Notes for Math 566 – Algebraic Combinatorics

Yiwei Fu

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Contents

Office hours: Tu, Fr 1:00 - 2:20 pm, 4868 East Hall.

Chapter 1

Graph and Trees

1.1 Linear Algebra Preliminaries

Let M be a $p \times p$ matrix with entries in $\mathbb{C}$. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$ are defined by

$$\det(t \,\mathrm{id} - M) = \prod_{i=1}^{p} (t - \lambda_i).$$

Taking coefficients of t^{p-1} on both sides we obtain

$$\operatorname{tr} M = \sum_k \lambda_k. \tag{1.1.1}$$

Lemma 1.1.1. *Let $f(t) \in \mathbb{C}[t]$. Then $f(M)$ have eigenvalues $f(\lambda_1), \ldots, f(\lambda_p)$.*

Proof. If M is diagonalizable, then the statement is clear: $f(M)$ has the same eigenvectors as M, with eigenvalues $f(\lambda_k)$. Then use a continuity argument. (Diagonalizable matrices are dense.) Alternative proof: use Jordan's normal form. ■

Combining (1.1.1) with the lemma, we have

$$\operatorname{tr} M^\ell = \sum_k \lambda_k^\ell. \tag{1.1.2}$$

PROBLEM: [A solution is given in Stanley's textbook.] Let $\alpha_1, \ldots, \alpha_r$ and $\beta_1, \ldots, \beta_r$ be nonzero complex numbers such that for *all* positive integer ℓ we have

$$\alpha_1^\ell + \ldots + \alpha_r^\ell = \beta_1^\ell + \ldots + \beta_r^\ell.$$

Show that this implies that $r = s$, and that α's are a permutation of β's.

In the majority of forthcoming applications, M is symmetric and real. Then it is diagonalizable, with real eigenvalues $\lambda_1, \ldots, \lambda_p$.

1.2 Counting Walk

Let G be a graph on the vertex set $\{1, \ldots, p\}$. (We allow loops and multiple edges.) Let $M = A(G)$ be its adjacency matrix.

<u>OBSERVATION</u> The number of walks of length ℓ from i to j is equal to $(M^\ell)_{ij}$.

In general, counting walks requires knowing the matrix M (equivalently, knowing both the eigenvalues λ_k and the corresponding eigenvectors). On the other hand, some enumerative information can be extracted from the eigenvalues alone:

Proposition 1.2.1. *The number of marked closed walks of length ℓ is equal to $\sum_{k=1}^p \lambda_k^\ell$.*

Here "marked" means that the starting location is fixed, as is a particular instance of passing through it, in case we do it several times.

Proof. By the last observation, the number of marked closed walks of length ℓ is equal to $\operatorname{tr} M^\ell$, which equals to $\sum_{k=1}^p \lambda_k^\ell$ by (1.1.2). ■

Example 1.2.1. Let $G = K_p$, the complete graph on p vertices. Let J denote the $p \times p$ matrix all of whose entries are 1. Let I denote the $p \times p$ identity matrix. Then $A(G) = J - I$. Obviously $\operatorname{rk} J = 1$ and $\operatorname{tr} J = p$. Hence the eigenvalues of J are $0, \ldots, 0, p$, and the eigenvalues of $A(G) = J - I$ are $-1, \ldots, -1, p-1$.

Corollary 1.2.1. *There are $(p-1)^\ell + (-1)^\ell(p-1)$ marked closed walks of length ℓ in K_p.*

<u>NOTE</u> This is the number of $(\ell+1)$-letter words in a p-letter alphabet in which no two consecutive letters are identical, and which begin and end by the same letter.

<u>PROBLEM</u> Show that the number of walks of length ℓ between two distinct vertices in K_p differs by 1 from the number of closed walks of length ℓ starting at a given vertex.

<u>RECALL</u>

$$\#\text{ of marked closed walks of length } \ell = \sum_{i=1}^p \lambda_i^\ell.$$

It can be used backwards: using counted walks to compute eigenvalues.

Example 1.2.2. Let $G = K_{n,m}$ a complete bipartite graph.

$$\# \text{ of marked closed works of length } \ell = \begin{cases} 0 & \ell = 2k+1 \\ 2n^{\ell/2}m^{\ell/2} & \ell = 2k \end{cases} = (\sqrt{nm})^\ell + (-\sqrt{nm})^\ell$$

$\xRightarrow{\text{Problem}}$ eigenvalues are $\sqrt{nm}, -\sqrt{nm}, 0, \ldots, 0$.

PROBLEM Prove that, for G connected, the $\operatorname{diam}(G) < \#$ of distinct eigenvalues.

Example 1.2.3. $K_p = 1 < 2, K_{n,m} = 2 < 3$.

