Problem 1.

(a).

$AA^+ =$ ~~$AA^+=$~~

Suppose there are two A^+, A_1^+ and A_2^+

$$AA_1^+ = AA_2^+AA_1^+ \qquad A_1^+A = A_1^+AA_2^+A$$

$$= (A_2^+)^T A^T (A_1^+)^T A^T \qquad = A^T (A_1^+)^T A^T (A_2^+)^T$$

$$= (A_2^+)^T (AA_1^+A)^T \qquad = (AA_1^+A)^T (A_2^+)^T$$

$$= (A_2^+)^T A^T \qquad = A^T (A_2^+)^T$$

$$= AA_2^+ \qquad = A_2^+ A$$

$$A_1^+AA_1^+ = A_1^+AA_2^+ = A_2^+AA_2^+ = A_2^+ = A_1^+$$

So A^+ is unique.

(b).

$$A(A^TA)^{-1}A^TA = AI = A$$

$$(A^TA)^{-1}A^TA(A^TA)^{-1}A^T = (A^TA)^{-1}IA^T = (A^TA)^{-1}A^T$$

$$(A(A^TA)^{-1}A^T)^T = A((A^TA)^{-1})^T)^TA^T = A(A^TA)^{-1}A^T$$

$$(A^TA)^{-1}A^TA = I \quad I^T = I$$

So $(A^TA)^{-1}A^T$ satisfies all the properties of a pseudoinverse of A.

$(A^TA)^{-1}A^TA = I$, so $(A^TA)^{-1}A^T$ is a left inverse of A.

(c).

$$AA^T(AA^T)^{-1}A = IA = A$$

$$A^T(AA^T)^{-1}AA^T(AA^T)^{-1} = A^T(AA^T)^{-1}I = A^T(AA^T)^{-1}$$

$$(AA^T(AA^T)^{-1})^T = I^T \cancel{0} = I = AA^T(AA^T)^{-1}$$

$$(A^T(AA^T)^{-1}A)^T = A^T(((AA^T)^{-1})^T)^T A = A^T(AA^T)^{-1}A$$

So $A^T(AA^T)^{-1}$ satisfies all the properties of pseudoinverse of A.

$$\cancel{AA^T} \; AA^T(AA^T)^{-1} = I$$

So $A^T(AA^T)^{-1}$ is a right inverse of A.

(d).

$$AA^{-1}A = AI = A \qquad A^{-1}AA^{-1} = A^{-1}I = A^{-1}$$

$$(AA^{-1})^T = I^T = I = AA^{-1}$$

$$(A^{-1}A)^T = I^T = I = A^{-1}A$$

So A^{-1} satisfies all the properties of a pseudoinverse of A.

(e). Since A is a projection matrix

$A^2 = A$ and A is symmetric

$$AAA = A^2A = AA = A^2 = A$$

$$(AA)^T = A^T = A = AA$$

So A satisfies all the properties of a pseudoinverse of A.

(f).

$$A^T(A^T)^+ = (AA^+A)^T(A^T)^+$$
$$= A^T(A^+)^T A^T(A^T)^+$$
$$= A^+A((A^T)^+)^T A$$
$$= A^+(A^T(A^T)^+A^T)^T$$
$$= A^+A$$
$$= A^T(A^+)^T$$

$$(A^T)^+A^T = (A^T)^+(AA^+A)^T$$
$$= (A^T)^+A^T(A^+)^T A^T$$
$$= A((A^T)^+)^T A A^+$$
$$= (A^T(A^T)^+A^T)^T A^+$$
$$= AA^+$$
$$= (A^+)^T A^T$$

$$(A^T)^+ A^T (A^T)^+ = (A^T)^+ = (A^T)^+ A^T (A^+)^T = (A^+)^T A^T (A^+)^T$$
$$= (A^+AA^+)^T = (A^+)^T$$

So $(A^T)^+ = (A^+)^T$

(g).

