

# Notes for Math 566 – Algebraic Combinatorics

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# Chapter 1

## Graph and Trees

### 1.1 Linear Algebra Preliminaries

Let  $M$  be a  $p \times p$  matrix with entries in  $\mathbb{C}$ . The eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_p$  are defined by

$$\det(t \operatorname{id} - M) = \prod_{i=1}^p (t - \lambda_i).$$

Taking coefficients of  $t^{p-1}$  on both sides we obtain

$$\operatorname{tr} M = \sum_k \lambda_k. \quad (1.1.1)$$

**Lemma 1.1.1.** *Let  $f(t) \in \mathbb{C}[t]$ . Then  $f(M)$  have eigenvalues  $f(\lambda_1), \dots, f(\lambda_p)$ .*

*Proof.* If  $M$  is diagonalizable, then the statement is clear:  $f(M)$  has the same eigenvectors as  $M$ , with eigenvalues  $f(\lambda_k)$ . Then use a continuity argument. (Diagonalizable matrices are dense.) Alternative proof: use Jordan's normal form. ■

Combining (1.1.1) with the lemma, we have

$$\operatorname{tr} M^\ell = \sum_k \lambda_k^\ell. \quad (1.1.2)$$

PROBLEM: [A solution is given in Stanley's textbook.] Let  $\alpha_1, \dots, \alpha_r$  and  $\beta_1, \dots, \beta_r$  be

nonzero complex numbers such that for *all* positive integer  $\ell$  we have

$$\alpha_1^\ell + \dots + \alpha_r^\ell = \beta_1^\ell + \dots + \beta_r^\ell.$$

Show that this implies that  $r = s$ , and that  $\alpha$ 's are a permutation of  $\beta$ 's.

In the majority of forthcoming applications,  $M$  is symmetric and real. Then it is diagonalizable, with real eigenvalues  $\lambda_1, \dots, \lambda_p$ .

## 1.2 Counting Walk

Let  $G$  be a graph on the vertex set  $\{1, \dots, p\}$ . (We allow loops and multiple edges.) Let  $M = A(G)$  be its adjacency matrix.

OBSERVATION The number of walks of length  $\ell$  from  $i$  to  $j$  is equal to  $(M^\ell)_{ij}$ .

In general, counting walks requires knowing the matrix  $M$  (equivalently, knowing both the eigenvalues  $\lambda_k$  and the corresponding eigenvectors). On the other hand, some enumerative information can be extracted from the eigenvalues alone:

**Proposition 1.2.1.** *The number of marked closed walks of length  $\ell$  is equal to  $\sum_{k=1}^p \lambda_k^\ell$ .*

Here "marked" means that the starting location is fixed, as is a particular instance of passing through it, in case we do it several times.

*Proof.* By the last observation, the number of marked closed walks of length  $\ell$  is equal to  $\text{tr } M^\ell$ , which equals to  $\sum_{k=1}^p \lambda_k^\ell$  by (1.1.2). ■

**Example 1.2.1.** Let  $G = K_p$ , the complete graph on  $p$  vertices. Let  $J$  denote the  $p \times p$  matrix all of whose entries are 1. Let  $I$  denote the  $p \times p$  identity matrix. Then  $A(G) = J - I$ . Obviously  $\text{rk } J = 1$  and  $\text{tr } J = p$ . Hence the eigenvalues of  $J$  are  $0, \dots, 0, p$ , and the eigenvalues of  $A(G) = J - I$  are  $-1, \dots, -1, p - 1$ .

**Corollary 1.2.1.** *There are  $(p - 1)^\ell + (-1)^\ell(p - 1)$  marked closed walks of length  $\ell$  in  $K_p$ .*

NOTE This is the number of  $(\ell + 1)$ -letter words in a  $p$ -letter alphabet in which no two consecutive letters are identical, and which begin and end by the same letter.

PROBLEM Show that the number of walks of length  $\ell$  between two distinct vertices in  $K_p$  differs by 1 from the number of closed walks of length  $\ell$  starting at a given vertex.

RECALL

$$\# \text{ of marked closed walks of length } \ell = \sum_{i=1}^p \lambda_i^\ell.$$

It can be used backwards: using counted walks to compute eigenvalues.

**Example 1.2.2.** Let  $G = K_{n,m}$  a complete bipartite graph.

$$\# \text{ of marked closed walks of length } \ell = \begin{cases} 0 & \ell = 2k + 1 \\ 2n^{\ell/2}m^{\ell/2} & \ell = 2k \end{cases} = (\sqrt{nm})^\ell + (-\sqrt{nm})^\ell$$

Problem  $\Rightarrow$  eigenvalues are  $\sqrt{nm}, -\sqrt{nm}, 0, \dots, 0$ .

PROBLEM Prove that, for  $G$  connected, the  $\text{diam}(G) < \#$  of distinct eigenvalues.

**Example 1.2.3.**  $K_p = 1 < 2, K_{n,m} = 2 < 3$ .

### 1.3 Inequalities for the Maximal Eigenvalue

**Definition 1.3.1.** Suppose  $G$  a graph with vertices  $= \{1, \dots, p\}$ . Let

$$\lambda_{\max} := \max_i |\lambda_i| = \max \lambda_i.$$

**Proposition 1.3.1.**

$$\lambda_{\max} \leq \max \deg(G)$$

*Proof.* For any vector  $X = (x_k) \in \mathbb{C}^p$ ,

$$\max_j |(A(G)X)_j| \leq \max \deg(G) \cdot \max_k |X_k|$$

Now suppose  $X$  is an eigenvector of  $A(G)$  with eigenvalue  $\lambda$ . Then

$$\max_j |(A(G)X)_j| = |\lambda| \max_k |X_k| \leq \max \deg(G) \cdot \max_k |X_k| \implies |\lambda| \leq \max \deg(G)$$

This holds for all eigenvalue  $\lambda_i$ , which proves our proposition. ■

ALTERNATE PROOF: by counting closed walks ( $\leq \sum \max \deg(G)^\ell$ .)

PROBLEM Prove that  $\lambda_{\max} \geq$  average degrees of the vertices of  $G$ .

HINT for symmetric real matrix  $M$  we have  $\lambda_{\max} = \max_{|x|=1} x^T M x$ .

**Corollary 1.3.1.** # of closed walk of length  $\ell$  grows exponentially in  $\ell$  with a rate  $\geq$  average degree.

## 1.4 Eigenvalue of Block Anti-diagonal Matrices

$$M = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \in \mathbb{R}_{n+m}$$

**Lemma 1.4.1.** *The non-zero eigenvalues (called "singular values" of  $B$ ) of  $M$  are  $\pm\sqrt{\mu_i}$  where  $\mu_i$  are nonzero eigenvalues of  $B^T B$  with multiplicities.*

Note that  $B^T B$  is positive definite.

*Proof.* Let  $F_X(t) = \det(t \text{id}_p - X)$ .

$$\begin{bmatrix} t \text{id}_n & -B \\ -B^T & t \text{id}_m \end{bmatrix} \begin{bmatrix} \text{id}_n & B \\ 0 & t \text{id}_m \end{bmatrix} = \begin{bmatrix} t \text{id}_n & 0 \\ -B & -B^T B + t^2 \text{id}_m \end{bmatrix}$$

$$F_M(t) \cdot t^m = t^n F_{B^T B}(t^2)$$

and the claim follows ■

So now we are equipped to compute the eigenvalue of bipartite graphs.

**Example 1.4.1.** Suppose  $G = K_{n,m}$ ,  $B^T B$  is  $m \times m$  matrix with all entries being  $n$ . So the eigenvalues of  $B^T B = nm, 0, 0, \dots$ . So eigenvalues of  $A(G)$  is  $\sqrt{mn}, -\sqrt{mn}, 0, 0, \dots$

PROBLEM Let  $G$  to be the graph obtained by removing  $n$  disjoint edges from  $K_{n,n}$ . Find the eigenvalue of  $G$ .

**Example 1.4.2.** Let  $G$  be a  $2n$ -cycle.  $M_{2n} = A(G) = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$ . The  $B^T B = 2I_n + M_n$  for an appropriate labeling.

So if the eigenvalue of  $n$ -cycle are  $\lambda_1, \dots, \lambda_n$ . Then the eigenvalues of  $2n$ -cycles are  $\pm\sqrt{\lambda_i + 2}$ .

## 1.5 Eigenvalues of Circulant Matrices

**Definition 1.5.1.** A circulant matrix is of the form

$$M = \begin{bmatrix} s_0 & s_1 & s_2 & \dots & s_{p-1} \\ s_{p-1} & s_0 & s_1 & \dots & s_{p-2} \\ \vdots & & & & \\ s_1 & s_2 & s_3 & \dots & s_0 \end{bmatrix}.$$

**Lemma 1.5.1.** *M has eigenvalues*

$$\lambda_k = \sum_{j=0}^{p-1} s_j e^{\frac{2\pi i}{p} jk}, k = 0, 1, \dots, p-1.$$

Notice that

$$\lambda_k = \sum_{j=0}^{p-1} s_j e^{\frac{2\pi i}{p} jk} = s \left( e^{\frac{2\pi i}{p} k} \right) \quad p\text{-th root of unity.}$$

where

$$s(x) = \sum_{j=0}^{p-1} s_j x^j.$$

*Proof.* Let

$$T = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}$$

We have that the eigenvalues of  $T$  and  $p$ -th roots of unity and characteristic polynomial is  $t^p - 1$ .

