ECE 269 - Solution to Homework 3

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Problem 1 It is given that $\mathcal{V}$ is a subspace of $\mathbb{R}^n$, $\mathcal{V}^\perp = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^T\mathbf{y} = 0, \forall \mathbf{y} \in \mathcal{V}\}$.

(a) Consider $\alpha, \beta \in \mathbb{R}, v_1, v_2 \in \mathcal{V}^\perp$. So $\forall u \in \mathcal{V}$:
$(\alpha v_1 + \beta v_2)^T u = \alpha v_1^T u + \beta v_2^T u = 0$ since $v_1^T u = 0 = v_2^T u$.
Also, $0 \in \mathcal{V}^\perp$ since $0^T u = 0, \forall u \in \mathcal{V}$. Thus, $\mathcal{V}^\perp$ is a subspace of $\mathbb{R}^n$.

(b) Using the given matrix $\mathbf{A}$, notice that:
$\mathcal{V} = \{\sum_{i=1}^k x_i \mathbf{v_i} : \forall x_1, ..., x_k \in \mathbb{R}\} =$
$\{\mathbf{A} \cdot (x_1, ..., x_k)^T : \forall x_1, ..., x_k \in \mathbb{R}\} =$
$\{\mathbf{Ax} : \mathbf{x} \in \mathbb{R}^k\} = \mathcal{R}(\mathbf{A})$. Meaning $\mathcal{V}$ is the range space of $\mathbf{A}$.
Similarly:
$\mathcal{V}^\perp = \{\mathbf{y} \in \mathbb{R}^n : \mathbf{y}^T \sum_{i=1}^k x_i \mathbf{v_i} = 0, \forall x_1, ..., x_k \in \mathbb{R}\}$
$= \{\mathbf{y} \in \mathbb{R}^n : (\sum_{i=1}^k x_i \mathbf{v_i}^T)\mathbf{y} = 0, \forall x_1, ..., x_k \in \mathbb{R}\}$
$= \{\mathbf{y} \in \mathbb{R}^n : \mathbf{x}^T\mathbf{A}^T\mathbf{y} = 0, \mathbf{x} \in \mathbb{R}^k\}$
$= \{\mathbf{y} \in \mathbb{R}^n : \mathbf{A}^T\mathbf{y} = 0\}$
$= \mathcal{N}(\mathbf{A}^T)$.
Meaning the orthogonal complement of the range space is the null space of the transpose.

(c) Consider $\mathbf{x} \in \mathcal{V}$. By definition of the orthogonal subspace $\mathbf{x}^T\mathbf{y} = 0, \forall \mathbf{y} \in \mathcal{V}^\perp \to \mathbf{x} \in (\mathcal{V}^\perp)^\perp$. Thus, $\mathcal{V} \subseteq (\mathcal{V}^\perp)^\perp$. Using the result of section (d), $dim(\mathcal{V}) + dim(\mathcal{V}^\perp) = n \to dim(\mathcal{V}^\perp) + dim((\mathcal{V}^\perp)^\perp) = n \to dim((\mathcal{V}^\perp)^\perp) = dim(\mathcal{V}) \to \mathcal{V} = (\mathcal{V}^\perp)^\perp$.

(d) Using section (b), we get that
$dim(\mathcal{V}) = dim(\mathcal{R}(\mathbf{A})) = rank(\mathbf{A}) = rank(\mathbf{A}^T)$. And,
$dim(\mathcal{V}^\perp) = dim(\mathcal{N}(\mathbf{A}^T))$.
Using the rank nullity theorem and the fact that $\mathbf{A}^T \in \mathbb{R}^{k \times n}$:
$n = rank(\mathbf{A}^T) + dim(\mathcal{N}(\mathbf{A}^T)) = dim(\mathcal{V}) + dim(\mathcal{V}^\perp)$.

