College of Engineering
Swansea University

Dynamics & Earthquake Analysis of Structures

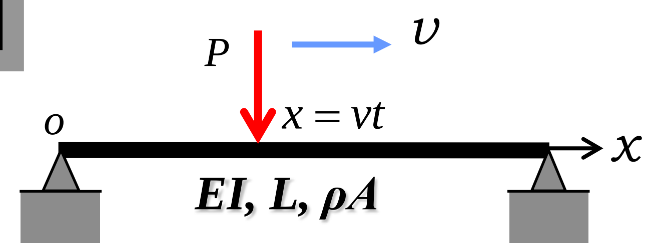
Question Sheet 2 – Solutions

(Q1, 2 and 3 only)

Prof. Y T Feng

Rayleigh's Method

Question 1



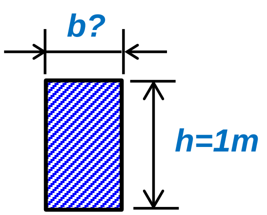
A point load ***P = 1 kN*** moves along with constant speed ***v = 10 m/s*** on a simply supported beam of length ***L= 10π m*** as shown in the figure. The beam is made of **concrete**, has a **rectangular section** of height ***1m*** and an average density of ***2,800 kg/m³***.

Determine: (a) the deflection of the beam as a function of time; (b) the dynamic magnification factor; and (c) the maximum bending moment at the central section.

The moving force: $P\delta(x - vt)$

The default shape function:

$$\psi(x) = \sin(\pi x / L)$$



- **From Q2(a) solution 1 in Sheet 1, the natural frequency:**

$$\omega^2 = \pi^4 \frac{EI}{\rho A L^4} = \pi^4 \frac{E\cancel{b}/12}{\rho \cancel{b} L^4} = \pi^4 \frac{14\times 10^9/12}{2800(10\pi)^4} = 41.7 \qquad \therefore \omega = 6.45 \ 1/s$$

Independent of the width!

$$A = bh = b$$

$$I = \frac{bh^3}{12} = \frac{b}{12}$$

- **The equivalent force:** $F(t) = P\sin(\pi v t / L) = \sin(t) \qquad (0 \le t \le L/v = \pi)$
- **The SDOF equation:** $m\ddot{u}_o + k u_o = \sin(t) \qquad (0 \le t \le L/v = \pi)$

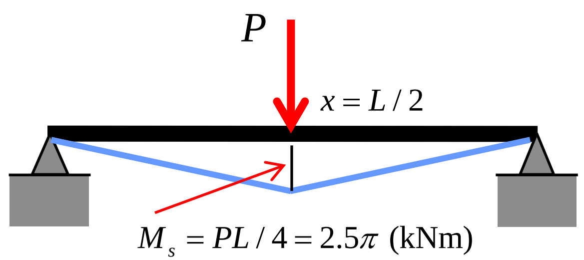
(a) The dynamic solution: See the expression on *Slide No 52*

(b) The dynamic amplification factor (*Slide No 52*): $D_f \approx 1.8$

(c) The maximum static bending moment M_s when *P* at *x=L/2*

The maximum dynamic bending moment M_d

$$M_d = D_f M_s \approx 14.237 \text{ kNm}$$



Rayleigh's Method

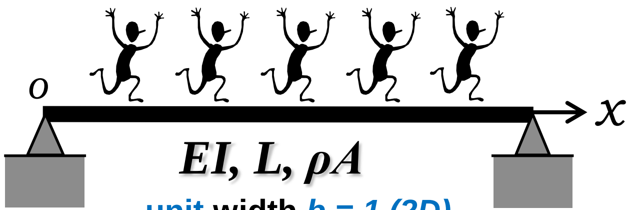
Question 2

Rhythmic Activities (BS 6399)

