C*-Algebra and K-theory

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Abstract

K-the theory is important in non-commutative geometry, and C^*-algebra is the main object researched by non-commutative geometry. In this report, we firstly introduce the basic concept, the spectrum of the elements in C^*-algebra, and function calculus on C^*-algebra. In the second part, we discuss unitary elements and projections which are special parts of C^*-algebra with special properties and the equivalences defined on them. Finally, we introduce the Grothendieck construction and we define the K_0 group by the Grothendieck construction.

Contents

1 C*-Algebra

1.1 Algebra

Definition 1.1 (Algebra)

Given a field $\mathbb{F}$, an algebra $\mathcal{A}$ over $\mathbb{F}$ is a set endowed with the following conditions:

- (linear stucture) $\mathcal{A}$ is a vector space over $\mathbb{F}$.
- (multiplicative structure) A multiplication on $\mathcal{A} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that,
  - $(xy)z = x(yz)$ for all $x, y, z \in \mathcal{A}$.
  - $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$.
  - $\lambda(xy) = (\lambda x)y = x(\lambda y)$ for all $x, y \in \mathcal{A}$, and $\lambda \in \mathbb{F}$.

Remark 1.2

The requirement for being an *algebra* is much stronger than that for a *vector space*, since it has an additional multiplicative structure.

Example 1.3 (Algebras)

- (Group Algebra)
- (Lie Algebra)

Definition 1.4 (C-Algebra)

When the field $\mathbb{F}$ is the complex number field $\mathbb{C}$, algebra $\mathcal{A}$ is called a complex algebra or *C-algebra* (*i.e.*, $\mathbb{F} = \mathbb{C}$).

Definition 1.5 (Normed Algebra)

The algebra $\mathcal{A}$ is called a *normed algebra* if

- $\mathcal{A}$ is a $\mathbb{C}$-algebra.
- There is a norm on its associated vector space structure (*i.e.*, $\mathcal{A}$ is a normed space), and satisfies

$$\|ab\| \leq \|a\| \cdot \|b\| \qquad \text{for any } a, b \in \mathcal{A}.$$

Remark **1.6**

In another word, there is a norm $\|\cdot\|$ over $\mathcal{A}$:

$$\begin{aligned} \mathcal{A} &\to \mathbb{R} \\ a &\mapsto \|a\|. \end{aligned}$$

Definition 1.7 (Banach Algebra)

A normed algebra $\mathcal{A}$ is a *Banach algebra* if it is complete under its norm $\|\cdot\|$.

Remark **1.8**

The normed algebra $\mathcal{A}$ is complete means, for any Cauchy sequence $\{x_n\}$ in $\mathcal{A}$, we have

$$\{x_n\} \longrightarrow x \in \mathcal{A} \qquad \text{converge in } \mathcal{A}$$

(*i.e.*, $\lim \{x_n\} = x$, its limit, is in A).

Definition 1.9 ($*$-Algebra)

An algebra $\mathcal{A}$ is called a $*$-algebra if

- $\mathcal{A}$ is a $\mathbb{C}$-algebra.
- ($*$-structure(conjugate)) $\mathcal{A}$ is endowed with a conjugate "$*$": for any $a, b \in \mathcal{A}, \alpha \in \mathbb{C}$,
  - $(a + b)^* = a^* + b^*$
  - $(\alpha a)^* = \overline{\alpha} a^*$
  - $a^{**} = a$
  - $(ab)^* = b^* a^*$.

Remark **1.10** (Adjoint)

For an element $a \in \mathcal{A}$, the conjugation $a^* \in \mathcal{A}$ is called an adjoint of a.

Definition 1.11 (C^*-Norm)

A norm on a $*$-algebra $\mathcal{A}$ is a C^*-norm if it satisfies

$$\|a^* a\| = \|a\|^2 \qquad \text{for all } a \in \mathcal{A}.$$

Definition 1.12 (C^*-Algebra)

The algebra $\mathcal{A}$ is called a C^*-algebra if

- $\mathcal{A}$ is a $*$-algebra
- It endow with a C^*-norm.
- The algebra $\mathcal{A}$ is complete under the C^*-norm.

Remark **1.13**

Equivalently $\mathcal{A}$ is a C^*-algebra $\Longleftrightarrow$ $\mathcal{A}$ is a Banach algebra with a $*$–structure and a C^*-norm.

Next we consider the subalgera of a C^*-algebra.

Definition 1.14

A $*$-algebra $\mathcal{A}'$ is called a $*$–subalgebra of a $*$– algebra $\mathcal{A}$ if it is closed under

- (linear operation) $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ by $(a, b) \longmapsto a + b$.
- (multiplication) $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ by $(a, b) \longmapsto a \cdot b$
- (adjoint) $\mathcal{A} \to \mathcal{A}$ by $a \longmapsto a^*$.
- (scalar multiplication) $\mathbb{C} \times \mathcal{A} \to \mathcal{A}$ by $(\alpha, a) \longmapsto \alpha a$.

Definition 1.15

A non-empty subset B of a C^*-algebra $\mathcal{A}$ is called a sub-C^*-algebra if B is a sub-*-algebra and also a C^*-algebra.
(*i.e.*, B is a sub-*-algebra and complete under the *-norm on $\mathcal{A}$).

