线性代数试题参考解答

(2013.11)

一、1．$-\dfrac{1}{8}$；　2．$\begin{pmatrix} \boldsymbol{O} & \boldsymbol{C}^{-1} \\ \boldsymbol{B}^{-1} & -\boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{C}^{-1} \end{pmatrix}$；　3．D；

4．$\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_4$(或$\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_3,\boldsymbol{\alpha}_4$)；　5．$n-2$；　6．$-1$，3；

7．B；　8．$t<0$．

二、$$D_n \xlongequal{\text{按第1行展开}} (a+b)D_{n-1} + 1\cdot(-1)^{1+2}\begin{vmatrix} ab & 1 & & & \\ & a+b & 1 & & \\ & ab & a+b & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & ab & a+b \end{vmatrix}_{n-1}$$

$$\xlongequal{\text{按第1列展开}} (a+b)D_{n-1} - abD_{n-2}$$

变形为

$$D_n - aD_{n-1} = b(D_{n-1} - aD_{n-2})$$

由于 $D_1 = a+b$， $D_2 = (a+b)^2 - ab = a^2 + ab + b^2$，利用上述递推公式得

$$D_n - aD_{n-1} = b(D_{n-1} - aD_{n-2}) = b^2(D_{n-2} - aD_{n-3}) = \cdots = b^{n-2}(D_2 - aD_1) = b^n$$

于是

$$D_n = aD_{n-1} + b^n = a^2 D_{n-2} + ab^{n-1} + b^n = \cdots = a^{n-1}D_1 + a^{n-2}b^2 + \cdots + ab^{n-1} + b^n$$

$$= a^n + a^{n-1}b + a^{n-2}b^2 + \cdots + ab^{n-1} + b^n = \frac{a^{n+1} - b^{n+1}}{a-b}$$

三、可求得 $\det \boldsymbol{A} = 4$，于是 $\boldsymbol{A}^* = 4\boldsymbol{A}^{-1}$，代入矩阵方程得

$$4\boldsymbol{A}^{-1}\boldsymbol{X}\boldsymbol{A} = 8\boldsymbol{A}^{-1}\boldsymbol{X} - 4\boldsymbol{E}$$

上式左乘 $\dfrac{1}{4}\boldsymbol{A}$ 得 $\boldsymbol{X}\boldsymbol{A} = 2\boldsymbol{X} - \boldsymbol{A}$，从而

$$\boldsymbol{X} = -\boldsymbol{A}(\boldsymbol{A} - 2\boldsymbol{E})^{-1} = -\boldsymbol{A}\begin{pmatrix} -1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}^{-1} = -\boldsymbol{A}\cdot(-\frac{1}{2})\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

四、设 $\boldsymbol{\beta} = x_1\boldsymbol{\alpha}_1 + x_2\boldsymbol{\alpha}_2 + x_3\boldsymbol{\alpha}_3 + x_4\boldsymbol{\alpha}_4$，比较分量得

$$\begin{cases} -x_2 + ax_3 = 0 \\ -ax_1 + x_2 + ax_4 = 1 \\ x_1 - x_3 - x_4 = 1 \\ x_2 - ax_3 = b \end{cases}$$

增广矩阵

$$\left(\begin{array}{cccc:c} 0 & -1 & a & 0 & 0 \\ -a & 1 & 0 & a & 1 \\ 1 & 0 & -1 & -1 & 1 \\ 0 & 1 & -a & 0 & b \end{array}\right) \xrightarrow{\text{初等行变换}} \left(\begin{array}{cccc:c} 1 & 0 & -1 & -1 & 1 \\ 0 & 1 & -a & 0 & 1+a \\ 0 & 0 & 0 & 0 & 1+a \\ 0 & 0 & 0 & 0 & b \end{array}\right)$$

当$\boldsymbol{\beta}$可由$\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3,\boldsymbol{\alpha}_4$线性表示时上述线性方程组有解，所以$1+a=0,b=0$，故$a=-1,b=0$．代入上式得同解方程组

$$\begin{cases} x_1 = 1 + x_3 + x_4 \\ x_2 = -x_3 \end{cases}$$

通解为

$$x_1 = 1 + k_1 + k_2，\quad x_2 = -k_1，\quad x_3 = k_1，\quad x_4 = k_2 \qquad k_1,k_2\text{任意}$$

于是

$$\boldsymbol{\beta} = (1+k_1+k_2)\boldsymbol{\alpha}_1 - k_2\boldsymbol{\alpha}_2 + k_1\boldsymbol{\alpha}_3 + k_2\boldsymbol{\alpha}_4 \qquad k_1,k_2\text{任意}$$

五、1) 因为

$$(\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\boldsymbol{\beta}_3) = (\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3)\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}，\quad (\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2,\boldsymbol{\gamma}_3) = (\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3)\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

所以

$$(\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2,\boldsymbol{\gamma}_3) = (\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\boldsymbol{\beta}_3)\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$= (\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\boldsymbol{\beta}_3)\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = (\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\boldsymbol{\beta}_3)\begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

故由基$\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\boldsymbol{\beta}_3$到基$\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2,\boldsymbol{\gamma}_3$的过渡矩阵为$\boldsymbol{C} = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$.

$$2)\quad \boldsymbol{\alpha} = (\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3)\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = (\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2,\boldsymbol{\gamma}_3)\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$$

$$= (\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2,\boldsymbol{\gamma}_3)\frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = (\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2,\boldsymbol{\gamma}_3)\begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$$

故向量$\boldsymbol{\alpha}$在基$\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2,\boldsymbol{\gamma}_3$下的坐标为$(3,-1,-2)^{\mathrm{T}}$.