$$(AA^T)(AA^T)^+ = AA^+AA^T(A^T)^+A^T(AA^T)^+$$
$$= \cancel{AA^+AA^T(A^+)^TA^T(AA^T)^+}\ AA^+A((A^T)^+)^T AA^T(AA^T)^+$$
$$= (A^+)^TA^T AA^+AA^T(AA^T)^+$$
$$= (A^+)^TA^T(A^+)^T A^T((AA^T)^+)^T AA^T$$
$$= (A^+)^TA^+ AA^T((AA^T)^+)^T AA^T$$
$$= (A^+)^TA^+(AA^T)^T$$
$$= AA^T(A^+)^TA^+$$

$(AA^T)^+ AA^T = (AA^T)^+ AA^+A\, A^T(A^T)^+A^T$

$= (AA^T)^+ AA^T (A^+)^T A^T A((A^T)^+)^T$

$= AA^T((AA^T)^+)^T A((A^T)^+)^T A((A^T)^+)^T$

$= AA^T((AA^T)^+)^T AA^T (A^+)^T((A^T)^+)^T$

$= (A^+A^T(AA^T)^+AA^T)^T ((A^T)^+A^+)^T$

$= (AA^T)^T((A^+)^TA^+)^T$

$= \cancel{(A^+)^TA^+ AA^+}\quad AA^T(A^+)^TA^+$

$= AA^T(A^T)^+A^+$

$= A((A^T)^+)^T AA^+$

$= A((A^+)^T)^T(A^+)^TA^T$

$= (A^+)^T A^T(A^T)^+A^T$

$= (A^+)^T((A^+)^+)^T AA^T$

$= (A^+)^TA^+AA^T$

$(AA^T)^+ AA^T(AA^T)^+ = (AA^T)^+ = (AA^T)^+AA^T(A^+)^TA^+ = (A^+)^TA^+AA^T(A^+)^TA^+$

$= (A^+)^TA^+AA^+AA^+$

$= (A^+)^TA^+AA^+ = (A^+)^TA^+$

So $(AA^T)^+ = (A^+)^TA^+$

Let $B = A^T$

$(BB^T)^+ = (B^+)^TB^+ \Leftarrow\Rightarrow (A^TA)^+ = A^+(A^T)^+ = A^+(A^+)^T$

(h).

Let $x \in R(A^+)$

$\cancel{A^+ x = y}$

$A^+ y = x$

$\Longleftrightarrow \quad A^+ A A^+ y = x$

$\Longleftrightarrow \quad A^T (A^+)^T A^+ y = x$

So $x \in R(A^T)$

Let $x \in R(A^T)$

$A^T y = x$

$\Longleftrightarrow \quad A^T (A^T)^+ A^T y = x$

$\Longleftrightarrow \quad [(A^T)^+]^T A A^T y = x$

$\Longleftrightarrow \quad [(A^+)^T]^T A A^T y = x$

$\Longleftrightarrow \quad A^+ A A^T y = x$

So $x \in R(A^+)$

Then $R(A^+) = R(A^T)$

Let $x \in N(A^+)$

$A^+ x = 0$

$A^T x = A^T (A^T)^+ A^T x$

$= A^T A A^+ x$

$= 0$

$x \in N(A^T)$

Let $x \in N(A^T)$

$A^T x = 0$

$A^+ x = A^+ A A^+ x$

$= A^+ (A^+)^T A^T x$

$= 0$

$x \in N(A^+)$

So $\cancel{R(A}$ $N(A^+) = N(A^T)$

(i). Since AA^+ and A^+A are already symmetric

we only need to show that

$(AA^+)^2 = AA^+$ and $(A^+A)^2 = A^+A$

$(AA^+)^2 = AA^+AA^+ = AA^+$ $\quad$ $(A^+A)^2 = A^+AA^+A = A^+A$

So P and Q are projection matrices.

