Homological Algebra

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Abstract

This report consists of four parts. The first part concludes with some definitions and useful properties in category theory, especially abelian categories and additive functors. The second part is about the projective modules and injective modules, and also proves the existence of projective resolution and injective resolution for any module which are used in the constructions of derived functors. In the third part, we first introduce the derived functors in abelian categories, and then we discuss the Tor and Ext as an example of derived functors. The last part is the application of homology, it shows the relation between the homology of the Koszul complex and the depth of an ideal I in the R-module.

Contents

1 Category Theory

Definition 1.1 (Category)

A category $\mathcal{C}$ consist of three parts,

- A class of objects, denoted by $\mathrm{Obj}(\mathcal{C})$
- Sets of morphisms, denoted by $\mathrm{Hom}(A, B)$, consist of the morphisms $f : A \to B$ where $A, B \in \mathrm{Obj}(\mathcal{C})$.
- A composition rule for morphisms, for any morphisms $f \in \mathrm{Hom}(A, B)$ and $g \in \mathrm{Hom}(B, C)$, there is a new morphism $g \circ f \in \mathrm{Hom}(A, C)$,

and satisfied the following condition

- For each object A, there is an identity morphism $1_A \in \mathrm{Hom}(A, A)$ such that $f 1_A = f$ and $1_B f = f$ for all $f : A \to B$.
- For any triple morphisms $f : A \to B$, $g : B \to C$, $h : C \to D$, we have

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Definition 1.2 (Isomorphism)

$\mathcal{A}$ is a category, and $f : A \to B$ is a morphism in $\mathcal{A}$, then A is isomorphic to B deneted by $A \simeq B$ if there is an inverse morphism $f^{-1} : B \to A$ such that

$$f \circ f^{-1} = 1_B \quad \text{and} \quad f^{-1} \circ f = 1_A$$

Next, we will show the definitions of functors between categories. The functors can be classified into two types which are covariant functors and contravariant functors. The difference between these two types is that the covariant functor preserves the direction of the morphisms, but the contravariant functor changes the direction of morphisms in the opposite.

Definition 1.3 (Covariant Functor)

$\mathcal{A}$ and $\mathcal{C}$ are categories, a functor $T : \mathcal{A} \to \mathcal{C}$ is consist of two parts,

- Mapping objects, $T : A \mapsto T(A)$, where $A \in \mathrm{Obj}(\mathcal{A})$ and $T(A) \in \mathcal{C}$.
- Mapping morphisms, for any morphism $f \in \mathrm{Hom}(A, B)$ where $A, B \in \mathrm{Obj}(\mathcal{A})$, then $T(f) \in \mathrm{Hom}(T(A), T(B))$ in $\mathcal{C}$.

and such that

- For each $A \in \mathrm{Obj}(\mathcal{A})$, $T(1_A) = 1_{T(A)}$.
- For every pair of morphisms $f : A \to B$ and $g : B \to C$ in category $\mathcal{A}$,

$$T(g \circ f) = T(g) \circ T(f) \in \mathrm{Hom}(T(A), T(C)).$$

Definition 1.4 (Contravariant Functor)

$\mathcal{A}$ and $\mathcal{C}$ are categories, a functor $T : \mathcal{A} \to \mathcal{C}$ is consist of two parts,

- Mapping objects, $T : A \mapsto T(A)$, where $A \in \mathrm{Obj}(\mathcal{A})$ and $T(A) \in \mathcal{C}$.
- Mapping morphisms, for any morphism $f \in \mathrm{Hom}(A, B)$ where $A, B \in \mathrm{Obj}(\mathcal{A})$, then $T(f) \in \mathrm{Hom}(T(B), T(A))$ in $\mathcal{C}$.

and such that

- For each $A \in \mathrm{Obj}(\mathcal{A})$, $T(1_A) = 1_{T(A)}$.
- For every pair of morphisms $f : A \to B$ and $g : B \to C$ in category $\mathcal{A}$,

$$T(g \circ f) = T(f) \circ T(g) \in \mathrm{Hom}(T(A), T(C)).$$

Example 1.5

- $\square \otimes_R B$, $A \otimes_R \square$ and $\mathrm{Hom}_R(A, \square)$ are covariant functors.
- $\mathrm{Hom}(\square, B)$ is a controvariant functor.

Definition 1.6 (Natural translation)

$T, T' : \mathcal{A} \to \mathcal{C}$ are covariant functors, a natural transformation $\tau : T \to T'$ is family of morphisms in $\mathcal{C}$ by

$$\tau = (\tau_A : TA \to T'A)_{A \in \mathrm{Obj}(\mathcal{A})}$$

such that the following diagram commute for all morphisms $f : A \to A'$ in $\mathcal{A}$

$$\begin{tikzcd} TA \arrow[r, "\tau_A"] \arrow[d, "Tf"'] & T'A \arrow[d, "T'f"] \\ TA' \arrow[r, "\tau_{A'}"'] & T'A' \end{tikzcd}$$

Remark 1.7

The definition of the natural transformation between contravariant functors are just change the direction of Tf and $T'f$ in opposite.

Definition 1.8 (Natural isomorphism)

$T, T' : \mathcal{A} \to \mathcal{C}$ are covariant functors, we call T is natural isomorphic to T' denoted by $T \cong T'$ if there is a natural translation $\tau : T \to T'$ and each τ_A is an isomorphism.

Remark 1.9

It is easy to verify that natural isomorphism is an equivalence.

To give the definition of Abelian Category, we need to define the additive category first.

Definition 1.10 (Additive Category)

$\mathcal{C}$ is an additive category, if

- For any $A, B \in \mathrm{Obj}(\mathcal{C})$, $\mathrm{Hom}(A, B)$ is an additive Abelian group.
- The distribution law is established, *i.e.*, for any morphisms $u \in \mathrm{Hom}(A, B)$, $f, g \in \mathrm{Hom}(B, C)$ and $v \in \mathrm{Hom}(C, D)$, we have

$$v \circ (f + g) = v \circ f + v \circ g \quad \text{and} \quad (f + g) \circ u = f \circ u + g \circ u$$

- $\mathcal{C}$ has a zero object. (Zero object is an object both initial and terminal).
- For any objects $A, B \in \mathrm{Obj}(\mathcal{C})$, both $A \sqcap B$ and $A \sqcup B$ are exist in $\mathrm{Obj}(\mathcal{C})$.

Remark 1.11

Since the last item of the definition is not used in the following sections, I do not give the exact definition about $A \sqcup B$ and $A \sqcap B$, and it has been showed in Rotman's book in Page 226.

Definition 1.12 (Additive Functor)

Given $\mathcal{A}$ and $\mathcal{C}$ are additive categories, a functor $T : \mathcal{A} \to \mathcal{C}$ is an additive functor if for any $A, B \in \mathrm{Obj}(\mathcal{A})$ and any morphisms $f, g \in \mathrm{Hom}(A, B)$, we have

$$T(f + g) = Tf + Tg$$

,*i.e.*, the functor $T : \mathrm{Hom}(A, B) \to \mathrm{Hom}(TA, TB)$ is a homomorphism of additive abelian groups.

Definition 1.13 (Abelian Category)

A category $\mathcal{C}$ is called an abelian category, if $\mathcal{C}$ is an additive category such that

- Every morphism has a kernel and a cokernel.
- Every monomorphism is a kernel and every epimorphism is a cokernel.

Remark 1.14

The kernel and cokernel in this definition is defned by category's language(see Rotman's book Page 305). However, by the Mitchell Embedding theorem (Page 316 Rotman's book), the kernel and cokernel of abelian category can be seen as the definition in abelian group.

Definition 1.15 (Left Exact Functor)

Given two abelian categories $\mathcal{A}$ and $\mathcal{C}$, an covariant additive functor $T : \mathcal{A} \to \mathcal{C}$ is called a left exact functor if for any $0 \to A' \to A \to A''$ is exact in $\mathcal{A}$ implies $0 \to TA' \to TA \to TA''$ is exact in $\mathcal{C}$.

Definition 1.16 (Right Exact Functor)

Given two abelian categories $\mathcal{A}$ and $\mathcal{C}$, an covariant additive functor $T : \mathcal{A} \to \mathcal{C}$ is called a left exact functor if for any $A' \to A \to A'' \to 0$ is exact in $\mathcal{A}$ implies $TA' \to TA \to TA'' \to 0$ is exact in $\mathcal{C}$.

Remark 1.17

If additive functor $T : \mathcal{A} \to \mathcal{C}$ is contravariant, then we define

- T is left exact if for any $A' \to A \to A'' \to 0$ is exact in $\mathcal{A}$ implies $0 \to TA'' \to TA \to TA'$ is exact in $\mathcal{C}$.
- T is right exact if for any $0 \to A' \to A \to A''$ is exact in $\mathcal{A}$ implies $TA'' \to TA \to TA' \to 0$ is exact in $\mathcal{C}$.

2 Modules

In this section, some conclusions are given without proofs and these conclusions are useful in the following sections especially in the constructions of derived functors.

2.1 Free Modules

Definition 2.1 (Free Module)

A R-module F is a free module if

$$F = \bigoplus_{b \in B} R_b \quad \text{where } R_b = \langle b \rangle \simeq R$$

(*i.e.*, F is isomorphic to the direct sum of copies of R).

Definition 2.2 (Free resolution)

A free resolution of an R-module A is an exact sequence

$$F : \cdots \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\delta} A \to 0$$

and the deleted free resolution of A is

$$F_A : \cdots \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to 0$$

Remark 2.3

The deleted free resolution may not be exact.

Claim 2.4

Every R-module is a quotient of a free R-module. (Rotman's book Page 58 Theorem 2.35[3]).

Theorem 2.5

Every R-module A has a free resolution F.

Proof. By **Claim 2.4**, A is a quotient of a free module, so there is an exact sequence

$$0 \to K_1 \xrightarrow{i_1} F_0 \xrightarrow{\pi} A \to 0 \quad \text{where } K_1 = \operatorname{Ker}\pi \text{ and } \pi \text{ is a quotient map.}$$

Then for K_1, there is also an exact sequence with free module F_1 according to **Claim 2.4**

$$0 \to K_2 \xrightarrow{i_2} F_1 \xrightarrow{\pi_1} K_1 \to 0$$

hence we can get,

$$\begin{tikzcd}
 & F_2 \arrow[rr, dashed, "d_2"] \arrow[dr, "\pi_2"'] & & F_1 \arrow[rr, dashed, "d_1"] \arrow[dr, "\pi_1"'] & & F_0 \arrow[r, "\pi"] & A \\
\cdots \arrow[ur, "i_3"'] & & K_2 \arrow[ur, "i_2"'] & & K_1 \arrow[ur, "i_1"'] & &
\end{tikzcd}$$

where $d_n = i_n \pi_{n+1}$. And the sequence $F = \cdots \xrightarrow{i_3} F_2 \xrightarrow{i_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\pi} A \to 0$ is an exact sequence since

$$\begin{aligned}
\operatorname{Im} d_{n+1} &= \operatorname{Im}(i_{n+1} \circ \pi_{n+1}) = \operatorname{Im} i_{n+1} \quad &&(\text{since } \pi_{n+1} \text{ is surjective}) \\
\operatorname{Ker} d_n &= \operatorname{Ker}(i_n \circ \pi_n) = \operatorname{Ker} \pi_n \quad &&(\text{since } i_n \text{ is injective})
\end{aligned}$$

and by the exactness of the sequence,

$$0 \to K_{n+1} \xrightarrow{i_{n+1}} F_n \xrightarrow{\pi_n} K_n \to 0$$

we get $\operatorname{Im} d_{n+1} = \operatorname{Im} i_{n+1} = \operatorname{Ker} \pi_n = \operatorname{Ker} d_n$. ■

2.2 Projective Module

Definition 2.6 (Projective module)

An R-module P is projective if for every surjective R-module homomorphism $f : N \to M$ and every R-module homomorphism $g : P \to M$,

$$\begin{tikzcd}
 & N \arrow[d, "f"] \\
P \arrow[ur, dotted, "h"] \arrow[r, "g"'] & M
\end{tikzcd}$$

there is a module homomorphism $h : P \to N$ such that $fh = g$.

And we can define the projective objects in abelian category by the same way. (See Rotman's book.[3]).

Definition 2.7 (Projective resolution)

$\mathcal{A}$ is an abelian category. A projective resolution of $A \in \operatorname{Obj}(\mathcal{A})$ is an exact sequence

$$P = \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \to 0$$

where P_n is projective for all n.

Remark 2.8

When the abelian category is the R-module category, a projective resolution of an R-module A is an exact sequence

$$P = \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \to 0$$

where P_n is a projective R-module for all n.

Claim 2.9

Every free R-module is also a projective R-module. (According to the Rotman's book Page 99, Theorem 3.1).

Proposition 2.10

Every R-module A has a projective resolution P.

Proof. By **Theorem 2.5**, there is a free resolution F of A, and by **Claim 2.9**, every free module is also a projective module. So F is also a projective resolution of A. ■

Corollary 2.11

If $\mathcal{A}$ is an abelian group with enough projectives, then for every $A \in \mathrm{Obj}(\mathcal{A})$, there exists a projective resolution of A. (Rotman's book Page 326 Corollary 6.3)

2.3 Injective Modules

Definition 2.12 (Injective module)

An R-module E is projective if for every injective module homomorphism $f : A \to B$ and every module homomorphism $f : A \to E$,

$$\begin{tikzcd} E & \\ A \arrow[u, "f"] \arrow[r, "i"'] & B \arrow[ul, dotted, "g"'] \end{tikzcd}$$

there is a module homomorphism $g : B \to E$ such that $gi = f$.

Definition 2.13 (Injective resolution)

$\mathcal{A}$ is an abelian group. An injective resolution of $A \in \mathrm{Obj}(\mathcal{A})$ is an exact sequence

$$E = 0 \to A \xrightarrow{\eta} E^0 \xrightarrow{d^0} R^1 \xrightarrow{d^1} E^2 \to \cdots$$

where each E^n is injective.

Claim 2.14

R is a ring, then every R-module M can be imbedded as a submodule of an injective R-module.(The details of proof can be seen in Rotman's book Page 123 Theorem 3.38)

Proposition 2.15

Every R-module A has an injective resolution E.

