College of Engineering
Swansea University

Dynamics & Earthquake Analysis of Structures

Question Sheet 3 – Solutions

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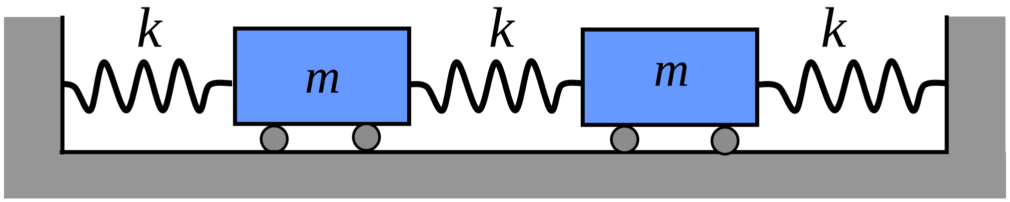
(Part III, p96)

Example

Q: Determine the two natural frequencies and modal shapes



Solution:

$$\mathbf{M}=\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} = \mathrm{m}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad \mathbf{K}=\begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} = k\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

I) Natural Frequencies

$$\det\left|\mathbf{K}-\omega^2\mathbf{M}\right| = \left| k\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \omega^2 m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \cancel{k^2} \left\| \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\| = 0$$

where $a=\omega^2 m/k$

$$\rightarrow \begin{vmatrix} 2-a & -1 \\ -1 & 2-a \end{vmatrix} = (2\text{-a})^2\text{-}(\text{-}1)^2 = (1\text{-a})(3\text{-}1) = 0 \rightarrow \therefore a_{1,2} = 1, 3$$

Two natural frequencies: $\omega_1 = \sqrt{\frac{a_1 k}{m}} = \sqrt{\frac{k}{m}}; \quad \omega_2 = \sqrt{\frac{a_2 k}{m}} = \sqrt{\frac{3k}{m}}$

Example

II) Modal Shapes

(Eigenvalue problem)

1) 1st mode: $\omega_1 = \sqrt{k/\mathrm{m}}$ **assume** $\mathbf{V}_1 = [\mathrm{v}_{11} \quad \mathrm{v}_{12}]^T$ $\qquad \left(\mathbf{K} - \omega_1^2 \mathbf{M}\right)\mathbf{v}_1 = 0$

$$\left(k\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \omega_1^2 m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{pmatrix} \mathrm{v}_{11} \\ \mathrm{v}_{12} \end{pmatrix} = 0 \rightarrow k\left(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - a_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{pmatrix} \mathrm{v}_{11} \\ \mathrm{v}_{12} \end{pmatrix} = 0$$

$$\begin{bmatrix} 2-a_1 & -1 \\ -1 & 2-a_1 \end{bmatrix}\begin{pmatrix} \mathrm{v}_{11} \\ \mathrm{v}_{12} \end{pmatrix} = 0 \rightarrow \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{pmatrix} \mathrm{v}_{11} \\ \mathrm{v}_{12} \end{pmatrix} = 0 \rightarrow \begin{matrix} \mathrm{v}_{11} - \mathrm{v}_{12} = 0 \\ \cancel{-\mathrm{v}_{11} + \mathrm{v}_{12} = 0} \end{matrix} \rightarrow \boxed{\mathrm{v}_{11} = \mathrm{v}_{12}}$$

Let $\mathrm{v}_{11} = 1;\ \mathrm{v}_{12} = 1$ **1st modal shape:** $\mathbf{V}_1 = [1 \quad 1]^T$ or $\mathbf{V}_1 = c_1[1 \quad 1]^T$ $(c_1 \neq 0)$

2) 2nd mode: $\omega_2 = \sqrt{3k/\mathrm{m}}$ **assume** $\mathbf{V}_2 = [\mathrm{v}_{21} \quad \mathrm{v}_{22}]^T$

