Notes for Math 597 – Real Analysis

Yiwei Fu

WN 2022

Contents

Office hour is Mon 12:30 - 1:30, Tue 12:30 - 1:30 in person EH 5838, Th 1 - 2 online.

Chapter 1

Abstract Measure

1.1 σ-Algebra

Definition 1.1. Let X be a set. A collection $\mathcal{M}$ of subsets of X is called a σ-algebra on X if

- $\emptyset \in \mathcal{M}$.
- $\mathcal{M}$ is closed under *complements*: $E \in \mathcal{M} \implies E^c \in \mathcal{M}$.
- $\mathcal{M}$ is closed under *countable unions*: $E_1, E_2, \ldots \in \mathcal{M} \implies \bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$.

<u>SIMPLE PROPERTIES:</u>

- $X = \emptyset^c \in \mathcal{M}$.
- $\bigcap_{i=1}^{\infty} E_i = \left(\bigcup_{i=1}^{n} E_i^c\right)^c \in \mathcal{M}$. It is closed under countable intersections.
- $\bigcup_{i=1}^{N} E_i = E_i \cup \ldots \cup E_n \cup \emptyset \cup \ldots$. It is closed under finite unions (similarly, intersections). sigma
- $E \setminus F = E \cap F^c \in \mathcal{M}, E \triangle F = (E \cap F^c) \cup (F \cap E^c) \in \mathcal{M}$.

Example 1.2. (a) $\mathcal{A} = \mathcal{P}(X)$ power algebra.

(b) $\mathcal{A} = \{\emptyset, X\}$ trivial algebra.

(c) Let $B \subset X, B \neq \emptyset, B \neq X. \mathcal{A} = \{\emptyset, B, B^c, X\}$.

Lemma 1.3. *(An intersection of σ-algebras is a σ-algebra) Let $\mathcal{A}_\alpha, \alpha \in I$, be a family a σ-algebras of X. Then $\bigcap_{\alpha \in I} A_\alpha$ is a σ-algebra. (I can be uncountable.)*

Proof. DIY ■

Definition 1.4. For $\mathcal{E} \subset \mathcal{P}(X)$ (not necessarily a σ-algebra), let $\langle \mathcal{E} \rangle$ be the intersection of all σ-algebras on X that contains $\mathcal{E}$. Call it the σ-algebra generated by $\mathcal{E}$.

- $\langle \mathcal{E} \rangle$ is the *smallest* σ-algebra containing $\mathcal{E}$ and is *unique*.
- $\{\emptyset, B, B^c, X\} = \langle \{B\} \rangle = \langle \{B^c\} \rangle = \langle \{\emptyset, B\} \rangle$.

The above definition gives us (potentially) lots of examples of σ-algebra on a set X

Lemma 1.5. *(a) Suppose $\mathcal{E} \subset \mathcal{P}(X), \mathcal{A}$ is a σ-algebra on X. $\mathcal{E} \in \mathcal{A} \implies \langle \mathcal{E} \rangle \in \mathcal{A}$.*

(b) $E \subset F \subset \mathcal{P}(X) \implies \langle \mathcal{E} \rangle \subset \langle \mathcal{F} \rangle$.

Proof. ■

Definition 1.6. For a topological space X, *the Borel σ-algebra $\mathcal{B}(X)$* is the σ-algebra generated by the collection of open sets.

Example 1.7. $(X = \mathbb{R})$ $\mathcal{B}(\mathbb{R})$ contains the following collections:

$$\begin{aligned} \mathcal{E}_1 &= \{(a,b) \mid a < b\}, & \mathcal{E}_2 &= \{[a,b] \mid a < b\}, \\ \mathcal{E}_3 &= \{(a,b] \mid a < b\}, & \mathcal{E}_4 &= \{[a,b) \mid a < b\}, \\ \mathcal{E}_5 &= \{(a,\infty) \mid a \in \mathbb{R}\}, & \mathcal{E}_6 &= \{[a,\infty) \mid a \in \mathbb{R}\}, \\ \mathcal{E}_7 &= \{(-\infty,a) \mid a \in \mathbb{R}\}, & \mathcal{E}_8 &= \{(-\infty,a] \mid a < b\}. \end{aligned}$$

Proposition 1.8. $\mathcal{B}(\mathbb{R}) = \langle \mathcal{E}_i \rangle$ *for each* $i = 1, \ldots, 8$.

Proof. Use 1.5. ■

Definition 1.9. $(X, \mathcal{A})$ is called a measurable space.

1.2 Measures

Definition 1.10. A measure on $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \to [0, \infty]$ $s.t.$

(a) $\mu(\emptyset) = 0$

(b) (countable additive) For $A_1, A_2, \ldots \in \mathcal{A}$ disjoint we have

$$\mu\left(\bigcup_1^\infty A_i\right) = \sum_{i=1}^\infty \mu(A_i).$$

$(X, \mathcal{A}, \mu)$ is then called a measure space.

Example 1.11. (a) For any $(X, \mathcal{A}), \mu(A) = \#A$ counting measure.

(b) For any $(X, \mathcal{A})$, let $x_0 \in X$. The Dirac measure at x_0 is

$$\mu(A) = \begin{cases} 1 & x_0 \in A, \\ 0 & x_0 \notin A. \end{cases}$$

(c) For $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, let $a_1, a_2, \ldots \in [0, \infty)$. $\mu(A) = \sum_{i \in A} a_i$ is a measure.

$(X, \mathcal{A})$ measurable space

$(X, \mathcal{A}, \mu)$ measure space

$\mu : \mathcal{A} \to [0, \infty]$ $s.t.$ $\mu(\emptyset) = 0$, countable additivity.

<u>NOTE</u>: $A, B \in \mathcal{A}, A \subset B$, then $\mu(B \setminus A) + \mu(A) = \mu(B) \implies \mu(B \setminus A) = \mu(B) - \mu(A)$ if $\mu(A) < \infty$.

Theorem 1.13. *Suppose $(X, \mathcal{A}, \mu)$ a measure space. Then*

(a) (monotonicity)

$$A, B \in \mathcal{A}, A \subset B \implies \mu(A) \leq \mu(B).$$

(b) (countable subadditivity)

$$A_1, A_2, \ldots, \in \mathcal{A}, \implies \mu\left(\bigcup_i^\infty A_i\right) \leq \sum_i^\infty \mu(A_i).$$

(c) (continuity from below/(MCT) from sets)

$$A_1, A_2, \ldots \in \mathcal{A}, A_1 \subset A_2 \subset A_3 \subset \ldots \implies \mu\left(\bigcup_i^\infty A_i\right) = \lim_{n \to \infty} \mu(A_n).$$

(d) (continuity from above)

$$A_1, A_2, \ldots \in \mathcal{A}, A_1 \supset A_2 \supset A_3 \supset \ldots, \mu(A_1) < \infty \implies \mu\left(\bigcap_i^\infty A_i\right) = \lim_{n \to \infty} \mu(A_n).$$

Proof. (a), (b), DIY.

For (c), let $B_1 = A_1, B_i = A_i \setminus A_{i-1}, i \geq 2. B_i \in \mathcal{A}$ and are disjoint.

$$\begin{aligned}
&\bigcup_i^\infty A_i = \bigcup_i^\infty B_i \\
\implies \mu\left(\bigcup_i^\infty A_i\right) = \mu\left(\bigcup_i^\infty B_i\right) &= \sum_i^\infty \mu(B_i) = \lim_{n\to\infty} \sum_i^n \mu(B_i) = \lim_{n\to\infty} \mu(A_n).
\end{aligned}$$

For (d), let $E_i = A_1 \setminus A_i$. Hence $E_i \in \mathcal{A}, E_1 \subset E_2 \subset \dots$ We have

$$\bigcup_i^\infty E_i = \bigcup_i^\infty (A_1 \setminus A_i) = A_1 \setminus \left(\bigcap_1^\infty A_i\right) \implies \bigcap_1^\infty A_i = A_1 \setminus \left(\bigcup_1^\infty E_i\right).$$

Hence

$$\mu\left(\bigcap_1^\infty A_i\right) = \mu(A_1) - \mu\left(\bigcup_1^\infty E_i\right) = \mu(A_1) - \lim_{n\to\infty} \mu(E_n) = \mu(A_1) - \lim_{n\to\infty} \mu(A_1) - \mu(A_n).$$

■

<u>NOTE:</u> the condition that $\mu(A_1) < \infty$ cannot be dropped.
For example, in $(\mathbb{N}, \mathcal{P}(N)$, counting measure), let $A_n = \{n, n+1, n+2\}, A_1 \supset A_2 \supset A_3 \supset \dots$ We have $\bigcap_1^\infty = \emptyset \implies \mu\left(\bigcap_1^\infty A_i\right) = 0$.

Definition 1.14. For $(X, \mathcal{A}, \mu)$ measure space,

- $A \subset X$ is a μ-null set if $A \in \mathcal{A}, \mu(A) = 0$.
- $A \subset X$ is a μ-subnull set if $\exists B$, μ-null set $A \subset B$.
- $(X, \mathcal{A}, \mu)$ is a complete measure space if every μ-subnull set is $\mathcal{A}$-measurable.

Definition 1.15. $(X, \mathcal{A}, \mu)$ measure space. A statement $P(x), x \in X$ holds μ-almost everywhere (a.e.) if the set $\{x \in X \mid P(x) \text{does not hold}\}$ is μ-null.

Definition 1.16. $(X, \mathcal{A}, \mu)$ measure space.

- μ is a *finite measure* is $\mu(X) < \infty$.
- μ is a *σ-finite measure* if $X = \bigcup_1^\infty X_n, X_n \in \mathcal{A}, \mu(X_n) < \infty$.

HW: every measure space can be "completed."

1.3 Outer Measures

Definition 1.17. An *outer measure* on X is $\mu^* : \mathcal{P}(X) \to [0, \infty]$ $s.t.$

- $\mu^*(\emptyset) = 0$
- (monotonicity) $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$.
- (countable subadditivity)

$$\forall A_1, A_2, \ldots \in X, \mu^*\left(\bigcup_i^\infty A_i\right) \leq \sum_i^\infty \mu^*(A_i).$$

Example 1.18. For $A \subset \mathbb{R}$,

$$\mu^*(A) = \inf\left\{\sum_{i=1}^\infty (b_i - a_i) \,\middle|\, \bigcup_1^\infty (a_i, b_i) \supset A\right\}.$$

is an outer measure due to the next proposition.

Proposition 1.19. *(1.19) Let $\mathcal{E} \in \mathcal{P}(X)$ s.t. $\emptyset, X \in \mathcal{E}$. Let $\rho : \mathcal{E} \to [0, \infty]$ s.t. $\rho(\emptyset) = 0$. Then*

$$\mu^*(A) = \inf\left\{\sum_{i=1}^\infty \rho(E_i) \,\middle|\, E_i \in \mathcal{E}, \forall i \in N, \bigcup_1^\infty E_i \supset A\right\}$$

is an outer measure on X.

Proof. (a) μ^* is well-defined (inf is taken over non-empty set.)

(b) $\mu^*(\emptyset) = 0$

(c) $A \subset B \implies \mu^*(A) \leq \mu^*(B)$.

We check the countable subadditivity.

Let $A_1, A_2, \ldots \subset X$. If one of $\mu^*(A_i) = \infty$, then the result holds. Suppose $\mu^*(A_n) < \infty, \forall n \in \mathbb{N}$.

"Give your self a room of epsilon":

Fix $\varepsilon > 0$. We will show

$$\mu^*\left(\bigcup_1^\infty A_n\right) \leq \sum_1^\infty \mu^*(A_i) + \varepsilon.$$

For each $n \in \mathbb{N}, \exists E_{n,1}, E_{n,2}, \ldots \in \mathcal{E}$ *s.t.*

$$\bigcup_{k=1}^\infty E_{n,k} \supset A_n \quad \text{and} \quad \mu^*(A_n) + \frac{\varepsilon}{2^n} \geq \sum_{k=1}^\infty \rho(E_{n,k}).$$

Then,

$$\bigcup_{1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} E_{n,k} = \bigcup_{(n,k)\in\mathbb{N}^2} E_{n,k}.$$

<u>RECALL:</u> Tonelli's thm for series. If $a_{ij} \in [0,\infty], \forall i,j \in \mathbb{N}$, then

$$\sum_{(i,j)\in\mathbb{N}^2} a_{ij} = \sum_{i=1}^{\infty} \sum_{j=1^{\infty}} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_i j.$$

Hence

$$\mu^* \left(\bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \rho(E_{k,n}) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \rho(E_{k,n}) \leq \sum_{n=1}^{\infty} \left(\mu^*(A_n) + \frac{\varepsilon}{2^n} \right) = \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon.$$

We have shown countable subadditivity. ■

Outer measure is very close to a measure. Here the textbooks diverge.

[Tao11] introduces Lebesgue measure on $\mathbb{R}$ using topological qualities of subsets of $\mathbb{R}$. [Fol99] introduces abstract method by Carathéodory and Kolmogorov.

Definition 1.20. Let μ^* be an outer measure on X. We say $A \subset X$ is Carathéodory measurable with respect to μ^* if $\forall E \subset X, \mu^*(E) = \mu^*(E \setminus A) + \mu^*(E \cap A)$.

Lemma 1.21. *Let μ^* be an outer measure on X. Suppose $B_1, B_2, \ldots, B_N$ are disjoint C-measurable sets. Then,*

$$\forall E \subset X, \mu^* \left(E \cap \left(\bigcup_{1}^{N} B_i \right) \right) = \sum_{i=1}^{n} \mu^*(E \cap B_i)$$

Proof.

$$\mu^* \left(E \cap \left(\bigcup_{1}^{N} B_i \right) \right) = \mu^*(E \cap B_1) + \mu^* \left(E \cap \left(\bigcup_{2}^{N} B_i \right) \right)$$

because B_1 is C-measurable. Then, iterate. ■

Improved version:

$B_1, B_2, \ldots$ C-measurable and *disjoint* $\implies \mu^*\left(E \cap \bigcup_1^\infty B_n\right) = \sum_1^\infty \mu^*\left(E \cap B_n\right), \forall E \subset X$.

Proof.

$$\begin{aligned}\sum_1^\infty \mu^*(E\cap B_n) &\geq \mu^*\left(E\cap\bigcup_1^\infty B_n\right)\\ &\geq \mu^*\left(E\cap\bigcup_1^N B_n\right) = \sum_1^N \mu^*(E\cap B_n.)\end{aligned}$$

Take $N\to\infty$ or note that $N\in\mathbb{N}$ is arbitrary we get the result. ∎

First big theorem:

Theorem 1.22 (Carathéodory extension theorem)**.** *Let μ^* be an outer measure on X. Let $\mathcal{A}$ be the collection of C-measurable sets with respect to μ^*. Then*

(a) $\mathcal{A}$ us a σ-algebra on X.

(b) $\mu=\mu^*|_{\mathcal{A}}$ is a measure on $(X,\mathcal{A})$.

(c) $(X,\mathcal{A},\mu)$ is a complete measure space.

Proof. (a) (1) $\emptyset\in\mathcal{A}$.

(2) $\mathcal{A}$ is closed under complements.

(3) To show $\mathcal{A}$ closed under countable unions.

- (finite union)
  <u>Claim</u> $A,B\in\mathcal{A}\implies A\cup B\in\mathcal{A}$.

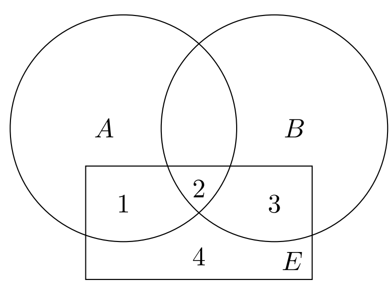

Figure 1.1: Venn diagram of A,B,E

Fix arbitrary $E\subset X$. We need to show

$$\mu^*(E)=\mu^*(E\cap(A\cup B))+\mu^*(E\setminus(A\cup B)).$$

i.e.

$$\mu^*(1\cup 2\cup 3\cup 4)=\mu^*(1\cup 2\cup 3)+\mu^*(4)$$

Since A is C-measurable, we have

$$\mu^*(1 \cup 2 \cup 3 \cup 4) = \mu^*(1 \cup 2) + \mu^*(3 \cup 4)$$

$$\mu^*(1 \cup 2 \cup 3) = \mu^*(1 \cup 2) + \mu^*(3)$$

Similarly since B is C-measurable, we have

$$\mu^*(3 \cup 4) = \mu^*(3) + \mu^*(4)$$

Hence

$$\begin{aligned} \mu^*(1 \cup 2 \cup 3 \cup 4) &= \mu^*(1 \cup 2) + \mu^*(3 \cup 4) \\ &= \mu^*(1 \cup 2 \cup 3) - \mu^*(3) + \mu^*(3) + \mu^*(4) \\ &= \mu^*(1 \cup 2 \cup 3) + \mu^*(4). \end{aligned}$$

- (countable disjoint unions)
  Let $A_1, A_2, \ldots \in \mathcal{A}$ and *disjoint*.

