Notes for Math 597 – Real Analysis

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Contents

Office hour is Mon 12:30 - 1:30, Tue 12:30 - 1:30 in person EH 5838, Th 1 - 2 online.

Chapter 1

Abstract Measure

1.1 σ-Algebra

Definition 1.1. Let X be a set. A collection $\mathcal{M}$ of subsets of X is called a σ-algebra on X if

- $\emptyset \in \mathcal{M}$.

- $\mathcal{M}$ is closed under <u>complements</u>: $E \in \mathcal{M} \implies E^c \in \mathcal{M}$.

- $\mathcal{M}$ is closed under <u>countable unions</u>: $E_1, E_2, \ldots \in \mathcal{M} \implies \bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$.

<u>SIMPLE PROPERTIES:</u>

- $X = \emptyset^c \in \mathcal{M}$.

- $\bigcap_{i=1}^{\infty} E_i = \left(\bigcup_{i=1}^{n} E_i^c \right)^c \in \mathcal{M}$. It is closed under countable intersections.

- $\bigcup_{i=1}^{N} E_i = E_i \cup \ldots \cup E_n \cup \emptyset \cup \ldots$. It is closed under finite unions (similarly, intersections). sigma

- $E \setminus F = E \cap F^c \in \mathcal{M}, E \triangle F = (E \cap F^c) \cup (F \cap E^c) \in \mathcal{M}$.

Example 1.2. (a) $\mathcal{A} = \mathcal{P}(X)$ power algebra.

(b) $\mathcal{A} = \{\emptyset, X\}$ trivial algebra.

(c) Let $B \subset X, B \neq \emptyset, B \neq X. \mathcal{A} = \{\emptyset, B, B^c, X\}$.

Lemma 1.3. *(An intersection of σ-algebras is a σ-algebra) Let $\mathcal{A}_\alpha, \alpha \in I$, be a family a σ-algebras of X. Then $\bigcap_{\alpha \in I} A_\alpha$ is a σ-algebra. (I can be uncountable.)*

Proof. DIY ∎

Definition 1.4. For $\mathcal{E} \subset \mathcal{P}(X)$ (not necessarily a σ-algebra), let $\langle \mathcal{E} \rangle$ be the intersection of all σ-algebras on X that contains $\mathcal{E}$. Call it the σ-algebra generated by $\mathcal{E}$.

- $\langle \mathcal{E} \rangle$ is the <u>smallest</u> σ-algebra containing $\mathcal{E}$ and is <u>unique</u>.

- $\{\emptyset, B, B^c, X\} = \langle \{B\} \rangle = \langle \{B^c\} \rangle = \langle \{\emptyset, B\} \rangle$.

The above definition gives us (potentially) lots of examples of σ-algebra on a set X

Lemma 1.5. *(a) Suppose $\mathcal{E} \subset \mathcal{P}(X), \mathcal{A}$ a σ-algebra on X. $\mathcal{E} \in \mathcal{A} \implies \langle \mathcal{E} \rangle \in \mathcal{A}$.*

(b) $E \subset F \subset \mathcal{P}(X) \implies \langle \mathcal{E} \rangle \subset \langle \mathcal{F} \rangle$.

Proof.   ∎

Definition 1.6. For a topological space X, <u>the Borel σ-algebra</u> $\mathcal{B}(X)$ is the σ-algebra generated by the collection of open sets.

Example 1.7. $(X = \mathbb{R})$ $\mathcal{B}(\mathbb{R})$ contains the following collections

$$\mathcal{E}_1 = \{(a,b) \mid a < b\}, \quad \mathcal{E}_2 = \{[a,b] \mid a < b\},$$

$$\mathcal{E}_3 = \{(a,b] \mid a < b\}, \quad \mathcal{E}_4 = \{[a,b) \mid a < b\},$$

$$\mathcal{E}_5 = \{(a,\infty) \mid a \in \mathbb{R}\}, \quad \mathcal{E}_6 = \{[a,\infty) \mid a \in \mathbb{R}\},$$

$$\mathcal{E}_7 = \{(-\infty,a) \mid a \in \mathbb{R}\}, \quad \mathcal{E}_8 = \{(-\infty,a] \mid a < b\}$$

Proposition 1.8. $\mathcal{B}(\mathbb{R}) = \langle \mathcal{E}_i \rangle$ *for each $i = 1, \ldots, 8$.*

Proof. 1.5.   ∎

Definition 1.9. $(X, \mathcal{A})$ is called a measurable space.

