

Notes for Math 597 – Real Analysis

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Office hour is Mon 12:30 - 1:30, Tue 12:30 - 1:30 in person EH 5838, Th 1 - 2 online.

Chapter 1

Abstract Measure

1.1 σ -Algebra

Definition 1.1. Let X be a set. A collection \mathcal{M} of subsets of X is called a σ -algebra on X if

- $\emptyset \in \mathcal{M}$.
- \mathcal{M} is closed under complements: $E \in \mathcal{M} \implies E^c \in \mathcal{M}$.
- \mathcal{M} is closed under countable unions: $E_1, E_2, \dots \in \mathcal{M} \implies \bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$.

SIMPLE PROPERTIES:

- $X = \emptyset^c \in \mathcal{M}$.
- $\bigcap_{i=1}^{\infty} E_i = (\bigcup_{i=1}^{\infty} E_i^c)^c \in \mathcal{M}$. It is closed under countable intersections.
- $\bigcup_{i=1}^N E_i = E_1 \cup \dots \cup E_N \cup \emptyset \cup \dots$. It is closed under finite unions (similarly, intersections). sigma
- $E \setminus F = E \cap F^c \in \mathcal{M}$, $E \Delta F = (E \cap F^c) \cup (F \cap E^c) \in \mathcal{M}$.

Example 1.2. (a) $\mathcal{A} = \mathcal{P}(X)$ power algebra.

(b) $\mathcal{A} = \{\emptyset, X\}$ trivial algebra.

(c) Let $B \subset X, B \neq \emptyset, B \neq X$. $\mathcal{A} = \{\emptyset, B, B^c, X\}$.

Lemma 1.3. (An intersection of σ -algebras is a σ -algebra) Let $\mathcal{A}_{\alpha}, \alpha \in I$, be a family a σ -algebras of X . Then $\bigcap_{\alpha \in I} \mathcal{A}_{\alpha}$ is a σ -algebra. (I can be uncountable.)

Proof. DIY

■

Definition 1.4. For $\mathcal{E} \subset \mathcal{P}(X)$ (not necessarily a σ -algebra), let $\langle \mathcal{E} \rangle$ be the intersection of all σ -algebras on X that contains \mathcal{E} . Call it the σ -algebra generated by \mathcal{E} .

- $\langle \mathcal{E} \rangle$ is the smallest σ -algebra containing \mathcal{E} and is unique.
- $\{\emptyset, B, B^c, X\} = \langle \{B\} \rangle = \langle \{B^c\} \rangle = \langle \{\emptyset, B\} \rangle$.

The above definition gives us (potentially) lots of examples of σ -algebra on a set X

Lemma 1.5. (a) Suppose $\mathcal{E} \subset \mathcal{P}(X)$, \mathcal{A} a σ -algebra on X . $\mathcal{E} \in \mathcal{A} \implies \langle \mathcal{E} \rangle \in \mathcal{A}$.

(b) $E \subset F \subset \mathcal{P}(X) \implies \langle \mathcal{E} \rangle \subset \langle \mathcal{F} \rangle$.

Proof. ■

Definition 1.6. For a topological space X , the Borel σ -algebra $\mathcal{B}(X)$ is the σ -algebra generated by the collection of open sets.

Example 1.7. ($X = \mathbb{R}$) $\mathcal{B}(\mathbb{R})$ contains the following collections

$$\begin{aligned}\mathcal{E}_1 &= \{(a, b) \mid a < b\}, & \mathcal{E}_2 &= \{[a, b] \mid a < b\}, \\ \mathcal{E}_3 &= \{(a, b] \mid a < b\}, & \mathcal{E}_4 &= \{[a, b) \mid a < b\}, \\ \mathcal{E}_5 &= \{(a, \infty) \mid a \in \mathbb{R}\}, & \mathcal{E}_6 &= \{[a, \infty) \mid a \in \mathbb{R}\}, \\ \mathcal{E}_7 &= \{(-\infty, a) \mid a \in \mathbb{R}\}, & \mathcal{E}_8 &= \{(-\infty, a] \mid a \in \mathbb{R}\}\end{aligned}$$

Proposition 1.8. $\mathcal{B}(\mathbb{R}) = \langle \mathcal{E}_i \rangle$ for each $i = 1, \dots, 8$.

Proof. Use 1.5. ■

Definition 1.9. (X, \mathcal{A}) is called a measurable space.

1.2 Measures

Definition 1.10. A measure on (X, \mathcal{A}) is a function $\mu : \mathcal{A} \rightarrow [0, \infty]$ s.t.

- (a) $\mu(\emptyset) = 0$
- (b) (countable additive) For $A_1, A_2, \dots \in \mathcal{A}$ disjoint we have

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

(X, \mathcal{A}, μ) is then called a measure space.

Example 1.11. (a) For any (X, \mathcal{A}) , $\mu(A) = \#A$ counting measure.

(b) For any (X, \mathcal{A}) , let $x_0 \in X$. The Dirac measure at x_0 is

$$\mu(A) = \begin{cases} 1 & x_0 \in A, \\ 0 & x_0 \notin A. \end{cases}$$

(c) For $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, let $a_1, a_2, \dots \in [0, \infty)$. $\mu(A) = \sum_{i \in A} a_i$ is a measure.

(X, \mathcal{A}) measurable space

(X, \mathcal{A}, μ) measure space

$\mu : \mathcal{A} \rightarrow [0, \infty]$ s.t. $\mu(\emptyset) = 0$, countable additivity.

NOTE: $A, B \in \mathcal{A}, A \subset B$, then $\mu(B \setminus A) + \mu(A) = \mu(B) \implies \mu(B \setminus A) = \mu(B) - \mu(A)$ if $\mu(A) < \infty$.

Theorem 1.13. Suppose (X, \mathcal{A}, μ) a measure space. Then

(a) (monotonicity)

$$A, B \in \mathcal{A}, A \subset B \implies \mu(A) \leq \mu(B).$$

(b) (countable subadditivity)

$$A_1, A_2, \dots \in \mathcal{A}, \implies \mu\left(\bigcup_i A_i\right) \leq \sum_i \mu(A_i).$$

(c) (continuity from below/(MCT) from sets)

$$A_1, A_2, \dots \in \mathcal{A}, A_1 \subset A_2 \subset A_3 \subset \dots \implies \mu\left(\bigcup_i A_i\right) = \lim_{n \rightarrow \infty} \mu(A_n).$$

(d) (continuity from above)

$$A_1, A_2, \dots \in \mathcal{A}, A_1 \supset A_2 \supset A_3 \supset \dots, \mu(A_1) < \infty \implies \mu\left(\bigcap_i A_i\right) = \lim_{n \rightarrow \infty} \mu(A_n).$$

Proof. (a), (b), DIY.

For (c), let $B_1 = A_1, B_i = A_i \setminus A_{i-1}, i \geq 2. B_i \in \mathcal{A}$ and are disjoint.

$$\begin{aligned} \bigcup_i^\infty A_i &= \bigcup_i^\infty B_i \\ \implies \mu\left(\bigcup_i^\infty A_i\right) &= \mu\left(\bigcup_i^\infty B_i\right) = \sum_i^\infty \mu(B_i) = \lim_{n \rightarrow \infty} \sum_i^n \mu(B_i) = \lim_{n \rightarrow \infty} \mu(A_n). \end{aligned}$$

For (d), let $E_i = A_1 \setminus A_i$. Hence $E_i \in \mathcal{A}, E_1 \subset E_2 \subset \dots$. We have

$$\bigcup_i^\infty E_i = \bigcup_i^\infty (A_1 \setminus A_i) = A_1 \setminus \left(\bigcap_1^\infty A_i\right) \implies \bigcap_1^\infty A_i = A_1 \setminus \left(\bigcup_1^\infty E_i\right).$$

Hence

$$\mu\left(\bigcap_1^\infty A_i\right) = \mu(A_1) - \mu\left(\bigcup_1^\infty E_i\right) = \mu(A_1) - \lim_{n \rightarrow \infty} \mu(E_n) = \mu(A_1) - \lim_{n \rightarrow \infty} \mu(A_1) - \mu(A_n).$$

■

NOTE: the condition that $\mu(A_1) < \infty$ cannot be dropped.

