Riemannian Geometry and Lie Group, Lie Algebra

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Abstract

In the first part, we will firstly introduce the construction of the Riemannian manifold by using the Riemannian metric and the representation of the Riemannian metric. Next, we concentrate on the concepts of connections and covariant derivatives and their relations, which are basic concepts to build the geodesics in the Riemannian manifold. Then we will introduce the concept of geodesics which is useful to describe the 'shape' of a Riemannian manifold. Finally, we introduce the Riemannian normal coordinates, and these coordinates have some good properties with geodesics. In the second part, we will just introduce the basic constructions of the Lie group and Lie algebra with some important examples.

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1 Riemannian Geometry

1.1 Riemannian Manifold

The Riemannian manifold is a smooth manifold endowed with a Riemannian metric, and we will firstly consider the concept of Riemannian metric and its representations.

1.1.1 Riemannian Metric

Definition 1.1 (Riemannian metric)

Given an smooth manifold M, and TM is its tangent bundle. Then a Riemannian metric on M is a smooth covariant 2-tensor field $g \in \mathcal{T}^2(M)$, (*i.e.*, for each point $p \in M$ and $v_1, V_2 \in T_pM$, the function $g_p(v_1, v_2)$ is bilinear and symmetric) such that $g_p(v, v) \geq 0$ for each point $p \in M$ and with equality if and only if $v = 0$[1].

Remark 1.2

For each point $p \in M$, the Riemannian metric (a 2-tensor g_p) also can be viewed as an inner product on each tangent linear space T_pM by

$$\langle v_1, v_2 \rangle_g = g_p(v_1, v_2) \qquad \text{for } v_1, v_2 \in T_pM$$

since the conditions of Riemannian metric are same to the conditions of inner product on each linear space T_pM, that is for any vectors $v, w, x \in T_pM$,

- (Symmetry) $\langle v, w \rangle_g = \langle w, v \rangle_g$.
- (Bilinear) $\langle \alpha v + \beta w, x \rangle_g = \alpha \langle v, x \rangle_g + \beta \langle w, x \rangle_g$
- (Positive) $\langle v, v \rangle_g \geq 0$ and $\langle v, v \rangle_g = 0$ if and only if $v = 0$.

Proposition 1.3

For every smooth manifold M, there also exists a Riemannian metric g on M.

Proof.
- We first consider a local coordinates system $(U_\alpha, x_\alpha)_{\alpha \in A}$ (A is an index set), where $\{U_\alpha\}_{\alpha \in A}$ is an open cover of manifold M and each $x_\alpha : U_\alpha \to \mathbb{R}^n$ is a diffeomorphism.
- Then according to the identity partition theorem [2], there is a smooth partition of unitary subordinate to $\{U_\alpha\}_{\alpha \in A}$, which is $\{\psi_\alpha\}_{\alpha \in A}$ such that
  - $0 \leq \psi_\alpha \leq 1$, for all $\alpha \in A$ and $x \in M$.
  - $\operatorname{supp} \psi_\alpha \subseteq U_\alpha$ for each $\alpha \in A$, *i.e.*, $\psi_\alpha(p) = 0$ for all $p \notin U_\alpha$.
  - Locally finited.
  - $\sum_{\alpha \in A} \psi_\alpha(x) = 1$ for all $x \in M$.
- Then we define a Riemannian metric by

$$g = \sum_{\alpha \in A} \psi_\alpha \cdot g_\alpha \qquad \text{where } g_\alpha \text{ is a smooth bilinear function on } TM|_{U_\alpha}.$$

  ($g_\alpha(v, v) \geq 0$ and $g_\alpha(v, v) = 0$ iff $v = 0$). Then since $supp\psi_\alpha \subseteq U_\alpha$, the function $g_\alpha \cdot \psi_\alpha$ is smooth in global M. Next we need to prove that such bilinear functions g_α are exists.
- For each local coordinate map x_α,

$$x_\alpha : U_\alpha \to \mathbb{R}^n \qquad \text{and} \qquad dx_\alpha : TM|_{U_\alpha} \to x_\alpha(U_\alpha) \times \mathbb{R}^n$$

  are both diffeomorphisms. And for each fibre T_pM, $p \in U_\alpha$, the map $dx_\alpha : T_pM \to x_\alpha(p) \times \mathbb{R}^n$ is a linear isomorphism. Then the metric of T_pM can be defined by the inner product of the Euclidean Space $\mathbb{R}^n$,

$$g_\alpha(v, w) = \langle dx_\alpha(v), dx_\alpha(w) \rangle \quad \text{where } v, w \in T_pM \text{ and } p \in U_\alpha$$

  so g_α is a smooth bilinear functions on $TM|_{U_\alpha}$(since inner product in $\mathbb{R}^n$ is smooth, bilinear, $\langle v, v \rangle \geq 0$, $\langle v, v \rangle = 0$ iff $v = 0$, and dx_α is a diffeomorphism).

■

Remark 1.4

The Riemannian metric which we have already constructed is independent of the choice of local coordinate system.[1].

By the definition of Riemannian metric, we can defined the Riemannian manifold now by adding an Riemannian metric on the smooth manifold.

Definition 1.5

A Riemannian manifold (M, g) is a smooth manifold M endowed with a Riemannian metric.

Example 1.6 (Euclidean Metric)

Euclidean manifold $(\mathbb{R}^n, \overline{g})$ is a Riemannian manifold. Since each $T_x\mathbb{R}^n \simeq \mathbb{R}^n$, and by the standard coordinate $(x^1, x^2, \cdots, x^n)$ of $\mathbb{R}^n$,

$$T_x\mathbb{R}^n = \text{Span}\{\partial_1|_x, \cdots, \partial_n|_x\},$$

then the Riemannian metric $\overline{g}$ is defined by for any $v = \sum_i v^i \partial_i|_x$ and $w = \sum_j w_j \partial_j|x$,

$$\langle v, w\rangle_{\overline{g}} = \sum_{i=1}^{n} v^i w^i.$$

Next we will introduce the concept of isometry, which is stronger than diffeomorphism, since diffeomorphism is an equivalence on smooth manifold, however isometry need to preserve the Riemannian metric additionally.

Definition 1.7 (Isometry)

A diffeomorphism $\varphi : M \to \widetilde{m}$ is an isometry, (*i.e.*, $\varphi : (M, g) \to (\widetilde{M}, \tilde{g})$), if

$$\varphi^* \tilde{g} = g,$$

(where $\varphi^* \tilde{g}(v, w) = \tilde{g}(d\varphi(v), d\varphi(w))$, $v, w \in TM$).

Remark 1.8

To be more specific, φ is an isometry if and only if for each $p \in M$, the map $d\varphi_p : T_pM \to T_{\varphi(p)}\widetilde{M}$ such that

$$g_p(v, w) = \tilde{g}_{\varphi(p)}(d\varphi_p(v), d\varphi_p(w)) \text{ for all } v, w \in T_pM$$

i.e., $d\varphi_p$ is a linear isometry between linear metric space for each $p \in M$.

Remark 1.9

Isometry is an equivalence since

- The inverse map φ^{-1} of a diffeomorphism φ is also a diffeomorphism, and because $d\varphi$ is a linear isometry, the differential of the inverse map $d\varphi^{-1}$ is also a linear isometry.
- Consider $M \xrightarrow{\varphi_1} \widetilde{M} \xrightarrow{\varphi_2} \widetilde{\widetilde{M}}$, if φ_1 and φ_2 are isometry, then it is easy to verify that $\varphi_2\varphi_1$ is also an isometry.

Definition 1.10 (Local isometry)

Given two Riemannian manifold (M, g) and $(\widetilde{M}, \tilde{g})$. A smooth map $\varphi : M \to \widetilde{M}$ is called a local isometry if for each $p \in M$, there exists a neighborhood $p \in U \subseteq M$ such that

$$\varphi|_U : U \to V \quad \text{is an isometry onto a open subset } V \subseteq \widetilde{M}.$$

Next we can define a special kind of Riemannian manifold.

Definition 1.11 (Flat Riemannian Manifold)

A Riemannian manifold is flat if it is locally isometric to the Euclidean space $(\mathbb{R}^n, \overline{g})$.

Remark 1.12

Every Riemannian manifold (M, g) is also diffeomorphic to the Euclidean space $\mathbb{R}^n$ (since it is also a smooth manifold), however (M, g) may not be a flat Riemannian manifold.

1.1.2 The Local Representation of Metric

Since a Riemannian metric on each fiber T_pM (a linear space) can be seen as an inner product. So we can use matrix to represent the Riemannian metric on each fiber, and since the inner product is a smooth 2-tensor field on M, the matrix should be 'change smoothly' on the global M, (which means a matrix-valued function).

Construction 1.13 (Matrix reperesentation of metric)

Suppose $(x^1, \cdots, x^n)$ is a local smooth coordinate of Riemannian manifold (M, g), then $dx^1, \cdots, dx^n$ are basis of T_PM. Any two vectors $v_p, w_p \in T_pM$ can be represented by

$$v_p = v_p^1 dx^1 + \cdots + v_p^n dx^n \quad \text{and} \quad w_p = w_p^1 + \cdots + w_p^n dx^n,$$

then since the inner product (Riemannian metric) is bilinear, we have

$$\begin{aligned}
\langle v_p, w_p \rangle_g &= \sum v_p^i w_p^j \langle v_p, w_p \rangle_g \\
&= \sum v_p^i w_p^j g_{ij} \\
&= \begin{pmatrix} v_p^1, \cdots, v_p^n \end{pmatrix} \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix} \begin{pmatrix} w_p^1 \\ \vdots \\ w_p^n \end{pmatrix}
\end{aligned}$$

where $g_{ij} = \langle dx^i, dx^j \rangle_g$ and each $g_{ij} = g_{ij}(p)$ is a smooth function on M. Hence, the Riemannian metric can be uniquely determined by the matrix-valued function,

$$G = \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix}$$

Remark 1.14

- Every g_{ij} in matrix G is smooth function of points p in M.
- The Riemannian metric is symmetrical, so $g_{ij} = \langle dx^i, dx^j \rangle_g = \langle dx^j, dx^i \rangle_g = g_{ji}$, which means that the represented matrix are always symmetric matrix.
- The Riemannian metric satisfies $\langle v_p, v_p \rangle \geq 0$ and with equality if and only if $v_p = 0$. Thus $g_{ii}(p) > 0$ for all i and $p \in M$, which means the represented matrix should always be non-singular.

Example 1.15 (Euclidean metric)

In an Euclidean space $\mathbb{R}^n$, the matrix representation of Euclidean metric $\overline{g}$ is the unit $n \times n$ matrix,

$$\overline{g} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cdots & \\ & & & 1 \end{pmatrix}$$

, *i.e.*, $\overline{g} = \delta_{ij} dx^i dx^j$ where

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

Given a Riemannian metric g on a smooth manifold, we have the inner product at the same times. Then by the inner product, we will have the orthogonal relation defined by $\langle v, w \rangle_g = 0$ and the norm defined by $|v| = \langle v, v \rangle_g$.

Definition 1.16 (Orthonormal frame)

A local frame for Riemannian manifold (M, g) is (E_i) on an open subset $U \subseteq M$, then (E_i) is an orthonormal frame if $E_1|_p, \cdots, E_n|_p$ are orthonormal in T_pM for each $p \in U$, *i.e.*,

$$\langle E_i, E_j \rangle_g = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

Proposition 1.17

(M, g) is a Riemannian n-manifold (with or without bound), and (X_i) is a smooth local frame on open set $U \subseteq M$, then there exist a local orthonormal frame (E_i) on U, *i.e.*, $E_1, \cdots, E_n$ are orthogonal, each $|E_i| = 1$ and

$$\text{Span}\{E_1|_p, \cdots, E_n|_p\} = \text{Span}\{X_1|_p, \cdots, X_n|_p\} = T_pM \quad \text{for each } p \in U.$$

In particular, for each point $p \in M$, there exists a smooth local orthonormal frame (E_i) on $TM|_U$ over a neighborhood $p \in U \subseteq M$.

Proof.

- The Gram-schmidt orthogonalization is a smooth translation on variety $p \in U$, then we get

$$\{X_1|_p, \cdots, X_n|_p\} \xrightarrow{\text{Gram-schmidt orthogonalization}} \{E_1|_p, \cdots, E_n|_p\}$$

  where each $E_i|_p$ is smooth on $p \in U$ and $\{E_1|_p, \cdots, E_n|_p\}$ is an orthonormal basis for all $p \in U$.
- For each point, there is a smooth local frame $\{dx^1, \cdots, dx^n\}$, where $(x^1, \cdots, x^n)$ is a local coordinate for a neighborhood $p \in U \subseteq M$. Then by applying Gram-schmidt orthogonalization, we get an orthonormal frame on neighborhood U.