Remark

$B \subset \mathcal{A}$ is a sub-C^*-algebra $\iff$ B is closed under algebraic operations and is norm-closed.

1.2 Spectrum

To define the spectrum of elements in C^*-algebra $\mathcal{A}$, we need to introduce the concept and properties of the invertible group of $\mathcal{A}$ first.

Definition 1.16 (Invertible group $\mathrm{GL}(\mathcal{A})$, as in [3])

$\mathcal{A}$ is an unital C^*-algebra, an element $a \in \mathcal{A}$ is called invertible if there is an element $b \in \mathcal{A}$ such that $ab = ba = 1_{\mathcal{A}}$. So we define the set of all invertible elements in $\mathcal{A}$ by

$$\mathrm{GL}(\mathcal{A}) = \{a \in \mathcal{A} : a \text{ is invertible } \}.$$

Remark 1.17

- The inversion b is unique, since if $ab_1 = ab_2$, then $(ab_1)b_2 = 1 \cdot b_2 = b_2 = b_1(ab_2) = b_1 \cdot 1 = b_1$, *i.e.*, $b_1 = b_2$.
- (Invertible group $\mathrm{GL}(\mathcal{A})$)
  $\mathrm{GL}(\mathcal{A})$ is a group under the multiplication. Since for any elements $a, b \in \mathrm{GL}(\mathcal{A})$, we have $a^{-1}, b^{-1} \in \mathcal{A}$, then $(ab) \cdot (b^{-1}a^{-1}) = a(bb^{-1})a^{-1} = 1$. Hence $(ab)^{-1} = b^{-1}a^{-1} \in \mathcal{A}$, *i.e.*, $ab \in \mathrm{GL}(\mathcal{A})$. Other conditions for group are obvious.
  In addition, if $a \in \mathrm{GL}(\mathcal{A})$, then $\alpha a \in \mathrm{GL}(\mathcal{A})$ for any $\alpha \in \mathbb{C}$, since $(\alpha a)(\frac{1}{\alpha}a^{-1}) = (\alpha\frac{1}{\alpha})(aa^{-1}) = 1$.

Definition 1.18 (Spectrum, as in [1])

$\mathcal{A}$ is an unital C^*-algebra, the spectrum of an element $a \in \mathcal{A}$ is the set of complex numbers $\lambda \in \mathbb{C}$ such that $a - \lambda 1_{\mathcal{A}}$ is not invertible in $\mathcal{A}$, *i.e.*,

$$\mathsf{Sp}(a) = \mathsf{Sp}_{\mathcal{A}}(a) = \{\lambda \in \mathbb{C} : a - \lambda 1_{\mathcal{A}} \notin \mathrm{GL}\,(A)\}.$$

Remark 1.19

- (Resolution) We define the *resolution* to be the complement of spectrum, *i.e.*,

$$\sigma(a) = \{\lambda \in \mathbb{C} : \lambda 1 - a \in \mathrm{GL}(\mathcal{A}), \textit{i.e.}, \text{ invertible } \}.$$

  and the resolution function by $\lambda \in \sigma(a)$ is in resolution (*i.e.*, $\lambda 1 - a \in \mathrm{GL}(\mathcal{A})$ is invertible), then

$$R(\lambda) = (\lambda - a)^{-1}.$$

1.3 The continuous function calculus for normal elements

A natural thinking about the C^* algebra is how to define the function calculus for an element in C^*-algebra.

Definition 1.20 ($C_0(X)$)

$C_0(X)$ is the C^*-algebra of all continuous functions $f : x \to \mathbb{C}$ which vanishing at infinity. (*i.e.*, for each $\epsilon > 0$, there is a compact subset K of X, such that $|f(x| \leq \epsilon$ for all $x \in X \setminus K$).

Remark

$C_0(X)$ is a C^*-algebra since,

- $C_0(X)$ is obviously closed under algebraic operation.
- $f^* = \overline{f(x)}$
- The norm is defined by $\|f\| = \sup\{|f(x)||x \in X\}$.

Remark

$C(X)$ is defined to be the set of all continuous function on X. And if X is compact, we deduce that $C_0(X) = C(X)$.

Definition 1.21 ($C^*(S)$)

$\mathcal{A}$ is a C^*-algebra and S is a subset of $\mathcal{A}$. Then $C^*(S)$ is defined to be the C^*-subalgebra generated by S. (*i.e.*, $C^*(S)$ is the smallest C^*-subalgebra contains subset S).

Now we will see the important theorem which can also define the continuous function calculus for normal elements in C^*-algebra.

Theorem 1.22 (Continuous function calculus)

$\mathcal{A}$ is an unital C^*-algebra and a is a normal element (*i.e.*, $a \cdot a^* = a^* \cdot a$). Then there is an isometric *-isometric isomorphism such that

$$C^*(a) \xleftrightarrow{\simeq} C_0(sp(a))$$

$$f(a) \xleftrightarrow{\text{correspondence}} f$$

$$a \xleftrightarrow{\text{correspondence}} id$$

The proof an be seen in [3] by using some analytical tools.
By this theorem and the correspondence, we can define the continuous function calculus on normal elements.