Proof. By **Claim 2.14**, there is an injective module E_0 and an injection $\eta : A \to E^0$, so there is an exact sequence

$$0 \to A \xrightarrow{\eta} E_0 \xrightarrow{\pi} V_0 \to 0$$

where π is a quotient map and $V_0 = \operatorname{Coker} \eta = E^0/\eta(A)$, and Π is the natural quotient map. Next according to **Claim 2.14** again, there is an exact sequence,

$$0 \to V^0 \xrightarrow{\eta^1} E^1 \xrightarrow{\pi^1} V_1 \to 0$$

where E^1 is injective. Hence we get

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\eta"] & E^0 \arrow[rr, dashed, "d^0"] \arrow[dr, "\pi"'] & & E^1 \arrow[rr, dashed, "d^1"] \arrow[dr, "\pi^1"'] & & E^2 \arrow[r, dashed] & {} \\
& & & V_0 \arrow[ur, "\eta^1"'] & & V_1 \arrow[ur, "\eta^2"'] & &
\end{tikzcd}$$

where $d_n = \eta^{n+1}\pi^n$, and

$$\operatorname{Im} d^n = \operatorname{Im}(\eta^{n+1} \circ \pi^n) = \operatorname{Im} \eta^{n+1} \quad \text{(since } \eta^{n+1} \text{ is injective and } \pi^n \text{ is surjective).}$$
$$\operatorname{Ker} d^{n+1} = \operatorname{Ker}(\eta^{n+2} \circ \pi^{n+1}) = \operatorname{Ker} \pi^{n+1} \quad \text{(since } \eta^{n+2} \text{ is injective).}$$

Hence by the exactness of

$$0 \to V_n \xrightarrow{\eta^{n+1}} E^n \xrightarrow{\pi^{n+1}} V_{n+1} \to 0,$$

we get $\operatorname{Im} d^n - \operatorname{Im} \eta^{n+1} = \operatorname{Ker} \pi^{n+1} = \operatorname{Ker} d^{n+1}$, which means $P = 0 \to A \xrightarrow{\eta} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \to \cdots$ is an injective resolution of A. ∎

Remark **2.16**

If $\mathcal{A}$ is an abelian category with enough injectives, this conclusion is also established on $\mathcal{A}$. (Rotman's book Page 326 Corollary 6.3[3])

3 Homology

3.1 Complexes and Homology

Definition 3.1 (Complex)

A complex in Abelian Category $\mathcal{A}$ is a sequence

$$C: \quad \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$

such that $d_n d_{n+1} = 0$ for all $n \in \mathbb{Z}$. Where we call the morphisms d_i differentials.

Definition 3.2 (Chain Map)

The chain map between complexes in Abelian category is a series of maps $f = (f_n) : C \to C'$ such that the diagram commute

$$\begin{tikzcd}
\arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & {} \\
\arrow[r] & C'_{n+1} \arrow[r, "d'_{n+1}"'] & C'_n \arrow[r, "d'_n"'] & C'_{n-1} \arrow[r] & {}
\end{tikzcd}$$

Remark 3.3

- We denote $Z_n(C) = \operatorname{Ker} d_n$ and $B_n(C) = \operatorname{Im} d_{n+1}$.
- In ${}_R\mathbf{Mod}$, the $d_n d_{n+1} = 0$ means $\operatorname{Im} d_{n+1} \subseteq \operatorname{Ker} d_n$.
- We can define a complex category $\mathbf{Comp}(\mathcal{A})$ consist of complexes on Abelian category and with morphisms defined by chain maps.

Definition 3.4 (Complexes isomorphism)

An isomorphism in category $\mathbf{Comp}(\mathcal{A})$ is a chain map $f = (f_n) : C \to C'$ which exists an inverse chain map $f^{-1} = (f_n^{-1}) : C' \to C$ such that $f^{-1}f = 1_C$ and $ff^{-1} = 1_{C'}$. We denote $C \simeq C'$ if C is isomorphic to C'.

Remark 3.5

Complexes $C \simeq C'$ if and only if there is a chain map $f = (f_n) : C \to C'$ such that each $f_n : C_n \to C'_n$ is an isomorphism.

Proof. We just need to verify that the sequence of inverse(f_n^{-1}) is a chian map. Consider

$$\begin{tikzcd}
\arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, shift right, "f_{n+1}"'] & C_n \arrow[r, "d_n"] \arrow[d, shift right, "f_n"'] & C_{n-1} \arrow[r] \arrow[d, shift right, "f_{n-1}"'] & {} \\
\arrow[r] & C'_{n+1} \arrow[r, "d_{n+1}"] \arrow[u, dotted, shift right, "f_{n+1}^{-1}"'] & C'_n \arrow[r, "d_n"] \arrow[u, dotted, shift right, "f_n^{-1}"'] & C'_{n-1} \arrow[r] \arrow[u, dotted, shift right, "f_{n-1}^{-1}"'] & {}
\end{tikzcd}$$

since $f = (f_n)$ is a chain map, $d_n f_n^{-1} = f_{n-1}^{-1} d'_n$. However each f_n is an isomorphism, then

$$d_n f_n^{-1} = f_{n-1}^{-1} f_{n-1} d_n f_n^{-1} = f_{n-1}^{-1} d'_n f_n f_n^{-1} = f_{n-1}^{-1} d'_n$$

, so $f^{-1} = (f_n^{-1})$ is a chian map. ■

Example 3.6 (Exact sequence)

Every exact sequence is also a complex, since $\operatorname{Im} d_{n+1} = \operatorname{Ker} d_n$ satisfied $d_{n+1} d_n = 0$. But a complex may not be an exact sequence.

Example 3.7 (The Singular Complex in Topology space)

In a topology space, $S_k(X)$ is the abelian group of singular complexes and differential ∂_k is the boundary map. (see Hatcher's book [2])

$$\cdots \to S_{n+1}(X) \xrightarrow{\partial_{n+1}} S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots$$

Proposition 3.8 (Additive functor preserve complexes)

Let $F : \mathcal{A} \to \mathcal{A}'$ be an additive functor where $\mathcal{A}$ and $\mathcal{A}'$ are Abelian categories. And let a complex

$$C = \cdots \to C_n \xrightarrow{d_n} C_{n-1} \to \cdots$$

in $\mathbf{Comp}(\mathcal{A})$, then $(FC, Fd) = \cdots \to F(C_n) \xrightarrow{Fd_n} F(C_{n-1}) \to \cdots$ is also a complex in $\mathbf{Comp}(\mathcal{A}')$.

Proof. Since F is an additive functor, $F(d_{n+1})F(d_n) = F(d_{n+1}d_n) = F(0) = 0$ for all n. Hence (FC, Fd) is a complex in $\mathbf{Comp}(\mathcal{A})$. ■

Definition 3.9 (Homology)

C is a complex in complex category $\mathbf{Comp}(\mathcal{A})$, and $\mathcal{A}$ is an Abelian category, then for each $n \in \mathbb{Z}$, we have the n-th homology defined by

$$H_n(C) = Z_n(C)/B_n(C) = \operatorname{Ker} d_n / \operatorname{Im} d_{n+1}.$$

Remark 3.10

For each $n \in \mathbb{Z}$, the n-th Homology $H_n(\mathcal{A})$ is an Abelian group.

Remark 3.11

A complex is an exact sequence if and only if $H_n(C) = 0$ for all n, since for each n we have

$$H_n(C) = 0 \iff Z_n(C)/B_n(C) = \operatorname{Ker} d_n / \operatorname{Im} d_{n+1} = 0 \iff \operatorname{Ker} d_n = \operatorname{Im} d_{n+1} = 0$$

(the second equivalence is because when C is a complex, $Z_n(C) = \operatorname{Ker} d_n = 0$ if and only if $B_n(C) = \operatorname{Im} d_{n+1} = 0$).

Definition 3.12 (Induced map)

For a chain map $f : C \to C'$, we have the induced map $H_n(f) : H_n(C) \to H_n(C')$ for each $n \in \mathbb{Z}$, and it is also denoted by f_* instead of $H_n(f)$.
More precisely, we defined the induced map f_* by,

$$\begin{aligned} f_* = H_n(f) : H_n(C) &\to H_n(C') \\ [z_n] &\mapsto [f_n(z_n)] \end{aligned}$$

Remark 3.13

The induced map is well-defined, since

- For any $[z_n] \in H_n(C)$, *i.e.*, $z_n \in Z_n(C) = \operatorname{Ker} d_n$. Then by the chain map, we have

$$d'_n(f_n z_n) = f_{n-1}(d_n z_n) = 0 \qquad , i.e., \quad f_n(z_n) \in \operatorname{Ker} d'_n = Z_n.$$

  So $[f_n(z_n)]$ makes sense in $H_n(C')$.
- If $[c_1] = [c_2]$ in $H_n(C)$, then $c_1 - c_2 \in B_n(C) = \operatorname{Im} d_{n+1}$. So there is an $u \in C_{n+1}$ such that $d_{n+1}(u) = c_1 - c_2$. Then by chain map,

$$f_n(c_1) - f_n(c_2) = f_n(c_1 - c_2) = f_n(d_{n+1}u) = d'_{n+1}(f_{n+1}u),$$

  we get $f_n(c_1) - f_n(c_2) \in \operatorname{Im} d'_{n+1} = B_n(C')$, *i.e.*, $[f_n(c_1)] = [f_n(c_2)]$ in $H_n(C')$.

Proposition 3.14

If a chain map $f : C \to C'$ is a complex isomorphism, then its induced maps $H_n(f)$ for all n are also isomorphisms. (*i.e.*, $C \simeq C' \Rightarrow H_n(C) \simeq H_n(C')$ for all n).

This proposition is easy to verify.

Proposition 3.15 (Additive functor)

The n-th homology functor $H_n : \mathbf{Comp}(\mathcal{A}) \to \mathcal{A}$ is an additive functor for each $n \in \mathbb{Z}$.

Proof. For any two chain maps $f, g : C \to C'$ in category $\mathbf{Comp}(\mathcal{A})$, pick $z \in Z_(C)$, $[z] \in H_n(C)$, then

$$H_n(f+g)[z] = [(f+g)z] = [f(z) + g(z)] = [f(z)] + [g(z)] = H_n(f)[z] + H_n(g)[z].$$

Hence H_n is an additive functor. ■

There is a dual concept of complex and homology.

Definition 3.16 (Cochain Complex)

A cochain complex in Abelian Category $\mathcal{A}$ is a sequence

$$C: \quad \longleftarrow C^{n+1} \xleftarrow{d^n} C^n \xleftarrow{d^{n-1}} C^{n-1} \longleftarrow \cdots$$

such that $d_n d_{n-1} = 0$ for all $n \in \mathbb{Z}$. Where we call the morphisms d_i differentials.

Definition 3.17 (Cohomology)

C is a cochain complex in Abelian category $\mathcal{A}$), then for each $n \in \mathbb{Z}$, the n-th cohomology is defined by

$$H^n(C) = \operatorname{Ker} d^n / \operatorname{Im} d^{n-1}.$$

Consider the structure of short exact sequence in complex category $\mathbf{Comp}(\mathcal{A})$,

$$0 \to C'_* \xrightarrow{i} C_* \xrightarrow{p} C''_* \to 0$$

where morphisms i, p are chain maps. And more precisely, we can write the diagram expending,

$$\begin{tikzcd}
 & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \\
0 \arrow[r] & C'_{n+1} \arrow[r, "i_{n+1}"] \arrow[d, "d'_{n+1}"'] & C_{n+1} \arrow[r, "p_{n+1}"] \arrow[d, "d_{n+1}"'] & C''_{n+1} \arrow[r] \arrow[d, "d''_{n+1}"'] & 0 \\
0 \arrow[r] & C'_n \arrow[r, "i_n"] \arrow[d, "d'_n"'] & C_n \arrow[r, "p_n"] \arrow[d, "d_n"'] & C''_n \arrow[r] \arrow[d, "d''_n"'] & 0 \\
0 \arrow[r] & C'_{n-1} \arrow[r, "i_{n-1}"] \arrow[d] & C_{n-1} \arrow[r, "p_{n-1}"] \arrow[d] & C''_{n-1} \arrow[r] \arrow[d] & 0 \\
 & \cdots & \cdots & \cdots &
\end{tikzcd}$$

Next we will construct a snake connection at first to define the connecting homomorphism. And by the connecting homomorphisms, for every short exact sequence in $\mathbf{Comp}(\mathcal{A})$, we can also get a long exact sequence like a snake.

Construction 3.18 (Snake connection)

Consider the upside diagram, for any $h \in H_n(C'')$, we can construct a snake connection (h, c'', c, c', g) from $H_n(C'')$ to $H_{n-1}(C')$ by

- For any $h \in H_n(C'')$, there is an element $c'' \in \operatorname{Im} d''_{n+1} \subseteq C''_n$ such that $[c''] = h$ in $H_n(C'')$.
- Then there is also an $c \in C_n$ such that $p_n(c) = c''$, since the short exact sequence, p_n is a surjective.
- Next we have $d_n(c) \in C_{n-1}$, and there is an element $c' \in C'_{n-1}$ such that $i_{n-1}(c') = d_n(c)$.
  Such element c' is exist, since p is a chain map, we have $d''_n(p_n(c)) = p_{n-1}(d_n(c))$ and $p_n(c) = c'' \in \operatorname{Im} d''_{n+1} \subseteq \ker d''_n$, so $p_{n-1}(d_n(c)) = d''_n(c'') = 0$, *i.e.*, $d_n(c) \in \ker p_{n-1} = \operatorname{Im} i_{n-1}$ (by the short exact sequence).
- Finally we have the corresponded homology class $[c'] = g \in H_{n-1}(C')$.
  Such corresponded Homological class is exist because $c' \in \ker d'_{n-1} = Z_{n-1}(C')$. Since i is a chain map, we have $i_{n-2}(d'_{n-1}(c')) = d_{n-1}(i_{n-1}(c')) = d_{n-1}(d_n(c)) = 0$, and i_{n-2} is an injection (by exactness) so $d'_{n-1}(c') = 0$, *i.e.*, $c' \in \ker d'_{n-1}$.

For any element $h \in H_n(C''_n)$ the existence of the snake connection begin at h is proved during our construction. According to the process of constructing snake connection, the step 3) and step 4) are uniquely decided since i is an injection. So the only two situations we need to varify are the independence of the choice of lifting and represented element in step 1) and step 2).

Lemma 3.19 (Independence of the choice of lifting and represented element)

The snake connection is a map $H_n(C'') \to H_{n-1}(C')$, *i.e.*, for any two snake connections started at a same point, will also end with the same point.