∎

1.1.3 Riemannian Submanifold

Lemma 1.18 (Induced Riemannian metric)

Given a Riemannian manifold $(\widetilde{M}, \tilde{g})$ (with or without boundary) and a smooth manifold M with a smooth map,

$$F : M \to \widetilde{M}.$$

Then the induced 2-tensor field $g = F^*\tilde{g}$ is a Riemannian metric on M if and only if F is an immersion. (Where $g(v, w) = \tilde{g}(dF(v), dF(w))$).

Proof.
- ($\Leftarrow$) If smooth map F is an immersion, then $dF_p : T_pM \to T_p\widetilde{M}$ is a linear isomorphism for each $p \in M$. So $\langle v, w\rangle_g = \langle dF(v), dF(w)\rangle_{\tilde{g}}$ is a well-defined inner product on each $T_pM, p \in M$ and dF is smooth on M. Hence $g = F^*\tilde{g}$ is a Riemannian metric on $\widetilde{M}$.
- ($\Rightarrow$) If smooth map F is not an immersion, then $dF_p : T_pM \to T_p\widetilde{M}$ is not injective for some $p' \in M$. Thus there are two different vectors v, w in $T_{p'}M$ such that $dF_{p'}(v) = dF_{p'}(w)$. And we consider the induced inner product

$$\begin{aligned}\langle v - w, v - w\rangle_g &= \langle dF_{p'}(v - w), dF_{p'}(v - w)\rangle_{\tilde{g}} \\ &= \langle dF_{p'}(v) - dF_{p'}(w), dF_{p'}(v) - dF_{p'}(w)\rangle_{\tilde{g}} \\ &= \langle 0, 0\rangle_{\tilde{g}} = 0.\end{aligned}$$

  However, $v - w \neq 0$ since $v \neq w$, so the induced inner is not well-defined. Hence, $F^*\tilde{g}$ is not a Riemannian metric on M.

∎

Remark 1.19

$F : (M, g) \to (\widetilde{M}, \tilde{g})$ is an isometry, where the metric g is induced by immersion F.

Thus by the Lemma 1.18, we can define the induced metric under an immersion.

Definition 1.20 (Induced map)

Given an immersion $F : M \to \widetilde{M}$ and $(\widetilde{M}, \tilde{g})$ is a Riemannian manifold. Then the metric g induced by F is

$$g = F^*\tilde{g} \quad \text{where} \quad g(v, w) = \tilde{g}(dF(v), dF(w))$$

Definition 1.21 (Isometric immersion and embedding)

Given two Riemannian manifold (M, g) and $(\widetilde{M}, \tilde{g})$,
- If the immersion $F : M \to \widetilde{M}$ satisfies $g = F^*\tilde{g}$, then F is called an isometric immersion.
- If the embedding $F : M \to \widetilde{M}$ satisfies $g = F^*\tilde{g}$, then F is called am isometric embedding.

Finally we can give the definition of Riemannian submanifold in an appropriate way.

Definition 1.22 (Riemannian Submanifold)

(M, g) is a Riemannian submanifold of $(\widetilde{M}, \tilde{g})$ if $M \subseteq \widetilde{M}$ is an immersed submanifold (where the inclusion map (immersion) is $\tau : M \to \widetilde{M}$) and $g = \tau^*\tilde{g}$.

We have already defined the Riemannian submanifold and by the Riemannian metric we have orthogonal relation in each fiber, so we can separate each fiber to the image linear subspace and normal space.

Definition 1.23 (Normal vectors)

Given a Riemannian manifold $(\widetilde{M}, \tilde{g})$ is a Riemannian manifold and $M \subseteq \widetilde{M}$ is a smooth submanifold ($F : M \to \widetilde{M}$ is an immersion). At a point $p \in M$, the linear injection

$$\begin{aligned}dF_p : T_pM &\longrightarrow T_{F(p)}\widetilde{M} \\ v &\longmapsto dF_p(v).\end{aligned}$$

Then a tangent vector $w \in T_p\widetilde{M}$ is normal to M if

$$\langle dF_p(v), w\rangle_{\tilde{g}} = 0 \quad \text{for all } v \in T_pM,$$

(*i.e.*, w is orthogonal to the image $dF_p(T_pM)$ in the linear space $T_{F(p)}\widetilde{M}$ under metric (inner product) $\tilde{g}$).

Definition 1.24 (Normal space)

The linear subspace of $T_{F(p)}\widetilde{M}$ consists of all tangent vectors $v \in T_pM$ which are normal to M at point $p \in M$ is called the normal space at point p, denoted by $N_p = (T_PM)^{\perp}$.

***Remark* 1.25**

Actually $N_p = (dF_p(T_pM))^{\perp}$ is a subspace of $T_{F(p)}\widetilde{M}$, and

$$T_{F(p)}\widetilde{M} = dF_p(T_pM) \oplus N_pM.$$

After consider the normal space on each fiber, we will consider the normal bundle and field which are defined on global M.

Definition 1.26 (Normal bundles)

The normal bundle of submainifold M in $(\widetilde{M}, \tilde{g})$ is defined by

$$NM = \bigcup_{p\in M} N_pM$$

(where the normal bundle NM is well defined (smooth on M), since $TM = \bigcup_{p\in U} T_pM$, $T\widetilde{M}$ are both smooth on M and dF_p is smooth.)

Definition 1.27 (Normal vector field)

Given a normal vector bundle, we smoothly select vectors $p \in T_pM$ for each $p \in M$, then we will get a normal vector field.

Proposition 1.28 (Relation between $T\widetilde{M}$ and NM)

$(\widetilde{M}, \tilde{g})$ is a Riemannian m-manifold and M is a submanifold of $\widetilde{M}$ with dimension n. Then the normal boundle NM is a subbundle of $T\widetilde{M}|_M$ with rank $m - n$. And there are two smooth boundle homomorphisms (like two projections of vector bundle),

$$\begin{tikzcd} & T\widetilde{M}|_M \arrow[dl, "\pi^{\top}"'] \arrow[dr, "\pi^{\perp}"] & \\ TM & & NM \end{tikzcd}$$

where $\pi^{\top}$ is called tangential and $\pi^{\perp}$ is called normal projection.
(Actually the symbol $T\widetilde{M}|_M$ means $T\widetilde{M}|_{F(M)}$ where $F : M \to \widetilde{M}$ is the immersion).

After discuss the induced Riemannian metric of a submersion, we also need to discuss the submersion under Riemannian metric. At first, considering a smooth immersion

$$\pi : \widetilde{M} \to M$$

where $(\widetilde{M}, \tilde{g})$ is a Riemannian manifold. Then for each $x \in \widetilde{M}$, each fiber $T_x\widetilde{M}$ of the tangent boundle $T\widetilde{M}$ can be separated by following objects.

Definition 1.29 (Vertical tangent space)

The vertical tangent space at point $x \in \widetilde{M}$ is defined by

$$V_x = \operatorname{Ker} d\pi_x = T_x(\widetilde{M}_{\pi(x)})$$

where $\widetilde{M}_{\pi(x)}$ means the fiber of the boundle $\widetilde{M}$ when the manifold $\widetilde{M}$ is seen as a boundle of M, *i.e.*, $\widetilde{M} = \pi^{-1}(M)$ And since by the property, each fiber $\pi^{-1}(x)$ is an embedded submanifold of $\widetilde{M}$.

***Remark* 1.30** (Bundle)

The vector bundle in here is more general, it is defined by $\pi^{-1}M$ and each fiber $\pi^{-1}(x)$ is a linear space with same dimension. However, the tangent bundle TM is a special case where each fiber is a tangent space.

Definition 1.31 (Horizontal tangent space)

The horizontal tangent space is defined by

$$T_x\widetilde{M} = (V_x)^{\perp}.$$

Proposition 1.32 (Orthogonal decompositiion)

$$T_x\widetilde{M} = V_x \oplus H_x.$$

Now we can define the Riemannian submersion between two Riemannian manifold.

Definition 1.33 (Riemannian submersion)

Given two Riemannian manifold $(\widetilde{M}, \tilde{g})$ and (M, g), then a smooth submersion

$$\pi : \widetilde{M} \to M$$

is called a Riemannian submersion if

$$d\pi_x|_{H_x} : H_x \to M$$

is a linear isometry for each $x \in \widetilde{M}$, *i.e.*, $\tilde{g}_x(v, w) = g_{\pi_x}(d\pi_x(v), d\pi_x(w))$ for any $v, w \in H_x$. .

1.1.4 The cotangent bundle of Riemannian Manifold

When we add a Riemannian metric g on a smooth manifold M, we can construct a smooth map $\hat{g}$ from the tangent bundle TM to the cotangent bundle T^*M by using the inner product on tangent space T_pM (induced by metric g). Actually we will see that this smooth map is a correspondence between vector fields of TM and covector fields of T^*M. For each point $p \in M$, the smooth map $\hat{g}$ is defined by

$$\begin{aligned} \hat{g}_p : T_pM &\longrightarrow T_p^*M \\ v_p &\longmapsto g_{p,v} \end{aligned}$$

where

$$g_{p,v}: T_pM \longrightarrow \mathbb{R}$$
$$w_p \longmapsto \langle v_p, w_p\rangle_g.$$

This is the describe of $\hat{g}$ in each point p. Then in global, the smooth map $\hat{g}$ is defined by

$$\hat{g}: TM \longrightarrow T^*M$$
$$X \longmapsto g_X$$

where X is any vector field in vector boundle TM and

$$g_X: TM \longrightarrow \mathbb{R}$$
$$Y \longmapsto \langle X, Y\rangle_g$$

Remark 1.34

Since the vector field X in tangent bundle TM is smooth on M and the inner product is smooth on M, so obviously our bundle homomorphism $\hat{g}$ defined on vector fields is smooth on M.

Construction 1.35

Consider the frame of TM and the coframe of T^*M. Given a local frame $E_1, \cdots, E_n$ of $TM|_U$ in a neighborhood $p \in U \subseteq M$, we have a corresponded local coframe $\varepsilon_1, \cdots, \varepsilon_n$ of $T^*M|_U$ such that

$$\varepsilon_i(E_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases} \tag{1}$$

Then we can represent the metric g by

$$g = g_{ij}\varepsilon_i\varepsilon_j,$$

since eq. (1), for each vector field $X = \sum X^i E_i \in TM$, the $\hat{g}$ certain a covector field by

$$\hat{g}(X) = \sum_{i,j} g_{ij}\varepsilon_i(X^1E_1 + \cdots + X^nE_n)\varepsilon_j \tag{2}$$
$$= \sum_{i,j} g_{ij}X^i\varepsilon_j \tag{3}$$
$$= \sum_j X_j\varepsilon_j \qquad \text{(where } X_j = \sum_i g_{ij}X^i \text{ is a coefficient function of } p) \tag{4}$$

In particular, for each vector $w \in TM$, the covector $\hat{g}(X): w \longmapsto \sum_{j=1}^n X_j\varepsilon_j(w)$ is a function on TM.

Definition 1.36 (X Flat)

Given a vector field $X \in TM$, the corresponded covector field by $\hat{g}$ is called X flat, and denote

$$X^\flat = \hat{g}(X) = \sum X_j\varepsilon_j$$

Actually, according to eq. (4), the smooth map $\hat{g}$ can be represented by the same matrix-valued function G of metric g,

$$M(\hat{g}) = G = \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_n & \cdots & g_{nn} \end{pmatrix}$$

where $G\begin{pmatrix} X^1 \\ \vdots \\ X^n \end{pmatrix} = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$ is the coordinate of $\hat{g}(X)$ under the corresponded coframe $\varepsilon_1, \cdots, \varepsilon_n$. And since $rank(M(\hat{g})) = rank(G) = n$, *i.e.*, $M(\hat{g})$ is full rank, then the smooth map $\hat{g}$ is an isomorphism. And it is easily to construct the inverse smooth map by

$$\begin{aligned} \hat{g}^{-1} : T^*M &\longrightarrow TM \\ w &\longmapsto \hat{g}^{-1}(w) = \sum w^i E_i \end{aligned}$$

where $w = \sum w^i \varepsilon_i$, $\begin{pmatrix} w^1 \\ \vdots \\ w^n \end{pmatrix} = G^{-1}\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$, and we also denote

$$(g^{ij}) = \begin{pmatrix} g^{11} & \cdots & g^{1n} \\ \vdots & \ddots & \vdots \\ g^{n1} & \cdots & g^{nn} \end{pmatrix} = \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_n & \cdots & g_{nn} \end{pmatrix}^{-1} = G^{-1}.$$

Now we can consider image of the inverse smooth map $\hat{g}^{-1} : T^*M \to TM$.