For example, in $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \text{counting measure})$, let $A_n = \{n, n+1, n+2\}, A_1 \supset A_2 \supset A_3 \supset \dots$. We have $\bigcap_1^\infty A_i = \emptyset \implies \mu\left(\bigcap_1^\infty A_i\right) = 0$.

Definition 1.14. For (X, \mathcal{A}, μ) measure space,

- $A \subset X$ is a μ -null set if $A \in \mathcal{A}, \mu(A) = 0$.
- $A \subset X$ is a μ -subnull set if $\exists B, \mu$ -null set $A \subset B$.
- (X, \mathcal{A}, μ) is a complete measure space if every μ -subnull set is \mathcal{A} -measurable.

Definition 1.15. (X, \mathcal{A}, μ) measure space. A statement $P(x), x \in X$ holds μ -almost everywhere (a.e.) if the set $\{x \in X \mid P(x) \text{ does not hold}\}$ is μ -null.

Definition 1.16. (X, \mathcal{A}, μ) measure space.

- μ is a finite measure is $\mu(X) < \infty$.
- μ is a σ -finite measure if $X = \bigcup_1^\infty X_n, X_n \in \mathcal{A}, \mu(X_n) < \infty$.

HW: every measure space can be "completed."

1.3 Outer Measures

Definition 1.17. An outer measure on X is $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$ s.t.

- $\mu^*(\emptyset) = 0$
- (monotonicity) $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$.
- (countable subadditivity)

$$\forall A_1, A_2, \dots \in X, \mu^* \left(\bigcup_i^\infty A_i \right) \leq \sum_i^\infty \mu^*(A_i).$$

Example 1.18. For $A \subset \mathbb{R}$,

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^\infty (b_i - a_i) \mid \bigcup_1^\infty (a_i, b_i) \supset A \right\}.$$

is an outer measure due to the next proposition.

Proposition 1.19. (1.19) Let $\mathcal{E} \in \mathcal{P}(X)$ s.t. $\emptyset, X \in \mathcal{E}$. Let $\rho : \mathcal{E} \rightarrow [0, \infty]$ s.t. $\rho(\emptyset) = 0$. Then

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^\infty \rho(E_i) \mid E_i \in \mathcal{E}, \forall i \in \mathbb{N}, \bigcup_1^\infty E_i \supset A \right\}$$

is an outer measure on X .

Proof. (a) μ^* is well-defined (inf is taken over non-empty set.)

(b) $\mu^*(\emptyset) = 0$

(c) $A \subset B \implies \mu^*(A) \leq \mu^*(B)$.

We check the countable subadditivity.

Let $A_1, A_2, \dots \subset X$. If one of $\mu^*(A_i) = \infty$, then the result holds. Suppose $\mu^*(A_n) < \infty, \forall n \in \mathbb{N}$.

"Give your self a room of epsilon":

Fix $\varepsilon > 0$. We will show

$$\mu^* \left(\bigcup_1^\infty A_n \right) \leq \sum_1^\infty \mu^*(A_i) + \varepsilon.$$

For each $n \in \mathbb{N}, \exists E_{n,1}, E_{n,2}, \dots \in \mathcal{E}$ s.t.

$$\bigcup_{k=1}^\infty E_{n,k} \supset A_n \quad \text{and} \quad \mu^*(A_n) + \frac{\varepsilon}{2^n} \geq \sum_{k=1}^\infty \rho(E_{n,k}).$$

Then,

$$\bigcup_1^\infty A_n \subset \bigcup_{n=1}^\infty \bigcup_{k=1}^\infty E_{n,k} = \bigcup_{(n,k) \in \mathbb{N}^2} E_{n,k}.$$

RECALL: Tonelli's thm for series. If $a_{ij} \in [0, \infty]$, $\forall i, j \in \mathbb{N}$, then

$$\sum_{(i,j) \in \mathbb{N}^2} a_{ij} = \sum_{i=1}^\infty \sum_{j=1}^\infty a_{ij} = \sum_{j=1}^\infty \sum_{i=1}^\infty a_{ij}.$$

Hence

$$\mu^* \left(\bigcup_{n=1}^\infty A_n \right) \leq \sum_{n=1}^\infty \rho(E_{k,n}) = \sum_{n=1}^\infty \sum_{k=1}^\infty \rho(E_{k,n}) \leq \sum_{n=1}^\infty \left(\mu^*(A_n) + \frac{\varepsilon}{2^n} \right) = \sum_{n=1}^\infty \mu^*(A_n) + \varepsilon.$$

We have shown countable subadditivity. ■

Outer measure is very close to a measure. Here the textbooks diverge.

[Tao11] introduces Lebesgue measure on \mathbb{R} using topological qualities of subsets of \mathbb{R} .

[Fol99] introduces abstract method by Carathéodory and Kolmogorov.

Definition 1.20. Let μ^* be an outer measure on X . We say $A \subset X$ is Carathéodory measurable with respect to μ^* if $\forall E \subset X, \mu^*(E) = \mu^*(E \setminus A) + \mu^*(E \cap A)$.

Lemma 1.21. Let μ^* be an outer measure on X . Suppose B_1, B_2, \dots, B_N are disjoint C -measurable sets. Then,

$$\forall E \subset X, \mu^* \left(E \cap \left(\bigcup_1^N B_i \right) \right) = \sum_{i=1}^n \mu^*(E \cap B_i)$$

Proof.

$$\mu^* \left(E \cap \left(\bigcup_1^N B_i \right) \right) = \mu^*(E \cap B_1) + \mu^* \left(E \cap \left(\bigcup_2^N B_i \right) \right)$$

because B_1 is C -measurable. Then, iterate. ■

Improved version:

B_1, B_2, \dots C -measurable and disjoint $\implies \mu^*(E \cap \bigcup_1^\infty B_n) = \sum_1^\infty \mu^*(E \cap B_n), \forall E \subset X$.

Proof.

$$\begin{aligned} \sum_1^\infty \mu^*(E \cap B_n) &\geq \mu^*\left(E \cap \bigcup_1^\infty B_n\right) \\ &\geq \mu^*\left(E \cap \bigcup_1^N B_n\right) = \sum_1^N \mu^*(E \cap B_n). \end{aligned}$$

Take $N \rightarrow \infty$ or note that $N \in \mathbb{N}$ is arbitrary we get the result. ■

First big theorem:

Theorem 1.22 (Carathéodory extension theorem). *Let μ^* be an outer measure on X . Let \mathcal{A} be the collection of C -measurable sets with respect to μ^* . Then*

- (a) \mathcal{A} is a σ -algebra on X .
- (b) $\mu = \mu^*|_{\mathcal{A}}$ is a measure on (X, \mathcal{A}) .
- (c) (X, \mathcal{A}, μ) is a complete measure space.

Proof. (a) (1) $\emptyset \in \mathcal{A}$.

(2) \mathcal{A} is closed under complements.

(3) To show \mathcal{A} closed under countable unions.

- (finite union)

CLAIM $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$.