  Fix $E \subset X$ arbitrary. Since μ^* is countably subadditive,

  $$\mu^*(E) \leq \mu^*\left(E \cap \bigcup_1^\infty\right) + \mu^*\left(E \setminus \bigcup_1^\infty A_n\right)$$

  Fix $n \in \mathbb{N}$.

  $$\begin{aligned} &\implies \bigcup_1^N A_n \in \mathcal{A} \\ &\implies \mu^*(E) = \mu^*\left(E \cap \bigcup_1^N\right) + \mu^*\left(E \setminus \bigcup_1^N A_n\right) \\ &\geq \sum_1^N \mu^*(E \cap A_n) + \mu^*\left(E \setminus \bigcup_1^\infty A_n\right) \text{ by lemma.} \end{aligned}$$

  Take $n \to \infty$.

- (countable unions)
  Let $A_1, A_2, \ldots \in \mathcal{A}$. Take $E_1 = A_1, E_n = A_n \setminus \left(\bigcup_1^{n-1} A_i\right)$ for $n \geq 2$. Then $\bigcup A_n = \bigcup E_n$ and E_n's are disjoint.

(b) Firstly we have $\mu(\emptyset) = \mu^*(\emptyset) = 0$.

Countable additvity of μ^* on $\mathcal{A}$ follows from the improved lemma with $E = X$.

(c) HW. ■

1.4 Hahn-Kolmogorov Theorem

<u>RECALL</u> 1.19 Let $\mathcal{E} \subset \mathcal{P}(X)$ $s.t.$ $\emptyset, X \in \mathcal{E}$. Let $\rho : \mathcal{E} \to [0, \infty]$ $s.t.$ $\rho(\emptyset) = 0$

$$(\mathcal{E}, \rho) \xrightarrow[1.19]{} (\mathcal{P}(X), \mu^*) \xrightarrow[\text{C-theorem}]{} (A, \mu)$$

<u>QUESTION</u> $\mathcal{E} \subset \mathcal{A}$ and $\mu|_{\mathcal{E}} = \rho$? No!

Definition 1.23. Let $\mathcal{A}_0$ be an algebra on X. We say $\mu_0 : \mathcal{A}_0 \to [0, \infty]$ is a *pre-measure* if

(a) $\mu_0(\emptyset) = 0$.

(b) (finite additivity)

$$\mu_0\left(\bigcup_1^N A_i 1\right) = \sum_1^N \mu_0(A_i) \text{ if } A_1, \ldots, A_N \in \mathcal{A}_0 \text{ are disjoint.}$$

(c) (countable additivity within the algebra) If $A \in \mathcal{A}_0$ and

$$A = \bigcup_1^\infty A_n, A_n \in \mathcal{A}_0 \text{ and are disjoint, then } \mu_0(A) = \sum_1^\infty \mu_0(A_n)$$

<u>NOTATION:</u> Folland uses $\mathcal{M}$ for σ-algebra and $\mathcal{A}$ for algebra. (Jinho) uses $\mathcal{A}$ for σ-algebra and $\mathcal{A}_0$ for alegbra.

Example 1.24. $\mathcal{A}_0$ finite disjoint unions of $(a, b]$.

$$\mu_0\left(\bigcup_1^\infty (a_i, b_i]\right) = \sum_1^\infty (b_i - a_i) \text{ or } b_i^n - a_i^n, e^{b_i} - e^{a_i}, \text{ etc.}$$

Lemma 1.25.
- *(a) + (c) $\implies$ (b).*
- *μ_0 is monotone.*

Theorem 1.26 (Hahn-Kolmogorov Theorem)**.** *Let μ_0 be a pre-measure on algebra $\mathcal{A}_0$ on X. Let μ^* be the outer measure induced by $(\mathcal{A}_0, \mu_0)$ in 1.19. Let $\mathcal{A}$ and μ be the Carathéodory σ-algebra and measure for μ^* $\implies$ $(\mathcal{A}, \mu)$ extends $(\mathcal{A}_0, \mu_0)$ $i.e.$ $\mathcal{A} \supset \mathcal{A}_0, \mu|_{\mathcal{A}_0} = \mu_0$.*

Proof. (a) ($\mathcal{A} \supset \mathcal{A}_0$) Let $A \in \mathcal{A}_0$.

Question: $A \in \mathcal{A}$? i.e. is A C-measurable? i.e. $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c), \forall E \subset$

X.

Fix $E \subset X$.

- (countable) subadditivity of $\mu^* \implies \mu^*(E) \le \mu^*(E \cap A) + \mu^*(E \cap A^c)$.
- If $\mu^*(E) = \infty$ then $\mu^*(E) = \infty \ge \mu^*(E \cap A) + \mu^*(E \cap A^c)$.
- If $\mu^*(E) < \infty$.

  Fix $\varepsilon > 0$. By the definition of $\mu^*, \exists B_1, B_2, \ldots \in \mathcal{A}_0$ $s.t.$ $\bigcup_1^\infty B_n \supset E$ and

  $$\mu^*(E) + \varepsilon \ge \sum_1^\infty \mu_0(B_n) = \sum_1^\infty \left(\mu_0(B_n \cap A) + \mu_0\left(B_n \cap A^c\right)\right).$$

  Note that

  $$\bigcup_1^\infty (B_n \cap A) \supset E \cap A, \quad \bigcup_1^\infty (B_n \cap A^c) \supset E \cap A^c \implies \ge$$

(b) Let $A \in \mathcal{A}_0$. We want to show that $\mu(A) = \mu_0(A)$.

By definition, $\mu(A) = \mu^*(A)$.

- Let $B_i = \begin{cases} A & i = 1, \\ \emptyset & i = 2 \end{cases} \in \mathcal{A}_0$ and $\bigcup_1^\infty B_i \supset A$.

  Hence $\mu^*(A) \le \sum_1^\infty \mu_0(B_i) = \mu_0(A)$.
- Let $B_i \in \mathcal{A}_0, \bigcup_1^\infty B_i \supset A$ an arbitrary collection of sets.

  Let $C_1 = A \cap B_1, C_i = A \cap B_i \setminus \left(\bigcup_{j=1}^{i-1} B_j\right)$. Then $A = \bigcup_1^\infty$ is a disjoint countable union. By countable additivitiy we have

  $$\mu_0(A) = \sum_1^\infty \mu_0(C_i) \implies \mu_0(A) \le \sum_1^\infty \mu_0(B_i).$$

Hence we have $\mu_0(A) = \mu^*(A) = \mu(A)$. We have completed our proof. ■

Definition 1.27. Such $(\mathcal{A}, \mu)$ is called the *Hahn-Kolmogorov extension* of $(\mathcal{A}_0, \mu_0)$, and is also called the *Carathéodory σ-algebra* for $(\mathcal{A}_0, \mu_0)$.

Theorem 1.28 (uniqueness of HK extension)**.** *Let $\mathcal{A}_0$ be an algebra on X, μ_0 be a pre-measure on $\mathcal{A}_0$, $(\mathcal{A}, \mu)$ be the Hahn-Kolmogorov extension of $(\mathcal{A}_0, \mu_0)$. And let $(\mathcal{A}', \mu')$ be another extension of $(\mathcal{A}_0, \mu_0)$.*

*If μ_0 is σ-*finite*, then $\mu \mid_{\mathcal{A} \cap \mathcal{A}'} = \mu' \mid_{\mathcal{A} \cap \mathcal{A}'}$.*

NOTE σ-finite means

$$\forall X, X = \bigcup_1^\infty X_n, X_n \in \mathcal{A}_0, \mu_0(X_n) < \infty.$$

Corollary 1.29. *Let μ_0 be a pre-measure on algebra $\mathcal{A}_0$ on X. Suppose μ_0 is σ-finite, then $\exists!$ measure μ on $\langle \mathcal{A}_0 \rangle$ that extends $\mathcal{A}_0$. Furthermore,*

(a) the completion of $(X, \langle \mathcal{A}_0 \rangle, \mu)$ is the HK extension of $(\mathcal{A}_0, \mu_0)$.

(b)
$$\mu(A) = \inf \left\{ \sum_{i=1}^\infty \mu_0(B_i) \mid B_i \subset A_0, \forall i \in \mathbb{N}, \bigcup_1^\infty B_i \supset A \right\}, \forall A \in \overline{\langle \mathcal{A}_0 \rangle}.$$

Proof of 1.28. Let $A \in \mathcal{A} \cap \mathcal{A}'$. We need to show $\mu(A) = \mu^*(A) = \mu'(A)$.

- $\mu^*(A) \geq \mu'(A)$ (HW)
- $\mu(A) \leq \mu'(A)$:

  (i) Assume $\mu(A) < \infty$. Fix $\varepsilon > 0$. Then $\exists B_i \in \mathcal{A}_0, \forall i \in \mathbb{N}, \bigcup_1^\infty B_i \supset A \; s.t.$

  $$\mu(A) + \varepsilon = \mu^*(A) + \varepsilon \geq \sum_1^\infty \mu_0(B_i) = \sum_1^\infty \mu(B_i) \geq \mu\left(\bigcup_1^\infty B_i\right) = \mu(B)$$

  Hence $\mu(B \setminus A) = \mu(B) - \mu(A) \leq \varepsilon$.

  On the other hand,

  $$\mu(B) = \lim_{N\to\infty} \mu\left(\bigcup_1^N B_i\right) = \lim_{N\to\infty} \mu'\left(\bigcup_1^N B_i\right) = \mu'(B)$$

  by continuity of measure from below.

  $$\mu(A) \leq \mu(B) = \mu'(B) = \mu'(A) + \mu'(B \setminus A) \leq \mu'(A) = \varepsilon.$$

  (ii) Assume $\mu(A) = \infty$.

  Since μ_0 is σ-finite, $X = \bigcup_1^\infty X_n, X_n \in \mathcal{A}_0, \mu_0(X_0) < \infty$. Replacing X_n by $X_1 \cup \ldots \cup X_n$, we may assume $X_1 \subset X_2 \subset \ldots$.

  $$\forall n \in N, \mu(A \cap X_n) < \infty \implies \mu(A \cap X_n) \leq \mu'(A \cap X_n).$$

Hence
$$\mu(A) = \lim_{N\to\infty} \mu(A \cap X_n) \le \lim_{N\to\infty} \mu'(A \cap X_n) = \mu'(A). \qquad \blacksquare$$

1.5 Borel Measures on $\mathbb{R}$

Definition 1.30. $F : \mathbb{R} \to \mathbb{R}$ is an *increasing* function if $F(x) \le F(y)$ for $x < y$. $F : \mathbb{R} \to \mathbb{R}$ is increasing and right-continuous $\implies$ F is distribution function.

Example 1.31.

- $F(x) = \begin{cases} 1, & x \ge 0 \\ 0, & x < 0. \end{cases}$
- $\mathbb{Q} = \{r_1, r_2, \ldots\}, F_n(x) = \begin{cases} 1 & x \ge r_n \\ 0 & x < r_n \end{cases}$. $F(x) = \displaystyle\sum_{n=1}^{\infty} \frac{F_n(x)}{2^n}$ is a distribution function.

<u>NOTE</u> If F is increasing, $F(\infty) := \lim_{x\to\infty} F(x), F(-\infty) := \lim_{x\to-\infty} F(x)$ exists in $[-\infty, \infty]$.

In probability theory, cumulative distribution function (CDF) is a distribution function with $F(\infty) = 1$ and $F(-\infty) = 0$.

There are distributions [Fol99, Ch.9], but these are different from *distribution functions*.

Definition 1.32. Suppose X a topological space. μ on $(X, \mathcal{B}(X))$ is called *locally finite* is $\mu(K) < \infty$ for any compact set $K \subset X$.

Lemma 1.33. *Let μ be a locally finite Borel measure on $\mathbb{R}$ $\implies$*

$$F_\mu(x) = \begin{cases} \mu((0, x]), & x > 0 \\ 0, & x = 0 \\ -\mu((x, 0]), & x < 0 \end{cases} \text{ is a distribution function.}$$

Proof. DIY. Use continuity of measure. ∎

Definition 1.34. h-intervals are $\emptyset, (a, b], (a, \infty), (-\infty, b], (\infty, \infty)$.

Lemma 1.35. *Let $\mathcal{H}$ be the collections of finite disjoint unions of h-intervals. Then $\mathcal{H}$ is an algebra on $\mathbb{R}$.*

Proof. DIY. ∎

Proposition 1.36 (Distribution function defines a pre-measure). *Let $F : \mathbb{R} \to \mathbb{R}$ be a distribution function. For an h-interval I, define*

$$\ell(I) = \ell_F(I) = \begin{cases} 0, & I = \emptyset \\ F(b) - F(a), & I = (a, b] \\ F(\infty) - F(a), & I = (a, \infty) \\ F(b) - F(\infty), & I = (-\infty, b] \\ F(\infty) - F(-\infty), & I = (-\infty, \infty). \end{cases}$$

Define $\mu_0 = \mu_{0,F} : \mathcal{H} \to [0, \infty]$ by

$$\mu_0(A) := \sum_{k=1}^{N} \ell(I_k) \quad \text{if } A = \bigcup_{k=1}^{N} I_k, \text{ finite disjoint union of } h\text{-intervals.}$$

Then μ_0 is a pre-measure.

Proof. (a) μ_0 is well-defined.

(b) μ_0 is finite additive.

(c) μ_0 is countable additive within $\mathcal{H}$.

Suppose $A \in \mathcal{H}$ and $A = \bigcup_1^\infty A_i$ a disjoint union, $A_i \in \mathcal{H}$. It is enough to consider the case $A = I, A_k = I_k$ all h-intervals. (Why?)

Focus on the case $I = (a, b]$: (HW: check other cases)
We have

$$(a, b] = \bigcup_1^\infty (a_n, b_n], \text{ a disjoint union.}$$

Check

$$F(b) - F(a) \stackrel{?}{=} \sum_1^\infty (F(b_n) - F(a_n))$$

$(a, b] \supset \bigcup_1^N (a_n, b_n] \implies F(b) - F(a) \geq \sum_1^N F(b_n) - F(a_n), \forall N \in \mathbb{N}$. (Arranging them in decreasing order) Take $N \to \infty$ we have

$$F(b) - F(a) \geq \sum_1^\infty (F(b_n) - F(a_n)).$$

Since F is right-continuous, $\exists a' > a \ s.t. \ F(a') - F(a) < \varepsilon$. For each $n \in \mathbb{N}$, $\exists b_n' >$

b_n $s.t.$ $F(b_n') - F(b_n) < \frac{\varepsilon}{2^n}$.

$$\begin{aligned}
&\implies [a', b] \subset \bigcup_1^\infty (a_n, b_n') \\
&\implies \exists N \in \mathbb{N} \; s.t. \; [a', b] \subset \bigcup_1^n (a_n, b_n') \\
&\implies F(b) - F(a') \leq \sum_1^N F(b_n') - F(a_n) \\
&\implies F(b) - F(a) \leq F(b) - F(a') + \varepsilon \leq \sum_1^\infty (F(b_n') - F(a_n)) + \varepsilon \\
&\quad \leq \sum_1^\infty \left(F(b_n) - F(a_n) + \frac{\varepsilon}{2^n} \right) + \varepsilon
\end{aligned}$$ ■

Once we have this pre-measure, HK theorem allows us to extended it to a measure.