1.2   Measures

Definition 1.10. A measure on $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \to [0, \infty]$ *s.t.*

1. $\mu(\emptyset) = 0$

2. (countable additive) For $A_1, A_2, \ldots \in \mathcal{A}$ disjoint we have

$$\mu\left(\bigcup_1^\infty A_i\right) = \sum_{i=1}^\infty \mu(A_i).$$

$(X, \mathcal{A}, \mu)$ is then called a measure space.

Example 1.11.   1. For any $(X, \mathcal{A}), \mu(A) = \#A$ counting measure.

2. For any $(X, \mathcal{A})$, let $x_0 \in X$. The Dirac measure at x_0 is

$$\mu(A) = \begin{cases} 1 & x_0 \in A, \\ 0 & x_0 \notin A. \end{cases}$$

3. For $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, let $a_1, a_2, \ldots \in [0, \infty)$. $\mu(A) = \sum_{i \in A} a_i$ is a measure.

$(X, \mathcal{A})$ measurable space

$(X, \mathcal{A}, \mu)$ measure space

$\mu : \mathcal{A} \to [0, \infty]$ $s.t.$ $\mu(\emptyset) = 0$, countable additivity.

<u>NOTE</u>: $A, B \in \mathcal{A}, A \subset B$, then $\mu(B \setminus A) + \mu(A) = \mu(B) \implies \mu(B \setminus A) = \mu(B) - \mu(A)$ if $\mu(A) < \infty$.

Theorem 1.13. $(X, \mathcal{A}, \mu)$ *measure space*

1. *(monotonicity)*
$$A, B \in \mathcal{A}, A \subset B \implies \mu(A) \leq \mu(B).$$

2. *(countable subadditivity)*

$$A_1, A_2, \ldots, \in \mathcal{A}, \implies \mu\left(\bigcup_i^{\infty} A_i\right) \leq \sum_i^{\infty} \mu(A_i).$$

3. *(continuity from below/(MCT) from sets)*

$$A_1, A_2, \ldots \in \mathcal{A}, A_1 \subset A_2 \subset A_3 \subset \ldots \implies \mu\left(\bigcup_i^{\infty} A_i\right) = \lim_{n \to \infty} \mu(A_n).$$

4. *(continuity from above)*

$$A_1, A_2, \ldots \in \mathcal{A}, A_1 \supset A_2 \supset A_3 \supset \ldots, \mu(A_1) < \infty \implies \mu\left(\bigcap_i^{\infty} A_i\right) = \lim_{n \to \infty} \mu(A_n).$$

Proof. 1, 2, DIY.

For 3, let $B_1 = A_1, B_i = A_i \setminus A_{i-1}, i \geq 2. B_i \in \mathcal{A}$ and are disjoint.

$$\bigcup_i^\infty A_i = \bigcup_i^\infty B_i$$

$$\implies \mu\left(\bigcup_i^\infty A_i\right) = \mu\left(\bigcup_i^\infty B_i\right) = \sum_i^\infty \mu(B_i) = \lim_{n\to\infty} \sum_i^n \mu(B_i) = \lim_{n\to\infty} \mu(A_n).$$

For 4, let $E_i = A_1 \setminus A_i$. Hence $E_i \in \mathcal{A}, E_1 \subset E_2 \subset \ldots$ We have

$$\bigcup_i^\infty E_i = \bigcup_i^\infty (A_1 \setminus A_i) = A_1 \setminus \left(\bigcap_1^\infty A_i\right) \implies \bigcap_1^\infty A_i = A_1 \setminus \left(\bigcup_1^\infty E_i\right).$$

Hence

$$\mu\left(\bigcap_1^\infty A_i\right) = \mu(A_1) - \mu\left(\bigcup_1^\infty E_i\right) = \mu(A_1) - \lim_{n\to\infty} \mu(E_n) = \mu(A_1) - \lim_{n\to\infty} \mu(A_1) - \mu(A_n). \qquad \blacksquare$$

NOTE: the condition that $\mu(A_1) < \infty$ cannot be dropped.

For example, in $(\mathbb{N}, \mathcal{P}(N), \text{counting measure})$, let $A_n = \{n, n+1, n+2\}, A_1 \supset A_2 \supset A_3 \supset \ldots$ We have $\bigcap_1^\infty = \emptyset \implies \mu\left(\bigcap_1^\infty A_i\right) = 0.$

Definition 1.14. For $(X, \mathcal{A}, \mu)$ measure space,

- $A \subset X$ is a μ-null set if $A \in \mathcal{A}, \mu(A) = 0$.

- $A \subset X$ is a μ-subnull set if $\exists B, \mu$-null set $A \subset B$.

- $(X, \mathcal{A}, \mu)$ is a complete measure space if every μ-subnull set is $\mathcal{A}$-measurable.

