The invariant distribution of a stochastic dynamical system

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The initial question is

$$\left\{\begin{array}{l} dx = vdt \\ dv = -\nabla V dt - \gamma(v - Ax)dt + Avdt + \sigma d\omega, \end{array}\right. \tag{1}$$

where

$$V = \frac{1}{2}x^T Bx.$$

This equation can be written as

$$d\begin{pmatrix} x \\ v \end{pmatrix} = M\begin{pmatrix} x \\ v \end{pmatrix} dt + \begin{pmatrix} O & O \\ O & I \end{pmatrix} \sigma d\omega,$$

where

$$M = \begin{pmatrix} O & I \\ \gamma A - B & A - \gamma I \end{pmatrix}.$$

We can get the solution of the initial form of the problem is

$$x(t) = e^{Mt}x(0) + \int_0^t e^{M(t-\tau)} \begin{pmatrix} O & O \\ O & I \end{pmatrix} d\omega_\tau.$$

Assumption.

$$\lim_{t\to+\infty} e^{Mt} = 0.$$

Then $x(t)$ will converge to a limit distribution when $t \to \infty$. It can be proved that it is a normal distribution $N(0,C)$, where

$$C = \sigma^2 \int_0^{+\infty} \exp(Mt) \begin{pmatrix} O & O \\ O & I \end{pmatrix} \exp\left(M^T t\right) dt.$$

So, we get a linear system below

$$
\begin{aligned}
MC+CM^T &= \sigma^2 \int_0^{+\infty} \left(\mathrm{M}\exp(Mt) \begin{pmatrix} O & O \\ O & I \end{pmatrix} \exp\left(M^T t\right) + \exp(Mt) \begin{pmatrix} O & O \\ O & I \end{pmatrix} \exp\left(M^T t\right) M^T \right) dt \\
&= \sigma^2 \int_0^{+\infty} \frac{d}{dt} \left(\exp(Mt) \begin{pmatrix} O & O \\ O & I \end{pmatrix} \exp\left(M^T t\right) \right) dt \\
&= -\sigma^2 \begin{pmatrix} O & O \\ O & I \end{pmatrix}.
\end{aligned}
.
$$

We arrange the content we talked about above, and get the theorem below

Theorem 1. Suppose that we have an equation

$$
\begin{cases} dx = vdt \\ dv = -\nabla V dt - \gamma(v-Ax)dt + Avdt + \sigma d\omega \end{cases} \tag{2}
$$

Let

$$
M = \begin{pmatrix} O & I \\ \gamma A - B & A - \gamma I \end{pmatrix}
$$

Suppose that

$$
\lim_{t \to +\infty} e^{Mt} = 0,
$$

then the limit distribution is $N(0,C)$ where

$$
MC + CM^T = -\sigma^2 \begin{pmatrix} O & O \\ O & I \end{pmatrix}.
$$

From this equation we can not only solve the matrix C easily but also get some corollaries. Here we give the most obvious one.

Corollary 1. Suppose that

$$
C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}.
$$

We have

(1) C_2 is a skew-symmetric matrix.

(2) the diagonal elements of C_2 are all 0.

Proof. From

$$
MC + CM^T = -\sigma^2 \begin{pmatrix} O & O \\ O & I \end{pmatrix}
$$

We can get

$$
C_2 + C_3 = O
$$

But C is the covariance matrix of a normal distribution, so C is symmetric. Therefore,

$$
C_3 = C_2^T
$$

We conclude

$$
C_2^T = -C_2
$$

So (1) is proved. (2) is a corollary of (1).

Finally we give two examples.

Example 1.

$$
\begin{aligned}
&n=2,\sigma=1,\gamma=1,A=\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix},B=I.\\
&M=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & b\gamma & -\gamma & b \\ 0 & -1 & 0 & -\gamma \end{pmatrix},\\
&C=\begin{pmatrix} c_{11} & c_{12} & 0 & c_{14} \\ c_{12} & c_{22} & -c_{14} & 0 \\ 0 & -c_{14} & c_{33} & c_{34} \\ c_{14} & 0 & c_{34} & c_{44} \end{pmatrix},\\
&D=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.\\
&MC+CM^T=D.
\end{aligned}
$$

We get

$$
c_{11}=\frac{1}{4}\left(3b^2+2\right),c_{12}=\frac{b}{2},c_{22}=\frac{1}{2},c_{33}=\frac{1}{2}\left(b^2+1\right),c_{34}=\frac{b}{4},c_{44}=\frac{1}{2},c_{14}=-\frac{b}{4}.
$$