2 Unitary Elements and Projections

2.1 Homotopy Class of Unitary Elements

In this part, we shall just consider the unitary elements in an unital C^*-algebra.

Definition 2.1 (Unitary Elements, as in [2])

Let $\mathcal{A}$ be a C^*-algebra, then an element $u \in \mathcal{A}$ is unitary if $u \cdot u^* = u^* \cdot u = 1$.

Remark 2.2

u is unitary $\Longrightarrow \|u\| = \|u^*\| = 1$ (since $\|uu^*\| = \|u\|^2 = 1$), however $\|u\| = 1 \not\Rightarrow u$ is unitary. This is different of the algebra of complex numbers, which has $|\alpha| = 1 \Longrightarrow \alpha\overline{\alpha} = |\alpha| = 1$.

Definition 2.3 (Unitary Group)

Denote the group of all unitary elements in C^*-algebra $\mathcal{A}$ as $U(\mathcal{A})$, *i.e.*,

$$U(\mathcal{A}) = \{u \in \mathcal{A} : uu^* = 1 = u^*u\}.$$

Remark 2.4

$U(\mathcal{A})$ is a group under the multiplication of the C^*-algebra $\mathcal{A}$, since if $u_1, u_2 \in \mathcal{A}$, then

- $(u_1u_2)(u_1u_2)^* = u_1u_2u_2^*u_1^* = 1 \Longrightarrow u_1u_2 \in U(\mathcal{A})$.
- $1_{\mathcal{A}} \in U(\mathcal{A})$.
- $(u_1u_2)u_3 = u_1(u_2u_3)$.
- there is $u_1^* \in U(\mathcal{A})$ such that $u_1u_1^* = u_1^*1u = 1$, *i.e.*, $u_1^{-1} = u_1^*$.

Hence $U(\mathcal{A})$ is a group under multiplication.

Now we can consider the topological structure on a C^*-algebra, more specifically, on the group of unitary elements.

2.2 Semigroup of Projections

Definition 2.5 (Semigroup of Projections)

Define the semigroup of projections for C^*-algebra matrixes, by

$$\mathcal{P}_\infty(\mathcal{A}) = \bigcup_{n=1}^{\infty} \mathcal{P}_n(\mathcal{A}) \qquad \text{where} \quad \mathcal{P}_n(\mathcal{A}) = \mathcal{P}(M_n(\mathcal{A}))$$

($\mathcal{P}(M_n(\mathcal{A}))$ means the set of projections in $M_n(\mathcal{A})$).

Definition 2.6 (Semigroup Operation)

Define the operation $\oplus$ on $\mathcal{P}_\infty(\mathcal{A})$ by

$$p \oplus q = \operatorname{diag}(p, q) = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$$

where $p \in \mathcal{P}_m(\mathcal{A}), q \in \mathcal{P}_n(\mathcal{A})$, and $p \oplus q \in M_{m+n}(\mathcal{A})$.

Remark 2.7

$\mathcal{P}_\infty(\mathcal{A})$ is a semigroup under the operation "$\oplus$", since

- If $p \in \mathcal{P}_m(\alpha), q \in \mathcal{P}_n(\mathcal{A})$, then $p \oplus q = \operatorname{diag}(p, q) = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \in \mathcal{P}_{m+n}(\mathcal{A})$, since

$$(p \oplus q)(p \oplus q) = \begin{pmatrix} p^2 & 0 \\ 0 & q^2 \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$$

  and $\begin{pmatrix} p^* & 0 \\ 0 & q^* \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$.

- Combination:

$$(p \oplus q) \oplus w = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \oplus w = \begin{pmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & w \end{pmatrix} = p \oplus \begin{pmatrix} q & 0 \\ 0 & w \end{pmatrix} = p \oplus (q \oplus w).$$

Definition 2.8 ($\sim_0$ Relation)

Define the relation $\sim_0$ on $\mathcal{P}_\infty(\mathcal{A})$ by $p \in \mathcal{P}_m(\mathcal{A}), q \in \mathcal{P}_n(\mathcal{A})$, then $p \sim_0 q$ if there is an element $v \in M_{m,n}(\mathcal{A})$ such that

$$p = v^*v \text{ and } q = vv^*$$

Remark 2.9

- The $M_{m,n}(\mathcal{A})$,
- vv^*, v^*v are the usual multiplication of matrixes.