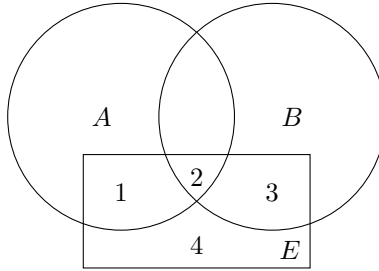


Figure 1.1: Venn diagram of A, B, E

Fix arbitrary $E \subset X$. We need to show

$$\mu^*(E) = \mu^*(E \cap (A \cup B)) + \mu^*(E \setminus (A \cup B)).$$

i.e.

$$\mu^*(1 \cup 2 \cup 3 \cup 4) = \mu^*(1 \cup 2 \cup 3) + \mu^*(4)$$

Since A is C -measurable, we have

$$\mu^*(1 \cup 2 \cup 3 \cup 4) = \mu^*(1 \cup 2) + \mu^*(3 \cup 4)$$

$$\mu^*(1 \cup 2 \cup 3) = \mu^*(1 \cup 2) + \mu^*(3)$$

Similarly since B is C -measurable, we have

$$\mu^*(3 \cup 4) = \mu^*(3) + \mu^*(4)$$

Hence

$$\begin{aligned} \mu^*(1 \cup 2 \cup 3 \cup 4) &= \mu^*(1 \cup 2) + \mu^*(3 \cup 4) \\ &= \mu^*(1 \cup 2 \cup 3) - \mu^*(3) + \mu^*(3) + \mu^*(4) \\ &= \mu^*(1 \cup 2 \cup 3) + \mu^*(4). \end{aligned}$$

- (countable disjoint unions)

Let $A_1, A_2, \dots \in \mathcal{A}$ and disjoint.

Fix $E \subset X$ arbitrary. Since μ^* is countably subadditive,

$$\mu^*(E) \leq \mu^*\left(E \cap \bigcup_1^\infty A_n\right) + \mu^*\left(E \setminus \bigcup_1^\infty A_n\right)$$

Fix $n \in \mathbb{N}$.

$$\begin{aligned} &\Rightarrow \bigcup_1^N A_n \in \mathcal{A} \\ &\Rightarrow \mu^*(E) = \mu^*\left(E \cap \bigcup_1^N A_n\right) + \mu^*\left(E \setminus \bigcup_1^N A_n\right) \\ &\geq \sum_1^N \mu^*(E \cap A_n) + \mu^*\left(E \setminus \bigcup_1^\infty A_n\right) \text{ by lemma.} \end{aligned}$$

Take $n \rightarrow \infty$.

- (countable unions)

Let $A_1, A_2, \dots \in \mathcal{A}$. Take $E_1 = A_1, E_n = A_n \setminus \left(\bigcup_1^{n-1} A_i\right)$ for $n \geq 2$. Then $\bigcup A_n = \bigcup E_n$ and E_n 's are disjoint.

(b) Firstly we have $\mu(\emptyset) = \mu^*(\emptyset) = 0$.

Countable additivity of μ^* on \mathcal{A} follows from the improved lemma with $E = X$.

(c) HW. ■

1.4 Hahn-Kolmogorov Theorem

RECALL 1.19 Let $\mathcal{E} \subset \mathcal{P}(X)$ s.t. $\emptyset, X \in \mathcal{E}$. Let $\rho : \mathcal{E} \rightarrow [0, \infty]$ s.t. $\rho(\emptyset) = 0$

$$(\mathcal{E}, \rho) \xrightarrow{1.19} (\mathcal{P}(X), \mu^*) \xrightarrow{\text{C-theorem}} (A, \mu)$$

QUESTION $\mathcal{E} \subset \mathcal{A}$ and $\mu|_{\mathcal{E}} = \rho$? No!

Definition 1.23. Let \mathcal{A}_0 be an algebra on X . We say $\mu_0 : \mathcal{A}_0 \rightarrow [0, \infty]$ is a pre-measure if

- (a) $\mu_0(\emptyset) = 0$.
- (b) (finite additivity)

$$\mu_0 \left(\bigcup_1^N A_i \right) = \sum_1^N \mu_0(A_i) \text{ if } A_1, \dots, A_N \in \mathcal{A}_0 \text{ are disjoint.}$$

- (c) (countable additivity within the algebra) If $A \in \mathcal{A}_0$ and

$$A = \bigcup_1^\infty A_n, A_n \in \mathcal{A}_0 \text{ and are disjoint, then } \mu_0(A) = \sum_1^\infty \mu_0(A_n)$$

NOTATION: Folland uses \mathcal{M} for σ -algebra and \mathcal{A} for algebra. (Jinho) uses \mathcal{A} for σ -algebra and \mathcal{A}_0 for algebra.

Example 1.24. \mathcal{A}_0 finite disjoint unions of $(a, b]$.

$$\mu_0 \left(\bigcup_1^\infty (a_i, b_i] \right) = \sum_1^\infty (b_i - a_i) \text{ or } b_i^n - a_i^n, e^{b_i} - e^{a_i}, \text{ etc.}$$

Lemma 1.25. • $(a) + (c) \implies (b)$.

- μ_0 is monotone.

Theorem 1.26 (Hahn-Kolmogorov Theorem). Let μ_0 be a pre-measure on algebra \mathcal{A}_0 on X . Let μ^* be the outer measure induced by (\mathcal{A}_0, μ_0) in 1.19. Let \mathcal{A} and μ be the Carathéodory σ -algebra and measure for $\mu^* \implies (\mathcal{A}, \mu)$ extends (\mathcal{A}_0, μ_0) i.e. $\mathcal{A} \supset \mathcal{A}_0, \mu|_{\mathcal{A}_0} = \mu_0$.

Proof. (a) $(\mathcal{A} \supset \mathcal{A}_0)$ Let $A \in \mathcal{A}_0$.

Question: $A \in \mathcal{A}$? i.e. is A C -measurable? i.e. $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c), \forall E \subset X$

X .

Fix $E \subset X$.

- (countable) subadditivity of $\mu^* \implies \mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c)$.
- If $\mu^*(E) = \infty$ then $\mu^*(E) = \infty \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$.
- If $\mu^*(E) < \infty$.

Fix $\varepsilon > 0$. By the definition of μ^* , $\exists B_1, B_2, \dots \in \mathcal{A}_0$ s.t. $\bigcup_1^\infty B_n \supset E$ and

$$\mu^*(E) + \varepsilon \geq \sum_1^\infty \mu_0(B_n) = \sum_1^\infty (\mu_0(B_n \cap A) + \mu_0(B_n \cap A^c)).$$

Note that

$$\bigcup_1^\infty (B_n \cap A) \supset E \cap A, \quad \bigcup_1^\infty (B_n \cap A^c) \supset E \cap A^c \implies \geq$$

(b) Let $A \in \mathcal{A}_0$. We want to show that $\mu(A) = \mu_0(A)$.

By definition, $\mu(A) = \mu^*(A)$.

- Let $B_i = \begin{cases} A & i = 1, \\ \emptyset & i = 2 \end{cases} \in \mathcal{A}_0$ and $\bigcup_1^\infty B_i \supset A$.

Hence $\mu^*(A) \leq \sum_1^\infty \mu_0(B_i) = \mu_0(A)$.

- Let $B_i \in \mathcal{A}_0, \bigcup_1^\infty B_i \supset A$ an arbitrary collection of sets.

Let $C_1 = A \cap B_1, C_i = A \cap B_i \setminus \left(\bigcup_{j=1}^{i-1} B_j \right)$. Then $A = \bigcup_1^\infty C_i$ is a disjoint countable union. By countable additivity we have

$$\mu_0(A) = \sum_1^\infty \mu_0(C_i) \implies \mu_0(A) \leq \sum_1^\infty \mu_0(B_i).$$

Hence we have $\mu_0(A) = \mu^*(A) = \mu(A)$. We have completed our proof. ■

Definition 1.27. Such (\mathcal{A}, μ) is called the Hahn-Kolmogorov extension of (\mathcal{A}_0, μ_0) , and is also called the Carathéodory σ -algebra for (\mathcal{A}_0, μ_0) .