Definition 1.37 (w sharp)
For any covector $w \in T^*M$, the w sharp is defined by

$$w^\sharp = \hat{g}^{-1}(w) \in TM.$$

Thus, the correspondence between TM and T^*M can be described by a musical isomorphism:

$$\begin{tikzcd} TM \arrow[r, bend left, "\flat \text{ by } \hat{g}"] \arrow[r, no head, "\simeq"] & T^*M \arrow[l, bend left, "\sharp \text{ by } \hat{g}^{-1}"] \end{tikzcd}$$

Definition 1.38 (Grad)
Given a Riemannian manifold (M, g), and $f : M \to \mathbb{R}$ is a smooth function on M, (*i.e.*, $f \in C^\infty(M)$). Then apply the differential operator d on it, the differential $df : TM \to \mathbb{R}$ is a linear function, *i.e.*, $f \in T^*M$. So df have the corresponded vector in TM by musical isomorphism, and define the grad of smooth function f by

$$\operatorname{grad} f = (df)^\sharp = \hat{g}^{-1}(df)$$

and in converse, every df can be characterized as

$$df_p(w) = \langle \operatorname{grad} f|_p, w\rangle_g \quad \text{for all } p \in M, w \in T_pM$$

In conclusion, the relations can be represented by,

$$C^\infty(M) \xrightarrow[differential]{d} T^*M \xrightarrow{\hat{g}} TM$$

$$f \xmapsto{d} df \underset{\flat}{\overset{\sharp}{\rightleftarrows}} \operatorname{grad} f$$

and it is similar to the Riez representation theorem in Hilbert space.

1.2 Connections and Covariant Derivatives

1.2.1 Connections

Recall 1.39 (Smooth section)

Given a vector bundle $\pi : E \to M$ of a smooth manifold M, the smooth section of E is denoted by $\varepsilon(M)$, which is consisted of all smooth vector fields in E.

Definition 1.40 (Connection in a vector bundle)

The connection in a vector bundle E is a bilinear smooth map,

$$\begin{aligned} \nabla : \mathcal{T}(M) \times \varepsilon(M) &\longrightarrow \varepsilon(M) \\ (X, Y) &\longmapsto \nabla_X Y \end{aligned}$$

such that

- (Linear over $C^\infty(M)$ on X)
  $\nabla_{fX_1+gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y$ for any $f, g \in C^\infty(M)$.
- (Linear over $\mathbb{R}$ on Y)
  $\nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2$.
- (Not linear over $C^\infty(M)$ on Y)

$$\begin{aligned} \nabla_X(fY) &= \nabla_{fX} Y + (Xf) \cdot Y \\ &= f \cdot \nabla_X Y + (Xf) \cdot Y \end{aligned}$$

(where fX means the multiplication but not composition).

The connection $\nabla_X Y$ can be seen as the covariant derivative of vector field Y on direction $X_p \in T_pM$ for each $p \in M$ smoothly, where our direction X_p is in tangent space T_pM and smoothly on M (like a vector field).

Example 1.41

Given a Riemannian manifold (M, g) and a vector field $M \times \mathbb{R}$ on it, defined by a smooth function $f \in C^\infty(M)$. Then

$$\langle \operatorname{grad} f, v \rangle \quad \text{where} \quad \operatorname{grad} f = df^\sharp \text{ and } v \in TM$$

is the covariant derivative of vector field $M \times \mathbb{R}$ on the direction v.

The connection of

Lemma 1.42

Given an vector bundle $\pi : E \to M$, select vector fields $X \in \mathcal{T}(M)$, $Y \in \varepsilon(M)$. Then for each $p \in M$, $\nabla_X Y|_p$ is only depended on the victors of vector fields X and Y on any neighborhood of point p. (More precisely, if $X = \widetilde{X}$ and $Y = \widetilde{Y}$ on a neighborhood U of point p, then $\nabla_X Y|_p = \nabla_X \widetilde{Y}|_p$).

Proof.
- At first, we fix a vector field $X \in \mathcal{T}(M)$, then consider the two vector fields Y and $\widetilde{Y}$. If there is a neighborhood $p \in U \subseteq M$ such that $Y_U = \widetilde{Y}_U$, then in this neighborhood U, we have

  $$Y - \widetilde{Y}|_U \equiv 0.$$

  So it is sufficient to consider the vector fields Y such that $Y|_U \equiv 0$.
  By the property, there is a function $\varphi \in C^\infty(M)$ such that $supp(\varphi) \subseteq U$ and $\varphi(p) = 1$. Since $Y|_U \equiv 0$, we have $\varphi Y \equiv 0$. Thus

  $$\nabla_X(\varphi Y) = \nabla_X(0 \cdot \varphi Y) = 0 \cdot \nabla_X \varphi Y = 0,$$

  and by the third rule of connection,

  $$\nabla_X(\varphi Y) = \varphi \cdot \nabla_X Y + (X\varphi) \cdot Y = 0$$

  where $(X\varphi)Y|_U = 0$ (since $Y|_U \equiv 0$), so $\varphi \cdot \nabla_X Y|_p = (\nabla_X Y)|_p$ (since $\varphi(p) = 1$). Hence, $\nabla_X Y|_p = 0$ when $Y|_U = 0$.
- Next we fix Y and consider two vector fields $X, \widetilde{X} \in \mathcal{T}(M)$ such that $X|_U = \widetilde{X}|_U$, *i.e.*, $X - \widetilde{X}|_U \equiv 0$. It is sufficient to only consider the vector field X such that $X|_U \equiv 0$. We select a similar function $\eta \in C^\infty(M)$ such that $supp(\eta) \subseteq U$ and $\eta(p) = 1$. Then we have

  $$\nabla_{\eta X} Y|_p = \eta(p) \cdot \nabla_X Y|_p = \nabla_X Y|_p$$

  where $\nabla_{\eta X} Y|U \equiv 0$ since $X|_U \equiv 0$ ($\nabla_{\eta X} Y|_U = \nabla_{0 \cdot \eta X} Y|_U = 0 \cdot \nabla_{\eta X} Y|_U = 0$). Hence $\nabla_X Y|_p = 0$. ■

Actually, similar to the concept of covariant derivative in calculus, the covariant derivative is depended on the neighborhood of Y, but is just depended on the vector of X at point p.

Proposition 1.43

The connection $\nabla_X Y|_p$ is just depended on following two parts,
- Vectors $Y|_U$ in neighborhood $p \in U \subseteq M$.
- Vector X_p at point p.

Proof. By **Lemma 1.42**, we just need to prove the second item now, which means that if $X_p = \widetilde{X}_p$, then $\nabla_X Y|_p = \nabla_{\widetilde{X}} Y|_p$. It is sufficient to consider $X_p = 0$. Then we select a local coordinate $(x_1, \cdot, x_n)$ on the neighborhood $p \in U \subseteq M$, the vector field can be presented by

$$X = X^1 \frac{\partial}{\partial x_1} + \cdots + X^n \frac{\partial}{\partial x_n} \quad \text{where} \quad X^1(p), \cdots, X^n(p) = 0$$

Then since connections are linear over $C^\infty(M)$ on X, we have

$$\nabla_X Y = \nabla_{X^1 \frac{\partial}{\partial x_1} + \cdots + X^n \frac{\partial}{\partial x_n}} Y = X^1 \nabla_{\frac{\partial}{\partial x_1}} Y + \cdots + X^n \nabla_{\frac{\partial}{\partial x_n}} Y,$$

so

$$\nabla_X Y|_p = X^1(p) \nabla_{\frac{\partial}{\partial x_1}} Y|_p + \cdots + X^n(p) \nabla_{\frac{\partial}{\partial x_n}} Y|_p = 0$$

Remark 1.44

By the proposition, the covariant derivative $\nabla_X Y|_p$ can be written as $\nabla_{X_p} Y$

Example 1.45 (Covariant derivatives on Euclidean space)

Given a smooth function f defined on $\mathbb{R}^n$. The vector field $Y = \{(x, f(x)v) : x \in \mathbb{R}^n\}$ is in the vector bundle $\mathbb{R}^n \times \mathbb{R}$. The covariant derivatives of Y at point p is only depended on the vector v_p at point p, and equal to the directional derivative $\frac{\partial}{\partial v_p} f$.

After considering the general situation of connections (Y is a vector field in any vector boundle $\varepsilon(M)$), we will discuss the special case when $\varepsilon(M) = \mathcal{T}(M)$.

Definition 1.46 (Linear connection)

A linear connection of a manifold M is a connection on $\mathcal{T}(M)$, *i.e.*,

$$\nabla : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M).$$

Remark 1.47

Linear connections are not tensor fields, since it is not linear over $f \in C^\infty(M)$.

Next we will consider the reperesentation of linear connections as we do for metric in previous by using the local frame.

Construction 1.48 (Christoffel symbols)

Pick a local frame $E^1, \cdots, E^n$ of TM on an openset U, (*i.e.*, $\mathrm{Span}\{E_1, \cdots, E_n\} = \mathcal{T}(M)|_U$). Then any vector field $X \in \mathcal{T}(M)|_U$ can be represented by

$$X = X^1 E_1 + \cdots + X^n E_n.$$

And for any vector field $Y = Y^1 E_1 + \cdots + Y^n E_n \in \mathcal{T}(M)|_U$, although $\nabla_X Y$ is not linear for $Y^i E_i$, we first consider

$$\nabla_{E_i} E_j = \sum_k \Gamma_{ij}^k E_k \quad \text{since} \quad \nabla_{E_i} E_j \in \mathcal{T}(M)|_U.$$

For all i, j, there are n^3 coefficient functions $\{\Gamma_{ij}^k\}$ on U and each $\nabla_{E_i} E_j$ is only determined by coefficient functions $\Gamma_{ij}^1, \cdots, \Gamma_{ij}^n$. These n^3 coefficient functions are called *Christoffel symbols*. Next we need to verify that the Christoffel symbols $\{\Gamma_{ij}^k\}$ can represent linear connections.

Consider

$$\begin{aligned}
\nabla_X Y &= \sum_j \nabla_X (Y^j E_j) \\
&= \sum_j (XY^j) E_j + \sum_{i,j} Y^j \nabla_{X^i E_j} E_j \\
&= \sum_j (XY^j) E_j + \sum_{i,j} X^i Y^j \nabla_{E_i} E_j \\
&= \sum_j (XY^j) E_j + \sum_{i,j,k} X^i Y^j \cdot \Gamma_{ij}^k E_k \\
&= \sum_{i,j,k} (XY^k + X^i Y^j \cdot \Gamma_{ij}^k) E_k.
\end{aligned}$$

Thus, every linear connections can be represented by a set of Christoffel symbols $\{\Gamma_{ij}^k\}$,

$$\nabla_X Y = (XY^k + X^i Y^j \cdot \Gamma_{ij}^k) E_k \tag{5}$$

where $XY^k = (\sum_i X^i E_i) Y^k = \sum_{i,k} X^i (E_i Y^k)$.

Example 1.49 (Euclidean connection)

The linear connection in Euclidean space $\mathbb{R}^n$ is represented by

$$\overline{\nabla}_X (Y^j \partial_j) = (XY^j) \partial_j \quad , i.e., \quad \Gamma_{ij}^k \equiv 0 \text{ for all } i, j, k$$

where $Y = Y^i \partial_i$ under coordinate frame.

Lemma 1.50 (Correspondence)

Given a single covered coordinate chart manifold M (*i.e.*, there is only one coordinate cover the global M). Then there is a correspondence between linear connections and Christoffel symbols,

$$\begin{tikzcd}
\text{linear connections } \nabla_X Y \arrow[rr, bend left] & & \text{Christoffal symbols } \{\Gamma_{ij}^k\} \arrow[ll, bend left, "\text{one-one correspondent}"]
\end{tikzcd}$$

by the reperesentation,

$$\nabla_X Y = (XY^k + X^i Y^j \cdot \Gamma_{ij}^k) \partial_k. \tag{6}$$

Proof. • (linear connections $\longrightarrow$ Christoffal symbols)
Consider a linear connection $\nabla_X Y$ on M, since M is a single covered coordinate chart manifold, we can select the general frame $\partial_1, \cdots, \partial_n$ on M. Then let $E^i = \partial^i$, we get eq. (6) by the eq. (5).

