Notes for Math 566 – Algebraic Combinatorics

Yiwei Fu

WN 2022

Contents

Office hours: Tu, Fr 1:00 - 2:20 pm, 4868 East Hall.

Chapter 1

Graph and Trees

1.1 Linear Algebra Preliminaries

Let M be a $p \times p$ matrix with entries in $\mathbb{C}$. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$ are defined by

$$\det(t\,\mathrm{id} - M) = \prod_{i=1}^{p} (t - \lambda_i).$$

Taking coefficients of t^{p-1} on both sides we obtain

$$\operatorname{tr} M = \sum_k \lambda_k. \tag{1.1.1}$$

Lemma 1.1.1. *Let $f(t) \in \mathbb{C}[t]$. Then $f(M)$ have eigenvalues $f(\lambda_1), \ldots, f(\lambda_p)$.*

Proof. If M is diagonalizable, then the statement is clear: $f(M)$ has the same eigenvectors as M, with eigenvalues $f(\lambda_k)$. Then use a continuity argument. (Diagonalizable matrices are dense.) Alternative proof: use Jordan's normal form. ∎

Combining (1.1.1) with the lemma, we have

$$\operatorname{tr} M^\ell = \sum_k \lambda_k^\ell. \tag{1.1.2}$$

PROBLEM: [A solution is given in Stanley's textbook.] Let $\alpha_1, \ldots, \alpha_r$ and $\beta_1, \ldots, \beta_r$ be nonzero complex numbers such that for *all* positive integer ℓ we have

$$\alpha_1^\ell + \ldots + \alpha_r^\ell = \beta_1^\ell + \ldots + \beta_r^\ell.$$

Show that this implies that $r = s$, and that α's are a permutation of β's.

In the majority of forthcoming applications, M is symmetric and real. Then it is diagonalizable, with real eigenvalues $\lambda_1, \ldots, \lambda_p$.

1.2 Counting Walk

Let G be a graph on the vertex set $\{1, \ldots, p\}$. (We allow loops and multiple edges.) Let $M = A(G)$ be its adjacency matrix.

OBSERVATION The number of walks of length ℓ from i to j is equal to $(M^\ell)_{ij}$.

In general, counting walks requires knowing the matrix M (equivalently, knowing both the eigenvalues λ_k and the corresponding eigenvectors). On the other hand, some enumerative information can be extracted from the eigenvalues alone:

Proposition 1.2.1. *The number of marked closed walks of length ℓ is equal to $\sum_{k=1}^p \lambda_k^\ell$.*

Here "marked" means that the starting location is fixed, as is a particular instance of passing through it, in case we do it several times.

Proof. By the last observation, the number of marked closed walks of length ℓ is equal to $\operatorname{tr} M^\ell$, which equals to $\sum_{k=1}^p \lambda_k^\ell$ by (1.1.2). ■

Example 1.2.1. Let $G = K_p$, the complete graph on p vertices. Let J denote the $p \times p$ matrix all of whose entries are 1. Let I denote the $p \times p$ identity matrix. Then $A(G) = J - I$. Obviously $\operatorname{rk} J = 1$ and $\operatorname{tr} J = p$. Hence the eigenvalues of J are $0, \ldots, 0, p$, and the eigenvalues of $A(G) = J - I$ are $-1, \ldots, -1, p-1$.

Corollary 1.2.1. *There are $(p-1)^\ell + (-1)^\ell (p-1)$ marked closed walks of length ℓ in K_p.*

NOTE This is the number of $(\ell + 1)$-letter words in a p-letter alphabet in which no two consecutive letters are identical, and which begin and end by the same letter.

PROBLEM Show that the number of walks of length ℓ between two distinct vertices in K_p differs by 1 from the number of closed walks of length ℓ starting at a given vertex.

1.3 Eigenvalues of Adjacency Matrices

RECALL

$$\# \text{ of marked closed walks of length } \ell = \sum_{i=1}^{p} \lambda_i^\ell.$$

It can be used backwards: using counted walks to compute eigenvalues.

Example 1.3.1. Let $G = K_{n,m}$ a complete bipartite graph.

$$\# \text{ of marked closed works of length } \ell = \begin{cases} 0 & \ell = 2k+1 \\ 2n^{\ell/2}m^{\ell/2} & \ell = 2k \end{cases} = (\sqrt{nm})^\ell + (-\sqrt{nm})^\ell$$

$\xRightarrow{\text{Problem}}$ eigenvalues are $\sqrt{nm}, -\sqrt{nm}, 0, \ldots, 0$.

PROBLEM Prove that, for G connected, the $\operatorname{diam}(G) < \#$ of distinct eigenvalues.

Example 1.3.2. $K_p = 1 < 2, K_{n,m} = 2 < 3$.