Remark 2.10

- "$\sim_0$" is a equivalence relation on semigroup $\mathcal{P}_\infty(\mathcal{A})$, since
  - (Reflexive) When $p \in \mathcal{P}_n(\mathcal{A}) \subset \mathcal{P}_\infty(\mathcal{A})$, then $p^* \in M_n(\mathcal{A})$ and $p^*p = pp^* = p$.
  - (Symmetry) If $p \in \mathcal{P}_n(\mathcal{A}), q \in \mathcal{P}_m(\mathcal{A}), p \sim_0 q$, *i.e.*, there is a $v \in M_{m,n}(\mathcal{A})$ such that $p = v^*v, q = vv^*$. Then let $a = v^* \in M_{n,m} * (\mathcal{A})$, we have $q = a^*a, p = aa^*$,*i.e.*, $q \sim_0 p$.
  - (Translative) If $p \in \mathcal{P}_{n_1}(\mathcal{A}), q \in \mathcal{P}_{n_2}(\mathcal{A}), z \in \mathcal{P}_{n_3}$ and $p \sim_0 q, q \sim_0 z$, *i.e.*, $p = v^*v, q = vv^* = w^*w, z = ww^*$. Then let $x = wv \in M_{n_3,n_1}(\mathcal{A})$, we have $p = x^*x, z = xx^*$, *i.e.*, $p \sim_0 z$.
- If $p, q \in \mathcal{P}_n(\mathcal{A}) \subset \mathcal{P}_\infty(\mathcal{A})$ for some n, then

$$p \sim_0 q \iff p \sim q \quad (\text{ Murry-von Neumann equivalence})$$

  since if $p \sim_0 q$, there is a $v \in M_n(\mathcal{A})$ such that $v^*v = p, vv^* = q$, *i.e.*, $p \sim q$.
- A projection $p \in \mathcal{P}_\infty(\mathcal{A})$ means p is a projection in $M_n(\mathcal{A})$, *i.e.*, $p \in \mathcal{P}_n(\mathcal{A})$ for some n.

Proposition 2.11 ($\sim_0$ under "$\oplus$")

Suppose elements p, q, r, p', q' are projections in $\mathcal{P}_\infty(\mathcal{A})$, where $\mathcal{A}$is a C^*-algebra, then

- $p \sim_0 p \oplus 0_n$, where 0_n is the zero matrix in $M_n(\mathcal{A})$.
- If $p \sim_0 p', q \sim_0 q'$, then we have $p \oplus q \sim_0 p' \oplus q'$.
- $p \oplus q \sim_0 q \oplus p$.
- $p, q \in \mathcal{P}_n(\mathcal{A})$ such that $pq = 0$, then we have $p + q \in \mathcal{P}_n(\mathcal{A})$ and $p + q \sim_0 p \oplus q$.

Proof.

- Let $p \in \mathcal{P}_m(\mathcal{A})$ and $0_n \in \mathcal{P}_n(\mathcal{A})$ then set $u = \begin{pmatrix} p \\ 0 \end{pmatrix} \in M_{m+n,m}(\mathcal{A})$, then

$$u^*u = \begin{pmatrix} p^* & 0 \end{pmatrix} \begin{pmatrix} p \\ 0 \end{pmatrix} = p^*p = p$$

  since $p \in \mathcal{P}_n(\mathcal{A})$ and

$$uu^* = \begin{pmatrix} p \\ 0 \end{pmatrix} \begin{pmatrix} p^* & 0 \end{pmatrix} = \begin{pmatrix} pp^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} = p \oplus 0_n.$$

  Hence $p \sim_0 p \oplus 0_n$.

- If $p \sim_0 p', q \sim_0 q'$, then there exists v, w such that

$$p = v^*v, \quad p' = vv^*, \quad q = w^*w, \quad q' = ww^*.$$

Set $u = \operatorname{diag}(v, w) = \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix}$, then we have

$$u^*u = \begin{pmatrix} v^*v & 0 \\ 0 & ww^* \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} = p \oplus q \quad , \quad uu^* = \begin{pmatrix} vv^* & 0 \\ 0 & ww^* \end{pmatrix} = \begin{pmatrix} p' & 0 \\ 0 & q' \end{pmatrix} = p' \oplus q'.$$

Hence we have $p \oplus q \sim_0 p' \oplus q'$.

- If $p \in \mathcal{P}_n(\mathcal{A}), q \in \mathcal{P}_m(\mathcal{A})$, then set $u = \begin{pmatrix} 0_{n,m} & q \\ p & 0_{m,n} \end{pmatrix} \in M_{m+n}(\mathcal{A})$, then we have

$$u^*u = \begin{pmatrix} 0_{m,n} & p^* \\ q^* & 0_{n,m} \end{pmatrix} \begin{pmatrix} 0_{n,m} & q \\ p & 0_{m,n} \end{pmatrix} = \begin{pmatrix} p^*p & 0 \\ 0 & q^*q \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} = p \oplus q$$

, and

$$uu^* = \begin{pmatrix} 0_{n,m} & q \\ p & 0_{m,n} \end{pmatrix} \begin{pmatrix} 0_{m,n} & p^* \\ q^* & 0_{n,m} \end{pmatrix} = \begin{pmatrix} qq^* & 0 \\ 0 & pp^* \end{pmatrix} = \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix} = q \oplus p.$$

Hence we get $p \oplus q \sim_0 q \oplus p$.

- We first consider a claim which will be used in our proof.