Proof. It is sufficient to prove that if a snake connection begin at $0 \in H_n(C'')$, then it must end at $0 \in H_{n-1}(C')$, *i.e.*, the snake connection also in the form $(0, c'', c, c', 0)$. Suppose $c'' \in C''_n$ such that $[c''] = 0$ in $H_{n-1}(C''_n)$, *i.e.*, $c'' \in B_n(C'') = \operatorname{Im} d''_{n+1}$, so there is an element $w \in C''_{n+1}$ such that $d''_{n+1}(w) = c''$, we can also select an element $v \in C_{n+1}$ such that $p_{n+1}(v) = w$(since p_{n+1} is surjection). By the commutative diagram, we also have

$$p_n(d_{n+1}(v)) = d''_{n+1}(p_{n+1}(v)) = d''_{n+1}(w) = c''.$$

And according to the step 2) of the construction, there is an element $c \in C_n$ such that $p_n(c) = c''$. Then consider $c - d_{n+1}(v) \in C_n$, we have $p_n(c - d_{n+1}(v)) = p_n(c) - p_n(d_{n+1}(v)) = c'' - c'' = 0$, *i.e.*, $c - d_{n+1}(v) \in \ker p_n$. Since i_n is an injection and $\ker p_n = \operatorname{Im} i_n$, there is an unique element $x \in C'_n$ such that $i_n(x) = c - d_{n+1}(v)$. So we have the following equivalence by commutative diagram,

$$i_{n-1}(d'_n(x)) = d_n(i_n(x)) = d_n(c - d_{n+1}(v)) = d_n(c) - d_n d_{n+1}(v) = d_n(c).$$

However, since i_{n-1} is also an injection, the element $c' \in C'_{n-1}$ such that $i_{n-1}(c') = d_n(c)$ is unique. Hence $c' = d'_n(x) \in \operatorname{Im} d'_n = B_{n-1}(C')$, $[c'] = 0$ in $H_{n-1}(C')$. ∎

Remark 3.20

As we see such snake connections are well-defined, and we called them "connection homomorphisms", denoted by $\partial_n : H_n(C'') \to H_{n-1}(C')$. Use these connection homomorphisms ∂, we will construct a long exact sequence by connect every short exact sequence

$$H_n(C') \xrightarrow{i_*} H_n(C) \xrightarrow{p_*} H_n(C'').$$

Proposition 3.21

The homological functor $H_n : \mathbf{Comp}(\mathcal{A}) \to \mathcal{A}$ for any $n \in \mathbb{Z}$ and an exact sequence $C' \xrightarrow{i} C \xrightarrow{p} C''$, then H_n preserves $\operatorname{Ker} p = \operatorname{Im} i$, (*i.e.*, $\operatorname{Ker} p_* = \operatorname{Im} i_*$).

Proof. Consider an exact sequence $C' \xrightarrow{i} C \xrightarrow{p} C''$, the image of the n-th Homological functor is $H_n(C') \xrightarrow{i_*} H_n(C) \xrightarrow{p_*} H_n(C'')$. Then

- ($\operatorname{Im} i_* \subseteq \operatorname{Ker} p_*$) For any $[c'] \in H_n(C')$, $c' \in Z_n(C') = \ker d'_n$, and $p_*(i_*([c'])) = p_*([i_n(c')]) = [p_n(i_n(c'))] = 0$, *i.e.*, $i_*([c']) \in \operatorname{Ker} p_*$. Hence $\operatorname{Im} i_* \subseteq \operatorname{Ker} p_*$.
- ($\operatorname{Ker} p_* \subseteq \operatorname{Im} i_*$) If $[c] \in \operatorname{Ker} p_*$, then $p_n(c) \in B_n(C'') = \operatorname{Im} d''_{n+1}$, *i.e.*, there is an element $u \in C''_{n+1}$ such that $d''_{n+1}(u) = p_n(c)$. Since p_{n+1} is surjection, there is an element $v \in C_{n+1}$ such that $p_{n+1}(v) = u$, so $d''_{n+1}(p_{n+1}(v)) = p_n(c)$. Then by the commmutative diagram, we have $p_n(d_{n+1}(v)) = d''_{n+1}(p_{n+1}(v)) = p_n(c)$, and consider $c - d_{n+1}(v)$, we have $p_n(c - d_{n+1}(v)) = p_n(c) - p_n(d_{n+1}(v)) = 0$, *i.e.*, $c - d_{n+1}(v) \in \operatorname{Ker} p_n$. By exactness, there is an $c' \in C'_n$ such that $i_n(c') = c - d_{n+1}(v)$, and $i_{n-1}(d'_n(c')) = d_n(i_n(c')) = d_n(c - d_{n+1}(v)) = d_n(c) - d_n(d_{n+1}(v)) = d_n(c) = 0$ (since $c \in Z_n(C) = \operatorname{Ker} d_n$). So $d'_n(c') = 0$, since i_{n-1} is injective, *i.e.*, $c' \in Z_n(C')$,*i.e.*, $[c']$ make sense in $H_n(C')$. Hence $i_*[c'] = [i_n(c')] = [c - d_{n+1}(v)] = [c]$ (since $d_{n+1}(v) \in B_n(C)$), *i.e.*, $[c] \in \operatorname{Im} i_*$. Thus we get $\operatorname{Ker} p_* \subseteq \operatorname{Im} i_*$.

■

Remark 3.22

- $[c] \in H_n(C)$ only make sense for $c \in Z_n(C)$.
- The Homological functor H_n may not preserve injection and surjection.

Theorem 3.23 (Long exact sequence)

According to the connection $\partial : H_n(C'') \to H_{n-1}(C')$ for each $n \in \mathbb{Z}$, we can get an long exact sequence

$$\begin{tikzcd}
H_{n+1}(C') \arrow[r, "i_*"] & H_{n+1}(C) \arrow[r, "p_*"] & H_{n+1}(C'') \arrow[dll, rounded corners, "\partial_{n+1}"] \\
H_n(C') \arrow[r, "i_*"] & H_n(C) \arrow[r, "p_*"] & H_n(C'') \arrow[dll, rounded corners, "\partial_n"] \\
H_{n-1}(C') \arrow[r, "i_*"] & H_{n-1}(C) \arrow[r, "p_*"] & H_{n-1}(C'')
\end{tikzcd}$$

or presented in such form,

$$\cdots \longrightarrow H_{n+1}(C'') \xrightarrow{\partial_{n+1}} H_n(C') \xrightarrow{i_*} H_n(C) \xrightarrow{p_*} H_n(C'') \xrightarrow{\partial_n} H_{n-1}(C') \longrightarrow \cdots$$

Proof. • According to **Proposition 3.21**, we have $\operatorname{Ker} p_* = \operatorname{Im} i_*$, for all $n \in \mathbb{Z}$.

- Next we will prove $\operatorname{Ker} \partial_n = \operatorname{Im} p_*$.
  - ($\operatorname{Ker} \partial_n \subseteq \operatorname{Im} p_*$) If $[c''] \in \operatorname{Ker} \partial_n \subseteq H_n(C)$, *i.e.*, $\partial_n([c'']) = [0]$, then $u \in B_{n-1}(C') = \operatorname{Im} d'_n$ where $[u] = \partial_n([c''])$, *i.e.*, there is $u = d'_n(c')$ for some $c' \in C'_n$. And we have a snake connection by the connection homomorphism $([c''], c'', c, u, [\partial_n(c'')])$, where $c'' = p_n(c)$ and $d_n(c) = i_{n-1}(u) = i_{n-1}(d'_n(c')) = d_n(i_n(c'))$. Then we get $d_n(c - i_n(c')) = 0$, *i.e.*, $c - i_n(c') \in Z_n(C)$, so $[c - i_n(c')] \in H_n(C)$ make sense. At the same time, $p_*[c - i_n(c')] = [p_n(c - i_n(c'))] = [p_n(c) - p_n(i_n(c'))] = [p_n(c)] = [c'']$. So $[c''] \in \operatorname{Im} p_*$.
  - ($\operatorname{Im} p_* \subseteq \operatorname{Ker} \partial_n$) For any $[c''] = p_*[c] = [p_n(c)] \in H_n(C')$ where $c \in Z_n(C) = \operatorname{Ker} d_n$, we have snake connection $([c''], p_n(c), c, c', \partial_n([c'']))$ where $[c'] = \partial_n([c''])$. And we have $i_{n-1}(c') = d_n(c) = 0$, so $c' = 0$ since i_{n-1} is injective. Hence $\partial_n([c'']) = [c'] = 0$, *i.e.*, $[c''] \in \operatorname{Ker} \partial_n$.
- Finally we will prove $\operatorname{Ker} i_* = \operatorname{Im} \partial_n$.

– ($\operatorname{Im}\partial_n \subseteq \operatorname{Ker} i_*$) For any $\partial_n([c''])\in \operatorname{Im}\partial_n \subseteq H_{n-1}(C')$ where $[c'']\in H_n(C'')$, we have an snake connection $([c''],c'',c,c',\partial_n([c'']))$ and $d_n(c) = i_{n-1}(c')$, *i.e.*, $i_{n-1}(c')\in B_{n-1}(C)$ where $[c'] = \partial_n([c''])$. So $i_*(\partial_n([c''])) = i_*[c'] = [i_{n-1}(c')] = 0$, *i.e.*, $\partial_n[c'']\in \operatorname{Ker} i_*$.
– ($\operatorname{Ker} i_* \subseteq \operatorname{Im}\partial_n$) If $[c']\in \operatorname{Ker} i_*$, *i.e.*, $i_*[c'] = [i_{n-1}(c')] = 0$, *i.e.*, $i_{n-1}(c')\in B_{n-1}(C) = \operatorname{Im} d_n$. So there is $c\in C_n$ such that $d_n(c) = i_{n-1}(c')$. Then by the chain map,

$$\begin{aligned} d''_n(p_n c) &= p_{n-1}(d_n c)\\ &= p_{n-1}(i_{n-1}(c))\\ &= 0 \qquad\qquad \text{by exactness.}\end{aligned}$$

Thus we get $p_n c\in \operatorname{Ker} d''_n = Z_n(C'')$, so $[p_n c]$ makes sense in $H_n(C'')$. Hence $[c'] = \partial_n[p_n c]$, *i.e.*, $[c']\in \operatorname{Im}\partial_n$.

∎

Theorem 3.24 (Naturality of connection ∂)

$\mathcal{A}$ is an abelian category. For any commutative diagram in category $\mathbf{Comp}(\mathcal{A})$, where both rows are short exact sequence,

$$\begin{tikzcd} 0 \arrow[r] & C' \arrow[r, "i"] \arrow[d, dashed, "f"] & C \arrow[r, "p"] \arrow[d, dashed, "g"] & C'' \arrow[r] \arrow[d, dashed, "h"] & 0 \\ 0 \arrow[r] & D' \arrow[r, "i'"] & D \arrow[r, "p'"] & D'' \arrow[r] & 0 \end{tikzcd}$$

then the following diagram is commutative and both rows are long exact sequence,

$$\begin{tikzcd} \cdots \arrow[r] & H_n(C') \arrow[r, "i_*"] \arrow[d, dashed, "f_*"] & H_n(C) \arrow[r, "p_*"] \arrow[d, dashed, "g_*"] & H_n(C'') \arrow[r, "\partial_n"] \arrow[d, dashed, "h_*"] & H_{n-1}(C') \arrow[r] \arrow[d, dashed, "f_*"] & \cdots \\ \cdots \arrow[r] & H_n(D') \arrow[r, "i'_*"] & H_n(D) \arrow[r, "p'_*"] & H_n(D'') \arrow[r, "\partial'_n"] & H_{n-1}(D') \arrow[r] & \cdots \end{tikzcd}$$

The details of proof can be founded in Rotman's book [3].

Definition 3.25 (Homotopy)

Two chain maps $f,g : (C,d)\to(C',d')$ are homotopic (denoted by $f\simeq g$) if for all $n\in\mathbb{Z}$, there is a map $s = (s_n) : C\to C'$ of degree $+1$

$$\begin{tikzcd} \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] \arrow[dl, dashed, "s_n"'] & C_{n-1} \arrow[r] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] \arrow[dl, dashed, "s_{n-1}"'] & {} \\ \arrow[r] & C'_{n+1} \arrow[r, "d'_{n+1}"] & C'_n \arrow[r, "d'_n"] & C'_{n-1} \arrow[r] & {} \end{tikzcd}$$

such that $f_n - g_n = d'_{n+1}s_n + s_{n-1}d_n$.

***Remark* 3.26** (Null-homotopic)

A chain map $f : C\to C'$ is Null-homotopic if $f\simeq 0$. (Where 0 is the zero chain map).

Theorem 3.27 (Induced homotopic morphisms are equal)

If chain maps $f, g : C \to C'$ are homotopic, *i.e.*, $f \simeq g$, then there indued morphisms f_{*n}, g_{*n} under homology functor $H_n : \mathbf{Comp} \to \mathcal{A}$ are same for all $n \in \mathbb{Z}$ (*i.e.*, $H_n(f) = H_n(g)$). More precisely, we have

$$f_{*n} = g_{*n} : H_n(C) \to H_n(C').$$

Proof. Consider $f_{*n} - g_{*n} : H_n(C) \to H_n(C')$, pick $[c] \in H_n(C)$ with $c \in Z_n(C)$, *i.e.*, $d_n(c) = 0$. Then we have

$$(f_{*n} - g_{*n})[c] = [f_n(c) - g_n(c)] = [d'_{n+1}s_n(c) + s_{n-1}d_n(c)] = [d'_{n+1}s_n(c)] = 0,$$

(since $d'_{n+1}s_n(c) \in \operatorname{Im} d'_{n+1} = B_n(C')$). Hence $f_{*n} = g_{*n}$. ∎

Corollary 3.28 (Additive functors preserve homotopy)

Let $T : \mathcal{C} \to \mathcal{C}'$ is an additive functor and $f, g : C \to C'$ are chain maps in $\mathbf{Comp}(\mathcal{C})$. If f and g are homotopic, then Tf and Tg are also homotopic, *i.e.*, $f \simeq g \Rightarrow Tf \simeq Tg$.

Proof. If $f \simeq g$, there is a map $s = (s_n) : C \to C'$ of degree +1 such that $f_n - g_n = d'_{n+1}s_n + s_{n-1}d_n$. Then since T is an additive functor, we have

$$T(f_n) - T(g_n) = T(f_n - g_n) = T(d'_{n+1}s_n + s_{n-1}d_n) = T(d'_{n+1}s_n) + T(s_{n-1}d_n) = Td'_{n+1}(Ts_n) + Ts_{n-1}(Td_n).$$

Hence $Tf \simeq Tg$. ∎

3.2 Derived Functors

In this section, we will assume that all abelian categories with enough injectives and projectives. And we also assume that all additive functors are covariant. The special case of controvariant functors will be discussed in the end of this chapter.

3.2.1 Left Derived Functors

We first consider the Comparison Theorem which will be used in the constructions of left and right derived functors.