Theorem 1.37 (Locally finite Borel measures on $\mathbb{R}$)**.**

(a) $F : \mathbb{R} \to \mathbb{R}$ *is a distribution function* $\implies$ $\exists!$ *locally finite Borel measure* μ_F *on* $\mathbb{R}$ *satisfying* $\mu_F((a, b]) = F(b) - F(a), \forall a, b, a < b$.

(b) Suppose $F, G : \mathbb{R} \to \mathbb{R}$ *are distribution functions. Then,* $\mu_F = \mu_G$ *on* $\mathcal{B}(\mathbb{R})$ *if and only if* $F - G$ *is a constant function.*

Proof. HW ■

1.6 Lebesgue-Stieltjes Measures on $\mathbb{R}$

F distribution function $\implies$ μ_F on Carathéodory σ-algebra $\mathcal{A}_{\mu_F}$.
Actually $(\mathcal{A}_{\mu_F}, \mu_F) = (\mathcal{B}(\mathbb{R}), \mu_F)$ (HW3).

Definition 1.38.

- μ_F on $\mathcal{A}_{\mu_F}$ is called the Lebesgue-Stieltjes measure corresponding to F.
- Special case: $F(x) = x \implies$ Lebesgue measure $(\mathcal{B}, m)$.

Example 1.39.

(a) $\mu_F((a, b]) = F(b) - F(a)$. F is right-continuous and increasing $\implies$ $F(x_-) \leq F(x) = F(x_+)$.

(HW) $\mu_F(\{a\}) = F(a) - F(a_-), \mu_F([a, b]) = F(b) - F(a_-), \mu_F((a, b)) = F(b_-) - F(a)$.

(b)

$$F(x) = \begin{cases} 1 & x \leq 0 \\ 0 & x < 0 \end{cases} \implies \mu_F(\{0\}) = 1, \mu_F(\mathbb{R}) = 1, \mu_F(\mathbb{R} \setminus \{0\}) = 0.$$

μ_F is the Dirac measure at 0.

(c)

$$\mathbb{Q} = \{r_1, r_2, \ldots\},\ F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n},\ F_n(x) = \begin{cases} 1 & x \leq r_n \\ 0 & x < r_n \end{cases}$$
$$\implies \mu_F(\{v\}) > 0, \forall v \in \mathbb{Q},\ \mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0.$$

(d) If F is continuous at $a, \mu_F(\{a\}) = 0$.

(e) $F(x) = x \implies m((a, b])) = m((a, b)) = m([a, b]) = b - a.$

(f) $F(x) = e^x, \implies \mu_F((a, b]) = \mu_F((a, b)) = e^b - e^a$

(a), (b) are examples of discrete measure.

Example 1.40 (Middle thirds Cantor set $\mathcal{C} = \bigcup_{n=1}^{\infty} K_n$**).**

$\mathcal{C}$ is uncountable set with $m(\mathcal{C}) = 0$.

$$x \in \mathcal{C} \implies x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}, a_n \in \{0, 2\}.$$

We are interested in the Cantor function F.

Example 1.41. Cantor function F is continuous and increasing. This defines the Cantor measure $\mu_F(\mathbb{R} \setminus \mathcal{C}) = 0, \mu_F(\mathcal{C}) = 1, \mu_F(\{a\}) = 0$. Compare with Lebesgue measure $m(\mathbb{R} \setminus \mathcal{C}) = \infty > 0, \mu(\mathcal{C}) = 0, m(\{a\}) = 0$.

1.7 Regularity Properties of Lebesgue-Stieltjes Measures

Lemma 1.42. *μ is Lebesgue-Stieltjes measure on $\mathbb{R}$ $\implies$*

$$\begin{aligned} \mu(A) &= \inf \left\{ \sum_1^{\infty} \mu((a_i, b_i]) \,\middle|\, \bigcup_1^{\infty} (a_i, b_i] \supset A \right\} \\ &= \inf \left\{ \sum_1^{\infty} \mu((a_i, b_i)) \,\middle|\, \bigcup_1^{\infty} (a_i, b_i) \supset A \right\} \end{aligned}$$

Proof. Using the continuity of measure. ■

Theorem 1.43. *μ is a Lebesgue-Stieltjes measure. Then $\forall A \in \mathcal{A}_\mu$,*

(a) (outer regularity)
$$\mu(A) = \inf\{\mu(O) \mid \textit{open } O \supset A\}.$$

(b) (inner regularity)
$$\mu(A) = \sup\{\mu(K) \mid \textit{compact } K \subset A\}.$$

Proof. (a) Followed from 1.42.

(b) Let $s = \sup\{\dots\}$. Monotonicity $\implies$ $\mu(A) \geq s$.

- (A bounded) $\overline{A} \in \mathcal{B}(\mathbb{R}) \subset \mathcal{A}_\mu$, $\overline{A}$ bounded $\implies$ $\mu(\overline{A}) < \infty$.

  Fix $\varepsilon > 0$. By 1, $\exists$ open $O \supset \overline{A} \setminus A$, $\mu(O) - \mu(\overline{A} \setminus A) = \mu(O \setminus (\overline{A} \setminus A)) \leq \varepsilon$.

  Let $K = \underbrace{A \setminus O}_{K \subset A} = \underbrace{\overline{A} \setminus O}_{\text{compact}}$. Show that $\mu(K) \geq \mu(A) - \varepsilon$.

- (A unbounded but $\mu(A) < \infty$) We have
$$A = \bigcup_1^\infty A_n,\ A_n = A \cap [-n, n],\ A_1 \subset A_2 \subset \dots$$
  Hence
$$\lim_{n\to\infty} \mu(A_n) = \mu(A) < \infty.$$

- ($\mu(A) = \infty$)
$$\lim_{n\to\infty} \mu(A_n) = \mu(A) = \infty.$$
  Fix $L > 0$. $\exists N\ s.t.\ \mu(A_N) \geq L$. ■

Definition 1.44. Suppose X a topological space.

A G_σ*-set* is $G = \displaystyle\bigcap_1^\infty O_i$, O_i open. An F_σ*-set* is $F = \displaystyle\bigcup_1^\infty F_i$, F_i closed.

Theorem 1.45. *Suppose μ a LS measure. Then the following statements are equivalent:*

(a) $A \in \mathcal{A}_\mu$.

(b) $A = G \setminus M$, G is a G_σ-set, and M is μ-null.

(c) $A = F \cup N$, F is an F_σ-set, and N is μ-null.

Proof. (b) $\implies$ (a) and (c) $\implies$ (a) are clear.

- (a) $\implies$ (c)
  (i) Assume $\mu(A) < \infty$. By inner regularity,
  $$\forall n \in \mathbb{N}, \exists \text{ compact } K_n \subset A \ s.t.\ \mu(K_n) + \frac{1}{n} \geq \mu(A).$$
  Let $F = \bigcup_1^\infty K_n$. Then $N = A \setminus F$ is μ-null.
  (ii) Assume $\mu(A) = \infty$. We construct
  $$A = \bigcup_{k \in \mathbb{Z}} A_k, A_k = A \cap (k, k+1].$$
  By (i), $\forall k \in \mathbb{Z}, A_k = F_k \cup N_k$. Hence
  $$A = \underbrace{\left(\bigcup_k F_k\right)}_{F\sigma} \cup \underbrace{\left(\bigcup_k N_k\right)}_{\mu\text{-null}}.$$
- (a) $\implies$ (b)
  $$A^c = F \cup N, A = F^c \cup N^c = F^c \setminus N.$$ ■

Proposition 1.46. *Suppose μ a LS measure, $A \in \mathcal{A}_\mu$, $\mu(A) < \infty$. Then*
$$\forall \varepsilon > 0, \exists I = \bigcup_1^{N=N(\varepsilon)} I_i, \text{ disjoint open intervals } s.t.\ \mu(A \triangle I) \leq \varepsilon.$$

Proof. DIY - use outer regularity. ■

Properties of Lebesgue measure

Theorem 1.47.
$$A \in \mathcal{L} \implies A + s \in \mathcal{L}, rA \in \mathcal{L}, \forall r, s \in \mathbb{R}.$$
In addition, $m(A + r) = m(A)$ and $m(rA) = rm(A)$.

Proof. DIY. ■

Example 1.48.

(a) $\mathbb{Q} = \{r_1\}_{i=1}^\infty$, which is dense in $\mathbb{R}$. Let $\varepsilon > 0$ and
$$O = \bigcup_{i=1}^\infty \left(r_i - \frac{\varepsilon}{2^i}, r_i + \frac{\varepsilon}{2^i}\right).$$

O is open and dense in $\mathbb{R}$. We have

$$m(O) \leq \sum_{i=1}^{\infty} \frac{2\varepsilon}{2^i} = 2\varepsilon, \partial O = \overline{O} \setminus O, m(O) = \infty.$$

(b) $\exists$ uncountable set A with $m(A) = 0$.

(c) $\exists A$ with $m(A) > 0$, but A contains no non-empty open interval.

(d) $\exists A \notin \mathcal{L}$ that is Vitali set.

(e) $\exists A \in \mathcal{L} \setminus \mathcal{B}(\mathbb{R})$. We will deal with that later.

Chapter 2

Integration

2.1 Measurable Functions

Definition 2.1. Suppose $(X, \mathcal{A})$, $(Y, \mathcal{B})$ two measurable spaces. $f : X \to Y$ is $(\mathcal{A}, \mathcal{B})$-measurable if

$$\forall B \in \mathcal{B}, f^{-1}(B) \in \mathcal{A}.$$

Lemma 2.2. *Suppose $\mathcal{B} = \langle \mathcal{E} \rangle$. Then*

$$f : X \to Y \text{ is } (\mathcal{A}, \mathcal{B})\text{-measurable} \iff \forall E \in \mathcal{E}, f^{-1}(E) \in \mathcal{A}.$$

Proof. $\implies$ clear

$\impliedby$ Let $\mathcal{D} = \{E \subset Y \mid f^{-1}(E) \in \mathcal{A}\}$. We have $\mathcal{E} \subset \mathcal{D}$ by assumption. In addition $\mathcal{D}$ is a σ-algebra $\implies \langle \mathcal{E} \rangle \subset \mathcal{D}$. ∎

Definition 2.3. Suppose $(X, \mathcal{A})$ a measurable space.

$$\left.\begin{array}{l} f : X \to \mathbb{R} \\ f : X \to \overline{\mathbb{R}} = [-\infty, \infty] \\ f : X \to \mathbb{C} \end{array}\right\} \text{ is } \mathcal{A}\text{-measurable if } \left\{\begin{array}{l} f \text{ is } (\mathcal{A}, \mathcal{B}(\mathbb{R}))\text{-measurable} \\ f \text{ is } (\mathcal{A}, \mathcal{B}(\overline{\mathbb{R}}))\text{-measurable} \\ \operatorname{Re} f, \operatorname{Im} f : X \to \mathbb{R} \text{ are } \mathcal{A}\text{-measurable.} \end{array}\right.$$

Here $\mathcal{B}(\overline{\mathbb{R}}) = \{E \subset \overline{\mathbb{R}} \mid E \cap R \in \mathcal{B}(\mathbb{R})\}$.

Lemma 2.4. *Suppose $f : X \to \mathbb{R}$. Then the followings are equivalent:*

(a) f is $\mathcal{A}$-measurable

(b) $\forall a \in \mathbb{R}, f^{-1}((a,\infty)) \in \mathcal{A}$.

(c) $\forall a \in \mathbb{R}, f^{-1}([a,\infty)) \in \mathcal{A}$.

(d) $\forall a \in \mathbb{R}, f^{-1}((-\infty,a)) \in \mathcal{A}$.

(e) $\forall a \in \mathbb{R}, f^{-1}((-\infty,a]) \in \mathcal{A}$.

For $f : X \to \overline{\mathbb{R}}$, change the interval to include $-\infty$ and ∞.

Proof. By 2.2. ■

Example 2.5. $\mathcal{A} = \mathcal{P}(X) \implies$ every function is $\mathcal{A}$ measurable.

$\mathcal{A} = \{\emptyset, X\} \implies$ only $\mathcal{A}$ functions are constant functions.

<u>PROPERTIES</u> Suppose $f, g : X \to \mathbb{R}$, $\mathcal{A}$-measurable functions.

(a) $\phi : \mathbb{R} \to \mathbb{R}$, $\mathcal{B}(\mathbb{R})$ measurable (i.e. Borel measurable) $\implies$ $\phi \circ f : X \to \mathbb{R}$ is $\mathcal{A}$-measurable.

(b) $-f, 3f, f^2, |f|$ are $\mathcal{A}$-measurable, $\frac{1}{f}$ is $\mathcal{A}$-measurable if $f(x) = 0, \forall x \in X$.

(c) $f + g$ is $\mathcal{A}$-measurable

$$(f+g)^{-1}((a,\infty)) = \bigcup_{r\in\mathbb{Q}} \left(f^{-1}((r,\infty)) \cap g^{-1}((a-r,\infty))\right).$$

(d) fg is $\mathcal{A}$-measurable

$$f(x)g(x) = \frac{1}{2}\left((f(x)+g(x))^2 - f(x)^2 - g(x)^2\right).$$

(e) $(f \wedge g)(x) = \min\{f(x), g(x)\}$, $(f \vee g)(x) = \max\{f(x), g(x)\}$ are $\mathcal{A}$-measurable.

(f) $f_n : X \to \overline{\mathbb{R}}$ are a sequence of $\mathcal{A}$-measurable functions $\implies$

$$\sup f_n, \inf f_n, \limsup_{n\to\infty} f_n, \liminf_{n\to\infty} f_n \text{ are } \mathcal{A}\text{-measurable.}$$

(g) If $f(x) = \lim_{n\to\infty} f_n(x)$ converges for every $x \in X$, then f is measurable.

Example 2.6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. Then f is Borel measurable $\implies$ f is Lebesgue measurable. (Preimage of an open set of a continuous function is open.)

Definition 2.7. For $f : X \to \overline{\mathbb{R}}$, let $f^+ = f \vee 0$, $f^- = (-f) \vee 0$.

<u>NOTE</u> $\operatorname{supp} f^+ \cap \operatorname{supp} f^- = \emptyset$. $f(x) = f^+(x) - f^-(x)$. f is $\mathcal{A}$-measurable $\iff$ f^+, f^- measurable.

Definition 2.8. For $E \subset X$, characteristic (indicator) funtion of E

$$\chi_E(x) = 1_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \in E^c. \end{cases}$$

1_E is $\mathcal{A}$-measurable $\iff E \in \mathcal{A}$.

Definition 2.9. Suppose $(X, \mathcal{A})$ a measurable space. A *simple function* $\phi : X \to \mathbb{C}$ that is $\mathcal{A}$-measurable and takes only finitely many values.

$$\phi(X) = \{c_1, \ldots, c_N\}, c_i \neq \pm\infty, E_i = \phi^{-1}(c_i) \in \mathcal{A} \implies \phi = \sum_{i=1}^{N} c_i 1_{E_i}.$$

Theorem 2.10. *Suppose $(X, \mathcal{A})$ a measurable space and $f : X \to [0, \infty]$. Then the followings are equivalent:*

(a) f is $\mathcal{A}$-measurable.

(b) $\exists$ simple functions $0 \leq \phi_1(x) \leq \phi_2(x) \leq \ldots \leq f(x)$ such that

$$\lim_{n\to\infty} \phi_n(x) = f(x),\ \forall x \in X.$$

(f is the pointwise upward limit of simple functions.)

Proof.

- (b) $\implies$ (a) is easy: $f(x) = \sup_{n\in\mathbb{N}} \phi_n(x)$.
- (a) $\implies$ (b): suppose f is $\mathcal{A}$-measurable.

  Fix $n \in \mathbb{N}$. Let $F_n = f^{-1}([2^n, \infty]) \in \mathcal{A}$. For

  $$0 \leq k \leq 2^{2n} - 1,\ E_{n,k} = f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right) \in \mathcal{A}.$$

  Let $\phi_n(x) = \sum_{k=0}^{2^{2n}-1} 1_{E_{n,k}} + 2^n 1_{F_n}$.

  This shows that

  - $0 \leq \phi_1(x) \leq \phi_2(x) \leq \ldots \leq f(x),\ \forall x \in X.$
  - $\forall x \in X \setminus F_n, 0 \leq f(x) - \phi_n(x) \leq \frac{1}{2^n}$.