1.4 Inequalities for the Maximal Eigenvalue

Definition 1.4.1. Suppose G a graph with vertices $= \{1, \ldots, p\}$. Let

$$\lambda_{\max} := \max_i |\lambda_i| = \max \lambda_i.$$

Proposition 1.4.1.

$$\lambda_{\max} \leq \max \deg(G)$$

Proof. For any vector $X = (x_k) \in \mathbb{C}^p$,

$$\max_j |(A(G)X)_j| \leq \max \deg(G) \cdot \max_k |X_k|$$

Now suppose X is an eigenvector of $A(G)$ with eigenvalue λ. Then

$$\max_j |(A(G)X)_j| = |\lambda| \max_k |X_k| \leq \max \deg(G) \cdot \max_k |X_k| \implies |\lambda| \leq \max \deg(G)$$

This holds for all eigenvalue λ_i, which proves our proposition. ■

ALTERNATE PROOF: by counting closed walks ($\leq \sum \max \deg(G)^\ell$.)

PROBLEM Prove that $\lambda_{\max} \geq$ average degrees of the vertices of G.
HINT for symmetric real matrix M we have $\lambda_{\max} = \max_{|x|=1} x^T M x$.

Corollary 1.4.1. *$\#$ of closed walk of length ℓ grows exponentially in ℓ with a rate $\geq$ average degree.*

1.5 Eigenvalue of Block Anti-diagonal Matrices

$$M = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \in \mathbb{R}_{n+m}$$

Lemma 1.5.1. *The non-zero eigenvalues (called "singular values" of B) of M are $\pm\sqrt{\mu_i}$ where μ_i are nonzero eigenvalues of $B^T B$ with multiplicities.*

Note that $B^T B$ is positive definite.

Proof. Let $F_X(t) = \det(t \,\mathrm{id}_p - X)$.

$$\begin{bmatrix} t\,\mathrm{id}_n & -B \\ -B^T & t\,\mathrm{id}_m \end{bmatrix} \begin{bmatrix} \mathrm{id}_n & B \\ 0 & t\,\mathrm{id}_m \end{bmatrix} = \begin{bmatrix} t\,\mathrm{id}_n & 0 \\ -B & -B^T B + t^2\,\mathrm{id}_m \end{bmatrix}$$

$$F_M(t) \cdot t^m = t^n F_{B^T B}(t^2)$$

and the claim follows ■

So now we are equipped to compute the eigenvalue of bipartite graphs.

Example 1.5.1. Suppose $G = K_{n,m}$, $B^T B$ is $m \times m$ matrix with all entries being n. So the eigenvalues of $B^T B = nm, 0, 0, \ldots$. So eigenvalues of $A(G)$ is $\sqrt{mn}, -\sqrt{mn}, 0, 0, \ldots$

<u>PROBLEM</u> Let G to be the graph obtained by removing n disjoint edges from $K_{n,n}$. Find the eigenvalue of G.

Example 1.5.2. Let G be a $2n$-cycle. $M_{2n} = A(G) = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$. The $B^T B = 2I_n + M_n$ for an appropriate labeling.

So if the eigenvalue of n-cycle are $\lambda_1, \ldots, \lambda_n$. Then the eigenvalues of $2n$-cycles are $\pm\sqrt{\lambda_i + 2}$.

1.6 Eigenvalues of Circulant Matrices

Definition 1.6.1. A circulant matrix is of the form

$$M = \begin{bmatrix} s_0 & s_1 & s_2 & \dots & s_{p-1} \\ s_{p-1} & s_0 & s_1 & \dots & s_{p-2} \\ \vdots & & & & \\ s_1 & s_2 & s_3 & \dots & s_0 \end{bmatrix}.$$

Lemma 1.6.1. *M has eigenvalues*

$$\lambda_k = \sum_{j=0}^{p-1} s_j e^{\frac{2\pi i}{p} jk}, k = 0, 1, \dots, p-1.$$

Notice that

$$\lambda_k = \sum_{j=0}^{p-1} s_j e^{\frac{2\pi i}{p} jk} = s\left(e^{\frac{2\pi i}{p} k}\right) \quad p\text{-th root of unity.}$$

where

$$s(x) = \sum_{j=0}^{p-1} s_j x^j.$$

Proof. Let

$$T = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}$$

We have that the eigenvalues of T and p-th roots of unity and characteristic polynomial is $t^p - 1$.

Key observation: $M = s(T)$. ■

Definition 1.6.2. A graph G is circulant if $A(G)$ is circulant, for some choice of vertex labeling.

Corollary 1.6.1. *The eigenvalue of p-cycle are*

$$2\cos\left(\frac{2\pi k}{p}\right), k = 0, 1, \dots, p-1.$$

Proof. By Lemma 1.6.1, we have that

$$\lambda_k = e^{\frac{2\pi i}{p}k} + e^{\frac{2\pi i}{p}(p-1)k} = e^{\frac{2\pi ik}{p}} + e^{-\frac{2\pi ik}{p}} = 2\cos\left(\frac{2\pi k}{p}\right). \quad \blacksquare$$

Remark. This formula is consistent with the formula linking the eigenvalues of a $2n$-cycle and an n-cycle: if $2\cos\alpha = \lambda$, then $2\cos\frac{\alpha}{2} = \pm\sqrt{2+\lambda}$.