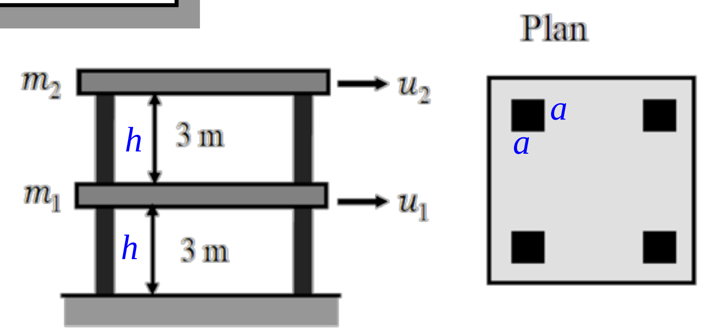
$$\left(\mathbf{K} - \omega_2^2 \mathbf{M}\right)\mathbf{v}_2 = 0 \rightarrow \begin{bmatrix} -1 & -1 \\ \cancel{-1} & \cancel{-1} \end{bmatrix}\begin{pmatrix} \mathrm{v}_{21} \\ \mathrm{v}_{22} \end{pmatrix} = 0 \rightarrow \boxed{\mathrm{v}_{21} = -\mathrm{v}_{22}}$$

Let $\mathrm{v}_{21} = 1;\ \mathrm{v}_{22} = -1$ **2nd modal shape:** $\mathbf{V}_2 = [1 \quad -1]^T$ or $\mathbf{V}_2 = c_2[1 \quad -1]^T$ $(c_2 \neq 0)$

Question 1

Solution:

One column: $I = \frac{bh^3}{12} = \frac{a^4}{12} = \frac{0.35^4}{12}$ Lateral stiffness: $k_p = \frac{12EI}{h^3}$



Lateral stiffness of one floor: $k_1 = k_2 = k = 4k_p = 31.12\times10^6\, N/m$

Mass matrix: $\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = 10^5 \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix} kg$ **Stiffness matrix:** $\mathbf{K} = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} = k \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

(a) Natural Frequencies

$$\det\left|\mathbf{K} - \omega^2\mathbf{M}\right| = \left| k\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - 10^5\omega^2 \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}\right| = \cancel{k}^2 \left|\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0$$

where $a = \omega^2 / 311.2$ or $\omega = \sqrt{311.2a}$

$$\rightarrow \begin{vmatrix} 2-1.5a & -1 \\ -1 & 1-a \end{vmatrix} = (2\text{-}1.5a)(1\text{-}a)\text{-}(\text{-}1)^2 = (3a\text{-}1)(a\text{-}1)/2 = 0 \rightarrow \therefore a_{1,2} = 1/3, 2$$

Natural frequencies: $\omega_1 = \sqrt{311.2/3} = 10.18(1/s)$; $\omega_2 = \sqrt{622.4} = 24.95(1/s)$

Question 1

(a) Modal Shapes

(Eigenvalue problem)

1) 1st mode $\omega_1 = 10.18$ assume $\mathbf{v}_1 = [\mathrm{v}_{11} \quad \mathrm{v}_{12}]^T$ $\left(\mathbf{K} - \omega_1^2 \mathbf{M}\right)\mathbf{v}_1 = 0$

$$\left(k\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - 10^5 \omega_1^2 \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{pmatrix} \mathrm{v}_{11} \\ \mathrm{v}_{12} \end{pmatrix} = 0 \rightarrow k\left(\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - a_1 \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{pmatrix} \mathrm{v}_{11} \\ \mathrm{v}_{12} \end{pmatrix} = 0$$

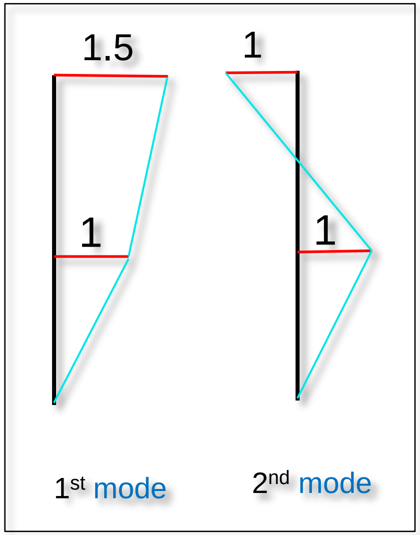
$$\begin{bmatrix} 2-1.5a_1 & -1 \\ -1 & 1-a_1 \end{bmatrix}\begin{pmatrix} \mathrm{v}_{11} \\ \mathrm{v}_{12} \end{pmatrix} = 0 \rightarrow \begin{bmatrix} 3/2 & -1 \\ -1 & 2/3 \end{bmatrix}\begin{pmatrix} \mathrm{v}_{11} \\ \mathrm{v}_{12} \end{pmatrix} = 0 \rightarrow \begin{matrix} 3\mathrm{v}_{11} - 2\mathrm{v}_{12} = 0 \\ \cancel{-3\mathrm{v}_{11} + 2\mathrm{v}_{12} = 0} \end{matrix} \rightarrow \boxed{\mathrm{v}_{12} = 3\mathrm{v}_{11}/2}$$