1.3 Inequalities for the Maximal Eigenvalue

Definition 1.3.1. Suppose G a graph with vertices $= \{1, \ldots, p\}$. Let

$$\lambda_{\max} := \max_i |\lambda_i| = \max \lambda_i.$$

Proposition 1.3.1.

$$\lambda_{\max} \leq \max \deg(G)$$

Proof. For any vector $X = (x_k) \in \mathbb{C}^p$,

$$\max_j |(A(G)X)_j| \leq \max \deg(G) \cdot \max_k |X_k|$$

Now suppose X is an eigenvector of $A(G)$ with eigenvalue λ. Then

$$\max_j |(A(G)X)_j| = |\lambda| \max_k |X_k| \leq \max \deg(G) \cdot \max_k |X_k| \implies |\lambda| \leq \max \deg(G)$$

This holds for all eigenvalue λ_i, which proves our proposition. ■

ALTERNATE PROOF: by counting closed walks ($\leq \sum \max \deg(G)^\ell$.)

PROBLEM Prove that $\lambda_{\max} \geq$ average degrees of the vertices of G.
HINT for symmetric real matrix M we have $\lambda_{\max} = \max_{|x|=1} x^T M x$.

Corollary 1.3.1. *$\#$ of closed walk of length ℓ grows exponentially in ℓ with a rate $\geq$ average degree.*

1.4 Eigenvalue of Block Anti-diagonal Matrices

$$M = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \in \mathbb{R}_{n+m}$$

Lemma 1.4.1. *The non-zero eigenvalues (called "singular values" of B) of M are $\pm\sqrt{\mu_i}$ where μ_i are nonzero eigenvalues of $B^T B$ with multiplicities.*

Note that $B^T B$ is positive definite.

Proof. Let $F_X(t) = \det(t \operatorname{id}_p - X)$.

$$\begin{bmatrix} t \operatorname{id}_n & -B \\ -B^T & t \operatorname{id}_m \end{bmatrix} \begin{bmatrix} \operatorname{id}_n & B \\ 0 & t \operatorname{id}_m \end{bmatrix} = \begin{bmatrix} t \operatorname{id}_n & 0 \\ -B & -B^T B + t^2 \operatorname{id}_m \end{bmatrix}$$

$$F_M(t) \cdot t^m = t^n F_{B^T B}(t^2)$$

and the claim follows ■

So now we are equipped to compute the eigenvalue of bipartite graphs.

Example 1.4.1. Suppose $G = K_{n,m}$, $B^T B$ is $m \times m$ matrix with all entries being n. So the eigenvalues of $B^T B = nm, 0, 0, \ldots$. So eigenvalues of $A(G)$ is $\sqrt{mn}, -\sqrt{mn}, 0, 0, \ldots$

PROBLEM Let G to be the graph obtained by removing n disjoint edges from $K_{n,n}$. Find the eigenvalue of G.

Example 1.4.2. Let G be a $2n$-cycle. $M_{2n} = A(G) = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$. The $B^T B = 2I_n + M_n$ for an appropriate labeling.

So if the eigenvalue of n-cycle are $\lambda_1, \ldots, \lambda_n$. Then the eigenvalues of $2n$-cycles are $\pm\sqrt{\lambda_i + 2}$.

1.5 Eigenvalues of Circulant Matrices

Definition 1.5.1. A circulant matrix is of the form

$$M = \begin{bmatrix} s_0 & s_1 & s_2 & \ldots & s_{p-1} \\ s_{p-1} & s_0 & s_1 & \ldots & s_{p-2} \\ \vdots & & & & \\ s_1 & s_2 & s_3 & \ldots & s_0 \end{bmatrix}.$$

Lemma 1.5.1. *M has eigenvalues*

$$\lambda_k = \sum_{j=0}^{p-1} s_j e^{\frac{2\pi i}{p} jk}, k = 0, 1, \ldots, p-1.$$

Notice that

$$\lambda_k = \sum_{j=0}^{p-1} s_j e^{\frac{2\pi i}{p} jk} = s\left(e^{\frac{2\pi i}{p} k}\right) \quad p\text{-th root of unity.}$$

where

$$s(x) = \sum_{j=0}^{p-1} s_j x^j.$$

Proof. Let

$$T = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & & \\ 1 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

We have that the eigenvalues of T and p-th roots of unity and characteristic polynomial is $t^p - 1$.

Key observation: $M = s(T)$. ■

Definition 1.5.2. A graph G is circulant if $A(G)$ is circulant, for some choice of vertex labeling.

Corollary 1.5.1. *The eigenvalue of p-cycle are*

$$2\cos\left(\frac{2\pi k}{p}\right), k = 0, 1, \ldots, p-1.$$

Proof. By Lemma 1.5.1, we have that

$$\lambda_k = e^{\frac{2\pi i}{p} k} + e^{\frac{2\pi i}{p}(p-1)k} = e^{\frac{2\pi i k}{p}} + e^{-\frac{2\pi i k}{p}} = 2\cos\left(\frac{2\pi k}{p}\right).$$ ■

Remark. This formula is consistent with the formula linking the eigenvalues of a $2n$-cycle and an n-cycle: if $2\cos\alpha = \lambda$, then $2\cos\frac{\alpha}{2} = \pm\sqrt{2+\lambda}$.

PROBLEM Find the eigenvalues of the graph obtains by removing n disjoint edges from K_{2n}.