PROBLEM Find the eigenvalues of the graph obtains by removing n disjoint edges from K_{2n}.

1.7 Eigenvalues of Cartesian Products

Definition 1.7.1. Suppose G, H are graphs with no loops. Define graph $G \times H$ where

$$V(G \times H) = \{(g, h) : g \in V(G), h \in V(H)\},$$

and we have two kinds of edges:

- $(g, h) - (g', h)$ for $g - g'$
- $(g, h) - (g, h')$ for $h - h'$

Example 1.7.1.

1. Grid graph = path $\times$ path
2. Discrete annulus (cylinder) = cycle $\times$ path
3. Discrete torus = cycle $\times$ cycle
4. n-cube graph

Proposition 1.7.1. *If G has eigenvalues $\lambda_1, \lambda_2, ldots$, H has eigenvalues $\mu_1, \mu_2, \ldots$ Then $G \times H$ has eigenvalues $\lambda_i + \mu_j$ for any pair i, j.*

Proof 1. (Tensor product) V_G, V_H are vector spaces formally spanned by vertices of G, H. Take $u = \sum \alpha_g g \in V_G, v = \sum \beta_h h \in V_H$. We have

$$u \otimes v = \sum_{g,h} \alpha_g \beta_h (g, h) \in V_{G\times H}.$$

The

$$A(G \times H)(u \otimes v) = (A(G)u) \otimes v + u \otimes (A(h)v)$$

Suppose u, v are eigenvectors $i.e.$ $A(G)u = \lambda u$, $A(H)v = \mu v$. Then we get

$$A(G \times H)(u \otimes v) = \lambda u \otimes v + u \otimes \mu v = (\lambda + \mu)(u \otimes v). \qquad \blacksquare$$

Proof 2. (Marked closed walk) Walk in $G \times H \xleftrightarrow{1-1}$ a <u>shuffle</u> of marked closed walks in $G \& H$.

$$\begin{aligned}
&\#\text{ of closed walks of length } \ell \text{ in } G \times H \\
=&\sum_k \binom{\ell}{k} \sum_i \lambda_i^k \sum_j \mu_j^{\ell-k} \\
=&\sum_i \sum_j \sum_k \binom{\ell}{k} \lambda_i^k \mu_j^{\ell-k} \\
=&\sum_{i,j} (\lambda_i + \mu_j)^\ell
\end{aligned}$$

This set of numbers are unique by problem in lecture 1, so they must be the eigenvalues of $G \times H$. $\blacksquare$

<u>PROBLEM</u> Take a 3×3 grid, find the number of marked closed walks of length ℓ.
<u>PROBLEM</u> Direct problem of 8-cycle and K_2.

<u>n-CUBE GRAPH:</u>

$$(K_2)^n = \underbrace{K_2 \times K_2 \times \cdots K_2}_{n \text{ times}}.$$

Example 1.7.2. When $n = 3$, we have a 3-D cube:

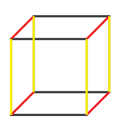


Figure 1.1: Cube graph $K_2 \times K_2 \times K_2$

K_2 has adjacency matrix $A(K_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ with eigenvalues $\pm 1 \implies$ eigenvalues of $(K_2)^n$ are

$$\lambda = \underbrace{\pm 1 \pm 1 \pm \ldots \pm 1}_{n \text{ times}}.$$

Proposition 1.7.2. *The eigenvalues of $(K_2)^n$ are of the form $n - 2k$ where $k = 0, 1, \ldots, n$, each* *with multiplicities* $\binom{n}{k}$ *i.e. the number of marked closed walks of length* ℓ *in the* $n-$*cube graph is*

$$\sum_{k=0}^{n}\binom{n}{k}(n-2k)^{\ell}$$

which is 0 *when* ℓ *is odd.*

1.8 Random Walks

Let G be a regular graph of degree d on p vertices.

Example 1.8.1. $G=(K_2)^n$ is regular with $d=n$.

A simple random walk on G originating at a vertex v is a random walk with equal probabilities for each adjacent vertices.

$$\begin{aligned} & \mathbb{P}\,(\text{walk is back at } v \text{ after } \ell \text{ steps}) \\ = & \frac{1}{d^{\ell}}\#\{\text{marked closed walks of length } \ell \text{ orginiating from } v\} \\ = & d^{-\ell}p^{-1}\sum_{1}^{p}\lambda_i^{\ell}. \end{aligned}$$

assuming that $\mathrm{Aut}(G)$ acts transitively on vertices.

Notice that an arbitrary regular G does not necessarily have that condition, but the converse is true.

