Notes for Math 597 – Real Analysis

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Contents

Office hour is Mon 12:30 - 1:30, Tue 12:30 - 1:30 in person EH 5838, Th 1 - 2 online.

Chapter 1

Abstract Measure

1.1 σ-Algebra

Definition 1.1. Let X be a set. A collection $\mathcal{M}$ of subsets of X is called a σ-algebra on X if

- $\emptyset \in \mathcal{M}$.
- $\mathcal{M}$ is closed under complements: $E \in \mathcal{M} \implies E^c \in \mathcal{M}$.
- $\mathcal{M}$ is closed under countable unions: $E_1, E_2, \ldots \in \mathcal{M} \implies \bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$.

SIMPLE PROPERTIES:

- $X = \emptyset^c \in \mathcal{M}$.
- $\bigcap_{i=1}^{\infty} E_i = \left(\bigcup_{i=1}^{n} E_i^c\right)^c \in \mathcal{M}$. It is closed under countable intersections.
- $\bigcup_{i=1}^{N} E_i = E_i \cup \ldots \cup E_n \cup \emptyset \cup \ldots$. It is closed under finite unions (similarly, intersections).
- $E \setminus F = E \cap F^c \in \mathcal{M}, E \triangle F = (E \cap F^c) \cup (F \cap E^c) \in \mathcal{M}$.

Example 1.2. (a) $\mathcal{A} = \mathcal{P}(X)$ power algebra.

(b) $\mathcal{A} = \{\emptyset, X\}$ trivial algebra.

(c) Let $B \subset X, B \neq \emptyset, B \neq X. \mathcal{A} = \{\emptyset, B, B^c, X\}$.

Lemma 1.3. *(An intersection of σ-algebras is a σ-algebra) Let $\mathcal{A}_\alpha, \alpha \in I$, be a family a σ-algebras of X. Then $\bigcap_{\alpha \in I} A_\alpha$ is a σ-algebra. (I can be uncountable.)*

Proof. DIY ∎

Definition 1.4. For $\mathcal{E} \subset \mathcal{P}(X)$ (not necessarily a σ-algebra), let $\langle \mathcal{E} \rangle$ be the intersection of all σ-algebras on X that contains $\mathcal{E}$. Call it the σ-algebra generated by $\mathcal{E}$.

- $\langle \mathcal{E} \rangle$ is the smallest σ-algebra containing $\mathcal{E}$ and is unique.
- $\{\emptyset, B, B^c, X\} = \langle \{B\} \rangle = \langle \{B^c\} \rangle = \langle \{\emptyset, B\} \rangle$.

The above definition gives us (potentially) lots of examples of σ-algebra on a set X

Lemma 1.5. *(a) Suppose $\mathcal{E} \subset \mathcal{P}(X)$, $\mathcal{A}$ a σ-algebra on X. $\mathcal{E} \in \mathcal{A} \implies \langle \mathcal{E} \rangle \in \mathcal{A}$.*

(b) $E \subset F \subset \mathcal{P}(X) \implies \langle \mathcal{E} \rangle \subset \langle \mathcal{F} \rangle$.

Proof. ■

Definition 1.6. For a topological space X, the Borel σ-algebra $\mathcal{B}(X)$ is the σ-algebra generated by the collection of open sets.

Example 1.7. ($X = \mathbb{R}$) $\mathcal{B}(\mathbb{R})$ contains the following collections

$$\mathcal{E}_1 = \{(a, b) \mid a < b\}, \quad \mathcal{E}_2 = \{[a, b] \mid a < b\},$$

$$\mathcal{E}_3 = \{(a, b] \mid a < b\}, \quad \mathcal{E}_4 = \{[a, b) \mid a < b\},$$

$$\mathcal{E}_5 = \{(a, \infty) \mid a \in \mathbb{R}\}, \quad \mathcal{E}_6 = \{[a, \infty) \mid a \in \mathbb{R}\},$$

$$\mathcal{E}_7 = \{(-\infty, a) \mid a \in \mathbb{R}\}, \quad \mathcal{E}_8 = \{(-\infty, a] \mid a < b\}$$

Proposition 1.8. $\mathcal{B}(\mathbb{R}) = \langle \mathcal{E}_i \rangle$ *for each* $i = 1, \ldots, 8$.