Let $\mathrm{v}_{11} = 1;\ \mathrm{v}_{12} = 1.5$ **1st modal shape:** $\mathbf{V}_1 = [1 \quad 1.5]^T$

2) 2nd mode $\omega_2 = 24.95$ assume $\mathbf{v}_2 = [\mathrm{v}_{21} \quad \mathrm{v}_{22}]^T$

$$\left(\mathbf{K} - \omega_2^2 \mathbf{M}\right)\mathbf{v}_2 = 0 \rightarrow \begin{bmatrix} -1 & -1 \\ \cancel{-1} & \cancel{-1} \end{bmatrix}\begin{pmatrix} \mathrm{v}_{21} \\ \mathrm{v}_{22} \end{pmatrix} = 0 \rightarrow \boxed{\mathrm{v}_{21} = -\mathrm{v}_{22}}$$

Let $\mathrm{v}_{21} = 1;\ \mathrm{v}_{22} = -1$ **2nd modal shape:** $\mathbf{v}_2 = [1 \quad -1]^T$



Question 1

(a) Check orthogonality

$$\mathbf{v}_1^T\mathbf{M}\mathbf{v}_2 = \mathbf{v}_2^T\mathbf{M}\mathbf{v}_1 = 0; \qquad \mathbf{v}_1^T\mathbf{K}\mathbf{v}_2 = \mathbf{v}_2^T\mathbf{K}\mathbf{v}_1 = 0$$

$$\mathbf{v}_1 = \begin{bmatrix}1 & 1.5\end{bmatrix}^T \qquad \mathbf{v}_2 = \begin{bmatrix}1 & -1\end{bmatrix}^T$$

$$\mathbf{M}\mathbf{v}_1 = 10^5\begin{bmatrix}1.5 & 0\\ 0 & 1\end{bmatrix}\begin{pmatrix}1\\1.5\end{pmatrix} = 10^5\begin{pmatrix}1.5\\1.5\end{pmatrix} \rightarrow \mathbf{v}_2^T\mathbf{M}\mathbf{v}_1 = 10^5\begin{bmatrix}1 & -1\end{bmatrix}\begin{pmatrix}1.5\\1.5\end{pmatrix} = 0$$

$$\mathbf{K}\mathbf{v}_1 = k\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{pmatrix}1\\1.5\end{pmatrix} = k\begin{pmatrix}0.5\\0.5\end{pmatrix} \rightarrow \mathbf{v}_2^T\mathbf{K}\mathbf{v}_1 = k\begin{bmatrix}1 & -1\end{bmatrix}\begin{pmatrix}0.5\\0.5\end{pmatrix} = 0$$

Modal masses & stiffnesses; frequencies

$$m_1 = \mathbf{v}_1^T\mathbf{M}\mathbf{v}_1 = 10^5\begin{bmatrix}1 & 1.5\end{bmatrix}\begin{pmatrix}1.5\\1.5\end{pmatrix} = 3.75\times10^5\,kg \qquad m_2 = \mathbf{v}_2^T\mathbf{M}\mathbf{v}_2 = 10^5\begin{bmatrix}1 & -1\end{bmatrix}\begin{pmatrix}1.5\\-1\end{pmatrix} = 2.5\times10^5\,kg$$

$$k_1 = \mathbf{v}_1^T\mathbf{k}\mathbf{v}_1 = k\begin{bmatrix}1 & 1.5\end{bmatrix}\begin{pmatrix}0.5\\0.5\end{pmatrix} = 1.25k \qquad k_2 = \mathbf{v}_2^T\mathbf{k}\mathbf{v}_2 = k\begin{bmatrix}1 & -1\end{bmatrix}\begin{pmatrix}3\\-2\end{pmatrix} = 5k$$

$$\omega_1 = \sqrt{k_1/m_1} = 10.18\ (1/s) \qquad \omega_2 = \sqrt{k_2/m_2} = 24.95\ (1/s)$$

Question 1

(Rayleigh-Ritz reduction Method)