Key observation:  $M = s(T)$ . ■

**Definition 1.5.2.** A graph  $G$  is circulant if  $A(G)$  is circulant, for some choice of vertex labeling.

**Corollary 1.5.1.** *The eigenvalue of  $p$ -cycle are*

$$2 \cos \left( \frac{2\pi k}{p} \right), k = 0, 1, \dots, p-1.$$

*Proof.* By Lemma 1.5.1, we have that

$$\lambda_k = e^{\frac{2\pi i}{p} k} + e^{\frac{2\pi i}{p} (p-1)k} = e^{\frac{2\pi i k}{p}} + e^{-\frac{2\pi i k}{p}} = 2 \cos \left( \frac{2\pi k}{p} \right). \quad \blacksquare$$

*Remark.* This formula is consistent with the formula linking the eigenvalues of a  $2n$ -cycle and an  $n$ -cycle: if  $2 \cos \alpha = \lambda$ , then  $2 \cos \frac{\alpha}{2} = \pm \sqrt{2 + \lambda}$ .

**PROBLEM** Find the eigenvalues of the graph obtains by removing  $n$  disjoint edges from  $K_{2n}$ .



## 1.6 Eigenvalues of Cartesian Products

**Definition 1.6.1.** Suppose  $G, H$  are graphs with no loops. Define graph  $G \times H$  where

$$V(G \times H) = \{(g, h) : g \in V(G), h \in V(H)\},$$

and we have two kinds of edges:

- $(g, h) - (g', h)$  for  $g - g'$
- $(g, h) - (g, h')$  for  $h - h'$

**Example 1.6.1.** 1. Grid graph = path  $\times$  path

2. Discrete annulus (cylinder) = cycle  $\times$  path

3. Discrete torus = cycle  $\times$  cycle

4.  $n$ -cube graph

**Proposition 1.6.1.** If  $G$  has eigenvalues  $\lambda_1, \lambda_2, \dots$ ,  $H$  has eigenvalues  $\mu_1, \mu_2, \dots$ . Then  $G \times H$  has eigenvalues  $\lambda_i + \mu_j$  for any pair  $i, j$ .

*Proof 1.* (Tensor product)  $V_G, V_H$  are vector spaces formally spanned by vertices of  $G, H$ . Take  $u = \sum \alpha_g g \in V_G, v = \sum \beta_h h \in V_H$ . We have

$$u \otimes v = \sum_{g,h} \alpha_g \beta_h (g, h) \in V_{G \times H}.$$

The

$$A(G \times H)(u \otimes v) = (A(G)u) \otimes v + u \otimes (A(H)v)$$

Suppose  $u, v$  are eigenvectors *i.e.*  $A(G)u = \lambda u, A(H)v = \mu v$ . Then we get

$$A(G \times H)(u \otimes v) = \lambda u \otimes v + u \otimes \mu v = (\lambda + \mu)(u \otimes v). \quad \blacksquare$$

*Proof 2.* (Marked closed walk) Walk in  $G \times H \xleftrightarrow{1-1}$  a *shuffle* of marked closed walks in  $G \& H$ .

# of closed walks of length  $\ell$  in  $G \times H$

$$= \sum_k \binom{\ell}{k} \sum_i \lambda_i^k \sum_j \mu_j^{\ell-k} = \sum_i \sum_j \sum_k \binom{\ell}{k} \lambda_i^k \mu_j^{\ell-k} = \sum_{i,j} (\lambda_i + \mu_j)^\ell.$$

1 This set of numbers are unique by problem in lecture 1, so they must be the eigenvalues of  $G \times H$ . ■

**PROBLEM** Take a  $3 \times 3$  grid, find the number of marked closed walks of length  $\ell$ .

**PROBLEM** Direct problem of 8-cycle and  $K_2$ .

**$n$ -CUBE GRAPH:**

$$(K_2)^n = \underbrace{K_2 \times K_2 \times \cdots \times K_2}_{n \text{ times}}.$$

**Example 1.6.2.** When  $n = 3$ , we have a 3-D cube:

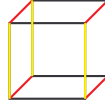


Figure 1.1: Cube graph  $K_2 \times K_2 \times K_2$

$K_2$  has adjacency matrix  $A(K_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  with eigenvalues  $\pm 1 \implies$  eigenvalues of  $(K_2)^n$  are

$$\lambda = \underbrace{\pm 1 \pm 1 \pm \cdots \pm 1}_{n \text{ times}}.$$

**Proposition 1.6.2.** The eigenvalues of  $(K_2)^n$  are of the form  $n - 2k$  where  $k = 0, 1, \dots, n$ , each with multiplicities  $\binom{n}{k}$  i.e. the number of marked closed walks of length  $\ell$  in the  $n$ -cube graph is

$$\sum_{k=0}^n \binom{n}{k} (n - 2k)^\ell$$

which is 0 when  $\ell$  is odd.

## 1.7 Random Walks

Let  $G$  be a regular graph of degree  $d$  on  $p$  vertices.

**Example 1.7.1.**  $G = (K_2)^n$  is regular with  $d = n$ .

A simple random walk on  $G$  originating at a vertex  $v$  is a random walk with equal probabilities for each adjacent vertices.

Assuming that  $\text{Aut}(G)$  acts transitively on vertices, we have the result

$$\mathbb{P}(\text{walk is back at } v \text{ after } \ell \text{ steps}) =$$

$$\frac{1}{d^\ell} \# \{\text{marked closed walks of length } \ell \text{ originating from } v\} = d^{-\ell} p^{-1} \sum_1^p \lambda_i^\ell.$$

Notice that an arbitrary regular  $G$  does not necessarily have that condition, but the converse is true.

**Example 1.7.2.** The probability that a simple random walk on  $(K_2)^n$  returns to its origin after  $\ell$  steps is

$$\frac{1}{n^\ell 2^n} \sum_{k=0}^n \binom{n}{k} (n - 2k)^\ell$$

## Chapter 2

# Tilings, Spanning Trees, and Electric Networks

### 2.1 Domino Tilings ("Dimers")

*If you sit in this classroom for a long enough period of time, you probably can figure out this (2.1.2) yourself. But I offer you a divine revelation.*

– Sergey Fomin

*If you can't solve this, it just means you're mere mortals. Even if you do it... I mean the grader will grade it, but...*

– Sergey Fomin on the perfect square problem at the end of the section

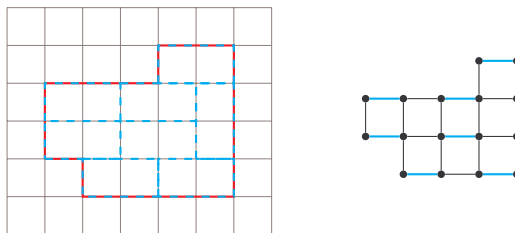
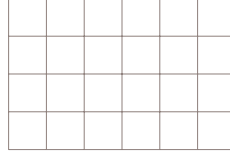


Figure 2.1: An example of domino tiling and perfect matching in its dual graph

A domino tiles decompose part of grids into  $1 \times 2$  rectangles.

Think of it another way: the "dual graph" where squares are vertices, and there exists an edge between two vertices iff the corresponding squares shares an edge. A tiling is a perfect matching between these vertices.

Special case:  $m \times n$  rectangular boards



Without loss of generality, assume that  $n$  is even. We denote the answer as  $T(m, n)$

The dual graph  $G$  is  $m$ -chain  $\times$   $n$ -chain. Notice that  $G$  is bipartite.

$M = A(G)$  has the form  $\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$  given appropriate labeling of vertices where  $B$  is a square matrix.

CLAIM  $T(m, n) = \text{the permanent of matrix } B$ .

Permanents do not have nice properties, thus they are hard to calculate. In order to better calculate the permanent of  $B$ , let  $\tilde{B}$  obtained from  $B$  by replacing the 1's by corresponding to *vertical* tiles by  $i$ 's where  $i^2 = -1$ .

**Proposition 2.1.1.**  $T(m, n) = \text{per}(B) = \pm \det(\tilde{B})$ .

**Lemma 2.1.1** (exercise). *Any two domino tilings of a rectangular board are related to each other via "flips" of the form (two horizontal  $\leftrightarrow$  two vertical)*

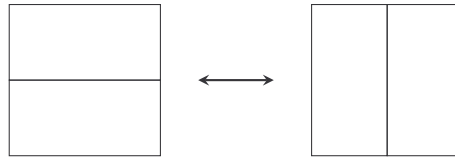


Figure 2.2: Example of a flip

*Proof of Prop.* This is equivalent to all nonzero terms in  $\det(\tilde{B})$  are equal and are  $\pm 1$ . The latter claim follows from the former, since the all-horizontal tiling contributes  $\pm 1$ .

Then it is enough to show that the contributions of two tilings that differ by a flip are equal to each other.

It means swapping two diagonal entries, thus change the sign of permutation, but one of them is  $1^2$  while the other being  $i^2$ , so the result does not change. ■

Now we can use some linear algebra to calculate the determinant.

Denote  $\tilde{M} = \begin{bmatrix} 0 & \tilde{B} \\ \tilde{B}^T & 0 \end{bmatrix}$ . Then  $\det(\tilde{M}) = \pm(\det(\tilde{B}))^2 = \pm(T(m, n))^2$ .

OBSERVATION We have

$$M = \text{id}_m \otimes A_n + A_m \otimes \text{id}_n,$$

where  $A_n, A_m$  are adjacency matrices of chain graphs. Similarly,

$$\tilde{M} = \text{id}_m \otimes A_n + iA_m \otimes \text{id}_n,$$

since  $\tilde{M}$  obtained by vertical tile with  $i$ 's. Hence the eigenvalues of  $\tilde{M}$  are  $\lambda_i + i\mu_k$ .