Theorem 1.28 (uniqueness of HK extension). *Let \mathcal{A}_0 be an algebra on X , μ_0 be a pre-measure on \mathcal{A}_0 , (\mathcal{A}, μ) be the Hahn-Kolmogorov extension of (\mathcal{A}_0, μ_0) . And let (\mathcal{A}', μ') be another extension of (\mathcal{A}_0, μ_0) .*

If μ_0 is σ -finite, then $\mu|_{\mathcal{A} \cap \mathcal{A}'} = \mu'|_{\mathcal{A} \cap \mathcal{A}'}$.

NOTE σ -finite means

$$\forall X, X = \bigcup_1^\infty X_n, X_n \in \mathcal{A}_0, \mu_0(X_n) < \infty.$$

Corollary 1.29. Let μ_0 be a pre-measure on algebra \mathcal{A}_0 on X . Suppose μ_0 is σ -finite, then $\exists!$ measure μ on $\langle \mathcal{A}_0 \rangle$ that extends \mathcal{A}_0 . Furthermore,

(a) the completion of $(X, \langle \mathcal{A}_0 \rangle, \mu)$ is the HK extension of (\mathcal{A}_0, μ_0) .

(b)

$$\mu(A) = \inf \left\{ \sum_{i=1}^\infty \mu_0(B_i) \mid B_i \subset A_0, \forall i \in \mathbb{N}, \bigcup_1^\infty B_i \supset A \right\}, \forall A \in \overline{\langle \mathcal{A}_0 \rangle}.$$

Proof of 1.28. Let $A \in \mathcal{A} \cap \mathcal{A}'$. We need to show $\mu(A) = \mu^*(A) = \mu'(A)$.

- $\mu^*(A) \geq \mu'(A)$ (HW)

- $\mu(A) \leq \mu'(A)$:

(i) Assume $\mu(A) < \infty$. Fix $\varepsilon > 0$. Then $\exists B_i \in \mathcal{A}_0, \forall i \in \mathbb{N}, \bigcup_1^\infty B_i \supset A$ s.t.

$$\mu(A) + \varepsilon = \mu^*(A) + \varepsilon \geq \sum_1^\infty \mu_0(B_i) = \sum_1^\infty \mu(B_i) \geq \mu\left(\bigcup_1^\infty B_i\right) = \mu(B)$$

Hence $\mu(B \setminus A) = \mu(B) - \mu(A) \leq \varepsilon$.

On the other hand,

$$\mu(B) = \lim_{N \rightarrow \infty} \mu\left(\bigcup_1^N B_i\right) = \lim_{N \rightarrow \infty} \mu'\left(\bigcup_1^N B_i\right) = \mu'(B)$$

by continuity of measure from below.

$$\mu(A) \leq \mu(B) = \mu'(B) = \mu'(A) + \mu'(B \setminus A) \leq \mu'(A) = \varepsilon.$$

(ii) Assume $\mu(A) = \infty$.

Since μ_0 is σ -finite, $X = \bigcup_1^\infty X_n, X_n \in \mathcal{A}_0, \mu_0(X_n) < \infty$. Replacing X_n by $X_1 \cup \dots \cup X_n$, we may assume $X_1 \subset X_2 \subset \dots$

$$\forall n \in \mathbb{N}, \mu(A \cap X_n) < \infty \implies \mu(A \cap X_n) \leq \mu'(A \cap X_n).$$

Hence

$$\mu(A) = \lim_{N \rightarrow \infty} \mu(A \cap X_n) \leq \lim_{N \rightarrow \infty} \mu'(A \cap X_n) = \mu'(A). \quad \blacksquare$$

1.5 Borel Measures on \mathbb{R}

Definition 1.30. $F : \mathbb{R} \rightarrow \mathbb{R}$ is an increasing function if $F(x) \leq F(y)$ for $x < y$. $F : \mathbb{R} \rightarrow \mathbb{R}$ is increasing and right-continuous $\implies F$ is distribution function.

Example 1.31.

- $F(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0. \end{cases}$
- $\mathbb{Q} = \{r_1, r_2, \dots\}$, $F_n(x) = \begin{cases} 1 & x \geq r_n \\ 0 & x < r_n \end{cases}$. $F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n}$ is a distribution function.

NOTE If F is increasing, $F(\infty) := \lim_{x \rightarrow \infty} F(x)$, $F(-\infty) := \lim_{x \rightarrow -\infty} F(x)$ exists in $[-\infty, \infty]$.

In probability theory, cumulative distribution function (CDF) is a distribution function with $F(\infty) = 1$ and $F(-\infty) = 0$.

There are distributions [Folland, Ch9], but these are different from distribution functions.

Definition 1.32. Suppose X a topological space. μ on $(X, \mathcal{B}(X))$ is called locally finite is $\mu(K) < \infty$ for any compact set $K \subset X$.

Lemma 1.33. Let μ be a locally finite Borel measure on $\mathbb{R} \implies$

$$F_\mu(x) = \begin{cases} \mu((0, x]), & x > 0 \\ 0, & x = 0 \\ -\mu((x, 0]), & x < 0 \end{cases} \text{ is a distribution function.}$$

Proof. DIY. Use continuity of measure. ■

Definition 1.34. h -intervals are $\emptyset, (a, b], (a, \infty), (-\infty, b], (-\infty, \infty)$.

Lemma 1.35. Let \mathcal{H} be the collections of finite disjoint unions of h -intervals. Then \mathcal{H} is an algebra on \mathbb{R} .

Proof. DIY. ■

Proposition 1.36 (Distribution function defines a pre-measure). *Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be a distribution function. For an h -interval I , define*

$$\ell(I) = \ell_F(I) = \begin{cases} 0, & I = \emptyset \\ F(b) - F(a), & I = (a, b] \\ F(\infty) - F(a), & I = (a, \infty) \\ F(b) - F(\infty), & I = (-\infty, b] \\ F(\infty) - F(-\infty), & I = (-\infty, \infty). \end{cases}$$

Define $\mu_0 = \mu_{0,F} : \mathcal{H} \rightarrow [0, \infty]$ by

$$\mu_0(A) := \sum_{k=1}^N \ell(I_k) \quad \text{if } A = \bigcup_{k=1}^N I_k, \text{ finite disjoint union of } h\text{-intervals.}$$

Then μ_0 is a pre-measure.

Proof. (a) μ_0 is well-defined.

(b) μ_0 is finite additive.

(c) μ_0 is countable additive within \mathcal{H} .

Suppose $A \in \mathcal{H}$ and $A = \bigcup_1^\infty A_i$ a disjoint union, $A_i \in \mathcal{H}$. It is enough to consider the case $A = I, A_k = I_k$ all h -intervals. (Why?)

Focus on the case $I = (a, b]$: (HW: check other cases)

We have

$$(a, b] = \bigcup_1^\infty (a_n, b_n], \text{ a disjoint union.}$$

Check

$$F(b) - F(a) \stackrel{?}{=} \sum_1^\infty (F(b_n) - F(a_n))$$

$(a, b] \supset \bigcup_1^N (a_n, b_n] \implies F(b) - F(a) \geq \sum_1^N (F(b_n) - F(a_n)), \forall N \in \mathbb{N}$. (Arranging them in decreasing order) Take $N \rightarrow \infty$ we have

$$F(b) - F(a) \geq \sum_1^\infty (F(b_n) - F(a_n)).$$

Since F is right-continuous, $\exists a' > a$ s.t. $F(a') - F(a) < \varepsilon$. For each $n \in \mathbb{N}$, $\exists b'_n >$

$$b_n \text{ s.t. } F(b'_n) - F(b_n) < \frac{\varepsilon}{2^n}.$$

$$\implies [a', b] \subset \bigcup_1^\infty (a_n, b'_n)$$

$$\implies \exists N \in \mathbb{N} \text{ s.t. } [a', b] \subset \bigcup_1^N (a_n, b'_n)$$

$$\implies F(b) - F(a') \leq \sum_1^N F(b'_n) - F(a_n)$$

$$\implies F(b) - F(a) \leq F(b) - F(a') + \varepsilon \leq \sum_1^\infty (F(b'_n) - F(a_n)) + \varepsilon$$

$$\leq \sum_1^\infty \left(F(b_n) - F(a_n) + \frac{\varepsilon}{2^n} \right) + \varepsilon \quad \blacksquare$$

Once we have this pre-measure, HK theorem allows us to extend it to a measure.