• (Christoffal symbols $\longrightarrow$ linear connection)
Given a Christoffal symbols $\{\Gamma_{ij}^k\}$ consists of n^3 smooth functions $\Gamma_{ij}^k \in C^\infty(M)$. Then follow the eq. (6), we get

$$\nabla_X Y = (X^i \partial_i Y^k + X^i Y^j \Gamma_{ij}^k) \partial_k.$$

Next we just need to verify that this formula is a linear connection on M.

– (Linear over $C^\infty(M)$ on X) Consider $fX = f\sum_i X^i\partial_i = (fX^i)\partial_i$, then

$$\begin{aligned}\nabla_X fY &= ((fX^i)\partial_i Y^k + (fX^i)Y^j\Gamma_{ij}^k)\partial_k \\ &= a(X^i\partial_i Y^k + X^iY^j\Gamma_{ij}^k)\partial_k \\ &= f\nabla_X Y\end{aligned}$$

– (Linear over $\mathbb{R}$ on Y)

$$\begin{aligned}\nabla_X(aY_1 + bY_2) &= (X^i\partial_i(aY_1^k + bY_2^k) + X^i(aY_1^j + bY_2^j)\Gamma_{ij}^k)\partial_k \\ &= a(X^i\partial_i Y_1^k + X^iY_1^j\Gamma_{ij}^k)\partial_k + b(X^i\partial_i Y_2^k + X^iY_2^j\Gamma_{ij}^k)\partial_k \\ &= a\nabla_X Y_1 + b\nabla_X Y_2\end{aligned}$$

– (Rule of fY)

$$\begin{aligned}\nabla_X(fY) &= (X^i\partial_i(fY^k) + X^ifY^j\Gamma_{ij}^k)\partial_k \\ &= (X^i(\partial_i f)Y^k + X^i(\partial_i Y^k)f + X^ifY^j\Gamma_{ij}^k)\partial_k \\ &= (X^i\partial_i f)\cdot Y^k\partial_k + f(X^i(\partial_i Y^k) + X^iY^j\Gamma_{ij}^k)\partial_k \\ &= (Xf)\cdot Y + f\nabla_X Y\end{aligned}$$

Hence eq. (6) is a well defined linear connection on M.

■

By using the partition of unity, we can get generalize this lemma to the global M.

Proposition 1.51 (The construction of a linear connection)

Every smooth manifold admits a linear connection.

Proof. Given a covered coordinate charts $\{U_\alpha\}_{\alpha\in\Lambda}$ of manifold M. By **Lemma 1.50**, there are connections ∇^α on each open subset U_α. By using the partition of unity, there are functions $\{\varphi_\alpha\}_{\alpha\in\Lambda}$ subordinate to open subsets $\{U_\alpha\}_{\alpha\in\Lambda}$, (where $\varphi_\alpha(p) = 0$ for all $p \notin U_\alpha$, and $\sum_{\alpha\in\Lambda}\varphi_\alpha(p) = 1$). Then we can define

$$\nabla_X Y = \sum_{\alpha\in\Lambda}\varphi_\alpha\cdot\nabla_X^\alpha Y \tag{7}$$

, it is obviously smooth and linear over $\mathbb{R}$ in Y and also linear over $C^\infty(M)$ in X. (Since each $\varphi_\alpha\cdot\nabla_X^\alpha Y$ is smooth on M and $\nabla_X^\alpha Y$ is linear over $\mathbb{R}$ in Y and linear over $C^\infty(M)$ in X). So we just need to verify the final rule of $\nabla_X(fY)$,

$$\begin{aligned}\nabla_X(fY) &= \sum_{\alpha\in\Lambda}\varphi_\alpha\cdot\nabla_X^\alpha(fY) \\ &= \sum_{\alpha\in\Lambda}\varphi_\alpha(Xf\cdot Y + f\cdot\nabla_X^\alpha Y) \\ &= \sum_{\alpha\in\Lambda}\varphi_\alpha\cdot Xf\cdot Y + \sum_{\alpha\in\Lambda}\varphi_\alpha\cdot f\nabla_X^\alpha Y \\ &= (\sum_{\alpha\in\Lambda}\varphi_\alpha)\cdot Xf\cdot Y + f\sum_{\alpha\in\Lambda}\varphi_\alpha\cdot\nabla_X^\alpha Y \\ &= (Xf)Y + f\cdot\nabla_X Y \qquad (\text{ since } \sum_{\alpha\in\Lambda}\varphi_\alpha = 1)\end{aligned}$$

Hence the linear connection eq. (7) is well defined. ■

Recall (Tensor field)

The tensor bundles are defined on the tangent bundle TM. And we denote,

$$T_k^l TM = \underbrace{TM \otimes \cdots \otimes TM}_{l} \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_{k}.$$

it can also be seen as a multi-linear map

$$T_k^l : \underbrace{T^*M \times \cdots \times T^*M}_{l} \times \underbrace{TM \times \cdots \times TM}_{k} \to \mathbb{R}.$$

In particular, the space of tensor fields in tensor bundle $T_k^l TM$ is denoted by $\Gamma(T_k^l TM)$.

Another part we need to consider is the connections of tensor fields, and actually we will see that given a Riemannian manifold (M, g), there is a correspondence between the connections $\nabla : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$ and connections of tensor fields $\nabla : \mathcal{T}(M) \times \Gamma(T_l^k TM) \to \gamma(T_l^k TM)$,

$$\begin{tikzcd}
\text{Christoffal symbols } \{\Gamma_{ij}^k\} \arrow[r, bend left] & \text{Linear connections } \nabla_X Y \arrow[l, bend left, "\text{one-one correspondence}"] \arrow[r, bend left] & \text{Connections on tensor field } \nabla \arrow[l, bend left, "\text{one-one correspondence}"]
\end{tikzcd}$$

i.e., given a set of Christoffal symbols $\{\Gamma_{ij}^k\}$, it can also determine a connection ∇ of tensor fields uniquely.

Construction 1.52

In a Riemannian manifold (M, g), given a set of Christoffal symbols $\{\Gamma_{ij}^k\}$, it determines a linear connection $\nabla : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$ on M (also determined a). Then we can construct a unique connection of tensor field $T_l^k M$

$$\nabla : \mathcal{T}(M) \times \Gamma(T_l^k TM) \to \Gamma(T_l^k TM)$$

by

- When tensor field is in $T^0 TM = M \times \mathbb{R}$, the connection ∇ is given by the ordinary ordinary differential of functions,

$$\nabla_X f = Xf.$$

- When tensor field is in $T^1 TM = T^1 TM$, the connection ∇ is given by the linear connection

$$\nabla : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$$

- Connection ∇ follow the tensor product by

$$\nabla_X (F \otimes G) = (\nabla_X F) \otimes G + F \otimes (\nabla_X G)$$

  where $X \in \mathcal{T}(M)$, $F \in \Gamma(T_{k_1}^{l_1} TM)$ and $G \in \Gamma(T_{k_2}^{l_2} TM)$.
- Commute of contractions:

$$\nabla_X (\operatorname{Tr} Y) = \operatorname{Tr}(\nabla_X Y)$$

Remark

In particular, consider smooth function $f \in C^\infty = \Gamma(T^0TM)$ and $Y \in \mathcal{T}(M) = \Gamma(T^0TM)$, fY can be seen as $f \otimes Y$, then

$$\nabla_X fY = \nabla_X f \otimes Y = (\nabla_X Y) \otimes f + (\nabla_X f) \otimes Y = f\nabla_X Y + (Xf)Y.$$

This is same to the third rule of connection in **Definition 1.40**.

The proof of the uniqueness can be seen [1].

1.2.2 Vector Fields along Curves

In this part we will discuss the curves as a submanifold of a manifold. We use a smooth map to represent a curve in a manifold M by

$$\gamma : I \to M \quad \text{where } I \subseteq \mathbb{R} \text{ is an interval}$$

and there are some useful fundamental definitions and symbols:

- (Act on functions) For any function $f \in C^\infty(M)$,

$$\dot{\gamma}(t)f = \frac{d}{dt}(f \circ \gamma)(t)$$

- (The velocity of γ)

$$\dot{\gamma}(t) = \gamma_*(\frac{d}{dt})$$

  In particular, under a coordinate $\gamma(t) = (\gamma^1(t), \gamma^2(t), \cdots, \gamma^n(t))$, then the representation of the velocity of γ is

$$\dot{\gamma}(t) = \sum_i \dot{\gamma}^i(t)\partial_i$$

  where $\dot{\gamma}^i = {\gamma^i}'(t)$ is the derivatives of t when the coordinate is normal $(x_1, \cdots, x_n)$.

Definition 1.53 (Vector fields along curve)

Given a curve $\gamma : I \to M$, a vector field along γ is a smooth map

$$V : I \to TM$$

such that $V(t) \in T_{\gamma(t)}M$ for all $t \in I$.

Remark 1.54

- We can see that the tangent vector field of a curve γ is depended on the choice of the smooth maps. A simple example are $\gamma_1 = \gamma(t)$ where $t \in [a, b]$ and $\gamma_2 = \gamma(\alpha t)$ where $t \in [a/\alpha, b/\alpha]$ looked same, however for each point $p = \gamma(t_0)$ on curve γ, the velocity at p are

$$\dot{\gamma}_1(t_0) = \dot{\gamma}(t_0) \quad \text{and} \quad \dot{\gamma}_2(\frac{t_0}{\alpha}) = \alpha\dot{\gamma}(t_0),$$

since under a normal coordinate,

$$\begin{aligned}\dot{\gamma_2}(t) &= (\frac{d}{dt}\gamma^1(\alpha t), \cdots, \frac{d}{dt}\gamma^n(\alpha t)) \\ &= (\alpha\gamma^{1'}(\alpha t), \cdots, \alpha\gamma^{n'}(\alpha t)) \\ &= \alpha(\gamma^{1'}(\alpha t), \cdots, \gamma^{n'}(\alpha t)) \\ &= \alpha\dot{\gamma}(\alpha t).\end{aligned}$$

Thus, the velocity vectors at point p have $v_{\gamma_2}(p) = \alpha v_{\gamma_1}(p)$.

- The vector fields along a curve γ is both depended on γ and its immersed manifold M.

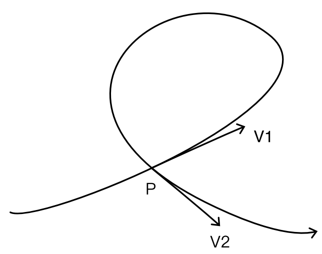
Definition 1.55 (Extendible vector fields along curve)

Given a curve $\gamma : I \to M$, the vector field $V : I \to TM$ is an extendible vector field along γ if there is a vector field $\widetilde{V}$ on a neighborhood of the image of γ, such that

$$V(t) = \widetilde{V}_{r(t)}$$

Remark **1.56**

Where the extendible means a vector field along curve γ can be extended to a vector field defined on an open subset contains $\mathrm{Im}\,\gamma$. And not all vector fields along curves can be extended, a simple example is



1.2.3 Covariant Derivatives along Curves

In this part, we are aim to define the covariant derivatives of any vector fields V along curves. Especially, when the vector fields V is extendible (as we have already discussed), the linear connection $\nabla_{\dot{\gamma}}V$ make sense (since by **Proposition 1.43**, $\dot{\gamma}$ is no need to be extendible, but V need), so we can define the covariant derivatives of V as $D_t V = \nabla_{\dot{\gamma}}V$ when V is extendible. However, actually we will use an appropriate way to generalize it to any vector fields V along γ (no need to be extendible).

Definition 1.57 (Covariant derivatives along curves)

Given a curve $\gamma : I \to M$ in manifold M, let ∇ be a linear connection on M, and $\mathcal{T}(\gamma)$ is the space of vector fields along the curve γ. Then the linear connection ∇ will induce a unique operator,

$$D_t : \mathcal{T}(\gamma) \to \mathcal{T}(\gamma)$$

such that for any two vector fields along γ,

- (Linear on $\mathbb{R}$) $D_t(aV + bW) = aD_t(V) + bD_t(W)$ where $a, b \in \mathbb{R}$.
- (Production rule) $D_t(fV) = \dot{f} \cdot V + fD_tV$, where $f \in C^\infty(I)$.
- If V is extendible, then for any extension $\widetilde{V}$ of V,

$$D_tV(t) = \nabla_{\dot{\gamma}(t)} \widetilde{V}.$$

The D_tV is called the covariant derivative of vector field $V \in \mathcal{T}(\gamma)$ along curve γ.