Now we only need to find the eigenvalues of chain graph. For a  $n$ -chain, we have

$$A_n = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

**Proposition 2.1.2.** *The eigenvalues of  $A_n$  are*

$$\lambda_k = 2 \cos \left( \frac{k\pi}{n+1} \right) \quad \text{for } k = 1, \dots, n.$$

*Proof.* An eigenvector  $u = (u_1, \dots, u_n)^T$  of  $A_n$  associated with eigenvalue  $\lambda$  satisfies

$$u_{j-1} + u_{j+1} = \lambda u_j, \quad 1 \leq j \leq n$$

with the convention that  $u_0 = u_{n+1} = 0$ .

A divine revelation: recall that

$$\sin \alpha + \sin \beta = 2 \cos \frac{\beta - \alpha}{2} \sin \frac{\alpha + \beta}{2}.$$

This suggests taking

$$u_j = \sin \left( \frac{\pi k j}{n+1} \right) \quad \text{for } j = 1, \dots, n.$$

with eigenvalue

$$\lambda_k = 2 \cos \left( \frac{k\pi}{n+1} \right).$$

■

**Example 2.1.1.**

$$n = 3, \det(t \text{id} - A_3) = t^3 - 2t = t(t - \sqrt{2})(t + \sqrt{2}).$$

So the eigenvalues are

$$\lambda_1 = \sqrt{2} = 2 \cos \left( \frac{1\pi}{4} \right), \lambda_2 = 0 = 2 \cos \left( \frac{2\pi}{4} \right), \lambda_3 = -\sqrt{2} = 2 \cos \left( \frac{3\pi}{4} \right).$$

Now

$$\begin{aligned} \det \tilde{M} &= \prod_{j=1}^n \prod_{k=1}^m \left( 2 \cos \frac{j\pi}{n+1} + i 2 \cos \frac{k\pi}{m+1} \right) \\ &= \prod_{j=1}^{n/2} \prod_{k=1}^m \left( 2 \cos \frac{j\pi}{n+1} + i 2 \cos \frac{k\pi}{m+1} \right) \left( 2 \cos \frac{(n+1-j)\pi}{n+1} + i 2 \cos \frac{k\pi}{m+1} \right) \\ &= \prod_{j=1}^{n/2} \prod_{k=1}^m \left( 2 \cos \frac{j\pi}{n+1} + i 2 \cos \frac{k\pi}{m+1} \right) \left( -2 \cos \frac{j\pi}{n+1} + i 2 \cos \frac{k\pi}{m+1} \right) \\ &= \pm \prod_{j=1}^{n/2} \prod_{k=1}^m \left( 4 \cos^2 \frac{j\pi}{n+1} + 4 \cos^2 \frac{k\pi}{m+1} \right) \end{aligned}$$

**Theorem 2.1.1** (P.Kasteleyn, M.Fisher, H.N.V.Temperley, 1961). *When  $m$  is even,*

$$T(m, n) = \prod_{j=1}^{n/2} \prod_{k=1}^{m/2} \left( 4 \cos^2 \frac{j\pi}{n+1} + 4 \cos^2 \frac{k\pi}{m+1} \right).$$

*When  $m$  is odd,*

$$T(m, n) = \prod_{j=1}^{n/2} 2 \cos \frac{j\pi}{n+1} \prod_{k=1}^{(m-1)/2} \left( 4 \cos^2 \frac{j\pi}{n+1} + 4 \cos^2 \frac{k\pi}{m+1} \right).$$

**Example 2.1.2.** For  $n = m = 8$ , we get  $T(8, 8) = 12,988,816 = 3604^2$ .

**PROBLEM\*** For any positive integer  $a \in \mathbb{Z}_{>0}$ ,  $T(4a, 4a)$  is a perfect square,  $T(4a-2, 4a-2)$  is twice a perfect square.

Asymptotics of  $T(n, n)$ : reasonable to expect  $T(n, n) \sim e^{cn^2}$ .

We take the natural log of  $T(n, n)$ :

$$\begin{aligned}\frac{\ln T(n, n)}{n^2} &= \frac{1}{n^2} \sum_{k=1}^{n/2} \sum_{j=1}^{n/2} \ln \left( 4 \cos^2 \frac{\pi k}{n+1} + 4 \cos^2 \frac{\pi j}{n+1} \right) \\ &\sim \frac{1}{\pi^2} \sum \sum \left( \frac{\pi}{n+1} \right)^2 \ln \left( 4 \cos^2 \frac{\pi k}{n+1} + 4 \cos^2 \frac{\pi j}{n+1} \right)\end{aligned}$$

Notice that the right-hand side is a Riemann sum of the function  $\ln(4 \cos^2 x + 4 \cos^2 y)$ . So the sum approaches to

$$\frac{1}{\pi^2} \int_0^{\pi/2} \int_0^{\pi/2} \ln(4 \cos^2 x + 4 \cos^2 y) dx dy = \frac{K}{\pi}$$

where  $K$  is Catalan's constant. As of today, it is not known whether it is irrational, nor transcendental.

So we have  $T(n, n) \approx 1.34^{n^2}$ .

Another way to define Catalan's constant:

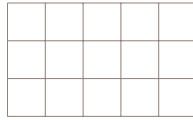
$$K = \beta(2) = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)^2} = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \dots$$

## 2.2 Spanning Tree in Grid Graphs, Planar Graphs

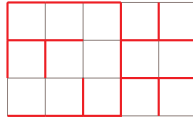
*He published this result when he was 60, which proved that aged people can still do mathematics.*

– Sergey Fomin on H.N.V. Temperley

Suppose a grid graph  $G$ :



We can keep some edges and discard others to obtain a connected acyclic subgraph of  $G$  (which is a spanning tree).



**Theorem 2.2.1** (H.N.V. Temperley, 1974). *Consider a rectangular board of odd size  $(2k-1) \times$*



$2\ell - 1$ ) with one corner removed. The number of domino tilings of the board is equal to the number of spanning trees in the  $k \times \ell$  grid.

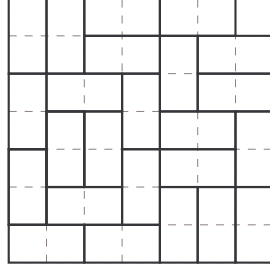


Figure 2.3: A domino tiling satisfying the condition

*Proof.* Find a bijection between domino tilings and spanning trees.

**PROBLEM** Prove that Temperley's map showed in Figure 2.4 produces a tree.

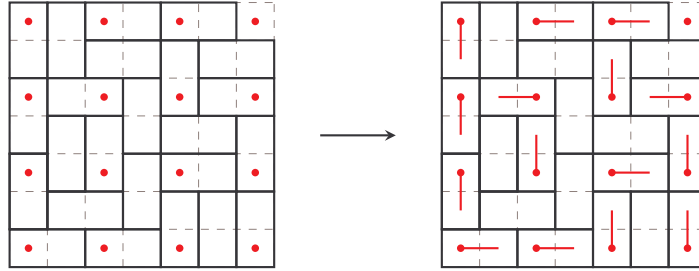


Figure 2.4: Converting domino tiling into trees

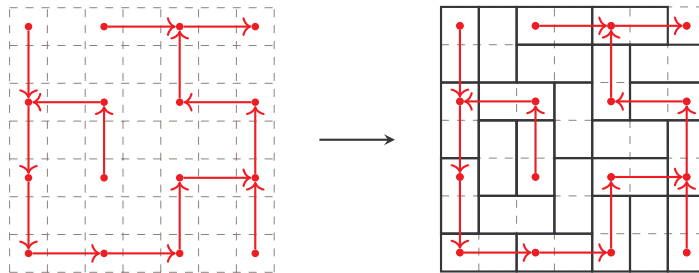


Figure 2.5: Converting domino tiling into trees

Now we have a forward map. We also need to obtain the inverse map from spanning trees to domino tiling. Fixing a border point as the root of the tree, we can make the tree a directed graph and add domino tiles accordingly. ■

**Corollary 2.2.1.**

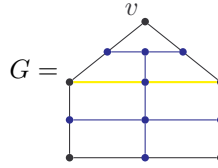
$$\# \text{ of spanning trees in a } k \times \ell \text{ grid} \approx \left( e^{\frac{4K}{\pi}} \right)^{k\ell} \approx 3.21^{k\ell}.$$

PROBLEM Prove that the number of domino tilings (if exist) of an odd-by-odd rectangle with a boundary box removed doesn't depend on which box we removed.

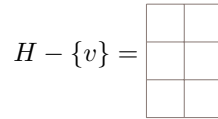
We now consider a similar problem:

Suppose  $P$  is a polygon,  $G$  a polygonal subdivision of  $P$ . Define  $H$  by adding midpoints and extra vertex in each bounded face and adding edges to connect them.

PROBLEM Show that the number of spanning trees in  $G$  is equal to the number of perfect matchings in  $H$  with one vertex that are also in  $P$  removed.

**Example 2.2.1.**

The number of spanning trees of  $G$  is  $4 + 4 + 3 = 11$ . If we take the vertex  $v$  specified above we have:



We can verify that it also has 11 matchings.

NOTE Here for arbitrary vertex  $v$  the result would be the same.