Definition 1.15. $(X, \mathcal{A}, \mu)$ measure space. A statement $P(x), x \in X$ holds μ-almost everywhere (a.e.) if the set $\{x \in X \mid P(x) \text{ does not hold}\}$ is μ-null.

Definition 1.16. $(X, \mathcal{A}, \mu)$ measure space.

- μ is a <u>finite measure</u> is $\mu(X) < \infty$.

- μ is a <u>σ-finite measure</u> if $X = \bigcup_1^\infty X_n, X_n \in \mathcal{A}, \mu(X_n) < \infty$.

HW: every measure space can be "completed."

1.3   Outer measures

Definition 1.17. An <u>outer measure</u> on X is $\mu^* : \mathcal{P}(X) \to [0, \infty]$ $s.t.$

- $\mu^*(\emptyset) = 0$

- (monotonicity) $\mu^*(A) \le \mu^*(B)$ if $A \subset B$.

- (countable subadditivity)

$$\forall A_1, A_2, \ldots \in X, \mu^* \left(\bigcup_i^{\infty} A_i \right) \le \sum_i^{\infty} \mu^*(A_i).$$

Example 1.18. For $A \subset \mathbb{R}$,

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} (b_i - a_i) \ \middle| \ \bigcup_1^{\infty} (a_i, b_i) \supset A \right\}.$$

is an outer measure due to the next proposition.

Proposition 1.19. *(1.19) Let $\mathcal{E} \in \mathcal{P}(X)$ s.t. $\emptyset, X \in \mathcal{E}$. Let $\rho : \mathcal{E} \to [0, \infty]$ s.t. $\rho(\emptyset) = 0$. Then*

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) \ \middle| \ E_i \in \mathcal{E}, \forall i \in N, \bigcup_1^{\infty} E_i \supset A \right\}$$

is an outer measure on X.

Proof.   1. μ^* is well-defined (inf is taken over non-empty set.)

2. $\mu^*(\emptyset) = 0$

3. $A \subset B \implies \mu^*(A) \le \mu^*(B)$.

We check the countable subadditivity.

Let $A_1, A_2, \ldots \subset X$. If one of $\mu^*(A_i) = \infty$, then the result holds. Suppose $\mu^*(A_n) < \infty, \forall n \in \mathbb{N}$.

"Give your self a room of epsilon":
Fix $\varepsilon > 0$. We will show

$$\mu^* \left(\bigcup_1^{\infty} A_n \right) \le \sum_1^{\infty} \mu^*(A_i) + \varepsilon.$$

For each $n \in \mathbb{N}, \exists E_{n,1}, E_{n,2}, \ldots \in \mathcal{E}$ s.t.

$$\bigcup_{k=1}^{\infty} E_{n,k} \supset A_n \quad \text{and} \quad \mu^*(A_n) + \frac{\varepsilon}{2^n} \ge \sum_{k=1}^{\infty} \rho(E_{n,k}).$$

Then,

$$\bigcup_1^\infty A_n \subset \bigcup_{n=1}^\infty \bigcup_{k=1}^\infty E_{n,k} = \bigcup_{(n,k)\in\mathbb{N}^2} E_{n,k}.$$

<u>RECALL:</u> Tonelli's thm for series. If $a_{ij} \in [0,\infty], \forall i,j \in \mathbb{N}$, then

$$\sum_{(i,j)\in\mathbb{N}^2} a_{ij} = \sum_{i=1}^\infty \sum_{j=1}^\infty a_{ij} = \sum_{j=1}^\infty \sum_{i=1}^\infty a_i j.$$

Hence

$$\mu^*\left(\bigcup_{n=1}^\infty A_n\right) \le \sum_{n=1}^\infty \rho(E_{k,n}) = \sum_{n=1}^\infty \sum_{k=1}^\infty \rho(E_{k,n}) \le \sum_{n=1}^\infty \left(\mu^*(A_n) + \frac{\varepsilon}{2^n}\right) = \sum_{n=1}^\infty \mu^*(A_n) + \varepsilon.$$

We have shown countable subadditivity.   ∎

Outer measure is very close to a measure. Here the textbooks diverge.

Tao: introduce Lebesgue measure on $\mathbb{R}$ using topological qualities of subsets of $\mathbb{R}$.
Folland: introduce abstract method by Carathéodory and Kolomogorov.

Definition 1.20. Let μ^* be an outer measure on X. We say $A \subset X$ is Carathéodory measurable with respect to μ^* if $\forall E \subset X, \mu^*(E) = \mu^*(E \setminus A) + \mu^*(E \cap A)$.