Example 1.8.2. The probability that a simple random walk on $(K_2)^n$ returns to its origin after ℓ steps is

$$\frac{1}{n^{\ell}2^n}\sum_{k=0}^{n}\binom{n}{k}(n-2k)^{\ell}$$

Chapter 2

Tilings, Spanning Trees, and Electric Networks

2.1 Domino Tilings ("Dimers")

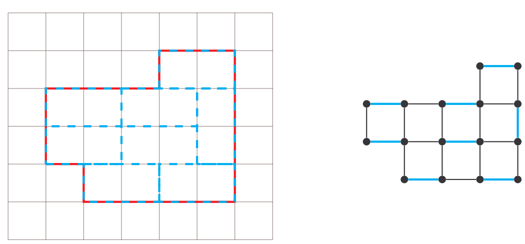
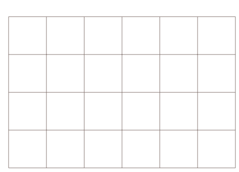


Figure 2.1: An example of domino tiling and perfect matching in its dual graph

A domino tiles decompose part of grids into 1×2 rectangles.

Think of it another way: the "dual graph" where squares are vertices, and there exists an edge between two vertices iff the corresponding squares shares an edge. A tiling is a perfect matching between these vertices.

Special case: $m \times n$ rectangular boards



Without loss of generality, assume that n is even. We denote the answer as $T(m, n)$

The dual graph G is m-chain $\times$ n-chain. Notice that G is bipartite.

$M = A(G)$ has the form $\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$ given appropriate labeling of vertices where B is a square matrix.

CLAIM $T(m, n) =$ the permanent of matrix B.

Permanents do not have nice properties, thus they are hard to calculate. In order to better calculate the permanent of B, let $\tilde{B}$ obtained from B by replacing the 1's by corresponding to vertical tiles by i's where $i^2 = -1$.

Proposition 2.1.1. $T(m, n) = \operatorname{per}(B) = \pm\det(\tilde{B})$.

Lemma 2.1.1 (exercise). *Any two domino tilings of a rectangular board are related to each other via "flips" of the form (two horizontal $\leftrightarrow$ two vertical)*

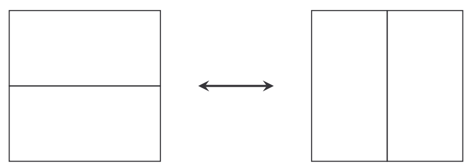


Figure 2.2: Example of a flip

Proof of Prop. This is equivalent to all nonzero terms in $\det(\tilde{B})$ are equal and are ± 1. The latter claim follows from the former, since the all-horizontal tiling contributes ± 1.

Then it is enough to show that the contributions of two tilings that differ by a flip are equal to each other.

It means swapping two diagonal entries, thus change the sign of permutation, but one of them is 1^2 while the other being i^2, so the result does not change. ■

Now we can use some linear algebra to calculate the determinant.

Denote $\tilde{M} = \begin{bmatrix} 0 & \tilde{B} \\ \tilde{B}^T & 0 \end{bmatrix}$. Then $\det(\tilde{M}) = \pm(\det(\tilde{B}))^2 = \pm(T(m, n))^2$.

OBSERVATION We have

$$M = \mathrm{id}_m \otimes A_n + A_m \otimes \mathrm{id}_n,$$

where A_n, A_m are adjacency matrices of chain graphs. Similarly,

$$\tilde{M} = \mathrm{id}_m \otimes A_n + iA_m \otimes \mathrm{id}_n,$$

since $\tilde{M}$ obtained by vertical tile with i's. Hence the eigenvalues of $\tilde{M}$ are $\lambda_i + i\mu_k$.

Now we only need to find the eigenvalues of chain graph. For a n-chain, we have

$$A_n = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

Proposition 2.1.2. *The eigenvalues of A_n are*

$$\lambda_k = 2\cos\left(\frac{k\pi}{n+1}\right) \quad \textit{for } k = 1, \dots, n.$$

Proof. An eigenvector $u = (u_1, \dots, u_n)^T$ of A_n associated with eigenvalue λ satisfies

$$u_{j-1} + u_{j+1} = \lambda u_j, \quad 1 \le i \le n$$

with the convention that $u_0 = u_{n+1} = 0$.