(b) Find the first mode using one Ritz vector:

$$\mathbf{r}_1 = \begin{bmatrix}1 & 2\end{bmatrix}^T$$

$$\hat{\mathrm{m}} = \mathbf{r}_1^T \mathbf{M} \mathbf{r}_1 = 10^5 \begin{bmatrix}1 & 2\end{bmatrix}^T \begin{bmatrix}1.5 & 0 \\ 0 & 1\end{bmatrix} \begin{bmatrix}1 \\ 2\end{bmatrix} = 5.5\times 10^5 (kg)$$

$$\hat{\mathrm{k}} = \mathbf{r}_1^T \mathbf{K} \mathbf{r}_1 = k \begin{bmatrix}1 & 2\end{bmatrix}^T \begin{bmatrix}2 & -1 \\ -1 & 1\end{bmatrix} \begin{bmatrix}1 \\ 2\end{bmatrix} = 2k = 62.24\times 10^6 (N/m)$$

→ **Approximation value:**

$$\omega_1 \approx \hat{\omega}_1 = \sqrt{\hat{k}/\hat{m}} = 10.64\ (1/s)$$

$$\mathbf{v}_1 \approx \mathbf{r}_1 = \begin{bmatrix}1 & 2\end{bmatrix}^T$$

Exact value:
$$\begin{cases} \omega_1 = 10.18\ (1/s) \\ \mathbf{v}_1 = \begin{bmatrix}1 & 1.5\end{bmatrix}^T \end{cases}$$

Next: consider to scale $\mathbf{r}_1$ by a non-zero factor c

$$\mathbf{r}_1 = c\begin{bmatrix}1 & 2\end{bmatrix}^T$$

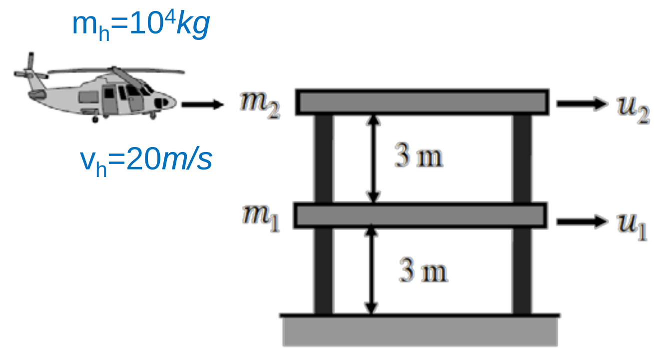
and repeat the procedure to see if any difference

Solution:

Question 2

(dynamic response)

From Q1, we know



Mass matrix: $$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = 10^5 \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix} kg$$

Stiffness matrix: $$\mathbf{K} = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} = k \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

1st mode: $\omega_1 = 10.18\ (1/s) \qquad \mathbf{v}_1 = [1 \quad 1.5]^T$

2nd mode: $\omega_2 = 24.95\ (1/s) \qquad \mathbf{v}_2 = [1 \quad -1]^T$

Initial conditions (after impact)

$$u_1(0) = u_2(0) = 0$$

$$\dot{u}_1(0) = \dot{u}_1^0 = ? \quad \dot{u}_2(0) = \dot{u}_2^0 = ?$$

Momentum conservation before and after the impact:

$$m_h \mathrm{v}_h = m_1 \dot{u}_1^0 + m_2 \dot{u}_2^0$$

(a) Consider two modes $\boxed{\dot{u}_1(0) = \dot{u}_1^0 = 0}$ $\rightarrow m_h \mathrm{v}_h = m_2 \dot{u}_2^0 \rightarrow \dot{u}_2^0 = 2m/s \quad \therefore \ \dot{\mathbf{u}}_0 = [0 \quad 2]^T$

SDOF equations

$$\ddot{x}_1 + 2\xi_1 \dot{x}_1 + \omega_1^2 x_1 = 0 \qquad x_1(0) = 0;\ \dot{x}_1(0) = \mathbf{v}_1^T \mathbf{M} \dot{\mathbf{u}}_0 / \tilde{m}_1 = 0.8 m/s \qquad \tilde{m}_1 \text{ (1st modal mass)}$$

$$\ddot{x}_2 + 2\xi_2 \dot{x}_2 + \omega_2^2 x_2 = 0 \quad x_2(0) = 0;\ \dot{x}_2(0) = \mathbf{v}_2^T \mathbf{M} \dot{\mathbf{u}}_0 / \tilde{m}_2 = -0.8 m/s \qquad \tilde{m}_2 \text{ (2nd modal mass)}$$