Therefore

$$
\begin{aligned}
&C=\begin{pmatrix} \frac{1}{4}\left(3b^2+2\right) & \frac{b}{2} & 0 & -\frac{b}{4} \\ \frac{b}{2} & \frac{1}{2} & \frac{b}{4} & 0 \\ 0 & \frac{b}{4} & \frac{1}{2}\left(b^2+1\right) & \frac{b}{4} \\ -\frac{b}{4} & 0 & \frac{b}{4} & \frac{1}{2} \end{pmatrix}.\\
&\rho=\frac{1}{\sqrt{|2\pi C|}}\exp\left(-\tfrac{1}{2}\left(q_1,q_2,p_1,p_2\right)C^{-1}\left(q_1,q_2,p_1,p_2\right)^T\right).
\end{aligned}
$$

From the Fokker-Planck equation

$$
\begin{aligned}
0=\frac{d\rho}{dt}=\sum_{i=1}^{n}-\frac{\partial}{\partial q_i}\left(p_i\rho\right)-\frac{\partial}{\partial p_i}\left[\left(-\frac{\partial V}{\partial q_i}-\gamma\left(p_i-\sum_j A_{ij}q_j\right)+\sum_j A_{ij}p_j\right)\rho\right]\\
+\frac{\sigma^2}{2}\frac{\partial^2}{\partial p_i^2}\rho,
\end{aligned}
$$

we can confirm this result.

Example 2.

$$
\begin{aligned}
&n=1.A=a,B=b\\
&M=\begin{pmatrix} 0 & 1 \\ \gamma a-b & a-\gamma \end{pmatrix}
\end{aligned}
$$

From

$$MC + CM^T = \begin{pmatrix} 0 & 0 \\ 0 & -\sigma^2 \end{pmatrix},$$

we can get

$$C = \begin{pmatrix} \frac{\sigma^2}{2(a-\gamma)(\gamma a-b)} & 0 \\ 0 & -\frac{\sigma^2}{2(a-\gamma)} \end{pmatrix}$$
$$\rho = \frac{1}{\sqrt{|2\pi C|}} \exp\left(-\tfrac{1}{2}(q,p)C^{-1}(q,p)^T\right).$$

From the Fokker-Planck equation

$$0 = \frac{d\rho}{dt} = \sum_{i=1}^{n} -\frac{\partial}{\partial q_i}(p_i\rho) - \frac{\partial}{\partial p_i}\left[\left(-\frac{\partial V}{\partial q_i} - \gamma\left(p_i - \sum_j A_{ij}q_j\right) + \sum_j A_{ij}p_j\right)\rho\right]$$
$$+\frac{\sigma^2}{2}\frac{\partial^2}{\partial p_i^2}\rho,$$

we have

$$MC + CM^T = -\sigma^2 \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

Assume $C' = \frac{1}{\sigma^2}C$, in this case, we get:

$$MC' + C'M^T = -\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

For conveniently writing, we denote C' as C in the following

$$MC + CM^T = -\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$$

where M is known, C is unknown based on simple calculation, we have the conclusion that the total account of the unknown variable is $2n \times 2n$.

The system is equivalent to a linear equation set $DX = y$, where $d \in R^{4n^2 \times 4n^2}, x \in R^{4n^2}, y \in R^{4n^2}$.

Since

$$C = \begin{pmatrix} C_1 & C_2 \\ -C_2 & C_4 \end{pmatrix}$$

where C_1 is a symmetric matrix, C_2 is a skew symmetric matrix, C_4 is a symmetric matrix, the amount of the unknown components, in fact, is only

$$\frac{1}{2}n(n+1) \times 2 + \frac{1}{2}n(n-1) = \frac{3}{2}n^2 + \frac{1}{2}n$$

Simplify it and we get

$$
\begin{aligned}
M &= \begin{pmatrix} 0 & I \\ \gamma A - B & A - \gamma I \end{pmatrix} \\
MC + CM^T &= \begin{pmatrix} 0 & I \\ P & Q \end{pmatrix}\begin{pmatrix} C_1 & C_2 \\ -C_2 & C_4 \end{pmatrix} + \begin{pmatrix} C_1 & C_2 \\ -C_2 & C_4 \end{pmatrix}\begin{pmatrix} 0 & P^T \\ I & Q^T \end{pmatrix} \\
&= \begin{pmatrix} -C_2 & C_4 \\ PC_1 - QC_2 & PC_2 + QC_4 \end{pmatrix} + \begin{pmatrix} C_2 & C_2P^T + C_2Q^T \\ C_4 & -C_2P^T + C_4Q^T \end{pmatrix} \\
&= \begin{pmatrix} 0 & C_4 + C_1p^T + C_2Q^T \\ C_4 + PC_1 - QC_2 & PC_2 + QC_4 - C_2P^T + C_4Q^T \end{pmatrix} \\
&= -\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}
\end{aligned}
$$

where $P = \gamma A - B, Q = A - \gamma I$

Thus we get:

$$
\begin{cases} C_4 + C_1P^T + C_2Q^T = 0 \\ PC_2 + QC_4 - C_2P^T + C_4Q^T = I \end{cases} \tag{3}
$$

