# Assignment 4: Brownian motions and option pricing

*Due on Mar. 14*

1. (**Josh's Barber Shop (An $M/M/1/3+M$ model)**)
   Josh operates a small barbershop in Raleigh. His barbershop has room for at most three customers, one in service and two waiting.

   - Potential customers arrive according to a Poisson process with rate $\lambda = 10$ per hour. If a potential arrival finds the barber shop full, with a customer in service and two other customers waiting, he (she) is blocked and lost (i.e., he will leave and will not affect future arrivals);
   - Successive service times are I.I.D. $\text{Exp}(\mu)$ r.v.'s with mean $1/\mu = 30$ minutes;
   - Waiting customers have limited patience, with each waiting customer being willing to wait only a random amount of time, if the customer has not started service by that time, the customer will abandon, leaving without receiving service. Customers' patience times are I.I.D. r.v.'s following $\text{Exp}(\gamma)$, with mean $1/\gamma = 20$ minutes.

   Using the first CTMC algorithm to simulate a path of the queue length process $\{Q(t), 0 \le t \le T\}$ (here $Q(t)$ denotes the total number of customers in the shop), $T = 10000$. Estimate

   (a) long-run average number of customers in the shop
   $$\hat{Q}_T \equiv \frac{1}{T}\int_0^T Q(t)dt.$$
   (b) long-run proportion of time the barber is busy
   $$\hat{B}_T \equiv \frac{1}{T}\int_0^T \mathbf{1}(Q(t) > 0)dt.$$
   (c) long-run proportion of time the shop is full
   $$\hat{F}_T \equiv \frac{1}{T}\int_0^T \mathbf{1}(Q(t) = 3)dt.$$
   (d) long-run average number of customers waiting in line
   $$\hat{L}_T \equiv \frac{1}{T}\int_0^T \max(Q(t) - 1, 0)dt.$$

   Note: Only one sample path of $Q(t)$ needs to be generated.

2. (**Option pricing under GBM**)
   The current price of a stock is \$100. Suppose the logarithm of the price of the stock changes according to a BM with drift coefficient $\mu$ and variance $\sigma^2 = 1$. Let the expiration time be $t = 4$ and assume the continuously compounded interest rate is $\alpha = 20\%$.

   (a) Give the Black-Scholes price of an European call option with strike prices (i) $K = \$100$, (ii) $K = \$120$ and (iii) $K = \$80$.

(b) Write a MATLAB program to price the European call option for all three cases in (a), compare your results to (a). (Simulate $N = 100000$ samples and use step size $\Delta t = 0.01$. Try to minimize the use of for-loops.)

(c) Write a MATLAB program to price the Asian call option under the setting of (b).

(d) Write a MATLAB program to price the lookback call option under the setting of (b).

3. (**Pricing a compound option**)
Consider a compound European call-on-call (CoC) option. Suppose that at time $T_1 = 2$, for a strike price $K_1 = \$80$ there is an option of buying a European call option with an expiration time $T_2 = 4$ and strike price $K_2 = \$120$.

   (a) Give pseudo-code for a Monte-Carlo algorithm to estimate the price of the compound European call option.

   (b) Write a MATLAB program to price the CoC option with BM parameters, current stock price, and interest rate the same as in Problem 1.

4. (**Option pricing under JDP**)
Now suppose that the stock price behaves as a GBM as in Problem 7 (with same parameters $\alpha$, $\sigma$ there) with random jumps, where the jumps occur according to a Poisson process $N(t)$ with rate $\lambda = 1$ and logarithm of price jump sizes are distributed as i.i.d. normal random variables with mean $a = 0$ and variance $b^2 = \frac{1}{2}$ (i.e., the Merton model). The jump times and sizes are independent with the GBM.

   (a) Determine the value of $\mu$ (in terms of other parameters: $\alpha$, $\lambda$, $a$, $b$, $\sigma$) under the "no-arbitrage" assumption for a stock price behaving according to a Merton JDP.

   (b) Write a MATLAB program to price an European call option with stock price under the JDP assumption for $t = 4$ and initial stock price \$100.