Proof. 1.5. ■

Definition 1.9. $(X, \mathcal{A})$ is called a measurable space.

1.2 Measures

Definition 1.10. A measure on $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \to [0, \infty]$ $s.t.$

1. $\mu(\emptyset) = 0$
2. (countable additive) For $A_1, A_2, \ldots \in \mathcal{A}$ disjoint we have

$$\mu\left(\bigcup_1^\infty A_i\right) = \sum_{i=1}^\infty \mu(A_i).$$

$(X, \mathcal{A}, \mu)$ is then called a measure space.

Example 1.11. 1. For any $(X, \mathcal{A}), \mu(A) = \#A$ counting measure.

2. For any $(X, \mathcal{A})$, let $x_0 \in X$. The Dirac measure at x_0 is

$$\mu(A) = \begin{cases} 1 & x_0 \in A, \\ 0 & x_0 \notin A. \end{cases}$$

3. For $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, let $a_1, a_2, \ldots \in [0, \infty)$. $\mu(A) = \sum_{i \in A} a_i$ is a measure.

$(X, \mathcal{A})$ measurable space

$(X, \mathcal{A}, \mu)$ measure space

$\mu : \mathcal{A} \to [0, \infty]$ $s.t.$ $\mu(\emptyset) = 0$, countable additivity.

<u>Note</u>: $A, B \in \mathcal{A}, A \subset B$, then $\mu(B \setminus A) + \mu(A) = \mu(B) \implies \mu(B \setminus A) = \mu(B) - \mu(A)$ if $\mu(A) < \infty$.

Theorem 1.13. $(X, \mathcal{A}, \mu)$ *measure space*

1. *(monotonicity)*
$$A, B \in \mathcal{A}, A \subset B \implies \mu(A) \leq \mu(B).$$

2. *(countable subadditivity)*
$$A_1, A_2, \ldots, \in \mathcal{A}, \implies \mu\left(\bigcup_i^\infty A_i\right) \leq \sum_i^\infty \mu(A_i).$$

3. *(continuity from below/(MCT) from sets)*
$$A_1, A_2, \ldots \in \mathcal{A}, A_1 \subset A_2 \subset A_3 \subset \ldots \implies \mu\left(\bigcup_i^\infty A_i\right) = \lim_{n\to\infty} \mu(A_n).$$

4. *(continuity from above)*
$$A_1, A_2, \ldots \in \mathcal{A}, A_1 \supset A_2 \supset A_3 \supset \ldots, \mu(A_1) < \infty \implies \mu\left(\bigcap_i^\infty A_i\right) = \lim_{n\to\infty} \mu(A_n).$$

Proof. 1, 2, DIY.

For 3, let $B_1 = A_1, B_i = A_i \setminus A_{i-1}, i \geq 2. B_i \in \mathcal{A}$ and are disjoint.

$$\bigcup_i^\infty A_i = \bigcup_i^\infty B_i$$
$$\implies \mu\left(\bigcup_i^\infty A_i\right) = \mu\left(\bigcup_i^\infty B_i\right) = \sum_i^\infty \mu(B_i) = \lim_{n\to\infty} \sum_i^n \mu(B_i) = \lim_{n\to\infty} \mu(A_n).$$

For 4, let $E_i = A_1 \setminus A_i$. Hence $E_i \in \mathcal{A}, E_1 \subset E_2 \subset \ldots$ We have

$$\bigcup_i^\infty E_i = \bigcup_i^\infty (A_1 \setminus A_i) = A_1 \setminus \left(\bigcap_1^\infty A_i\right) \implies \bigcap_1^\infty A_i = A_1 \setminus \left(\bigcup_1^\infty E_i\right).$$

Hence

$$\mu\left(\bigcap_1^\infty A_i\right) = \mu(A_1) - \mu\left(\bigcup_1^\infty E_i\right) = \mu(A_1) - \lim_{n\to\infty} \mu(E_n) = \mu(A_1) - \lim_{n\to\infty} \mu(A_1) - \mu(A_n).$$

■

NOTE: the condition that $\mu(A_1) < \infty$ cannot be dropped.
For example, in $(\mathbb{N}, \mathcal{P}(N)$, counting measure), let $A_n = \{n, n+1, n+2\}, A_1 \supset A_2 \supset A_3 \supset \ldots$ We have $\bigcap_1^\infty = \emptyset \implies \mu\left(\bigcap_1^\infty A_i\right) = 0$.