Lemma 3.29 (Comparison Theorem)

For an abelian category $\mathcal{A}$, and a morphism $f : A \to A'$ in $\mathcal{A}$, if there are two exact sequences $\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \to 0$ and $\cdots \to P'_2 \xrightarrow{d'_2} P'_1 \xrightarrow{d'_1} P'_0 \xrightarrow{\epsilon'} A' \to 0$ where P_i, P'_i are projective for all $i \geq 0$, then

$$\begin{tikzcd}
{} \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d, dashed, "\tilde{f}_2"'] & P_1 \arrow[r, "d_1"] \arrow[d, dashed, "\tilde{f}_1"'] & P_0 \arrow[r, "\epsilon"] \arrow[d, dashed, "\tilde{f}_0"'] & A \arrow[r] \arrow[d, "f"'] & 0 \\
{} \arrow[r] & P'_2 \arrow[r, "d'_2"] & P'_1 \arrow[r, "d'_1"] & P'_0 \arrow[r, "\epsilon'"] & A' \arrow[r] & 0
\end{tikzcd}$$

there is an chain map $\tilde{f} = (\tilde{f}_n) : P_A \to P'_{A'}$ such that the diagram commute. And any two such chain maps are homotopic.

Proof. See Rotamn's book [3]. ∎

Remark 3.30 (Chain map over f)

If P_A and $P'_{A'}$ are deleted projective resolutions of A and A',such chain map $\tilde{f} : P_A \to P'_{A'}$ is called the chain map over morphism $f : A \to A'$.

Actually when we consider the dual situation (projections are replaced by injections), such conclusion is also established.

Lemma 3.31 (Dual Comparison Theorem)

a morphism $f : A \to A'$ in $\mathcal{A}$, if there are complexes $0 \to A \xrightarrow{\eta} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \to$ where E^i are all injective and $0 \to A' \xrightarrow{\eta'} E'^0 \xrightarrow{d'^0} E'^1 \xrightarrow{d'^1} E'^2 \to$ is exact, then

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\eta"] \arrow[d, dashed, "f"] & E^0 \arrow[r, "d_2"] \arrow[d, dashed, "\tilde{f}_0"'] & E^1 \arrow[r, "d_1"] \arrow[d, dashed, "\tilde{f}_1"'] & E^2 \arrow[r] \arrow[d, dashed, "\tilde{f}_2"'] & {} \\
0 \arrow[r] & A' \arrow[r, "\eta'"] & E'^0 \arrow[r, "d'_2"] & E'^1 \arrow[r, "d'_1"] & E'^2 \arrow[r] & {}
\end{tikzcd}$$

there is a chain map $\tilde{f} = (\tilde{f}_n) : E^A \to E^{A'}$ such that the diagram commute, and any two wuch chain maps are homotopic.

Proof. The proof of this dual situation is similar as the proof of **Lemma 3.29**. ∎

Construction 3.32 (Left Derived Functors)

Consider a right exact additive functor $T : \mathcal{A} \to \mathcal{C}$ where $\mathcal{A}, \mathcal{C}$ are Abelian Categories. Then we can construct the left derived functor $L_n T : \mathcal{A} \to \mathcal{C}$ of T by the folloing steps.

- Consider objects $A, A' \in \mathcal{A}$ and morphism $f : A \to A'$, *i.e.*, $f \in \mathrm{Hom}(A, A')$. Then by the existance property of projective resolution, we have projective resolutions for both A and A',

$$P = \cdots \to P_2 \to P_1 \to P_0 \to A \to 0 \qquad \text{and} \qquad P' = \cdots \to P'_2 \to P'_1 \to P'_0 \to A' \to 0.$$

  Then according to the Comparison Theorem(**Lemma 3.29**), there exists a chain map $\tilde{f} : P \to P'$ over f, *i.e.*, there are morphisms $(\tilde{f}_i)$ such that the following diagram commute,

$$\begin{tikzcd}
P = & \cdots \arrow[r] & P_2 \arrow[r] \arrow[d, dashed, "\tilde{f}_2"'] & P_1 \arrow[r] \arrow[d, dashed, "\tilde{f}_1"'] & P_0 \arrow[r] \arrow[d, dashed, "\tilde{f}_0"'] & A \arrow[r] \arrow[d, dashed, "f"'] & 0 \\
P' = & \cdots \arrow[r] & P'_2 \arrow[r] & P'_2 \arrow[r] & P'_0 \arrow[r] & A' \arrow[r] & 0
\end{tikzcd}$$

- Next consider the deleted resolutions P_A and $P'_{A'}$, they are both complexes and the chain map $\tilde{f} = (\tilde{f}_n)$ where $n \geq 0$.

$$\begin{tikzcd}
P_A = \arrow[d, "\tilde{f}"'] & \cdots \arrow[r] & P_2 \arrow[r] \arrow[d, dashed, "\tilde{f}_2"'] & P_1 \arrow[r] \arrow[d, dashed, "\tilde{f}_1"'] & P_0 \arrow[r] \arrow[d, dashed, "\tilde{f}_0"'] & 0 \\
P'_{A'} = & \cdots \arrow[r] & P'_2 \arrow[r] & P'_2 \arrow[r] & P'_0 \arrow[r] & 0
\end{tikzcd}$$

- Then apply functor T to this diagram,

$$\begin{tikzcd}
TP_A = \arrow[d, "T\tilde{f}"'] & \cdots \arrow[r, "Td_3"] & T(P_2) \arrow[r, "Td_2"] \arrow[d, dashed, "T\tilde{f}_2"'] & T(P_1) \arrow[r, "Td_1"] \arrow[d, dashed, "T\tilde{f}_1"'] & T(P_0) \arrow[r] \arrow[d, dashed, "T\tilde{f}_0"'] & 0 \\
TP'_{A'} = & \cdots \arrow[r, "Td'_3"] & T(P'_2) \arrow[r, "Td'_2"] & T(P'_2) \arrow[r, "Td'_1"] & T(P'_0) \arrow[r] & 0
\end{tikzcd}$$

by **Proposition 3.8** and T is an additive functor, both TP_A and $TP'_{A'}$ are complexes in $\mathbf{Comp}(C)$, such diagram is commutative and $T\tilde{f}$ is a chain map in $\mathbf{Comp}(C)$.

- Finally, we can do homology on TP_A and $TP'_{A'}$, where $H_n(TP_A)$ and $H_n(TP'_{A'})$ are both in category $\mathcal{C}$. And the induced morphism $T\tilde{f}_{n*}$ is in $\mathrm{Hom}(H_n(TP_A), H_n(TP'_{A'}))$. Hence, we can define the n-th left derived functor from abelian category $\mathcal{A}$ to $\mathcal{C}$.

Definition 3.33 (Left Derived Functors)

The left derived functor of $T : \mathcal{A} \to C$ is defined according to our construction by,

$$\begin{aligned} L_nT : \mathcal{A} &\longrightarrow \mathcal{C} \\ A &\mapsto H_n(TP_A) \\ f &\mapsto H_n(T\tilde{f}) = T\tilde{f}_{*n} \end{aligned}$$

where the left derived functor's image of morphisms are look like

$$\begin{aligned} L_nT(f) = T\tilde{f}_{*n} : H_n(TP_A) &\longrightarrow H_n(TP'_{A'}) \\ [z] &\mapsto [T\tilde{f}_n(z)] \end{aligned}$$

To varify our definition of left derived functor is well-defined, we need to verify that it is both well-defined on morphisms f and objects $A \in obj(\mathcal{A})$, *i.e.*, the definition of L_nT is independent of the choice of chain map $\tilde{f}$ over f and the choice of projective resolutions.

***Remark* 3.34** (L_nT is well-defined on morphisms)

If $g : P_A \to P'_{A'}$ is another chain map over f, according to the **Lemma 3.29**, we have $g \simeq \tilde{f}$. Then since T is an additive functor and by **Corollary 3.28**, we have $Th \simeq T\tilde{f}$. Finally, by the **Theorem 3.27** we have $H_n(T\tilde{h}) = H_n(T\tilde{f})$, *i.e.*, $L_nT(h) = L_nT(f)$.

Theorem 3.35 (L_nT is independent of the choice of resolutions)

Let $T : \mathcal{A} \to \mathcal{C}$ be an additive functor, and if there is an another choice of deleted projective resolution $\tilde{P}_A$, and we can get the corresponded left induced functor $\widetilde{L_n}T$ (as we do in **Construction 3.32**). Then we get,

$$(L_nT)A \cong (\widetilde{L_n}T)A.$$

(*i.e.*, the left induced map L_nT is independent of the choice of projective resolution).

Proof. Consider the following diagram,

$$\begin{array}{ccccccccc} P = & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_0} & & \downarrow{\scriptstyle 1_A} & & \\ \tilde{P} = & \longrightarrow & \widetilde{P_1} & \longrightarrow & \widetilde{P_0} & \longrightarrow & A & \longrightarrow & 0 \end{array},$$

by the comparison theorem **Lemma 3.29** there is a chain map $\tau : P_A \to \widetilde{P_A}$ over 1_A and also a chain map $\sigma : \widetilde{P_A} \to P_A$ given by

$$\begin{array}{ccccc} P_A & \longrightarrow & A & \longrightarrow & 0 \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle 1_A} & & \\ \widetilde{P_A} & \longrightarrow & A & \longrightarrow & 0 \end{array}, \qquad \begin{array}{ccccc} \widetilde{P_A} & \longrightarrow & A & \longrightarrow & 0 \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle 1_A} & & \\ P_A & \longrightarrow & A & \longrightarrow & 0 \end{array}.$$

Next consider the chain map $\sigma\tau : P_A \to P_A$ which is also over 1_A and $1_{P_A} : P_A \to P_A$ is also a chain map over 1_A, so by the comparison theorem, we can get $\sigma\tau \simeq 1_{P_A}$. By the same way, we can also get $\tau\sigma \simeq 1_{\widetilde{P_A}}$. Thus by **Theorem 3.27**, we have

$$(T\sigma)_*(T\tau)_* = T(\sigma\tau)_* = 1_{(\widetilde{L_n})A} \quad \text{and} \quad (T\tau)_*(T\sigma)_* = T(\tau\sigma)_* = 1_{(L_nT)A}.$$

Hence $\tau_A = (T\tau)_* : L_n(A) \to \widetilde{L_n}T(A)$ is an isomorphism.
Next we need to prove that isomorphism τ_A is natural, *i.e.*, if $f : A \to B$ is a morphism in $\mathcal{A}$, then the following diagram commute

$$\begin{tikzcd}
(L_nT)A \arrow[r, "\tau_A"] \arrow[d, "(L_nT)f"'] & (\widetilde{L_n}T)A \arrow[d, "(\widetilde{L_n}T)f"] \\
(L_nT)B \arrow[r, "\tau_B"'] & (\widetilde{L_n}T)B
\end{tikzcd} .$$

Consider

$$P_A \xrightarrow{\tau} \widetilde{P_A} \xrightarrow{\tilde{f}'} P_B \quad \text{and} \quad P_A \xrightarrow{\tilde{f}} P_B \xrightarrow{\tau} \widetilde{P_B},$$

we can see that $\tilde{f}'\tau : P_A \to P_B$ is a chain map over $f1_A = f$ and $\tau\tilde{f} : P_A \to \widetilde{P_B}$ is a chain map over $1_Bf = f$. Then by the comparison theorem, they are homotopic, *i.e.*, $\tilde{f}'\tau \simeq \tau\tilde{f}$. And after functor T and homology we have

$$(\widetilde{L_n}T)f \circ \tau_A = \tau_B \circ (L_n)f$$

which also means that the diagram commute.
Hence τ_A is a natural isomorphism, *i.e.*, $(L_nT)A \cong (\widetilde{L_n}T)A$. ■

So according to ***Remark* 3.34** and **Theorem 3.35**, the left derived functor L_nT is well-defined.

Proposition 3.36 (L_nT is an Additive Functor)

If $T : \mathcal{A} \to \mathcal{C}$ is an additive functor, then its left derived functor $L_nT : \mathcal{A} \to \mathcal{C}$ is also an additive covariant functor for any n.

Proof. By our assumption, functor T is additive and by **Proposition 3.15** the homology functor H_n is additive. Hence $L_nT =$ is an additive functor. ■

Lemma 3.37 (Horseshoe Lemma)

For any short exact sequence $0 \to A' \xrightarrow{i} A \xrightarrow{p} A'' \to 0$ in Abelian category $\mathcal{A}$, and select any projective resolutions P' of A' and P'' of A'',

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r, "i"] & A \arrow[r, "p"] & A'' \arrow[r] & 0 \\
0 \arrow[r, dotted] & P'_0 \arrow[r, dotted] \arrow[u, "\gamma'"] & P_0 \arrow[r, dotted] \arrow[u, dashed, "\gamma"] & P''_0 \arrow[r, dotted] \arrow[u, "\gamma''"] & 0 \\
0 \arrow[r, dotted] & P'_1 \arrow[r, dotted] \arrow[u] & P_1 \arrow[r, dotted] \arrow[u, dashed] & P''_1 \arrow[r, dotted] \arrow[u] & 0 \\
 & \arrow[u] & \arrow[u, dashed] & \arrow[u] &
\end{tikzcd}$$

then there exist a projective resolution P of A and chain maps such that $0 \to P' \to P \to P'' \to 0$ is an exact sequence of complex in $\mathbf{Comp}(\mathcal{A})$.

Proof. The proof is given by Rotman's book [3] ■

Lemma 3.38 (Dual Horseshoe Lemma)

For any short exact sequence $0 \to A' \xrightarrow{i} A \xrightarrow{p} A'' \to 0$ in Abelian category $\mathcal{A}$, and we select any injective resolutions E' of A' and E'' of A'',

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r, "I"] \arrow[d, "\eta'"] & A \arrow[r, "p"] \arrow[d, dashed, "\eta"] & A'' \arrow[r] \arrow[d, "\eta''"] & 0 \\
0 \arrow[r, dotted] & E'^0 \arrow[r, dotted] \arrow[d] & E^0 \arrow[r, dotted] \arrow[d, dashed] & E'^0 \arrow[r, dotted] \arrow[d] & 0 \\
0 \arrow[r, dotted] & E'^1 \arrow[r, dotted] \arrow[d] & E^1 \arrow[r, dotted] \arrow[d, dashed] & E'^1 \arrow[r, dotted] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

then there exist an injective resolution E of A and chain maps such that $0 \to E' \to E \to E'' \to 0$ is an exact sequence of complex in $\mathbf{Comp}(\mathcal{A})$.