Theorem 1.37 (Locally finite Borel measures on \mathbb{R}).

- (a) $F : \mathbb{R} \rightarrow \mathbb{R}$ is a distribution function $\implies \exists!$ locally finite Borel measure μ_F on \mathbb{R} satisfying $\mu_F((a, b]) = F(b) - F(a), \forall a, b, a < b$.
- (b) Suppose $F, G : \mathbb{R} \rightarrow \mathbb{R}$ are distribution functions. Then, $\mu_F = \mu_G$ on $\mathcal{B}(\mathbb{R})$ if and only if $F - G$ is a constant function.

Proof. HW ■

1.6 Lebesgue-Stieltjes Measures on \mathbb{R}

F distribution function $\implies \mu_F$ on Carathéodory σ -algebra \mathcal{A}_{μ_F} .

Actually $(\mathcal{A}_{\mu_F}, \mu_F) = (\mathcal{B}(\mathbb{R}), \mu_F)$ (HW3).

Definition 1.38. • μ_F on \mathcal{A}_{μ_F} is called the Lebesgue-Stieltjes measure corresponding to F .

- Special case: $F(x) = x \implies$ Lebesgue measure (\mathcal{B}, m) .

Example 1.39.

- (a) $\mu_F((a, b]) = F(b) - F(a)$. F is right-continuous and increasing $\implies F(x_-) \leq F(x) = F(x_+)$.

$$\text{(HW)} \quad \mu_F(\{a\}) = F(a) - F(a_-), \mu_F([a, b]) = F(b) - F(a_-), \mu_F((a, b)) = F(b_-) - F(a).$$

(b)

$$F(x) = \begin{cases} 1 & x \leq 0 \\ 0 & x < 0 \end{cases} \implies \mu_F(\{0\}) = 1, \mu_F(\mathbb{R}) = 1, \mu_F(\mathbb{R} \setminus \{0\}) = 0.$$

μ_F is the Dirac measure at 0.

(c)

$$\mathbb{Q} = \{r_1, r_2, \dots\}, F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n}, F_n(x) = \begin{cases} 1 & x \leq r_n \\ 0 & x < r_n \end{cases} \\ \implies \mu_F(\{v\}) > 0, \forall v \in \mathbb{Q}, \mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0.$$

(d) If F is continuous at a , $\mu_F(\{a\}) = 0$.(e) $F(x) = x \implies m((a, b]) = m((a, b)) = m([a, b]) = b - a$.(f) $F(x) = e^x, \implies \mu_F((a, b]) = \mu_F((a, b)) = e^b - e^a$

(a), (b) are examples of discrete measure.

Example 1.40 (Middle thirds Cantor set $\mathcal{C} = \bigcup_{n=1}^{\infty} K_n$). \mathcal{C} is uncountable set with $m(\mathcal{C}) = 0$.

$$x \in \mathcal{C} \implies x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}, a_n \in \{0, 2\}.$$

We are interested in the Cantor function F .

Example 1.41. Cantor function F is continuous and increasing. This defines the Cantor measure $\mu_F(\mathbb{R} \setminus \mathcal{C}) = 0, \mu_F(\mathcal{C}) = 1, \mu_F(\{a\}) = 0$. Compare with Lebesgue measure $m(\mathbb{R} \setminus \mathcal{C}) = \infty > 0, \mu(\mathcal{C}) = 0, m(\{a\}) = 0$.

1.7 Regularity Properties of Lebesgue-Stieltjes Measures

Lemma 1.42. μ is Lebesgue-Stieltjes measure on $\mathbb{R} \implies$

$$\begin{aligned} \mu(A) &= \inf \left\{ \sum_1^{\infty} \mu((a_i, b_i]) \mid \bigcup_1^{\infty} (a_i, b_i] \supset A \right\} \\ &= \inf \left\{ \sum_1^{\infty} \mu((a_i, b_i)) \mid \bigcup_1^{\infty} (a_i, b_i) \supset A \right\} \end{aligned}$$

Proof. Using the continuity of measure. ■

Theorem 1.43. μ is a Lebesgue-Stieltjes measure. Then $\forall A \in \mathcal{A}_\mu$,

(a) (outer regularity)

$$\mu(A) = \inf\{\mu(O) \mid \text{open } O \supset A\}.$$

(b) (inner regularity)

$$\mu(A) = \sup\{\mu(K) \mid \text{compact } K \subset A\}.$$

Proof. (a) Followed from 1.42.

(b) Let $s = \sup\{\dots\}$. Monotonicity $\implies \mu(A) \geq s$.

- (A bounded) $\bar{A} \in \mathcal{B}(\mathbb{R}) \subset \mathcal{A}_\mu$, \bar{A} bounded $\implies \mu(\bar{A}) < \infty$.

Fix $\varepsilon > 0$. By 1, \exists open $O \supset \bar{A} \setminus A$, $\mu(O) - \mu(\bar{A} \setminus A) = \mu(O \setminus (\bar{A} \setminus A)) \leq \varepsilon$.

Let $K = \underbrace{A \setminus O}_{K \subset A} = \underbrace{\bar{A} \setminus O}_{\text{compact}}$. Show that $\mu(K) \geq \mu(A) - \varepsilon$.

- (A unbounded but $\mu(A) < \infty$) We have

$$A = \bigcup_{n=1}^{\infty} A_n, \quad A_n = A \cap [-n, n], \quad A_1 \subset A_2 \subset \dots$$

Hence

$$\lim_{n \rightarrow \infty} \mu(A_n) = \mu(A) < \infty.$$

- ($\mu(A) = \infty$)

$$\lim_{n \rightarrow \infty} \mu(A_n) = \mu(A) = \infty.$$

Fix $L > 0$. $\exists N$ s.t. $\mu(A_N) \geq L$. ■

Definition 1.44. Suppose X a topological space.

A $G\sigma$ -set is $G = \bigcup_{i=1}^{\infty} O_i$, O_i open. An $F\sigma$ -set is $F = \bigcup_{i=1}^{\infty} F_i$, F_i closed.

Theorem 1.45. Suppose μ a LS measure. Then the following statements are equivalent:

- (a) $A \in \mathcal{A}_\mu$.
- (b) $A = G \setminus M$, G is a $G\sigma$ -set, and M is μ -null.
- (c) $A = F \cup N$, F is a $F\sigma$ -set, and N is μ -null.

Proof. (b) \implies (a) and (c) \implies (a) are clear.

- (a) \implies (c)

(i) Assume $\mu(A) < \infty$. By inner regularity,

$$\forall n \in \mathbb{N}, \exists \text{ compact } K_n \subset A \text{ s.t. } \mu(K_n) + \frac{1}{n} \geq \mu(A).$$

Let $F = \bigcup_1^\infty K_n$. Then $N = A \setminus F$ is μ -null.