Definition 1.14. For $(X, \mathcal{A}, \mu)$ measure space,

- $A \subset X$ is a μ-null set if $A \in \mathcal{A}, \mu(A) = 0$.
- $A \subset X$ is a μ-subnull set if $\exists B$, μ-null set $A \subset B$.
- $(X, \mathcal{A}, \mu)$ is a complete measure space if every μ-subnull set is $\mathcal{A}$-measurable.

Definition 1.15. $(X, \mathcal{A}, \mu)$ measure space. A statement $P(x), x \in X$ holds μ-almost everywhere (a.e.) if the set $\{x \in X \mid P(x)$does not hold$\}$ is μ-null.

Definition 1.16. $(X, \mathcal{A}, \mu)$ measure space.

- μ is a finite measure is $\mu(X) < \infty$.
- μ is a σ-finite measure if $X = \bigcup_1^\infty X_n, X_n \in \mathcal{A}, \mu(X_n) < \infty$.

HW: every measure space can be "completed."

1.3 Outer measures

Definition 1.17. An outer measure on X is $\mu^* : \mathcal{P}(X) \to [0, \infty]$ $s.t.$

- $\mu^*(\emptyset) = 0$
- (monotonicity) $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$.
- (countable subadditivity)

$$\forall A_1, A_2, \ldots \in X, \mu^*\left(\bigcup_i^\infty A_i\right) \leq \sum_i^\infty \mu^*(A_i).$$

Example 1.18. For $A \subset \mathbb{R}$,

$$\mu^*(A) = \inf\left\{\sum_{i=1}^\infty (b_i - a_i) \,\middle|\, \bigcup_1^\infty (a_i, b_i) \supset A\right\}.$$

is an outer measure due to the next proposition.

Proposition 1.19. *(1.19) Let $\mathcal{E} \in \mathcal{P}(X)$ s.t. $\emptyset, X \in \mathcal{E}$. Let $\rho : \mathcal{E} \to [0, \infty]$ s.t. $\rho(\emptyset) = 0$. Then*

$$\mu^*(A) = \inf\left\{\sum_{i=1}^\infty \rho(E_i) \,\middle|\, E_i \in \mathcal{E}, \forall i \in N, \bigcup_1^\infty E_i \supset A\right\}$$

is an outer measure on X.

Proof. 1. μ^* is well-defined (inf is taken over non-empty set.)

2. $\mu^*(\emptyset) = 0$
3. $A \subset B \implies \mu^*(A) \leq \mu^*(B)$.

We check the countable subadditivity.

Let $A_1, A_2, \ldots \subset X$. If one of $\mu^*(A_i) = \infty$, then the result holds. Suppose $\mu^*(A_n) < \infty, \forall n \in \mathbb{N}$.

"Give your self a room of epsilon":
Fix $\varepsilon > 0$. We will show

$$\mu^*\left(\bigcup_1^\infty A_n\right) \leq \sum_1^\infty \mu^*(A_i) + \varepsilon.$$

For each $n \in \mathbb{N}, \exists E_{n,1}, E_{n,2}, \ldots \in \mathcal{E}$ s.t.

$$\bigcup_{k=1}^\infty E_{n,k} \supset A_n \quad \text{and} \quad \mu^*(A_n) + \frac{\varepsilon}{2^n} \geq \sum_{k=1}^\infty \rho(E_{n,k}).$$

Then,

$$\bigcup_{1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty} E_{n,k} = \bigcup_{(n,k)\in\mathbb{N}^2} E_{n,k}.$$