Claim 2.12

Projections $p, q \in \mathcal{P}(\mathcal{A})$ is said to be orthogonal and sign $p \perp q$ when $pq = 0$. Then we have the relation

$$p \perp q \iff p + q \in \mathcal{P}(\mathcal{A}) \iff p + q \leq 1.$$

So if $p, q \in \mathcal{P}_n(\mathcal{A})$ and $pq = 0$, *i.e.*, $p \perp q$. According to the claim, we have $p + q$ is also a projection, *i.e.*,$p + q \in \mathcal{P}_n\mathcal{A}$. Next we set $u = \begin{pmatrix} p \\ q \end{pmatrix} \in M_{2n,n}(\mathcal{A})$, we can get

$$u^*u = \begin{pmatrix} p^* & q^* \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = p^*p + q^*q = p + q \quad , \quad uu^* = \begin{pmatrix} p \\ q \end{pmatrix} \begin{pmatrix} p^* & q^* \end{pmatrix} = \begin{pmatrix} pp^* & pq^* \\ qp^* & qq^* \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} = p \oplus q.$$

Hence we get $p + q \sim_0 p \oplus q$.

∎

Claim 2.13

If $p \in \mathcal{P}(\mathcal{A})$, then $p \in \mathcal{A}^+$.

Proof. Since $p \in \mathcal{P}(\mathcal{A})$ is a projection, and the proposition of spectrum DNF , we have $sp(p)^2 = sp(p^2) = sp(p)$. Hecne for all elements $t \in sp(p) = sp(p)^2$, $t \geq 0$, *i.e.*, $p \geq 0$, *i.e.*, $p \in \mathcal{A}^+$. ∎

***Remark* 2.14** (The proof of the Claim 2.12)

- (1) $\Rightarrow$ (2): $p+q \in \mathcal{P}(\mathcal{A})$ since $(p+q)^2 = p^2+q^2+pq+qp = p^2+q^2 = p+q$, and $(p+q)^* = p^*+q^* = p+q$.
- (2) $\Rightarrow$ (3): $P + q \in \mathcal{P}(\mathcal{A})$, then by **Claim 2.13**, $p + q \in \mathcal{A}^+$, *i.e.*, $p + q \geq 0$. And $1 - (p + q) \in \mathcal{A}^+$ since $1 - (p + q)$ is adjoint and for all $t \in sp(1 - (p + q)) = 1 - sp(p + q)$, $t \geq 0$, (this is because $r(p + q) = \|p + q\| = 1$ by $p + q \in \mathcal{P}(\mathcal{A})$). Hence we have $1 - (p + q) \geq 0$, *i.e.*, $p + q \leq 1$.

- (3) $\Rightarrow$ (1): If $p+q \leq 1$. then $p(p+q)p = p^3 + pqp = p + pqp \leq p^2 = p$. So $pqp = p(qp) = 0$, we get $qp = 0$, *i.e.*, $p \perp q$.

Naturally we are condering about the equivalent class for the semigroup of projections under the equivalence $\sim_0$.

Definition 2.15 (Semigroup $\mathcal{D}(\mathcal{A})$)

- Consider the semigroup of projections $(\mathcal{P}_\infty(\mathcal{A}), \oplus)$ and the equivalence $\sim_0$ on it, then we can define
$$\mathcal{D}(\mathcal{A}) = \mathcal{P}_\infty(\mathcal{A}) / \sim_0,$$
and for each element $p \in \mathcal{P}_\infty(\mathcal{A})$, we denote the corresponded equivalet class by $[p]_{\mathcal{D}} \in \mathcal{D}(\mathcal{A})$.
- Then we define the addition "+" on $\mathcal{D}(\mathcal{A})$ by
$$[p]_{\mathcal{D}} + [q]_{\mathcal{D}} = [p \oplus q]_{\mathcal{D}} \qquad \text{where } p, q \in \mathcal{P}_\infty(\mathcal{A})$$

Remark **2.16**

- The addition "+" is well defined on $\mathcal{D}(\mathcal{A})$, since for any $p' \in [p]_{\mathcal{D}}$ and $q' \in [q]_{\mathcal{D}}$ (*i.e.*, $p' \sim_0 p, q' \sim_0 q$), by **Proposition 2.11** we have $p' \oplus q' \sim_0 p \oplus q$, *i.e.*, $[p']_{\mathcal{D}} + [q']_{\mathcal{D}} = [p]_{\mathcal{D}} + [q]_{\mathcal{D}}$.
- $(\mathcal{D}(\mathcal{A}), +)$ is a Abelian semigroup, since
$$[p]_{\mathcal{D}} + [q]_{\mathcal{D}} = [p \oplus q]_{\mathcal{D}} = [q \oplus p]_{\mathcal{D}} = [q]_{\mathcal{D}} + [p]_{\mathcal{D}},$$
the second equility is because $p \oplus q \sim_0 q \oplus p$ by the property of $\sim_0$ equivalence. Hence $(\mathcal{D}(\mathcal{A}), +)$ is an Abelian semigroup.

3 K_0-Group

This part is based on the construction of Grothendieck which is a way to translate an Abelian semigroup to an Abelian Group. And we wiil use his construction on the Abelian semigroup $\mathcal{D}(\mathcal{A})$ which we have already defined to get the Abelian group K_0 group.