According to the first line of (3) :

$$
C_4 = -C_1p^T - C_2Q^T
$$

Besides, we know C_4 is symmetric matrix, which means $C_4^T = C_4$ Thus we get:

$$
\begin{aligned}
C_4 &= -C_1P^T - C_2Q^T \\
&= -\left(C_1P^T - C_2Q^T\right)^T \\
&= -PC_1 - QC_2^T \\
&= -PC_1 - QC_2
\end{aligned} \tag{4}
$$

Substitute (2) into the left hand side of the second line of (1) we get :

$$
PC_2 + Q\left(-C_1P^T\right) - C_2P^T + \left(PC_1 + QC_2\right)Q^T = -I
$$

and

$$
PC_2 - PC_1Q^T - C_2P^T - QC_1P^T = -I
$$

Thus calculating the covariance matrix C which is equivalent to solving (3) is finally equivalent to solving the equations

$$
\begin{cases} C_4 + C_1P^T + C_2Q^T = 0 \\ P\left(C_2 - C_1Q^T\right) + \left(C_2 - C_1Q^T\right)^T P^T = -I \end{cases} \tag{5}
$$

without the integral.

$$
n = 2, \sigma = 1, \gamma = 1, A = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, B = I.
$$

so

$$P=\begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}, Q=\begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix}.$$

We denote C_1 by C_{ij}^1, C_2 by C_{ij}^2 and C_4 by C_{ij}^4. According to symmetry or skew-symmetry of C_1, C_2, C_4 and (3), we get the linear equations:

$$\begin{cases} -2C_{11}^1+2bC_{12}^1-2C_{11}^2+2b\left(C_{21}^1-bC_{22}^1+C_{21}^2\right)=-1 \\ -C_{12}^1-C_{21}^1+bC_{22}^1-C_{12}^2-C_{21}^2+b\left(C_{22}^1+C_{22}^2\right)=0 \\ -C_{11}^1+bC_{12}^1-C_{11}^2+bC_{12}^2+C_{11}^4=0 \\ -C_{21}^1+bC_{22}^1-C_{21}^2+bC_{22}^2+C_{21}^4=0 \\ -C_{12}^1-C_{12}^2+C_{12}^4=0 \\ -C_{22}^1-C_{22}^2+C_{22}^4=0 \\ -2C_{22}^1-2C_{22}^2=-1 \\ C_{12}^4-C_{21}^4=0 \\ C_{12}^1-C_{21}^1=0 \\ C_{12}^2+C_{21}^2=0 \\ C_{11}^2=0 \\ C_{22}^2=0 \end{cases}$$

Solving the equations, we get C :

$$C=\begin{pmatrix} \frac{1}{4}\left(2+3b^2\right) & \frac{b}{2} & 0 & -\frac{b}{4} \\ \frac{b}{2} & \frac{1}{2} & \frac{b}{4} & 0 \\ 0 & \frac{b}{4} & \frac{1}{2}\left(1+b^2\right) & \frac{b}{4} \\ -\frac{b}{4} & 0 & \frac{b}{4} & \frac{1}{2} \end{pmatrix}$$

Fokker-Planck Equation helps us to check the solution

$$\frac{d\rho}{dt}=\sum_{i=1}^{n}-\frac{\partial}{\partial q_i}(p_i\rho)-\frac{\partial}{\partial p_i}\left[\left(-\frac{\partial V}{\partial q_i}-\gamma\left(p_i-\sum_j A_{ij}q_j\right)+\sum_j A_{ij}p_j\right)\rho\right]+\frac{\sigma^2}{2}\frac{\partial^2}{\partial p_i^2}\rho=0$$

It shows that the solution is correct at the same time, and our derivation is also correct.