A divine revelation: recall that

$$\sin\alpha + \sin\beta = 2\cos\frac{\beta - \alpha}{2}\sin\frac{\alpha+\beta}{2}.$$

This suggests taking

$$u_j = \sin\left(\frac{\pi k j}{n+1}\right) \quad \text{for } j = 1, \dots, n.$$

with eigenvalue

$$\lambda_k = 2\cos\left(\frac{k\pi}{n+1}\right).$$

■

Example 2.1.1.

$$n = 3, \det(t\,\mathrm{id} - A_3) = t^3 - 2t = t(t-\sqrt{2})(t+\sqrt{2}).$$

So the eigenvalues are

$$\lambda_1 = \sqrt{2} = 2\cos\left(\frac{1\pi}{4}\right), \lambda_2 = 0 = 2\cos\left(\frac{2\pi}{4}\right), \lambda_2 = -\sqrt{2} = 2\cos\left(\frac{3\pi}{4}\right).$$

Now

$$\begin{aligned}
\det \tilde{M} &= \prod_{j=1}^{n}\prod_{k=1}^{m}\left(2\cos\frac{j\pi}{n+1} + i2\cos\frac{k\pi}{m+1}\right)\\
&= \prod_{j=1}^{n/2}\prod_{k=1}^{m}\left(2\cos\frac{j\pi}{n+1} + i2\cos\frac{k\pi}{m+1}\right)\left(2\cos\frac{(n+1-j)\pi}{n+1} + i2\cos\frac{k\pi}{m+1}\right)\\
&= \prod_{j=1}^{n/2}\prod_{k=1}^{m}\left(2\cos\frac{j\pi}{n+1} + i2\cos\frac{k\pi}{m+1}\right)\left(-2\cos\frac{j\pi}{n+1} + i2\cos\frac{k\pi}{m+1}\right)\\
&= \pm\prod_{j=1}^{n/2}\prod_{k=1}^{m}\left(4\cos^2\frac{j\pi}{n+1} + 4\cos^2\frac{k\pi}{m+1}\right)
\end{aligned}$$

Theorem 2.1.1 (P.Kasteleyn, M.Fisher, H.N.V.Temperley, 1961)**.** *When m is even,*

$$T(m,n) = \prod_{j=1}^{n/2}\prod_{k=1}^{m/2}\left(4\cos^2\frac{j\pi}{n+1} + 4\cos^2\frac{k\pi}{m+1}\right).$$

When m is odd,

$$T(m,n) = \prod_{j=1}^{n/2} 2\cos\frac{j\pi}{n+1}\prod_{k=1}^{(m-1)/2}\left(4\cos^2\frac{j\pi}{n+1} + 4\cos^2\frac{k\pi}{m+1}\right).$$

Example 2.1.2. For $n = m = 8$, we get $T(8,8) = 12,988,816 = 3604^2$.

PROBLEM* For any positive integer $a \in \mathbb{Z}_{>0}, T(4a,4a)$ is a perfect square, $T(4a-2,4a-2)$ is twice a perfect square.

Asymptotics of $T(n,n)$: reasonable to expect $T(n,n) \sim e^{cn^2}$.

We take the natural log of $T(n,n)$:

$$\begin{aligned}
\frac{\ln T(n,n)}{n^2} &= \frac{1}{n^2}\sum_{k=1}^{n/2}\sum_{j=1}^{n/2}\ln\left(4\cos^2\frac{\pi k}{n+1} + 4\cos^2\frac{\pi j}{n+1}\right)\\
&\sim \frac{1}{\pi^2}\sum\sum\left(\frac{\pi}{n+1}\right)^2\ln\left(4\cos^2\frac{\pi k}{n+1} + 4\cos^2\frac{\pi j}{n+1}\right)
\end{aligned}$$

Notice that the right-hand side is a Riemann sum of the function $\ln(4\cos^2 x + 4\cos^2 y)$.

So the sum approaches to

$$\frac{1}{\pi^2}\int_0^{\pi/2}\int_0^{\pi/2}\ln(4\cos^2 x+4\cos^2 y)dxdy=\frac{K}{\pi}$$

where K is Catalan's constant. As of today, it is not known whether it is irrational, nor transcendental.

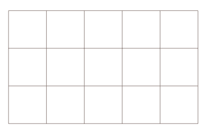
So we have $T(n,n)\approx 1.34^{n^2}$.

Another way to define Catalan's constant:

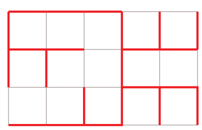
$$K=\beta(2)=\sum_{i=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\dots.$$

2.2 Spanning Tree in Grid Graphs

Suppose a grid graph G:



We can keep some edges and discard others to obtain a connected acyclic subgraph of G (which is a spanning tree).



Theorem 2.2.1 (H.N.V. Temperley, 1974)**.** *Consider a rectangular board of odd size* $(2k-1\times 2\ell-1)$ *with one corner removed. The number of domino tilings of the board is equal to the number of spanning trees in the* $k\times\ell$ *grid.*

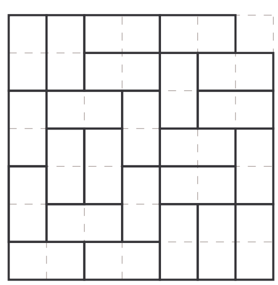


Figure 2.3: A domino tiling satisfying the condition

Proof. Find a bijection between domino tilings and spanning trees.

PROBLEM Prove that Temperley's map showed in Figure 2.4 produces a tree.

