Solutions for problems of shannon theory

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Definition

Quantum Channel Capacity

$$Q(\mathcal{E}) = \lim_{m\to\infty} \sup \frac{logd}{m}$$

where m, d satiesfies: $\exists d$ dimension subspace $S \subseteq H_{input}^{\otimes n}$

$$\int_{|\phi\rangle \in S} \langle\phi| \mathcal{E}(|\phi\rangle\langle\phi|) |\phi\rangle \, d(|\phi\rangle) \geq 1 - \varepsilon$$

Remark:

- The average fidelity criterion, which is equavelant to several other fidelity critierion, we'll use this definition throughout this solution
- Basically, the definition means the maximal ratio we can reliably transform quantum information through the quantum channel.

Definition

Coherent Information

$Q^{(1)}(\mathcal{E}) = \max_{\phi_{AA^\star}} H(\rho_B) - H(\rho_{AB}) \quad \text{where} H(\rho) = -tr(\rho \log_2 \rho)$
$\rho_{AB} = \mathcal{E}_{A^\star \to B}(\phi_{AA^\star})$ and $A = A^\star$ are isomorphic.

Lloyd, Shor, and Devetak (LSD) showed $Q^{(1)}(\mathcal{E})$ can be achieved, i.e.

$$Q(\mathcal{E}) \geq Q^{(1)}(\mathcal{E})$$

Problem

Show that $Q^{(n)}(\mathcal{E}) = \frac{1}{n} Q^{(1)}(\mathcal{E}^{\otimes n})$ is achievable.

- Step 1. $Q^{(1)}(\mathcal{E}^{\otimes n})$ is achievable for $\mathcal{E}^{\otimes n}$
- Step 2. By definition of quantum channel capacity, we can prove $Q(\mathcal{E}) \geq \frac{1}{n} Q^{(1)}(\mathcal{E}^{\otimes n}) = Q^{(n)}(\mathcal{E})$

Problem

Show $\sup_{n \in N} Q^{(n)}(\mathcal{E}) = \lim_{n \to \infty} Q^{(n)}(\mathcal{E})$

Coherent information have subadditive property, i.e.

Lemma

$$Q^{(m+n)}(\mathcal{E}) \geq Q^{(m)}(\mathcal{E}) + Q^{(n)}(\mathcal{E})$$

Subadditivity of coherent information

$$Q^{(m+n)}(\mathcal{E}) \geq Q^{(m)}(\mathcal{E}) + Q^{(n)}(\mathcal{E})$$

Proof: Recall

$$Q^{(1)}(\varepsilon) = \max_{\phi_{AA^\star}} H(\rho_B) - H(\rho_{AB})$$

and

$$\rho_{AB} = \mathcal{E}_{A^\star \to B}(\phi_{AA^\star})$$

simply take

$$\phi_{A_{m+n}A^\star_{m+n}} = \phi_{A_m A^\star_m \otimes A_n A^\star_n}$$

by additivity of von Neumann entropy, we can conclude

$$Q^{(m+n)}(\mathcal{E}) \geq Q^{(m)}(\mathcal{E}) + Q^{(n)}(\mathcal{E}) \quad \square$$

Disprove the additivity conjucture

Depolarizing Channel

$$D_p(\rho) = p\rho + (1-p)\frac{I}{2}, \textit{ for } 0 < p < 1$$

Problem

Calculate $Q^{(1)}(D_p)$

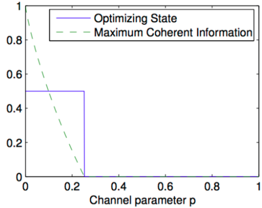
Answer: The depolarizing channel is unitarily covariant, i.e.

$$D_p(U^\dagger \rho U) = U^\dagger D_p(\rho) U$$

Consider, the form $\sqrt{\mu}\,|00\rangle + \sqrt{1-\mu}\,|11\rangle$, we can derive that maximally entangled state $\mu = 1/2$ is optimal for all values of the depolarizing parameter p for which the coherent information is non-negative, and for all other values, a product state with $\mu = 0$ is optimal.

When $\mu = 1/2$, $H(\rho_B) = 1$, since $\rho_B = I/2$.
Similarly, we can derive

$$H(\rho_{AB}) = -(1 - \frac{3p}{4})log(1 - \frac{3p}{4}) - \frac{3p}{4}log(\frac{p}{4})$$



For more on this problem, see Chapter 24.8 on
`https://arxiv.org/pdf/1106.1445.pdf`

Entanglement-breaking Channel

Independent of the input state $\phi_{AA^\star}$, the output state $\rho_{AB} = \mathcal{E}_{A^\star \to B}(\phi_{AA^\star})$ can be written in the form

$$\rho_{AB} = \sum_k p(k) \rho_k^A \otimes \rho_k^B$$

Problem

$$Q^{(n)}(\mathcal{E}) = 0 \quad \forall n \in N$$

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Proof: W.L.O.G, we can expand ρ_k^A and ρ_k^B in orthonormal basis, i.e

$$\rho_{AB} = \sum_{x,y} p(x,y)\rho_A^x \otimes \rho_B^y$$

where ρ_A^x and ρ_B^y orthonormal to each other.
Lemma: $\rho = \sum_x p_x \rho_x$ and ρ_x are orthonormal to each other,

$$H(\rho) = H(X) + \sum_{p_x} p_x H(\rho_x)$$

As a consequence, we have

$$\begin{aligned}
H(\rho_{AB}) &= H(X,Y) + \sum_{x,y} p_{xy}(S(\rho_x) + H(\rho_y)) \\
&= H(X,Y) + \sum_x p_x S(\rho_x) + \sum_y p_y S(\rho_y) \\
&\geq H(X,Y) + \sum_x p_x S(\rho_x) \\
&= H(\rho_B)
\end{aligned}$$

Proof of Lemma:

$$\begin{aligned}
H(\rho) &= -tr(\rho \log \rho) \\
&= -tr(\sum_x p_x \rho_x \log \sum_x p_x \rho_x) \\
&= -tr(\sum_x p_x \rho_x \sum_x \log p_x \rho_x) \\
&= -tr(\sum_x p_x \rho_x (\log p_x + \log \rho_x)) \\
&= -\sum_x p_x \log p_x \, tr(\rho_x) - \sum p_x \, tr(\rho_x \log \rho_x) \\
&= H(X) + \sum_x p_x H(\rho_x)
\end{aligned}$$

$\square$

Thanks for Listening!