Lemma 1.21. *Let μ^* be an outer measure on X. Suppose $B_1, B_2, \ldots, B_N$ are disjoint C-measurable sets. Then,*

$$\forall E \subset X, \mu^*\left(E \cap \left(\bigcup_1^N B_i\right)\right) = \sum_{i=1}^n \mu^*(E \cap B_i)$$

Proof.

$$\mu^*\left(E \cap \left(\bigcup_1^N B_i\right)\right) = \mu^*(E \cap B_1) + \mu^*\left(E \cap \left(\bigcup_2^N B_i\right)\right)$$

because B_1 is C-measurable. Then, iterate.   ∎

Improved version:

$B_1, B_2, \ldots$ C-measurable and <u>disjoint</u> $\implies \mu^*\left(E \cap \bigcup_1^\infty B_n\right) = \sum_1^\infty \mu^*(E \cap B_n), \forall E \subset X$.

Proof.

$$\sum_1^\infty \mu^*(E \cap B_n) \geq \mu^* \left(E \cap \bigcup_1^\infty B_n \right)$$

$$\geq \mu^* \left(E \cap \bigcup_1^N B_n \right) = \sum_1^N \mu^*(E \cap B_n.)$$

Take $N \to \infty$ or note that $N \in \mathbb{N}$ is arbitrary we get the result. ∎

First big theorem:

Theorem 1.22 (Carathéodory extension theorem)**.** *Let μ^* be an outer measure on X. Let $\mathcal{A}$ be the collection of C-measurable sets with respect to μ^*. Then*

1. *$\mathcal{A}$ us a σ-algebra on X.*

2. *$\mu = \mu^*|_{\mathcal{A}}$ is a measure on $(X, \mathcal{A})$.*

3. *$(X, \mathcal{A}, \mu)$ is a complete measure space.*

Proof.   1.   (a) $\emptyset \in \mathcal{A}$.

      (b) $\mathcal{A}$ is closed under complements.

      (c) To show $\mathcal{A}$ closed under countable unions.

         • (finite union)

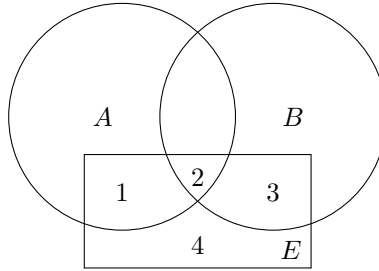   <u>Claim</u> $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$.

Figure 1.1: Venn diagram of A, B, E

Fix arbitrary $E \subset X$. We need to show

$$\mu^*(E) = \mu^*(E \cap (A \cup B)) + \mu^*(E \setminus (A \cup B)).$$

i.e.

$$\mu^*(1 \cup 2 \cup 3 \cup 4) = \mu^*(1 \cup 2 \cup 3) + \mu^*(4)$$

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Since A is C-measurable, we have

$$\mu^*(1 \cup 2 \cup 3 \cup 4) = \mu^*(1 \cup 2) + \mu^*(3 \cup 4)$$

$$\mu^*(1 \cup 2 \cup 3) = \mu^*(1 \cup 2) + \mu^*(3)$$

Similarly since B is C-measurable, we have

$$\mu^*(3 \cup 4) = \mu^*(3) + \mu^*(4)$$

Hence

$$\begin{aligned}
\mu^*(1 \cup 2 \cup 3 \cup 4) &= \mu^*(1 \cup 2) + \mu^*(3 \cup 4) \\
&= \mu^*(1 \cup 2 \cup 3) - \mu^*(3) + \mu^*(3) + \mu^*(4) \\
&= \mu^*(1 \cup 2 \cup 3) + \mu^*(4).
\end{aligned}$$

- (countable disjoint unions)

  Let $A_1, A_2, \ldots \in \mathcal{A}$ and $\underline{\text{disjoint}}$.

  Fix $E \subset X$ arbitrary. Since μ^* is countably subadditive,

  $$\mu^*(E) \le \mu^*\left(E \cap \bigcup_1^\infty\right) + \mu^*\left(E \setminus \bigcup_1^\infty A_n\right)$$

  Fix $n \in \mathbb{N}$.

  $$\implies \bigcup_1^N A_n \in \mathcal{A}$$

  $$\implies \mu^*(E) = \mu^*\left(E \cap \bigcup_1^N\right) + \mu^*\left(E \setminus \bigcup_1^N A_n\right)$$

  $$\ge \sum_1^N \mu^*(E \cap A_n) + \mu^*\left(E \setminus \bigcup_1^\infty A_n\right) \text{ by lemma.}$$

  Take $n \to \infty$.