(ii) Assume $\mu(A) = \infty$. We construct

$$A = \bigcup_{k \in \mathbb{Z}} A_k, A_k = A \cap (k, k+1].$$

By (i), $\forall k \in \mathbb{Z}, A_k = F_k \cup N_k$. Hence

$$A = \underbrace{\left(\bigcup_k F_k \right)}_{F_\sigma} \cup \underbrace{\left(\bigcup_k N_k \right)}_{\mu\text{-null}}.$$

- (a) \implies (b)

$$A^c = F \cup N, A = F^c \cup N^c = F^c \setminus N. \quad \blacksquare$$

Proposition 1.46. Suppose μ a LS measure, $A \in \mathcal{A}_\mu$, $\mu(A) < \infty$. Then

$$\forall \varepsilon > 0, \exists I = \bigcup_1^{N=N(\varepsilon)} I_i, \text{ disjoint open intervals s.t. } \mu(A \Delta I) \leq \varepsilon.$$

Proof. DIY - use outer regularity. ■

Properties of Lebesgue measure

Theorem 1.47.

$$A \in \mathcal{L} \implies A + s \in \mathcal{L}, rA \in \mathcal{L}, \forall r, s \in \mathbb{R}.$$

In addition, $m(A + r) = m(A)$ and $m(rA) = rm(A)$.

Proof. DIY. ■

Example 1.48.

(a) $\mathbb{Q} = \{r_1\}_{i=1}^\infty$, which is dense in \mathbb{R} . Let $\varepsilon > 0$ and

$$O = \bigcup_{i=1}^\infty \left(r_i - \frac{\varepsilon}{2^i}, r_i + \frac{\varepsilon}{2^i} \right).$$

O is open and dense in \mathbb{R} . We have

$$m(O) \leq \sum_{i=1}^{\infty} \frac{2\varepsilon}{2^i} = 2\varepsilon, \partial O = \overline{O} \setminus O, m(O) = \infty.$$

- (b) \exists uncountable set A with $m(A) = 0$.
- (c) $\exists A$ with $m(A) > 0$, but A contains no non-empty open interval.
- (d) $\exists A \notin \mathcal{L}$ that is Vitali set.
- (e) $\exists A \in \mathcal{L} \setminus \mathcal{B}(\mathbb{R})$. We will deal with that later.

Chapter 2

Integration

2.1 Measurable Functions

Definition 2.1. Suppose (X, \mathcal{A}) , (Y, \mathcal{B}) two measurable spaces. $f : X \rightarrow Y$ is $(\mathcal{A}, \mathcal{B})$ -measurable if

$$\forall B \in \mathcal{B}, f^{-1}(B) \in \mathcal{A}.$$

Lemma 2.2. Suppose $\mathcal{B} = \langle \mathcal{E} \rangle$. Then

$$f : X \rightarrow Y \text{ is } (\mathcal{A}, \mathcal{B})\text{-measurable} \iff \forall E \in \mathcal{E}, f^{-1}(E) \in \mathcal{A}.$$

Proof. \implies clear

\Leftarrow Let $\mathcal{D} = \{E \subset Y \mid f^{-1}(E) \in \mathcal{A}\}$. We have $\mathcal{E} \subset \mathcal{D}$ by assumption. In addition \mathcal{D} is a σ -algebra $\implies \langle \mathcal{E} \rangle \subset \mathcal{D}$. ■

Definition 2.3. Suppose (X, \mathcal{A}) a measurable space.

$$\left. \begin{array}{l} f : X \rightarrow \mathbb{R} \\ f : X \rightarrow \overline{\mathbb{R}} = [-\infty, \infty] \\ f : X \rightarrow \mathbb{C} \end{array} \right\} \text{ is } \mathcal{A}\text{-measurable if } \left\{ \begin{array}{l} f \text{ is } (\mathcal{A}, \mathcal{B}(\mathbb{R}))\text{-measurable} \\ f \text{ is } (\mathcal{A}, \mathcal{B}(\overline{\mathbb{R}}))\text{-measurable} \\ \Re f, \Im f : X \rightarrow \mathbb{R} \text{ are } \mathcal{A}\text{-measurable.} \end{array} \right.$$

Here $\mathcal{B}(\overline{\mathbb{R}}) = \{E \subset \overline{\mathbb{R}} \mid E \cap \mathbb{R} \in \mathcal{B}(\mathbb{R})\}$.

Lemma 2.4. Suppose $f : X \rightarrow \mathbb{R}$. Then the followings are equivalent:

(a) f is \mathcal{A} -measurable

- (b) $\forall a \in \mathbb{R}, f^{-1}((a, \infty)) \in \mathcal{A}$.
- (c) $\forall a \in \mathbb{R}, f^{-1}([a, \infty)) \in \mathcal{A}$.
- (d) $\forall a \in \mathbb{R}, f^{-1}((-\infty, a)) \in \mathcal{A}$.
- (e) $\forall a \in \mathbb{R}, f^{-1}((-\infty, a]) \in \mathcal{A}$.

For $f : X \rightarrow \overline{\mathbb{R}}$, change the interval to include $-\infty$ and ∞ .

Proof. By 2.2. ■

Example 2.5. $\mathcal{A} = \mathcal{P}(X) \implies$ every function is \mathcal{A} measurable.

$\mathcal{A} = \{\emptyset, X\} \implies$ only \mathcal{A} functions are constant functions.

PROPERTIES Suppose $f, g : X \rightarrow \mathbb{R}$, \mathcal{A} -measurable functions.

- (a) $\phi : \mathbb{R} \rightarrow \mathbb{R}$, $\mathcal{B}(\mathbb{R})$ measurable (i.e. Borel measurable) $\implies \phi \circ f : X \rightarrow \mathbb{R}$ is \mathcal{A} -measurable.
- (b) $-f, 3f, f^2, |f|$ are \mathcal{A} -measurable, $\frac{1}{f}$ is \mathcal{A} -measurable if $f(x) \neq 0, \forall x \in X$.
- (c) $f + g$ is \mathcal{A} -measurable

$$(f + g)^{-1}((a, \infty)) = \bigcup_{r \in \mathbb{Q}} (f^{-1}((r, \infty)) \cap g^{-1}((a - r, \infty))).$$

- (d) fg is \mathcal{A} -measurable

$$f(x)g(x) = \frac{1}{2} ((f(x) + g(x))^2 - f(x)^2 - g(x)^2).$$

- (e) $(f \wedge g)(x) = \min\{f(x), g(x)\}$, $(f \vee g)(x) = \max\{f(x), g(x)\}$ are \mathcal{A} -measurable.
- (f) $f_n : X \rightarrow \overline{\mathbb{R}}$ are a sequence of \mathcal{A} -measurable functions \implies

$$\sup f_n, \inf f_n, \limsup_{n \rightarrow \infty} f_n, \liminf_{n \rightarrow \infty} f_n \text{ are } \mathcal{A}\text{-measurable.}$$

- (g) If $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ converges for every $x \in X$, then f is measurable.

Example 2.6. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous. Then f is Borel measurable $\implies f$ is Lebesgue measurable. (Preimage of an open set of a continuous function is open.)

Definition 2.7. For $f : X \rightarrow \overline{\mathbb{R}}$, let $f^+ = f \vee 0$, $f^- = (-f) \vee 0$.

NOTE $\text{supp } f^+ \cap \text{supp } f^- = \emptyset$. $f(x) = f^+(x) - f^-(x)$. f is \mathcal{A} -measurable $\iff f^+, f^-$ measurable.

Definition 2.8. For $E \subset X$, characteristic (indicator) function of E

$$\chi_E(x) = 1_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \in E^c. \end{cases}$$

1_E is \mathcal{A} -measurable $\iff E \in \mathcal{A}$.

