Assignment 4: Generating Random Variables (Part II)

Due on Feb. 19

1. For generating a binomial distribution with $n = 5$ and $p = 0.5$, give the following three algorithms.

   (a) **Finite decimal method:** Give the pseudo codes and compute required constants in the setup stage. (Approximate all probabilities to the third decimal point, i.e., $q = 3$.)

   (b) **Refined finite decimal method:** Give the pseudo codes and compute required constants in the setup stage. (Approximate all probabilities to the third decimal point, i.e., $q = 3$.)

   (c) **Alias method.** Give the pseudo codes and compute required constants in the setup stage.

2. To develop a *ratio-of-uniforms* algorithm for generating a Gamma(α, 1) distribution with $\alpha > 1$, having a PDF

$$f(x) = \frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)}, \quad x > 0,$$

   we proceed in the following steps:

   (a) With $p = \Gamma(\alpha)$, give the region S.

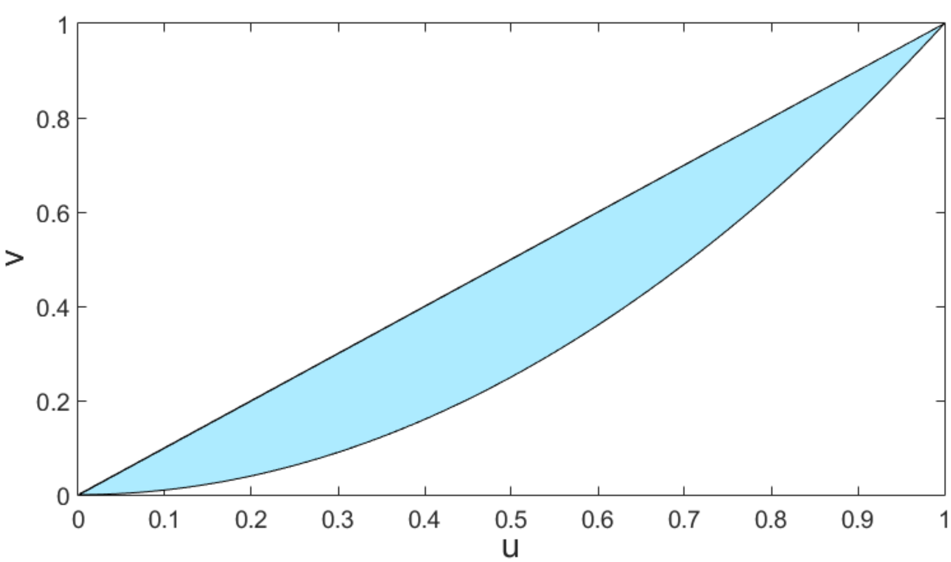
   (b) To obtain a majorizing rectangle T, compute the boundaries:

$$u^* = \sup_z u(z), \quad v_* = \inf_z v(z), \quad v^* = \sup_z v(z).$$

   (c) Give the ratio-of-uniforms algorithm.

3. Prove the multivariate lognormal mean and covariance formulas on p.72 of the notes.

4. Consider the ratio-of-uniforms example covered in class (p.42 of notes), we now hope to give an algorithm to directly generate $(U, V) \in S$ uniformly (without using a majorizing rectangle $T = \{(u, v) : 0 \le u \le 1,\ 0 \le v \le 1\}$), with the shaded region S given in the figure below. We follow several steps.



(a) Give the joint density of (U, V): $f_{U,V}(u, v)$.

(b) Compute the marginal density of U: $f_U(u)$. Does U follows Uniform(0,1)?

(c) Give an effective algorithm to simulate U.

(d) Use (a) and (b) to compute the conditional density $f_{V|U}(v|u)$.

(e) Give a complete algorithm to generate (U, V) uniformly in S.

5. **Gaussian copulas**

We hope to generate two correlated random variables X and Y for the following cases:

(i) $X \sim \text{Exp}(\lambda)$, $Y \sim \text{Weibull}(\alpha, \beta)$, with $\lambda = 2$, $\alpha = 2$, $\beta = 3$.

(ii) $X \sim \text{Gamma}(\alpha = 6.7, \lambda = 3)$, $Y \sim \text{Lognormal}(\mu = 0.1, \sigma = 0.5)$.

Do the following for both cases with three values for the correlation: $\rho_{X,Y} = 0.8, 0.2$ and -0.8:

(a) Use a Gaussian copula to generate $N = 10^5$ independent (X, Y) pairs and estimate the correlation using the simulated data.

(b) Plot all (X, Y) pairs. Verify that the marginal distributions look appropriate. (In MATLAB use command "histogram(X)" or "scatterhist(X,Y)")

Hint: In case (i), both distributions can be obtained by inverse-transform method. However, in case (ii), CDF's of Lognormal and Gamma are not invertible explicitly. In MATLAB, use command "makedist" to build the PDFs and command "icdf" to numerically invert the CDFs.

Sample MATLAB codes for case (i) is given below:

```
lambda = 2; alpha = 2; beta = 3;
rho = .2; N = 100000;
Z = mvnrnd([0 0],[1 rho; rho 1], N);
U = normcdf(Z);
X = (-1/lambda)*log(U(:,1));  % Inverse Transform for Exponential
Y = beta*(-log(U(:,2))).^(1/alpha);  % Inverse Transform for Weibull
corr(X,Y)
scatterhist(X,Y,'Direction','out')
```