow A - I \geq 0 \Rightarrow A - I$ is PSD

$\Rightarrow \lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_n - 1 \geq 0 \Rightarrow \lambda_1, \lambda_2, \ldots, \lambda_n \geq 1$

$\Rightarrow A^{-1}$ exists

$\therefore A \geq I \Rightarrow AA^{-1} \geq IA^{-1} \Rightarrow I \geq A^{-1}$

(e) $A \geq B > 0$

$\because A > 0$ and $B > 0$

$\Rightarrow$ A is positive definite and B is also positive definite

$\Rightarrow$ A^{-1} and B^{-1} exist

Thus, $A \geq B > 0$

$$\Rightarrow A^{-1}A \geq A^{-1}B > A^{-1}0$$

$$\Rightarrow I \geq A^{-1}B > 0$$

$$\Rightarrow B^{-1} \geq A^{-1}BB^{-1} > 0B^{-1}$$

$$\Rightarrow B^{-1} \geq A^{-1} > 0$$