Remark 1.58

The covariant derivative is well-defined and

$$\begin{tikzcd}
\text{Christoffal symbols } \{\Gamma_{ij}^k\} \arrow[r, bend left] & \text{Linear connections } \nabla_X Y \arrow[l, bend left, "\text{one-one correspondence}"] \arrow[r, bend left] & \text{Covariant derivatives } D_t \arrow[l, bend left, "\text{one-one correspondence}"]
\end{tikzcd}$$

- (Uniqueness) Choose a normal local coordinate (x, U) of manifold M, where U is the neighborhood of a point $\gamma(t_0)$. Then $V(t) = V^j(t)\partial_i$ under the selected coordinate. By second rule of D_t which we have already defined,

$$\begin{aligned}
D_tV(t_0) &= D_k(V^j(t_0)\partial_j) \\
&= \dot{V}^j(t_0)\partial_j + V^j(t_0)D_t\partial_j \\
&= \dot{V}^j(t_0)\partial_j + V^j(t_0)\nabla_{\dot{\gamma}(t_0)}\partial_j && \text{since each } \partial_i \text{ is extensible} \\
&= \dot{V}^j(t_0)\partial_j + V^j(t_0)\dot{\gamma}^i(t_0)\Gamma_{ij}^k(\gamma(t_0))\partial_k && \text{by } eq. \text{ (10)} \\
&= (\dot{V}^k(t_0) + V^j(t_0)\dot{\gamma}^i(t_0)\Gamma_{ij}^k(\gamma(t_0)))\partial_k
\end{aligned}$$

  where

$$\begin{aligned}
\nabla_{\dot{\gamma}(t_0)}\partial_j &= (\dot{\gamma}^i\partial_i(1) + \dot{\gamma}^i \cdot 1 \cdot \Gamma_{ij}^k)\partial_j \qquad (8) \\
&= (0 + \dot{\gamma}^i \cdot \Gamma_{ij}^k)\partial_j \qquad (9) \\
&= \dot{\gamma}^i\Gamma_{ij}^k\partial_j \qquad (10)
\end{aligned}$$

  Thus the covariant derivative D_t is also determined uniquely by the Christoffal symbols $\{\Gamma_{ij}^k\}$ (like the linear connection).
- (Existance) Given a set of Christoffal symbols $\{\Gamma_{ij}^k\}$, the corresponded covariant derivative is defined by

$$D_tV(t_0) = (\dot{V}^k(t_0) + V^j(t_0)\dot{\gamma}^i(t_0)\Gamma_{ij}^k(\gamma(t_0)))\partial_k,$$

  it obviously satisfies the three rules in **Definition 1.57**.

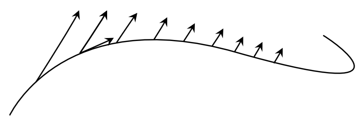
An important step is that ∂_i is extendible for any i, this is obviously since $\partial_i|_\gamma \subset \partial_i|_U$ is also an extending by the local coordinate and frame.

Proposition 1.59

Let $\gamma : (\varepsilon, \varepsilon) \to M$ is a curve in manifold M, where $\gamma(0) = p$ and $\dot{\gamma}(0) = X_p$. Given two vector fields $Y, \widetilde{Y} \in \mathcal{T}(M)$agree along curve γ (*i.e.*, $Y|_\gamma = \widetilde{Y}|_\gamma$). Then

$$\nabla_{X_p}Y = \nabla_{X_p}\widetilde{Y}.$$

(That is, the connection $\nabla_{X_p} Y$ is only depended on the values of Y along any curve tangent to X_p).



Proof. We first consider a vector field Γ along the curve γ,

$$\Gamma : (\varepsilon, \varepsilon) \to TM \qquad \text{where} \quad \Gamma = Y|_\gamma,$$

then Y is an extension of Γ. By the same way, we get $\tilde{\gamma} = \widetilde{Y}|_\gamma$ where $\widetilde{Y}$ is an extension of $\tilde{\Gamma}$. Thus, we have

$$\begin{aligned} \nabla_{X_p} Y &= D_0 \Gamma(t) \qquad \text{since } X_p = \dot{\gamma}(0) \\ \nabla_{X_p} \widetilde{Y} &= D_0 \tilde{\Gamma}(t) \qquad \text{since } X_p = \dot{\gamma}(0) \end{aligned}$$

However, since Y and $\widetilde{Y}$ agree along curve γ,

$$\tilde{\Gamma} = \widetilde{Y}|_\gamma = Y|_\gamma = \Gamma.$$

We have $\nabla_{X_p} Y = D_0 \Gamma(t) = D_0 \tilde{\Gamma}(t) = \nabla_{X_p} \widetilde{Y}$. ∎

1.3 Geodesics and Normal Coordinates

1.3.1 Geodesics

Now we begin to discuss the important part of the Riemannian geometry. The Geodesics can be seen as a description of the manifold's 'shape'. We have defined the velocity of a curve in a manifold, we can also define the acceleration of a curve in a manifold. By using the acceleration, we can define the geodesics which has constant velocities.

Definition 1.60 (Acceleration)

Given a manifold M with a defined linear connection ∇, and $\gamma : I \to M$ is a curve in M. Then the acceleration of the curve γ is defined to be the vector field

$$D_t \dot{\gamma} \qquad \text{along the } \gamma.$$

Definition 1.61 (Geodesics)

A curve in manifold M is called a geodesic with respect to linear connection ∇ if its acceleration is zero under ∇. (*i.e.*, $D_t \dot{\gamma} \equiv 0$ for all $t \in I$).

Theorem 1.62 (Existance and Uniqueness of Geodesics)

Given a manifold M with a defined linear connection ∇, then for any point $p \in M$, vector $v \in T_pM$ and $t_0 \in \mathbb{R}$, there exists an open interval I containing t_0 and a Geodesic,

$$\gamma : I \to M \qquad \text{where } \gamma(t_0) = p \text{ and } \dot{\gamma}(t_0) = v$$

Proof. This theorem is natural in geometry, but it is proved by using the uniqueness of PDE which is not important in geometry. The proof can be seen in Lee's book, page 58 [3]. ∎

Definition 1.63 (Maximal Geodesics)

For any point $p \in M$, and vector $v \in T_pM$, a geodesic initial with vector v, *i.e.*,

$$\gamma : I \to M \qquad \text{where} \quad \gamma(0) = p \text{ and } \dot{\gamma}(0) = v$$

is maximal, if the geodesic γ can not be extended to any other interval $I \subsetneq I'$. And we also just called it the geodesic with initial point p and initial velocity v.

Remark **1.64**

For each point $p \in M$, and any vector $v \in T_pM$, there always exists a unique maximal geodesic agree at (p, v).

Example 1.65

If a curve $\gamma_1 : (0, \alpha) \to M$ is a geodesic with initial point $p = \gamma_1(0)$ and velocity $v = \dot{\gamma_1}(0)$, then the new curve $\gamma_2 : (0, \frac{\alpha}{2}) \to M$ is a geodesic initial with point p and velocity v, but γ_2 is not a maximal geodesic with initial p and v.

As we have defined the geodesics by $D_t\dot{\gamma} \equiv 0$, which means that the tangent vector field $\dot{\gamma}(t)$ is parallel along γ. We will generalize this concept to any vector fields along γ (but not just velocities $\dot{\gamma}$) which is called parallel.

Definition 1.66 (Parallel)

A vector field along a curve $\gamma : I \to M$ is parallel along γ with respect to ∇, if $D_tV \equiv 0$ for all $t \in I$.

Definition 1.67 (Parallel in Global)

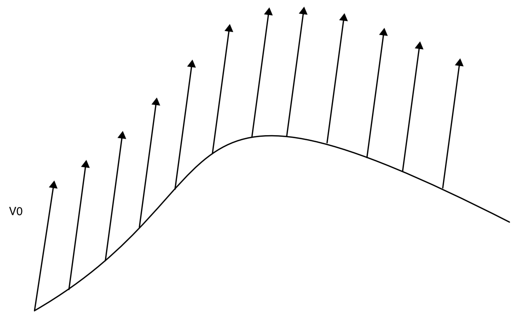
A global vector field V on manifold M is parallel if V is parallel along every curve in M.
(*i.e.*, for every curve $\gamma : I \to M$, $V|_\gamma$ is parallel along curve γ).

Theorem 1.68 (Parallel translation)

Given a curve $\gamma : I \to M$ and a vector $v_0 \in T_{\gamma(t_0)}M$ where $t_0 \in I$. Then there is a unique parallel vector field V along curve γ, such that $V(t_0) = v_0$.

Proof. The proof of this theorem is given by ODE theorey, see [3]. ∎

The Parallel translation theorem **Theorem 1.68** is said that for any point $p \in M$ and vector $v \in T_pM$, given any curve γ through point p, we can also parallel translate the selected vector v along the curve γ to form a parallel vector field along γ.



Example 1.69 (Geodesics in Euclidean space)

The geodesics on $\mathbb{R}^n$ with respect to Euclidean connection are exactly straight lines with constant velocity.

Proof. Consider a curve $\gamma : I \to \mathbb{R}^n$ under the normal coordinate by

$$\gamma(t) = (x_1(t), \cdots, x_n(t)),$$

then $\dot{\gamma}(t) = x_1'(t)\partial_1 + \cdots + x_n'(t)\partial_n$, so

$$\begin{aligned} D_t(\dot{\gamma}) &= D_t(x_1'(t)\partial_1 + \cdots + x_n'(t)\partial_n) \\ &= \sum_i D_t(x_i'(t)\partial_i) \\ &= \sum_i \dot{x}_i'(t) \cdot \partial_i + x_i'(t) D_t \partial_i \\ &= \sum_i x_i''(t)\partial_i + x_i'(t)\nabla_{\dot{\gamma}(t)}\partial_i \qquad \text{since } \partial_i \text{ is extendible} \\ &= \sum_i x_i''(t)\partial_i \qquad\qquad \text{by Euclidean connection } \nabla_{\dot{\gamma}(t)}\partial_i = 0 \end{aligned}$$

Thus, γ is a geodesic in $\mathbb{R}^n \iff D_t\dot{\gamma} \equiv 0 \iff x_i''(t) \equiv 0$, that is, $x'(t)$ is a constant, *i.e.*, γ is a straight line with constant speed. ∎

1.3.2 Riemannian Geodesics

In this part, we will consider the Riemannian connection which is a special connection such that it is compatible with metirc and symmetric (two important properties). To define the Riemannian connection, we will firstly consider the manifold $M \subseteq \mathbb{R}^n$ as an embedded submanifold in Euclidean space $(\mathbb{R}^n, \overline{g})$, then according to the property of smooth manifold [2], any vector field on M can be extended to the global $\mathbb{R}^n$. This property make it possible to define a special linear connection on M induced by the Euclidean connection $\overline{\nabla}$. And we will finally show that it has two properties as we need.

Definition 1.70 (Tangential connection)

Let $M \subseteq \mathbb{R}^n$ be an embedded submainifold of an Euclidean space. The tangential connection is a bilinear smooth map

$$\nabla^\top : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$$

defined by

$$\nabla^{\top}_X Y := \pi^{\top}(\overline{\nabla}_{\widetilde{X}}\widetilde{Y}),$$

where $\widetilde{X}$ and $\widetilde{Y}$ are any vector fields on $\mathbb{R}^n$ extended by X and Y. And $\pi^{\top} : T_p\mathbb{R}^n \to T_pM$ is the tangential as we discussed in **Proposition 1.28**.

It is easy to verify that $\nabla^{\top}$ satisfies the three rules in **Definition 1.40**, which means $\nabla^{\top}$ is a well-defined linear connection[3]. However, there is an important theorem **Lemma 1.71** proved by John Nash which lets the tangential connection make sense for any Riemannian manifold with respect to its Riemannian metric.

Lemma 1.71 (Embedding theorem)

Any Riemannian metric g on a manifold M can be seen as the induced metric of some embedding in a Euclidean space.