1.6 Eigenvalues of Cartesian Products

Definition 1.6.1. Suppose G, H are graphs with no loops. Define graph $G \times H$ where

$$V(G \times H) = \{(g, h) : g \in V(G), h \in V(H)\},$$

and we have two kinds of edges:

- $(g, h) - (g', h)$ for $g - g'$
- $(g, h) - (g, h')$ for $h - h'$

Example 1.6.1. 1. Grid graph = path $\times$ path

2. Discrete annulus (cylinder) = cycle $\times$ path
3. Discrete torus = cycle $\times$ cycle
4. n-cube graph

Proposition 1.6.1. *If G has eigenvalues $\lambda_1, \lambda_2, ldots$, H has eigenvalues $\mu_1, \mu_2, \ldots$ Then $G \times H$ has eigenvalues $\lambda_i + \mu_j$ for any pair i, j.*

Proof 1. (Tensor product) V_G, V_H are vector spaces formally spanned by vertices of G, H. Take $u = \sum \alpha_g g \in V_G, v = \sum \beta_h h \in V_H$. We have

$$u \otimes v = \sum_{g,h} \alpha_g \beta_h (g, h) \in V_{G \times H}.$$

The

$$A(G \times H)(u \otimes v) = (A(G)u) \otimes v + u \otimes (A(h)v)$$

Suppose u, v are eigenvectors $i.e.$ $A(G)u = \lambda u, A(H)v = \mu v$. Then we get

$$A(G \times H)(u \otimes v) = \lambda u \otimes v + u \otimes \mu v = (\lambda + \mu)(u \otimes v).$$ ■

Proof 2. (Marked closed walk) Walk in $G \times H \xleftrightarrow{1-1}$ a <u>shuffle</u> of marked closed walks in $G \& H$.

\# of closed walks of length ℓ in $G \times H$

$$= \sum_k \binom{\ell}{k} \sum_i \lambda_i^k \sum_j \mu_j^{\ell-k} = \sum_i \sum_j \sum_k \binom{\ell}{k} \lambda_i^k \mu_j^{\ell-k} = \sum_{i,j} (\lambda_i + \mu_j)^\ell.$$

1 This set of numbers are unique by problem in lecture 1, so they must be the eigenvalues of $G \times H$. ■

PROBLEM Take a 3×3 grid, find the number of marked closed walks of length ℓ.
PROBLEM Direct problem of 8-cycle and K_2.

n-CUBE GRAPH:

$$(K_2)^n = \underbrace{K_2 \times K_2 \times \cdots K_2}_{n \text{ times}}.$$

Example 1.6.2. When $n = 3$, we have a 3-D cube:

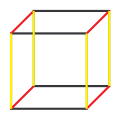


Figure 1.1: Cube graph $K_2 \times K_2 \times K_2$

K_2 has adjacency matrix $A(K_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ with eigenvalues $\pm 1 \implies$ eigenvalues of $(K_2)^n$ are

$$\lambda = \underbrace{\pm 1 \pm 1 \pm \ldots \pm 1}_{n \text{ times}}.$$

Proposition 1.6.2. *The eigenvalues of $(K_2)^n$ are of the form $n - 2k$ where $k = 0, 1, \ldots, n$, each with multiplicities $\binom{n}{k}$ i.e. the number of marked closed walks of length ℓ in the $n-$cube graph is*

$$\sum_{k=0}^{n} \binom{n}{k} (n - 2k)^{\ell}$$

which is 0 when ℓ is odd.

1.7 Random Walks

Let G be a regular graph of degree d on p vertices.

Example 1.7.1. $G = (K_2)^n$ is regular with $d = n$.

A simple random walk on G originating at a vertex v is a random walk with equal probabilities for each adjacent vertices.

Assuming that $\mathrm{Aut}(G)$ acts transitively on vertices, we have the result

$$\mathbb{P}\,(\text{walk is back at } v \text{ after } \ell \text{ steps}) =$$
$$\frac{1}{d^{\ell}} \#\{\text{marked closed walks of length } \ell \text{ orginiating from } v\} = d^{-\ell} p^{-1} \sum_{1}^{p} \lambda_i^{\ell}.$$

Notice that an arbitrary regular G does not necessarily have that condition, but the converse is true.

Example 1.7.2. The probability that a simple random walk on $(K_2)^n$ returns to its origin after ℓ steps is

$$\frac{1}{n^\ell 2^n} \sum_{k=0}^{n} \binom{n}{k} (n-2k)^\ell$$

Chapter 2

Tilings, Spanning Trees, and Electric Networks

2.1 Domino Tilings ("Dimers")

If you sit in this classroom for a long enough period of time, you probably can figure out this (2.1.2) yourself. But I offer you a divine revelation.

– Sergey Fomin

If you can't solve this, it just means you're mere mortals. Even if you do it... I mean the grader will grade it, but...