Construction 3.1 (Grothendieck Construction)

The Grothendieck construction is a construction to translate an Abelian semigroup to Abelian group.

$$\text{Abelian Semigroup} \longrightarrow \text{Abelian Group}$$
$$S \longrightarrow G(S)$$

Consider an Abelian Semigroup$(S, +)$, we first define an equivalence "$\sim$" on $S \times S$ by

$$(x_1, y_1) \sim (x_2, y_2) \quad \text{if } x_1 + y_2 + z = x_2 + y_1 + z \text{ for some } z \in S$$

Remark **3.2**

The equivalence "$\sim$" is well defined on $S \times S$ since

- (Reflexive) $(x, y) \sim (y, x)$ since $x + y = x + y$.
- (Symmmetry) If $(x_1, y_1) \sim (x_2, y_2)$, we get $x_1 + y_2 + z = x_2 + y_1 + z$ for some $z \in S$. Then $x_2 + y_1 + z = x_1 + y_2 + z$, *i.e.*, $(x_2, y_2) \sim (x_1, y_1)$.
- (Translative) If $(x_1, y_1) \sim (x_2, y_2)$ and $(x_2, y_2) \sim (x_3, y_3)$, we can get
$$x_1 + y_2 + z_1 = x_2 + y_1 + z_1 \quad , \quad x_2 + y_3 + z_2 = x_3 + y_2 + z_2 \quad \text{for some } z_1, z_2 \in S$$

So we have $x_1 + y_2 + z_1 + y_3 + z_2 = x_2 + y_1 + z_1 + y_3 + z_2 = x_3 + y_2 + z_2 + y_1 + z_1$, *i.e.*,

$$x_1 + y_3 + (y_2 + z_1 + z_2) = x_3 + y_1 + (y_2 + z_2 + z_1).$$

where $y_2 + z_1 + z_2 \in S$. Hence we get $(x_1, y_1) \sim (x_3, y_3)$.

Definition 3.3 (Grouthendieck Group)

Define the $G(S)$ as the Grothendieck group of S by the quotient,

$$G(S) = (S \times S)/\sim.$$

Denote $\langle x, y\rangle$ to be the equivalence class in $G(S)$, *i.e.*, $\langle x, y\rangle = \{(p, q) \sim (x, y) : (p, q) \in S \times S\}$. And the group operation "+" on $G(S)$ is defined by

$$\langle x_1, y_1\rangle + \langle x_2, y_2\rangle = \langle x_1 + x_2, y_1 + y_2\rangle.$$

(Notice that $-\langle x, y\rangle = \langle y, x\rangle$ and $\langle x, x\rangle = 0$ for all $x \in G(S)$, this ensure the identity and inverse elemenets in $G(S)$).

Remark 3.4

- ("+" is well defined) If $(p_1, q_1) \in \langle x_1, y_1\rangle$ and $(p_2, q_2) \in \langle x_2, y_2\rangle$, then $p_1 + y_1 + z = x_1 + q_1 + z$, $p_2 + y_2 + w = x_2 + q_2 + w$ for some $z, w \in S$. So we have

$$(p_1 + p_2) + (y_1 + y_2) + (z + w) = (x_1 + x_2) + (q_1 + q_2) + (z + w) \qquad \text{where } z + w \in S.$$

  Hence $(p_1 + p_2, q_1 + q_2) \sim (x_1 + x_2, y_1 + y_2)$, *i.e.*, $\langle p_1 + p_2, q_1 + q_2\rangle = \langle x_1 + x_2, y_1 + y_2\rangle$. The "+" is well defined on $G(S)$.
- ($(G(S), +)$ is an Abelian group) We have $\langle x_1, y_1\rangle + \langle x_2, y_2\rangle = \langle x_1 + x_2, y_1 + y_2\rangle = \langle x_2, y_2\rangle + \langle x_1, y_1\rangle$, since S is Abelian.

Remark 3.5

Becareful, when we consider about such formula

$$\langle x_1, y_1\rangle - \langle x_2, y_2\rangle \quad \text{in} \quad G(S),$$

it means $\langle x_1, y_1\rangle - \langle x_2, y_2\rangle = \langle x_1, y_1\rangle + (-\langle x_2, y_2\rangle) = \langle x_1, y_1\rangle + \langle y_2, x_2\rangle = \langle x_1 + y_2, x_2 + y_1\rangle$. And since S is a semigroup, an element $x \in S$ may not have the inverse elment $-x$.

Next there is a map from Abelian semigroup S to its Grothendieck group $G(S)$ like an embedding map, which is named Grothendieck map.

Definition 3.6

The Grothendieck map is defined by

$$\begin{aligned} \gamma_S : S &\longrightarrow G(S) \\ x &\longmapsto \langle x + y, y\rangle \qquad \text{for any } y \in S \end{aligned}$$

Remark 3.7

- (The Grothendieck map is well defined) For any $x \in S$ and $y_1, y_2 \in S, y_1 \neq y_2$, consider $\langle x + y_1$ and $\langle x + y_2, y_2\rangle$. Then we get $\langle x + y_1, y_2\rangle \sim \langle x + y_2, y_2\rangle$, since $x + y_1 + y_2 = x + y_2 + y_1$, *i.e.*, $\gamma_S(x)$ is independent of the selection of y in S. The Grothendieck map γ_S is well defined.