  Since $F_1 \supset F_2 \supset \ldots$ and $\bigcap_1^\infty F_n = f^{-1}(\{\infty\})$, we have

- $x \in f^{-1}([0,\infty)) = X \setminus \left(\bigcap_1^\infty F_n\right) \implies \lim_{n\to\infty} \phi_n(x) = f(x)$.
- $x \in f^{-1}(\{\infty\}) = \bigcap_1^\infty X_n \implies \phi_n(x) \geq 2^n \implies \lim_{n\to\infty} \phi_n(x) = \infty = f(x)$. ■

Corollary 2.11. *If f is bounded on a set $A \subset \mathbb{R}$ (i.e. $\exists L > 0$ s.t. $|f(x)| \leq L,\ \forall x \in A$) then $\phi_n \to f$ uniformly on A.*

Proof. DIY. ■

Corollary 2.12. *$f : X \to \mathbb{C}$, measurable function $\iff$ $\exists$ simple functions $\phi_n : X \to \mathbb{C}$ s.t. $0 \leq |\phi_1| \leq |\phi_2| \leq \ldots \leq |f|$ and ϕ_n converges to f pointwise. (Again, if f is bounded the convergence can be uniform.)*

2.2 Integration of Nonnegative Functions

Definition 2.13. Suppose $(X, \mathcal{A}, \mu)$ a measure space and $\phi = \sum_{i=1}^N c_i 1_{E_i} : X \to [0,\infty]$ a simple function. Let

$$\int \phi = \int \phi \,\mathrm{d}\mu = \int_X \phi \,\mathrm{d}\mu = \sum_1^N c_i \mu(E_i).$$

Proposition 2.14. *Suppose $\phi, \psi \geq 0$ are simple functions. Then,*

- *2.13 is well-defined.*
- $\int c\phi = c \int \phi, c \in [0,\infty)$.
- $\int (\phi + \psi) = \int \phi + \int \psi$.
- $\phi(x) \geq \psi(x),\ \forall x \implies \int \phi \geq \int \psi$.
- $\nu(A) = \int_A \phi \,\mathrm{d}\mu$ *is a measure on* $(X, \mathcal{A})$.

Proof. DIY. ■

Definition 2.15. Suppose $(X, \mathcal{A}, \mu)$, $f : X \to [0,\infty]$ is $\mathcal{A}$-measurable.

Define

$$\int f = \int f \,\mathrm{d}\mu = \sup\left\{ \int \phi \mid 0 \leq \phi \leq f, \phi \text{ simple} \right\}.$$

Proposition 2.16.

- *If f is a simple function then two definitions are the same.*
- $\displaystyle\int cf = c\int f$.
- $\displaystyle f \geq g \geq 0 \implies \int f \geq \int g$.
- $\displaystyle\int f+g = \int f + \int g$. *(A bit harder to check)*

Theorem 2.17 (Monotone convergence theorem)**.** *Suppose $(X, \mathcal{A}, \mu)$ a measure space and*

- $f : X \to [0, \infty]$ *is $\mathcal{A}$-measurable,* $\forall n \in \mathbb{N}$.
- $0 \leq f_1(x) \leq \dots$.
- $\displaystyle\lim_{n\to\infty} f_n(x) = f(x)$.

Then

$$\int f = \lim_{n\to\infty}\int f_n.$$

Proof. Note that $\lim_{n\to\infty} f_n(x)$ converges $\forall x \in X$ and $\lim_{n\to\infty} f_n(x)$ converges.

- $\displaystyle f_n \leq f \implies \int f_n \leq \int f \implies \lim_{n\to\infty}\int f_n \leq \int f$.
- Fix simple function $0 \leq \phi \leq f$. Enough to show that $\displaystyle\lim_{n\to\infty}\int f_n \geq \int \phi$.

  Now fix $\alpha \in (0,1)$. Enough to prove that $\displaystyle\lim_{n\to\infty}\int f_n \geq \alpha\int\phi$.

  Let $A_n = \{x \mid f_n(x) \geq \alpha\phi(x)\}$.

  - $A_n \in \mathcal{A}$.
  - $A_1 \subset A_2 \subset \dots$
  - $\displaystyle\bigcup_{n=1}^{\infty} A_n = X$. (check!)

  So we have

  $$\int f_n \geq \int f_n 1_{A_n} \geq \int \alpha\phi 1_{A_n} = \alpha\nu(A_n)$$

  where $\nu(A) = \int_A \phi$ is a measure.

  $$\implies \lim_{n\to\infty}\int f_n \geq \lim_{n\to\infty}\nu(A_n) = \alpha\nu(x) = \alpha\int\phi.$$

■

Corollary 2.18. $f, g \geq 0$ *measurable* $\displaystyle\implies \int f+g = \int f + \int g$.

Proof. $\exists$ simple functions $0 \le \phi_1 \le \phi_2 \le \dots, \phi_n \to f$ pointwise and $0 \le \psi_1 \le \psi_2 \le \dots, \psi_n \to g$ pointwise.

By MCT, we have

$$\int (f+g) = \lim_{n\to\infty} \int (\phi_n + \psi_n) = \lim_{n\to\infty} \int \phi_n + \int \psi_n = \int f + \int g. \qquad \blacksquare$$

Corollary 2.19 (Tonelli's theorem for series and integrals). *Given $s_n \ge 0, \forall n \in \mathbb{N}$ measurable functions. Then*

$$\int \sum_{n=1}^{\infty} s_n = \sum_{n=1}^{\infty} \int s_n.$$

Proof. Let $f_N = \sum_{n=1}^{N} s_n, 0 \le f_1 \le f_2 \le \dots$.

$$\lim_{N\to\infty} f_N(x) = \sum_{n=1}^{\infty} s_n(x)$$

By MCT, we have

$$\lim_{N\to\infty} \sum_{1}^{N} s_n = \sum_{1}^{\infty} s_n$$

$\blacksquare$

Theorem 2.20 (Fatou's lemma). *Suppose $f_n \ge 0$ measurable. Then*

$$\int \liminf_{n\to\infty} f_n \le \liminf_{n\to\infty} \int f_n.$$

Recall that

$$\liminf_{n\to\infty} f_n := \lim_{k\to\infty} \inf_{n\ge k} f_n = \sup_{k\in\mathbb{N}} \inf_{n\ge k} f_n,$$

and

$$\lim_{n\to\infty} a_n \text{ exists} \iff \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n.$$

Proof. Let $g_k = \inf_{n\ge k} f_n \implies s_k$ measurable and $0 \le g_1 \le g_2 \le \dots$. By MCT, we have

$$\int \liminf_{n\to\infty} = \int \lim_{k\to\infty} s_k = \lim_{k\to\infty} \int s_k = \lim_{k\to\infty} \int \inf_{n\ge k} f_n$$

$$\begin{aligned}
&\inf_{n\geq k} f_n \leq f_m, \forall m \geq k \\
\implies &\int \inf_{n\geq k} f_n \leq \int f_m, \forall m \geq k \\
\implies &\int \inf_{n\geq k} f_n \leq \inf_{m\geq k} \int f_m
\end{aligned}$$

■

Example 2.21. Suppose $(\mathbb{R}, \mathcal{L}, m)$

(a) (escape to horizontal infinity) $f_n = 1_{(n,n+1)}$.

We see that $f_n \to 0 = f$ pointwise and $\int f_n = 1, \forall n, \int f = 0$.

(b) (escape to width infinity) $f_n = \frac{1}{n} 1_{(0,n)}$.

(c) (escape to vertical infinity) $f_n = n 1_{(0,1/n)}$.

Lemma 2.22 (Markov's inequality). *$f \geq 0$ is measurable* $\implies$

$$\forall c \in (0, \infty),\ \mu\left(\{x \mid f(x) \geq c\}\right) \leq \frac{1}{c} \int f.$$

Proof. Let $E = \{x \mid f(x) \geq c\}$. Then

$$f(x) \geq c 1_E(x) \implies \int f \geq c \int 1_E = c\mu(E).$$

■

Proposition 2.23. *Suppose $f \geq 0$ measurable. Then $\int f = 0 \iff f = 0$ almost everywhere (a.e.)*

$$\int f \,\mathrm{d}\mu = \mu(A) = 0,\ A = \{x \mid f(x) > 0\} = f^{-1}((0, \infty])$$

Proof. (a) Assume $f = \phi$ a simple function. We may assume

$$\phi = \sum_{i=1}^{N} c_i 1_{E_i},\ c_i \in (0, \infty),\ E_i\text{'s are disjoint.}$$

$$\begin{aligned}
&\int \phi = \sum_{i=1}^{N} c_i \mu(E_i) = 0 \\
\iff &\mu(E_1) = \ldots = \mu(E_N) = 0 \\
\iff &\mu(A) = 0,\ A = \bigcup_{i=1}^{N} E_i.
\end{aligned}$$

(b) General $f \geq 0$.

(1) Assume $\mu(A) = 0$ (i.e. $f = 0$ a.e.)

Let $0 \leq \phi \leq f, \phi$ is simple.

$$\begin{aligned} &\implies \phi(x) = 0, \ \forall x \in A^c \\ &\implies \phi = 0 \text{ a.e.} \\ &\implies \int \phi = 0 \end{aligned}$$

Then $\int f = 0$ by the definition of $\int f$.

(2) Assume $\inf f = 0$. Let $A_n = f^{-1}\left(\left[\frac{1}{n}, \infty\right]\right)$

$$\begin{aligned} &\implies A_1 \subset A_2 \subset \dots \\ &\quad \bigcup_1^\infty A_n = f^{-1}\left(\bigcup_1^\infty \left[\frac{1}{n}, \infty\right]\right) = f^{-1}((0,\infty)) = A \\ &\quad \mu(A_n) = \mu\left(\left\{x \mid f(x) \geq \frac{1}{n}\right\}\right) \leq n \int f = 0 \\ &\implies \mu(A) = \lim_{n\to\infty} \mu(A_n) = 0 \end{aligned}$$

by the continuity of measure from below. ■

Corollary 2.24. *$f, g \geq 0$ are measurable, $f = g$ a.e. $\implies \int f = \int g$.*

Proof. Let $A = \{x \mid f(x) \geq g(x)\}$. A is measurable (why?). By assumption $\mu(A) = 0$. Hence $f1_A = 0$ a.e.

$$\begin{aligned} \int f &= \int f(1_A + 1_{A^c}) \\ &= \int f1_A + \int f1_{A^c} \\ &= \int f1_{A^c} \\ &= \int g1_{A^c} = \int g1_A + \int g1_{A^c} = \int g. \end{aligned}$$ ■

Corollary 2.25. *$f_n \geq 0$ measurable. Then*

(a)

$$\left.\begin{aligned} 0 \leq f_1 \leq f_2 \leq \dots \leq f \text{ a.e.} \\ \lim_{n\to\infty} f_n = f \text{ a.e.} \end{aligned}\right\} \implies \lim_{n\to\infty} f_n = \int f.$$

(b)

$$\lim_{n\to\infty} f_n = f \ a.e \implies \int f \le \liminf_{n\to\infty} \int f_n.$$

2.3 Integration of Complex Functions

I was afraid that you are bored.

— Jinho Baik on homework

Definition 2.26. $(X, \mathcal{A}, \mu)$ measure space.

- $f : X \to \overline{\mathbb{R}}$ or $f : X \to \mathbb{C}$ measurable functions is called *integrable* if $\int |f| < \infty$. Then

$$\int f = \int f^+ - \int f^- \text{ or } \int f = \int u^+ - \int u^- + i\left(\int v^+ - \int v^-\right).$$

- Suppose $f : X \to \overline{\mathbb{R}}$. Define

$$\int f = \begin{cases} \infty & \int f^+ = \infty, \int f^- < \infty, \\ -\infty & \int f^+ < \infty, \int f^- = \infty. \end{cases}$$

Lemma 2.27. *Suppose $f, g : x \to \overline{\mathbb{R}} \to \mathbb{C}$ integrable. Assume $f(x) + g(x)$ is well-defined $\forall x \in X$. (i.e. $\infty + (-\infty)$, $-\infty + \infty$ do not occur)*

(a) *$f + g$, cf, $c \in \mathbb{C}$ are integrable.*

(b) $\int f + g = \int f + \int g$.

(c) $\left|\int f\right| \le \int |f|$. *(This is essentially triangle inequality.)*

Proof. Check [Fol99, p.53]. ■

Lemma 2.28. *$(X, \mathcal{A}, \mu)$ measure space and f* integrable *function on X.*

(a) *f is finite a.e. (i.e. $\{x \in X : |f(x)| = \infty\}$ is a null set)*

(b) *The set $\{x \in X : f(x) \neq 0\}$ is σ-finite.*

Proof. HW5Q8. ■

Proposition 2.29. *Suppose $(X, \mathcal{A}, \mu)$ a measure space.*

(a) If h is integrable on X, then

$$\int_E h = 0, \forall E \in \mathcal{A} \iff \int |h| = 0 \iff h = 0 \text{ a.e.}$$

(b) If f, g are integrable on X then

$$\int_E f = \int_E g, \forall E \in \mathcal{A} \iff f = g \text{ a.e.}$$

Proof. (a) $\int |h| = 0 \iff h = 0$ is shown in 2.23.

$$\int |h| = 0 \implies \left| \int_E h \right| \leq \int_E |h| \leq \int |h| = 0.$$

On the other hand, assume $\int_E h = 0, \forall E \in \mathcal{A}$. $h = u + iv = u^+ - u^- + i(v^+ - v^-)$. Let $B = \{x \mid u^+(x) > 0\}$.

$$0 = \operatorname{Re} \int_B h = \int_B u = \int_B u^+ = \int_B u^+ + \int_{B^c} u^+ = \int u^+ \implies u^+ = 0 \text{ a.e.}$$

Similarly, we get $u^-, v^+, v^- = 0$ a.e..

(b) follows from (a). ■

Theorem 2.30 (Dominated convergence theorem)**.** *Suppose $(X, \mathcal{A}, \mu)$ a measure space and*

(a) f_n integrable *on X, $\forall n \in \mathbb{N}$.*

(b) $\lim_{n\to\infty} f_n(x) = f(x)$ *a.e. (pointwise)*

(c) $\exists g : X \to [0, \infty]$ *s.t.*

- *g is integrable.*
- *$|f_n(x)| \leq g(x)$ a.e., $\forall n \in \mathbb{N}$.*

Then

$$\lim_{n\to\infty} \int f_n = \int f.$$

Proof. Let F be the countable union of null sets on which (a)-(c) may fail. Modifying the def of f_n, f, g on F we may assume (a)-(c) hold everywhere. (b)+(c) $\implies$ f is integrable.

We consider $\overline{\mathbb{R}}$-valued case only. ($\mathbb{C}$-valued case follows)

$$\begin{aligned}
& g + f_n \geq 0, g - f_n \geq 0 \\
\xRightarrow{\text{Fatou}} & \int g + f \leq \liminf_{n\to\inf} \int g + f_n, \quad \int g - f \leq \liminf_{n\to\inf} \int g - f_n \\
\implies & \int g + \int f \leq \int g + \liminf_{n\to\infty} \int f_n, \quad \int g - \int f \leq \int g - \limsup_{n\to\infty} \int f_n \\
\xRightarrow{\int g<\infty} & \int f \leq \liminf_{n\to\infty} \int f_n, \quad -\int f \leq -\limsup_{n\to\infty} \int f_n. \\
\implies & \int f \leq \liminf_{n\to\infty} \int f_n \leq \limsup_{n\to\infty} \int f_n \leq \int f
\end{aligned}$$

So we should have
$$\int f = \liminf_{n\to\infty} \int f_n = \limsup_{n\to\infty} \int f_n.$$
■

Next we investigate the question:

$$\int \sum_1^\infty f_n \stackrel{?}{=} \sum_1^\infty \int f_n.$$

Tonelli: yes if $f_n \geq 0$. Fubini:

Corollary 2.31 (Fubini's theorem for series and integrals)**.**

$$\left.\begin{array}{c} f_n \textit{ integrable} \\ \sum_1^\infty \int |f_n| < \infty \end{array}\right\} \implies \int \sum_1^\infty f_n = \sum_1^\infty \int f_n.$$

Proof. $G(x) = \sum_1^\infty |f_n(x)| \geq |F_N(x)|, F_N(x) = \sum_1^N f_n(x).$ ■

2.4 $\boldsymbol{L^1}$ space

Definition 2.32. Suppose V is a vector space over field $\mathbb{R}$ or $\mathbb{C}$. A *seminorm* on V is $\|\cdot\| : V \to [0,\infty)$ *s.t.*

- $\|cv\| = |c|\|v\|, \forall v \in V, \forall c$ scalar
- $\|v + w\| \leq \|v\| + \|w\|$, triangle inequality

A *norm* is a seminorm such that $\|v\| \iff v = 0$.