2

Solutions

$$\dot{\mathbf{u}}(t) = \mathbf{v}_1 \dot{x}_1(t) + \mathbf{v}_2 \dot{x}_2(t) \qquad \mathbf{u}(t) = \mathbf{v}_1 x_1(t) + \mathbf{v}_2 x_2(t)$$

$$\dot{\mathbf{u}}_{\max} = \mathbf{v}_1 \dot{x}_{1,\max} + \mathbf{v}_2 \dot{x}_{2,\max} \qquad \mathbf{u}_{\max} = \mathbf{v}_1 x_{1,\max} + \mathbf{v}_2 x_{2,\max}$$

$$\boxed{\text{SDOF}:\ \dot{u}_{\max} = \omega u_{\max}}$$

top floor

$$\dot{u}_{2,\max} = \mathrm{v}_{12} \dot{x}_{1,\max} + \mathrm{v}_{22} \dot{x}_{2,\max}$$

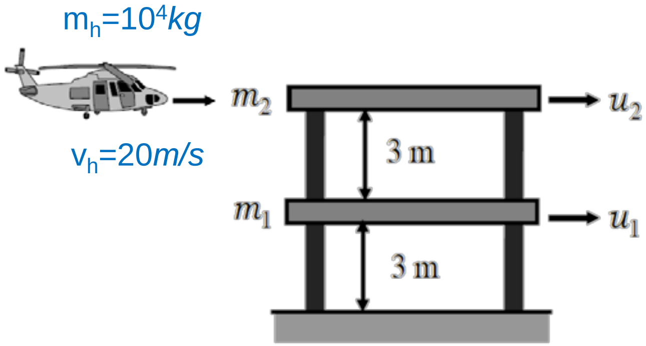
$$= 1.5 * .8 - 1 * (-.8) = 2.0 m/s$$

$$u_{2,\max} = \mathrm{v}_{12} x_{1,\max} + \mathrm{v}_{22} x_{2,\max} = \mathrm{v}_{12} \dot{x}_{1,\max} / \omega_1 + \mathrm{v}_{22} \dot{x}_{2,\max} / \omega_2$$

$$= 1.5 * .8 / 10.18 - 1 * (-0.8) / 24.95 = 0.1580 m$$

Question 2

(dynamic response)



(b) Consider one mode (from Q1(b))

$$\omega_1 \approx \hat{\omega}_1 = \sqrt{\hat{k}/\hat{m}} = 10.64\ (1/s)$$

$$\mathbf{v}_1 \approx \mathbf{r}_1 = [1 \quad 2]^T$$

$$\dot{\mathbf{u}}(t) = \mathbf{v}_1 \dot{x}_1(t) \qquad \dot{\mathbf{u}}(0) = \mathbf{v}_1 \dot{x}_1(0)$$

$$\boxed{\dot{u}_1(0) = \mathrm{v}_{11}\dot{x}_1^0;\ \dot{u}_2(0) = \mathrm{v}_{12}\dot{x}_1^0}$$

$$m_h \mathrm{v}_h = m_1\dot{u}_1^0 + m_2\dot{u}_2^0 \rightarrow m_h \mathrm{v}_h = (m_1\mathrm{v}_{11} + m_2\mathrm{v}_{12})\dot{x}_1^0 \rightarrow \dot{x}_1^0 = 4/7 m/s$$

SDOF equation

$$\ddot{x}_1 + 2\xi_1\dot{x}_1 + \omega_1^2 x_1 = 0 \qquad x_1(0) = 0;\ \dot{x}_1(0) = 4/7 m/s$$

Solutions

$$\dot{\mathbf{u}}(t) = \mathbf{v}_1\dot{x}_1(t) \qquad \mathbf{u}(t) = \mathbf{v}_1 x_1(t)$$

$$\dot{\mathbf{u}}_{\max} = \mathbf{v}_1\dot{x}_{1,\max} \qquad \mathbf{u}_{\max} = \mathbf{v}_1 x_{1,\max}$$

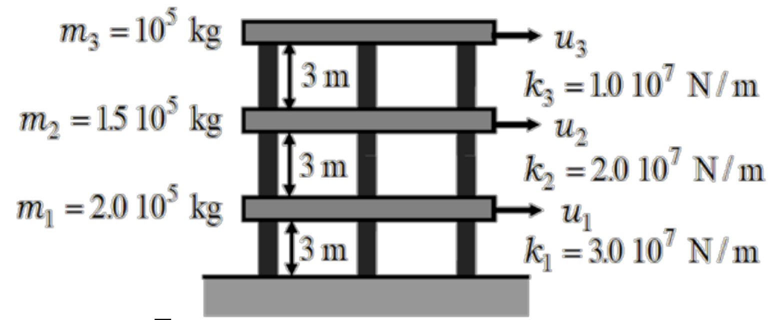
$$\dot{u}_{2,\max} = \mathrm{v}_{12}\dot{x}_{1,\max} = 2*4/7 = 1.142 m/s$$

$$u_{2,\max} = \mathrm{v}_{12}x_{1,\max} = \mathrm{v}_{12}\dot{x}_{1,\max}/\omega_1 = \dot{u}_{2,\max}/\omega_1 = 1.143/10.64 = 0.1074 m$$