Consider the possible solution

$$\rho=e^{-(x'x/2+V(y))}, x=C_1p+C_2q, y=Dq \tag{6}$$

of the Fokker-Planck Equation

$$0=\frac{d\rho}{dt}=\sum_{i=1}^{n}-\frac{\partial}{\partial q_i}(p_i\rho)-\frac{\partial}{\partial p_i}\left[\left(-\frac{\partial V}{\partial q_i}-\gamma\left(p_i-\sum_j A_{ij}q_j\right)+\sum_j A_{ij}p_j\right)\rho\right]+\frac{\sigma^2}{2}\frac{\partial^2}{\partial p_i^2}\rho,$$

where $V(y) = \frac{1}{\|y\|}$

$$
\begin{aligned}
-\tfrac{\partial}{\partial q_i}(p_i\rho) &= p_i\rho\left(x'\tfrac{\partial x}{\partial q_i} + \tfrac{\partial}{\partial q_i}V(y)\right) \\
-\tfrac{\partial}{\partial p_i}\left(-\tfrac{\partial V}{\partial q_i}\rho\right) &= \tfrac{\partial V(q)}{\partial q_i}\rho\left(-x'\tfrac{\partial x}{\partial p_i}\right) \\
-\tfrac{\partial}{\partial p_i}\left(-\gamma\left(p_i - \textstyle\sum_j A_{ij}q_j\right)\right)\rho &= \gamma\rho + \gamma\left(p_i - \textstyle\sum_j A_{ij}q_j\right)\rho\left(-x'\tfrac{\partial x}{\partial p_i}\right) \\
-\tfrac{\partial}{\partial p_i}\left(\textstyle\sum_j A_{ij}p_j\rho\right) &= -A_{ii}\rho + \textstyle\sum_j A_{ij}p_j\rho\left(x'\tfrac{\partial x}{\partial p_i}\right) \\
\tfrac{\sigma^2}{2}\tfrac{\partial^2}{\partial p_i^2}\rho &= \tfrac{\sigma^2}{2}\rho\left[\left(-x'\tfrac{\partial x}{\partial p_i}\right)^2 - \tfrac{\partial x'}{\partial p_i}\tfrac{\partial x}{\partial p_i}\right]
\end{aligned}
$$

Then consider the adjoint equation

$$
\sum_{i=1}^{n} -\frac{\partial}{\partial q_i}(p_i\rho) - \frac{\partial}{\partial p_i}\left[\left(-\frac{\partial V_1}{\partial q_i} - \gamma\left(p_i - \sum_j A_{ij}q_j\right) + \sum_j A_{ij}p_j\right)\rho\right] + \frac{\sigma^2}{2}\frac{\partial^2}{\partial p_i^2}\rho = 0 \qquad (7)
$$

where $V_1(y) = \frac{1}{2}y'By$ We get similar results

$$
\begin{aligned}
-\tfrac{\partial}{\partial q_i}(p_i\rho) &= p_i\rho\left(x'\tfrac{\partial x}{\partial q_i} + \tfrac{\partial}{\partial q_i}V_1\right) \\
-\tfrac{\partial}{\partial p_i}\left(-\tfrac{\partial V_1}{\partial q_i}\rho\right) &= \tfrac{\partial V_1}{\partial q_i}\rho\left(-x'\tfrac{\partial x}{\partial p_i}\right) \\
-\tfrac{\partial}{\partial p_i}\left(-\gamma\left(p_i - \textstyle\sum_j A_{ij}q_j\right)\right)\rho &= \gamma\rho + \gamma\left(p_i - \textstyle\sum_j A_{ij}q_j\right)\rho\left(-x'\tfrac{\partial x}{\partial p_i}\right) \\
-\tfrac{\partial}{\partial p_i}\left(\textstyle\sum_j A_{ij}p_j\rho\right) &= -A_{ii}\rho + \textstyle\sum_j A_{ij}p_j\rho\left(x'\tfrac{\partial x}{\partial p_i}\right)
\end{aligned}
$$

$$
\frac{\sigma^2}{2}\frac{\partial^2}{\partial p_i^2}\rho = \frac{\sigma^2}{2}\rho\left[\left(-x'\frac{\partial x}{\partial p_i}\right)^2 - \frac{\partial x'}{\partial p_i}\frac{\partial x}{\partial p_i}\right]
$$

For equation (5), we can solve it. And we know the solution is in the form

$$
\rho = \frac{1}{Z}e^{-\frac{x'x}{2}}
$$

Set B=0, the equation (5) is corresponding to some parts of the equation (4). So we use the same x solved in (5) in equation (4), and get

$$
\sum_{i=1}^{n} \frac{\partial V(y)}{\partial q_i}p_i - \frac{\partial V(q)}{\partial q_i}\left(x'\frac{\partial x}{\partial p_i}\right) = 0
$$

But for general V , the equation has no solution.