- (countable unions)

  Let $A_1, A_2, \ldots \in \mathcal{A}$. Take $E_1 = A_1, E_n = A_n \setminus \left(\bigcup_1^{n-1} A_i\right)$ for $n \ge 2$. Then $\bigcup A_n = \bigcup E_n$ and E_n's are disjoint.

2. Firstly we have $\mu(\emptyset) = \mu^*(\emptyset) = 0$.

   Countable additvity of μ^* on $\mathcal{A}$ follows from the improved lemma with $E = X$.

3. HW.   ∎

1.4   Hahn-Kolmogorov Theorem

<u>RECALL</u> Prop 1.19 Let $\mathcal{E} \subset \mathcal{P}(X)$ s.t. $\emptyset, X \in \mathcal{E}$. Let $\rho : \mathcal{E} \to [0, \infty]$ s.t. $\rho(\emptyset) = 0$

$$(\mathcal{E}, \rho) \xrightarrow[1.19]{} (\mathcal{P}(X), \mu^*) \xrightarrow[\text{C-theorem}]{} (A, \mu)$$

<u>QUESTION</u> $\mathcal{E} \subset \mathcal{A}$ and $\mu|_{\mathcal{E}} = \rho$? No!

Definition 1.23. Let $\mathcal{A}_0$ be an algebra on X. We say $\mu_0 : \mathcal{A}_0 \to [0, \infty]$ is a <u>pre-measure</u> if

1. $\mu_0(\emptyset) = 0$.

2. (finite additivity)

$$\mu_0 \left(\bigcup_1^N A_i 1 \right) = \sum_1^N \mu_0(A_i) \text{ if } A_1, \dots, A_N \in \mathcal{A}_0 \text{ are disjoint.}$$

3. (countable additivity within the algebra) If $A \in \mathcal{A}_0$ and

$$A = \bigcup_1^\infty A_n, A_n \in \mathcal{A}_0 \text{ and are disjoint, then } \mu_0(A) = \sum_1^\infty \mu_0(A_n)$$

<u>NOTATION:</u> Folland uses $\mathcal{M}$ for σ-algebra and $\mathcal{A}$ for algebra. (Jinho) uses $\mathcal{A}$ for σ-algebra and $\mathcal{A}_0$ for alegbra.

Example 1.24. $\mathcal{A}_0$ finite disjoint unions of $(a, b]$.

$$\mu_0 \left(\bigcup_1^\infty (a_i, b_i] \right) = \sum_1^\infty (b_i - a_i) \text{ or } b_i^n - a_i^n, e^{b_i} - e^{a_i}, \text{ etc.}$$

Lemma 1.25.   • *$1 + 3 \implies 2$.*

   • *μ_0 is monotone.*

Theorem 1.26 (Hahn-Kolmogorov Theorem)**.** *Let μ_0 be a pre-measure on algebra $\mathcal{A}_0$ on X. Let μ^* be the outer measure induced by $(\mathcal{A}_0, \mu_0)$ in Prop 1.19. Let $\mathcal{A}$ and μ be the Carathéodory σ-algebra and measure for $\mu^* \implies (\mathcal{A}, \mu)$ extends $(\mathcal{A}_0, \mu_0)$ i.e. $\mathcal{A} \supset \mathcal{A}_0, \mu|_{\mathcal{A}_0} = \mu_0$.*

Proof.   1. ($\mathcal{A} \supset \mathcal{A}_0$) Let $A \in \mathcal{A}_0$.

Question: $A \in \mathcal{A}$? i.e. is A C-measurable? i.e. $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c), \forall E \subset$

X.

Fix $E \subset X$.

- (countable) subadditivity of $\mu^* \implies \mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c)$.

- If $\mu^*(E) = \infty$ then $\mu^*(E) = \infty \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$.

- If $\mu^*(E) < \infty$.

  Fix $\varepsilon > 0$. By the definition of $\mu^*, \exists B_1, B_2, \ldots \in \mathcal{A}_0$ $s.t.$ $\bigcup_1^\infty B_n \supset E$ and

  $$\mu^*(E) + \varepsilon \geq \sum_1^\infty \mu_0(B_n) = \sum_1^\infty \left(\mu_0(B_n \cap A) + \mu_0\left(B_n \cap A^c\right)\right).$$

  Note that

  $$\bigcup_1^\infty (B_n \cap A) \supset E \cap A, \quad \bigcup_1^\infty (B_n \cap A^c) \supset E \cap A^c \implies \geq$$

2. Let $A \in \mathcal{A}_0$. We want to show that $\mu(A) = \mu_0(A)$.

   By definition, $\mu(A) = \mu^*(A)$.