Definition 2.9. Suppose (X, \mathcal{A}) a measurable space. A simple function $\phi : X \rightarrow \mathbb{C}$ that is \mathcal{A} -measurable and takes only finitely many values.

$$\phi(X) = \{c_1, \dots, c_N\}, c_i \neq \pm\infty, E_i = \phi^{-1}(c_i) \in \mathcal{A} \implies \phi = \sum_{i=1}^N c_i 1_{E_i}.$$

Theorem 2.10. Suppose (X, \mathcal{A}) a measurable space and $f : X \rightarrow [0, \infty]$. Then the followings are equivalent:

- (a) f is \mathcal{A} -measurable.
- (b) \exists simple functions $0 \leq \phi_1(x) \leq \phi_2(x) \leq \dots \leq f(x)$ such that

$$\lim_{n \rightarrow \infty} \phi_n(x) = f(x), \forall x \in X.$$

(f is the pointwise upward limit of simple functions.)

Proof. • (b) \implies (a) is easy: $f(x) = \sup_{n \in \mathbb{N}} \phi_n(x)$.

- (a) \implies (b): suppose f is \mathcal{A} -measurable.

Fix $n \in \mathbb{N}$. Let $F_n = f^{-1}([2^n, \infty]) \in \mathcal{A}$. For

$$0 \leq k \leq 2^{2n} - 1, E_{n,k} = f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right) \in \mathcal{A}.$$

$$\text{Let } \phi_n(x) = \sum_{k=0}^{2^{2n}-1} 1_{E_{n,k}} + 2^n 1_{F_n}.$$

We have

$$0 \leq \phi_1(x) \leq \phi_2(x) \leq \dots \leq f(x)$$

and

$$\forall x \in X, 0 \leq f(x) - \phi_n(x) \leq \frac{1}{2^n}.$$

■

Corollary 2.11. *If f is bounded on a set $A \subset \mathbb{R}$ (i.e. $\exists nL > 0$ s.t. $|f(x)| \leq L, \forall x \in A$) then $\phi_n \rightarrow f$ uniformly on A .*

Proof. DIY. ■

Corollary 2.12. $f : X \rightarrow \mathbb{C}$, measurable function $\iff \exists$ simple functions $\phi_n : X \rightarrow \mathbb{C}$ s.t.

2.2 Integration of Nonnegative Functions

Definition 2.13. Suppose (X, \mathcal{A}, μ) a measure space and $\phi = \sum_{i=1}^N c_i 1_{E_i} : X \rightarrow [0, \infty]$ a simple function. Let

$$\int \phi = \int \phi d\mu = \int_X \phi d\mu = \sum_1^N c_i \mu(E_i).$$

Proposition 2.14. *Suppose $\phi, \psi \geq 0$ are simple functions. Then,*

- 2.13 is well-defined.
- $\int c\phi = c \int \phi, c \in [0, \infty)$.
- $\int (\phi + \psi) = \int \phi + \int \psi$.
- $\phi(x) \geq \psi(x), \forall x \implies \int \phi \geq \int \psi$.
- $\nu(A) = \int_A \phi d\mu$ is a measure on (X, \mathcal{A}) .

Proof. DIY. ■

Definition 2.15. Suppose $(X, \mathcal{A}, \mu), f : X \rightarrow [0, \infty]$ is \mathcal{A} -measurable.

Define

$$\int f = \int f d\mu = \sup \left\{ \int \phi \mid 0 \leq \phi \leq f, \phi \text{ simple} \right\}.$$

Proposition 2.16.

- *If f is a simple function then two definitions are the same.*
- $\int cf = c \int f$.
- $f \geq g \geq 0 \implies \int f \geq \int g$.

- $\int f + g = \int f + \int g$. (A bit harder to check)

Theorem 2.17 (Monotone convergence theorem). Suppose (X, \mathcal{A}, μ) a measure space and

- $f : X \rightarrow [0, \infty]$ is \mathcal{A} -measurable, $\forall n \in \mathbb{N}$.
- $0 \leq f_1(x) \leq \dots$
- $\lim_{n \rightarrow \infty} f_n(x) = f(x)$.

Then

$$\int f = \lim_{n \rightarrow \infty} \int f_n.$$

Proof. TBC ■

Corollary 2.18. $f, g \geq 0$ measurable $\implies \int f + g = \int f + \int g$.

Corollary 2.19 (Tonelli's theorem for series and integrals). Given $s_n \geq 0, \forall n \in \mathbb{N}$ measurable functions. Then

$$\int \sum_{n=1}^{\infty} s_n = \sum_{n=1}^{\infty} \int s_n.$$

Proof. Let $f_N = \sum_{n=1}^N s_n, 0 \leq f_1 \leq f_2 \leq \dots$

$$\lim_{N \rightarrow \infty} f_N(x) = \sum_{n=1}^{\infty} s_n(x)$$

By MCT, we have

$$\lim_{N \rightarrow \infty} \sum_{n=1}^N s_n = \sum_{n=1}^{\infty} s_n$$
■

Theorem 2.20 (Fatou's lemma). Suppose $f_n \geq 0$ measurable. Then

$$\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n.$$

Recall that

$$\liminf_{n \rightarrow \infty} f_n := \lim_{k \rightarrow \infty} \inf_{n \geq k} f_n = \sum_{k \in \mathbb{N}} \inf_{n \geq k} f_n,$$

and

$$\lim_{n \rightarrow \infty} a_n \text{ exists} \iff \limsup_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n.$$

Proof. Let $g_k = \inf_{n \geq k} f_n \implies s_k$ measurable and $0 \leq g_1 \leq g_2 \leq \dots$. By MCT, we have

$$\int \liminf_{n \rightarrow \infty} = \int \lim_{k \rightarrow \infty} s_k = \lim_{k \rightarrow \infty} \int s_k = \lim_{k \rightarrow \infty} \int \inf_{n \geq k} f_n$$

$$\begin{aligned} & \inf_{n \geq k} f_n \leq f_m, \forall m \geq k \\ \implies & \int \inf_{n \geq k} f_n \leq \int f_m, \forall m \geq k \\ \implies & \int \inf_{n \geq k} f_n \leq \inf_{m \geq k} \int f_m \end{aligned}$$

■

Example 2.21. Suppose $(\mathbb{R}, \mathcal{L}, m)$

(a) (escape to horizontal infinity) $f_n = 1_{(n, n+1)}$.

We see that $f_n \rightarrow 0 = f$ pointwise and $\int f_n = 1, \forall n, \int f = 0$.

(b) (escape to width infinity) $f_n = \frac{1}{n} 1_{(0, n)}$.

(c) (escape to vertical infinity) $f_n = n 1_{(0, 1/n)}$.

Lemma 2.22 (Markov's inequality). $f \geq 0$ is measurable \implies

$$\forall c \in (0, \infty), \mu(\{x \mid f(x) \geq c\}) \leq \frac{1}{c} \int f.$$

Proof. Let $E = \{x \mid f(x) \geq c\}$. Then

$$f(x) \geq c 1_E(x) \implies \int f \geq c \int 1_E = c \mu(E).$$

■

Proposition 2.23. Suppose $f \geq 0$ measurable. Then $\int f = 0 \iff f = 0$ almost everywhere (a.e.)

$$\int f d\mu = \mu(A) = 0, A = \{x \mid f(x) > 0\} = f^{-1}((0, \infty])$$

Proof. (a) Assume $f = \phi$ a simple function. We may assume

$$\phi = \sum_{i=1}^N c_i 1_{E_i}, c_i \in (0, \infty), E_i\text{'s are disjoint.}$$

$$\begin{aligned}
\int \phi &= \sum_{i=1}^N c_i \mu(E_i) = 0 \\
&\iff \mu(E_1) = \dots = \mu(E_N) = 0 \\
&\iff \mu(A) = 0, \quad A = \bigcup_{i=1}^N E_i.
\end{aligned}$$

(b) General $f \geq 0$.

(1) Assume $\mu(A) = 0$ (i.e. $f = 0$ a.e.)

Let $0 \leq \phi \leq f$, ϕ is simple.

$$\begin{aligned}
&\implies \phi(x) = 0, \quad \forall x \in A^c \\
&\implies \phi = 0 \text{ a.e.} \\
&\implies \int \phi = 0
\end{aligned}$$

Then $\int f = 0$ by the definition of $\int f$.

