Question: 4. Problem 4: Spectral Norm. (a) Show that ||AH A || = || A||2. (b) Show that the spectral norm i...

4. **Problem 4: Spectral Norm.**

   (a) Show that $\|A^H A\| = \|A\|^2$.

   (b) Show that the spectral norm is *unitarily invariant*, namely, $\|UAV\| = \|A\|$ for any unitary matrices U and V.

   (c) Show that

$$\left\| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right\| = \max(\|A\|, \|B\|).$$

Expert Answer

Solution:-

Spectral Norm:-

The spectral norm 2-Norm is denoted by $\|A\|_2$

and it is given as

$$\|A\|_2 = \sqrt{\rho(A^T A)}$$

Where,

ρ = The spectral radius

So the square root of the spectral radius is the spectral norm.

(a) $\|A^H A\| = \|A\|^2$

The spectral radius formula holds for any matrix and any

norm: $\|A_n\|^{1/n} \to \rho(A)$

The spectral norm of the matrix A is defined by

$$\|A\|_2 = \bar{\sigma}(A)$$

(c) we first show that $\|Ax\| \le \|A\|\,\|x\|$.

suppose that this is not the case. then

$$\|Ax\| > \|A\|\,\|x\|$$

$$\Rightarrow \frac{1}{\|x\|} > \|A\|\,\|x\| \cdot > \|A\|$$

$$\Rightarrow \left\| A \frac{x}{\|x\|} \right\| > \|A\|$$

but $\frac{x}{\|x\|}$ is a vector of unit norm. The contradicts the definition of $\|A\|$.

Now we proceed to prove the

$$\|AB\| = \max_{\|x\| \leq 1} \|ABx\| \leq \max_{\|x\| \leq 1} \|A$$

(b) $\|UAV\| = \|A\|$

For any $A \in C^{m,n}$ and any unita

$U \in C^{m,m}$ and $V \in C^{n,n}$

If U and V are unitary then $U(A$

where, $\|F\| = \|E\|$