Lemma 2.33. *A normed vector space is a metric space with metric $\rho(v, w) = \|v - w\|$.*

Proof. (DIY)

- $\rho(v, w) = 0 \iff \|v - w\| = 0 \iff v - w = 0 \iff v = w$.
- $\rho(v, w) = \|v - w\| = \|{-1}(w - v)\| = |-1| \, \|w - v\| = \rho(w, v)$.
- $\rho(v, w) + \rho(w, z) = \|v - w\| + \|w - z\| \geq \|v - w + w - z\| = \|v - z\| = \rho(v, z)$. ■

Example 2.34. $\mathbb{R}^d$ with $\|x\|_p = \begin{cases} \left(\sum_1^d |x_i|^p \right)^{1/p} & p \in [1, \infty) \\ \max\limits_{1 \leq i \leq d} |x_i| & p = \infty \end{cases}$ is a normed vector space.

Unit ball $\{x : \|x\|_p < 1\}$.

All $\|\cdot\|_p$ norm induce the same topology i.e. if U is open in p-norm then it is open in p'-norm. This implies that a sequence converging under p-norm also converges under p'-norm.

<u>Recall</u> f is integrable $\implies \int |f| < \infty$. $f = g$ a.e. $\implies \int f = \int g$.

Definition 2.35. Suppose $(X, \mathcal{A}, \mu)$ a measure space.
$f \in L^1(X, \mathcal{A}, \mu) = L^1(X, \mu) = L^1(X) = L^1(\mu)$ means f is an integrable function on X.

Lemma 2.36. *$L^1(X, \mathcal{A}, \mu)$ is a vector space with seminorm* $\|f\|_1 = \displaystyle\int |f|$.

Definition 2.37. Define $f \sim g$ if $f = g$ a.e. $L^1(X, \mathcal{A}, \mu)/_\sim = L^1(X, \mathcal{A}, \mu)$. " $=$ " is just a notation for convenience!

With new definition we have $L^1(X, \mathcal{A}, \mu)$ is a normed vector space. $\rho(f, g) = \displaystyle\int |f - g|$.

Something interesting to discuss is what are the dense subsets of L^1.

Theorem 2.38.

(a) $\{$ integrable simple functions $\}$ is dense in $L^1(X, \mathcal{A}, \mu)$ (with respect to L^1 metric)

(b) $(X, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{A}_\mu, \mu)$, μ is Lebesgue-Stieltjes measure $\implies$ $\{$ integrable step functions $\}$ is dense in $L^1(X, \mathcal{A}, \mu)$

(c) $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R}, \mathcal{L}, m)$.

Definition 2.39.

- A step function on $\mathbb{R}$ is $\psi + \sum_1^N c_i 1_{I_i}$, where I_i is an interval.
- $C_c(\mathbb{R})$ is the collection of continuous functions with compact support $\operatorname{supp}(f) = \overline{\{x \in \mathbb{R} \mid f(x) \neq 0\}}$.

Proof. (a) $\exists$ simple functions $0 \leq |\phi_1| \leq |\phi_2| \leq \ldots \leq |f|$, $\phi_n \to f$ pointwise $\implies$

$\lim\limits_{n\to\infty} \displaystyle\int |\phi_n - f| = 0$ by DCT. ($|\phi_n - f| \le |\phi_n| + |f| \le 2|f|$)

(b) 1_E approx by $\sum_1^N c_i 1_{I_i}$? Regularity theorem for Lebesgue-Stieltjes measure $\implies$ $\forall \varepsilon' > 0, \exists I = \bigcup_1^N I_i \; s.t. \; \mu(E \triangle I) < \varepsilon'$.

(c) Suppose $1_{(a,b)}, g \in C_c(\mathbb{R})$. $\displaystyle\int |1_{(a,b)} - g| \, \mathrm{d}m \le 1 \cdot \frac{\varepsilon}{2} + 1 \cdot \frac{\varepsilon}{2} = \varepsilon$. ■

2.5 Riemann Integrability

Suppose $P = \{a = t_0 < t_1 < \ldots < t_k = b\}$ a partition of $[a, b]$. Lower Riemann sum of f using P

$$L_P = \sum_{i=1}^{k} \left(\inf_{[t_{i-1}, t_i]} f \right) (t_i - t_{i-1})$$

and upper Riemann sum

$$U_p = \sum_{i=1}^{k} \left(\sup_{[t_{i-1}, t_i]} f \right) (t_i - t_{i-1})$$

Lower Riemann integral of $f = \underline{I} = \sup_P L_P$. Upper Riemann integral of $f = \overline{I} = \inf_P U_P$.

Definition 2.40. A *bounded* function $f : [a, b] \to \mathbb{R}$ is called Riemann (Darboux) integrable if $\underline{I} = \overline{I}$. (If so, $\underline{I} = \overline{I} = \int_a^b f(x) \, \mathrm{d}x$.)

<u>NOTE</u>

- If $P \subset P'$, then $L_P \le L_{P'}, U_{P'} \le U_P$.
- Recall that continuous functions on $[a, b]$ are Riemann integrable on $[a, b]$.

Theorem 2.41. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function.*

(a) If f is Riemann integrable, then f is Lebesgue measurable. (thus Lebesgue integrable) and $\displaystyle\int_a^b f(x) \, \mathrm{d}x = \int_{[a,b]} f \, \mathrm{d}m$.

(b) f is Riemann integrable $\iff$ f is continuous Lebesgue a.e.

Proof. $\exists$partitions $P_1 \subset P_2 \subset P_3 \subset \ldots \; s.t. \; L_{P_n} \nearrow \underline{I}, U_{P_n} \searrow \overline{I}$.

Define simple (step) functions

$$\phi_n = \sum_{i=1}^{k} \left(\inf_{[t_{i-1}, t_i]} \right) 1_{(t_{i-1}, t_i]}$$

$$\psi_n = \sum_{i=1}^{k} \left(\sup_{[t_{i-1},t_i]} \right) 1_{(t_{i-1},t_i]}$$

Define $\phi = \sup_n \phi_n$, $\psi = \inf_n \psi_n$. Then ϕ, ψ are Lebesgue measurable functions.

<u>NOTE</u>

- $\exists M > 0\ s.t.\ |\phi_n|, |\psi_n| \leq M1[a,b], \forall n \in \mathbb{N}$.
- $\int \phi_n \,\mathrm{d}m = L_{P_n}, \int \psi_n \,\mathrm{d}m = U_{P_n}$.

By DCT, $\underline{I} = \lim\limits_{n\to\infty} \displaystyle\int \phi_n \,\mathrm{d}m = \int \phi \,\mathrm{d}m, \overline{I} = \int \psi \,\mathrm{d}m$.

Thus, f is Riemann integrable $\iff \int \phi = \int \psi \iff \int (\phi - \psi) = 0 \iff \phi = \psi$ Lebesgue a.e.

Recall that $\phi \leq f \leq \psi, \forall x \in (a,b]$. So $f = \phi$ a.e. Since $(\mathbb{R}, \mathcal{L}, \mu)$ is complete, f is Lebesgue measurable (see HW). The second statement hence follows. ■

2.6 Modes of Convergence

Suppose $f_n, f : X \to \mathbb{C}, S \subset X$.

- $f_n \to f$ *pointwise* on S: $\forall x \in S, \forall \varepsilon > 0, \exists N \in \mathbb{N}\ s.t.\ \forall n \geq \mathbb{N}, |f_n(x) - f(x)| < \varepsilon$.
- $f_n \to f$ *uniformly* on S: $\forall \varepsilon > 0, \exists N \in \mathbb{N}\ s.t.\ \forall x \in X, \forall n \geq \mathbb{N}, |f_n(x) - f(x)| < \varepsilon$.

We can change $\forall \varepsilon > 0$ to $\forall k \in \mathbb{N}$ and bound the distance by $\frac{1}{k}$.

Lemma 2.42. *Let $B_{n,k} = \{x \in X \mid |f_n(x) - f(x)| < \frac{1}{k}\}$.*

(a) $f_n \to f$ pointwise on $S \iff S \subset \displaystyle\bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} B_{n,k}$.

(b) $f_n \to f$ uniformly on $S \iff \exists N_1, N_2, \ldots \in \mathbb{N}\ s.t.\ S \subset \displaystyle\bigcap_{k=1}^{\infty} \bigcap_{n=N_k}^{\infty} B_{n,k}$.

Definition 2.43. Suppose $(X, \mathcal{A}, \mu)$ a measure space.

(a) $f_n \to f$ a.e means $\exists$ null set $E\ s.t.\ f_n \to f$ pointwise on E^c.

(b) $f_n \to f$ in L^1 means $\lim\limits_{n\to\infty} \|f_n - f\| = 0$.

Example 2.44. $(\mathbb{R}, \mathcal{L}, \mu)$. $f = 0$.

(a) $f_n = 1_{(n,n+1)}, f_n = \frac{1}{n}1_{(0,n)}, f_n = n1_{\left(0,\frac{1}{n}\right)}$. All of $f_n \to f$ pointwise but $\not\to f$ in L^1.

(b) Typewriter functions: $f_n \to f$ in L^1. $f_n \not\to f$ a.e.

Proposition 2.45 (Fast L^1 convergence $\implies$ a.e. convergence)**.** *Suppose $(x, \mathcal{A}, \mu)$ measure* *space. f_n, f measurable function on X.*

$$\sum_1^\infty \|f_n - f\|_1 < \infty \implies f_n \to f \text{ a.e.}$$

Proof. RECALL Markov's inequality.

Let $E = \bigcup_{k=1}^\infty \bigcap_{N=1}^\infty \bigcup_{n=N}^\infty B^c_{n,k} = \{x \mid f_n(x) \not\to f(x)\}$. By Markov we have

$$\begin{aligned}
&\forall k, \forall N, \mu(B^c_{n,k}) \leq k \int |f_n - f| \\
&\implies \forall k, \mu\left(\bigcap_{n=N}^\infty B^c_{n,k}\right) \leq \sum_{n=N}^\infty k \|f_n - f\|_1 \to 0 \text{ as } n \to 0 \\
&\implies \forall k, \mu\left(\bigcap_{N=1}^\infty \bigcap_{n=N}^\infty B^c_{n,k}\right) = \lim_{N\to\infty} \mu\left(\bigcap_{n=N}^\infty B^c_{n,k}\right) = 0 \\
&\implies \mu(E) = 0.
\end{aligned}$$

■

Corollary 2.46. *$f_n \to f$ in $L^1 \implies \exists$subsequence $f_{n_j} \to f$ a.e.*

Proof. $\forall j \in \mathbb{N}, \exists n_j \in \mathbb{N}$ *s.t.* $\left\|f_{n_j} - f\right\|_1 < \frac{1}{j^2}$. Then $\sum_{j=1}^\infty \left\|f_{n_j} - f\right\|_1 < \infty$. ■

Definition 2.47. f_n, f measurable functions on $(X, \mathcal{A}, \mu)$. $f_n \to f$ *in measure* means

$$\forall \varepsilon > 0, \lim_{n\to\infty} \mu\left(\{x \in X \mid |f_n(x) - f(x)| \geq \varepsilon\}\right) = 0.$$

Example 2.48.

- $f_n = n1_{\left(0, \frac{1}{n}\right)}, f = 0$.

$$\forall \varepsilon > 0, \{x \mid |f_n(x) - f(x)| > \varepsilon\} = \left(0, \frac{1}{n}\right).$$

(Recall that $f_n \not\to 0$ in L^1.)

- Typewriter function. (Recall that $f_n \not\to 0$ a.e.)

We can easily check that $f_n \to f$ in $L^1 \implies f_n \to f$ in measurable. But the converse is not true.

$f_n \to f$ in measure $\implies \exists f_{n_j} \to f$ a.e. (Check [Fol99])

We have then the following diagram:

$$\begin{tikzcd}
f_n \to f \text{ fast } L^1 \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & f_n \to f \text{ in } L^1 \arrow[r, shift left, Rightarrow] \arrow[d, shift right, Rightarrow, "/" marking] & f_n \to f \text{ in measure} \arrow[l, shift left, Rightarrow, "/" marking] \arrow[d, Rightarrow] \\
& f_n \to f \text{ a.e.} \arrow[u, shift right, Rightarrow, "/" marking] & \exists f_{n_j} \to f \text{ a.e.}
\end{tikzcd}$$

Definition 2.49. f_n, f measurable functions on $(X, \mathcal{A}, \mu)$.

(a) $f_n \to f$ *uniformly a.e* means $\exists$ null set F $s.t.$ $f_n \to f$ uniformly on F^c.

(b) $f_n \to f$ *almost uniformly* means $\forall \varepsilon > 0, \exists F \in \mathcal{A},\ s.t.\ \mu(F) < \varepsilon, f_n \to f$ uniformly on F^c.

Recall 2.42.

Theorem 2.50 (Egoroff)**.** *f_n, f measurable on $(X, \mathcal{A}, \mu)$. Suppose $\mu(X) < \infty$. Then $f_n \to f$ a.e $\iff$ $f_n \to f$ almost uniformly.*

Proof. " $\Longleftarrow$ ": DIY

" $\Longrightarrow$ ": Fix $\varepsilon > 0$.

$f_n \to f$ a.e $\implies \mu\left(\bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c\right) = 0 \implies \forall k, \mu\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c\right) = 0.$

By the continuity of measure from above and since $\mu(X) < \infty$,

$$\forall k, \lim_{N \to \infty} \mu\left(\bigcup_{n=N}^{\infty} B_{n,k}^c\right) = 0 \implies \forall k, \exists N_k \in \mathbb{N}, \mu\left(\bigcup_{n=N_k}^{\infty} B_{n,k}^c\right) < \frac{\varepsilon}{2^k}.$$

Let $F = \bigcup_{k=1}^{\infty} \bigcup_{n=N_k}^{\infty} B_{n,k}^c \implies \mu(F) < \varepsilon, f_n \to F$ uniformly on F^c. ■

Chapter 3

Product Measures

(p.22 - 36, section 1.2 and section 2.5, 2.6 of [Fol99])
The ultimate goal is to prove Fubini's theorem. This is also related to probability in in the sense that a series of events is in product measure.

3.1 Product σ-algebra

- Product space $X = \prod_{\alpha \in I} X_\alpha, x = (x_\alpha)_{\alpha \in I}$.
- Coordinate map $\pi_\alpha : X \to X_\alpha$.

Definition 3.1. $(X_\alpha, \mathcal{A}_\alpha)$ measurable space. $\forall \alpha \in I$, the *product σ-algebra* on $X = \prod\limits_{\alpha \in I} X_\alpha$ is

$$\bigotimes_{\alpha \in I} \mathcal{A}_\alpha = \left\langle \bigcup_{\alpha \in I} \pi_\alpha^{-1}\left(\mathcal{A}_\alpha\right) \right\rangle$$

where

$$\pi_\alpha^{-1}\left(A_\alpha\right) = \{\pi_\alpha^{-1}(E) | E \in \mathcal{A}_\alpha\}.$$

<u>Notation</u>

$$I = \{1, \ldots, d\} \implies X = \prod_{i=1}^{d} X_i, x = (x_1, \ldots, x_d), \bigotimes_{i=1}^{d} \mathcal{A}_i = \mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_d.$$

Lemma 3.2. *If I is countable, then*

$$\bigotimes_{\alpha \in I} \mathcal{A}_\alpha = \left\langle \left\{ \prod_{i=1}^{\infty} E_i \mid E_i \in \mathcal{A}_i \right\} \right\rangle$$

Lemma 3.3. *Suppose $\mathcal{A}_\alpha = \langle \mathcal{E}_\alpha \rangle, \forall \alpha \in I$.*

(a) $\pi_\alpha^{-1}(\mathcal{A}_\alpha) = \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle$.