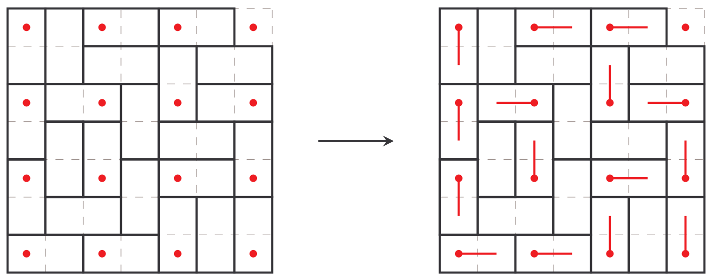


Figure 2.4: Converting domino tiling into trees

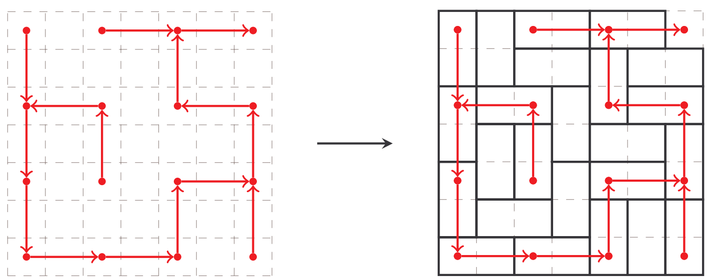


Figure 2.5: Converting domino tiling into trees

Now we have a formard map. We also need to obtain the inverse map from spanning trees to domino tiling. Fixing a border point as the root of the tree, we can make the tree a directed graph and addign domino tiles accordingly. ■

Corollary 2.2.1.

$$\#\text{ of spanning trees in a } k \times \ell \text{ grid} \approx \left(e^{\frac{4K}{\pi}}\right)^{k\ell} \approx 3.21^{k\ell}.$$

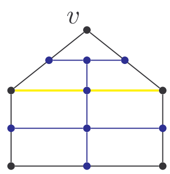
PROBLEM Prove that the number of domino tilings (if exist) of an odd-by-odd rectangle with a boundary box removed doesn't depend on which box we removed.

2.3 Spanning Trees of Planar Graphs

Suppose P is a polygon, G a polygonal subdivision of P. Define H by adding midpoints and extra vertex in each bounded face and adding edges to connect them.

<u>PROBLEM</u> Show that the number of spanning trees in G is equal to the number of perfect matchings in H with one vertex that are also in P removed.

Example 2.3.1.

$G =$ 

The number of spanning trees of $= 4+4+3 = 11$. If we take the vertex v specified above we have:

$H - \{v\} =$ 

We can verify that it also has 11 matchings.

<u>NOTE</u> Here the for arbitrary vertex v the result would be the same.

2.4 The Diamond Lemma

Definition 2.4.1. A <u>one-player game</u> is define by:

- the set of positions $\mathcal{S}$
- for each $s \in \mathcal{S}$ a set of positions $s' \neq s$ into which the player can from from from s. Denote as $s \rightsquigarrow s'$.

If the latter set is empty, then S is called <u>terminal</u>.

A <u>play sequence</u> is a sequence

$$s \rightsquigarrow s' \rightsquigarrow s'' \rightsquigarrow \dots$$

A game is <u>terminating</u> is $\nexists$ infinite play sequences.

A game is <u>confluent</u> is its outcome is uniquely determined by initial position.

Lemma 2.4.1 (The Diamond Lemma for terminating games)**.** *For a one-player game, assume that*

- *the game is terminating*
- ◇ *(diamond condition) $\forall s \in \mathcal{S}$, $\forall s \rightsquigarrow s'$, $s \rightsquigarrow s''$, $\exists$ some position that can be reach from both s' and s''. (You never say goodbye forever!)*

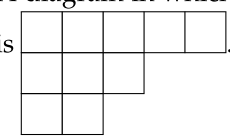
Then the game is confluent.

Proof. Color the position:

- Green is the terminal position reachable from this position is unique.
- Red otherwise.

Assume a red position exists. Starts at the red position until no move into red position exists.

For each green position, there is a unique terminal position. Since it starts from red there need to be two distinct ones, but that is a contradiction, since all green position will have a common successor, which would have color green. ■

Definition 2.4.2 (Young diagrams)**.** A diagram in which the number of boxes on a row is decreasing. An example of which is .

We define a one-player game where: Position = {Young diagrams}

Move = Removal of a domino tile from the SE rim that also results in a Young diagram.

<u>Claim</u> This game is <u>confluent</u>.

Note: the remaining shape would always be a staircase. If we color the blocks black and white alternatively, we can determine the final shape by the difference between white and black boxes.

<u>Problem</u> Consider a similar game but we are removing border strips consisting of p boxes ($p \in \mathbb{N}$). Prove that the game is confluent.