Remark (Uniqueness)

We notice that the tangential connection of a Riemannian manifold (m, g) is unique. Because there is a smallest n such that the embedding $M \subseteq \mathbb{R}^n$ and g is induced by Euclidean matirc $\overline{g}$ and for any $\mathbb{R}^m$ with $m \geq n$, $\mathbb{R}^m = \mathbb{R}^n \oplus \mathbb{R}^{m-n}$, and $M \subset \mathbb{R}^n$, so $T(\mathbb{R}^m)|_M = T(\mathbb{R}^n \oplus R^{m-n})|_M = T\mathbb{R}^n|_M \oplus M \times \mathbb{R}^{m-n}$. Thus $\pi^{\top}(T\mathbb{R}^m|_M) = \pi^{\top}(T\mathbb{R}^n|_M)$.

We have shown that tangential connection is a well-defined linear connection, and it is unique and it exists for any Riemannian manifold (M, g) by Embedding theorem **Lemma 1.71**. Now, we discuss the two important properties of the tangential connections.

Definition 1.72 (Compatibe with matirc)

Given a Riemannian manifold (M, g), a linear connection is said to be compatible with metric g, if it satisfies the following product rule for all vector fields $X, Y, Z \in \mathcal{T}(M)$,

$$\nabla_X \langle Y, Z\rangle_g = \langle \nabla_X Y, Z\rangle_g + \langle Y, \nabla_X Z\rangle_g.$$

Remark

Where the inner product $\langle Y, Z\rangle_g$ is a 2-tensor field, *i.e.*, $\langle Y, Z\rangle_g \in \Gamma(T^2TM)$, so it follows by the connection of tensor fields **Construction 1.52**.

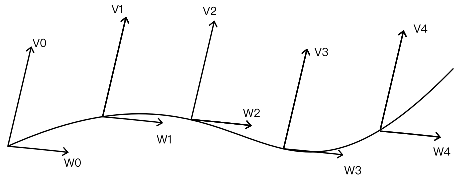
Lemma 1.73 (Equivalence of Compatibe)

The followings are equivalent,

1) The connection ∇ is compatible with metric g.
2) $\nabla_g \equiv 0$, where $\nabla_g(Y, Z, X) = \nabla_X\langle Y, Z\rangle_g$, *i.e.*, $\nabla_X\langle Y, Z\rangle_g \equiv 0$ for any vector fields $X, Y, Z \in \mathcal{T}(M)$.
3) Given vector fields V, W both along a curve γ, then

$$\frac{d}{dt}\langle V, W\rangle_g = \langle D_t V, W\rangle_g + \langle V, D_t W\rangle_g$$

4) Given a pair of vector fields parallel along a curve γ, then the inner product $\langle V_{\gamma(t)}, W_{\gamma(t)}\rangle_g$ is a constant.



i.e., the parallel translations induced by connection ∇

$$P_{t_0,t_1} : T_{\gamma(t_0)}M \to T_{\gamma(t_1)}M$$

is an isometry for any curve γ and t_0, t_1.

Definition 1.74

A linear connection on a manifold M is called symmetric if

$$\nabla_X Y - \nabla_Y X \equiv [X, Y]$$

where the $[X, Y]$ is the Lie bracket which means $[X, Y] = XY - YX$.

Proposition 1.75 (Tangential connection)

Given a Riemannian submainifold (M, g) embedded in an Euclidean space, (*i.e.*, $M \subseteq \mathbb{R}^n$ and g is induced by $\overline{g}$), then the tangential connection $\nabla^\top$ of M is compatible with the induced matirc g and is symmetric.

Actually, we will see such tangential connection is the unique connection of a Riemannian manifold which is both symmetric and compatible with the metric induced by Euclidean metric.

Theorem 1.76 (Fundamental lemma of Riemannian geometry)

Given a Riemannian manifold (M, g), then there is a unique linear connection ∇ of M such that

- ∇ is compatible with g.
- ∇ is symmetric.

So by the **Lemma 1.71**, **Proposition 1.75** and **Theorem 1.76**, we can conclude that the tangential connection of a Riemannian manifold (M, g) is the unique connection which is both symmetric and compatible with g.

$$\begin{tikzcd}
\text{Riemannian Manifold } (M,g) \arrow[r, "\text{Embedd theorem}"] \arrow[dr, "\text{unique}", "\text{Fundamental lemma}"'] & \text{Embedd Riemannian submanifold } M \subseteq \mathbb{R}^n, g = E^*\overline{g} \arrow[d, "\text{induced by } \overline{\nabla}"] \\
& \text{Tangental connection } \nabla^\top
\end{tikzcd}$$

Definition 1.77 (Riemannian connection)

We define such unique connection (tangential connection) of a Riemannian manifold (M, g) is its Riemannian connection.

Now under the Riemannian connection, we can construct the Riemannian geodesics of a Riemannian manifold. And we will just use the Riemannian connection now.

Definition 1.78 (Speed of curves)

The speed of a curve $\gamma : I \to M$ is the length of its velocity vector, *i.e.*, $|\dot{\gamma}(t)|$.

Remark

- Curve γ is constant speed if $|\dot{\gamma}(t)|$ is independent of t.
- The speed of curve γ is unit speed if $|\dot{\gamma}(t)| \equiv 1$.

Example

The circle $c : [0, \pi/2] \to \mathbb{R}^2$ by $c(t) = (\cos t, \sin t)$, its speed is constant and is unit speed.

Lemma 1.79 (Riemannian geodesics)

All Riemannian geodesics are constant speed curves.

Proof. Suppose γ is a Riemannian geodesic, then the vectors $\dot{\gamma}(t)$are parallel along the curve γ. Then by since Riemannian connection is compatible with metric g, by **Lemma 1.73**, the inner product of two parallel vector fields $\dot{\gamma}(t)$ is constant,

$$|\dot{\gamma}(t)| = \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_g = c$$

is independent of t. Hence, γ is constant speed. ∎

Proposition 1.80 (Naturality of Riemannian connections)

Given an isometry $\varphi : (M, g) \to (\widehat{M}, \hat{g})$, then

1) φ naturally take Riemannian connection ∇ of metric g to the Riemannian connection $\widehat{\nabla}$ of $\hat{g}$, that is

$$d\varphi(\nabla_X Y) = \widehat{\nabla}_{\varphi_* X}(d\varphi Y).$$

2) γ is a curve in M and V is a vector field along γ, then

$$d\varphi D_t V = \widehat{D}_t(d\varphi V)$$

3) φ : geodesics in M $\to$ geodesics in $\widehat{M}$. If curve γ in M is a geodesic with initial point p and vector v, then $\varphi\gamma$ is a geodesic in $\widehat{M}$ with initial point $\varphi(p)$ and initial vector $d\varphi(v)$.

Proof. See [1] ∎

In conclusion,

$$\begin{array}{c}
\text{Riemannian manifold } (M, g) \\
\Big\downarrow{\scriptstyle \text{unique determine}} \\
\text{Riemannian connection} \nabla_X Y \\
\Big\downarrow{\scriptstyle \text{unique induce}} \\
\text{Covariant Derivatives } D_t \\
\Big\downarrow{\scriptstyle D_t \dot{\gamma} \equiv 0} \\
\text{Geodesics in } M (|\dot{\gamma}(t)| \text{ is constant})
\end{array}$$

1.3.3 The Exponential Map

Definition 1.81 (The domain of exponential map)

Define the subset of TM,

$$\varepsilon := \{v \in TM : \gamma_v \text{ is defined on an interval } I \text{ contains } [0,1]\}$$

where γ_v is the geodesic with the initial vector $v \in TM$.

Example

Not all vectors in TM are in subset ε, since not all geodesics can be defined on an interval I contains $[0,1]$, such as if there is an initial vector $v \in TM$ such that $\gamma_v : [0,m] \to M$ and $m < 1$. Then if we represent this curve γ_v on $[0,1]$, then we translate $\gamma_v = (x_1(t), \cdots, x_n(t))$ to $\widehat{\gamma_v} = (x_1'(t) = x_1(ct), \cdots, x_n'(t) = x_n(ct))$ where $c = m$, however $\dot{\tilde{\gamma}}_v(0) = (cx_1'(0), \cdots, cx_n'(0)) = cv \neq v$.

Remark

For any vector $v \in TM$ and its geodesic with initial v, there exists a small enough non-negative constant c, such that γ_{cv} is defined on an interval I contains $[0,1]$. (Use the method in example.)
i.e., for any vector $v \in TM$, there exists a small enough non-negative constant c, such that $cv \in \varepsilon$.

Definition 1.82 (Exponential map)

The exponential map $\exp : \varepsilon \to M$ is defined by

$$\exp(v) = \gamma_v(1).$$

Definition 1.83 (Restricted exponential map)

The restricted exponential map $\exp_p : \varepsilon_p \to M$ is defined by

$$\exp_p(v) = \exp(v) \quad \text{for} \quad v \in \varepsilon_p$$

where $\varepsilon_p = \varepsilon \cap T_pM$.

Lemma 1.84 (Rescaling lemma)

For any $v \in TM$ and $c, t \in \mathbb{R}$,

$$\gamma_{cv}(t) = \gamma_v(ct),$$

whenever either side is defined.

Proof. Suppose the domain of γ_v is an intervel $I \subset \mathbb{R}$. Let $\gamma = \gamma_v$, define the curve $\tilde{\gamma} = \tilde{\gamma}(t) = \gamma(ct)$ with domain $\frac{1}{c}I = \{t : ct \in I\}$. Next we need to show that $\tilde{\gamma}$ is a geodesic with initial point p and initial vector cv.

- By the definition of geodesic, $\tilde{F}(0) = \gamma(0) = p$, then under a coordinate $(\gamma^1(t), \cdots, \gamma^n(t))$, we have
$$\dot{\tilde{\gamma}}(t) = \frac{d}{dt}\gamma^i(ct) = c \cdot \dot{\gamma}^i(ct),$$
so when $t = 0$, we have
$$\dot{\tilde{\gamma}}(0) = c \cdot \dot{\gamma}(0) = c \cdot v.$$

- Next we will prove that $\tilde{\gamma}$ is a geodesic. Consider D_t along γ and $\widetilde{D}_t$ along $\tilde{\gamma}$,

$$\begin{aligned}
\widetilde{D}_t \dot{\tilde{\gamma}}(t) &= (\frac{d}{dt}\dot{\tilde{\gamma}}^k(t) + \Gamma_{ij}^k(\tilde{\Gamma}(t))\dot{\tilde{\gamma}}^i(t)\dot{\tilde{\Gamma}}^j(t))\partial_k \\
&= (\frac{d}{dt}c \cdot \dot{\gamma}^k(ct) + \Gamma_{ij}^k(\tilde{\Gamma}(ct)) \cdot c \cdot \dot{\gamma}^i(ct) \cdot c \cdot \dot{\gamma}^j(ct))\partial_k \\
&= (c^2\ddot{\gamma}^k(ct) + c^2\Gamma_{ij}^k(\tilde{\gamma}(ct))\dot{\gamma}^i(ct)\dot{\gamma}^j(ct))\partial_k \\
&= c^2(\ddot{\gamma}^k(ct) + \Gamma_{ij}^k(\tilde{\gamma}(ct))\dot{\gamma}^i(ct)\dot{\gamma}^j(ct))\partial_k \\
&= c^2 D_t\dot{\gamma}(ct) = 0
\end{aligned}$$

Thus $\tilde{\gamma}$ is a geodesic, and by the uniqueness of the geodesic with initial point p and initial vector cv, we conclude that

$$\tilde{\gamma} = \gamma_v(ct) = \gamma_{cv}(t).$$

■

Proposition 1.85

1) ε is an open set of TM containing the zero section and each ε_p is star-sharped with respect ti zero.
2) For each $v \in TM$, the geodesic γ_v is given by

$$\gamma_v(t) = \exp(tv)$$

for all t such that either side is defined (which means that $tv \in \varepsilon$ and t is in the domain of γ_v).
3) The exponential map is smooth.

Remark (Star-sharped)

A subset S in a vector space is star-sharped with respect to $x \in S$, if for any $y \in S$, the line segment from x to y also in S.

Proof. The proof can be seen in [3] ■

We next need to verify naturality as we do in previous.

Proposition 1.86 (Naturality of the exponential map)

Suppose $\varphi : (M, g) \to (\widetilde{M}, \tilde{g})$ is an isometry. Then for any $p \in M$,

$$\begin{tikzcd}
T_pM \arrow[r, "d\varphi"] \arrow[d, "\exp_p"'] & T_{\varphi(p)}\widetilde{M} \arrow[d, "\exp_{\varphi(p)}"] \\
M \arrow[r, "\varphi"'] & \widetilde{M}
\end{tikzcd}$$

the diagram is commute.