A concrete ribbed slab floor spans 9m and has an average mass of 500 kg/m². The floor is simply supported on either side and has a natural frequency of vibration of 6.3 Hz. The floor is to be used for aerobics and other similar rhythmic activities at frequencies ranging from 1.5 Hz to 2.5 Hz and with contact ratios α between 0.5 and 1. During these activities the average imposed load will remain below 0.75 kN/m² (before applying dynamic magnification) and the damping ratio is taken to be 3%.

a) Determine the maximum possible resonant displacement and the resulting peak acceleration and bending moment per unit width;

b) If the floor has been designed for a service load of 5kN/m², determine its suitability for the proposed use.



unit width $b = 1$ (2D)

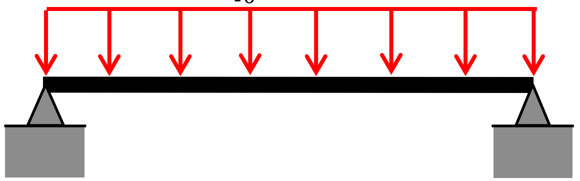
The natural frequency:

$$f = 6.3Hz;\ \omega = 2\pi f = 39.58\ 1/\text{s}$$

$$\rho A = 500kg/m^2;\ \xi = 3\%$$

The force imposed by the activities

$$\text{Static}:\ q_0 = 0.75kN/m$$



- The force imposed by the activities is consider as a UDL.
- The force frequency: $\Omega_f = 1.5 \sim 2.5Hz;\ T_f = 2\pi/\Omega_f$
- The contact ratio: $\alpha = 0.5 \sim 1$
- For a given force frequency Ω_f, the frequencies of harmonic forces are:

$$\Omega_f, 2\Omega_f, 3\Omega_f, \dots, n\Omega_f, \dots$$

When one of these frequencies is equal to the natural frequency f, a resonance happens!

Continued

Question 2

$f = 6.3Hz$
$\Omega_f = 1.5 \sim 2.5Hz$

- **The resonance condition:**

$$f = n\Omega_f \quad (n=1,2,3,\ldots)$$

$n=1: \quad \Omega_f = 6.3Hz > 2.5Hz;$ Too large ✗

$n=2: \quad \Omega_f = 3.15Hz > 2.5Hz;$ Too large ✗

$n \geq 5: \quad \Omega_f < 1.5Hz;$ too small ✗

$n=3: \quad \Omega_f = 2.1Hz;$ valid ✓

$n=4: \quad \Omega_f = 1.575Hz;$ valid ✓

Two possible cases

- **Compute D_f for Case 1: $\Omega_f = 2.1Hz$**

Taken a particular contact ratio: $\alpha = 2/3$

$$D_f = \sqrt{1+\left(\frac{F_1}{F_0}H_1\right)^2+\left(\frac{F_2}{F_0}H_2\right)^2+\ldots+\left(\frac{F_n}{F_0}H_n\right)^2+\ldots} = \sqrt{1+\sum_{n=1}^{\infty}\left(\frac{F_n}{F_0}H_n\right)^2} = \sqrt{1+\sum_{n=1}^{\infty}D_{f,n}^2}$$

(Slide No. 57)

From the 1st row of the Table on *Slide No. 57:*

$$H_n = \frac{1}{\sqrt{(1-\gamma_n^2)^2+4\xi^2\gamma_n^2}}; \quad \gamma_n = \frac{n\Omega}{\omega} = \frac{n\Omega_f}{f}$$

$$\frac{F_1}{F_0} = 1.29; \quad \frac{F_2}{F_0} = 0.16; \quad \frac{F_3}{F_0} = 0.13; \quad \frac{F_4}{F_0} = 0.04$$

$$\gamma_1 = \frac{\Omega_f}{f} = \frac{1}{3}; \quad \gamma_2 = \frac{2\Omega_f}{f} = \frac{2}{3}; \quad \gamma_3 = \frac{3\Omega_f}{f} = 1; \quad \gamma_4 = \frac{4\Omega_f}{f} = \frac{4}{3}$$