   - Let $B_i = \begin{cases} A & i = 1, \\ \emptyset & i = 2 \end{cases} \in \mathcal{A}_0$ and $\bigcup_1^\infty B_i \supset A$.

     Hence $\mu^*(A) \leq \sum_1^\infty \mu_0(B_i) = \mu_0(A)$.

   - Let $B_i \in \mathcal{A}_0, \bigcup_1^\infty B_i \supset A$ an arbitrary collection of sets.

     Let $C_1 = A \cap B_1, C_i = A \cap B_i \setminus \left(\bigcup_{j=1}^{i-1} B_j \right)$. Then $A = \bigcup_1^\infty$ is a disjoint countable union. By countable additivitiy we have

     $$\mu_0(A) = \sum_1^\infty \mu_0(C_i) \implies \mu_0(A) \leq \sum_1^\infty \mu_0(B_i).$$

   Hence we have $\mu_0(A) = \mu^*(A) = \mu(A)$. We have completed our proof. ∎

Definition 1.27. Such $(\mathcal{A}, \mu)$ is called the <u>Hahn-Kolmogorov extension</u> of $(\mathcal{A}_0, \mu_0)$, and is also called the <u>Carathéodory σ-algebra</u> for $(\mathcal{A}_0, \mu_0)$.

Theorem 1.28 (uniqueness of HK extension). *Let $\mathcal{A}_0$ be an algebra on X, μ_0 be a pre-measure on $\mathcal{A}_0$, $(\mathcal{A}, \mu)$ be the Hahn-Kolmogorov extension of $(\mathcal{A}_0, \mu_0)$. And let $(\mathcal{A}', \mu')$ be another extension of $(\mathcal{A}_0, \mu_0)$.*

If μ_0 is <u>σ-finite</u>, then $\mu = \mu' = \mathcal{A} \cap \mathcal{A}'$.

NOTE σ-finite means

$$\forall X, X = \bigcup_1^\infty X_n, X_n \in \mathcal{A}_0, \mu_0(X_n) < \infty.$$

Corollary 1.29. *Let μ_0 be a pre-measure on algebra $\mathcal{A}_0$ on X. Suppose μ_0 is σ-finite, then $\exists!$ mreasure μ on $\langle \mathcal{A}_0 \rangle$ that extends $\mathcal{A}_0$. Furthermore,*

1. the completion of $(X, \langle \mathcal{A}_0 \rangle, \mu)$ is the HK extension of $(\mathcal{A}_0, \mu_0)$.

2.
$$\mu(A) = \inf \left\{ \sum_{i=1}^\infty \mu_0(B_i) \mid B_i \subset A_0, \forall i \in \mathbb{N}, \bigcup_1^\infty B_i \supset A \right\}, \forall A \in \overline{\langle \mathcal{A}_0 \rangle}.$$

Proof of 1.28. Let $A \in \mathcal{A} \cap \mathcal{A}'$. We need to show $\mu(A) = \mu^*(A) = \mu'(A)$.

- $\mu^*(A) \geq \mu'(A)$ (HW)

- $\mu(A) \leq \mu'(A)$:

  (i) Assume $\mu(A) < \infty$. Fix $\varepsilon > 0$. Then $\exists B_i \in \mathcal{A}_0, \forall i \in \mathbb{N}, \bigcup_1^\infty B_i \supset A$ s.t.

  $$\mu(A) + \varepsilon = \mu^*(A) + \varepsilon \geq \sum_1^\infty \mu_0(B_i) = \sum_1^\infty \mu(B_i) \geq \mu\left(\bigcup_1^\infty B_i \right) = \mu(B)$$

  Hence $\mu(B \setminus A) = \mu(B) - \mu(A) \leq \varepsilon$.

  On the other hand,

  $$\mu(B) = \lim_{N \to \infty} \mu\left(\bigcup_1^N B_i \right) = \lim_{N \to \infty} \mu'\left(\bigcup_1^N B_i \right) = \mu'(B)$$

  by continuity of measure from below.

  $$\mu(A) \leq \mu(B) = \mu'(B) = \mu'(A) + \mu'(B \setminus A) \leq \mu'(A) = \varepsilon.$$

  (ii) Assume $\mu(A) = \infty$.