## 2.3 The Diamond Lemma

*This should really be explained to six-graders... Then they will have of joy of discovering this result.*

– Sergey Fomin on the diamond lemma

**Definition 2.3.1.** A one-player game is defined by:

- the set of positions  $\mathcal{S}$
- for each  $s \in \mathcal{S}$  a set of positions  $s' \neq s$  into which the player can move from  $s$ . Denote as  $s \rightsquigarrow s'$ .

If the latter set is empty, then  $s$  is called *terminal*.

A *play sequence* is a sequence

$$s \rightsquigarrow s' \rightsquigarrow s'' \rightsquigarrow \dots$$

A game is *terminating* if  $\nexists$  infinite play sequences.

A game is *confluent* if its outcome is uniquely determined by initial position.

**Lemma 2.3.1** (The Diamond Lemma for terminating games). *For a one-player game, assume that*

- *the game is terminating*
- ◊ *(diamond condition)  $\forall s \in \mathcal{S}, \forall s \rightsquigarrow s', s \rightsquigarrow s'', \exists$  some position that can be reached from both  $s'$  and  $s''$ . (You never say goodbye forever!)*

*Then the game is confluent.*

*Proof.* Color the position:

- Green is the terminal position reachable from this position is unique.
- Red otherwise.

Assume a red position exists. Starts at the red position until no move into red position exists.

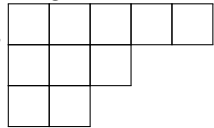
For each green position, there is a unique terminal position. Since it starts from a red position, there need to be two distinct outcomes. But that is a contradiction, since all green position are obtained from a certain red position, which means any two of them will have a common successor, then from now on the outcome should be the same. ■

## 2.4 Using the Diamond Lemma

*I play Wordle twice a day. Once in English, once in Russian.  
The Russian version is simpler because there are much fewer  
5-letter words in Russian.*

– Sergey Fomin

**Definition 2.4.1** (Young diagrams). A diagram in which the number of boxes on a row is decreasing. An example of which is



We define a one-player game where: Position = {Young diagrams}

Move = Removal of a domino tile from the SE rim that also results in a Young diagram.

CLAIM This game is *confluent*.

Note: the remaining shape would always be a staircase. If we color the blocks black and white alternatively, we can determine the final shape by the difference between white and black boxes.

PROBLEM Consider a similar game, but we are removing border strips consisting of  $p$  boxes ( $p \in \mathbb{N}$ ). Prove that the game is confluent.

**Definition 2.4.2** (Young tableaux). Take a Young diagram and fill it with numbers so that each row and column is in increasing order. Such diagram is called a standard Young tableau (SYT).

Or take a skew shape where a Young diagram is taken away from the top left corner of another Young diagram. Then filling it the same way we obtain standard skew tableau (Skew SYT).

1	2	5	8
3	6	7	
4			

			3	6
	1	4	8	9
2	5	7	10	

Figure 2.6: A standard Young tableau (left) and skew tableau (right)

JEU DE TAQUIN [M.-P. Schützenberger] Given a skewed tableau, choose a top-left corner piece and move the blocks one at a time so that after a series of moves we also get a skewed tableau.

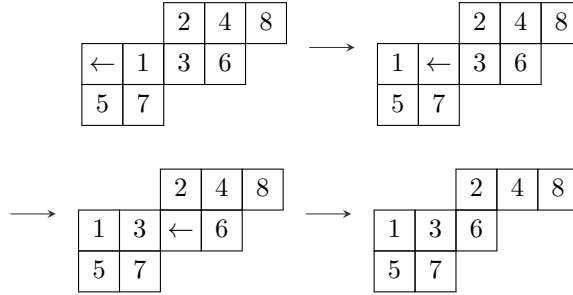


Figure 2.7: One step in a jeu de taquin game

The game ends on a SYT, called a *rectification* of  $T$ .

PROBLEM The rectification is unique. (Jeu de taquin is confluent.)

**Definition 2.4.3** (Tutte Polynomial).  $T_G(x, y)$  of a graph  $G$  is defined recursively as follows:

- $G$  has no edges  $\implies T_G = 1$ .

- $e$  edge in  $G \implies$

$$T_G = \begin{cases} xT_{G-e} & e \text{ is a bridge} \\ yT_{G-e} & e \text{ is a loop} \\ T_{G-e} + T_{G/e} & \text{otherwise} \end{cases}$$

This is a two variable generalization of the chromatic polynomial.

PROBLEM Use the diamond lemma to show that  $T_G$  is well defined.

For non-terminating games, the diamond lemma does not necessarily hold:

**Example 2.4.1** (Naive counterexample). Suppose a game:

$$1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow \dots, n \rightarrow \infty, \forall n.$$

Then there are two outcomes for any given starting position ( $\infty$  or non-terminating).

**Theorem 2.4.1** (Diamond Lemma for Non-terminating Games). *Suppose a one-player game.  $\forall s \in \mathcal{S}, \forall s \rightsquigarrow s', s \rightsquigarrow s'', \exists$  some position that can be reach from both  $s'$  and  $s''$  in the same number of steps. Then the game is confluent.*

*Moreover, if the game terminates for a given initial position, then it does so in a fixed number of steps.*

*Proof.* Left as PROBLEM. ■

## 2.5 Loop-erased Walks

*He took my class shortly before he discovered this algorithm (on generating uniform spanning trees)... After that he worked at Microsoft Research for 10 years until they decided they don't need mathematicians anymore and fired everybody.*

– Sergey Fomin on D.B. Wilson

**Definition 2.5.1** (G. Lawler, 1980). Suppose  $G$  a connected graph. Let  $\pi$  be a (finite) walk in  $G$ .  $\text{LE}(\pi)$  "loop erasure" of  $\pi$  is defined by Algorithm 1.

### STACKS & CYCLE POPPING

Recall: Markov chains.

"Running a Markov chain with stacks": at each state, decide on transition choices *in*

**Algorithm 1** Loop erasure

---

```

1: function LE( $\pi$ )
2:   if  $\pi$  does not intersect itself then
3:     return  $\pi$ 
4:   else
5:     Remove the first cycle of  $\pi$  to get  $\pi'$ 
6:     return LE( $\pi'$ )
7:   end if
8: end function

```

---

*advance.*

$v$  is a vertex,  $u(v) = (u_1, u_2, u_3, \dots)$  where  $u_k$  denotes the vertex we move to after visiting  $v$  for the  $k$ -th time. They are i.i.d RV's.

Assume: the Markov chain arrives with probability 1 at an absorbing state (where the stack is empty).

Given a collection of partially depleted stacks, we get a graph (A subgraph of  $G$ , the underlying oriented graph of Markov chain) determined by the top of each stack. The out degrees of each non-absorbing vertex in the graph is equal to 1. The removing of cycles from this graph is called *cycle-popping*.

LOOP-ERASED RANDOM WALK

Start at vertex  $s$ , and stop upon arriving at some absorbing state  $t$ .

OBSERVATION LERW is obtained by popping some cycle, leaving a path from  $s$  to  $t$ .

Define a game where positions are collections of stacks of at each vertex and the moves are cycle poppings.

CLAIM This game is confluent (by diamond lemma for non-terminating games).

The outcome of the game (after all cycles have been popped) is a rooted forest that is oriented towards absorbing states.

WILSON'S ALGORITHM [D.B. Wilson, 1996]

Input: connected loopless graph  $G$ . Output: random spanning tree  $T$  in  $G$ .

**Theorem 2.5.1.** *This algorithm, shown in Algorithm 2, outputs a uniformly distributed spanning tree of  $G$ .*

*Proof.* Make  $r$  an absorbing state. Run Wilson's algorithm with stacks, each time designating the vertices in  $T$  as absorbing states. Loop erasure = cycle popping.

**Algorithm 2** Wilson's algorithm

---

```

1: function WILSON( $G = (V(G), E(G))$ )
2:    $T = (V(T), E(T)) := (\{r\}, \emptyset)$  for some  $r \in V(G)$ .
3:   while  $V(T) \neq V(G)$  do                                ▷ Continue until  $T$  covers all vertices
4:     Pick  $v \in V(G) \setminus V(T)$ 
5:     Run a simple random walk  $\pi$  from  $v$  until it hits  $T$ 
6:      $V(T) \leftarrow V(T) \cup \{v, \text{LE}(\pi) \text{ includes vertex } v\}$ 
7:      $E(T) \leftarrow E(T) \cup \{e, \text{LE}(\pi) \text{ includes edge } e\}$ 
8:   end while
9:   return  $T$ 
10: end function

```

---

The algorithm terminates with probability 1, revealing the tree lying underneath all pop-able cycles.

Need: the output tree  $T$  is uniformly distributed.

Suppose  $H$  = heap of cycles.  $\mathbb{P}(T, H)$  denotes probability of getting  $T$  after removing  $H$ . We have

$$\mathbb{P}(T, H) = \left( \prod_{v \in T - \{r\}} \deg(v)^{-1} \right) \cdot \left( \prod_{v \in H} \deg(v)^{-1} \right),$$

while

$$\mathbb{P}(T) = \sum_H \mathbb{P}(T, H).$$

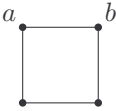
All the expressions above have the same values regardless of  $T$ , so the distribution is uniform. ■

The art of computer programming, vol 4. (generate random combinatorial objects that satisfies certain distribution)

**PROBLEM** Let  $a, b$  be two vertices in a connected graph  $G$ . Show that the following two constructions produce the same distributions on the set of self-avoiding walks from  $a$  to  $b$ .