– Sergey Fomin on the perfect square problem at the end of the section

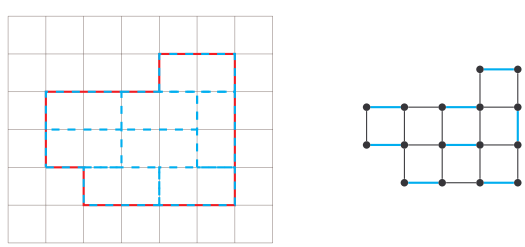
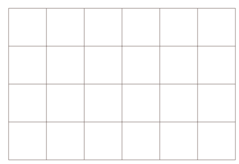


Figure 2.1: An example of domino tiling and perfect matching in its dual graph

A domino tiles decompose part of grids into 1×2 rectangles.

Think of it another way: the "dual graph" where squares are vertices, and there exists an edge between two vertices iff the corresponding squares shares an edge. A tiling is a perfect matching between these vertices.

Special case: $m \times n$ rectangular boards



Without loss of generality, assume that n is even. We denote the answer as $T(m, n)$

The dual graph G is m-chain $\times$ n-chain. Notice that G is bipartite.

$M = A(G)$ has the form $\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$ given appropriate labeling of vertices where B is a square matrix.

CLAIM $T(m, n) =$ the permanent of matrix B.

Permanents do not have nice properties, thus they are hard to calculate. In order to better calculate the permanent of B, let $\tilde{B}$ obtained from B by replacing the 1's by corresponding to vertical tiles by i's where $i^2 = -1$.

Proposition 2.1.1. $T(m, n) = \operatorname{per}(B) = \pm \det(\tilde{B})$.

Lemma 2.1.1 (exercise). *Any two domino tilings of a rectangular board are related to each other via "flips" of the form (two horizontal $\leftrightarrow$ two vertical)*

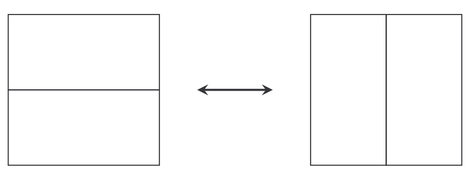


Figure 2.2: Example of a flip

Proof of Prop. This is equivalent to all nonzero terms in $\det(\tilde{B})$ are equal and are ± 1. The latter claim follows from the former, since the all-horizontal tiling contributes ± 1.

Then it is enough to show that the contributions of two tilings that differ by a flip are equal to each other.

It means swapping two diagonal entries, thus change the sign of permutation, but one of them is 1^2 while the other being i^2, so the result does not change. ∎

Now we can use some linear algebra to calculate the determinant.

Denote $\tilde{M} = \begin{bmatrix} 0 & \tilde{B} \\ \tilde{B}^T & 0 \end{bmatrix}$. Then $\det(\tilde{M}) = \pm(\det(\tilde{B}))^2 = \pm(T(m,n))^2$.

<u>OBSERVATION</u> We have

$$M = \mathrm{id}_m \otimes A_n + A_m \otimes \mathrm{id}_n,$$

where A_n, A_m are adjacency matrices of chain graphs. Similarly,

$$\tilde{M} = \mathrm{id}_m \otimes A_n + iA_m \otimes \mathrm{id}_n,$$

since $\tilde{M}$ obtained by vertical tile with i's. Hence the eigenvalues of $\tilde{M}$ are $\lambda_i + i\mu_k$.

Now we only need to find the eigenvalues of chain graph. For a n-chain, we have

$$A_n = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

Proposition 2.1.2. *The eigenvalues of A_n are*

$$\lambda_k = 2\cos\left(\frac{k\pi}{n+1}\right) \quad \textit{for } k = 1, \dots, n.$$

Proof. An eigenvector $u = (u_1, \dots, u_n)^T$ of A_n associated with eigenvalue λ satisfies

$$u_{j-1} + u_{j+1} = \lambda u_j, \quad 1 \le i \le n$$

with the convention that $u_0 = u_{n+1} = 0$.

A divine revelation: recall that

$$\sin\alpha + \sin\beta = 2\cos\frac{\beta - \alpha}{2}\sin\frac{\alpha + \beta}{2}.$$

This suggests taking

$$u_j = \sin\left(\frac{\pi k j}{n+1}\right) \quad \text{for } j = 1, \dots, n.$$

with eigenvalue

$$\lambda_k = 2\cos\left(\frac{k\pi}{n+1}\right). \qquad \blacksquare$$

Example 2.1.1.

$$n = 3, \det(t \operatorname{id} - A_3) = t^3 - 2t = t(t - \sqrt{2})(t + \sqrt{2}).$$

So the eigenvalues are

$$\lambda_1 = \sqrt{2} = 2\cos\left(\frac{1\pi}{4}\right), \lambda_2 = 0 = 2\cos\left(\frac{2\pi}{4}\right), \lambda_2 = -\sqrt{2} = 2\cos\left(\frac{3\pi}{4}\right).$$