<u>RECALL:</u> Tonelli's thm for series. If $a_{ij} \in [0,\infty], \forall i, j \in \mathbb{N}$, then

$$\sum_{(i,j)\in\mathbb{N}^2} a_{ij} = \sum_{i=1}^{\infty}\sum_{j=1\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_i j.$$

Hence

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty}\rho(E_{k,n}) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\rho(E_{k,n}) \le \sum_{n=1}^{\infty}\left(\mu^*(A_n) + \frac{\varepsilon}{2^n}\right) = \sum_{n=1}^{\infty}\mu^*(A_n) + \varepsilon.$$

We have shown countable subadditivity. ■

Outer measure is very close to a measure. Here the textbooks diverge.

Tao: introduce Lebesgue measure on $\mathbb{R}$ using topological qualities of subsets of $\mathbb{R}$.
Folland: introduce abstract method by Carathéodory and Kolomogorov.

Definition 1.20. Let μ^* be an outer measure on X. We say $A \subset X$ is Carathéodory measurable with respect to μ^* if $\forall E \subset X, \mu^*(E) = \mu^*(E \setminus A) + \mu^*(E \cap A)$.

Lemma 1.21. *Let μ^* be an outer measure on X. Suppose $B_1, B_2, \ldots, B_N$ are disjoint C-measurable sets. Then,*

$$\forall E \subset X, \mu^*\left(E \cap \left(\bigcup_{1}^{N} B_i\right)\right) = \sum_{i=1}^{n}\mu^*(E \cap B_i)$$

Proof.

$$\mu^*\left(E \cap \left(\bigcup_{1}^{N} B_i\right)\right) = \mu^*(E \cap B_1) + \mu^*\left(E \cap \left(\bigcup_{2}^{N} B_i\right)\right)$$

because B_1 is C-measurable. Then, iterate. ■

Improved version:

$B_1, B_2, \ldots$ C-measurable and <u>disjoint</u> $\implies \mu^*\left(E \cap \bigcup_1^\infty B_n\right) = \sum_1^\infty \mu^*\left(E \cap B_n\right), \forall E \subset X$.

Proof.

$$\begin{aligned}\sum_1^\infty \mu^*(E\cap B_n) &\geq \mu^*\left(E\cap\bigcup_1^\infty B_n\right)\\ &\geq \mu^*\left(E\cap\bigcup_1^N B_n\right)=\sum_1^N \mu^*(E\cap B_n.)\end{aligned}$$

Take $N\to\infty$ or note that $N\in\mathbb{N}$ is arbitrary we get the result. ■

First big theorem:

Theorem 1.22 (Carathéodory extension theorem)**.** *Let μ^* be an outer measure on X. Let $\mathcal{A}$ be the collection of C-measurable sets with respect to μ^*. Then*

1. *$\mathcal{A}$ us a σ-algebra on X.*
2. *$\mu=\mu^*|_{\mathcal{A}}$ is a measure on $(X,\mathcal{A})$.*
3. *$(X,\mathcal{A},\mu)$ is a complete measure space.*

Proof. 1. (a) $\emptyset\in\mathcal{A}$.

(b) $\mathcal{A}$ is closed under complements.

(c) To show $\mathcal{A}$ closed under countable unions.

- (finite union)
  CLAIM $A,B\in\mathcal{A}\implies A\cup B\in\mathcal{A}$.

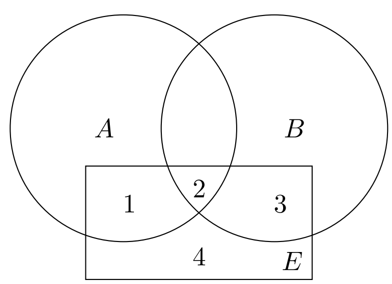

Figure 1.1: Venn diagram of A,B,E

Fix arbitrary $E\subset X$. We need to show

$$\mu^*(E)=\mu^*(E\cap(A\cup B))+\mu^*(E\setminus(A\cup B)).$$

i.e.