(2) Assume $\inf f = 0$. Let $A_n = f^{-1}([\frac{1}{n}, \infty])$

$$\begin{aligned}
&\implies A_1 \subset A_2 \subset \dots \\
&\bigcup_1^\infty A_n = f^{-1}\left(\bigcup_1^\infty \left[\frac{1}{n}, \infty\right]\right) = f^{-1}((0, \infty)) = A \\
&\mu(A_n) = \mu\left(\left\{x \mid f(x) \geq \frac{1}{n}\right\}\right) \leq n \int f = 0 \\
&\implies \mu(A) = \lim_{n \rightarrow \infty} \mu(A_n) = 0
\end{aligned}$$

by the continuity of measure from below. ■

Corollary 2.24. $f, g \geq 0$ are measurable, $f = g$ a.e. $\implies \int f = \int g$.

Proof. Let $A = \{x \mid f(x) \geq g(x)\}$. A is measurable (why?). By assumption $\mu(A) = 0$.

Hence $f1_A = 0$ a.e.

$$\begin{aligned}
 \int f &= \int f(1_A + 1_{A^c}) \\
 &= \int f1_A + \int f1_{A^c} \\
 &= \int f1_{A^c} \\
 &= \int g1_{A^c} = \int g1_A + \int g1_{A^c} = \int g.
 \end{aligned}$$

■

Corollary 2.25. $f_n \geq 0$ measurable. Then

(a)

$$\left. \begin{aligned} 0 \leq f_1 \leq f_2 \leq \dots \leq f \text{ a.e.} \\ \lim_{n \rightarrow \infty} f_n = f \text{ a.e.} \end{aligned} \right\} \implies \lim_{n \rightarrow \infty} \int f_n = \int f.$$

(b)

$$\lim_{n \rightarrow \infty} \int f_n = \int f \text{ a.e.} \implies \int f \leq \liminf_{n \rightarrow \infty} \int f_n.$$

2.3 Integration of Complex Functions

I was afraid that you are bored.

— Jinho Baik on homework

Definition 2.26. (X, \mathcal{A}, μ) measure space.

- $f : X \rightarrow \overline{\mathbb{R}}$ or $f : X \rightarrow \mathbb{C}$ measurable functions is called integrable if $\int |f| < \infty$.
Then

$$\int f = \int f^+ - \int f^- \text{ or } \int f = \int u^+ - \int u^- + i \left(\int v^+ - \int v^- \right).$$

- Suppose $f : X \rightarrow \overline{\mathbb{R}}$. Define

$$\int f = \begin{cases} \infty & \int f^+ = \infty, \int f^- < \infty, \\ -\infty & \int f^+ < \infty, \int f^- = \infty. \end{cases}$$

Lemma 2.27. Suppose $f, g : x \rightarrow \overline{\mathbb{R}} \rightarrow \mathbb{C}$ integrable. Assume $f(x) + g(x)$ is well-defined $\forall x \in X$. (i.e. $\infty + (-\infty)$, $-\infty + \infty$ do not occur)

- (a) $f + g, cf, c \in \mathbb{C}$ are integrable.
 (b) $\int f + g = \int f + \int g$.
 (c) $\left| \int f \right| \leq \int |f|$. (This is essentially triangle inequality.)

Proof. Check [Fol99, p.53]. ■

Lemma 2.28. (X, \mathcal{A}, μ) measure space and f integrable function on X .

- (a) f is finite a.e. (i.e. $\{x \in X : |f(x)| = \infty\}$ is a null set)
 (b) The set $\{x \in X : f(x) \neq 0\}$ is σ -finite.

Proof. HW5Q8. ■

Proposition 2.29. Suppose (X, \mathcal{A}, μ) a measure space.

- (a) If h is integrable on X , then

$$\int_E h = 0, \forall E \in \mathcal{A} \iff \int |h| = 0 \iff h = 0 \text{ a.e.}$$

- (b) If f, g are integrable on X then

$$\int_E f = \int_E g, \forall E \in \mathcal{A} \iff f = g \text{ a.e.}$$

Proof. (a) $\int |h| = 0 \iff h = 0$ is shown in 2.23.

$$\int |h| = 0 \implies \left| \int_E h \right| \leq \int_E |h| \leq \int |h| = 0.$$

On the other hand, assume $\int_E h = 0, \forall E \in \mathcal{A}$. $h = u + iv = u^+ - u^- + i(v^+ - v^-)$.

Let $B = \{x \mid u^+(x) > 0\}$.

$$0 = \operatorname{Re} \int_B h = \int_B u = \int_B u^+ = \int_B u^+ + \int_{B^c} u^+ = \int u^+ \implies u^+ = 0 \text{ a.e.}$$

Similarly, we get $u^-, v^+, v^- = 0$ a.e..

- (b) follows from (a). ■

Theorem 2.30 (Dominated convergence theorem). Suppose (X, \mathcal{A}, μ) a measure space and

- (a) f_n integrable on $X, \forall n \in \mathbb{N}$.

(b) $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ a.e. (pointwise)

(c) $\exists g : X \rightarrow [0, \infty]$ s.t.

- g is integrable.
- $|f_n(x)| \leq g(x)$ a.e., $\forall n \in \mathbb{N}$.

Then

$$\lim_{n \rightarrow \infty} \int f_n = \int f.$$

Proof. Let F be the countable union of null sets on which (a)-(c) may fail. Modifying the def of f_n, f, g on F we may assume (a)-(c) hold everywhere. (b)+(c) $\implies f$ is integrable.

We consider \mathbb{R} -valued case only. (\mathbb{C} -valued case follows)

$$\begin{aligned} & g + f_n \geq 0, g - f_n \geq 0 \\ & \xRightarrow{\text{Fatou}} \int g + f \leq \liminf_{n \rightarrow \infty} \int g + f_n, \quad \int g - f \leq \liminf_{n \rightarrow \infty} \int g - f_n \\ & \implies \int g + \int f \leq \int g + \liminf_{n \rightarrow \infty} \int f_n, \quad \int g - \int f \leq \int g - \limsup_{n \rightarrow \infty} \int f_n \\ & \xRightarrow{\int g < \infty} \int f \leq \liminf_{n \rightarrow \infty} \int f_n, \quad -\int f \leq -\limsup_{n \rightarrow \infty} \int f_n. \\ & \implies \int f \leq \liminf_{n \rightarrow \infty} \int f_n \leq \limsup_{n \rightarrow \infty} \int f_n \leq \int f \end{aligned}$$

So we should have

$$\int f = \liminf_{n \rightarrow \infty} \int f_n = \limsup_{n \rightarrow \infty} \int f_n. \quad \blacksquare$$

Next we investigate the question:

$$\int \sum_1^\infty f_n \stackrel{?}{=} \sum_1^\infty \int f_n.$$

Tonelli: yes if $f_n \geq 0$. Fubini:

Corollary 2.31 (Fubini's theorem for series and integrals).

$$\left. \sum_1^\infty \int |f_n| < \infty \right\} \implies \int \sum_1^\infty f_n = \sum_1^\infty \int f_n.$$

$$\text{Proof. } G(x) = \sum_1^\infty |f_n(x)| \geq |F_N(x)|, F_N(x) = \sum_1^N f_n(x). \quad \blacksquare$$

Bibliography

- [Fol99] G. B. Folland. *Real Analysis: Modern Techniques and Their Applications*. Pure and Applied Mathematics. Wiley, New York, 2nd ed edition, 1999. "A Wiley-Interscience publication."
- [Tao11] Terence Tao. *An Introduction to Measure Theory*. Number v. 126 in Graduate Studies in Mathematics. American Mathematical Society, Providence, R.I, 2011.