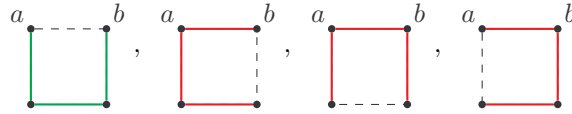
- Run a simple random walk  $\pi$  that starts at  $a$  and stops at  $b$ . Output  $\text{LE}(\pi)$ .
- Choose uniformly at random a spanning tree in  $G$ . Output the walk from  $a$  to  $b$  in the spanning tree.

**Corollary 2.5.1.** *This distribution does not change if we swap  $a$  and  $b$ .*

**Example 2.5.1.**  $G =$  . Using the first method we have

$$\mathbb{P} \left( \begin{array}{c} \text{---} \bullet \text{---} \bullet \text{---} \\ | \quad | \\ \text{---} \bullet \text{---} \bullet \text{---} \end{array} \right) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}, \quad \mathbb{P} \left( \begin{array}{c} \text{---} \bullet \text{---} \bullet \text{---} \\ | \quad | \\ \text{---} \bullet \text{---} \bullet \text{---} \end{array} \right) = \frac{1}{4}.$$

For the spanning tree method, we have these spanning trees:



## 2.6 Flows

(Making analogies using distilling of Whiskey)...

"I actually don't drink any alcohol at all."

— Sergey Fomin

**Definition 2.6.1.** Suppose  $G$  is connected loopless graph with  $a, b$  designated as *boundary* vertices and all other vertices called *interior* vertices.

A *flow*  $f$  assigns a number  $f(e, u, v)$  to each edge  $e$  with endpoint  $u, v$ , so that

- $f(e, u, v) = -f(e, v, u)$ .
- $\forall$  interior vertex  $u$ ,  $\sum_{v \in V(G), u \neq v \in E(G)} f(e, u, v) = 0$ . ("Conservation of flow")

It follows that  $\exists$  number  $|f|$ , called the *total flow* from  $a$  to  $b$ , such that

$$\sum_{v \in V(G), u \neq v \in E(G)} f(e, u, v) \begin{cases} 0 & u \text{ interior} \\ |f| & u = a \\ -|f| & u = b. \end{cases} \quad (2.6.1)$$

We basically created a weighted graph in some sense:  $w : E(G) \rightarrow \mathbb{R}_{>0}$ .

**Definition 2.6.2.** For any vertex  $u$ ,  $w(u) := \sum_{u \neq v \in E(G)} w(e)$ .

A "weighted version" of a simple random walk is a Markov chain with transition prob-

ability from  $u$  to  $v$  being  $\sum_{u \neq v \in E(G)} \frac{w(e)}{w(u)}$ .



PROBLEM\* Generalize Wilson's algorithm to weighed graphs.

### RESISTOR NETWORKS

**Definition 2.6.3.** Resistor network = weighted graph, conductance = edge weights, conductance =  $\frac{1}{\text{resistance}}$ .

(2.6.1) is the first Kirchhoff law. (conversation of charge/current)

The second Kirchhoff law expresses the conversation of energy: given a cycle

$$u_0 \xrightarrow{e_1} u_1 \xrightarrow{e_2} u_2 \xrightarrow{e_3} \dots \xrightarrow{e_k} u_k = u_0$$

we have

$$\sum_{i=1}^k \frac{f(e_i, u_{i-1}, u_i)}{w(e_i)} = 0. \quad (2.6.2)$$

(2.6.1) and (2.6.2) are a linear system of equations in  $f(e, u, v)$ .

**Theorem 2.6.1.** *Kirchhoff's equations (2.6.1) and (2.6.2) have a unique solution.*

## 2.7 Potentials

*"He is a wonderful mathematician."*

*(He is also a great lecturer...)*

*"Well some people have them both."*

– Jeffrey C. Lagarias on Sergey Fomin

**Definition 2.7.1.** Suppose  $f$  satisfies (2.6.2). Define the potential  $p = p_f$ , a function on the vertices of  $G$ , as follows:

- assign an arbitrary value to  $p(a)$
- for any walk that starts at  $a = u_0 \xrightarrow{e_1} u_1 \xrightarrow{e_2} u_2 \xrightarrow{e_3} \dots \xrightarrow{e_k} u_k = u$ , set

$$p(u) := p(a) + \sum_{i=1}^k \frac{f(e_i, u_{i-1}, u_i)}{w(e_i)}.$$

**Lemma 2.7.1.** *The function  $u \mapsto p(u)$  is well-defined.*

**Proposition 2.7.1** (Ohm's Law). *For an edge  $u \xrightarrow{e} v$ , we have*

$$\frac{f(e, u, v)}{w(e)} = p(v) - p(u)$$

This shows us how to recover flow from potential. (Doesn't it look like FTC?)

Condition (2.6.1) becomes

$$\sum_{u \in v} w(e)(p(v) - p(u)) = 0 \quad (2.7.1)$$

given a fixed interior vertex  $u$ .

$p$  is a discrete harmonic function with respect to weight  $w$  in the sense that

$$p(u) = \sum_{u \in v} \frac{w(e)}{w(u)} p(v) \quad (2.7.2)$$

(2.7.2) should be satisfied at each interior vertex  $u$ . For a *conservative* flow (no boundary), (2.7.2) is satisfied everywhere.

## 2.8 The Maximum Principle

*...no, the graph is finite. We are doing combinatorics.*

– Sergey Fomin

**Lemma 2.8.1.** *Suppose  $p$  is a discrete harmonic real function on the vertices of a connected weighted finite graph with positive valued weights. Then  $p$  attains its maximum and minimum values at the boundary (where harmonicity conditions does not hold). In the absence of the boundary,  $p$  must be constant.*

We can now prove the uniqueness of a solution to Kirchhoff equations.

*Proof.* Suppose  $f_1, f_2$  satisfy (2.6.1) and (2.6.2). For a given total flow value  $|f_1| = |f_2|$ . Then  $f = f_1 - f_2$  is a conservative flow satisfying (2.6.2). Now the potential  $p = p_f$  is harmonic everywhere. By the maximum principle  $p$  is constant. Now by Ohm's Law,  $f = 0$  everywhere  $\implies f_1 = f_2$ . ■

Now we need the existence of the solution to Kirchhoff equations. We give a method to construct a solution the Kirchhoff equations using random walks.

Take  $\pi$  random walk on  $G$  with

- starts at  $a$
- stops at  $b$
- proceeds from  $u$  along  $u \in v$  with probability  $\frac{w(e)}{w(u)}$ .

Denote  $\pi(t)$  the vertex occupies by  $\pi$  at time  $t \in \mathbb{Z}_{\geq 0}$ . Define the Bernoulli random variables

$$\xi_t(e, u, v) \begin{cases} 1 & \text{if } \pi(t) = u \text{ and } \pi \text{ then moves along } e \text{ to } v \\ 0 & \text{otherwise} \end{cases}$$

Consider the expectation

$$\sum_t \mathbb{E} \xi_t(e, u, v) = \frac{w(e)}{w(u)} \sum_t \mathbb{P}(\pi(t) = u) < \infty.$$

**Proposition 2.8.1.** Define  $f$  by

$$f(e, u, v) = \sum_t \mathbb{E} \xi_t(e, u, v) - \sum_t \mathbb{E} \xi_t(e, v, u).$$

Then  $f$  satisfies (2.6.1) and (2.6.2) with  $|f| = 1$ .

*Proof.*

- We have

$$\begin{aligned} \sum_{u \in v} f(e, u, v) &= \sum_{u \in v} \sum_t \mathbb{E} \xi_t(e, u, v) - \sum_{u \in v} \sum_t \mathbb{E} \xi_t(e, v, u) \\ &= E \# \{ \text{departures from } u \} - E \# \{ \text{arrivals from } u \} \\ &= \begin{cases} 0 & u \text{ internal,} \\ 1 & u = a, \\ -1 & u = b. \end{cases} \end{aligned}$$

This shows  $f$  satisfies (2.6.1).

- Suppose a cycle  $u_0 \xrightarrow{e_1} u_1 \xrightarrow{e_2} u_2 \xrightarrow{e_3} \dots \xrightarrow{e_k} u_k = u_0$ . Then

$$\begin{aligned} \sum_{i=1}^k \frac{f(e_i, u_{i-1}, u_i)}{w(e_i)} &= \sum_t \sum_i \frac{\mathbb{E} \xi_t(e_i, u_{i-1}, u_i)}{w(e_i)} - \sum_t \sum_i \frac{\mathbb{E} \xi_t(e_i, u_i, u_{i-1})}{w(e_i)} \\ &= \sum_t \sum_i \frac{\mathbb{P}(\pi(t) = u_{i-1})}{w(u_{i-1})} - \sum_t \sum_i \frac{\mathbb{P}(\pi(t) = u_i)}{w(u_i)} \\ &= 0. \text{ (observe that } u_k = u_0) \end{aligned}$$

This shows  $f$  satisfies (2.6.2). ■

**Theorem 2.8.1** (G. Kirchhoff, 1847). Suppose  $G$  contains an edge  $a \stackrel{e_0}{\sim} b$  ( $a, b$  are boundary ver-

tices). Run a unit current (i.e. flow)  $f$  in  $G$  from  $a$  to  $b$ . Then

$$f(e_0, a, b) = \mathbb{P}(\text{random spanning tree in } G \text{ contains } e_0)$$

where  $\mathbb{P}(T)$  is proportional to  $\prod_{e \in T} w(e)$ .