Now

$$\begin{aligned}
\det \tilde{M} &= \prod_{j=1}^{n}\prod_{k=1}^{m}\left(2\cos\frac{j\pi}{n+1} + i2\cos\frac{k\pi}{m+1}\right) \\
&= \prod_{j=1}^{n/2}\prod_{k=1}^{m}\left(2\cos\frac{j\pi}{n+1} + i2\cos\frac{k\pi}{m+1}\right)\left(2\cos\frac{(n+1-j)\pi}{n+1} + i2\cos\frac{k\pi}{m+1}\right) \\
&= \prod_{j=1}^{n/2}\prod_{k=1}^{m}\left(2\cos\frac{j\pi}{n+1} + i2\cos\frac{k\pi}{m+1}\right)\left(-2\cos\frac{j\pi}{n+1} + i2\cos\frac{k\pi}{m+1}\right) \\
&= \pm\prod_{j=1}^{n/2}\prod_{k=1}^{m}\left(4\cos^2\frac{j\pi}{n+1} + 4\cos^2\frac{k\pi}{m+1}\right)
\end{aligned}$$

Theorem 2.1.1 (P.Kasteleyn, M.Fisher, H.N.V.Temperley, 1961)**.** *When m is even,*

$$T(m,n) = \prod_{j=1}^{n/2}\prod_{k=1}^{m/2}\left(4\cos^2\frac{j\pi}{n+1} + 4\cos^2\frac{k\pi}{m+1}\right).$$

When m is odd,

$$T(m,n) = \prod_{j=1}^{n/2} 2\cos\frac{j\pi}{n+1}\prod_{k=1}^{(m-1)/2}\left(4\cos^2\frac{j\pi}{n+1} + 4\cos^2\frac{k\pi}{m+1}\right).$$

Example 2.1.2. For $n = m = 8$, we get $T(8,8) = 12,988,816 = 3604^2$.

<u>PROBLEM</u>* For any positive integer $a \in \mathbb{Z}_{>0}, T(4a,4a)$ is a perfect square, $T(4a-2,4a-2)$ is twice a perfect square.

Asymptotics of $T(n,n)$: reasonable to expect $T(n,n) \sim e^{cn^2}$.

We take the natural log of $T(n,n)$:

$$\begin{aligned}\frac{\ln T(n,n)}{n^2} &= \frac{1}{n^2}\sum_{k=1}^{n/2}\sum_{j=1}^{n/2}\ln\left(4\cos^2\frac{\pi k}{n+1}+4\cos^2\frac{\pi j}{n+1}\right)\\ &\sim \frac{1}{\pi^2}\sum\sum\left(\frac{\pi}{n+1}\right)^2\ln\left(4\cos^2\frac{\pi k}{n+1}+4\cos^2\frac{\pi j}{n+1}\right)\end{aligned}$$

Notice that the right-hand side is a Riemann sum of the function $\ln(4\cos^2 x+4\cos^2 y)$. So the sum approaches to

$$\frac{1}{\pi^2}\int_0^{\pi/2}\int_0^{\pi/2}\ln(4\cos^2 x+4\cos^2 y)dxdy=\frac{K}{\pi}$$

where K is Catalan's constant. As of today, it is not known whether it is irrational, nor transcendental.

So we have $T(n,n)\approx 1.34^{n^2}$.

Another way to define Catalan's constant:

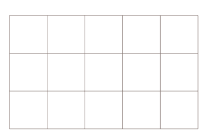
$$K=\beta(2)=\sum_{i=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\ldots.$$

2.2 Spanning Tree in Grid Graphs, Planar Graphs

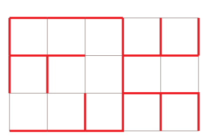
He published this result when he was 60, which proved that aged people can still do mathematics.

– Sergey Fomin on H.N.V. Temperley

Suppose a grid graph G:



We can keep some edges and discard others to obtain a connected acyclic subgraph of G (which is a spanning tree).



Theorem 2.2.1 (H.N.V. Temperley, 1974)**.** *Consider a rectangular board of odd size* $(2k-1\times$

$2\ell - 1)$ *with one corner removed. The number of domino tilings of the board is equal to the number of spanning trees in the* $k \times \ell$ *grid.*

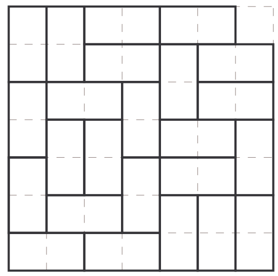


Figure 2.3: A domino tiling satisfying the condition

Proof. Find a bijection between domino tilings and spanning trees.

PROBLEM Prove that Temperley's map showed in Figure 2.4 produces a tree.