Proof. The proof is similar to **Lemma 3.37** by considering the dual situation. ■

Next for an additive functor $T : \mathcal{A} \to \mathcal{C}$ we will consider the short exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$ and we will see that after operated by the left derived functor L_nT, we can also get a long exact sequence.(similar to the **Theorem 3.23** of homology functor, but the difference is the domain of H_n is $\mathbf{Comp}(\mathcal{A})$, but the domian of L_nT is $\mathcal{A}$ itself).

Construction 3.39 (Long exact sequence of L_nT)

For an additive functor $T : \mathcal{A} \to \mathcal{C}$, and an exact sequence $0 \to A' \xrightarrow{i} A \xrightarrow{p} A'' \to 0$ in category $\mathcal{A}$. We can first construct two projective resolutions P' of A' and P'' of A''. Then by **Lemma 3.37** (Harseshoe Lemma), there is a projective resolution $\tilde{P}$ of A and chain maps j (over i), q (over p) such that

$$0 \to P' \xrightarrow{j} \tilde{P} \xrightarrow{q} P'' \to 0 \tag{1}$$

is an exact sequence of complexes. Then for the deleted projective resolutions,

$$0 \to P'_{A'} \xrightarrow{j} \tilde{P}_A \xrightarrow{q} P''_{A''} \to 0$$

is also an exact sequence of complexes. Next apply additive functor T on it, the sequence

$$0 \to TP'_{A'} \xrightarrow{Tj} T\tilde{P}_A \xrightarrow{Tq} TP''_{A''} \to 0$$

is exact either, since each $0 \to P'_n \xrightarrow{j_n} \tilde{P}_n \xrightarrow{q_n} P''_n \to 0$ in eq. (1) is a split exact sequence (by P''_n is projective[3]) and additive functor preserve split short exact sequence(by [3]). Hence we apply homology on it, and we can get the long exact sequence by **Theorem 3.23**,

$$\cdots \to H_n(TP'_{A'}) \xrightarrow{(Tj)_*} H_n(T\tilde{P}_A) \xrightarrow{(Tq)_*} H_n(TP''_{A''}) \xrightarrow{\partial_n} H_{n-1}(TP'_{A'}) \to \cdots$$

, *i.e.*,

$$\cdots \to (L_nT)A' \xrightarrow{(Tj)_*} (\widetilde{L_n}T)A \xrightarrow{(Tq)_*} (L_nT)A'' \xrightarrow{\partial_n} (L_{n-1}T)A' \to \cdots \tag{2}$$

(where the connection morphisms ∂_n are from the homology of exact sequence of complexes as we discussed in previous).

Remark 3.40

The choice of projective resolution $\tilde{P}$ of A is not arbitary in our construction. However the choice of projective resolution of A can not influence the long exact sequence which we get.
In the long exact sequence eq. (2), the exactness for the only one part we need to discuss is

$$(L_nT)A' \to (L_nT)A \to (L_nT)A'' \quad , i.e., \quad H_n(TP'_{A'}) \to H_n(TP_A) \to H_n(TP''_{A''}),$$

where P is an arbitary projective resolution of A instead of $\tilde{P}$. If we select an arbitary projective resolution P of A, then by the comparison theorem, there are chain maps $\tau : P_A \to \tilde{P}_A$ and $\sigma : \tilde{P}_A \to p_A$, both τ and σ are over 1_A. Then by the same way in **Theorem 3.35**, we can see that $(T\tau)_*(T\sigma)_* = 1_{(\tilde{L}_nT)A}$ and $(T\sigma)_*(T\tau)_* = 1_{(L_nT)A}$ which means that $(T\tau) : L_nT \to \tilde{L}_nT$ is an isomorphism, and $(T\sigma)_*$ is its inverse. Now we need to check the exactness of the part after replaced.
From the above results, we have the following diagram

$$\begin{tikzcd}
P'_{A'} \arrow[r, "j"] \arrow[dr, "\tilde{i}"'] & \tilde{P}_A \arrow[r, "p"] \arrow[d, dashed, shift left, "\sigma"] & P''_{A''} \\
& P_A \arrow[u, dashed, shift left, "\tau"] \arrow[ur, "\tilde{p}"'] &
\end{tikzcd}
\xrightarrow{\quad T \quad}
\begin{tikzcd}
TP'_{A'} \arrow[r, "Tj"] \arrow[dr, "T\tilde{i}"'] & T\tilde{P}_A \arrow[r, "Tq"] \arrow[d, dashed, shift left, "T\sigma"] & TP''_{A''} \\
& TP_A \arrow[u, dashed, shift left, "T\tau"] \arrow[ur, "T\tilde{p}"'] &
\end{tikzcd}$$

We have $T\tilde{i} \simeq T\sigma Tj$ and $T\tilde{p} \simeq TqT\tau$, since both $T\sigma Tj$, $T\tilde{i}$ are chain maps over $Ti : TA' \to TA$, and both $TqT\tau$, $T\tilde{p}$ are chain maps over $Tj : TA \to TA''$. (But these diagram may not commute since we only have the homotopy relations "$\simeq$", but not equal "="). However after apply homology on the diagram, we have

$$\begin{tikzcd}
H_n(TP'_{A'}) \arrow[r, "(Tj)_*"] \arrow[dr, "(T\tilde{i})_*"'] & H_n(T\tilde{P}_A) \arrow[r, "(Tq)_*"] \arrow[d, dashed, shift left, "(T\sigma)_*"] & H_n(TP''_{A''}) \\
& H_n(TP_A) \arrow[u, dashed, shift left, "(T\tau)_*"] \arrow[ur, "(T\tilde{p})_*"'] &
\end{tikzcd}$$

and by **Theorem 3.27**, the induced maps

$$(T\tau)_*(Tj)_* = (T\tau Tj)_* = (T\tilde{i})_* \quad , \quad (Tq)_*(T\tau)_* = (TqT\tau)_* = (T\tilde{p})_*$$

, *i.e.*, the diagram commute.

Claim 3.41

If $B' \xrightarrow{j} C \xrightarrow{q} B''$ is exact, *i.e.*, $\mathsf{Ker}\, q = \mathsf{Im}\, j$, map $k : B \to C$ is an isomorphism (l is its inverse map), and the following diagram commute,

$$\begin{tikzcd}
B' \arrow[r, "j"] \arrow[dr, "i"'] & C \arrow[r, "q"] \arrow[d, dotted, shift right, "l"'] & B'' \\
& B \arrow[u, dotted, shift right, "k"'] \arrow[ur, "p"'] &
\end{tikzcd}.$$

Then $B' \xrightarrow{i} B \xrightarrow{p} B''$ is also exact, *i.e.*, $\mathsf{Ker}\, p = \mathsf{Im}\, i$.

Proof. The diagram is commutative, so $j = ki, q = pl$. Then $\mathsf{Ker}(p) = l(\mathsf{Ker}(q))$ since l is a bijection. At the same time $\mathsf{Im}(j) = k(\mathsf{Im}(i))$, so $\mathsf{Im}(i) = k^{-1}(\mathsf{Im}(j)) = l(\mathsf{Im}(j))$ since l, k are bijections. We now have $\mathsf{Im}(i) = \mathsf{Ker}(p)$ by $\mathsf{Ker}(q) = \mathsf{Im}(j)$ (since $B' \longrightarrow C \longrightarrow B''$ is exact). Hence $B' \longrightarrow B \longrightarrow B''$ is exact. ■

Thus by **Claim 3.41**,

$$H_n(TP'_{A'}) \to H_n(TP_A) \to H_n(TP''_{A''}) \quad , i.e., \quad (L_nT)A' \to (L_nT)A \to (L_nT)A''$$

is exact.
(It is sufficient to verify $\mathsf{Ker}(L_nT)p = \mathsf{Im}(L_nT)i$, since $\mathsf{Ker}(L_nT)i = \mathsf{Ker}(L_nT)j$ and $\mathsf{Im}(L_nT)p = \mathsf{Im}(L_nT)q$ by $(\tilde{L}_nT)A \cong (L_nT)A$).
Hence there is a long exact sequence,

$$\cdots \to (L_nT)A' \xrightarrow{(L_nT)i} (L_nT)A \xrightarrow{(L_nT)p} (L_nT)A'' \xrightarrow{\partial_n} \cdots$$
$$\cdots$$
$$\xrightarrow{\partial_1} (L_0T)A' \xrightarrow{(L_0T)i} (L_0T)A \xrightarrow{(L_0T)p} (L_0T)A'' \to 0$$

Corollary 3.42

The left derived functor L_nT is a right exact functor.

Proof. According to our construction of long exact sequence (**Construction 3.39**), the sequence $(L_0T)A' \xrightarrow{(L_0T)i} (L_0T)A \xrightarrow{(L_0T)p} (L_0T)A'' \to 0$ is exact, *i.e.*, L_0T preserve the exactness of $A' \xrightarrow{i} A \xrightarrow{p} A'' \to 0$. ■

Theorem 3.43 ($T \cong L_0T$)

If an additive functor $T : \mathcal{A} \to \mathcal{C}$ is right exact, then T is naturally isomorphic to L_0T, *i.e.*, $T \cong L_0T$.

Proof. For $A \in obj(\mathcal{A})$, consider its projective resolution $P = \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \to 0$. Then we have

$$(L_0T)A = H_0(TP_A) = TP_0 / \mathsf{Im}\, Td_1 = cokerTd_1. \tag{3}$$

Since P is a projective resolution, sequence $P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \to 0$ is exact. And

$$TP_1 \xrightarrow{Td_1} TP_0 \xrightarrow{T\epsilon} TA \to 0$$

is also an exact sequence, since functor T is right exact. Then $T\epsilon$ is surjective, so there is a natural isomorphism $TP_0 / \mathsf{Ker}\, T\epsilon \simeq TA$ induced by $T\epsilon$ (according to the first isomorphism theorem), such as

$$\varphi_A : (L_0T)A \to TA$$
$$[z] \mapsto T\epsilon(z)$$

(because $TP_0/\operatorname{Ker} T\epsilon = TP_0/\operatorname{Im} Td_1 = \operatorname{Coker} Td_1 = (L_0T)A$, by eq. (3)).
Next we will proof the isomorphism $\varphi = (\varphi_A)_{A\in\mathrm{Obj}(\mathcal{A})} : L_0T \to T$ is natural, and it is sufficient to prove the following diagrams commute for all n,

$$\begin{tikzcd} (L_0T)A \arrow[r, "\varphi_A"] \arrow[d, "(L_0T)f"'] & TA \arrow[d, "Tf"] \\ (L_0T)B \arrow[r, "\varphi_B"] & TB \end{tikzcd}, i.e., \quad \begin{tikzcd} H_0(TA) \arrow[r, "\varphi_A"] \arrow[d, "(Tf)_*"'] & TA \arrow[d, "Tf"] \\ H_0(TB) \arrow[r, "\varphi_B"] & TB \end{tikzcd}.$$

For a morphism $f : A \to B$ in $\mathcal{A}$, we have diagrams,

$$\begin{tikzcd} {} \arrow[r] & P_0^A \arrow[r, "\epsilon_A"] \arrow[d, dashed, "f_0"] & A \arrow[r] \arrow[d, dashed, "f"] & 0 \\ {} \arrow[r] & P_0^B \arrow[r, "\epsilon_B"] & B \arrow[r] & 0 \end{tikzcd} \xrightarrow{T} \begin{tikzcd} {} \arrow[r] & TP_0^A \arrow[r, "T\epsilon_A"] \arrow[d, "Tf_0"'] & TA \arrow[r] \arrow[d, "Tf"] & 0 \\ {} \arrow[r] & TP_0^B \arrow[r, "T\epsilon_B"] & TB \arrow[r] & 0 \end{tikzcd}$$

both commute, *i.e.*, $Tf \circ T\epsilon_A = T\epsilon_B \circ Tf_0$. Then since for any $[z] \in (L_0T)A = H_0(TA)$, we have

$$Tf \circ \varphi_A[z] = Tf \circ T\epsilon_A(z) \quad \text{and} \quad \varphi_B \circ (Tf_0)_*[z] = \varphi_B[Tf_0(z)] = T\epsilon_B \circ Tf_0(z).$$

so we get $Tf \circ \varphi_A = \varphi_B(Tf_0)_*$,*i.e.*, $Tf \circ \varphi_A = \varphi_B \circ (L_0T)f$, the diagram commute. Hence $\varphi = (\varphi_A)_{A\in obj(\mathcal{A})} : L_0T \to T$ is a natural isomorphism. ∎

Corollary 3.44

If an additive functor $T : \mathcal{A} \to \mathcal{C}$ is right exact, then we have the following long exact sequence,

$$\cdots \longrightarrow (L_nT)A' \longrightarrow (L_nT)A \longrightarrow (L_nT)A'' \xrightarrow{\partial_n} \cdots$$

$$\cdots \longrightarrow (L_1T)A' \longrightarrow (L_1T)A \longrightarrow (L_1T)A'' \xrightarrow{\partial_1}$$

$$TA' \longrightarrow TA \longrightarrow TA'' \longrightarrow 0$$

Proof. According to **Theorem 3.43**, we get $(L_0T)A' \cong TA'$, $(L_0T)A \cong TA$ and $(L_0T)A'' \cong TA''$. Then we first applied **Claim 3.41** on $(L_0T)A'$,

$$\begin{tikzcd} (L_1T)A'' \arrow[r, "\partial_1"] \arrow[dr, dashed, "\varphi_{A'}\partial_1"'] & (L_0T)A' \arrow[r, "(Tj)_*"] \arrow[d, shift left, "\varphi_{A'}"] & (L_0T)A \\ & TA' \arrow[u, shift left, "\varphi_{A'}^{-1}"] \arrow[ur, dashed, "(Tj)_*\varphi_{A'}^{-1}"'] & \end{tikzcd},$$

so we get the long exact sequence $\cdots \to (L_1T)A'' \xrightarrow{\partial_1} TA' \xrightarrow{\varphi_{A'}\partial_1} (L_0T)A \xrightarrow{Tq_*} (L_0T)A'' \to 0$. Next we applied **Claim 3.41** on $(L_0T)A$ again,

$$\begin{tikzcd}
TA' \arrow[r, "(Tj)_*\varphi_{A'}^{-1}"] \arrow[dr, dashed, "\varphi_A(Tj)_*\varphi_{A'}^{-1}"'] & (L_0T)A \arrow[r, "(Tq)_*"] \arrow[d, shift left, "\varphi_A"] & (L_0T)A'' \\
 & TA' \arrow[u, shift left, "\varphi_A^{-1}"] \arrow[ur, dashed, "(Tq)_*\varphi_A^{-1}"'] &
\end{tikzcd},$$

we get the long exact sequence $\cdots \to (L_1T)A'' \xrightarrow{\partial_1} TA' \xrightarrow{\varphi_A(Tj)_*\varphi_{A'}^{-1}} TA' \xrightarrow{(Tq)_*\varphi_A^{-1}} (L_0T)A \to 0$. Finally, we applied **Claim 3.41** on $(L_0T)A$, we get the long exact sequence

$$\cdots \to (L_1T)A'' \xrightarrow{\partial_1} TA' \xrightarrow{\varphi_A(Tj)_*\varphi_{A'}^{-1}} TA' \xrightarrow{(Tq)_*\varphi_A^{-1}} TA \to 0.$$

■

Actually, we can give a definition of the derived functors of T with long exact sequence property in general.