*Proof.* Use weighted version of Wilson's Algorithm (which is published 150 years after this result) to generate  $T$ , using  $b$  as the root and a random walk  $\pi$  starting at  $a$  as the first step. We have

$$\mathbb{P}(T \text{ contains } e_0) = \mathbb{P}(\text{LE}(\pi) \text{ contains } e_0) = \mathbb{P}(\pi \text{ contains } e_0).$$

On the other hand,

$$f(e_0, a, b) = \sum_t \mathbb{E} \xi_t(e, a, b) - \underbrace{\sum_t \mathbb{E} \xi_t(e, b, a)}_{=0} = \mathbb{P}(\pi \text{ contains } e_0). \quad \blacksquare$$

## 2.9 Effective Conductance

*This is like MATH 454 or something... I don't know. I didn't take that class.*

– Sergey Fomin

**Definition 2.9.1.** Define effective conductance as

$$w(G, a, b) := \frac{|f|}{p(b) - p(a)}$$

where  $f$  is a flow in  $G$  of total value  $|f|$  and  $p = p_f$ .

**Corollary 2.9.1.** The potential  $p = p_f$  of a flow  $f$  is uniquely determined, up to a constant, by:

1.  $p$  is discrete harmonic in the interior of  $G$ ;
2.  $p(b) - p(a) = \frac{|f|}{w(G, a, b)}$ .

*Proof.* Suppose  $p, p'$  are two functions satisfying the conditions above. Let  $\Delta p = p - p'$ . Then  $\Delta p$  is harmonic outside  $\{a, b\}$  and moreover  $\Delta p(a) - \Delta p(b) = 0$ . By the maximum principle,  $\Delta p$  must be constant.  $\blacksquare$

Probabilistic distribution of potentials

PROBLEM Let  $f$  be a unit flow and  $p = p_f$  its potential. Suppose  $u$  is a vertex and  $\pi_u$

random walk starting at  $u$ . Prove that

$$p(u) - p(a) = \frac{\mathbb{P}(\pi_u \text{ visits } b \text{ before } a)}{w(G, a, b)}.$$

**Corollary 2.9.2.** *For any flow  $f$  with potential  $p$ ,*

$$\frac{p(u) - p(a)}{p(b) - p(a)} = \mathbb{P}(\pi_u \text{ visits } b \text{ before } a)$$

**PROBLEM** Assume that  $G \subset H$  and  $G$  coincides with  $H$  at  $a, b$ , in the sense that if there are edges in  $H$  from vertices except  $a$  and  $b$  that are both in  $G$  and  $H$ , it should be contained in  $G$ .

$H'$  is the weight graph obtained from  $H$  by replacing  $G$  by a single edge  $a \overset{e_0}{\longleftrightarrow} b$  of weight  $w(e_G) = W(G, a, b)$ . Show that the solutions to Kirchhoff equations for  $H, H'$  coincide outside  $G$ .

**Theorem 2.9.1** (G. Kirchhoff, 1847). *Suppose  $G$  is a resistor network with unit conductances. Then*

$$w(G, a, b) = \frac{\#\{\text{spanning trees in } G\}}{\#\{\text{spanning trees in } G \text{ with two components containing } a \text{ and } b \text{ respectively}\}}.$$

Alternatively,

$$w(G, a, b) = \frac{\#\{\text{spanning trees in } G\}}{\#\{\text{spanning trees in } G_{ab}\}},$$

where  $G_{ab}$  is obtained from  $G$  by gluing  $a$  and  $b$ .

**Example 2.9.1.**

1.  $n$ -chain graph (series circuit):



$\#\{\text{spanning trees in } G\} = 1, \#\{\text{spanning trees in } G_{ab}\} = n$ . So effective conductance  $w(G, a, b) = \frac{1}{n}$ .

2. Parallel circuit:  $\#\{\text{spanning trees in } G\} = n, \#\{\text{spanning trees in } G_{ab}\} = 1$ . So effective conductance  $w(G, a, b) = n$ .

*Proof.* Add an edge  $e$  between  $a, b$  with  $w(e) = 1$  and get  $H$ . Run a unit current in  $H$

from  $a$  to  $b$ . By Kirchhoff's 2nd law,

$$\frac{f(e, a, b)}{w(e, a, b)} = \frac{1 - f(e, a, b)}{w(G, a, b)} \implies f(e, a, b) = \frac{1}{1 + w(G, a, b)}.$$

By 2.8.1,

$$f(e, a, b) = \frac{\#\{\text{spanning trees in } H \text{ containing } e\}}{\#\{\text{spanning trees in } H\}}.$$

Hence

$$\begin{aligned} w(G, a, b) &= \frac{\#\{\text{spanning trees in } H \text{ not containing } e\}}{\#\{\text{spanning trees in } H \text{ containing } e\}} \\ &= \frac{\#\{\text{spanning trees in } G\}}{\#\{\text{spanning trees in } G_{ab}\}}. \end{aligned} \quad \blacksquare$$

PROBLEM Consider the  $n$ -cube graph  $G = (K_2)^n$ . Suppose  $a, b$  are antipodal vertices in  $G$ . Find  $w(G, a, b)$ .

PROBLEM Suppose  $G$  has  $n$  vertices and  $m$  edges. Assume that each edge belongs to the same number of spanning trees. Find the effective conductance between a pair of adjacent vertices.

Application: find the effective conductance between adjacent vertices of a dodecahedron.

Dessert!

"Squaring a square" – partitioning a rectangle into distinct squares. [L. Brooks, C. Smith, A. Stone, W. Tutte, 1938]

Horizontal bars are made of an ideal conductor. Separate the square plates so that square plates are resistors.

Want: potential of a vertex = its height.

PROBLEM Find another such tiling of 9 distinct squares of a rectangle.

For squares, [www.squaring.net](http://www.squaring.net)

## Chapter 3

# More Algebraic Graph Theory

### 3.1 Laplace Expansion

**Definition 3.1.1.** Suppose a  $m \times n$  matrix  $X$  and a set  $S$  of indices  $\leq n$ . Then  $X[S] = (x_{ij})_{j \in S}$  i.e.  $X[S]$  is the matrix selected from the  $j$ -th column of  $X$  where  $j \in S$ .

**Theorem 3.1.1** (Laplace expansion formula). Let  $Z = \begin{bmatrix} X \\ Y \end{bmatrix}$ , where  $X$  is an  $n \times (n + m)$  matrix and  $Y$  is an  $m \times (n + m)$  matrix. Then

$$\det(Z) = \sum_{\substack{S \subseteq [1, n+m] \\ |S|=n}} \operatorname{sgn}(S) \det X[S] \det Y[\bar{S}]$$

where

$$\bar{S} = [1, n + m] \setminus S, \quad \operatorname{sgn}(S) = (-1)^{\#\{\bar{s} < s \mid s \in S, \bar{s} \in \bar{S}\}}.$$

**Corollary 3.1.1.** Let  $A$  and  $B$  be matrices of sizes  $n \times m$  and  $m \times n$ , respectively, with  $n \leq m$ . Then

$$\det \begin{bmatrix} 0 & A \\ B & I_m \end{bmatrix} = (-1)^n \sum_{\substack{|S|=n \\ S \subseteq [1, m]}} \det A[S] \cdot \det B^T[S].$$

**Theorem 3.1.2** (Binet-Cauchy formula). Let  $A$  and  $B$  be matrices of sizes  $n \times m$  and  $m \times n$ , respectively, with  $n \leq m$ . Then

$$\det(AB) = \sum_{\substack{S \subseteq [1, m] \\ |S|=n}} \det A[S] \det B^T[\bar{S}].$$

*Proof.* ■

**Definition 3.1.2.** Let  $Z$  be a square matrix of size  $n$ . The  $(i, j)$  *minor* refers to the determinant of the  $(n - 1) \times (n - 1)$  submatrix formed by deleting the  $i$ -th row and  $j$ -th column from  $Z$ . The corresponding cofactor is the signed minor

$$Z_{ij} = (-1)^{i+j} \det(z_{kl})_{\substack{k \neq i \\ l \neq j}}$$

Given the definition of cofactors we have a formula of  $\det Z$ :

$$\det Z = \sum_j z_{aj} Z_{aj} = \sum_i z_{ib} Z_{ib}.$$

**Lemma 3.1.1.** Let  $Z = (z_{ij})$  be a square matrix where row sums are all 0. The all cofactors for a given row are equal to each other.

*Proof.* Let  $Z'$  be obtained from  $Z$  by adding 1 to the entry  $z_{i1}$ . Adding all columns of  $Z$  to column  $k$  and then expanding the determinant with respect to the column, we get that  $\det(Z') = Z_{ik}$ . Hence  $Z_{i1} = Z_{i2} = \dots = \det(Z')$ . ■

## 3.2 Matrix Representation of Direct Graphs

**Example 3.2.1.**

**Lemma 3.2.1.** Let  $T$  be a tree and  $M_0$  be the incidence matrix of an orientation  $T$  with one of its rows removed. Then  $\det(M_0) = \pm 1$ .

*Proof.* Exactly one term in the expression of the determinant is nonzero, which corresponds to mapping each edge to its endpoint that is farther from the root. ■

**Corollary 3.2.1.** Let  $G$  be an undirected graph and  $M_0$  denote the incidence matrix of an orientation of  $G$ , with one row (corresponds to some vertex  $v$ ) removed. Then the maximal minors of

$$M_0 \text{ are given by } \det M_0[S] = \begin{cases} \pm 1 & S \text{ forms a spanning tree} \\ 0 & \text{otherwise} \end{cases}.$$

*Proof.* The first case is already done.

