Constructions with straightedge and compass

Start with two points:

Build from those:

- points, lines, circles
- Given 2 points, can construct the line through them and the circle centered at one point and passing through the other
- Given 2 lines (or 2 circles, or 1 line 1 circle) can construct their intersection.

Say a number $\ell \in \mathbb{R}$ is *constructible* if, with straightedge and compass, we can construct a point C on the line through A to B such that the (signed) distance from A to C is ℓ times the (signed) distance from A to B.

Theorem. *A number $\ell \in \mathbb{R}$ is constructible if and only if $\ell \in K_n$ where $\mathbb{Q} = K_0 \subset K_1 \subset K_2 \subset \ldots \subset K_n$ and $K_i = K_{i-1}(\sqrt{\alpha_i})$ with $\alpha_i \in K_{i-1} \cap \mathbb{R}_{>0}$.*

Corollary. *If $\ell \in \mathbb{R}$ is constructible then $[\mathbb{Q}(\ell) : \mathbb{Q}] = 2^m$, $m \in \mathbb{Z}_{\geq 0}$.*

Remark. The converse is not true.

Three classic problems:

- Squaring a circle
- Trisect an angle
- Duplicate a cube

Under straightedge and compass construction,

1. It is impossible to duplicate a cube that has twice the volume as the original cube, since $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$.
2. It is impossible to squaring a circle, i.e. construct a square whose area is that of a given circle, i.e. $\sqrt{\pi}$ is not constructible, since $[\mathbb{Q}(\sqrt{\pi}) : \mathbb{Q}] = \infty$.
3. It is impossible to trisect an arbitrary angle. ($\cos 60° = \frac{1}{2}$ is constructible but $\cos 20°$ is not)

$$\begin{aligned} &\cos 3\theta = 4\cos^3\theta - 3\cos\theta \\ \implies &\frac{1}{2} = 4(\cos 20°)^3 - 3\cos 20° \\ \implies &\cos 20° \text{ is a root of } 4x^3 - 3x - \frac{1}{2}. \end{aligned}$$

Let $x = \frac{y}{2}$, then $8x^3 - 6x - 1 = y^3 - 3y - 1$ is irreducible in $\mathbb{Z}/2\mathbb{Z}[y]$. So $8x^3 - 6x - 1$

is irreducible in $\mathbb{Z}[x] \implies$ irreducible in $\mathbb{Q}[y] \implies 4x^3 - 3x - \frac{1}{2}$ is irreducible in $\mathbb{Q}[x] \implies [\mathbb{Q}(\cos 20°) : \mathbb{Q}] = 3$.

So we can construct a 17-gon since

$$\cos\frac{2\pi}{17} = \frac{1}{16} + \frac{\sqrt{17}}{16} + \frac{\sqrt{34 - 2\sqrt{17}}}{16} + \frac{1}{8}\sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}}}.$$

$$\mathbb{Q} \to \mathbb{Q}(\sqrt{17}) \to \mathbb{Q}\left(\sqrt{34 + 2\sqrt{17}}\right)\left(\ni \sqrt{34 - 2\sqrt{17}} = \frac{8\sqrt{17}}{\sqrt{34 + 2\sqrt{17}}}\right) \to \mathbb{Q}\left(\cos\frac{2\pi}{17}.\right)$$

<u>Claim</u> Can construct a regular n-gon if and only if can construct $\cos\frac{2\pi}{n}$.

Corollary. *If p is prime and a regular p-gon is constructible then $p = 2^k + 1$. ($\implies k = 2^\ell$)*

$\alpha = e^{2\pi i/p} = \cos\frac{2\pi}{p} + i\sin\frac{2\pi}{p}$.

$\frac{\alpha + \frac{1}{\alpha}}{2} = \cos\frac{2\pi}{p}, \alpha^2 - (2\cos\frac{2\pi}{p})\alpha + 1 = 0 \implies \left[\mathbb{Q}\left(\cos\frac{2\pi}{p}, \alpha\right) : \mathbb{Q}\left(\cos\frac{2\pi}{p}\right)\right] \leq 2$.

Since $\alpha \notin \mathbb{R} \ni \cos\frac{2\pi}{p}$, this is equal. But $\mathrm{minpol}_{\mathbb{Q}}(\alpha) = x^{p-1} + x^{p-2} + \ldots + 1 \implies [\mathbb{Q}(\alpha) : \mathbb{Q}] = p - 1$.

Then $\mathbb{Q}(\alpha) \xrightarrow{2} \mathbb{Q}\left(\cos\frac{2\pi}{p}\right) \xrightarrow{\frac{p-1}{2}} \mathbb{Q} \implies$ [] so if p-gon is constructible then $p - 1 = 2^k$.

A line through 2 points with coordinates in a field K has an equation over K. The intersection of two lines with equations over K is either $\emptyset$ or a point with coordinates in K. The intersection of a line over K with a circle in K is either $\emptyset$, a point with coordinate in K, two point with coordinates in K, or two points with coordinates in $K(\sqrt{\alpha})$ for some $\alpha \in K, \alpha > 0$.

Intersecting two circles: $x^2 + y^2 = r_1$, $(x-a)^2 + (y-b)^2 = r_2 \implies -2ax + a^2 - 2by + b^2 = r_2 - r_1$. If $(a, b) \neq (0, 0)$ then it defines a line over K. So: intersection of two circles over K is either $\emptyset$, a point over K, two points in K, or two points over $K(\sqrt{\alpha})$.

Some geometry:

1. Given 2 points, we can construct perpendicular bisector of the segment between them.
2. Given a line ℓ and a point p, can construct a line through p which is perpendicular to ℓ.
3. Given a line ℓ and a point p, we construct a line through p that is parallel to ℓ.
4. Given points p, q, r and a line ℓ containing r, can construct a point s on ℓ such that the segments rs and pq gave the same length.

5. Can construct an angle iff can construct $\cos\theta$.
6. If a, b are constructible then so are $a+b, -a, ab$ (through similar triangles), $\frac{1}{a}$. (So constructible numbers is a field.) Also $\sqrt{a}$.