Proof. Consider a point p and a vector $v \in \varepsilon \subset T_pM$. Then there is a unique geodesic γ_v, next according to the Naturality property of Riemannian connections(**Proposition 1.80**) and φ is a isometry, so $\varphi(\gamma_v)$ is also the geodesic in $(\widetilde{M}, \tilde{g})$ with initial point $\varphi(p)$, initial vector $d\varphi(v) \in T_{\varphi(p)}\widetilde{M}$. Thus

$$\exp_p(v) = \gamma_v(1) \quad \text{and} \quad \exp_{\varphi(p)}(d\varphi(v)) = \varphi(\gamma_v(1)),$$

we get $\varphi \exp_p(v) = \exp_{\varphi(p)} d\varphi(v)$, *i.e.*, the diagram commute. ■

1.3.4 Normal Neighborhoods and Normal Coordinates

Lemma 1.87 (Normal neighborhood lemma)

Consider the map $\exp_p : \varepsilon \subset T_pM \to M$. Then there is a neighborhood's relation between T_pM and M: there is a neighborhood V of origin $(p = 0, v = 0)$ in T_pM and a neighborhood U of p in M such that,

$$\exp_p : V \to U \qquad \text{is a diffeomorphism}$$

Proof. According to the **Proposition 1.85**, the exponential map is smooth, then by the inverse function theorem, we just need to show thet $d\exp_p$ is invertible at $v = 0$ in T_pM.
Since the vector space T_pM can be seen as a manifold, we have $T_0(T_pM) = T_pM$. Next we will show that

$$d\exp_p : T_0(T_pM) \to T_pM$$

is an identity map. For any $v \in T_pM = T_0(T_pM)$, choose a curve τ in T_pM with initial point 0 and initial vector v, (*i.e.*, $\tau(0) = 0, \dot{\tau} = v$), then we have

$$d\exp_p(v) = \frac{d}{dt}\exp_p \circ\tau(t)|_{t=0}.$$

Choose a line $\tau(t) = tv$ with initial point 0 and initial vector v. Then

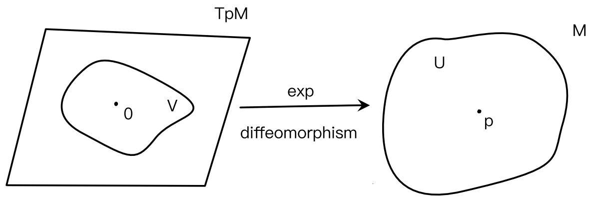
$$d\exp_p(v) = \frac{d}{dt}\exp_p(tv)|_{t=0} = \frac{d}{dt}\gamma_v(t)|_{t=0} = \dot{\gamma}_v(0) = v.$$

Hence, $d\exp_p$ is an identity map. ■

Based on **Lemma 1.87**, we have several definitions.

Definition 1.88 (Normal neighborhood)

Any open neighborhood U of a point p in M is called a normal neighborhood of p, if there is a star-sharped open neighborhood V of vector 0 in T_pM, such that $U = \exp_p(V)$ is a diffeomorphism.



Definition 1.89 (Geodesic ball)

For $\varepsilon > 0$, the ball $B_\varepsilon(0) \subset T_pM$ such that $\exp_p$ is a diffeomorphism on $B_\varepsilon(0)$, then the image of diffeomorphism $\exp_p(B_\varepsilon(0))$ is called a geodesic ball in M.

Remark

The norm of T_pM is defined by $|v| = \langle v, v\rangle_g$, and the ball B_ε is defined by this norm.

Definition 1.90 (Geodesic sphere)

- (Closed ball) If a closed ball $\overline{B}_\varepsilon(0)$ is contained in an open set $v \subset T_pM$ such that $\exp_p : V \to \exp_p(V)$ is a diffeomorphism, then $\overline{B}_\varepsilon(0)$ is called a closed geodesic ball.
- (Sphere) The image $\exp_p(\partial\overline{B}_\varepsilon(0))$ is called a geodesic sphere, where $\overline{B}_\varepsilon(0)$ is a closed geodesic ball.

1.3.5 Riemannian Normal Coordinates

After preparing the useful definitions and properties of Riemannian geodesics, we come to construct the Riemannian normal coordinate $\varphi : U \to \mathbb{R}^n$ which can let all the geodesics in Riemannian manifold to be reperesented by straight lines with constant speed in Euclidean space.

Definition 1.91 (Riemannian normal coordinates)

Consider an orthonormal basis $\{E_i\}$ for T_pM, then there is an isomorphism $E : \mathbb{R}^n \to T_pM$ by $E(x^1, \cdots, x^n) = x^1E_1 + \cdots + x^nE_n$. If $U = \exp_p(V)$ (*i.e.*, U is an image of $\exp_p$) is a normal neighborhood of p, then by the diffeomorphism $\exp_p$, we have a smooth coordinate by

$$\varphi := E^{-1} \circ \exp_p^{-1} : U \to \mathbb{R}^n$$

called *Riemannian normal coordinate* centered at p, *i.e.*,

$$\begin{tikzcd}
V \subset T_pM \arrow[d, shift left, "E^{-1}"] \arrow[r, dashed, shift right, "\exp_p"'] & U \subset M \arrow[l, shift right, "\exp_p^{-1}"'] \arrow[dl, dotted, "E^{-1}\exp_p^{-1}"] \\
E^{-1}(V) \subset \mathbb{R}^n \arrow[u, dashed, shift left, "E"] &
\end{tikzcd}$$

where $\exp_p$ is an diffeomorphism and E^{-1} is a linear isomorphism.

Definition 1.92 (Radical distance function)

For any normal coordinate centered at point p ($p \in U$), define the radical distance function $r : U \to \mathbb{R}$ under the normal coordinate by

$$r(x) = \sqrt{\sum_i (x^i)^2}$$

where $x = (x^1, \cdots, x^n)$ is the coordinate of x under the Riemannian normal coordinate centered at p.

Definition 1.93 (Unit radical vector field)

The unit radical vector field is defined by

$$\frac{\partial}{\partial r} := \sum_i \frac{x^i}{r} \frac{\partial}{\partial x_i}$$

where $r = r(x)$ and $x = (x_1, \cdots, x_n)$ under the Riemannian normal coordinate. (It is easy to see that $\frac{\partial}{\partial r}$ is the unit vector field tangent to the straight lines through the origin).

Remark

The value of distance function $r(x)$ is the distance from origin to x in U, *i.e.*, the distance from p to x. Hence by Riemannian normal coordinate, we can define the distance on Riemannian manifold M induced by a Euclidean metric.(where $d(p, x) = r(x) = \sqrt{\sum_i x^i} = d(\varphi(p) = 0, \varphi(x))$ by the Euclidean metirc of $\mathbb{R}^n$)

Proposition 1.94 (Properties of normal coordinates)

Let $(U, (x^i))$ be a Riemannian normal coordinate chart centered at point p, then

1) For any vector $v = v^i \partial_i \in T_pM$, the geodesic γ_v with initial point p and initial vector v is represented by a straight line with constant speed in $\mathbb{R}^n$ under the Riemannian normal coordinate,

$$\gamma_v(t) = (tv^1, \cdots, tv^n)$$

as long as $\gamma_v \subset U$.
2) The component of the metric at p agree with Euclidean metric, *i.e.*, $g_{ij} = \delta_{ij}$.
3) For any point $q \in U$ and $q \neq p$, its unit radical vector field $\frac{\partial}{\partial r}$ is the velocity victor of the unit speed geodesic from p to q.
4) Any Euclidean ball $\{x : r(x) < \varepsilon\}$ contained in U is a geodesic ball in M.

Definition 1.95 (Uniformly normal)

An open set $W \subset M$ is uniformly normal if there exists a $\delta > 0$ such that, W is contained in a geodesic ball $B_\varepsilon(p)$ for all $p \in W$.

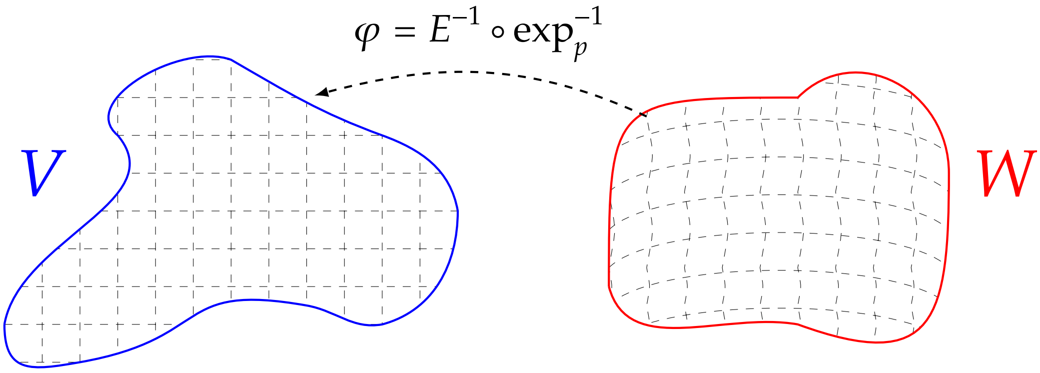
We will see that every point $p \in M$ has a uniformly normal neighborhood.

Lemma 1.96 (Uniformly normal neighborhood lemma)

Given a point $p \in M$ and any neighborhood U of p, there exists a uniformly normal neighborhood W of p contained in U, *i.e.*, $W \subset U$.

Proof. The proof can be seen in [1]. ■

By the Uniformly normal neighborhood lemma (**Lemma 1.96**) and item (1) in properties of normal coordinates (**Proposition 1.94**), we see that for any point $p \in M$, there is a uniformly normal neighborhood W of g, such that



all geodesics in W are represented by straight line in $V \subset \mathbb{R}^n$, where $V = \varphi(W)$ under the normal coordinate.

2 Lie Groups and Lie Algebras

2.1 Lie Groups

The Lie groups are defined on a smooth manifold, which means that it is a group construction on a smooth manifold. Thus, the first thing that we need to define is the group operation on a smooth manifold. To be general, we will firstly introduce the group structures on topological spaces.

Definition 2.1 (Topologiacl group)

The topological space G is a topological space G with the group operations defined by the following continuous maps,

- $(a, b) \longmapsto ab$ from $G \times G$ to G.
- $a \longmapsto a^{-1}$ from G to G.

Next we can consider the concept of Lie group which is a spceial case of topological group G, since the manifold G should be smooth.

Definition 2.2 (Lie Group)

A Lie group G is a smooth manifold G with group operations defined by following smooth maps,

- $(x, y) \longmapsto xy$
- $x \longmapsto x^{-1}$

Remark

To verify that xy and x^{-1} are both smooth map, it is sufficient to verify that xy^{-1} is also a smooth map.(*i.e.*, xy, $x^{-1} \in C^\infty(M) \circledast xy^{-1} \in C^\infty$).

Example

- $(\mathbb{R}^n, +)$ is a Lie group where the operation $+$ is defined by,
  - $(x_1, \cdots, x_n) + (y_1, \cdots, y_n) = (x_1 + y_1, \cdots, x_n + y_n)$
  - $-(x_1, \cdots, x_n) = (-x_1, \cdots, -x_n)$

  and both map are obviously smooth.

- The circle S^1 can be considered as $\mathbb{R}/\mathbb{Z}$ since for any point $s \in S^1$, there is a local coordinate by two smooth functions on $\mathbb{R}/Z$,

$$x \longmapsto \cos 2\pi x \quad \text{and} \quad x \longmapsto \sin 2\pi x$$

Thus S^1 has group operation under local coordinate, by

$$\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$$

$$([x],[y]) \longmapsto x-y \longmapsto ab^{-1} = [x-y]$$

where $a = ([x])$ amd $b = ([y])$. In particular,
- $(x,y) \longmapsto \cos 2\pi(x-y) = \cos 2\pi x \cos 2\pi y + \sin 2\pi x \sin 2\pi y$ is smooth,
- $(x,y) \longmapsto \sin 2\pi(x-y) = \sin 2\pi x \cos 2\pi x - \cos 2\pi x \sin 2\pi y$ is smooth,

and the quotient map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is a smooth map, thus the map $(a,b) \longmapsto ab^{-1}$ is smooth.