We need to show the second case. The subgraph formed by  $S$  must be connected. One of the components doesn't contain  $v$ . The sum of the corresponding rows of  $M_0[S]$  is equal to 0. ■



**Definition 3.2.1** (Discrete Laplacian). Suppose  $G$  is a loopless graph on vertices  $1, \dots, n$ . The *Laplacian* of  $G$  is the matrix  $L = L(G) = (L_{ij})_{i,j=1}^n$  defined by

$$L_{ij} := \begin{cases} \deg(i) & i = j \\ -k & i - j \text{ with } k \text{ edges.} \end{cases}$$

This is the matrix is a system of linear equations that satisfied by (discrete) harmonic functions on  $G$ .

**Example 3.2.2.**  $G = 4\text{-cycle}$ ,  $L(G) = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$ .

REMARK: The row (respectively, column) sums to 0 in  $L(G)$ .

**Corollary 3.2.2.** All cofactors of  $L(G)$  are equal to each other.

**Lemma 3.2.2.** Suppose  $M$  is an incidence matrix of an orientation of  $G$ , then  $L(G) = MM^T$ .

*Proof.* Direct inspection. ■

Relation to PDE's.  $\nabla \nabla^T = \Delta$ .

### 3.3 The Matrix-Tree Theorem

*Not sure this is the best terminology, but it is the established one, so I'll use this.*

– Sergey Fomin

(implicit in G. Kirchhoff)

**Theorem 3.3.1.** # of spanning trees in  $G = (\text{any})$  cofactor of  $L(G)$ .

**Example 3.3.1.**  $G = 4\text{-cycle}$  then

$$\det \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix} = 8 - 2 - 2 = 4.$$

*Proof.* Let  $L_0 = L(G)$  without last row and column. Then, by Binet-Cauchy,

$$\det(L_0) = \det(M_0 M_0^T) = \sum_{|S|=n-1} (\det M_0[S])^2 = \# \text{ of spanning trees.} \quad \blacksquare$$

## Cofactors and Eigenvalues

**Proposition 3.3.1.**  $Z = n \times n$  matrix with zero row and column sums. Let  $\lambda_1, \dots, \lambda_{n-1}, \lambda_n = 0$  be eigenvalues of  $Z$ . Then any cofactor of  $Z$  is equal to  $\frac{1}{n} \lambda_1 \cdot \dots \cdot \lambda_{n-1}$ .

*Proof.*  $\det(Z - tI_n) = \det(Z) - t \cdot nZ_{11} + \text{some higher order terms}$ . Also  $= \prod(\lambda_i - t) = -\lambda_1 \lambda_2 \dots \lambda_{n-1} t + \text{some higher order terms}$ . ■

**Corollary 3.3.1.**  $G$  is (connected) loopless graph on  $n$  vertices.  $\lambda_1, \dots, \lambda_{n-1}, \lambda_n = 0$  are eigenvalues of  $L(G)$ .

Then the spanning trees in  $G = \frac{1}{n} \lambda_1 \dots \lambda_{n-1}$

Spanning trees in regular graphs.  $G$  is a regular graph of degree  $d$ . OBSERVATION:  $L(G) = dI - A(G) \implies \lambda_i = d - \mu_i$ .

**Example 3.3.2.**  $G = (K_2)^d$ .  $n = 2^d$  and eigenvalues of  $A(G) : d - 2k$  with multiplicities  $\binom{d}{k}, 0 \leq k \leq d \implies$  eigenvalues of  $L(G) : 2k$  with multiplicities  $\binom{d}{k}, 0 \leq k \leq d$ .

**Corollary 3.3.2.** # of spanning trees in  $(K_2)^d$  is equal to

$$\frac{1}{2^d} \prod_{k=1}^d (2k)^{\binom{d}{k}} = 2^{2^d - 1 - d} 1^{\binom{d}{1}} 2^{\binom{d}{2}} \dots d^{\binom{d}{d}}$$

For  $k = 3$ , this gives 384.

PROBLEM Find the number of spanning trees in the Peterson graph.

### 3.4 Cayley's Formula

**Theorem 3.4.1.** The number of trees in a fixed  $n$ -element set of vertices is  $n^{n-2}$ .

**Example 3.4.1.**  $n = 2$  :

$n = 3$  :

Basically we are counting spanning trees of a complete graph.

*Proof.* Eigenvalues of  $A(K_n) : -1, -1, \dots, n-1$ . Eigenvalues of  $L(K_n) : n, n, n, \dots, 0$ .

So the number of spanning trees:  $\frac{1}{n} n^{n-1} = n^{n-2}$ . ■

Directed Matrix-Tree theorem.

**Definition 3.4.1.** An *arborescence* rooted at a vertex  $v$  is a tree whose edges are directed away from  $v$ .

"spanning arborescence".

$\tau(D, v) = \#\text{spanning arborescences rooted at } v$ .

**Example 3.4.2.**

**Theorem 3.4.2.**  $\tau(D, v)$  = any cofactor in the row labeled by  $v$  of the matrix  $L(D) = (L_{ij})$  defined by

$$L_{ij} = \begin{cases} \text{indeg}(i) & i = j \\ \#\text{edges } j \rightarrow i & i \neq j, \end{cases}$$

where  $\text{indeg}(i) = \#\text{edges arriving at vertex } i$ .

*Proof.* Left as PROBLEM. ■

**Example 3.4.3.**

## 3.5 Eulerian Tours

**Definition 3.5.1.** An Eulerian tour is a closed walk that visits all vertices and traverses each edge exactly once. If such a tour exists, then the graph is *Eulerian*.

**Theorem 3.5.1** (L. Euler, 1735). *An undirected graph is Eulerian  $\iff$  it is connected with all degrees even.*

**Definition 3.5.2.** A directed graph  $D$  is *balanced* if for any  $v$ ,  $\text{indeg}(v) = \text{outdeg}(v)$ .

**Theorem 3.5.2.** *For a directed graph  $D$ , LEAE (the following are equivalent)*

- (a)  $D$  is Eulerian
- (b)  $D$  is connected (disregarding orientation) and balanced.
- (c)  $D$  is strongly connected and balanced.

**Definition 3.5.3.** A directed graph  $D$  is *strongly connected* if  $\forall u, v, \exists$  a walk  $u \rightsquigarrow v$ .

*Proof.* (a)  $\implies$  (c), (c)  $\implies$  (b) is clear.

Need (b)  $\implies$  (a). We prove by induction on  $\#\text{edges}$ . vertices.

Starting from  $v_0$ , build a closed walk in an arbitrary way. Then apply induction hypothesis to smaller connected components when the edges of the closed walk is not considered. Attach such walks to the original closed walk. ■

Suppose  $D$  an Eulerian graph. Let  $\varepsilon(D)$  denote the number of Eulerian tours that start with a fixed edge  $e$ .

**Theorem 3.5.3** (The BEST Theorem, N. De Bruijn, T. Ehrenfest (1951), C. Smith, W. Tutte).  
In an Eulerian directed graph  $D$ ,

$$\varepsilon(D) = \tau(D, v) \prod_v (\text{indeg}(v) - 1)!$$

*Proof.* Fix an edge  $e$  with arrives at  $v$ . #Eulerian tours with  $e$  is  $\varepsilon(D)$ . For each such tour and each vertex  $u \neq v$ , consider the edges of first arrival at  $u$ .

OBSERVATION these edges form a spanning arborescence rooted at  $v$ .

For a given Eulerian tour, order the  $\text{indeg}(u)$  edges coming into each vertex  $u$  in the order in which they are traversed by the tour. We get a collection of linear orderings of incoming edges (at each  $u$ ) such that

- (a) the unique edges  $e \in T$  is the first one in the ordering
- (b) at  $v$  the edge  $e$  is the last one in the ordering.

We got a correspondence between Eulerian tours ending with  $e$  with arborescence  $T$  rooted at  $v$ , collection of linear ordering satisfying (a), (b). ■

*This is all kind of clear if you just close your eyes and meditate.*

## 3.6

Postman Routes

$G$  is connected  $\implies D(G)$  is Eulerian.

"Postman route" = Eulerian Tour

## 3.7 De Bruijn Sequences

**Definition 3.7.1.** A binary de Bruijn sequence of degree  $n$  is  $(0, 1)$ -sequence that is  $2^n$ -periodic and whose list of contiguous segments of length  $n$  contains all binary strings of length  $n$ .

Application: "rotating drum" random number generator

This can be generalized to larger alphabets

The de Bruijn graph vertices = binary strings of length  $n - 1$

edges =  $\{(a_1, \dots, a_{n-1}) \rightarrow (a_2, \dots, a_{n-1}, a_n)\}$ .