Definition 3.45 (Homological ∂-functor)

A sequence of additive functors $(T_n : \mathcal{A} \to \mathcal{C})$ is called a homological ∂-functor if for any exact sequence $0 \to A' \to A \to A'' \to 0$ in category $\mathcal{A}$, there is a long exact sequence,

$$\begin{array}{c}
\xrightarrow{\partial_{n+1}} T_n(A') \longrightarrow T_n(A) \longrightarrow T_n(A'') \xrightarrow{\partial_n} \cdots \\[2em]
\cdots \xrightarrow{\partial_1} T_0(A') \longrightarrow T_0(A) \longrightarrow T_0(A'') \longrightarrow 0
\end{array}$$

where $\partial = (\partial_n : T_n(A'') \to T_{n-1}(A'))$ is a sequence of natural connecting morphisms.

From the **Theorem 3.43** and **Corollary 3.44**, the long exact sequence of L_nT is seems like an extension of the exact sequence $TA' \to TA \to TA'' \to 0$, and we can define the homological ∂-functors with this property in general.

Definition 3.46 (Homological Extension)

For a additive functor $F : \mathcal{A} \to \mathcal{C}$, a homological ∂-functor (T_n) is called a homological extension of F if $F \cong T_0$, *i.e.*, F is natural isomophic to T_0.

Example 3.47

- For an additive functor $T : \mathcal{A} \to \mathcal{C}$, its left derived functor L_nT is a homological ∂-functor by **Construction 3.39**.
- If the additive functor $T : \mathcal{A} \to \mathcal{C}$ is right exact, its left derived functor L_nT is a homological extension of T by **Theorem 3.43**.

The **Construction 3.39** and **Theorem 3.43** show that if additive functor T is right exact, then it always exists a homological exatention L_nT. However there is a stronger theorem in general which will deduce that L_nT is the unique homological extension of T such that the image equal to zero for all projective P and $n \geq 1$. We just give this strong theorem without prove.

Theorem 3.48

If an additive functor $T : \mathcal{A} \to \mathcal{C}$ is right exact, then there is a unique homological extension (T_n) of T such that $T_n(P) = 0$ for all projective P and all $n \geq 1$.

Claim 3.49

If an additive functor $T : \mathcal{A} \to \mathcal{C}$ is right exact, then its left derived functor $L_nT(P) = 0$ for all projective P and all $n \geq 1$.
Since we can first select a projective resolution $\to 0 \to 0 \to P \to P \to 0$, and finnally we have $L_nT(P) = 0$ for all $n \geq 1$.

Corollary 3.50

By **Claim 3.49**, for an additive right exact functor T, its left derived functor L_nT is the unique homological extension of T which equal to zero for all projective P and all $n \geq 1$.

3.2.2 Right Derived Functor

The construction of the right derived functor is dual of the construction of the left derived functor.

Construction 3.51 (Convariant right derived functor)

For a right exact additive functor $T : \mathcal{A} \to \mathcal{C}$, and $B \in obj(\mathcal{A})$, we first consider the injective resolution of B,

$$E = 0 \to B \xrightarrow{\eta} E^0 \xrightarrow{d_0} E^1 \xrightarrow{d_1} \cdots$$

then we get the deleted resolution

$$\begin{array}{ccccccccc} E^B = 0 & \longrightarrow & E^0 & \xrightarrow{d^0} & E^1 & \xrightarrow{d^1} & E^2 & \xrightarrow{d^2} & \\ {\scriptstyle T}\downarrow & & & & & & & & . \\ TE^B = 0 & \longrightarrow & TE^0 & \xrightarrow{Td^0} & TE^1 & \xrightarrow{Td^1} & TE^2 & \xrightarrow{Td^2} & \end{array}$$

We apply homology on TE^B, then we define the right derived functor by

$$(R^nT)B = H^n(TE^B) = \mathrm{Ker}\, Td^n / \mathrm{Im}\, Td^{n-1}.$$

And for a morphism $f : B \to B'$, the right derived functor act on f, which is

$$\begin{aligned} R^nT(f) = T\tilde{f}_{*n} : H_n(TE^B) &\longrightarrow H_n(TE'^{B'}) \\ [z] &\mapsto [T\tilde{f}_n(z)] \end{aligned}$$

(according to the dual comparison theorem (**Lemma 3.31**), the chain map $\tilde{f} : E^B \to E'^{B'}$ over f is always existed and $\tilde{f}$ under homotopy is unique). We can see that the construction of right derived functor is dual to the construction of left derived functor, since they use the dual resolutions (injective resolution E and projective resolution P). So it is easy to verify the right derived functor is well-defined by the dual way.

Then we can get the following propositions of the right derived functors R^nT dual to the propositions of the left derived functors L_nT.

Proposition 3.52 (*R^nT is an additive functor*)

If $T : \mathcal{A} \to \mathcal{C}$ is an additive functor, then the right derived functor $R^nT : \mathcal{A} \to \mathcal{C}$ is also an additive functor for all n.

Proposition 3.53 (Long exact sequence)

If $0 \to B' \xrightarrow{i} B \xrightarrow{p} B'' \to 0$ is an exact sequence in abelian category $\mathcal{A}$, and $T : \mathcal{A} \to \mathcal{C}$ is an additive functor. Then there is a long exact sequence

$$0 \longrightarrow (R^0T)B' \xrightarrow{(R^0T)i} (R^0T)B \xrightarrow{(R^0T)p} (R^0T)B'' \xrightarrow{\partial_0} \cdots$$

$$\cdots \xrightarrow{\partial_{n-1}} (R^nT)B' \xrightarrow{(R^nT)i} (R^nT)B \xrightarrow{(R^nT)p} (R^nT)B'' \xrightarrow{\partial_n} \cdots$$

Proposition 3.54 ($T \cong R^0T$)

If the additive functor $f : \mathcal{A} \to \mathcal{C}$ is left exact, then T is naturally isomorphic to R^0T, *i.e.*, $T \cong R^0T$.

Corollary 3.55

The functor R^0T is a left exact functor, since $0 \to (R^0T)B' \to (R^0T)B \to (R^0T)B''$ is exact.

Corollary 3.56

$0 \to B' \xrightarrow{i} B \xrightarrow{p} B'' \to 0$ is an exact sequence in abelian category $\mathcal{A}$. If the additive functor $T : \mathcal{A} \to \mathcal{C}$ is right exact, then we have the following long exact sequence,

$$0 \longrightarrow TB' \xrightarrow{Ti} TB \xrightarrow{Tp} TB'' \xrightarrow{\partial_0} \cdots$$

$$\cdots \xrightarrow{\partial_{n-1}} (R^nT)B' \xrightarrow{(R^nT)i} (R^nT)B \xrightarrow{(R^nT)p} (R^nT)B'' \xrightarrow{\partial_n} \cdots$$

Since right derived functor is dual to the left derived functor, we also have the dual definition of homological ∂-functor and homological extension.

Definition 3.57 (Cohomological ∂-functor)

A sequence of additive functors $(T^n : \mathcal{A} \to \mathcal{C})$ is called a cohomological ∂-functor if for any exact sequence $0 \to B' \to B \to B'' \to 0$ in category $\mathcal{A}$, there is a long exact sequence,

$$0 \longrightarrow T^0(B') \longrightarrow T^0(B) \longrightarrow T^0(B'') \xrightarrow{\partial_0} \cdots$$

$$\cdots \xrightarrow{\partial_{n-1}} T^n(B') \longrightarrow T^n(B) \longrightarrow T^n(B'') \xrightarrow{\partial_n} \cdots$$

where $\partial = (\partial_n : T^n(B'') \to T^{n-1}(B'))$ is a sequence of natural connecting morphisms.

Definition 3.58 (Cohomological extension)

For a additive functor $F : \mathcal{A} \to \mathcal{C}$, a cohomological ∂-functor (T^n) is called a homological extension of F if $F \cong T^0$, *i.e.*, F is natural isomophic to T^0.

Example 3.59

- For an additive functor $T : \mathcal{A} \to \mathcal{C}$, its right derived functor R^nT is a cohomological ∂-functor by **Proposition 3.53**.

- If the additive functor $T : \mathcal{A} \to \mathcal{C}$ is left exact, its right derived functor $R^n T$ is a cohomological extension of T by **Proposition 3.54**.

And there is also a theorem of existence and uniqueness similar to the homological extension.

Theorem 3.60

If an additive functor $T : \mathcal{A} \to \mathcal{C}$ is left exact, then there is a unique cohomological extension (T^n) of T such that $T^n(E) = 0$ for all injective E and all $n \geq 1$.

Claim 3.61

If an additive functor $T : \mathcal{A} \to \mathcal{C}$ is left exact, then its right derived functor $R^n T(E) = 0$ for all injective E and all $n \geq 1$.
Since we can first select an injective resolution $0 \to E \to E \to 0 \to 0 \to \cdots$ of E, and finnally we have $R^n T(E) = 0$ for all $n \geq 1$.

Corollary 3.62

By **Claim 3.61**, for an additive left exact functor T, its right derived functor $R^n T$ is the unique cohomological extension of T which equal to zero for all injective E and all $n \geq 1$.

3.2.3 Tor and Ext

In this part we will consider two special case of left and right derived functors in R-module category. We will get the left derived functors of additive covarient functors $\square \otimes_R B$, $A \otimes_R \square$ and the right derived functor of $\mathrm{Hom}_R(A, \square)$.
Firstly, we consider two examples of additive functors

- $A \otimes_R \square : \quad {}_R\mathbf{Mod} \to \mathbf{Ab}$ by

$$B \mapsto A \otimes_R B \quad , \quad g \mapsto 1_A \otimes g$$

- $\square \otimes_R B : \quad \mathbf{Mod}_R \to \mathbf{Ab}$ by

$$A \mapsto A \otimes_R B \quad , \quad f \mapsto f \otimes 1_B$$

Lemma 3.63 (Right exactness)

Both functors $A \otimes_R \square$ and $\square \otimes_R B$ are right exact functors.

Proof. Consider the exact sequence $B \xrightarrow{i} C \xrightarrow{p} D \to 0$ in ${}_R\mathbf{Mod}$, apply $A \otimes_R \square$ on it we get

$$A \otimes_R B \xrightarrow{1_A \otimes i} A \otimes_R C \xrightarrow{1_A \otimes p} A \otimes_R D \to 0$$

where $1_A \otimes p$ is surjective since p is surjective, and $\mathsf{Ker}(1_A \otimes p) = A \otimes_R \mathsf{Ker}\, p$, $\mathsf{Im}(1_A \otimes i) = A \otimes_R \mathsf{Im}\, i$, so $\mathsf{Ker}(1_A \otimes p) = \mathsf{Im}(1_A \otimes i)$, the sequence is exact. And we can also prove that $\square \otimes_R B$ is a right exact functor by the same way. ∎

Next we will define **Tor** and **tor** to be the left derived functors of $\square \otimes_R B$ and $A \otimes_R \square$.

Definition 3.64 ($\mathbf{Tor}_n^R(\square, B)$)

Let functor $T = \square \otimes_R B$, then we define

$$\mathbf{Tor}_n^R(\square, B) = L_n T : \mathbf{Mod}_R \to \mathbf{Ab},$$

more precisely, we firstly choose a projective resolution of $A \in \mathrm{Obj}(\mathbf{Mod}_R)$,

$$P = \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \to 0$$

we let functor $T = \Box \otimes_R B$ act on deleted projective resolution P_A,

$$TP_A = P_A \otimes_R B = \to P_2 \otimes_R B \xrightarrow{d_2 \otimes 1_B} P_1 \otimes_R B \xrightarrow{d_1 \otimes 1_B} P_0 \otimes_R B \to 0,$$

finally we take homology and get

$$\mathbf{Tor}_n^R(A, B) = H_n(P_A \otimes_R B) = \frac{\mathsf{Ker}(d_n \otimes 1_B)}{\mathsf{Im}(d_{n+1} \otimes 1_B)}$$

Next we consider the left derived functor $L_n T$ when $T = A \otimes_R \Box$.

Definition 3.65 ($\mathbf{tor}_n^R$)
Let the additive functor $T = A \otimes_R \Box$, then we define

$$\mathbf{tor}_n^R(A, \Box) = L_n T :_R \mathbf{Mod} \to \mathbf{Ab}$$

more precisely, we firstly choose a projective resolution of $B \in \mathrm{Obj}({}_R\mathbf{Mod})$,

$$P = \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} B \to 0$$

and we let T act on P_B,

$$TP_B = A \otimes_R P_B = \to A \otimes_R P_2 \xrightarrow{1_A \otimes d_2} A \otimes_R P_1 \xrightarrow{1_A \otimes d_1} A \otimes_R P_0 \to 0$$

after taking homology, we get

$$\mathbf{tor}_n^R(A, B) = L_n T = \frac{\mathsf{Ker}(1_A \otimes d_n)}{\mathsf{Im}(1_A \otimes d_{n+1})}.$$

Then according to the properties of left derived functors $L_n T$, both **Tor** and **tor** have the following properties.

Proposition 3.66
Both $\mathbf{Tor}_n^R(A, B)$ and $\mathbf{tor}_n^R(A, B)$ are independent of the selection of projective resolutions.

Proposition 3.67
$\mathbf{Tor}_0^R(\Box, B)$ and $\mathbf{tor}_0^R(A, \Box)$ are naturally isomorphic to $\Box \otimes_R$ and $A \otimes_R \Box$, *i.e.*,

$$\mathbf{Tor}_0^R(\Box, B) \cong \Box \otimes_R B \quad \text{and} \quad \mathbf{tor}_0^R(A, \Box) \cong A \otimes_R \Box$$

(since both $\Box \otimes_R$ and $A \otimes_R \Box$ are right exact functors).

Proposition 3.68
Both $\mathbf{Tor}_n^R(A, B)$ and $\mathbf{tor}_n^R(A, B)$ are additive functors.