$$\mu^*(1\cup 2\cup 3\cup 4)=\mu^*(1\cup 2\cup 3)+\mu^*(4)$$

Since A is C-measurable, we have

$$\mu^*(1 \cup 2 \cup 3 \cup 4) = \mu^*(1 \cup 2) + \mu^*(3 \cup 4)$$

$$\mu^*(1 \cup 2 \cup 3) = \mu^*(1 \cup 2) + \mu^*(3)$$

Similarly since B is C-measurable, we have

$$\mu^*(3 \cup 4) = \mu^*(3) + \mu^*(4)$$

Hence

$$\begin{aligned}\mu^*(1 \cup 2 \cup 3 \cup 4) &= \mu^*(1 \cup 2) + \mu^*(3 \cup 4) \\ &= \mu^*(1 \cup 2 \cup 3) - \mu^*(3) + \mu^*(3) + \mu^*(4) \\ &= \mu^*(1 \cup 2 \cup 3) + \mu^*(4).\end{aligned}$$

- (countable disjoint unions)
  Let $A_1, A_2, \ldots \in \mathcal{A}$ and disjoint.

  Fix $E \subset X$ arbitrary. Since μ^* is countably subadditive,

  $$\mu^*(E) \leq \mu^*\left(E \cap \bigcup_1^\infty\right) + \mu^*\left(E \setminus \bigcup_1^\infty A_n\right)$$

  Fix $n \in \mathbb{N}$.

  $$\begin{aligned} &\implies \bigcup_1^N A_n \in \mathcal{A} \\ &\implies \mu^*(E) = \mu^*\left(E \cap \bigcup_1^N\right) + \mu^*\left(E \setminus \bigcup_1^N A_n\right) \\ &\geq \sum_1^N \mu^*(E \cap A_n) + \mu^*\left(E \setminus \bigcup_1^\infty A_n\right) \text{ by lemma.}\end{aligned}$$

  Take $n \to \infty$.

- (countable unions)
  Let $A_1, A_2, \ldots \in \mathcal{A}$. Take $E_1 = A_1, E_n = A_n \setminus \left(\bigcup_1^{n-1} A_i\right)$ for $n \geq 2$. Then $\bigcup A_n = \bigcup E_n$ and E_n's are disjoint.

2. Firstly we have $\mu(\emptyset) = \mu^*(\emptyset) = 0$.

   Countable additvity of μ^* on $\mathcal{A}$ follows from the improved lemma with $E = X$.

3. HW. ∎

1.4 Hahn-Kolmogorov Theorem

RECALL Prop 1.19 Let $\mathcal{E} \subset \mathcal{P}(X)$ $s.t.$ $\emptyset, X \in \mathcal{E}$. Let $\rho : \mathcal{E} \to [0, \infty]$ $s.t.$ $\rho(\emptyset) = 0$

$$(\mathcal{E}, \rho) \xrightarrow[1.19]{} (\mathcal{P}(X), \mu^*) \xrightarrow[\text{C-theorem}]{} (A, \mu)$$

QUESTION $\mathcal{E} \subset \mathcal{A}$ and $\mu|_{\mathcal{E}} = \rho$? No!

Definition 1.23. Let $\mathcal{A}_0$ be an algebra on X. We say $\mu_0 : \mathcal{A}_0 \to [0, \infty]$ is a pre-measure if

1. $\mu_0(\emptyset) = 0$.
2. (finite additivity)

$$\mu_0\left(\bigcup_1^N A_i 1\right) = \sum_1^N \mu_0(A_i) \text{ if } A_1, \ldots, A_N \in \mathcal{A}_0 \text{ are disjoint.}$$

3. (countable additivity within the algebra) If $A \in \mathcal{A}_0$ and

$$A = \bigcup_1^\infty A_n, A_n \in \mathcal{A}_0 \text{ and are disjoint, then } \mu_0(A) = \sum_1^\infty \mu_0(A_n)$$

NOTATION: Folland uses $\mathcal{M}$ for σ-algebra and $\mathcal{A}$ for algebra. (Jinho) uses $\mathcal{A}$ for σ-algebra and $\mathcal{A}_0$ for alegbra.