- (The Grothendieck map is additive) Consider $\gamma_S(x_1 + x_2) = \langle x_1 + x_2 + y, y\rangle$ for any $y \in S$, and $\gamma_S(x_1) + \gamma_S(x_2) = \langle x_1 + y/2, y/2\rangle + \langle x_2 + y/2, y/2\rangle = \langle x_1 + x_2 + y, y\rangle = \gamma_S(x_1 + x_2)$. Hence γ_S is an additive map.

Definition 3.8

A semigroup $(S, +)$ has the cancellation property if for $x, y, z \in S$ such that $x + z = y + z$, we can get $x = y$.

In Abelian group the cancellation property is also established, and we will see later that when a Abelian semigroup is an "operation closed" subset of a Abelian group, it also has the cancellation property.

Proposition 3.9

1) (Universal property) If $\phi : S \to H$ is an additive map and S is an Abelian semigroup, H is an Abelian group. Then there is an unique group homomorphism $\psi : G(S) \to H$ such that the following diagram commute,

$$\begin{tikzcd} S \arrow[rr, "\varphi"] \arrow[dr, "\gamma_S"'] & & H \\ & G(S) \arrow[ur, dashed, "\psi"'] & \end{tikzcd}$$

2) (Functoriality) Given an additive map $\varphi : S \to T$, where S and T are both Abelian semigroup. Then there is an unique group homomorphism $G(\varphi)$ induced by φ such that the diagram commute,

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, "\gamma_S"'] & T \arrow[d, "\gamma_T"] \\ G(S) \arrow[r, dashed, "G(\varphi)"] & G(T) \end{tikzcd}$$

3) The Grothendieck group $G(S)$can be exactly presented by using Grothendieck map, $G(S) = \{\gamma_S(x) - \gamma_S(y) : x, y \in S\}$,*i.e.*,

$$\langle x, y\rangle = \gamma_S(x) - \gamma_S(y)$$

4) For any $x, y \in S$, then

$$\gamma_S(x) = \gamma_S(y) \iff x + z = y + z \quad \text{for some } z \in S.$$

5) The Grothendieck map γ_S is injective if and only if the Abelian semigroup has the cancellation property.
6) If $(H, +)$ is an Abelian group, $S \subseteq H$ is a non-empty subset of H and closed under "+". Then $(S, +)$ is an Abelian semigroup with cancellation property. And there is an isomorphism

$$G(S) \simeq H_0 = \{x - y : x, y \in S\}$$

(Where H_0 is the semigroup of H generated by subset S).

Proof. We first consider the basic conclusion 3),

- (Proof of 3)) For any $\langle x, y\rangle \in G(S)$, we have

$$\langle x, y\rangle = \langle x + y, y\rangle - \langle x + y, x\rangle = \gamma_S(x) - \gamma_S(y).$$

(where $\langle x,y\rangle = \langle x+y,y\rangle - \langle x+y,x\rangle$ is because $\langle x+y,y\rangle - \langle x+y,x\rangle = \langle x+y,y\rangle + \langle x,x+y\rangle = \langle (x+y)+x,(x+y)+y\rangle = \langle x+y,x+y\rangle + \langle x,y\rangle = 0 + \langle x,y\rangle = \langle x,y\rangle$)

- (Proof of 4))
  - ($\Leftarrow$): If $x+z = y+z$ for some $z \in S$, then we have $x+y+(x+z) = x+y+(y+z)$,*i.e.*, $(x+y)+x+z = (x+y)+y+z$, *i.e.*, $(x+y,y) \sim (x+y,x)$ in $S \times S$, *i.e.*, $\langle x+y,y\rangle = \langle x+y,x\rangle$. Hence $\gamma_S(x) = \gamma_S(y)$.
  - ($\Rightarrow$): If $\gamma_S(x) = \gamma_S(y)$,*i.e.*, $\langle x+y,y\rangle = \langle x+y,x\rangle$, so we have $(x+y,y) \sim (x+y,x)$, *i.e.*, $x+y+x+w = x+y+y+w$ for some $w \in S$. Then we set $z = x+y+w$, we get $x+z = y+z$.
- (Proof of 5))
  - ($\Rightarrow$): Suppose the Grothendieck map $\gamma_S : S \to G(S)$ is injective, then if $x+z = y+z$ in S, by 4) we have $\gamma_S(x) = \gamma_S(y) \Rightarrow x = y$ (since γ_S is injective).
  - ($\Leftarrow$): If S has the cancellation property, then by 4) and cancellation property, we have

$$\gamma_S(x) = \gamma_S(y) \Rightarrow x+z = y+z, \text{ for some } z \in S \Rightarrow x = y,$$

    which means the Grothendieck map γ_S is injective.
- (Proof of 1))
  - According to 3), every element in $G(S)$ has the form $\gamma_s(x) - \gamma_s(y)$. Then since γ_s is additive, $\gamma_s(x) - \gamma_s(y) = \gamma_s(x-y)$, so $\varphi(x-y) = \psi(\gamma_s(x-y))$. However, since φ is additive,

$$\varphi(x-y) = \varphi(x) - \varphi(y) = \psi(\gamma_s(x)) - \psi(\gamma_s(y)) = \psi(\gamma_s(x) - \gamma_s(y)),$$

    so ψ is additive. Then for any elements $\gamma_s(x) - \gamma_s(y)$ in $G(S)$, we have

$$\psi(\gamma_s(x) - \gamma_s(y)) = \psi(\gamma_s(x)) - \psi(\gamma_s(y)) = \varphi(x) - \varphi(y).$$