(e) Assuming $\mathcal{V} \subseteq \mathcal{W}$ implies:
$\forall \mathbf{v} \in \mathcal{V} \to \mathbf{v} \in \mathcal{W}$, so taking $\mathbf{u} \in \mathcal{W}^\perp$:
$\mathbf{u}^T\mathbf{w} = 0, \forall \mathbf{w} \in \mathcal{W} \to \mathbf{u}^T\mathbf{w} = 0, \forall \mathbf{w} \in \mathcal{V} \to \mathbf{u} \in \mathcal{V}^\perp \to \mathcal{W}^\perp \subseteq \mathcal{V}^\perp$.

(f) We know that if $\mathbf{v}$ is a projection of $\mathbf{x}$ on $\mathcal{V}$, the $\mathbf{v}^\perp = \mathbf{x} - \mathbf{v}$ is orthogonal to all vectors in $\mathcal{V}$, meaning $\mathbf{v}^\perp \in \mathcal{V}^\perp$. Let us assume $\exists \mathbf{v} \neq \mathbf{u}, \mathbf{v}, \mathbf{u} \in \mathcal{V}$ such that $\mathbf{x} = \mathbf{v} + \mathbf{v}^\perp = \mathbf{u} + \mathbf{u}^\perp$, which means $\mathbf{v}^\perp, \mathbf{u}^\perp \in \mathcal{V}^\perp$. So $\mathbf{v} - \mathbf{u} = \mathbf{u}^\perp - \mathbf{v}^\perp$. However, since $\mathcal{V}, \mathcal{V}^\perp$ are sub spaces, we get that $\mathbf{v}-\mathbf{u} \in \mathcal{V}, \mathbf{u}^{\perp}-\mathbf{v}^{\perp} \in \mathcal{V}^{\perp}$, so $\mathbf{v}-\mathbf{u} \in \mathcal{V} \cap \mathcal{V}^{\perp} = \mathbf{0} \to \mathbf{v} = \mathbf{u}$ in contradiction to the assumption. Thus, the representation is unique.

Problem 2 $\mathbf{A} \in \mathbb{R}^{4\times 3}, \mathbf{B} \in \mathbb{R}^{3\times 3}$. $Rank(\mathbf{A}) = 2, Rank(\mathbf{B}) = 3$.

(a) We will show that in this question, $Rank(\mathbf{A}) = Rank(\mathbf{AB})$.
Consider $\mathbf{v} \in \mathcal{R}(\mathbf{A})$. So $\exists \mathbf{u} \in \mathbb{R}^3$ s.t. $\mathbf{Au} = \mathbf{v}$. Since $\mathbf{B}$ is full rank, $\exists \mathbf{x} \in \mathbb{R}^3$ s.t. $\mathbf{Bx} = \mathbf{u}$. So $\mathbf{A}(\mathbf{Bx}) = (\mathbf{AB})\mathbf{x} = \mathbf{v} \to \mathbf{v} \in \mathcal{R}(\mathbf{AB}) \to \mathcal{R}(\mathbf{A}) \subseteq \mathcal{R}(\mathbf{AB})$.
Similarly, consider $\mathbf{v} \in \mathcal{R}(\mathbf{AB})$, so $\exists \mathbf{u} \in \mathbb{R}^3$ s.t. $(\mathbf{AB})\mathbf{u} = \mathbf{v} \to \mathbf{A}(\mathbf{Bu}) = \mathbf{v} \to \mathbf{v} \in \mathcal{R}(\mathbf{A}) \to \mathcal{R}(\mathbf{AB}) \subseteq \mathcal{R}(\mathbf{A}) \to \mathcal{R}(\mathbf{AB}) = \mathcal{R}(\mathbf{A}) \to Rank(\mathbf{A}) = Rank(\mathbf{AB})$. So, $r_{min} = r_{max} = 2$.
Consider:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{B} = \mathbf{I}_{3\times 3}, \mathbf{AB} = \mathbf{A}.$$

(b) The example above satisfies the requirements.