Example 1.24. $\mathcal{A}_0$ finite disjoint unions of $(a, b]$.

$$\mu_0\left(\bigcup_1^\infty (a_i, b_i]\right) = \sum_1^\infty (b_i - a_i) \text{ or } b_i^n - a_i^n, e^{b_i} - e^{a_i}, \text{ etc.}$$

Lemma 1.25.
- *1 + 3 $\implies$ 2.*
- *μ_0 is monotone.*

Theorem 1.26 (Hahn-Kolmogorov Theorem)**.** *Let μ_0 be a pre-measure on algebra $\mathcal{A}_0$ on X. Let μ^* be the outer measure induced by $(\mathcal{A}_0, \mu_0)$ in Prop 1.19. Let $\mathcal{A}$ and μ be the Carathéodory σ-algebra and measure for μ^* $\implies$ $(\mathcal{A}, \mu)$ extends $(\mathcal{A}_0, \mu_0)$ $i.e.$ $\mathcal{A} \supset \mathcal{A}_0, \mu|_{\mathcal{A}_0} = \mu_0$.*

Proof. 1. ($\mathcal{A} \supset \mathcal{A}_0$) Let $A \in \mathcal{A}_0$.

Question: $A \in \mathcal{A}$? i.e. is A C-measurable? i.e. $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c), \forall E \subset$

X.

Fix $E \subset X$.

- (countable) subadditivity of $\mu^* \implies \mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c)$.
- If $\mu^*(E) = \infty$ then $\mu^*(E) = \infty \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$.
- If $\mu^*(E) < \infty$.

  Fix $\varepsilon > 0$. By the definition of $\mu^*, \exists B_1, B_2, \ldots \in \mathcal{A}_0\ s.t.\ \bigcup_1^\infty B_n \supset E$ and

  $$\mu^*(E) + \varepsilon \geq \sum_1^\infty \mu_0(B_n) = \sum_1^\infty \left(\mu_0(B_n \cap A) + \mu_0\left(B_n \cap A^c\right)\right).$$

  Note that

  $$\bigcup_1^\infty (B_n \cap A) \supset E \cap A, \quad \bigcup_1^\infty (B_n \cap A^c) \supset E \cap A^c \implies \geq$$

2. Let $A \in \mathcal{A}_0$. We want to show that $\mu(A) = \mu_0(A)$.

   By definition, $\mu(A) = \mu^*(A)$.

   - Let $B_i = \begin{cases} A & i = 1, \\ \emptyset & i = 2 \end{cases} \in \mathcal{A}_0$ and $\bigcup_1^\infty B_i \supset A$.

     Hence $\mu^*(A) \leq \sum_1^\infty \mu_0(B_i) = \mu_0(A)$.
   - Let $B_i \in \mathcal{A}_0, \bigcup_1^\infty B_i \supset A$ an arbitrary collection of sets.

     Let $C_1 = A \cap B_1, C_i = A \cap B_i \setminus \left(\bigcup_{j=1}^{i-1} B_j\right)$. Then $A = \bigcup_1^\infty$ is a disjoint countable union. By countable additivitiy we have

     $$\mu_0(A) = \sum_1^\infty \mu_0(C_i) \implies \mu_0(A) \leq \sum_1^\infty \mu_0(B_i).$$

   Hence we have $\mu_0(A) = \mu^*(A) = \mu(A)$. We have completed our proof. ■

Definition 1.27. Such $(\mathcal{A}, \mu)$ is called the Hahn-Kolmogorov extension of $(\mathcal{A}_0, \mu_0)$, and is also called the Carathéodory σ-algebra for $(\mathcal{A}_0, \mu_0)$.

Theorem 1.28 (uniqueness of HK extension)**.** *Let $\mathcal{A}_0$ be an algebra on X, μ_0 be a pre-measure on $\mathcal{A}_0$, $(\mathcal{A}, \mu)$ be the Hahn-Kolmogorov extension of $(\mathcal{A}_0, \mu_0)$. And let $(\mathcal{A}', \mu')$ be another extension of $(\mathcal{A}_0, \mu_0)$.*

If μ_0 is σ-finite, then $\mu = \mu' = \mathcal{A} \cap \mathcal{A}'$.