(j) Let $w \in R(A)$, $w = Ab$

$\langle x - Px, w \rangle$

$= w^T x - w^T P x$

$= w^T x - w^T A A^+ x$

$= b A^T x - b A^T A A^+ x$

$= b^T A^T x - b^T A^T (A^+)^T A^T x$

$= b^T A^T x - b^T A^T (A^T)^+ A^T x$

$= b^T A^T x - b^T A^T x$

$= 0$

for every b

So $y = Px$ are the projection of x onto $R(A)$

Let $w \in R(A^T)$, $w = A^T b$

$\langle x - Qx, w \rangle$

$= w^T x - w^T Qx$

$= b^T A x - b^T A A^+ A x$

$= b^T A x - b^T A x$

$= 0$

for every b

So $y = Qx$ are the projection of x onto $R(A^T)$

(k).

$$\|Ax-b\|_2^2 = \|Ax - Ax^* + Ax^* - b\|_2^2$$

$$= \|Ax - Ax^*\|_2^2 + \underbrace{\langle A(x-x^*), Ax^*-b\rangle + \langle Ax^*-b, A(x-x^*)\rangle} + \|Ax^*-b\|_2^2$$

$$= 2(x-x^*)^T A^T (Ax^*-b)$$

$$= 2(x-x^*)^T (A^T A A^+ b - A^T b)$$

$$= 2(x-x^*)^T (A^T (A^+)^T A^T b - A^T b)$$

$$= 2(x-x^*)^T (A^T b - A^T b)$$

$$= 0$$

So $\|Ax-b\|_2^2 = \|Ax-Ax^*\|_2^2 + \|Ax^*-b\|_2^2 \geq \|Ax^*-b\|_2^2$

$\|Ax-b\|_2 \geq \|Ax^*-b\|_2$

(l). $\|x\| = \|x - x^* + x^*\| = \underbrace{\langle x - x^*, x^* \rangle + \langle x^*, x - x^* \rangle}_{} + \langle x - x^*, x - x^* \rangle + \langle x^*, x^* \rangle$

$$= 2(x^*)^T (x - x^*)$$

$$= 2b^T (A^+)^T (x - A^+ b)$$

$$= 2b^T (A^T)^+ (x - A^+ b)$$

$$= 2b^T (A^T)^+ A^T (A^T)^+ (x - A^+ b)$$

$$= 2b^T (A^T)^+ (A^+ A x - A^+ A A^+ b)$$

$$= 2b^T (A^T)^+ (A^+ b - A^+ b)$$

$$= 0$$

So $\|x\| = \|x - x^*\| + \|x^*\| \geq \|x^*\|$ for every x.

Problem 2.

(a). Since A has n distinct eigenvalues

$\det(A-\lambda I)$ canbe written in a polynomial of λ of degree n

$$\det(A-\lambda I) = (\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_n-\lambda)$$

Let $\lambda = 0$

$$\det(A-0\cdot I) = \det(A) = \lambda_1\lambda_2\cdots\lambda_n$$

(b). Since $\det(A^T) = \det(A)$

$$\det(A^T-\lambda I) = \det((A-\lambda I)^T) = \det(A-\lambda I)$$

So A^T and A have the same set of λ.

(c).

$Av=\lambda_i v, \quad i\in 1,\cdots n$

$AAv = A\lambda_i v = \lambda_i Av = \lambda_i^2 v$

$\cdots$

$A^k v = \lambda_i^k v$

So λ_i^k is the eigenvalue of A^k

(d).

Since $\det(A) = \lambda_1 \cdots \lambda_n$

If A is invertible, then A is nonsingular

From 6.1.13 we know

$\det(A) \neq 0$ if and only if A is nonsingular

So $\lambda_1 \cdots \lambda_n \neq 0$

which means A does not have a zero eigenvalues.

(e).

$Av = \lambda_i v \quad i\in 1,\cdots n$

$\Leftrightarrow A^{-1}Av = \lambda_i A^{-1} v$

$\Leftrightarrow v = \lambda_i A^{-1} v$

$\lambda_i^{-1} v = A^{-1} v$

So λ_i^{-1} is the eigenvalues of A^{-1}

(f).

$$\det(AB) = \det(A)\det(B)$$

$$\det(T^{-1})\det(A-\lambda I)\det(T)$$

$$= \det(T^{-1}A - T^{-1}\lambda I)\det(T)$$

$$= \det(T^{-1}AT - \lambda T^{-1}T) = \det(T^{-1}AT - \lambda I) = \det(A-\lambda I)(\det(T^{-1}T)$$

$$= \det(A-\lambda I)$$

So A and $T^{-1}AT$ have the same set of eigenvalues.