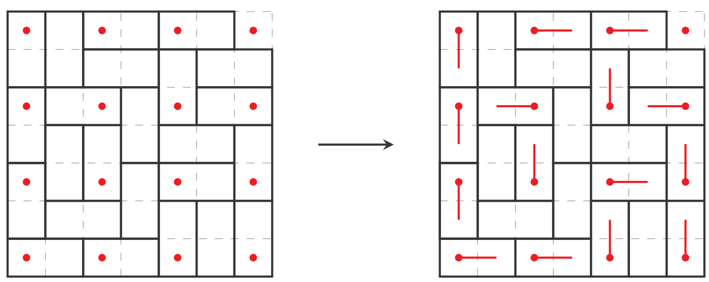


Figure 2.4: Converting domino tiling into trees

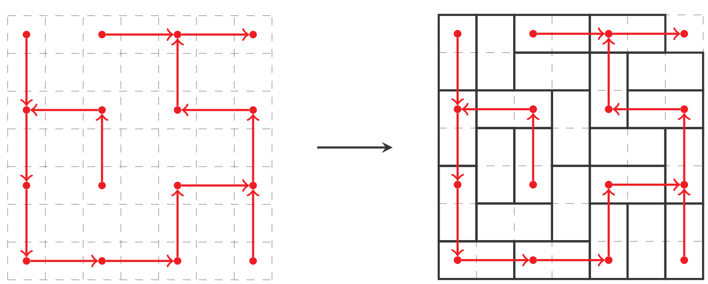


Figure 2.5: Converting domino tiling into trees

Now we have a formard map. We also need to obtain the inverse map from spanning trees to domino tiling. Fixing a border point as the root of the tree, we can make the tree a directed graph and addign domino tiles accordingly. ■

Corollary 2.2.1.

$$\#\text{ of spanning trees in a } k \times \ell \text{ grid} \approx \left(e^{\frac{4K}{\pi}}\right)^{k\ell} \approx 3.21^{k\ell}.$$

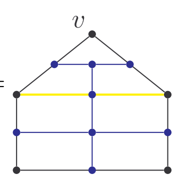
<u>PROBLEM</u> Prove that the number of domino tilings (if exist) of an odd-by-odd rectangle with a boundary box removed doesn't depend on which box we removed.

We now consider a similar problem:

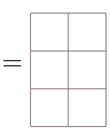
Suppose P is a polygon, G a polygonal subdivision of P. Define H by adding midpoints and extra vertex in each bounded face and adding edges to connect them.

<u>PROBLEM</u> Show that the number of spanning trees in G is equal to the number of perfect matchings in H with one vertex that are also in P removed.

Example 2.2.1.

$G =$ 

The number of spanning trees of $= 4+4+3 = 11$. If we take the vertex v specified above we have:

$H - \{v\} =$ 

We can verify that it also has 11 matchings.

<u>NOTE</u> Here the for arbitrary vertex v the result would be the same.

2.3 The Diamond Lemma

This should really be explained to six-graders... Then they will have of joy of discovering this result.

– Sergey Fomin on the diamond lemma

Definition 2.3.1. A <u>one-player game</u> is defined by:

- the set of positions $\mathcal{S}$
- for each $s \in \mathcal{S}$ a set of positions $s' \neq s$ into which the player can from from from s. Denote as $s \rightsquigarrow s'$.

If the latter set is empty, then S is called <u>terminal</u>.

A play sequence is a sequence
$$s \rightsquigarrow s' \rightsquigarrow s'' \rightsquigarrow \dots$$

A game is terminating is $\nexists$ infinite play sequences.

A game is confluent is its outcome is uniquely determined by initial position.

Lemma 2.3.1 (The Diamond Lemma for terminating games)**.** *For a one-player game, assume that*

- *the game is terminating*
- ⋄ *(diamond condition)* $\forall s \in \mathcal{S}$, $\forall s \rightsquigarrow s'$, $s \rightsquigarrow s''$, $\exists$ *some position that can be reach from both* s' *and* s''*. (You never say goodbye forever!)*

Then the game is confluent.

Proof. Color the position:

- Green is the terminal position reachable from this position is unique.
- Red otherwise.

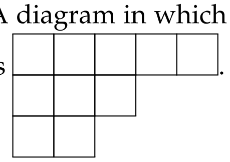
Assume a red position exists. Starts at the red position until no move into red position exists.

For each green position, there is a unique terminal position. Since it starts from a red position, there need to be two distinct outcomes. But that is a contradiction, since all green position are obtained from a certain red position, which means any two of them will have a common successor, then from now on the outcome should be the same. ■

2.4 Using the Diamond Lemma

I play Wordle twice a day. Once in English, once in Russian. The Russian version is simpler because there are much fewer 5-letter words in Russian.

– Sergey Fomin

Definition 2.4.1 (Young diagrams)**.** A diagram in which the number of boxes on a row is decreasing. An example of which is .

We define a one-player game where: Position = {Young diagrams}

Move = Removal of a domino tile from the SE rim that also results in a Young diagram.

CLAIM This game is confluent.

Note: the remaining shape would always be a staircase. If we color the blocks black and white alternatively, we can determine the final shape by the difference between white and black boxes.

PROBLEM Consider a similar game, but we are removing border strips consisting of p boxes ($p \in \mathbb{N}$). Prove that the game is confluent.

Definition 2.4.2 (Young tableaux)**.** Take a Young diagram and fill it with numbers so that each row and column is in increasing order. Such diagram is called a standard Young tableau (SYT).

Or take a skew shape where a Young diagram is taken away from the top left corner of another Young diagram. Then filling it the same way we obtain standard skew tableau (Skew SYT).