  Since μ_0 is σ-finite, $X = \bigcup_1^\infty X_n, X_n \in \mathcal{A}_0, \mu_0(X_0) < \infty$. Replacing X_n by $X_1 \cup \ldots \cup X_n$, we may assume $X_1 \subset X_2 \subset \ldots$.

  $$\forall n \in N, \mu(A \cap X_n) < \infty \implies \mu(A \cap X_n) \leq \mu'(A \cap X)n).$$

Hence
$$\mu(A) = \lim_{N \to \infty} \mu(A \cap X_n) \le \lim_{N \to \infty} \mu'(A \cap X_n) = \mu'(A). \qquad \blacksquare$$

1.5 Borel Measures on $\mathbb{R}$

Definition 1.30. $F : \mathbb{R} \to \mathbb{R}$ is an increasing function if $F(x) \le F(y)$ for $x < y$. $F : \mathbb{R} \to \mathbb{R}$ is increasing and right-continuous $\implies$ F is distribution function.

Example 1.31. •
$$F(x) = \begin{cases} 1, & x \ge 0 \\ 0, & x < 0. \end{cases}$$

• $\mathbb{Q} = \{r_1, r_2, \dots\}, F_n(x) = \begin{cases} 1 & x \ge r_n \\ 0 & x < r_n. \end{cases}$

$$F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n} \text{ is a distribution function.}$$

<u>NOTE</u> If F is increasing, $F(\infty) := \lim_{x \to \infty} F(x), F(-\infty) := \lim_{x \to -\infty} F(x)$ exists in $[-\infty, \infty]$.

In probability theory, cumulative distribution function (CDF) is a distribution function with $F(\infty) = 0$ and $F(-\infty) = 0$.

There are distributions [Folland, Ch9], but these are different from <u>distribution</u> functions.

Definition 1.32. Suppose X a topological space. μ on $(X, \mathcal{B}(X))$ is called <u>locally finite</u> is $\mu(K) < \infty$ for any compact set $K \subset X$.

Lemma 1.33. *Let μ be a locally finite Borel measure on $\mathbb{R}$ $\implies$*

$$F_\mu(x) = \begin{cases} \mu((0, x]), & x > 0 \\ 0, & x = 0 \text{ is a distribution function.} \\ -\mu((x, 0]), & x < 0 \end{cases}$$

Proof. DIY. Use continuity of measure. $\qquad \blacksquare$

Definition 1.34. h-intervals are $\emptyset, (a, b], (a, \infty), (-\infty, b], (\infty, \infty)$.

Lemma 1.35. *Let $\mathcal{H}$ be the collections of finite disjoint unions of h-intervals. Then $\mathcal{H}$ is an*

algebra on $\mathbb{R}$.

Proof. DIY.   ∎

Proposition 1.36 (Distribution function defines a pre-measure). *Let $F : \mathbb{R} \to \mathbb{R}$ be a distribution function. For an h-interval I, define*

$$\ell(I) = \ell_F(I) = \begin{cases} 0, & I = \emptyset \\ F(b) - F(a), & I = (a, b] \\ F(\infty) - F(a), & I = (a, \infty) \\ F(b) - F(\infty), & I = (-\infty, b] \\ F(\infty) - F(-\infty), & I = (-\infty, \infty). \end{cases}$$

Define $\mu_0 = \mu_{0,F} : \mathcal{H} \to [0, \infty]$ by

$$\mu_0(A) := \sum_{k=1}^{N} \ell(I_k) \quad \text{if } A = \bigcup_{k=1}^{N} I_k, \text{ finite disjoint union of } h\text{-invervals.}$$

Then μ_0 is a pre-measure.

Proof.   1. μ_0 is well-defined.

    2. μ_0 is finite additive.

    3. μ_0 is countable additive within $\mathcal{H}$.

      Suppose $A \in \mathcal{H}$ and $A = \bigcup_1^\infty A_i$ a disjoint union, $A_i \in \mathcal{H}$. It is enough to consider the case $A = I, A_k = I_k$ all h-intervals. (Why?)

      Focus on the case $I = (a, b]$: (HW: check other cases)
      We have
$$(a, b] = \bigcup_1^\infty (a_n, b_n], \text{ a disjoint union.}$$

      Check
$$F(b) - F(a)? = \sum_1^\infty (F(b_n) - F(a_n))$$

      $(a, b] \supset \bigcup_1^N (a_n, b_n] \implies F(b) - F(a) \geq \sum_1^N F(b_n) - F(a_n), \forall N \in \mathbb{N}$. (Arranging them in decreasing order) Take $N \to \infty$ we have

$$F(b) - F(a) \geq \sum_1^\infty (F(b_n) - F(a_n)).$$

Since F is right-continuous, $\exists a' > a$ s.t. $F(a') - F(a) < \varepsilon$. For each $n \in \mathbb{N}$, $\exists b'_n > b_n$ s.t. $F(b'_n) - F(b_n) < \frac{\varepsilon}{2^n}$.

$$\implies [a', b] \subset \bigcup_1^\infty (a_n, b'_n)$$

$$\implies \exists N \in \mathbb{N} \text{ s.t. } [a', b] \subset \bigcup_1^n (a_n, b'_n)$$

$$\implies F(b) - F(a') \le \sum_1^N F(b'_n) - F(a_n)$$

$$\implies F(b) - F(a) \le F(b) - F(a') + \varepsilon \le \sum_1^\infty (F(b'_n) - F(a_n)) + \varepsilon$$

$$\le \sum_1^\infty \left(F(b_n) - F(a_n) + \frac{\varepsilon}{2^n} \right) + \varepsilon \qquad \blacksquare$$

Once we have this pre-measure, HK theorem allows us to extended it to a measure.