Construction 2.3 (Product of Lie groups)

Given two Lie groups G and H, then $G \times H$ is also a Lie group. The group operation in Lie group $G \times H$ is by

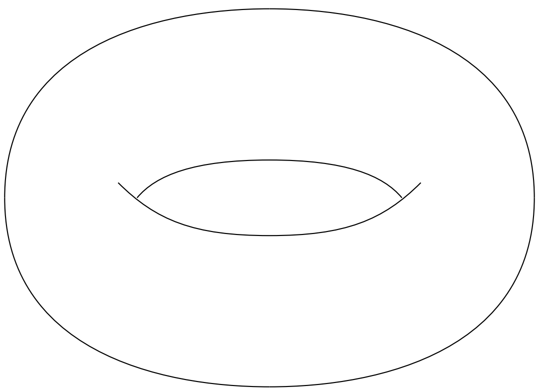
$$\begin{aligned}(G \times H) \times (G \times H) &\longrightarrow G \times H \\ ((g_1,h_1),(g_2,h_2)) &\longmapsto (g_1g_2,h_1h_2)\end{aligned}$$

where $(g_1,h_1),(g_2,h_2) \in G \times H$. And it is smooth since $(g,h) \mapsto gh$ is smooth. And the inverse operation is defined by

$$\begin{aligned}G \times H &\longrightarrow G \times H \\ (g,h) &\longmapsto (g^{-1},h^{-1})\end{aligned}$$

Example

The torus $T^1 = S^1 \times S^1$ is a Lie group, since S^1 is a Lie group.



And $S^1 \times S^1$ also can be represented by a quotient map $(\mathbb{R} \times \mathbb{R})/(\mathbb{Z} \times \mathbb{Z}) = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ under local coordinate.

There is a important examples of Lie groups.

Recall 2.4

The general linear group is a group consists of all non-singular (*i.e.*, $\det A \neq 0$) real $n \times n$ matrixs.

Example 2.5 ($\mathrm{GL}(n, \mathbb{R})$)

The general linear group $\mathrm{GL}(n, \mathbb{R})$ is a Lie group.
The general linear group $\mathrm{GL}(n, \mathbb{R})$ can be seen as a subset of Euclidean space $\mathbb{R}^{n \times n}$. Consider the determinant function

$$\det : \mathbb{R}^{n \times n} \to \mathbb{R}$$

is continuous and smooth (a polynomial map), then $\mathbb{R} \backslash \{0\}$ is an open set in $\mathbb{R}$ deduce that $\mathrm{GL}(n, \mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$ is an open set in $\mathbb{R}^{n \times n}$. Hence, $\mathrm{GL}(n, \mathbb{R})$ is a smooth manifold and we can construct a C^∞-structure on group $\mathrm{GL}(n, \mathbb{R})$ by,

- $(A, B) \longmapsto AB$ is smooth, since every entry of AB is a polynomial.
- $A \longmapsto A^{-1}$ is smooth, since $(A^{-1})_{ji} = \det A^{ij} / \det A$ is smooth.

Remark

According to the **Example 2.5**, we see that we can construct a Lie group from a smooth manifold endowed with a smooth group operation. And also a Lie group can be constructed from a group which has the structure of smooth manifold and its group operations are continuous under the topology.

Definition 2.6 (Invariant vector field)

Given a Lie group G, a vector field $X \in \mathcal{T}(M)$ is left-invariant if

$$L_{a*}X = X \qquad \text{for all } a \in G$$

(where $L_a b = ab$), that is $L_a : b \longmapsto ab$ is a diffeomorphism and $L_{a*} : X_b \longmapsto X_{ab}$ is an isomorphism, *i.e.*, $L_{a*}X_b = X_{ab}$ for all $a, b \in G$. The set of all left-invariant vector fields on G is denoted by $\mathcal{L}(G)$.

Remark

A vector field X on a Lie group G is left invariant if and only if $L_{a*}X_e = X_a$ for all elements $a \in G$, where e is the identity element of Lie group G. Since if $L_{a*}X_e = X_a$ for all elements $a \in G$, then for any $a, b \in G$,

$$L_{a*}X_b = L_{a*}(L_{b*}X_e) = (L_{a*} \circ L_{b*})X_e = L_{ab*}X_e = X_{ab}$$

(where $L_{a*} \circ L_{b*} = L_{ab*}$, because both L_{a*} and L_{b*} are differentials of L_a and L_b, respectively). Hence, fix a vector $X_e \in G_e = T_eG$, there is a unique left invariant vector field X which has value X_e at point e. Thus, we have the following correspondence

$$\text{Tangent space } G_e \xleftrightarrow{\text{correspondence}} \text{Set of left invariant vector fields}$$

$$v_e \;\overset{\longleftarrow}{\underset{X_a = L_{a*}v_e}{\longmapsto}}\; X \text{ with } X_e = v_e$$

However, we need to verify that every left invariant vector fields induced by vector v_e is a smooth vector field.

Proposition 2.7

Every left invariant vector field X on a Lie group G is smooth.

Proof. Select a vector $v_e \in G_e = T_eG$, for any point $a \in G$ and a neighborhood $a \in U$, there is a diffeomorphism

$$L_a : V \longrightarrow U$$

where V is a neighborhood of e. So it is sufficient to prove that the left invariant vector fields X are all continuous at point e. Consider the local coordinate x, U of point e, choose a neighborhood $e \in V$, such that $ab \in V$ and $ab^{-1} \in U$. Then for any $a \in V$, we have

$$X_{x^i(a)} = L_{a*}X_e(x^i) = X_e(x^i \circ L_a).$$

Then since $(a, b) \longmapsto ab$ is smooth, we have

$$x^i(ab) = x^i(L_ab) = f^i(x^1(a), x^2(a), \cdots, x^1(b), \cdots, x^n(b))$$

where f is a smooth binary function in$a$ and b, (since every coordinate maps x^i is smooth, ab is smooth, so each $x^i(ab)$ is smooth with respect to a and b). Thus,

$$\begin{aligned} X_{x^i(a)} &= X_e(x^i \circ L_a) \\ &= \sum_{j=1}^{n} c_j \frac{\partial}{\partial x^j}|_e(x^i \circ L_a) \\ &= \sum_{j=1}^{n} c_j D_{n+j} f^i(x(a), x(e)) \quad \text{is smooth.} \end{aligned}$$

Hence, X is smooth. ∎

2.2 Lie Algebra

Given a Lie algebra G, we can define a Lie algebra on $\mathcal{L}(G)$. Firstly, we will define the Lie algebra in general.

Definition 2.8 (Lie algebra)

Given a Lie group G, the set $\mathcal{L}(G)$ together wih a binary operation $[\square, \square]$ is a Lie algebra, if the bracket operation satisfied,

- (Alternativity) $[X, X] = 0$
- (Bilinear) $[ax + by, z] = a[x, z] + b[y, z]$ and $[z, ax + by] = a[z, x] + b[z, y]$.
- (Jocobi identity) $[[X, Y], Z] + [[Y, Z], X] + [[Z + X], Y] = 0$.

Remark

The alternativity of the bracket shows that $[X+Y, X+Y] = [X+Y, X]+[X+Y, Y] = [X, Y]+[X, X]+[Y, X] + [Y, Y] = [X, Y] + [Y, X] = 0$, so we also have anticommutativity,

$$[X, Y] = -[Y, X]$$

Definition 2.9 (Lie bracket)

The Lie bracket is defined by

$$[X, Y] = XY - YX$$

It is easy to verify that the Lie bracket satisfied the three rules in Definition 2.8.

Remark 2.10 (Lie bracket in coordinate)

Given two smooth vector fields $[X, Y]$, under the coordinate $(x^1, \cdots, x^n)$, they are represented by

$$X = X^i \frac{\partial}{\partial_x^i} \quad , \quad Y = Y^j \frac{\partial}{\partial x^j}$$

then their Lie bracket

$$[X, Y] = (X^i \frac{\partial Y^j}{\partial X^i} - Y^j \frac{\partial X^j}{\partial x^i}) \frac{\partial}{\partial x^j} = (XY^j - YX^j) \frac{\partial}{\partial X^j}$$

Given a Lie subgroup H of G, we can also construct the Lie subalgebra of G, which is the Lie algebra of H.

Construction 2.11 (Lie subalgebra)

Consider a Lie subgroup H of a Lie group G, set the inclusion map $i : H \to G$, then the induced map $i_* : H_e \to i_*(H_e) \subset G_e$ is an inclusion. Hence, H_e is a subspace of linear space G_e. On the other hand, any vector $X \in H_e$ can be extended to a left-invariant vector field $\widetilde{X}$ on H and a left-invariant vector field $\widetilde{\widetilde{X}}$ on G. Then we have

$$i_* \circ \widetilde{X}(a) = i_* L_{a*} X = L_{a*}(i_* X) = \widetilde{\widetilde{X}}(a),$$

so $\widetilde{X}$ and $\widetilde{X}$ are i-related (i is an isomorphism). Thus for any $Y \in H_e$,

$$[\widetilde{X}, \widetilde{Y}](e) = i_*([\widetilde{\widetilde{X}}, \widetilde{\widetilde{Y}}](e)).$$

Hence H_e is a subalgebra of G_e, *i.e.*, H is a subalgebra of G.

There is an important example of Lie algebra, which is defined on the Lie group in Example 2.5.

Example (The Lie algebra structure of $\mathrm{GL}(n, \mathbb{R})$)

In $\mathrm{GL}(n, \mathbb{R})$, by the standard coordinate x^{ij} on $\mathbb{R}^{n \times n}$, given a matrix $M \in \mathrm{GL}(n, \mathbb{R})$, the tangent vector at the identity matrix I can be represented by,

$$M_I = \sum_{i,j} M_{ij} \frac{\partial}{\partial x^{ij}} |_I$$

- Consider the left-invariant vector field $\widetilde{M}$ on $\mathrm{GL}(n, \mathbb{R})$ induced by vector M_I. Then since $\widetilde{M}(A) = \sum_{i,j} \widetilde{M}_{ij}(A) \frac{\partial}{\partial x^{ij}} |_A$, we have

$$\widetilde{M}_{kl} = \widetilde{M} x^{kl}$$

  (where for any $k = i, l = j$, $\frac{\partial x^{kl}}{\partial x^{ij}} = 0$).
- Then for any $A \in \mathrm{GL}(n, \mathbb{R})$, we have

$$\widetilde{M} x^{kl}(A) = \widetilde{M}_A(x^{kl}) = L_{A*} M_I(x^{kl}) = M_I(x^{kl} \circ L_A).$$

- Consider $x^{kl} \circ L_A : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$, for any matrix $B \in \mathrm{GL}(n, \mathbb{R})$, we have

$$(x^{kl} \circ L_A)(B) = x^{kl}(AB) = (AB)_{kl} = \sum_{\alpha=1}^{n} A_{k\alpha} B_{\alpha l}$$

where $\frac{\partial}{\partial x^{ij}}(x^{kl} \circ L_A) = A_{ki}$ when $l = j$ and equal to zero when $l \neq j$ (since only $B_{ij} = x^{ij}$). Hence,

$$\widetilde{M} x^{kl}(A) = \widetilde{M}_{kl} = M_I(x^{kl} \circ L_A) = \sum_{ij} M_{ij} \frac{\partial}{\partial x^{ij}}|_I (x^{kl} \circ L_A) = \sum_{\alpha=1}^{n} M_{\alpha l} A_{k\alpha}.$$

- Then consider $\frac{\partial}{\partial x^{ij}} \widetilde{M} x^{kl} = M_{jl}$ when $k = i$ and equal to zero when $k \neq i$. Thus, for another tangent vector N_I at I, we have

$$\begin{aligned} N_I(\widetilde{M} x^{kl}) &= \sum_{i,j} N_{ij} \frac{\partial}{\partial x_{ij}} (\widetilde{M} x^{kl}) \\ &= \sum_{j=1}^{n} N_{kj} M_{jl} \\ &= (NM)_{kl} \end{aligned}$$

- Finally in conclusion,

$$\begin{aligned} [\widetilde{M}, \widetilde{N}]_I &= \sum_{i,j} (\widetilde{M}(\widetilde{N}^{ij}) - \widetilde{N}(\widetilde{M}^{ij}))|_I \frac{\partial}{\partial x^{ij}} \\ &= \sum_{i,j} ((MN)_{ij} - (NM)_{ij}) \frac{\partial}{\partial x^{ij}} \end{aligned}$$

that is $[M, N] = MN - NM$ where MN and NM are the multiplication of matrixes.

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