1	2	5	8
3	6	7	
4			

			3	6
	1	4	8	9
2	5	7	10	

Figure 2.6: A standard Young tableau (left) and skew tableau (right)

JEU DE TAQUIN [M.-P. Schützenberger] Given a skewed tableau, choose a top-left corner piece and move the blocks one at a time so that after a series of moves we also get a skewed tableau.

		2	4	8
←	1	3	6	
5	7			

$\longrightarrow$

		2	4	8
1	←	3	6	
5	7			

$\longrightarrow$

		2	4	8
1	3	←	6	
5	7			

$\longrightarrow$

		2	4	8
1	3	6		
5	7			

Figure 2.7: One step in a jeu de taquin game

The game ends on a SYT, called a rectification of T.

PROBLEM The rectification is unique. (Jeu de tauqin is confluent.)

Definition 2.4.3 (Tutte Polynomial)**.** $T_G(x, y)$ of a graph G is defined recursively as follows:

- G has no edges $\implies T_G = 1$.
- e edge in $G \implies$

$$T_G = \begin{cases} xT_{G-e} & e \text{ is a bridge} \\ yT_{G-e} & e \text{ is a loop} \\ T_{G-e} + T_{G/e} & \text{otherwise} \end{cases}$$

This is a two variable generalization of the chromatic polynomial.

PROBLEM Use the diamond lemma to show that T_G is well defined.

For non-terminating games, the diamond lemma does not necessarily hold:

Example 2.4.1 (Naive counterexample)**.** Suppose a game:

$$1 \to 2 \to 3 \to 4 \to \ldots, n \to \infty, \forall n.$$

Then there are two outcomes for any given starting position (∞ or non-terminating).

Theorem 2.4.1 (Diamond Lemma for Non-terminating Games)**.** *Suppose a one-player game. $\forall s \in \mathcal{S}$, $\forall s \rightsquigarrow s'$, $s \rightsquigarrow s''$, $\exists$ some position that can be reach from both s' and s'' in the same number of steps. Then the game is confluent.*

Moreover, if the game terminates for a given initial position, then it does so in a fixed number of steps.

Proof. Left as PROBLEM. ■

2.5 Loop-erased Walks

He took my class shortly before he discovered this algorithm (on generating uniform spanning trees)... After that he worked at Microsoft Research for 10 years until they decided they don't need mathematicians anymore and fired everybody.

– Sergey Fomin on D.B. Wilson

Definition 2.5.1 (G. Lawler, 1980)**.** Suppose G a connected graph. Let π be a (finite) walk in G. $\mathrm{LE}(\pi)$ "loop erasure" of π is defined by Algorithm 1.

STACKS & CYCLE POPPING

Recall: Markov chains.

"Running a Markov chain with stacks": at each state, decide on transition choices in advance.

Algorithm 1 Loop erasure

```
procedure LE(π)
    if π does not intersect itself then
        return π
    else
        Remove the first cycle of π to get π′
        return LE(π′)
    end if
end procedure
```

v is a vertex, $u(v) = (u_1, u_2, u_3, \ldots)$ where u_k denotes the vertex we move to after visiting v for the k-th time. They are i.i.d RV's.

Assume: the Markov chain arrives with probability 1 at an absorbing state (where the stack is empty).

Given a collection of partially depleted stacks, we get a graph (A subgraph of G, the underlying oriented graph of Markov chain) determined by the top of each stack. The out degrees of each non-absorbing vertex in the graph is equal to 1. The removing of cycles from this graph is called cycle-popping.

LOOP-ERASED **RANDOM** WALK

Start at vertex s, and stop upon arriving at some absorbing state t.

OBSERVATION LERW is obtained by popping some cycle, leaving a path from s to t.

Define a game where positions are collections of stacks of at each vertex and the moves are cycle poppings.

CLAIM This game is confluent (by diamond lemma for non-terminating games).

The outcome of the game (after all cycles have been popped) is a rooted forest that is oriented towards absorbing states.

WILSON'S ALGORITHM [D.B. Wilson, 1996]

Input: connected loopless graph G. Output: random spanning tree T in G.

Theorem 2.5.1. *This algorithm, shown in Algorithm 2, outputs a uniformly distributed spanning tree of G.*

Proof. Make r an absorbing state. Run Wilson's algorithm with stacks, each time designating the vertices in T as absorbing states. Loop erasure = cycle popping.

The algorithm terminates with probability 1, revealing the tree lying underneath all pop-

Algorithm 2 Wilson's algorithm

procedure WILSON($G = (V(G), E(G))$)
  $T = (V(T), E(T)) := (\{r\}, \emptyset)$ for some $r \in V(G)$.
  while $V(T) \neq V(G)$ **do** ▷ Continue until T covers all vertices
    Pick $v \in V(G) \setminus V(T)$
    Run a simple random walk π from v until it hits T
    $V(T) \leftarrow V(T) \cup \{v, \mathrm{LE}(\pi) \text{ includes vertex } v\}$
    $E(T) \leftarrow E(T) \cup \{e, \mathrm{LE}(\pi) \text{ includes edge } e\}$
  end while
  return T
end procedure

able cycles.