$$H_1 = 1.5; \quad H_2 = 3.0; \quad H_3 = 16.67; \quad H_4 = 0.34$$

$D_f = 3.1$ ✓

- **Compute D_f for Case 2: $\Omega_f = 1.575Hz$**

$D_f = ?$ ✗

So Case 1 is the worse case: $D_f = 3.1$

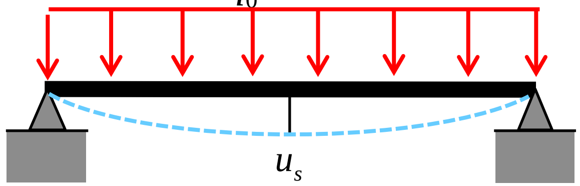
Continued

Question 2

(a) **The maximum displacement:** $u_{\max} = D_f u_s$

$$u_s = \frac{5 q_0 L^4}{384 EI} \qquad EI = ?$$

Static: $q_0 = 0.75 kN/m$



From Q2(a) solution 1 in Sheet 1, the natural frequency

$$\omega^2 = \pi^4 \frac{EI}{\rho A L^4} \Rightarrow \frac{L^4}{EI} = \frac{\pi^4}{\rho A \omega^2} \Rightarrow$$

The maximum static displacement

$$u_s = \frac{5\pi^4 q_0}{384 \rho A \omega^2} = \underline{1.2mm}$$

The maximum displacement: $u_{\max} = D_f u_s = \underline{3.72mm}$

The maximum acceleration: $a_{\max} = \omega^2 u_{\max} = \underline{5.826 m/s^2}$

The maximum static bending moment

$$M_s = \frac{q_0 L^2}{8} = 7.59 kNm$$

The maximum bending moment: $M_{\max} = D_f M_s = \underline{23.54 kNm}$

(b) **The suitability of the structure – Safety check**

The dynamic UDL load:

$$q_d = D_f q_0 = 3.1 \times 0.75 = 2.33 kN/m < q_{\max} = \underline{5kN/m}$$

Safe! ✓

Alternatively, compute the maximum bending moment under the designed service load q_{max} and compare it with M_{max}

Rayleigh's Method

Question 3

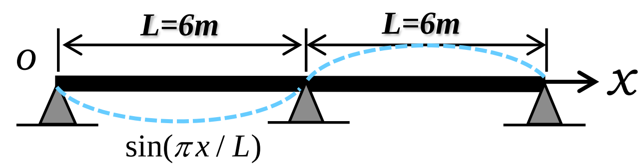
Multiple Moving Loads

A 12m long rail bridge has the continuous beam configuration shown in the figure, an average flexural stiffness $EI=5GNm^2$, a mass per unit length of 100,000 kg/m and a damping ratio of 2%.

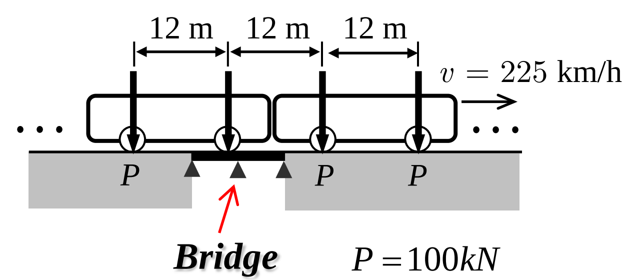
a) Using Rayleigh's method with an appropriate sinusoidal function, obtain its fundamental frequency of vibration.

b) Determine the equivalent modal force that results from the motion of a train consisting of an 'infinite' number of 100kN point loads separated by equal distances of 12m and travelling at a constant speed of 225 km/h.

c) Obtain the dynamic amplification factor for the above train of loads.