Problem 3 It is given that the columns of $\mathbf{U} \in \mathbb{R}^{n\times k}$ are orthonormal.
Also, notice that: $\mathbf{x}, \mathbf{UU}^T\mathbf{x} \in \mathbb{R}^n$.
Let $\mathbf{UU}^T\mathbf{x}$ be the projection of $\mathbf{x}$ onto $\mathcal{R}(\mathbf{U})$. So:
$0 \leq \|\mathbf{x} - \mathbf{UU}^T\mathbf{x}\|_2^2 = \|(\mathbf{I} - \mathbf{UU}^T)\mathbf{x}\|_2^2 = \mathbf{x}^T(\mathbf{I} - \mathbf{UU}^T)^T(\mathbf{I} - \mathbf{UU}^T)\mathbf{x} = \mathbf{x}^T(\mathbf{I} - \mathbf{UU}^T)\mathbf{x} = \mathbf{x}^T\mathbf{x} - \mathbf{x}^T\mathbf{UU}^T\mathbf{x} = \|\mathbf{x}\|_2^2 - \|\mathbf{U}^T\mathbf{x}\|_2^2$.
Thus, $\|\mathbf{U}^T\mathbf{x}\|_2 \leq \|\mathbf{x}\|_2$.

Problem 4 $\mathbf{Q} = \mathbf{I} - 2\mathbf{uu}^T$
For unit vector $\mathbf{u} \in \mathbb{R}^n$. Notice that $\mathbf{Q} \in \mathbb{R}^{n\times n}$.
Also, notice that if a matrix $\mathbf{Q} \in \mathbb{R}^{n\times n}$ is orthogonal then by definition $\mathbf{Q}^T\mathbf{Q} = \mathbf{QQ}^T = \mathbf{I}$.

(a) $\mathbf{QQ}^T = (\mathbf{I}-2\mathbf{uu}^T)(\mathbf{I}-2\mathbf{uu}^T)^T = (\mathbf{I}-2\mathbf{uu}^T)(\mathbf{I}-2\mathbf{uu}^T) = \mathbf{I}-2\mathbf{uu}^T - 2\mathbf{uu}^T + 4\mathbf{uu}^T\mathbf{uu}^T = \mathbf{I} - 4\mathbf{uu}^T + 4\mathbf{uu}^T = \mathbf{I}$.

(b) For unit vector $\mathbf{u} \in \mathbb{R}^n$:
$\mathbf{Qu} = (\mathbf{I} - 2\mathbf{uu}^T)\mathbf{u} = \mathbf{u} - 2\mathbf{uu}^T\mathbf{u} = \mathbf{u} - 2\mathbf{u} = -\mathbf{u}$.
For any $\mathbf{u} \perp \mathbf{v}$:
$\mathbf{Qv} = (\mathbf{I} - 2\mathbf{uu}^T)\mathbf{v} = \mathbf{v} - 2\mathbf{uu}^T\mathbf{v} = \mathbf{v} - 0 = \mathbf{v}$.
Any $\mathbf{x} \in \mathbb{R}^n$ can be written as $\mathbf{x} = (\mathbf{u}^T\mathbf{x})\mathbf{u} + \mathbf{v}$, where $\mathbf{u}^T\mathbf{v} = \mathbf{u}^T(\mathbf{x} - (\mathbf{u}^T\mathbf{x})\mathbf{u}) = 0$, meaning $\mathbf{v} \perp \mathbf{u}$. Thus, $\mathbf{Qx} = -(\mathbf{u}^T\mathbf{x})\mathbf{u} + \mathbf{v}$ which can be interpreted as the reflection of $\mathbf{x}$ through a hyperplane with normal vector $\mathbf{u}$.

(c) Since $\mathbf{Q}$ is symmetric and orthogonal, $\mathbf{Q}^{-1} = \mathbf{Q}^T = \mathbf{Q}$. Hence, $\mathbf{x} = \mathbf{Q}^{-1}\mathbf{y} = \mathbf{Qy}$ which is the reflection back from $\mathbf{y}$.