Need: the output tree T is uniformly distributed.

Suppose $H =$ heap of cycles. $\mathbb{P}(T, H)$ denotes probability of getting T after removing H. We have

$$\mathbb{P}(T, H) = \left(\prod_{v \in T - \{r\}} \deg(v)^{-1} \right) \cdot \left(\prod_{v \in H} \deg(v)^{-1} \right),$$

while

$$\mathbb{P}(T) = \sum_{H} \mathbb{P}(T, H).$$

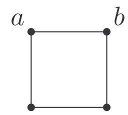
All the expressions above have the same values regardless of T, so the distribution is uniform. ■

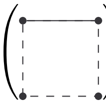
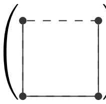
The art of computer programming, vol 4. (generate random combinatorial objects that satisfies certain distribution)

<u>PROBLEM</u> Let a, b be two vertices in a connected graph G. Show that the following two constructions produce the same distributions on the set of self-avoiding walks from a to b.

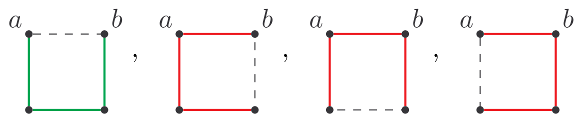
- Run a simple random walk π that starts at a and stops at b. Output $\mathrm{LE}(\pi)$.
- Choose uniformly at random a spanning tree in G. Output the walk from a to b in the spanning tree.

Corollary 2.5.1. *This distribution does not change if we swap a and b.*

Example 2.5.1. $G =$  . Using the first method we have

$\mathbb{P}\Big($  $\Big) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4},\ \mathbb{P}\Big($  $\Big) = \frac{1}{4}.$

For the spanning tree method, we have these spanning trees:



2.6 Flows

(Making analogies using distilling of Whiskey)...
"I actually don't drink any alcohol at all."

— Sergey Fomin

Definition 2.6.1. Suppose G is connected loopless graph with a, b designated as <u>boundary</u> vertices and all other vertices called <u>interior</u> vertices.

A <u>flow</u> f assigns a number $f(e, u, v)$ to each edge e with endpoint u, v, so that

- $f(e, u, v) = -f(e, v, u)$.
- $\forall$ interior vertex u, $\displaystyle\sum_{v \in V(G), (u,v) \in E(G)} f(e, u, v) = 0$. ("Conservation of flow")

It follows that $\exists$ number $|f|$, called the <u>total flow</u> from a to b, such that

$$\sum_{v \in V(G), (u,v) \in E(G)} f(e, u, v) \begin{cases} 0 & u \text{ interior} \\ |f| & u = a \\ -|f| & u = b. \end{cases} \tag{2.6.1}$$

We basically created a weighted graph in some sense: $w : E(G) \to \mathbb{R}_{>0}$.

Definition 2.6.2. For any vertex u, $w(u) := \displaystyle\sum_{e=(u,v) \in E(G)} w(e)$.

A "weighted version" of a simple random walk is a Markov chain with transition probability from u to v being $\displaystyle\sum_{e=(u,v) \in E(G)} \frac{w(e)}{w(u)}$.

PROBLEM* Generalize Wilson's algorithm to weighed graphs.

RESISTOR NETWORKS

Definition 2.6.3. Resistor network = weighted graph, conductance = edge weights, conductance = $\frac{1}{\text{resistence}}$.
(2.6.1) is the first Kirchhoff law. (conversation of charge/current)

The second Kirchhoff law expresses the conversation of energy: given a cycle

$$u_0 \overset{e_1}{—} u_1 \overset{e_2}{—} u_2 \overset{e_3}{—} \dots \overset{e_k}{—} u_k = u_0,$$

we have

$$\sum_{i=1}^{k} \frac{f(e_i, u_{i-1}, u_i)}{w(e_i)} = 0. \tag{2.6.2}$$

(2.6.1) and (2.6.2) are a linear system of equations in $f(e, u, v)$.

Theorem 2.6.1. *Kirchhoff's equations* (2.6.1) *and* (2.6.2) *have a unique solution.*

2.7 Potentials

"He is a wonderful mathematician."
(He is also a great lecturer...)
"Well some people have them both."
– Jeffrey C. Lagarias on Sergey Fomin

Definition 2.7.1. Suppose f satisfies (2.6.2). Define the potential $p = p_f$, a function on the vertices of G, as follows:

- assign an arbitrary value to $p(a)$
- for any walk that starts at $a = u_0 \overset{e_1}{—} u_1 \overset{e_2}{—} u_2 \overset{e_3}{—} \dots \overset{e_k}{—} u_k = u$, set

$$p(u) := p(a) + \sum_{i=1}^{k} \frac{f(e_i, u_{i-1}, u_i)}{w(e_i)}.$$

Lemma 2.7.1. *The function $u \mapsto p(u)$ is well-defined.*