$EI = 5GNm^2;\ \rho A = 10^5 kg/m;\ \xi = 2\%$



$P = 100kN$

a) **The shape function:** $\psi(x) = \sin(\pi x / L)$ (*Check it satisfies the boundary conditions*)

$$m = \int_0^{2L} \rho[\psi(x)]^2 A dx = \rho A \int_0^{2L} \sin^2(\frac{\pi x}{L}) dx \qquad k = \int_0^{2L} EI[\psi''(x)]^2 dx$$

$$\omega^2 = \pi^4 \frac{EI}{\rho A L^4} \quad \Rightarrow \quad \omega = 61.3\ 1/s$$

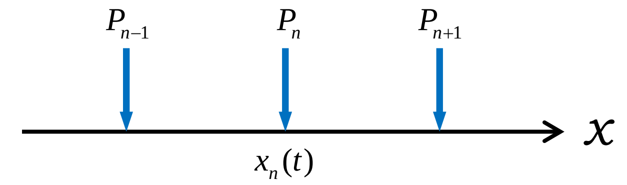
Question 3

b) The equivalent force

The *n*th point force $P_n(t) = \mathrm{P}\delta(x - x_n(t))$

The *equivalent* force $F_n(t) = \mathrm{P}\sin(\pi x_n / L)$

The total *equivalent* force $F(t) = \sum_n F_n(t) = \mathrm{P}\sum_n \sin(\pi x_n / L)$



- **1st Force: Starts at the left end at** $t = 0$

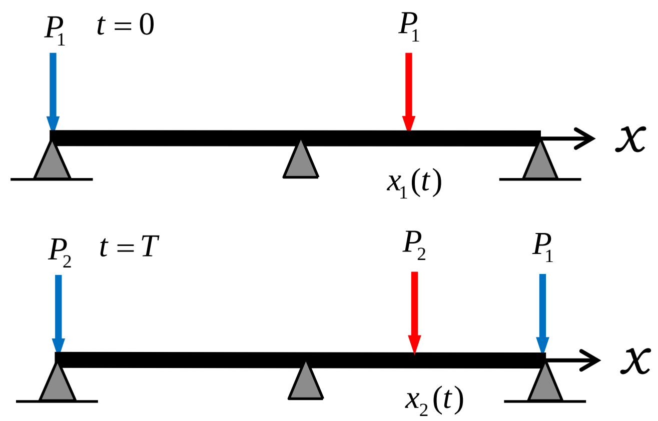
  $x_1(t) = vt \qquad 0 \le t \le T = 2L/v$

- **2nd Force: Enters the bridge at** $t = T$ ***while 1st force is about to leave***

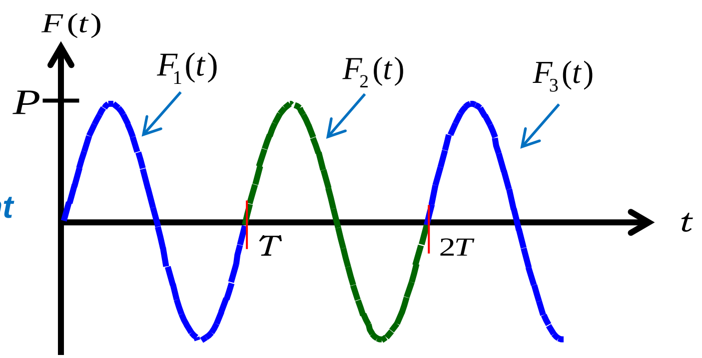
  $x_2(t) = v(t - T) \qquad T \le t \le 2T$

- ***n*th Force**

  $x_n(t) = v[t - (n-1)T] \qquad (n-1)T \le t \le nT$



The *equivalent* force



A Harmonic Force

$$F(t) = \sum_n F_n(t) = \mathrm{P}\sin(\pi vt / L)$$

$$\Omega = \pi v / L = 32.72 \text{ 1/s}$$

c) **The dynamic amplification factor**

$$D_f = H = \frac{1}{\sqrt{(1-\gamma^2)^2 + 4\xi^2\gamma^2}}$$

$$\gamma = \frac{\Omega}{\omega} = \frac{32.72}{61.3} = 0.5338$$

$$D_f = 1.3978$$

The Harmonic Force

$$F(t) = \sum_n F_n(t) = \mathrm{P}\sin(\pi v t / L)$$

$$\Omega = \pi v / L = 32.72 \text{ 1/s}$$