NOTE σ-finite means

$$\forall X, X = \bigcup_1^\infty X_n, X_n \in \mathcal{A}_0, \mu_0(X_n) < \infty.$$

Corollary 1.29. *Let μ_0 be a pre-measure on algebra $\mathcal{A}_0$ on X. Suppose μ_0 is σ-finite, then $\exists!$ mreasure μ on $\langle \mathcal{A}_0 \rangle$ that extends $\mathcal{A}_0$. Furthermore,*

1. *the completion of $(X, \langle \mathcal{A}_0 \rangle, \mu)$ is the HK extension of $(\mathcal{A}_0, \mu_0)$.*
2. $$\mu(A) = \inf \left\{ \sum_{i=1}^\infty \mu_0(B_i) \mid B_i \subset A_0, \forall i \in \mathbb{N}, \bigcup_1^\infty B_i \supset A \right\}, \forall A \in \overline{\langle \mathcal{A}_0 \rangle}.$$

Proof of 1.28. Let $A \in \mathcal{A} \cap \mathcal{A}'$. We need to show $\mu(A) = \mu^*(A) = \mu'(A)$.

- $\mu^*(A) \geq \mu'(A)$ (HW)
- $\mu(A) \leq \mu'(A)$:
  (i) Assume $\mu(A) < \infty$. Fix $\varepsilon > 0$. Then $\exists B_i \in \mathcal{A}_0, \forall i \in \mathbb{N}, \bigcup_1^\infty B_i \supset A \ s.t.$

  $$\mu(A) + \varepsilon = \mu^*(A) + \varepsilon \geq \sum_1^\infty \mu_0(B_i) = \sum_1^\infty \mu(B_i) \geq \mu\left(\bigcup_1^\infty B_i\right) = \mu(B)$$

  Hence $\mu(B \setminus A) = \mu(B) - \mu(A) \leq \varepsilon$.

  On the other hand,

  $$\mu(B) = \lim_{N \to \infty} \mu\left(\bigcup_1^N B_i\right) = \lim_{N \to \infty} \mu'\left(\bigcup_1^N B_i\right) = \mu'(B)$$

  by continuity of measure from below.

  $$\mu(A) \leq \mu(B) = \mu'(B) = \mu'(A) + \mu'(B \setminus A) \leq \mu'(A) = \varepsilon.$$

  (ii) Assume $\mu(A) = \infty$.

  Since μ_0 is σ-finite, $X = \bigcup_1^\infty X_n, X_n \in \mathcal{A}_0, \mu_0(X_0) < \infty$. Replacing X_n by $X_1 \cup \ldots \cup X_n$, we may assume $X_1 \subset X_2 \subset \ldots$.

  $$\forall n \in N, \mu(A \cap X_n) < \infty \implies \mu(A \cap X_n) \leq \mu'(A \cap X)n).$$

Hence

$$\mu(A) = \lim_{N\to\infty} \mu(A \cap X_n) \leq \lim_{N\to\infty} \mu'(A \cap X_n) = \mu'(A). \qquad \blacksquare$$

1.5 Borel Measures on $\mathbb{R}$

Definition 1.30. $F : \mathbb{R} \to \mathbb{R}$ is an increasing function if $F(x) \leq F(y)$ for $x < y$. $F : \mathbb{R} \to \mathbb{R}$ is increasing and right-continuous $\implies$ F is distribution function.

Example 1.31.

- $$F(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

- $\mathbb{Q} = \{r_1, r_2, \ldots\}$, $F_n(x) = \begin{cases} 1 & x \geq r_n \\ 0 & x < r_n. \end{cases}$

$$F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n} \text{ is a distribution function.}$$

NOTE If F is increasing, $F(\infty) := \lim_{x\to\infty} F(x), F(-\infty) := \lim_{x\to-\infty} F(x)$ exists in $[-\infty, \infty]$.