(d) Using the given property: $det(\mathbf{Q}) = det(\mathbf{I}-2\mathbf{uu}^T) = det(1-2\mathbf{u}^T\mathbf{u}) = det(1-2) = -1$.

(e) The question is asking to find the vector $\mathbf{u}$ such that $\mathbf{x}$ and $\alpha\mathbf{y}$ for some α are reflections of each other through the hyper plane $\{\mathbf{z} : \mathbf{u}^T\mathbf{z} = 0\}$. Since $\mathbf{Q}$ is given to be in the span of $\mathbf{y}$ we can write $\mathbf{Qx} = \alpha\mathbf{y}$ for some constant α, or $\|\mathbf{x}\| = \|\mathbf{Qx}\| = \|\alpha\mathbf{y}\|$, which implies that $\alpha = \pm\frac{\|\mathbf{x}\|}{\|\mathbf{y}\|}$. We consider the case $\mathbf{Qx} = (\frac{\|\mathbf{x}\|}{\|\mathbf{y}\|})\mathbf{y} = \tilde{\mathbf{x}}$, but the other case works equally well and provides an alternative answer. Now note that if the hyperplane $\{\mathbf{z} : \mathbf{u}^T\mathbf{z} = 0\}$ reflects $\mathbf{x}$ to $\tilde{\mathbf{x}}$ and vice versa, then $\mathbf{x} - \tilde{\mathbf{x}}$ is normal to the hyperplane, or equivalently:
$\mathbf{u} = \frac{\mathbf{x}-\tilde{\mathbf{x}}}{\|\mathbf{x}-\tilde{\mathbf{x}}\|} = \frac{\mathbf{x}-(\frac{\|\mathbf{x}\|}{\|\mathbf{y}\|})\mathbf{y}}{\|\mathbf{x}-(\frac{\|\mathbf{x}\|}{\|\mathbf{y}\|})\mathbf{y}\|} = \frac{\|\mathbf{y}\|\mathbf{x}-\|\mathbf{x}\|\mathbf{y}}{\|\|\mathbf{y}\|\mathbf{x}-\|\mathbf{x}\|\mathbf{y}\|}$
which is the desired unit normal vector.
Alternatively, if we take $-\tilde{\mathbf{x}}$ as the reflection of $\mathbf{x}$, then $\mathbf{u} = \frac{\mathbf{x}+\tilde{\mathbf{x}}}{\|\mathbf{x}+\tilde{\mathbf{x}}\|} = \frac{\|\mathbf{y}\|\mathbf{x}+\|\mathbf{x}\|\mathbf{y}}{\|\|\mathbf{y}\|\mathbf{x}+\|\mathbf{x}\|\mathbf{y}\|}$.

Problem 5 A symmetric matrix $\mathbf{P} = \mathbf{P}^T \in \mathbb{R}^{n\times n}$ is said to be a projection matrix if $\mathbf{P} = \mathbf{P}^2$.

(a) $(\mathbf{I}-\mathbf{P})^T = \mathbf{I} - \mathbf{P}^T = \mathbf{I}-\mathbf{P}$ and so $\mathbf{I}-\mathbf{P}$ is symmetric. Also, $(\mathbf{I}-\mathbf{P})^2 = (\mathbf{I}-\mathbf{P})(\mathbf{I}-\mathbf{P}) = \mathbf{I}-\mathbf{P}-\mathbf{P}+\mathbf{P}^2 = \mathbf{I}-2\mathbf{P}+\mathbf{P} = \mathbf{I}-\mathbf{P}$. Hence, $\mathbf{I}-\mathbf{P}$ is symmetric and $\mathbf{I}-\mathbf{P} = (\mathbf{I}-\mathbf{P})^2$, and thus it is a projection matrix.

(b) Notice that $\mathbf{UU}^T = (\mathbf{U}^T)^T\mathbf{U} = \mathbf{UU}^T$ and $(\mathbf{UU}^T)^2 = \mathbf{UU}^T\mathbf{UU}^T = \mathbf{UIU}^T = \mathbf{UU}^T$.