Theorem 1.37 (Locally finite Borel measures on $\mathbb{R}$)**.**

1. $F : \mathbb{R} \to \mathbb{R}$ *is a distribution function* $\implies$ $\exists!$ *locally finite Borel measure* μ_F *on* $\mathbb{R}$ *satisfying* $\mu_F((a, b]) = F(b) - F(a), \forall a, b, a < b$.

2. *Suppose* $F, G : \mathbb{R} \to \mathbb{R}$ *are distribution functions. Then,* $\mu_F = \mu_G$ *on* $\mathcal{B}(\mathbb{R})$ *if and only if* $F - G$ *is a constant function.*

Proof. HW $\blacksquare$

1.6 Lebesgue-Stieltjes measures on $\mathbb{R}$

F distribution function $\implies$ μ_F on Carathéodory σ-algebra $\mathcal{A}_{\mu_F}$.
Actually (HW3) $(\mathcal{A}_{\mu_F}, \mu_F) = (\mathcal{B}(\mathbb{R}), \mu_F)$.

Definition 1.38. • μ_F on $\mathcal{A}_{\mu_F}$ is called the Lebesgue-Stieltjes measure corresponding to F.

- Special case: $F(x) = x \implies$ Lebesgue measure $(\mathcal{B}, m)$.

Example 1.39. 1. $\mu_F((a, b]) = F(b) - F(a)$. F is right-continuous and increasing $\implies$ $F(x_-) \le F(x) = F(x_+)$.

(HW) $\mu_F(\{a\}) = F(a) - F(a_-), \mu_F([a, b]) = F(b) - F(a_-), \mu_F((a, b)) = F(b_-) - F(a)$.

2.

$$F(x) = \begin{cases} 1 & x \le 0 \\ 0 & x < 0 \end{cases} \implies \mu_F(\{0\}) = 1, \mu_F(\mathbb{R}) = 1, \mu_F(\mathbb{R} \setminus \{0\}) = 0.$$

μ_F is the Dirac measure at 0.

3.

$$\mathbb{Q} = \{r_1, r_2, \ldots\}, \; F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n}, \; F_n(x) = \begin{cases} 1 & x \le r_n \\ 0 & x < r_n \end{cases}$$

$$\implies \mu_F(\{v\}) > 0, \forall v \in \mathbb{Q}, \; \mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0.$$

4. $F(x) = x \implies m((a, b])) = m((a, b)) = m([a, b]) = b - a$.

5. $F(x) = e^x, \implies \mu_F((a, b]) = \mu_F((a, b)) = e^b - e^a$

1, 2 are examples of discrete measure.

If F is continuous at a, $\mu_F(\{a\}) = 0$.

Example 1.40 (Middle thirds Cantor set $\mathcal{C} = \bigcup_{n=1}^{\infty} K_n$).

$\mathcal{C}$ is uncountable set, with $m(\mathcal{C}) = 0$.

$x \in \mathcal{C} \implies x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}, a_n \in \{0, 2\}$

We are interested in the Cantor function F.

Example 1.41. Cantor function F is continuous and increasing. This defines the Cantor measure $\mu_F(\mathbb{R} \setminus \mathcal{C}) = 0, \mu_F(\mathcal{C}) = 1, \mu_F(\{a\}) = 0$. Compare with Lebesgue measure $m(\mathbb{R} \setminus \mathcal{C}) = \infty > 0, \mu(\mathcal{C}) = 0, m(\{a\}) = 0$.

1.7   Regularity Properties of Lebesgue-Stieltjes Measures

Lemma 1.42. μ *is LS measure on* $\mathbb{R}$ $\implies$

$$\mu(A) = \inf \left\{ \sum_1^{\infty} ((a_i, b_i]) \mid \bigcup_1^{\infty} (a_i, b_i] \supset A \right\}$$
$$= \inf \left\{ \sum_1^{\infty} ((a_i, b_i)) \mid \bigcup_1^{\infty} (a_i, b_i) \supset A \right\}$$

using the continuity of measure.