Question 3

(Rayleigh-Ritz reduction Method)

Solution:



Mass matrix:

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} = 10^5 \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} kg$$

Stiffness matrix:

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} = 10^7 \begin{bmatrix} 5 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} (\mathrm{N}/m)$$

(a) Find the first mode using one Ritz vector: $\mathbf{r}_1 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$

$$\hat{\mathrm{m}} = \mathbf{r}_1^T \mathbf{M} \mathbf{r}_1 = 17 \times 10^5 (kg)$$

$$\hat{\mathrm{k}} = \mathbf{r}_1^T \mathbf{K} \mathbf{r}_1 = 6 \times 10^7 (N/m)$$

→ **Approximation value:**

$$\omega_1 \approx \hat{\omega}_1 = \sqrt{\hat{k}/\hat{m}} = 5.94\ (1/s)$$

$$\mathbf{v}_1 \approx \mathbf{r}_1 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$$

Question 3

(b) Find the first TWO modes using two Ritz vectors:

$$\mathbf{r}_1 = [1 \quad 2 \quad 3]^T \qquad \mathbf{r}_2 = [1 \quad 4 \quad 9]^T$$

$$\hat{\mathbf{M}} = [\mathbf{r}_1 \quad \mathbf{r}_2]^T \mathbf{M} [\mathbf{r}_1 \quad \mathbf{r}_2] = \begin{bmatrix} \mathbf{r}_1^T\mathbf{M}\mathbf{r}_1 & \mathbf{r}_1^T\mathbf{M}\mathbf{r}_2 \\ \mathbf{r}_2^T\mathbf{M}\mathbf{r}_1 & \mathbf{r}_2^T\mathbf{M}\mathbf{r}_2 \end{bmatrix} = \begin{bmatrix} 17 & 41 \\ 41 & 107 \end{bmatrix} \times 10^5 (kg)$$

$$\hat{\mathbf{K}} = [\mathbf{r}_1 \quad \mathbf{r}_2]^T \mathbf{K} [\mathbf{r}_1 \quad \mathbf{r}_2] = \begin{bmatrix} \mathbf{r}_1^T\mathbf{K}\mathbf{r}_1 & \mathbf{r}_1^T\mathbf{K}\mathbf{r}_2 \\ \mathbf{r}_2^T\mathbf{K}\mathbf{r}_1 & \mathbf{r}_2^T\mathbf{K}\mathbf{r}_2 \end{bmatrix} = \begin{bmatrix} 6 & 14 \\ 14 & 46 \end{bmatrix} \times 10^7 (N/m)$$

Solve the following eigenvalue problem (2 DOFs):

$$\hat{\mathbf{K}}\mathbf{x}_i = \hat{\omega}_i^2 \hat{\mathbf{M}}\mathbf{x}_i \quad (i = 1,2)$$

first mode:

$$\omega_1 \approx \hat{\omega}_1 = 5.93 \ (1/s) \quad \mathbf{x}_1 = [0.9522 \quad 0.0478]^T$$

$$\mathbf{v}_1 \approx [\mathbf{r}_1 \quad \mathbf{r}_2]\mathbf{x}_1 = [1 \quad 2.096 \quad 3.287]^T$$

second mode:

$$\omega_2 \approx \hat{\omega}_2 = 12.84 \ (1/s) \quad \mathbf{x}_2 = [1.6978 \quad -0.6978]^T$$

$$\mathbf{v}_2 \approx [\mathbf{r}_1 \quad \mathbf{r}_2]\mathbf{x}_2 = [1 \quad 0.6045 \quad -1.1866]^T$$

Exact solutions

$\omega_1 = 5.928/s;\ \mathbf{V}_1 = [1 \quad 2.149 \quad 3.313]^T$

$\omega_2 = 12.68/s;\ \mathbf{V}_2 = [1 \quad 0.893 \quad -1.473]^T$