$Q(x) = \frac{1}{2}x^T Bx + g^T x + c \qquad \Omega = B_r(x_0) = \{ x \in \mathbb{R}^n \mid \|x - x_0\| \leq r \}$

$B \in \mathbb{R}^{n\times n}$, $g \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, $c \in \mathbb{R}$

frobenius 范数： $A = (a_{ij})_{n\times n}$. $\|A\|_F = \left(\sum_{i,j}(a_{ij})^2\right)^{\frac{1}{2}}$

$$
\begin{aligned}
|Q|^2_{H^0(B_r)} &= \int_{B_r(x_0)} (Q(x))^2 \, dx = \int_{B_r(x_0)} \left(\tfrac{1}{2}x^T Bx + g^T x + c\right)^2 dx \\
&= \int_{B_r(0)} \left(\tfrac{1}{2}(x+x_0)^T B(x+x_0) + g^T(x+x_0) + c\right)^2 dx \\
&= \int_{B_r(0)} \left(\tfrac{1}{2}(x^T Bx + x^T Bx_0 + x_0^T Bx + x_0^T Bx_0) + g^T x \right. \\
&\quad \left. + g^T x_0 + c\right)^2 dx \\
&= \int_{B_r(0)} \left(\tfrac{1}{4}(x^T Bx + 2x^T Bx_0 + x_0^T Bx_0)^2 \,① + (g^T x)^2 \,② \right. \\
&\quad + (g^T x_0)^2 \,③ + c^2 \,④ + (x^T Bx + 2x^T Bx_0 + x_0^T Bx_0)(g^T x) \,⑤ \\
&\quad + (x^T Bx + 2x^T Bx_0 + x_0^T Bx_0)(g^T x_0 + c) \,⑥ + 2g^T x(g^T x_0 + c) \,⑦ \\
&\quad \left. + 2cg^T x_0 \,⑧ \right) dx
\end{aligned}
$$

先设 $x_0 = 0$；则

① $= \int_{B_r(0)} \frac{1}{4}(x^T Bx)^2 dx = \frac{1}{4} I \left(2\|B\|_F^2 + Tr^2 B\right)$

$+ \frac{1}{4}(J - 3I)\|D\|_F^2$.

其中 $I = \int_{B_r(0)} x_1^2 x_2^2 \, dx$, $J = \int_{B_r(0)} x_1^4 \, dx$

$K = \int_{B_r(0)} x_1^2 \, dx \qquad \|D\|_F = \left(\sum_i b_{ii}^2\right)^{\frac{1}{2}}$

② $= \int_{B_r(0)} (g^T x)^2 dx = K\|g\|_2^2$

③ $= 0$

④ $= V_2 r^n c^2$

⑤ $= \int_{B_r(0)} (x^T Bx)(g^T x) \, dx = 0$

⑥ $= \int_{B_r(0)} (x^T Bx)(c) = cK \, Tr B$

⑦ $= \int_{B_r(0)} 2cg^T x \, dx = 0$

⑧ $= 0$

$\therefore \ |Q|^2_{H^0(B_r(0))} = \frac{1}{4} I \left(2\|B\|_F^2 + Tr^2 B\right) + \frac{1}{4}(J - 3I)$
$\|D\|_F^2 + K\left(\|g\|_2^2 + c\, Tr B\right) + V_2 r^n c^2 .$

$$I = \frac{\Gamma(\frac{3}{2})^2 \ \Gamma(\frac{n}{2})}{(\Gamma(\frac{1}{2}))^2 \ \Gamma(\frac{n}{2}+2)} \cdot \frac{n}{n+4} V_2 r^{n+4}$$

$$= \frac{1}{4} \times \frac{1}{(\frac{n}{2}+1)(\frac{n}{2})} \times \frac{n}{n+4} V_2 r^{n+4}$$

$$= \frac{1}{(n+4)(n+2)} V_2 r^{n+4}$$

$$J = \frac{\Gamma(\frac{5}{2})\ \Gamma(\frac{1}{2})\ \Gamma(\frac{n}{2})}{(\Gamma(\frac{1}{2}))^2 \ \Gamma(\frac{n}{2}+2)} \cdot \frac{n}{n+4} V_2 r^{n+4}$$

$$= \frac{3}{2} \times \frac{1}{2} \times \frac{1}{(\frac{n}{2}+1)(\frac{n}{2})} \cdot \frac{n}{n+4} V_2 r^{n+4}$$

$$= \frac{3}{(n+4)(n+2)} V_2 r^{n+4} = 3I$$

$$K = \frac{\Gamma(\frac{3}{2})\ \Gamma(\frac{1}{2})\ \Gamma(\frac{n}{2})}{(\Gamma(\frac{1}{2}))^2 \ \Gamma(\frac{n}{2}+1)} \cdot \frac{n}{n+2} V_2 r^{n+2}$$

$$= \frac{1}{2} \times \frac{2}{n} \times \frac{n}{n+2} V_2 r^{n+2}$$

$$= \frac{1}{n+2} V_2 r^{n+2}$$

$\therefore \ |Q|^2_{H^0(B_r(0))} = V_2 r^n \left(\frac{r^4}{4(n+4)(n+2)} \left(2\|B\|_F^2 + Tr^2 B\right) \right.$
$\left. + c^2 + \frac{r^2}{n+2}\left(\|g\|_2^2 + c\, Tr B\right)\right)$

(半范数)

$$|Q|^2_{H^1(B_r(0))} = \int_{B_r(0)} \|Bx + g\|_2^2 \, dx$$

$$= \int_{B_r(0)} \left(x^T B^2 x + \|g\|_2^2\right) dx$$

$$= K\|B\|_F^2 + \|g\|_2^2 \cdot V_2 r^n$$

$$= V_2 r^n \left(\frac{r^2}{n+2} \|B\|_F^2 + \|g\|_2^2 \right)$$

$$|Q|_{H^2(B_r(0))}^2 = \int_{B_r(0)} \|B\|_F^2 \, dx = V_2 r^n \|B\|_F^2 .$$
(半范数)

H^1 范数：$|Q|_{H^1}^2 = V_2 r^n \left(\left(\frac{r^4}{2(n+4)(n+2)} + \frac{r^2}{n+2} \right) \|B\|_F^2 + \left(\frac{r^2}{n+2} + 1 \right) \|g\|_2^2 + \frac{r^2}{4(n+4)(n+2)} \mathrm{Tr}^2 B + c^2 + \frac{cr^2}{n+2} \mathrm{Tr} B \right)$

H^2 范数：$|Q|_{H^2}^2 = V_2 r^n \left(\left(\frac{r^4}{2(n+4)(n+2)} + \frac{r^2}{n+2} + 1 \right) \|B\|_F^2 + \left(\frac{r^2}{n+2} + 1 \right) \|g\|_2^2 + \frac{r^2}{4(n+4)(n+2)} \mathrm{Tr}^2 B + c^2 + \frac{cr^2}{n+2} \mathrm{Tr} B \right)$