(b) $\displaystyle\bigotimes_{\alpha} \mathcal{A}_\alpha = \left\langle \bigcup_{\alpha} \pi_\alpha^{-1}(\mathcal{E}_\alpha) \right\rangle$.

(c) If I is countable, then $\displaystyle\bigotimes_{\alpha \in I} \mathcal{A}_\alpha = \left\langle \left\{ \prod_{i=1}^{\infty} E_i \mid E_i \in \mathcal{E}_i \right\} \right\rangle$.

Proof.

(a)
- $f : Y \to Z$, $\mathcal{B}$ a σ-algebra on $Z \implies f^{-1}(\mathcal{B})$ is a σ-algebra since set union commutes with preimage. Hence $\pi_\alpha^{-1}(\mathcal{A}_\alpha)$ is a σ-algebra on X. Since $\pi_\alpha^{-1}(\mathcal{E}_\alpha) \subset \pi_\alpha^{-1}(\mathcal{A}_\alpha) \implies \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle \subset \pi_\alpha^{-1}(\mathcal{A}_\alpha)$.
- Let $\mathcal{M} = \{B \subset X_\alpha \mid \pi_\alpha^{-1}(B) \in \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle\}$. We show that $\mathcal{A}_\alpha \subset \mathcal{M}$.
  - $\mathcal{M}$ is a σ-algebra. (easy)
  - $\mathcal{E}_\alpha \subset \mathcal{M}$. (by definition)

  So $\mathcal{A}_\alpha = \langle \mathcal{E}_\alpha \rangle \subset \mathcal{M}$. Hence, if $E \in \mathcal{A}_\alpha$, $E \subset \mathcal{M} \implies \pi_\alpha^{-1}(E) \in \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle$ i.e. $\mathcal{A}_\alpha \subset \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle$.

(b, c) DIY. ∎

Theorem 3.4. *Suppose $X_1, \ldots, X_d$ metric spaces. Let $X = \prod_1^d X_i$ with product metric $\rho(x, y) = \sum_{i=1}^d \rho_i(x, y)$. Then*

(a) $\displaystyle\bigotimes_{i=1}^d \mathcal{B}(X_i) \subset \mathcal{B}(X)$.

(b) If, in addition, each X_i has a countable dense subset, then $\displaystyle\bigotimes_{i=1}^d \mathcal{B}(X_i) = \mathcal{B}(X)$.

Proof. DIY. ∎

As a consequence, we have $\mathcal{B}(\mathbb{R}^d) = \mathcal{B}(\mathbb{R}) \otimes \ldots \otimes \mathcal{B}(\mathbb{R})$.

Suppose $f = u + iv : X \to \mathbb{C}$. f is measurable $\iff$ $u^{-1}(E) \in \mathcal{A}, v^{-1}(E) \in \mathcal{A}, \forall E \in \mathcal{B}(\mathbb{R}) \iff f^{-1}(F) \in \mathcal{A}, \forall F \in \mathcal{B}(\mathbb{C}) = \mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

p.65. Let's focus on finite product.

You like Minecraft right? It's all rectangles.

Definition 3.5. Suppose X, Y sets.

(a) For a $E \subset X \times Y$, $E_x = \{y \in Y \mid (x, y) \in E\}$ and $E^y = \{x \in X \mid (x, y) \in E\}$.

(b) For $f : X \times Y \to \mathbb{C}$, define $f_x : Y \to \mathbb{C}, f^y : X \to \mathbb{C}$ by $f_x(y) = f(x, y) = f^y(x)$.

(c)

Example 3.6. $(1_E)_x = 1_{E_x}$. $(1_E)^y = 1_{E^y}$.

Proposition 3.7. $(X, \mathcal{A}), (Y, \mathcal{B})$ *measurable spaces.*

(a) $E \in \mathcal{A} \otimes \mathcal{B} \implies E_x \in \mathcal{B}, E^y \in \mathcal{A}, \forall x \in X, y \in Y$.

(b) $f : X \times Y \to \mathbb{C}$ *is* $\mathcal{A} \otimes \mathcal{B}$*-measurable* $\implies$ f_x *is* $\mathcal{B}$*-measurable,* f^y *is* $\mathcal{A}$*-measurable,* $\forall x \in X, y \in Y$.

Proof. (a) Let $\mathcal{F} = \{E \subset X \times Y \mid \text{(a) holds}\}$.

- $\mathcal{F}$ is a σ-algebra (easy)
- $\mathcal{R}_0 := \{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\} \subset \mathcal{F}$ (easy) $\implies \mathcal{A} \otimes \mathcal{B} = \langle \mathcal{R}_0 \rangle \subset \mathcal{F}$

(b) DIY. ∎

MIDTERM is up till here.

3.2 Product Measures

Definition 3.8. Suppose $(X, \mathcal{A}), (Y, \mathcal{B})$. A (measurable) rectangle is $R = A \times B, A \in \mathcal{A}, b \in \mathcal{B}$.

Let $\mathcal{R}_0 := \{R = A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$.

$$\mathcal{R} := \left\{ \bigcup_1^N R_i \mid N \in \mathbb{N}, R_1, \dots, R_N \text{ disjoint rectangles} \right\}.$$

Lemma 3.9. *$\mathcal{R}$ is an algebra.* $\langle \mathcal{R}_0 \rangle = \langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B}$.

Theorem 3.10. *Suppose* $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ *measure spaces.*

(a) $\exists$ measure $\mu \times \nu$ on $\mathcal{A} \otimes \mathcal{B}$ satisfying $(\mu \times \nu)(A \otimes B) = \mu(A)\nu(B), \forall A \in \mathcal{A}, B \in \mathcal{B}$.

(b) If μ, ν are σ-finite, then $\mu \times \nu$ is unique.

Proof. (a) Define $\pi : \mathcal{R} \to [0, \infty]$ by $\pi(A \times B) = \mu(A)\nu(B)$ and extend linearly.

CLAIM π is a pre-measure on $\mathcal{R}$.

Enough to check $\pi(A \times B) = \sum_1^\infty \pi(A_n \times B_n)$ if $A \times B = \bigcup_1^\infty (A_n \times B_n)$ disjoint union.

Since $A_n \times B_n$ are disjoint,

$$1_{A \times B}(x, y) = \sum_1^\infty 1_{A_n \times B_n}(x, y),\ 1_A(x)1_B(y) = \sum_1^\infty 1_{A_n}(x)1_{B_n}(y).$$

By Tonelli's theorem for series and integrals, we have

$$\begin{aligned} \mu(A)1_B(y) &= \int_x 1_A(x)1_B(y)\ \mathrm{d}\mu(x) \\ &= \sum_1^\infty \int_x 1_{A_n}(x)1_{B_n}(y)\ \mathrm{d}\mu(x) = \sum_1^\infty \mu(A_n)1_{B_n}(y). \end{aligned}$$

We then integrate with respect to y to complete the claim.

By HK theorem, $\exists \mu \otimes \nu$ on $\langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B}$ extending π on $\mathcal{R}$.

(b) μ, ν σ-finite $\implies$ π is σ-finite on $\mathcal{R}$ $\implies$ HK uniqueness them applies. ∎

So we have a measure

$$(\mu \times \nu)(E) = \inf \left\{ \sum_1^\infty \mu(A_u)\nu(B_i) \middle| E \subset \bigcup_1^\infty A_i \times B_i, A_i \in \mathcal{A}, B_i \in \mathcal{B} \right\}.$$

Then one questions naturally arises: suppose $f : X \times Y \to \mathbb{C}$,

$$\int_{X \times Y} f\ \mathrm{d}(\mu \times v) \stackrel{?}{=} \int_y \left(\int_x f\ \mathrm{d}\mu \right)\ \mathrm{d}\nu.$$

3.3 Monotone Class Lemma

Definition 3.11. Suppose X is a set, $\mathcal{C} \subset \mathcal{P}(X)$. $\mathcal{C}$ is a *monotone class* on X if

- closed under *countable increasing unions*
  (i.e. $E_n \in \mathcal{C}, E_1 \subset E_2 \subset \dots \implies \bigcup_1^\infty C_i \in \mathcal{C}$.)

- closed under *countable decreasing intersections*
  (i.e. $E_n \in \mathcal{C}, E_1 \supset E_2 \supset \dots \implies \bigcap_1^\infty C_i \in \mathcal{C}$.)

Example 3.12.

- σ-algebra is a monotone class.
- $\bigcap\limits_\alpha \mathcal{C}_\alpha$ is a monotone class $\implies$ if $\mathcal{E} \in \mathcal{P}(X)$, there is unique smallest monotone class containing $\mathcal{E}$.

The importance of this definition shows up in the following theorem:

Theorem 3.13. *Suppose $\mathcal{A}_0$ is an* algebra *on X. Then $\langle \mathcal{A}_0 \rangle$ is the monotone class generated by $\mathcal{A}_0$.*

Proof. Let $\mathcal{A} = \langle \mathcal{A}_0 \rangle$, $\mathcal{C} =$ monotone class generated by $\mathcal{A}_0$.

(a) $\mathcal{A}$ is a σ-algebra $\implies$ $\mathcal{A}$ is a monotone class containing $\mathcal{A}_0 \implies \mathcal{A} \supset \mathcal{C}$.

(b) To show that $\mathcal{C} \supset \mathcal{A}$, we show that $\mathcal{C}$ is a σ-algebra.

(1) $\emptyset \subset \mathcal{A}_0 \subset \mathcal{C}$.

(2) Let $\mathcal{C}' = \{E \subset X \mid E^c \subset \mathcal{C}\}$.

- $\mathcal{C}'$ is a monotone class (easy)
- $\mathcal{A}_0 \subset \mathcal{C}'$ since $(E \in \mathcal{A}_0 \implies E^c \in \mathcal{A}_0 \subset \mathcal{C})$.

These two show that $\mathcal{C} \subset \mathcal{C}'$. So $E \in \mathcal{C} \implies E \in \mathcal{C}' \implies E^c \in \mathcal{C}$. So $\mathcal{C}$ is closed under complements.

(3) For $E \subset X$, let $\mathcal{D}(E) = \{F \in \mathcal{C} \mid E \cup F \in \mathcal{C}\}$.

- $\mathcal{D}(E) \subset \mathcal{C}$ by definition.
- $\mathcal{D}(E)$ is a monotone class (easy). $E \cup (\bigcup_1^\infty F_n) = \bigcap_1^\infty (E \cup F_n)$.
- If $E \in \mathcal{A}_0$, then $\mathcal{A}_0 \subset \mathcal{D}(E)$. ($F \in \mathcal{A}_0 \implies E \cup F \in \mathcal{A}_0 \subset \mathcal{C}$.)

These show that $\mathcal{C} = \mathcal{D}(E)$ if $E \in \mathcal{A}_0$.

(4) Let $\mathcal{D} = \{E \in \mathcal{C} \mid \mathcal{D}(E) = \mathcal{C}\} = \{E \in \mathcal{C} \mid E \cup F \in \mathcal{C}, \forall F \in \mathcal{C}\}$.

- $\mathcal{A}_0 \subset \mathcal{D}$ by (3).
- $\mathcal{D}$ is a monotone class (easy).
- $\mathcal{D} \subset \mathcal{C}$ by definition.

So we conclude that $\mathcal{D} = \mathcal{C}$. Now we have $\mathcal{C}$ is closed under finite unions.

(5) $\mathcal{C}$ is closed under finite unions and countable increasing unions $\implies$ $\mathcal{C}$ is closed under countable unions. (check) ■

<u>RECALL</u> $E \in \mathcal{A} \otimes \mathcal{B} \implies E_x \in \mathcal{B}, E^y \in \mathcal{A}, \forall x \in X, y \in Y$. However, the inverse is not necessarily true.

Now comes the main thing:

3.4 Fubini-Tonelli Theorem

Theorem 3.14 (Tonelli for characteristic functions)**.** *Suppose $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ are σ-finite measure spaces. Suppose $E \in \mathcal{A} \otimes \mathcal{B}$. Then*

(a) $\alpha(x) := \nu(E_x) : X \to [0, \infty]$ *is a $\mathcal{A}$-measurable function.*

(b) $\beta(y) := \mu(E^y) : Y \to [0, \infty]$ *is a $\mathcal{B}$-measurable function.*

(c) $(\mu \times \nu)(E) = \displaystyle\int_X \nu(E_x) \,\mathrm{d}\mu(x) = \int_Y \mu\left(E^y\right) \,\mathrm{d}\nu(y)$.

Proof. (a) Assume μ, ν are finite measures. Let

$$\mathcal{C} = \{E \in \mathcal{A} \otimes \mathcal{B} \mid \text{(a), (b), (c) hold}\}.$$

Enough to prove that $\langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B} \subset \mathcal{C}$.

Because of monotone class lemma and that $\mathcal{R}$ is a σ-algebra, it is enough to show that $\mathcal{R} \subset \mathcal{C}$ and $\mathcal{C}$ is a monotone class.

- Show that $\mathcal{R} \subset \mathcal{C}$.

$$\alpha(x) = \nu((A \times B)_x) = \begin{cases} \nu(B) & x \in A \\ 0 & x \notin A \end{cases} = \nu(B) 1_A(x).$$

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$$
$$\iff \int_X \nu((A \times B)_x) \,\mathrm{d}\mu(x) = \nu(B)\mu(A)$$

- Show that $\mathcal{C}$ is a monotone class.

(1) Let $E_n \in \mathcal{C}, E_1 \subset E_2 \subset \dots$. Need to show that $E = \bigcup_1^\infty E_n \in \mathcal{C}$.

$$\begin{aligned} & E_n \in \mathcal{C}, E_1 \subset E_2 \subset \dots \\ \implies & E_x = \bigcup_1^\infty (E_n)_x, (E_1)_x \subset (E_2)_x \subset \dots \\ \implies & \alpha(x) = \nu(E_x) = \lim_{n\to\infty} \nu\left((E_n)_x\right), \forall x \in X, \quad \alpha_n(x)\ \mathcal{A}\text{-measurable} \end{aligned}$$

This satisfies (a), (b). For (c), we have

$$\begin{aligned} (\mu\times\nu)(E) &= \lim_{n\to\infty}(\mu\times\nu)(E_n) \\ &= \lim_{n\to\infty}\int_X \nu\left((E_n)_x\right)\,\mathrm{d}\mu(x) \overset{MCT}{=} \int_X \nu(E_x)\,\mathrm{d}\mu(X). \end{aligned}$$

So we have shown countable increasing unions.

(2) Let $F_n \in \mathcal{C}$, $F_1 \supset F_2 \supset \dots$. Need to show that $F \bigcup_1^\infty F_n \in \mathcal{C}$. Using continuity of measure from above instead of below, DCT instead of MCT, we obtained a similar result.

(b) Now assume that μ, ν are σ-finite. Since $X \times Y = \bigcup_1^\infty (X_n \times Y_n)$, where $X_1 \subset X_2 \dots, Y_1 \subset Y_2 \subset \dots$ with $\mu(X_k), \nu(Y_k)$ finite. Apply results from then finite case. (DIY) ■

Theorem 3.15 (Fubini-Tonelli)**.** *Suppose $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are σ-finite measure spaces.*

(a) (Tonelli) If $f : X \times Y \to [0,\infty]$ is $\mathcal{A}\otimes\mathcal{B}$-measurable then

(1) $g(x) := \int_Y f(x,y)\,\mathrm{d}\nu(y) : X \to [0,\infty]$ is a $\mathcal{A}$-measurable function.

(2) $h(y) := \int_X f(x,y)\,\mathrm{d}\mu(x) : Y \to [0,\infty]$ is a $\mathcal{B}$-measurable function.

(3) We have the iterated integral formula

$$\begin{aligned} \int_{X\times Y} f\,\mathrm{d}(\mu\times\nu) &= \int_X \left[\int_Y f(x,y)\,\mathrm{d}\nu(y)\right]\,\mathrm{d}\mu(x) \\ &= \int_X \left[\int_X f(x,y)\,\mathrm{d}\mu(x)\right]\,\mathrm{d}\nu(y). \end{aligned}$$

(b) (Fubini) If $f \in L^1(X\times Y, \mu\times\nu)$, then

(1) $f_x \in L^1(Y,\nu)$ for μ-a.e x and $g(x)$ (which is defined μ-a.e) $\in L^1(X,\mu)$.

