Assignment 5: Generating Stochastic Processes

Due on Feb. 28

1. Bus arrive at the Oracle Arena for the NBA final according to a Poisson process with rate 5 per hour. Each bus is equally likely to containn either 20, 21, ..., 40, with the numbers in the different buses being independent. Write a MatLab program to simulate a sample path of the total number of customer (not bus) arrivals in the interval [0, 2].

2. Consider an NHPP $\{N(t), t \geq 0\}$ with rate $\lambda(t) = 1 + 0.6\sin(t)$. Write MatLab programs to implement the following two algorithms to simulate sample paths of the NHPP in $[0, 20]$:

   (a) The **naive algorithm:**

      i. $t = 0$; $I = 0$.

      ii. Generate $U \sim$ Unif(0,1);

      iii. Set $t = t - \frac{1}{\lambda(t)} \log U$. If $t > T$, stop; else go to Step (iv).

      iv. $I = I + 1$; $S(I) = t$; Go to Step (ii).

   (b) One of the 2 NHPP algorithms introduced in class.

   For each algorithm, simulate 100 independent sample paths of $N(t)$ in [0, 20] to estimate (i) $\mathbb{E}[N(t)]$, (ii) $\mathrm{Var}(N(t))$, and plot a 95% confidence interval for $\mathbb{E}[N(t)]$ for all $0 \leq t \leq 10$ (so a variance band). Also, graph $\Lambda(t) = \int_0^t \lambda(u)du$ and compare to your estimates in both cases. This should show you that the "naive approach" is incorrect.

3. Consider a DTMC $\{X_n, n = 0, 1, \ldots\}$ with states $\{0, 1, 2\}$ and transition probability

$$\mathbf{P} = \begin{array}{c} (0) \\ (1) \\ (2) \end{array} \begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 1/3 & 2/3 \\ 1/2 & 0 & 1/2 \end{bmatrix}.$$

   (a) Let $X_0 = 0$. Simulate a sample path of $\{X_n\}$ for $n = 0, 1, \ldots, N$, with $N = 1000$. Estimate the long-run proportion of time (steps) the DTMC is in state 0, 1, 2:

$$\hat{\pi}_k = \frac{1}{N} \sum_{k=0}^{N} \mathbf{1}_{\{X_k = i\}}, \qquad i = 0, 1, 2.$$

   Compute the exact values of the steady states and compare them to your estimations.

   (b) Estimate the long-run average of the DTMC

$$\hat{X} = \frac{1}{N} \sum_{k=0}^{N} X_k.$$

   Compute the exact value and compare them to your estimations.

4. (**Josh's Barber Shop (An $M/M/1/3+M$ model)**)
   Josh operates a small barbershop in Raleigh. His barbershop has room for at most three customers, one in service and two waiting.

- Potential customers arrive according to a Poisson process with rate $\lambda = 10$ per hour. If a potential arrival finds the barber shop full, with a customer in service and two other customers waiting, he (she) is blocked and lost (i.e., he will leave and will not affect future arrivals);
- Successive service times are I.I.D. Exp(μ) r.v.'s with mean $1/\mu = 30$ minutes;
- Waiting customers have limited patience, with each waiting customer being willing to wait only a random amount of time, if the customer has not started service by that time, the customer will abandon, leaving without receiving service. Customers' patience times are I.I.D. r.v.'s following Exp(γ), with mean $1/\gamma = 20$ minutes.

Using the second CTMC algorithm (i.e., uniformization) to simulate a path of the queue length process $\{Q(t), 0 \le t \le T\}$ (here $Q(t)$ denotes the total number of customers in the shop), $T = 10000$. Estimate

(a) long-run average number of customers in the shop

$$\hat{Q}_T \equiv \frac{1}{T}\int_0^T Q(t)dt.$$

(b) long-run proportion of time the barber is busy

$$\hat{B}_T \equiv \frac{1}{T}\int_0^T \mathbf{1}(Q(t) > 0)dt.$$

(c) long-run proportion of time the shop is full

$$\hat{F}_T \equiv \frac{1}{T}\int_0^T \mathbf{1}(Q(t) = 3)dt.$$

(d) long-run average number of customers waiting in line

$$\hat{L}_T \equiv \frac{1}{T}\int_0^T \max(Q(t) - 1, 0)dt.$$

Note: Only one sample path of $Q(t)$ needs to be generated.