Definition 2.4.3 (Young tableaux)**.** Take a Young diagram and fill it with numbers so that each row and column is in increasing order. Such diagram is called a standard Young tableau (SYT).

Or take a skew shape where a Young diagram is taken away from the top left corner of another Young diagram. Then filling it the same way we obtain standard skew tableau (Skew SYT).

1	2	5	8
3	6	7	
4			

			3	6
	1	4	8	9
2	5	7	10	

Figure 2.6: A standard Young tableau (left) and skew tableau (right)

<u>Jeu De Taquin</u> [M.-P. Schützenberger] Given a skewed tableau, choose a top-left corner

piece and move the blocks one at a time so that after a series of moves we also get a skewed tableau.

$$
\begin{array}{ccccc}
 & & 2 & 4 & 8 \\
\leftarrow & 1 & 3 & 6 & \\
5 & 7 & & &
\end{array}
\longrightarrow
\begin{array}{ccccc}
 & & 2 & 4 & 8 \\
1 & \leftarrow & 3 & 6 & \\
5 & 7 & & &
\end{array}
$$

$$
\longrightarrow
\begin{array}{ccccc}
 & & 2 & 4 & 8 \\
1 & 3 & \leftarrow & 6 & \\
5 & 7 & & &
\end{array}
\longrightarrow
\begin{array}{ccccc}
 & & 2 & 4 & 8 \\
1 & 3 & 6 & & \\
5 & 7 & & &
\end{array}
$$

Figure 2.7: One step in a jeu de taquin game

The game ends on a SYT, called a rectification of T.

PROBLEM The rectification is unique. (Jeu de tauqin is confluent.)

Definition 2.4.4 (Tutte Polynomial). $T_G(x, y)$ of a graph G is defined recursively as follows:

- G has no edges $\implies T_G = 1$.
- e edge in $G \implies$

$$
T_G = \begin{cases} xT_{G-e} & e \text{ is a bridge} \\ yT_{G-e} & e \text{ is a loop} \\ T_{G-e} + T_{G/e} & \text{otherwise} \end{cases}
$$

This is a two variable generalization of the chromatic polynomial.

PROBLEM Use the diamond lemma to show that T_G is well defined.

For non-terminating games, the diamond lemma does not necessarily hold:

Example 2.4.1 (Naive counterexample). Suppose a game:

$$1 \to 2 \to 3 \to 4 \to \ldots, n \to \infty, \forall n.$$

Then there are two outcomes for any given starting position (∞ or non-terminating).

Theorem 2.4.1 (Diamond Lemma for Non-terminating Games). *Suppose a one-player game. $\forall s \in \mathcal{S}$, $\forall s \rightsquigarrow s'$, $s \rightsquigarrow s''$, $\exists$ some position that can be reach from both s' and s'' in the same number of steps. Then the game is confluent.*

Moreover, if the game terminates for a given initial position, then it does so in a fixed number of steps.

Proof. Left as PROBLEM. ■

2.5 Loop-erased Walks

Definition 2.5.1 (G.Lawler, 1980)**.** Suppose G a connected graph. Let π be a (finite) walk in G. LE(π) "loop erasure" of π is defined by:

IF π doesn't intersect itself THEN LE(π) $:= \pi$ ELSE remove the 1st cycle that π get π'; LE(π) $:=$ LE(π')

STACKS & CYCLE POPPING

Recall: Markov chains.

"Running a Markov chain with stacks": at each state, decide on transition choices in advance.

v is a vertex, $u(v) = (u_1, u_2, u_3, \ldots)$ where u_k denotes the vertex we move to after visiting v for the k-th time. They are i.i.d RV's.

Assume: the Markov chain arrives with probability 1 at an absorbing state (where the stack is empty).

Given a collection of partially depleted stacks, we get a graph (A subgraph of G, the underlying oriented graph of Markov chain) determined by the top of each stack. The out degrees of each non-absorbing vertex in the graph is equal to 1. The removing of cycles from this graph is called cycle-popping.

LOOP-ERASED **RANDOM** WALK

Start at vertex s, and stop upon arriving at some absorbing state t.

OBSERVATION LERW is obtained by popping some cycle, leaving a path from s to t.

Define a game where positions are collections of stacks of at each vertex and the moves are cycle poppings.

CLAIM This game is confluent (by diamond lemma for non-terminating games).

The outcome of the game (after all cycles have been popped) is a rooted forest that is oriented towards absorbing states.

WILSON'S ALGORITHM [D.B. Wilson, 1996]

Input: connected loopless graph G. Output: random spanning tree T in G. fix a vertedx in G (a root) $T := r$.

WHILE T does not cover all vertices in G DO

pick a vertex $v \notin T$ (using any rule)
run a simple random walk π from v until it hits T.
$T \leftarrow T \cup \{v : v \in \mathrm{LE}(\pi)\}$.

Theorem 2.5.1. *This algorithm outputs a uniformly distributed spanning tree of G.*

Proof. Make r an absorbing state. Run Wilson's algorithm with stacks, each time designating the vertices in T as absorbing states. Loop erasure = cycle popping.