(2) $f^y \in L^1(X,\mu)$ for ν-a.e y and $h(y)$ (which is defined ν-a.e) $\in L^1(Y,\nu)$.

(3) *The iterated integral formula from (a).(3) hold.*

Usually we apply Tonelli to $|f|$ to show $f \in L^1(X \times Y, \mu \times \nu)$ and then apply Fubini to evaluate.

Proof. See [Fol99]. ■

3.5 Lebesgue Measure on $\mathbb{R}^d$

Example 3.16 (($\mathbb{R}^2, \mathcal{L} \otimes \mathcal{L}, m \times m$) is not complete)**.** Let $A \in \mathcal{L}, A \neq \emptyset, m(A) = 0$. Let $B \subset [0,1], B \notin \mathcal{L}$ (e.g. Vitali set). Then let $E = A \times B, F = A \times [0,1]$. We can see that $E \subset F$ and $F \in \mathcal{L} \otimes \mathcal{L}, (m \times m)(F) = m(A)m([0,1]) = 0$.

So E is a subnull set but not $\mathcal{L} \otimes \mathcal{L}$-measurable. (otherwise each section of E is measurable, a contradiction.)

Definition 3.17. Let $(\mathbb{R}^d, \mathcal{L}^d, m^d)$ be the *completion* of $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), m \times \ldots \times m)$, which is same(check!) as the *completion* of $(\mathbb{R}^d, \mathcal{L} \otimes \ldots \otimes \mathcal{L}, m \times \ldots \times m)$.

So how do we compute m^d?

A *rectangle* in $\mathbb{R}^d$ is $R = \prod_{i=1}^{d} E_i$, $E_i \in \mathcal{B}(\mathbb{R})$. Then

$$m^d(E) = \inf\left\{ \sum_1^\infty m^d R_k \,\middle|\, E \subset \bigcup_1^\infty R_k, R_k \text{ rectangle} \right\}.$$

Theorem 3.18. *Let $E \in \mathcal{L}^d$.*

(a) $m^d(E) = \inf\left\{m^d(O) \mid \textit{open } O \supset E\right\} = \sup\left\{m^d(K) \mid \textit{compact } K \subset E\right\}$.

(b) $E = \underbrace{A_1}_{F\sigma} \cup \underbrace{N_1}_{\textit{null}} = \underbrace{A_2}_{G\sigma} \setminus \underbrace{N_2}_{\textit{null}}$.

(c) *If* $m^d(E) < \infty, \forall \varepsilon > 0, \exists R_1, \ldots, R_m$ *rectangles whose sides are* intervals *such that* $m^d\left(E \triangle \left(\bigcup_1^m R_i\right)\right) < \varepsilon$.

Proof. Similar to $d = 1$ case. ■

Theorem 3.19. *Integrable "step functions" and $C_c(\mathbb{R}^d)$ are dense in $L^1(\mathbb{R}^d, \mathcal{L}^d, m^d)$.*

Theorem 3.20. *Lebesgue measure in $\mathbb{R}^d$ is translation-invariant.*

Theorem 3.21. *"Effect of linear transformations on Lebesgue measure"*

Skip p. 71-81 of [Fol99] except 3.21.

Chapter 4

Differentiation on Euclidean Space

Suppose $f : [a, b] \to \mathbb{R}$. There are two versions of fundamental theorem of Calculus:

- $\displaystyle\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)$.
- $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)dt = f(x)$.

We focus on the second statement, which implies that

$$\lim_{r\to 0^+}\frac{1}{r}\int_x^{x+r} f(t)\,\mathrm{d}t = \lim_{r\to 0^+}\frac{1}{r}\int_{x-r}^{x} f(t)\,\mathrm{d}t$$

Write $f(x) = \displaystyle\frac{1}{r}\int_x^{x+r} f(x)\,\mathrm{d}t$, then

$$\lim_{r\to 0^+}\frac{1}{r}\int_x^{x+r} (f(t) - f(x))\,\mathrm{d}t = \lim_{r\to 0^+}\frac{1}{r}\int_{x-r}^{x} (f(t) - f(x))\,\mathrm{d}t.$$

This generalizes well in $\mathbb{R}^d$:

$$f : \mathbb{R}^d \to \mathbb{R}, \quad \lim_{r\to 0^+}\frac{1}{v(B(x,r))}\int_{B(x,r)} f(t) - f(x)\,\mathrm{d}t = 0.$$

QUESTION to what extent does this hold?

Start from [Fol99, 3.4].

4.1 Hardy-Littlewood Maximal Function

Suppose an open ball in $\mathbb{R}^d$, $B = B(a, r)$. Denote $cB = B(a, cr), c > 0$.

Lemma 4.1 (Vitali-type covering lemma)**.** *Let $B_1, \ldots, B_k$ be a finite collection of open balls in R^d. Then $\exists$ a sub-collection $B'_1, \ldots, B'_m$ of* disjoint *open balls such that*

$$\bigcup_1^m (3B'_j) \supset \bigcup_1^k B_i.$$

Proof. Greedy algorithm. ■

<u>NOTATION</u>: $\int_E f \, \mathrm{d}m = \int_E f(x) \, \mathrm{d}x$.

Definition 4.2. $f : \mathbb{R}^d \to \mathbb{C}$ is Lebesgue measurable. f is *locally integrable* if

$$\int_K |f| \, \mathrm{d}m < \infty, \forall \text{ compact } K \subset \mathbb{R}^d.$$

We write $f \in L^1_{\text{loc}}(\mathbb{R}^d)$.

Example 4.3. $f(x) = x^2 \in L^1_{\text{loc}}(\mathbb{R}^d)$. (in fact all continuous functions $\in L^1_{\text{loc}}(\mathbb{R}^d)$).

Definition 4.4. For $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, define Hardy-Littlewood maximal function for f

$$Hf(x) = \sup\{A_r(x) \mid r > 0\}, \quad A_r(x) = \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)| \, \mathrm{d}y.$$

Lemma 4.5. *Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$. Then,*

(a) $A_r(x)$ is jointly continuous for $(x, r) \in \mathbb{R}^d \times (0, \infty)$.

(b) $Hf(x)$ is Borel measurable.

Proof.

(a) $(x, r) \to (x^*, r^*) \implies A_r(x) \to A_{r^*}(x^*)$.

Let (x_n, r_n) be any sequence $\to (x^*, r^*)$.

$$A_{r_n}(x_n) \leq \int |f(y)| 1_{B(x_n, r_n)}(y).$$

Apply DCT.

(b) $(Hf)^{-1}((a, \infty)) = \bigcup_{r>0} A_r^{-1}((a, \infty))$ is open.

■

<u>RECALL</u> Markov inequality

$$m\left(\{x \mid |f(x)| \geq c\}\right) \leq \frac{1}{c}\int |f(x)|\,\mathrm{d}x$$

Theorem 4.6 (Hardy-Littlewood maximal inequality)**.** $\exists C_d > 0$ $s.t.$ $\forall f \in L^1_{\text{loc}}(\mathbb{R}^d), \forall \alpha > 0$,

$$m\left(\{x \mid Hf(x) > \alpha\}\right) \leq \frac{C_d}{\alpha}\int |f(x)|\,\mathrm{d}x.$$

Proof. Fix $f \in L^1$ and $\alpha > 0$. Let $E\{x \mid (Hf)(x) > \alpha\}$. E is a Borel measurable set. Then

$$x \in E \implies \exists r_x > 0,\ s.t.\ A_{r_x}(x) > \alpha \implies m(B(x, r_x)) < \frac{1}{\alpha}\int_{B(x,r_x)} |f(y)|\,\mathrm{d}y.$$

By inner regularity, we have $m(E) = \sup\{m(K) \mid \text{ compact } K \subset E\}$. Let $K \subset E$ be compact. Then

$$\begin{aligned}
&K \subset \bigcup_{x \in K} B(x, r_x)\\
\Longrightarrow\ & K \subset \bigcup i = 1^N B_i\\
\Longrightarrow\ & K \subset \bigcup_{j=1}^{m}(3B'_j), B'_1, \ldots, B'_m \text{ disjoint}\\
\Longrightarrow\ & m(K) \leq \sum_{j=1}^{n} m(3B'_j) = 3^d \sum_{j=1}^{n} m(B'_j)\\
\Longrightarrow\ & m(K) \leq \frac{3^d}{\alpha}\sum_{j=1}^{N}\int_{B'_j} |f(y)|\,\mathrm{d}y\\
\Longrightarrow\ & m(K) \leq \frac{3^d}{\alpha}\int_{\mathbb{R}^d} |f(y)|\,\mathrm{d}y.
\end{aligned}$$

■

4.2 Lebesgue Differentiation Theorem

Theorem 4.7. *Let $f \in L^1(\mathbb{R}^d)$. Then*

$$\lim_{r \to 0} \frac{1}{m(B(x,r))}\int_{B(x,r)} |f(y) - f(x)|\,\mathrm{d}y = 0 \textit{ for a.e } x.$$

Proof. (a) The result is holds for $f \in C_c(\mathbb{R}^d)$ (check!)

(b) Let $f \in L^1(\mathbb{R}^d)$. Fix $\varepsilon > 0$. $\exists g \in C_c(\mathbb{R}^d)$ $s.t.$ $\|f-g\|_1 < \varepsilon$. Then

$$\begin{aligned}&\int_{B(x,r)} |f(y)-f(x)| \, \mathrm{d}y \\ &\leq \int_{B(x,r)} |f(y)-g(y)| \, \mathrm{d}y + \int_{B(x,r)} |g(y)-g(x)| \, \mathrm{d}y + \int_{B(x,r)} |g(x)-f(x)| \, \mathrm{d}y.\end{aligned}$$

Let $Q(x) = \limsup_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)-f(x)| \, \mathrm{d}y$. WTS that $m(\{x \mid Q(x) > 0\}) = m\left(\bigcup_{n=1}^{\infty} \left\{x \mid Q(x) > \frac{1}{n}\right\}\right) = 0$.

Enough to show that $m(E_\alpha) = 0, \forall \alpha > 0, E_\alpha = \{x \mid Q(x) > \alpha\}$.

But $Q(x) \leq (H(f-g))(x) + 0 + |g(x)-f(x)|$,

$\{x \mid Q(x) > \alpha \subset \left\{x \mid H(f-g)(x) > \frac{\alpha}{2}\right\} \bigcup \left\{x \mid |g(x)-f(x)| > \frac{\alpha}{2}\right\}$.

$m(\{x \mid Q(x) > \alpha\}) \leq \frac{2C_d}{\alpha} \|f-g\|_1 + \frac{2}{\alpha}\|f-g\|_1 \leq \frac{2(C_d+1)}{\alpha}\varepsilon$. ■

Corollary 4.8. *This also holds for $f \in L^1_{\text{loc}}(\mathbb{R}^d)$.*

Proof. DIY. ■

Corollary 4.9. *For $f \in L^1_{\text{loc}}(\mathbb{R}^d)$,*

$$\lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y) \, \mathrm{d}y = 0 \text{ for a.e } x.$$

Proof. DIY. ■

Definition 4.10. Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$. The point $x \in \mathbb{R}^d$ is called a *Lebesgue point* of f if

$$\lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)-f(x)| = 0.$$

$f \in L^1_{\text{loc}}(\mathbb{R}^d) \implies$ a.e point is a Lebesgue point of f.

Definition 4.11. $\{E_r\}_{r>0}$ *shrinks nicely* to x as $r \to 0$ means $E_r \subset B(x,r)$ and $\exists c > 0$ $s.t.$ $cm(B(x,n)) \leq m(E_r)$.

Corollary 4.12 (Lebesgue differentiation theorem)**.**

$$\left.\begin{array}{r} E_r \textit{ shrinks nicely to } 0 \\ f \in L^1_{\text{loc}}(\mathbb{R}^d) \\ x \textit{ a Lebesgue point of } f \end{array}\right\} \implies \lim_{r\to 0} \frac{1}{m(E_r)} \int_{E_r+x} |f(y)-f(x)| \, \mathrm{d}y = 0.$$

Proof. DIY. ■

Corollary 4.13. $f \in L^1_{\text{loc}}(\mathbb{R}^d) \implies F(x) = \int_0^x f(y)\,\mathrm{d}y$ *is differentiable and* $F'(x) = f(x)$ *a.e.*

Rest of [Fol99, Ch.3] will be covered later.

Chapter 5

Normed Vector Spaces

Topological spaces $\supset$ metric spaces $\supset$ normed spaces $\supset$ inner product spaces.

Let's start with metric spaces. [Fol99, 5.1, 6.1, 6.2]

5.1 Metric Spaces and Normed Spaces

Definition 5.1. Suppose Y is a set. A *metric* of Y is $\rho : Y \times Y \to [0, \infty)$ $s.t.$

(a) $\rho(x, y) = \rho(y, x)$

(b) $\rho(x, y) \le \rho(x, z) + \rho(z, y)$

(c) $\rho(x, y) = 0 \iff x = y$.

Example 5.2.

(a) $\mathbb{Q}, \rho(x, y) = |x - y|$.

(b) $\mathbb{R}, \rho(x, y) = |x - y|$.

(c) $\mathbb{R}_+, \rho(x, y) = \left|\ln\left(\frac{y}{x}\right)\right|$.

(d) $\mathbb{R}^d, \rho_1(x, y) = \sum_{i=1}^{d} |x_i - y_i|, \rho_2(x, y) = \left(\sum_{i=1}^{d} |x_i - y_i|^2\right)^{1/2}, \rho_\infty(x, y) = \max_{1 \le i \le d} |x_i - y_i|$.

(e)

Definition 5.3 (Recall 2.32)**.** Suppose V is a vector space over field $\mathbb{R}$ or $\mathbb{C}$. A *seminorm* on V is $\|\cdot\| : V \to [0, \infty)$ $s.t.$

- $\|cv\| = |c|\|v\|, \forall v \in V, \forall c$ scalar

- $\|v+w\| \le \|v\| + \|w\|$, triangle inequality

A *norm* is a seminorm such that $\|v\| \iff v = 0$.

Norm gives rise to a metric where $\rho(v,w) = \|v-w\|$.

$v_n \to v \iff \lim_{n\to\infty} \|v_n - v\| = 0$.

Example 5.4. (a) $L^1(X, \mathcal{A}, \mu)$

(b) $C([0,1]), \|f\|_1 = \int_0^1 |f(x)| \, \mathrm{d}x, \|f\|_\infty \max_{0\le x\le 1} |f(x)|$.

(c) $\mathbb{R}^d, \|x\|_2 = \sqrt{\sum_1^d |x_i|^2}, \|x\|_1 = \sum_1^d |x_i, \|x\|_\infty \max_{1\le i\le d} |x_i|$.

5.2 L^p Spaces

Definition 5.5. Suppose $(X, \mathcal{A}, \mu)$ a measure space. f is measurable function. For $0 < p < \infty$, define $\|f\|_p = \left(\int_X |f|^p \, \mathrm{d}\mu\right)^{1/p}$.

Define $L^p(X, \mathcal{A}, \mu) = \left\{ f \mid \|f\|_p < \infty \right\}$.

Example 5.6.

Definition 5.7. $\ell^p = \ell^p(N) = \{a = (a_1, a_2, \dots) \mid \|a\|_p = (\sum_1^\infty |a_i|^p)^{1/p} < \infty\}$.

Lemma 5.8. *L^p is a vector space, $\forall p \in (0,\infty)$.*

Proof.

$$\left(\int |cf|^p\right)^{1/p} = |c| \, \|f\|_p$$

$$(\alpha+\beta)^p \le (2\max(|\alpha|,|\beta|))^p = 2^p \max\left(|\alpha|^p, |\beta|^p\right) \le 2^p(|\alpha|^p + |\beta|^p)$$

∎

Theorem 5.9 (Hölder's Inequality). *Let $p < \infty, q = \frac{p}{p-1}$ so $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\|fg\|_1 \le \|f\|_p \|g\|_q$$

Proof.

$$t \le \frac{t^p}{p} + 1 - \frac{1}{p}, \forall t \ge 0$$

(Take $F(t) = t - \frac{t^p}{p}$)

$$\alpha\beta \le \frac{\alpha^p}{p} + \frac{\beta^q}{q}, \forall \alpha, \beta \ge 0 \text{ (Young's inequality)} \tag{5.1}$$

WLOG assume $0 \le \|f\|_p, \|g\|_q < \infty$. Let $F(x) = \frac{f(x)}{\|f\|_p}, G(x) = \frac{g(x)}{\|g\|_q}$.