Proposition 3.69

If $0 \to A' \xrightarrow{i} A \xrightarrow{p} A'' \to 0$ is an exact sequence in $\mathbf{Mod}_R$(or ${}_R\mathbf{Mod}$), then we have the long exact sequences

$$\longrightarrow \mathbf{Tor}_n^R(A',B) \longrightarrow \mathbf{Tor}_n^R(A,B) \longrightarrow \mathbf{Tor}_n^R(A'',B) \xrightarrow{\partial_n} \cdots$$

$$\cdots \xrightarrow{\partial_2} \mathbf{Tor}_1^R(A',B) \longrightarrow \mathbf{Tor}_1^R(A',B) \longrightarrow \mathbf{Tor}_1^R(A',B) \xrightarrow{\partial_1}$$

$$A' \otimes_R B \xrightarrow{(i\otimes 1_B)_*} A \otimes_R B \xrightarrow{(p\otimes 1_B)_*} A'' \otimes_R B \longrightarrow 0$$

and same as **tor**.

Example 3.70

According to **Proposition 3.69**, both **Tor** and **tor** are homological ∂-functors, and they are also the unique homological extensions of $\square \otimes_R B$ and $A \otimes_R \square$.

Except these properties same to L_nT which we have mentioned. There is a relation between **Tor** and **tor**.

Theorem 3.71

Let $A \in obj(\mathbf{Mod}_R)$, $B \in obj({}_R\mathbf{Mod})$, and suppose

$$P = \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \to 0 \quad , \quad Q = \to Q_1 \xrightarrow{d_1'} Q_0 \xrightarrow{\epsilon'} B \to 0$$

are projective resolutions of A and B. Then $H_n(P_A \otimes_R B) \cong H_n(A \otimes_R Q_B)$ for all n, *i.e.*,

$$\mathbf{Tor}_n^R(A,B) \cong \mathbf{tor}_n^R(A,B).$$

Proof. See Rotman's book [3]. ∎

Then we consider about the right derived functor of $\mathrm{Hom}_R(A,\square)$.

Lemma 3.72

$\mathrm{Hom}_R(A,\square)$ is a left exact functor.

Proof. Consider an exact sequence $0 \to B \xrightarrow{i} C \xrightarrow{p} D$ in ${}_R\mathbf{Mod}$, and we apply the functor $\mathrm{Hom}_R(A,\square)$,

$$0 \to \mathrm{Hom}_R(A,B) \xrightarrow{i_*} \mathrm{Hom}_R(A,C) \xrightarrow{p_*} \mathrm{Hom}_R(A,D)$$

, for any morphism $f \in \mathrm{Hom}_R(A,B)$, $i_*f = i \circ f = 0$ if and only if $f = 0$(since i is injective). Thus i_* is injective. And $\mathsf{Ker}\, p_* = \mathrm{Hom}_R(A, \mathsf{Ker}\, p)$, $\mathsf{Im}\, i_* = \mathrm{Hom}_R(A, \mathsf{Im}\, i)$, so $\mathsf{Ker}\, p_* = \mathsf{Im}\, i_*$ (since $\mathsf{Ker}\, p = \mathsf{Im}\, i$). ∎

Definition 3.73 (Ext)

Let the additive functor $T = \mathrm{Hom}_R(A,\square)$, then we define

$$\mathbf{Ext}_R^n(A,\square) = R^nT,$$

more precisely, we choose an injective resolution E of $B \in \mathbf{Mod}_R$

$$E = 0 \to B \xrightarrow{\eta} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \cdots,$$

then let T act on E^B and take homology, we get

$$\mathbf{Ext}_R^n(A,B) = H_n(\mathrm{Hom}_R(A,E^B)) = \frac{\operatorname{Ker} d_*^n}{\operatorname{Im} d_*^{n-1}}$$

The right derived functor $\mathbf{Ext}_R^n(A,\cdot)$ have the all properties of R^nT which we have mentioned.

3.3 Contravariant Right Derived Functor and ext

In this part we will consider the right derived functor of the additive functor $\mathrm{Hom}_R(\square,B) :_R \mathbf{Mod} \to \mathbf{Ab}$. However, the difference between $\mathrm{Hom}_R(\square,B)$ and $\mathrm{Hom}_R(A,\square)$ is that $\mathrm{Hom}_R(\square,B)$is a contravariant functor. So we need to discuss the contravariant right derived functor in general first. And then we can define the contravariant derived functor of $\mathrm{Hom}_R(\square,B)$ to be $\mathbf{ext}_R^n(\square,B)$.

Recall 3.74 (Contravariant functor)

A functor $T : \mathcal{A} \to \mathcal{C}$ is called a contravariant functor if for every morphism $f : A \to A'$ in category $\mathcal{A}$,

$$f : A \to A' \mapsto T(f) : T(A') \to T(A)$$

i.e., $T \in \mathrm{Hom}_R(A,A') \to T(f) \in \mathrm{Hom}_R(TA',TA)$.

Example 3.75

The functor $\mathrm{Hom}_R(\square,B)$ is an additive contravariant functor,

$$\begin{aligned} \mathrm{Hom}_R(\square,B) :_R \mathbf{Mod} &\to \mathbf{Ab} \\ A &\mapsto \mathrm{Hom}_R(A,B) \end{aligned}$$

where the morphisms between the functor,

$$f : A \to A' \longmapsto f_* : \mathrm{Hom}_R(A',B) \to \mathrm{Hom}_R(A,B).$$

Construction 3.76 (Contravariant right derived functor)

Different from the construction of covariant right derived functor, for a contravariant additive functor $T : \mathcal{A} \to \mathcal{C}$, and any $A \in \mathrm{Obj}(\mathcal{A})$ we first select a projective resolution but not an injective resolution.(this is because after acting by contravariant functor, the complex's morphisms will goes in the opposite direction).

$$P = \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \to 0,$$

the we act functor T on the deleted projective resolution,

$$TP_A = 0 \to TP_0 \xrightarrow{Td_1} TP_1 \xrightarrow{Td_2} \cdots$$

(notice that the index of the morphisms of TP_A is +1 compared with the usual complex at same position). Finally we do homology on it, and we can define the right derived functor by

$$(R^nT)A = H^n(TP_A) = \frac{\operatorname{Ker} Td_{n+1}}{\operatorname{Im} Td_n}$$

and for the morphisms $f : A \to A'$ in the abelian category $\mathcal{A}$,

$$(R^nT)f = f_{*n} : (R^nT)A \to (R^nT)A'$$

(where the chain map $(f_n) : P_A \to P'_{A'}$ between the deleted projective resolutions of A and A' is induced by morphism f, and the $(f_* n)$is the induced morphism of $T(f_n) = (Tf_n)$ under homology.)

Similar to the covariant right derived functor, the contravariant derived functor R^nT is also well defined and independent of the choice of projective resolutions, and also have the following properties.

Proposition 3.77 (long exact sequence)

Given a short exact sequence $0 \to A' \xrightarrow{i} A \xrightarrow{p} A'' \to 0$ in Abelian category $\mathcal{A}$ and $T : \mathcal{A} \to \mathcal{C}$ is an additive contravariant functor, then there is a long exact sequence,

$$0 \longrightarrow (R^0T)A'' \xrightarrow{(R^0T)p} (R^0T)A \xrightarrow{(R^0T)i} (R^0T)A' \xrightarrow{\partial_0} \cdots$$

$$\cdots \xrightarrow{\partial_{n-1}} (R^nT)A'' \xrightarrow{(R^nT)p} (R^nT)A \xrightarrow{(R^nT)i} (R^nT)A' \xrightarrow{\partial_n} \cdots$$

Remark 3.78

Since T is a contravariant additive functor,

$$\begin{array}{c} 0 \longrightarrow P_{A'} \xrightarrow{i_*} P_A \xrightarrow{p_*} P_{A''} \longrightarrow 0 \\ \Big\Downarrow T \\ 0 \longrightarrow TP_{A''} \xrightarrow{Tp_*} TP_A \xrightarrow{Ti_*} TP_{A'} \longrightarrow 0 \end{array}$$

where $P_{A'}, P_A, P_{A''}$ are deleted projective resolutions of A', A, A'' and the chain maps i_*, p_* are induced by morphisms i, p. So the direction of the long exact sequnce is opposite to the direction of the given short exact sequence.

Proposition 3.79

By **Proposition** 3.77, the additive contravariant functor R^0T is left exact.

Proposition 3.80

If the additive contravariant functor $T : \mathcal{A} \to \mathcal{C}$ is left exact, then $T \cong R^0T$.

Proposition 3.81

If the additive contravariant functor T is left exact, then there is a long exact sequence,

$$0 \longrightarrow TA'' \xrightarrow{Tp} TA \xrightarrow{Ti} TA' \xrightarrow{\partial_0} \cdots$$

$$\cdots \xrightarrow{\partial_{n-1}} (R^nT)A'' \xrightarrow{(R^nT)p} (R^nT)A \xrightarrow{(R^nT)i} (R^nT)A' \xrightarrow{\partial_n} \cdots$$

Proposition 3.82

The contravariant right derived functor $R^nT(P) = 0$ for all projective P and $n \geq 0$, since we can also select a projective resolution $\to 0 \to 0 \to P \to P \to 0$.

Next we will consider the special case of contravariant right derived functor where $T = \mathrm{Hom}_R(\square, B)$ and $R^nT = \mathbf{ext}_R^n(\square, B)$.

Construction 3.83 ($\mathbf{ext}_R^n$)

Consider the contravariant additive functor $T = \mathrm{Hom}_R(\square, B) :_R \mathbf{Mod} \to \mathbf{Ab}$, then for any $A \in_R \mathbf{Mod}$ we can select a projective resolution

$$P = \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \to 0$$

and deleted resolution $P_A = \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to 0$. Next we act functor $T = \mathrm{Hom}_R(\square, B)$ on it,

$$TP_A = \mathrm{Hom}_R(P_A, B) = 0 \to \mathrm{Hom}_R(P_0, B) \xrightarrow{d^{1*}} \mathrm{Hom}_R(P_1, B) \xrightarrow{d^{2*}} \mathrm{Hom}_R(P_2, B) \xrightarrow{d^{3*}}$$

finally we take homology then we can define

$$\mathbf{ext}_R^n(A, B) = R^nT(A) = H^n(\mathrm{Hom}_R(P_A, B)) = \frac{\mathrm{Ker}\, d^{n+1*}}{\mathrm{Im}\, d^{n*}}$$

where d^{n*} is the induce morphism

$$\begin{aligned} d^{n*} : \mathrm{Hom}_R(P_{n-1}, B) &\to \mathrm{Hom}_R(P_A, B) \\ f &\mapsto f \circ d_n \end{aligned}$$

Since $\mathrm{Hom}_R(\square, B)$ is an additive contravariant functor, $\mathbf{ext}_R^n(\square, B)$ has all properties of contravariant right derived functor. And also $\mathrm{Hom}_R(\square, B)$ is left exact, so $\mathbf{ext}_R^0(\square, B) \cong \mathrm{Hom}_R(\square, B)$ by **Proposition 3.80**. At the same time, there is a relation between $\mathbf{Ext}_R^n(A, \square)$ and $\mathbf{ext}_R^n(\square, B)$, like $\mathbf{Tor}_n^R(A, B) \cong \mathbf{tor}_n^R(A, B)$.

Theorem 3.84

For any left R-modules A, B, *i.e.*, $A, B \in_R \mathbf{Mod}$, we have

$$\mathbf{Ext}_R^n(A, B) \cong \mathbf{ext}_R^n(A, B) \quad \text{natural isomorphism}$$

4 Homological Methods

4.1 Regular Seqences and M-sequences

In this section, we will assume that the all rings considered are Noetherian and the all modules considered are finitely generated module.

Definition 4.1 (R-sequence)

R is a ring, then a sequence of elements $x_1, \cdots, x_n$ in R is called a regular sequence (or R-sequence) if

- $(x_1, \cdots, x_n) \neq R$, (*i.e.*, $(x_1, \cdots, x_n)$ is a proper ideal in R).
- x_i is a nonzerodivisor on $R/(x_1, \cdots, x_{i-1})$ for all $1 \leq i \leq n$, (*i.e.*, for any elements $r \notin (x_1, \cdots x_{i-1})$, $x_i r \notin (x_1, \cdots, x_{i-1})$).

We can also give the similar definition on R-module.

Definition 4.2 (M-Sequence)

R is a ring, let M be an R- module. Then a sequence of elements $x_1, x_2, \cdots, x_n$ in R is called a regular sequence on M (or M-sequence) if

- $(x_1, x_2, \cdots, x_n)M \neq M$,

- x_i is a nonzerodivisor on $M/(x_1, \cdots, x_{i-1})M$ for all $1 \leq i \leq n$,
  (*i.e.*, for any elements $m \notin (x_1, \cdots, x_{i-1})M$ in M, $x_i m \notin (x_1, \cdots, x_{i-1})M$)

Example 4.3

Consider the polynomial ring $R = K[x_1, \cdots, x_n]$ where K is a field, then $x_1, \cdots, x_n$ is an $R - sequence$ on R, since for all $1 \leq i \leq n$, if $p \notin (x_1, \cdots, x_{i-1})$, then $x_i p \notin (x_1, \cdots, x_{i-1})$.

4.2 Kazual Complex

Next we will construct the Koszul complex which is an important tools, and we will have a criterion of the regular sequence by using Koszul complex.

Definition 4.4 (Koszul Complex)

R is a ring and there is a sequnce $x_1, \cdots, x_n$ in R, its Koszul complex is defined by

$$\mathbf{K}.(x_1, \cdots, x_n) = 0 \to F_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} 0.$$

where $F_0, F_1, \cdots, F_n$ are free R-modules with $\binom{n}{i}$ generators and for each $F_i = \bigoplus_{\lambda \in \bigwedge_i} R\lambda$, the set of generators is

$$\bigwedge_i = \{Z_{\alpha_1} \wedge Z_{\alpha_2} \wedge \cdots \wedge Z_{\alpha_i} : 1 \leq \alpha_1 < \alpha < \cdots < \alpha_i \leq n\}.$$

($Z_{\alpha_1}, \cdots, Z_{\alpha_i}$ are just symbols to represent the generators in wedge algebra).
The morphisms $\partial_i : F_i \to F_{i-1}$ in Koszul complex $\mathbf{K}.(x_1, \cdots x_n)$ is defined by for each generator $Z_{\alpha_1} \wedge \cdots \wedge Z_{\alpha_i}$ in F_i,

$$\partial_i(Z_{\alpha_1} \wedge Z_{\alpha_2} \wedge \cdots Z_{\alpha_i}) = \sum_{j=1}^{i} (-1)^j x_{\alpha_j} (Z_{\alpha_1} \wedge \cdots \wedge Z_{\alpha_{j-1}} \wedge \widehat{Z_{\alpha_j}} \wedge Z_{\alpha_{j+1}} \wedge \cdots Z_{\alpha_i}),$$

where $\widehat{Z_{\alpha_j}}$ means this term is omitted and obviously every $Z_{\alpha_1} \wedge \cdots \wedge Z_{\alpha_{j-1}} \wedge \widehat{Z_{\alpha_j}} \wedge Z_{\alpha_{j+1}} \wedge \cdots Z_{\alpha_i}$ is a generator of F_{i-1}. The morphisms can also be seen as an R-linear translation between free R-modules and it can be represented by matrix.