六、线性无关．设

$$k_1\boldsymbol{\alpha}_1+k_2\boldsymbol{\alpha}_2+k_3\boldsymbol{\alpha}_3+k_4(b\boldsymbol{\beta}_1+\boldsymbol{\beta}_2)=\mathbf{0}$$

由题设$\boldsymbol{\beta}_1=l_1\boldsymbol{\alpha}_1+l_2\boldsymbol{\alpha}_2+l_3\boldsymbol{\alpha}_3$，代入上式并整理得

$$(k_1+bl_1k_4)\boldsymbol{\alpha}_1+(k_2+bl_2k_4)\boldsymbol{\alpha}_2+(k_3+bl_3k_4)\boldsymbol{\alpha}_3+k_4\boldsymbol{\beta}_2=\mathbf{0}$$

由于$\boldsymbol{\beta}_2$不能由$\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3$线性表示，所以$\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3,\boldsymbol{\beta}_2$线性无关，故由上式得

$$\begin{cases} k_1 + bl_1k_4=0 \\ k_2+bl_2k_4=0 \\ k_3+bl_3k_4=0 \\ k_4=0 \end{cases}$$

该齐次线性方程组的系数行列式

$$\det\boldsymbol{A}=\begin{vmatrix} 1 & 0 & 0 & bl_1 \\ 0 & 1 & 0 & bl_2 \\ 0 & 0 & 1 & bl_3 \\ 0 & 0 & 0 & 1 \end{vmatrix}=1$$

故只有$k_1=k_2=k_3=k_4=0$，从而向量组$\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3,b\boldsymbol{\beta}_1+\boldsymbol{\beta}_2$线性无关.

七、二次型$f=x^2+3y^2+z^2+2axy+2xz+2yz$的矩阵

$$\boldsymbol{A}=\begin{pmatrix} 1 & a & 1 \\ a & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}\text{正交相似于对角矩阵}\begin{pmatrix} 0 & & \\ & 1 & \\ & & 4 \end{pmatrix}$$

所以$\boldsymbol{A}$的特征值为$\lambda_1=0$，$\lambda_2=1$，$\lambda_3=4$，再由$\det\boldsymbol{A}=-(a-1)^2=\lambda_1\lambda_2\lambda_3=0$得$a=1$.

可求得对应的特征向量分别为

$$\boldsymbol{p}_1=(-1,0,1)^{\mathrm{T}},\quad \boldsymbol{p}_2=(1,-1,1)^{\mathrm{T}},\quad \boldsymbol{p}_3=(1,2,1)^{\mathrm{T}}$$

单位化得正交矩阵$\boldsymbol{Q}=\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}$.

八、1) $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{P}=\begin{pmatrix} \boldsymbol{A}-\boldsymbol{B}\boldsymbol{C}^{-1}\boldsymbol{B}^{\mathrm{T}} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{C} \end{pmatrix}=\boldsymbol{D}$；(2 分)

2) 由$\boldsymbol{M}$正定知$\boldsymbol{A}$和$\boldsymbol{C}$均为正定矩阵，从而它们是对称矩阵，于是

$$(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{C}^{-1}\boldsymbol{B}^{\mathrm{T}})^{\mathrm{T}}=\boldsymbol{A}^{\mathrm{T}}-\boldsymbol{B}(\boldsymbol{C}^{-1})^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}=\boldsymbol{A}-\boldsymbol{B}\boldsymbol{C}^{-1}\boldsymbol{B}^{\mathrm{T}}$$

即$\boldsymbol{A}-\boldsymbol{B}\boldsymbol{C}^{-1}\boldsymbol{B}^{\mathrm{T}}$是对称矩阵.(2 分)

又由 1)知$\boldsymbol{M}$与$\boldsymbol{D}$合同，从而由$\boldsymbol{M}$正定知$\boldsymbol{D}$为正定矩阵，且对任意m维非零列向量$\boldsymbol{x}$和n维零向量$\mathbf{0}$有

$$\boldsymbol{x}^{\mathrm{T}}(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{C}^{-1}\boldsymbol{B}^{\mathrm{T}})\boldsymbol{x}=(\boldsymbol{x}^{\mathrm{T}},\mathbf{0}^{\mathrm{T}})\begin{pmatrix} \boldsymbol{A}-\boldsymbol{B}\boldsymbol{C}^{-1}\boldsymbol{B}^{\mathrm{T}} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{C} \end{pmatrix}\begin{pmatrix} \boldsymbol{x} \\ \mathbf{0} \end{pmatrix}=(\boldsymbol{x}^{\mathrm{T}},\mathbf{0}^{\mathrm{T}})\boldsymbol{D}\begin{pmatrix} \boldsymbol{x} \\ \mathbf{0} \end{pmatrix}>0$$

故$\boldsymbol{A}-\boldsymbol{B}\boldsymbol{C}^{-1}\boldsymbol{B}^{\mathrm{T}}$为正定矩阵.(2 分)