The algorithm terminates with probability 1, revealing the tree lying underneath all popable cycles.

Need: the output tree T is uniformly distributed.

Suppose $H =$ heap of cycles. $\mathbb{P}(T, H)$ denotes probability of getting T after removing H. We have

$$\mathbb{P}(T, H) = \left(\prod_{v \in T - \{r\}} \deg(v)^{-1} \right) \cdot \left(\prod_{v \in H} \deg(v)^{-1} \right),$$

while

$$\mathbb{P}(T) = \sum_{H} \mathbb{P}(T, H).$$

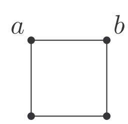
All the expressions above have the same values regardless of T, so the distribution is uniform. ∎

The art of computer programming, vol 4. (generate random combinatorial objects that satisfies certain distribution)

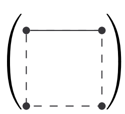
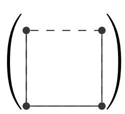
PROBLEM Let a, b be two vertices in a connected graph G. Show that the following two constructions produce the same distributions on the set of self-avoiding walks from a to b.

- Run a simple random walk π that starts at a and stops at b. Output $\mathrm{LE}(\pi)$.
- Choose uniformly at random a spanning tree in G. Output the walk from a to b in the spanning tree.

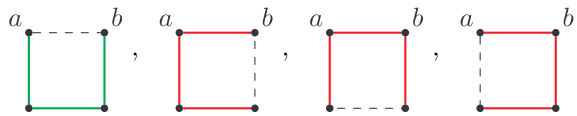
Corollary 2.5.1. *This distribution does not change if we swap a and b.*

Example 2.5.1. $G =$ 

. Using the first method we have

$\mathbb{P}$  $= \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, $\mathbb{P}$  $= \frac{1}{4}$.

For the spanning tree method, we have these spanning trees:



2.6 Flows

I actually don't drink any alcohol at all.

— Sergey Fomin

Definition 2.6.1. Suppose G is connected loopless graph with a, b designated as boundary vertices and all other vertices called interior vertices.

A flow f assigns a number $f(e, u, v)$ to each edge e with endpoint u, v, so that

- $f(e, u, v) = -f(e, v, u)$.
- $\forall$ interior vertex u, $\displaystyle\sum_{v \in V(G), (u,v) \in E(G)} f(e, u, v) = 0$. ("Conservation of flow")

It follows that $\exists$ number $|f|$, called the total flow from a to b, such that

$$\sum_{v \in V(G), (u,v) \in E(G)} f(e, u, v) \begin{cases} 0 & u \text{ interior} \\ |f| & u = a \\ -|f| & u = b. \end{cases} \tag{2.6.1}$$

We basically created a weighted graph in some sense: $w : E(G) \to \mathbb{R}_{>0}$.

Definition 2.6.2. For any vertex u, $w(u) := \displaystyle\sum_{e=(u,v) \in E(G)} w(e)$.

A "weighted version" of a simple random walk is a Markov chain with transition probability from u to v being $\displaystyle\sum_{e=(u,v) \in E(G)} \frac{w(e)}{w(u)}$.

PROBLEM* Generalize Wilson's algorithm to weighed graphs.

RESISTOR NETWORKS

Definition 2.6.3. Resistor network = weighted graph, conductance = edge weights, conductance = $\dfrac{1}{\text{resistence}}$.

(2.6.1) is the first Kirchhoff law. (conversation of charge/current)

The second Kirchhoff law expresses the conversation of energy: given a cycle

$$u_0 \overset{e_1}{\text{---}} u_1 \overset{e_2}{\text{---}} u_2 \overset{e_3}{\text{---}} \dots \overset{e_k}{\text{---}} u_k = u_0,$$

we have

$$\sum_{i=1}^{k} \frac{f(e_i, u_{i-1}, u_i)}{w(e_i)} = 0. \tag{2.6.2}$$

(2.6.1) and (2.6.2) are a linear system of equations in $f(e, u, v)$.

Theorem 2.6.1. *Kirchhoff's equations* (2.6.1) *and* (2.6.2) *have a unique solution.*

2.7 Potentials

Definition 2.7.1. Suppose f satisfies (2.6.2). Define the potential $p = p_f$, a function on the vertices of G, as follows:

- assign an arbitrary value to $p(a)$
- for any walk that starts at $a = u_0 \overset{e_1}{\text{---}} u_1 \overset{e_2}{\text{---}} u_2 \overset{e_3}{\text{---}} \dots \overset{e_k}{\text{---}} u_k = u$, set

$$p(u) := p(a) + \sum_{i=1}^{k} \frac{f(e_i, u_{i-1}, u_i)}{w(e_i)}.$$

Lemma 2.7.1. *The function $u \mapsto p(u)$ is well-defined.*