In probability theory, cumulative distribution function (CDF) is a distribution function with $F(\infty) = 0$ and $F(-\infty) = 0$.

There are distributions [Folland, Ch9], but these are different from distribution functions.

Definition 1.32. Suppose X a topological space. μ on $(X, \mathcal{B}(X))$ is called locally finite is $\mu(K) < \infty$ for any compact set $K \subset X$.

Lemma 1.33. *Let μ be a locally finite Borel measure on $\mathbb{R}$ $\implies$*

$$F_\mu(x) = \begin{cases} \mu((0, x]), & x > 0 \\ 0, & x = 0 \\ -\mu((x, 0]), & x < 0 \end{cases} \textit{ is a distribution function.}$$

Proof. DIY. Use continuity of measure. $\blacksquare$

Definition 1.34. h-intervals are $\emptyset, (a, b], (a, \infty), (-\infty, b], (\infty, \infty)$.

Lemma 1.35. *Let $\mathcal{H}$ be the collections of finite disjoint unions of h-intervals. Then $\mathcal{H}$ is an* *algebra on $\mathbb{R}$.*

Proof. DIY. ∎

Proposition 1.36 (Distribution function defines a pre-measure)**.** *Let $F : \mathbb{R} \to \mathbb{R}$ be a distribution function. For an h-interval I, define*

$$\ell(I) = \ell_F(I) = \begin{cases} 0, & I = \emptyset \\ F(b) - F(a), & I = (a, b] \\ F(\infty) - F(a), & I = (a, \infty) \\ F(b) - F(\infty), & I = (-\infty, b] \\ F(\infty) - F(-\infty), & I = (-\infty, \infty). \end{cases}$$

Define $\mu_0 = \mu_{0,F} : \mathcal{H} \to [0, \infty]$ by

$$\mu_0(A) := \sum_{k=1}^{N} \ell(I_k) \quad \textit{if } A = \bigcup_{k=1}^{N} I_k, \textit{ finite disjoint union of } h\textit{-inververals.}$$

Then μ_0 is a pre-measure.

Proof.

1. μ_0 is well-defined.
2. μ_0 is finite additive.
3. μ_0 is countable additive within $\mathcal{H}$.

   Suppose $A \in \mathcal{H}$ and $A = \bigcup_1^\infty A_i$ a disjoint union, $A_i \in \mathcal{H}$. It is enough to consider the case $A = I, A_k = I_k$ all h-intervals. (Why?)

   Focus on the case $I = (a, b]$: (HW: check other cases)
   We have
   $$(a, b] = \bigcup_1^\infty (a_n, b_n], \text{ a disjoint union.}$$

   Check
   $$F(b) - F(a)? = \sum_1^\infty (F(b_n) - F(a_n))$$

   $(a, b] \supset \bigcup_1^N (a_n, b_n] \implies F(b) - F(a) \geq \sum_1^N F(b_n) - F(a_n), \forall N \in \mathbb{N}$. (Arranging them in decreasing order) Take $N \to \infty$ we have

   $$F(b) - F(a) \geq \sum_1^\infty (F(b_n) - F(a_n)).$$

Since F is right-continuous, $\exists a' > a\ s.t.\ F(a') - F(a) < \varepsilon$. For each $n \in \mathbb{N}$, $\exists b_n' > b_n\ s.t.\ F(b_n') - F(b_n) < \frac{\varepsilon}{2^n}$

$\implies [a', b] \subset \bigcup_1^\infty (a_n, b_n') \implies \exists N \in \mathbb{N}\ s.t.\ [a', b] \subset \bigcup_1^n (a_n, b_n') \implies F(b) - F(a') \leq \sum_1^N F(b_n') - F(a_n) \implies F(b) - F(a) \leq F(b) - F(a') + \varepsilon \leq \sum_1^\infty (F(b_n') - F(a_n)) + \varepsilon \leq \sum_1^\infty (F(b_n) - F(a_n) + \frac{\varepsilon}{2^n}) + \varepsilon$

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