    (*i.e.*, $\psi(\langle x,y\rangle) = \varphi(x) - \varphi(y)$). Hence ψ is unique.
  - $\psi(\langle x,y\rangle) = \varphi(x) - \varphi(y)$ is well-defined, since if $\langle x_1,y_1\rangle = \langle x_2,y_2\rangle$, then $x_1+y_2+z = x_2+y_2+z$ for some $z \in S$. Then we have

$$\varphi(x_1+y_2+z) = \varphi(x_1)+\varphi(y_2)+\varphi(z) = \varphi(x_2+y_1+z) = \varphi(x_2)+\varphi(y_1)+\varphi(z)$$

    in H. Since H is an Abelian group, we can use the cancellation, so $\varphi(x_1) - \varphi(y_1) = \varphi(x_2) - \varphi(y_2)$, *i.e.*, $\psi(\langle x,y\rangle) = \psi(\langle x_2,y_2\rangle)$, ψ is well-defined.
- (Proof of 2)) According to 1), we can regard the diagram as
-

$$\begin{tikzcd} S \arrow[r, "\gamma_T \circ S"] \arrow[d, "\gamma_s"'] & G(T) \\ G(S) \arrow[ur, dashed, "G(\varphi)"'] & \end{tikzcd}$$

  where $\gamma_T \circ S$ is an additive map , (since γ_T, S are both additive). $G(T)$ is an Abelian group, so we can apply 1), and we get a unique group homomorphism $G(\varphi)$.
- A non-empty subset S of Abelian group H which is closed under+ is obviously an Abelian semigroup with cancellation property (since if $x+z = y+z$, we deduce that $x = y$ in $S \subset H$). Then

consider the diagram

$$\begin{tikzcd} S \arrow[r, "\tau"] \arrow[d, "\gamma_s"'] & H \\ G(S) \arrow[ur, dashed, "\psi"'] & \end{tikzcd}$$

where H is an Abelian group, τ is the induced map (automatically additive). Hence, according to 1), we have a unique group homomorphism $\psi : G(S) \to H$ such that the diagram commute, *i.e.*, $\psi(r_s(x)) = \tau(x) = x \in H$ for all $x \in S$. And the image of map $\psi : G(S) \to H$ is

$$\operatorname{Im}\psi - \{\psi(r_s(x) - r_s(y)) = \psi(\gamma_s(x)) - \psi(\gamma_s(y)) = x - y | x, y \in S\},$$

by 3). Thus, $\operatorname{Im}\psi = H_0$ and if $\psi(\gamma_s(x) - \gamma_s(y)) = 0$, we deduce that $x = y$, *i.e.*, $\langle x, y\rangle = 0$. Hence ψ is injective, $\psi : G(S) \to H_0 = \{x - y | x, y \in S\}$ is an isomorphism.

■

Example 3.10

The Grothendieck group of $(\mathbb{Z}^+, +)$ is $(\mathbb{Z}, +)$, *i.e.*, $G(\mathbb{Z}^+) = \mathbb{Z}$.
Since $(\mathbb{Z}^+, +)$ is an Abelian semigroup and $(\mathbb{Z}, +)$ is an Abelian group, so $G(\mathbb{Z}^+) \simeq H_0 = \{x-y | x, y \in \mathbb{Z}^+\}$. And $H_0 = \mathbb{Z}$ since $H_0 \subset \mathbb{Z}$ and for any $z \in \mathbb{Z}$, if $z > 0$, then $z = z \in \mathbb{Z}^+$ and if $z < 0$, then $z = 0 - z \in H_0$.

Now we can given the definition about the K_0 group by using Grothendieck's construction.

Definition 3.11 (The K_0-group for a unital C^*-algebra)

Let $\mathcal{A}$ be a C^*-algebra, and let $(\mathcal{D}(\mathcal{A}), +)$ be the semigroup by $\mathcal{D}(\mathcal{A}) = \mathcal{P}_\infty(\mathcal{A})/\sim_0$. Then $K_0(\mathcal{A})$ is defined to be the Grothendieck group of $\mathcal{D}(\mathcal{A})$, *i.e.*,

$$K_0(\mathcal{A}) = G(\mathcal{D}(\mathcal{A})).$$

Remark

We have the map $[\cdot]_0 : \mathcal{P}_\infty(\mathcal{A}) \to K_0(\mathcal{A})$ and in particular,

$$\begin{array}{rccccc} [\cdot]_0 : & \mathcal{P}_\infty(\mathcal{A}) & \longrightarrow & \mathcal{D}(\mathcal{A}) & \longrightarrow & K_0(\mathcal{A}) \\ & p & \longmapsto & [p]_{\mathcal{D}} & \longmapsto & \gamma([p]_{\mathcal{D}}) \end{array}$$

by $[p]_0 = \gamma([p]_{\mathcal{D}})$ where γ is the Grothendieck map.

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