Problem 3.

(a).

$$\det(A-\lambda I) = (\lambda_1-\lambda)\cdots(\lambda_n-\lambda)$$
$$= (\lambda_1+\lambda_2+\cdots\lambda_n)\lambda^{n-1} + \cdots$$

$$\det(A-\lambda I) = \sum_p \sigma(p)\, a_{1p_1}\cdots a_{np_n}$$

$$= (A_{11}-\lambda)(A_{22}-\lambda)\cdots(A_{nn}-\lambda) + \sum_{p\neq(1,2\cdots n)} \sigma(p)\, a_{1p_1}\cdots a_{np_n}$$

$$= (A_{11}+A_{22}+\cdots A_{nn})\lambda^{n-1} + \cdots + \sum_{p\neq(1,2\cdots n)} \sigma(p)\, a_{1p_1}\cdots a_{np_n}$$

So $(\lambda_1+\lambda_2+\cdots\lambda_n) = A_{11}+A_{22}+\cdots A_{nn} = Tr(A)$

(b).

Since $\lambda_1^k\cdots\lambda_n^k$ is the eigen value of A^k

Let $B=A^k$

$(\lambda_1^k+\cdots\lambda_n^k) = Tr(B) = Tr(A^k)$

Problem 4.

The situation in problem (3) is consistent when $A \in C^{n\times n}$.

$$tr(A^2) = |\lambda_1|^2 + |\lambda_2|^2 + \cdots |\lambda_n|^2 = \sum_{i=1}^{n} |\lambda_i|^2$$

$$\cancel{\sum_{i=1}^{n} |\lambda_i|^2 \le tr(A^2)}$$

$$|A|^2 = \sum_{i=1}^{n} |a_{ii}|^2 + \cancel{\sum_{i=1}^{n}\sum_{j=1}^{n} 2|a_{ij}||a_{ji}|} \le \sum_{i=1}^{n} |a_{ii}|^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^2 + |a_{ji}|^2$$

$$\sum_{i\neq j}^{n} |a_{ij}|^2 |a_{ji}|$$

Problem 4.

The situation in problem (3) is consisten for $|A|$ since $|A| \in R^{n\times n}$

$$tr(|A|^2) = |\lambda_1|^2 + |\lambda_2|^2 + \cdots |\lambda_n|^2 = \sum_{i=1}^{n} |\lambda_i|^2$$

$$|A|^2_{ii} = |a_{ii}|^2 + \sum_{j=1}^{n} |a_{ij}||a_{ji}|$$

$$tr(|A|^2) = \sum_{i=1}^{n} |A|^2_{ii} = \sum_{i=1}^{n} |a_{ii}|^2 + \sum_{j=i+1}^{n} 2|a_{ij}||a_{ji}| \le \sum_{i=1}^{n} |a_{ii}|^2 + \sum_{i=1}^{n}\sum_{j=i+1}^{n} |a_{ij}|^2 + |a_{ji}|^2$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} |A_{ij}|^2$$

Problem 5.

First we find the eigenvalue and ~~vectors~~ eigenvectors of A by

$$\det|A-\lambda I| = 0$$

$$\begin{pmatrix} 5-\lambda & -\frac{8}{5} \\ 12 & -\frac{19}{5}-\lambda \end{pmatrix} = 0$$

$$\lambda = 1 \text{ or } \frac{1}{5}$$

eigenvector $v = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Let $B = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$

$$B^{-1} = \frac{1}{\det(B)} \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} = \cancel{\begin{pmatrix} 3 & 4 \\ -5 & -2 \end{pmatrix}} \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} = B \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{pmatrix} B^{-1}$$

$$A^n = B \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{pmatrix}^n B^{-1}$$

$$\lim_{n\to\infty} A^n = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & -2 \\ 15 & -5 \end{pmatrix}$$