$\implies \|F\|_p = 1 = \|G\|_q$.

By (5.1),

$$\int |F(x)G(x)| \le \int \frac{|F(x)|^p}{p} + \int \frac{|G(x)|^q}{q}$$

$$\frac{\int |f(x)g(x)|}{\|f\|_p \|g\|_q} \le \frac{1}{p} + \frac{1}{q} = 1.$$

■

Theorem 5.10 (Minkowski's inequality)**.** *Let* $1 \le p < \infty$*. For* $f, g \in L^p$, $\|f+g\|_p \le \|f\|_p + \|g\|_p$.

Proof. $p = 1$ is easy.

Assume $1 < p < \infty$. WLOG assume $\|f+g\|_p \neq 0$. We have

$$\begin{aligned}
\int |f(x)+g(x)|^p &\le \int |f(x)+g(x)|^{p-1}(|f(x)|+|g(x)|) \\
&\le \left(\int (|f+g|^{p-1})^q\right)^{1/q}\left(\int |f|^p\right)^{1/p} + \left(\int (|f+g|^{p-1})^q\right)^{1/q}\left(\int |g|^p\right)^{1/p} \\
&\le \left(\int (|f+g|^{p-1})^q\right)^{1/q}\left[\left(\int |f|^p\right)^{1/p} + \left(\int |g|^p\right)^{1/p}\right] \\
&\le \left(\int (|f+g|^{p-1})^q\right)^{1/q}\left[\|f\|_p + \|g\|_p\right]
\end{aligned}$$

Since $q(p-1) = p$, divide by $\left(\int (|f+g|^{p-1})^q\right)^{1/q}$ on both sides we have

$$\left(\int |f(x)+g(x)|^p\right)^{1-1/q} \le \|f\|_p + \|g\|_p .$$

■

Hölder: $\|fg\|_1 \le \|f\|_p \|g\|_q, \frac{1}{p} + \frac{1}{q} = 1$.

Minkowski: $\|f+g\|_p \le \|f\|_p + \|g\|_p, 1 \le p < \infty$.

Definition 5.11. For a measurable function f on $(X, \mathcal{A}, \mu)$, let

$$S = \{\alpha \ge 0 \mid \mu(\{x \mid |f(x)| > \alpha\}) = 0\} = \{\alpha \ge 0 \mid f(x) \le \alpha \text{ a.e}\}.$$

Define $\|f\|_\infty = \begin{cases} \inf S & S \neq \emptyset \\ \infty & S = \emptyset. \end{cases}$. Let $L^\infty(X, \mathcal{A}, \mu) = \{f \mid \|f\|_\infty < \infty\}$.

Example 5.12.

- $(\mathbb{R}, \mathcal{L}, m)$, $f(x) = \frac{1}{x}1_{(0,\infty)}(x) \neq L^\infty$, $f(x) = x1_{\mathbb{Q}}(x) + \frac{1}{1+x^2} \in L^\infty$.
- If f is *continuous* on $(\mathbb{R}, \mathcal{L}, m)$, $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$. For $a \in \ell^\infty$, $\|a\|_\infty = \sup_{i\in\mathbb{N}} |a_i|$. ($\ell^\infty = \{a = (a_1, a_2, \dots) \mid \|a\|_\infty < \infty\} = \{a \mid \exists M \geq 0 \; s.t. \; |a_i| \leq M_i, \forall i\}$)

Lemma 5.13. *(a) For $\alpha \geq \|f\|_\infty, \mu(\{x \mid |f(x)| > \alpha\}) = 0$. For $\alpha < \|f\|_\infty, \mu(\{x \mid |f(x)| > \alpha\}) > 0$.*

(b) $|f(x)| \leq \|f\|_\infty$ a.e.

(c) $f \in L^\infty \iff \exists$ bounded *measurable function g such that $f = g$ a.e.*

Proof. DIY. ∎

Theorem 5.14.

(a) $\|fg\|_1 \leq \|f\|_1 \|g\|_\infty$.

(b) $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

(c) $f_n \to f$ in $L^\infty \iff f_n \to f$ uniformly a.e.

Proof. DIY For (c): $\implies$ Let $A_n = \{x \mid |f_n(x) - f(x)| > \|f_n - f\|_\infty\}$. Then $\mu(A_n) = 0$.

Let $A = \bigcup_1^\infty A_n, \mu(A_n) = 0$. $\forall x \in A^c = \bigcap_1^\infty A_n^c, \forall n, |f_n(x) - f(x)| \leq \|f_n - f\|_\infty$. The latter converges to 0 by assumption.

Given $\varepsilon > 0, \exists N \; s.t. \; \|f_n - f\|_\infty < \varepsilon, \forall n \geq N$. So $\forall x \in A^c, \forall n \geq N, |f_n(x) - f(x)| \leq \|f_n - f\|_\infty < \varepsilon$. ∎

Proposition 5.15.

(a) For $1 \leq p < \infty$, the collection of simple functions with finite measure support is dense in $L^p(X, \mathcal{A}, \mu)$.

(b) For $1 \leq p < \infty$, the collection of step functions (by definition they have finite measure support) is dense in $L^p(\mathbb{R}, \mathcal{L}, m)$. So is $C_c(\mathbb{R})$.

(c) For $p = \infty$, the collection of simple functions is dense in $L^\infty(X, \mathcal{A}, \mu)$.

Proof. DIY ∎

NOTE: $C_c(\mathbb{R})$ is *not dense* in $L^\infty(\mathbb{R}, \mathcal{L}, m)$.

5.3 Embedding Properties of L^p spaces

Definition 5.16. Two norms $\|\cdot\|, \|\cdot\|'$ on the same spaces V are said to be *equivalent* if $\exists c_1, c_2 > 0\ s.t.\ c_1 \|v\| \le \|v\|' \le c_2 \|v\|, \forall v \in V$.

So we have same open sets, same convergence.

Example 5.17.

- For $\mathbb{R}^d$, $\|\cdot\|_p$, $1 \le p \le \infty$ are equivalent.
- For $1 \le p, q \le \infty p \ne q$, $L^p(\mathbb{R}, m)$-norm and $L^q(\mathbb{R}, m)$-norm are *not equivalent*. $L^p(\mathbb{R}, m) \not\subset L^q(\mathbb{R}, m), L^p(\mathbb{R}, m) \not\supset L^q(\mathbb{R}, m)$.

Proposition 5.18. $\forall 0 < p < q < r \le \infty$, $L^p \cap L^r \subset L^q$.

5.4 Banach Spaces

Theorem 5.19. *Suppose $(V, \|\cdot\|)$ a normed space. Then it is complete $\iff$ Every absolutely convergent series is convergent (i.e. if $\sum_1^\infty \|v_n\| < \infty$ then $\exists s \in V\ s.t.\ \sum_1^N v_n \to s$ as $N \to \infty$)*

Proof. $\implies$: DIY. (partial sums form a Cauchy Sequence)

$\impliedby$: Suppose $v_n, n \in \mathbb{N}$ is a Cauchy sequence. $\forall j \in \mathbb{N}, \exists N_j \in \mathbb{N}\ s.t.\ \|v_n - v_m\| < \frac{1}{2^j}, \forall n, m \ge N_j$.

WLOG we may assume $N_1 < N_2 < \ldots$. Let $w_1 = v_{N_1}, w_j = v_{N_j} - v_{N_{j-1}}, \forall j \ge 2 \implies \sum_1^\infty \|w_j\| \le \|v_{N_1}\| + \sum_{j=2}^\infty \frac{1}{2^{j-1}} < \infty \implies \sum_1^k w_j \to \exists s \in V$.

Thus $V_{N_k} \to s$ as $k \to \infty$. v_n is Cauchy $\implies v_n \to s$ as $n \to \infty$. ■

5.5 Bounded Linear Transformation

Definition 5.20. Suppose $(V, \|\cdot\|), (W, \|\cdot\|')$ two normed spaces. A linear map $T : V \to W$ is said to be a *bounded map* is $\exists c \ge 0\ s.t.\ \|T_v\|' \le C \|v\|, \forall v \in V$.

Proposition 5.21. *Suppose $T : (V, \|\cdot\|) \to (W, \|\cdot\|')$ is a linear map. Then the followings are equivalent:*

(a) T is continuous

(b) T is continuous at 0

(c) T is a bounded map

Proof. (a) $\implies$ (b) is clear.

(b) $\implies$ (c): For $\varepsilon = 1$, $\exists \delta > 0$ $s.t.$ $\|Tu\|' < \varepsilon = 1$ if $\|u\| < \delta$. Suppose $v \in V, v \neq 0$. Let $u = \frac{\delta}{2\|v\|}v \implies \|u\| = \frac{\delta}{2} < \delta \implies \|Tu\|' < 1 \implies \frac{\delta}{2\|v\|}\|Tv\|' < 1 \implies \|Tu\|' < \frac{2}{\delta}\|v\|$.

(c) $\implies$ (a): Fix $v_0 \in V$. $\|Tv - Tv_0\|' = \|T(v - v_0)\|' \le C\|v - v_0\|$. ■

Example 5.22. (a) $T : \ell^1 \to \ell^1, Ta = (a_2, a_3, \ldots)$, $\|Ta\|_1 \le \|a\|_1$. T is BLT.

(b) $T : (C([-1,1]), \|\cdot\|_1) \to \mathbb{C}, Tf = f(0)$. This is not continuous.

(c) $T : (C([-1,1]), \|\cdot\|_\infty) \to \mathbb{C}, Tf = f(0)$ is BLT.

(d) Let A be a $n \times m$ matrix. $T : \mathbb{R}^n \to \mathbb{R}^m, v \mapsto Av$ is BLT.

(e) Let $K(x, y)$ be a continuous function on $[0,1] \times [0,1]$.

$$T : (C_[0,1], \|\cdot\|_\infty) \to (C[0,1], \|\cdot\|_\infty), Tf = \int_0^1 K(x,y)f(y)\,\mathrm{d}y$$

is a BLT.

(f) $T : L^1(\mathbb{R}) \to (C(\mathbb{R}), \|\cdot\|_\infty), (Tf)(t) = \displaystyle\int_{-\infty}^{\infty} e^{-itx}f(x)\,\mathrm{d}x$ (Fourier transform of f)

(g) $T : (C^\infty([0,1]), \|\cdot\|_\infty) \to (C^\infty([0,1]), \|\cdot\|_\infty), (Tf)(x) = f'(x)$ is not bounded.

Definition 5.23. Let $L(V, W) = \{T : V \to W \mid T \text{ is BLT}\}$. For $T \in L(V, W)$, the *operator norm* of T is

$$\begin{aligned}\|T\| &:= \inf\{c \ge 0 \mid \|Tv\|' \le c\|v\|, \forall v \in V\}\\ &= \sup\left\{\frac{\|Tv\|'}{\|v\|}\,\middle|\, v \neq 0, v \in V\right\}\\ &= \sup\left\{\|Tv\|' \mid \|v\| = 1\right\}.\end{aligned}$$

Lemma 5.24. *(a) Above three definitions are equivalent.*

(b) It is indeed a normed space.

Proof. DIY. ■

5.6 Dual of L^p Spaces

Chapter 6

Signed and Complex Measures

[Fol99, Ch.3].

<u>RECALL</u> Suppose $(X, \mathcal{A}, \mu)$ a measure space. $f : X \to [0, \infty]$ measurable. Let $\nu(E) = \int_E f \, \mathrm{d}\mu, E \in \mathcal{A} \implies \nu$ is a measure on $(X, \mathcal{A})$.

6.1 Signed Measures

Definition 6.1. Suppose $(X, \mathcal{A})$ a measurable space. A signed measure is $\nu : \mathcal{A} \to [-\infty, \infty)$ or $\nu : \mathcal{A} \to (-\infty, \infty]$ such that

- $\nu(\emptyset) = 0$.
- $A_1, A_2, \ldots \in \mathcal{A}$, A_i disjoint $\implies \nu\left(\bigcup_1^\infty A_i\right) = \sum_1^\infty \nu(A_i)$ where the series converges absolutely if $\nu\left(\bigcup_1^\infty A_i\right) \in (-\infty, \infty)$.

Example 6.2.

- ν positive measure $\implies \nu$ is a signed measure.
- μ_1, μ_2 positive measures such that either $\nu_1(X) < \infty$ or $\nu_2(X) < \infty \implies \nu = \mu_1 - \mu_2$ a signed measure.
- $f : X \to \bar{\mathbb{R}}$ $s.t.$ $\int_X f^+ \, \mathrm{d}\mu < \infty$ or $\int_X f^- \, \mathrm{d}\mu < \infty \implies \nu(E) = \int_E f \, \mathrm{d}\mu$.

<u>NOTE</u>:

(a) $A \subset B \not\Rightarrow \nu(A) \leq \nu(B)$ since $\nu(B) = \nu(A) + \nu(B \setminus A)$.

(b) $A \subset B, \nu(A) = \infty \implies \nu(B) = \infty$.

Lemma 6.3. *ν is a signed measure on $(X, \mathcal{A})$. Then*

- $E_n \in \mathcal{A}, E_1 \subset E_2 \subset \dots \implies \nu\left(\bigcup_1^\infty E_n\right) = \lim_{n\to\infty} \nu(E_n)$.
- $E_n \in \mathcal{A}, E_1 \supset E_2 \supset \dots, -\infty < \nu(E_1) < \infty \implies \nu\left(\bigcap_1^\infty E_n\right) = \lim_{N\to\infty} \nu(E_n)$.

Definition 6.4. ν is a signed measure on $(X, \mathcal{A})$. Let $E \in \mathcal{A}$. We say

(a) E is *positive* for ν (a positive set for ν) if $\forall F \subset E, F \in \mathcal{A}$, $\nu(F) \geq 0$.

(b) E is *negative* for ν (a negative set for ν) if $\forall F \subset E, F \in \mathcal{A}$, $\nu(F) \leq 0$.

(c) E is *null* for ν (a null set for ν) if $\forall F \subset E, F \in \mathcal{A}$, $\nu(F) = 0$.

<u>NOTE</u> E positive set, $F \subset E \implies \nu(F) \leq \nu(E)$. E negative set, $F \subset E \implies \nu(F) \geq \nu(E)$.

Theorem 6.5 (The Hahn decomposition theorem)**.** *Suppose ν is a signed measure of $(X, \mathcal{A})$. Then $\exists P, N \in \mathcal{A}$ $s.t.$ $P \cap N = \emptyset, P \cup N = X$, P is positive for ν, and N is negative for ν.*

If P', N' are another such pair, then $P \triangle P' (= N \triangle N')$ is null for ν.

Definition 6.6. μ, ν signed measure on $(X, \mathcal{A})$.

$\nu \perp \nu$ (singular to each other) means $\exists E, F \in \mathcal{A}$ $s.t.$ $E \cap F = \emptyset, E \cup F = X$, F is null for μ, E is null for ν.

Example 6.7. For $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$,

(a) Lebesgue measure m

(b) Cantor measure $\mu_C((a, b])$.

(c) Discrete measure $\mu_D = \delta_1 + 2\delta_{-1}$.

For (a), (c), take $E = \mathbb{R} \setminus \{-1, 1\}$, $F = \{-1, 1\}$. For (a), (b), take the cantor set K, $E = \mathbb{R} \setminus K, F = K$.

Bibliography

[Fol99] G. B. Folland. *Real Analysis: Modern Techniques and Their Applications*. Pure and Applied Mathematics. Wiley, New York, 2nd ed edition, 1999. "A Wiley-Interscience publication.".

[Tao11] Terence Tao. *An Introduction to Measure Theory*. Number v. 126 in Graduate Studies in Mathematics. American Mathematical Society, Providence, R.I, 2011.