***Remark* 4.5**

$\mathbf{K}.(x_1, \cdots, x_n)$ is a complex, since for each $i \geq 1$

$$\begin{aligned}
\partial_{i-1}\partial_i(Z_{\alpha_1} \wedge \cdots \wedge Z_{\alpha_i}) &= \partial_{i-1}(\sum_{j=1}^{i} (-1)^j x_{\alpha_j} (Z_{\alpha_1} \wedge \cdots \wedge \widehat{Z_{\alpha_{j-1}}} \wedge \cdots \wedge Z_{\alpha_i})) \\
&= \sum_{j=1}^{i} (-1)^j x_{\alpha_j} \partial_{i-1}(Z_{\alpha_1} \wedge \cdots \wedge \widehat{Z_{\alpha_j}} \wedge \cdots \wedge Z_{\alpha_i}) \\
&= 0
\end{aligned}$$

(*i.e.*, $\partial_{i-1}\partial_i = 0$).

Definition 4.6 (Koszul Complex on R-Module)

M is a finitely generated R-module, and a sequence $x_1, \cdots, x_n$ in R, then we define the Koszul complex on M by

$$\mathbf{K}.(x_1, \cdots, x_n; M) := \mathbf{K}.(x_1, \cdots, x_n) \otimes_R M.$$

Remark 4.7

More precisely, $\mathbf{K}.(x_1, \cdots, x_n) = 0 \to F_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} 0$, then

$$\mathbf{K}.(x_1, \cdots, x_n; M) = 0 \to F_n \otimes_R M \xrightarrow{\partial_n \otimes 1_M} F_{n-1} \otimes_R M \xrightarrow{\partial_{n-1} \otimes 1_M} \cdots \xrightarrow{\partial_1 \otimes 1_M} F_0 \otimes_R M \xrightarrow{\partial_0 \otimes 1_M} 0$$

where the induced morphisms,

$$\begin{aligned} \partial_i \otimes 1_M : F_i \otimes_R M &\longrightarrow F_{i-1} \otimes_R M \\ v \otimes m &\longmapsto \partial_i(v) \otimes m, \end{aligned}$$

and the homology on $\mathbf{K}.(x_1, \cdots, x_n; M)$ is

$$\begin{aligned} H_i(\mathbf{K}.(x_1, \cdots, x_n; M)) &= H_i(\mathbf{K}.(x_1, \cdots, x_n) \otimes_R M) \\ &= \frac{\mathsf{Ker}(\partial_i \otimes 1_M)}{\mathsf{Im}(\partial_{i+1} \otimes 1_M)} \end{aligned}$$

Example 4.8

$\mathbf{K}.(x, y, z) = 0 \to F_3 \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} 0$ where

- $F_1 \simeq R$ generated by $Z_x \wedge Z_y \wedge Z_z$.
- $F_2 \simeq R \oplus R \oplus R$ generated by $Z_x \wedge Z_y$, $Z_x \wedge Z_z$ and $Z_y \wedge Z_z$.
- $F_3 \simeq R \oplus R \oplus R$ generated by Z_x, Z_y and Z_z.

And the morphisms are

- $\partial_3(Z_x \wedge Z_y \wedge Z_z) = -xZ_y \wedge Z_z + yZ_x \wedge Z_z - zZ_x \wedge Z_y$, *i.e.*, $\partial_3 : r \mapsto (-rz, ry, -rx))$ and represented by matrix $(-z, y, -x)$.
- – $\partial_2(Z_x \wedge Z_y) = -xZ_y + yZ_x$
  – $\partial_2(Z_x \wedge Z_z) = -xZ_z + zZ_x$
  – $\partial_2(Z_y \wedge Z_z) = -yZ_z + zZ_x$

  i.e., $\partial_2 : (r_1, r_2, r_3) \mapsto (r_1 y + r_2 z, r_3 z - r_1 x, -r_2 x - r_3 y)$ and represented by matrix $\begin{pmatrix} y & z & o \\ -x & 0 & z \\ 0 & -x & -y \end{pmatrix}$
- $\partial_1(Z_x) = -x$, $\partial_1(Z_y) = -y$, $\partial_1(Z_z) = -z$, *i.e.*, $\partial_1 : (r_1, r_2, r_3) \mapsto -(r_1 x + r_2 y + r_3 z)$ and represented by matrix $(-x, -y, -z)$.

Definition 4.9 (Tensor product of complexes)

Given two complexes $\mathcal{F}$ and $\mathcal{G}$ of R-modules,

$$\begin{aligned} \mathcal{F} : \cdots \to F_i \xrightarrow{\varphi_i} F_{i-1} \xrightarrow{\varphi_{i-1}} \cdots \\ \mathcal{G} : \cdots \to G_i \xrightarrow{\psi_i} G_{i-1} \xrightarrow{\psi_{i-1}} \cdots \end{aligned}$$

their tensor product is

$$\mathcal{F} \otimes \mathcal{G} : \cdots \to \sum_{i+j=k} F_i \otimes_R G_j \xrightarrow{d_k} \sum_{i+j=k-1} F_i \otimes_R G_j \xrightarrow{d_{k-1}} \cdots$$

where the morphisms

$$d_k = \begin{cases} F_i \otimes G_j \to F_{i-1} \otimes G_j, & \text{by } \varphi_i \otimes 1 \\ F_i \otimes G_j \to F_i \otimes G_{j-1}, & \text{by } (-1)^i 1 \otimes \psi_i \\ \quad \text{else} \quad , & \text{by} \quad 0 \end{cases} \tag{4}$$

,*i.e.*, for any element $f_i \otimes g_j \in F_i \otimes G_j$,

$$d_k(f_i \otimes g_j) = \varphi_i(f_i) \otimes g_j + (-1)^i f_i \otimes \psi_j(g_j)$$

Remark 4.10

The Koszul complex $\mathbf{K}.(x_1, \cdots, x_n; R) \simeq \mathbf{K}.(x_1; R) \otimes \mathbf{K}.(x_2; R) \otimes \cdots \otimes \mathbf{K}.(x_n; R)$, thus

$$\mathbf{K}.(x_1, \cdots, x_n) \simeq \mathbf{K}.(x_1, \cdots x_i) \otimes \mathbf{K}.(x_{i+1}, \cdots, x_n) \qquad \text{for any} \quad 1 \leq i \leq n,$$

(the details of vanishing this isomorphism can be found in Eisenbud's book in propsition 19 by using the properties of skew-commutative algebras).

Claim 4.11

Given two complex

$$\mathcal{F} : \cdots \to F_i \xrightarrow{\varphi_i} F_{i-1} \xrightarrow{\varphi_{i-1}} \cdots$$
$$\mathcal{G} : \cdots \to G_i \xrightarrow{0} G_{i-1} \xrightarrow{0} \cdots$$

where the all morphisms of $\mathcal{G}$ are zero. Then $H_k(\mathcal{F} \otimes \mathcal{G}) \simeq \sum_{i+j=k} H_i(\mathcal{F}) \otimes G_j$

Proof. Since $d_k(f_i \otimes g_j) = \varphi_i(f_i) \otimes g_j + (-1)^i f_i \otimes 0 = \varphi_i(f_i) \otimes g_j$, we have $\mathsf{Ker}\, d_k = \sum_{i+j=k} \mathsf{Ker}\, \varphi_i \otimes G_j$ and $\mathsf{Im}\, d_{n+1} = \sum_{i+j=k+1} \mathsf{Im}(\varphi_i) \otimes G_j$, so by the properties of tensor product, the homology is

$$H_k(\mathcal{F} \otimes \mathcal{G}) = \frac{\mathsf{Ker}\, d_k}{\mathsf{Im}_{d_{k+1}}} \simeq \sum_{i+j=k} H_i(\mathcal{F}) \otimes G_j$$

■

4.3 Depth

Next we will discuss the deep relation between regular sequence and the Koszul complex.

Definition 4.12 (Depth)

I is an ideal of ring R and M is a finitely generated R-module with $IM \neq M$, then the depth of I on M is defined by

$$\mathit{depth}(I, M) = \text{ the length of the maximal } M\text{-sequence in } I$$

Remark 4.13

To make this definition be well defined, we need verify that every M-sequence in I has same length. We will finally prove this by using the Koszul complex in **Theorem 4.18**.

Lemma 4.14

If M is an R-module, $y_1, \cdots, y_r$ are elements of the ideal $(x_1, \cdots, x_n)$ where $x_1, \cdots, x_n \in R$, then for each $1 \le i \le n + r$,

$$H_i(\mathbf{K}.(x_1, \cdots, x_n, y_1, \cdots, y_r) \otimes M) \simeq \sum_{i=j+k} H_j(\mathbf{K}.(x_1, \cdots, x_n) \otimes M) \otimes \bigwedge^k R^r$$

(where $\bigwedge^k R^r$ is a free R-module generated by $\{Z_{\alpha_1} \wedge \cdots \wedge Z_{\alpha_k} : 1 \le \alpha_1 < \cdots < \alpha_k \le r\}$).

Proof. First consider the free module, $R^n \oplus R^r$, and suppose each $y_i = \sum_j a_{ij} x_j$ (since each $y_i \in (x_1, \cdots, x_n)$) let matrix $A = (a_ij)$, then there is an automorphism represented by matrix,

$$\begin{pmatrix} I & 0 \\ -A & I \end{pmatrix} : (x_1, \cdots, x_n, y_1 \cdots, y_r) \longmapsto (x_1, \cdots, x_n, 0, \cdots, 0),$$

(this is an automorphism since the matrix is full rank). So by the functoriality of Koszul complex we have

$$\mathbf{K}.(x_1, \cdots, x_n, y_1, \cdots, y_n) \simeq \mathbf{K}.(x_1, \cdots, x_n, 0, \cdots, 0),$$

then according to the *Remark* 4.10, we have

$$\mathbf{K}.(x_1, \cdots, x_n, y_1, \cdots, y_n) \simeq \mathbf{K}.(x_1, \cdots, x_n, 0, \cdots, 0) \simeq \mathbf{K}.(x_1, \cdots, x_n) \otimes \mathbf{K}.(0, \cdots, 0).$$

and the Koszul complex of $0, \cdots, 0$ is

$$\mathbf{K}.(0, \cdots, 0) = 0 \to \bigwedge^r R^r \xrightarrow{0} \cdots \xrightarrow{0} \bigwedge^1 R^r \xrightarrow{0} \bigwedge^0 R^r \xrightarrow{0} 0$$

so by the **Claim 4.11** we get

$$\begin{aligned} H_i(\mathbf{K}.(x_1, \cdots, x_n, y_1, \cdots, y_r) \otimes M) &\simeq H_i(\mathbf{K}.(x_1, \cdots, x_n) \otimes M \otimes \mathbf{K}.(0, \cdots, 0)) \\ &\simeq \sum_{i=j+k} H_j(\mathbf{K}.(x_1, \cdots, x_n) \otimes M) \otimes \bigwedge^k R^r \end{aligned}$$

∎

Corollary 4.15

$H_i(\mathbf{K}.(x_1, \cdots, x_n, y_1, \cdots, y_r) \otimes M) = 0$ if and only if

$$H_j(\mathbf{K}.(x_1, \cdots, x_n) \otimes M) = 0 \quad \text{for all } i - r \le j \le i$$

Proof. The proof is naturally, by **Lemma 4.14** and $H_j(\mathbf{K}.(x_1, \cdots, x_n) \otimes M) \otimes \bigwedge^k R^r = 0$ if and only if $H_j(\mathbf{K}.(x_1, \cdots, x_n) \otimes M)$. ∎

Now we will introduce the important theorem which explain the relation between the depth and Koszul complex. We first give two claims but without proof, the details of proof can be found in Eisenbud's book [1].

Claim 4.16

If $x_1, \cdots, x_r$ is an M-sequence in ideal $I = (x_1, \cdots, x_n)$, then $H_k(\mathbf{K}.(x_1, \cdots, x_n) \otimes M) = 0$ for all $k > n - r$. In particular, if $x_1, \cdots, x_r$ is a maximal M-sequence in ideal I, and $IM \neq M$, then $H_{n-r}(\mathbf{K}.(x_1, \cdots, x_n) \otimes M) \neq 0$

Claim 4.17

If $(x_1, \cdots, x_n)M = M$, then $H_i(\mathbf{K}.(x_1, \cdots, x_n) \otimes M) = 0$ for all i.

Theorem 4.18

Given an R-module M, if

$$H_i(\mathbf{K}.(x_1, \cdots, x_n) \otimes M) = 0 \quad \text{for all } i > r$$

and

$$H_r(\mathbf{K}.(x_1, \cdots, x_n) \otimes M) \neq 0,$$

then every maximal M-sequence in ideal $I = (x_1, \cdots, x_n)$ has length $n - r$, *i.e.*, $depth(I, M) = n - r$.

Proof. Suppose $y_1, \cdots, y_m$ is an maximal M-sequence in ideal $I = (x_1, \cdots, x_n)$. And by our hypothesis r is the biggest integer such that $H_i(\mathbf{K}.(x_1, \cdots, x_n) \otimes M) \neq 0$, then by **Corollary 4.15**, $r + m$ is the biggest integer such that $H_i(\mathbf{K}.(x_1, \cdots, x_n, y_1, \cdots, y_m) \otimes M) \neq 0$. Hence according to **Claim 4.17** $(x_1, \cdots, x_n)M \neq M$, then by **Claim 4.16**, $H_i(\mathbf{K}.(x_1, \cdots, x_n, y_1, \cdots, y_m) \otimes M) = 0$ for all $i \geq m + n - m = n$, and $H_n(\mathbf{K}.(x_1, \cdots, x_n) \otimes M) \neq 0$, *i.e.*, n is the largest integer such that $H_i(\mathbf{K}.(x_1, \cdots, x_n, y_1, \cdots, y_m) \otimes M) \neq 0$. Thus $n = r + m$, we get $m = n - r$, *i.e.*, the every maximal M-sequence has length $n - r$. ∎

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