(c) First of all, $\mathbf{A}^T\mathbf{A}$ is invertible for a full-rank tall matrix $\mathbf{A}$ (from problem 6 in HW 2). For symmetry, consider
$(\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T)^T = (\mathbf{A}^T)^T((\mathbf{A}^T\mathbf{A})^{-1})^T\mathbf{A}^T = \mathbf{A}((\mathbf{A}^T\mathbf{A})^{-1})^T\mathbf{A}^T = \mathbf{A}((\mathbf{A}^T\mathbf{A})^T)^{-1}\mathbf{A}^T = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$.
Also, consider
$(\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T)^2 = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$.

(d) It suffices to show that $\|\mathbf{x}-\mathbf{v}\|$ is minimized over all $\mathbf{v} \in \mathcal{R}(\mathbf{P})$ by $\mathbf{v}^* = \mathbf{Px}$. For any $\mathbf{v} \in \mathcal{R}(\mathbf{P})$:
$\|\mathbf{x}-\mathbf{v}\|^2 = \|\mathbf{x}-\mathbf{Px}+\mathbf{Px}-\mathbf{v}\|^2 = \|\mathbf{x}-\mathbf{Px}\|^2+\|\mathbf{Px}-\mathbf{v}\|^2+2(\mathbf{x}-\mathbf{Px})^T(\mathbf{Px}-\mathbf{v})$
$= \|\mathbf{x}-\mathbf{Px}\|^2 + \|\mathbf{Px}-\mathbf{v}\|^2 + 2(\mathbf{x}^T-\mathbf{x}^T\mathbf{P})(\mathbf{Px}-\mathbf{v})$
$= \|\mathbf{x}-\mathbf{Px}\|^2 + \|\mathbf{Px}-\mathbf{v}\|^2 + 2(\mathbf{x}^T\mathbf{Px}-\mathbf{x}^T\mathbf{v}-\mathbf{x}^T\mathbf{P}^2\mathbf{x}+\mathbf{x}^T\mathbf{Pv})$
$= \|\mathbf{x}-\mathbf{Px}\|^2 + \|\mathbf{Px}-\mathbf{v}\|^2 + 2(\mathbf{x}^T\mathbf{Px}-\mathbf{x}^T\mathbf{v}-\mathbf{x}^T\mathbf{Px}+\mathbf{x}^T\mathbf{v})$
$= \|\mathbf{x}-\mathbf{Px}\|^2 + \|\mathbf{Px}-\mathbf{v}\|^2 \geq \|\mathbf{x}-\mathbf{Px}\|^2$.
Where we used $\mathbf{P} = \mathbf{P}^2$ and $\mathbf{Pv} = \mathbf{v}, \forall\mathbf{v} \in \mathcal{R}(\mathbf{P})$. To achieve equality in the last inequality, we need $\|\mathbf{Px}-\mathbf{v}\| = 0 \to \mathbf{v} = \mathbf{Px}$. Thus, $argmin_{\mathbf{v}\in\mathcal{R}(\mathbf{P})}\|\mathbf{x}-\mathbf{v}\| = \mathbf{Px}$.

(e) Consider $\mathbf{P} = \mathbf{uu}^T$. Since $\mathbf{P} = \mathbf{P}^T$ and $\mathbf{P} = \mathbf{P}^2$, $\mathbf{P}$ is a projection matrix. Since $\mathcal{R}(\mathbf{P}) = span(\mathbf{u})$, using part (d) $\mathbf{Px}$ is the projection of $\mathbf{x}$ onto $span(\mathbf{u})$. This can also be directly verified since $(\mathbf{u}^T\mathbf{x})\mathbf{u} = \mathbf{uu}^T\mathbf{x} = \mathbf{Px}$ is the component of $\mathbf{x}$ in the direction for the unit vector $\mathbf{u}$.