The de Bruijn sequence is cryptomorphic to Eulerian tours in this graph since the de Bruijn graph is connected and balanced

Enumeration of de Bruijn sequences

**Corollary 3.7.1.** *# dB sequences of degree  $n$  = any cofactor of the directed Laplacian of the de Bruijn graph.*

*Proof.*



### 3.8 Construction of de Bruijn Sequences

## Chapter 4

# $q$ -binomial coefficients

### 4.1 Partitions and Tableaux

### 4.2 More Partition Identities

PROBLEM Suppose  $k, n > 0$ . Prove that

$$\begin{aligned} \#\{\text{ptns of } n \text{ into parts NOT divisible by } k\} \\ = \#\{\text{ptns of } n \text{ s.t. the multiplicity of each part is } < k\}. \end{aligned}$$

**Example 4.2.1.**  $n = 6, k = 3$ . We have partition

$$\{51, 42, 411, 222, 2211, 21111, 111111\}, \{6, 51, 42, 411, 33, 321, 2211\}.$$

PROBLEM\* Prove that

$$\sum_{n \geq 0} \frac{x^{n^2}}{((1-x)(1-x^2) \dots (1-x^n))^2} = \prod_{k \geq 1} \frac{1}{1-x^k}.$$

### 4.3 The Dominance Order

*I should've assigned homework last Thursday but I took pity on you.*

– Sergey Fomin

**Definition 4.3.1.** The *dominance order* is the partial order on  $\text{Par}(n)$  defined by

$$\mu \leq \lambda \iff \forall i, \mu_1 + \dots + \mu_i \leq \lambda_1 + \dots + \lambda_i.$$

**Example 4.3.1.** For  $n = 6$  we have the following Hasse diagram:

One would expect it to be graded, but here is a counterexample: for  $n = 7$ ,

PROBLEM Show that the dominance order is a *lattice* (any two partitions of  $n$  have a least upper bound and greatest lower bound)

PROBLEM Show that the dominance order is *self-dual* under conjugation:  $\mu \leq \lambda \implies \lambda' \leq \mu'$ .

**Theorem 4.3.1** (No proof given). Denote the set of  $n \times n$  nilpotent matrices whose Jordan blocks have sizes  $\lambda_1, \lambda_2, \dots$  as  $N_\lambda$ . Then

$$\overline{N_\lambda} = \bigcup_{\lambda \leq \mu} N_\mu.$$

**Dominance in  $\mathbb{R}^n$ .** Suppose  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n) \in \mathbb{R}^n$  and  $\mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_n) \in \mathbb{R}^n$ . Let  $P_\lambda$  denote the convex hull of  $S_n$ -orbit of  $\lambda$ . (Here  $S_n$  is the symmetry group  $n$ )

PROBLEM Let  $\lambda, \mu$  defined as above (weakly decreasing coordinates). Show that for  $\mu, \lambda$  s.t.  $\mu_1 + \dots + \mu_n = \lambda_1 + \dots + \lambda_n$ . We have  $P_\mu \leq P_\lambda \iff \mu \leq \lambda$ . (Dominance extended to  $\mathbb{R}^n$ .)

**Example 4.3.2.** Let  $n = 3$ .

**Theorem 4.3.2** (The Schur-Horn Theorem). Suppose  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n) \in \mathbb{R}^n$  and  $\mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_n) \in \mathbb{R}^n$ . The followings are equivalent:

- $\exists$  Hermitian matrix with eigenvalues  $\mu_1, \dots, \mu_n$  and diagonal entries  $\lambda_1, \dots, \lambda_n$ .
- $\lambda_1 + \dots + \lambda_n = \mu_1 + \dots + \mu_n$  and  $\lambda \leq \mu$ . (dominance)

I. Schur [1923] proved  $\Downarrow$ , A. Horn [1954] proved  $\Uparrow$ .

## 4.4 The Lattices $L(m, n)$

Let  $L(m, n)$  be the partitions whose shape (is Young diagram) fits into the  $m \times n$  box. (i.e.  $\lambda_1 \leq n, \# \text{parts} \leq m$ )

$L(m, n)$  is partially ordered by *containment*. It is an easy exercise to check that  $L(m, n)$  is a *distributive lattice*.

OBSERVATION  $\#L(m, n) = \binom{m+n}{n} = \binom{m+n}{m}$ .

Let

$$g(m, n, a) := \#\{\text{ptns of } a \text{ in } L(m, n)\}$$

and

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_q \begin{bmatrix} m+n \\ n \end{bmatrix}_q := \sum_{a=0}^{mn} g(m, n, a) q^a$$

## 4.5 Sperner's Theorem

*Once you get stepped on by an elephant, you remain 2-dimensional for the rest of your life.*

**Proposition 4.5.1.** Boolean lattice  $\mathbb{B}_n$  is Sperner ( $|\text{the largest antichain}| = |\text{largest level set}|$ )

*Proof.* Let  $P_k = \{S \mid |S| = k, S \subset [1, n]\} \subset \mathbb{B}_n$ . So  $|P_k| = \binom{n}{k}$ . Suppose  $k < \frac{n}{2}$ .

Construct an injection  $\phi_k : P_k \rightarrow P_{k+1}$  s.t.  $\forall S \in P_k, \phi_k(S) > S$ . Similarly, for  $k > \frac{n}{2}$  we would get injections  $\psi_k : P_k \rightarrow P_{k-1}$  s.t.  $\psi_k(S) < S$ .

Stitching these injections together would give a partition of  $\mathbb{B}_n$  into  $\max |P_k| = \binom{n}{\lfloor n/2 \rfloor}$  chains, thus proving the theorem.

We prove the existence of such injections.

Let

$$\mathbb{R}P_k := \text{vector space formally spanned by } P_k.$$

Suppose we can build  $U_k : \mathbb{R}P_k \rightarrow \mathbb{R}P_{k+1}$  s.t.

- $U_k$  is injective:  $\text{rk}(U_k) = |P_k|$ ,
- $\forall x, U_k(x) \in \text{span}\{y \in P_{k+1}, y > x\}$ ,

then we are done.



Set

$$U_k(x) = \sum_{\substack{y \in P_{k+1} \\ y > x}} y, D_k(x) = \sum_{\substack{y \in P_{k-1} \\ y < x}} y.$$

We have that

$$D_{k+1} = U_k^*.$$

We introduce a lemma first.

**Lemma 4.5.1.**  $\forall k \in [0, n], D_{k+1}U_k - U_{k-1}D_k = (n - 2k)I.$

*Proof of lemma.* The off-diagonal entries of LHS are all 0. And we have

$$(D_{k+1}U_k - U_{k-1}D_k)x = (n - k - k)x + \dots$$

Some stuff ■

Unimodality of Gaussian coefficients

A sequence is unimodal if it has a (weakly) unique max. A polynomial is unimodal if its coefficients are unimodal.

*I accomplished with scarcely an effort a task which I had  
believed lay outside the range of human power.*

– James Joseph Sylvester

*The proof is very technical and un-illuminating*

– Sergey Fomin

**Theorem 4.5.1** (J. Sylvester, 1878).  $\left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right]_q$  is unimodal.

First combinatorial proof is found by Kathleen O'Hara in 1990.

**Theorem 4.5.2** (R.P. Stanley, 1980).  $L(m, n)$  is Sperner.

This is a stronger result (of which the proof) implies Sylvester's theorem.

## 4.6 Rank-Unimodality of the Lattice $L(m, n)$

**Definition 4.6.1.** Let  $P$  be a poset.  $\lambda \subset P$  is an *order ideal* iff  $\forall b \in \lambda, \forall c < b, c \in \lambda$ .

**Example 4.6.1.**  $P = [1, m] \times [1, n]$ , the product of  $m$  chain  $\{1, 2, \dots, m\}$  and  $n$  chain.

**Definition 4.6.2.**  $J(P) :=$  the set of finite order ideals of  $P$ , viewed as a poset ordered by containment.

**Example 4.6.2.** (a)  $P = [1, m] \times [1, n]$ ,  $J(P) = L(m, n)$ .

(b)  $P = \{n\text{-element anti-chain}\}$ ,  $J(P) = \mathbb{B}(n) = [0, 1]^n$ .

Any  $J(P)$  is a distributive lattice. In fact, every finite distributive lattice is isomorphic to  $J(P)$  for some  $P$ .

A key lemma:

**Lemma 4.6.1.** Let  $P$  be a finite poset. For  $k \in \mathbb{Z}_{\geq 0}$ ,  $J(P)_k$  be the order ideals  $\lambda \subset P$  s.t.  $|\lambda| = k$ .

For  $\lambda \in J(P)$ , denote

$$\begin{aligned} \text{add}(\lambda) &= \{x \notin \lambda \mid \lambda \cup \{x\} \in J(P)\} \\ \text{del}(\lambda) &= \{x \in \lambda \mid \lambda - \{x\} \in J(P)\}. \end{aligned}$$

Assume  $w : P \rightarrow \mathbb{R}_{\geq 0}$  satisfies  $\forall \lambda \in J(P)_k$ ,

$$\delta(\lambda) := \sum_{x \in \text{add}(\lambda)} w(x) - \sum_{x \in \text{del}(\lambda)} w(x) > 0.$$

Then  $\exists$  an order respecting injection  $J(P)_k \rightarrow J(P)_{k+1}$ .

NOTE: the converse is not necessarily true.

*Proof.* Let  $V_k = \mathbb{R}J(P)_k$  a formal linear combination of order ideal of size  $k$ .

Define linear maps

$$U_j : V_j \rightarrow V_{j+1}, \quad D_j : V_j \rightarrow V_{j-1}$$

by

$$\begin{aligned} U_j(\lambda) &= \sum_{x \in \text{add}(\lambda)} \sqrt{w(x)} (\lambda \cup \{x\}), \\ D_j(\lambda) &= \sum_{x \in \text{del}(\lambda)} \sqrt{w(x)} (\lambda - \{x\}). \end{aligned}$$

CLAIM:  $D_{k+1}U_k - U_{k+1}D_k = U_k^*U_k - U_{k-1}^*U_{k-1}$  has diagonal matrix with  $(\lambda, \lambda)$  entry described by  $\delta(\lambda)$ . ■

## 4.7 The Hooklength Formula

**Definition 4.7.1.** Young lattice  $\mathbb{Y} := J(\mathbb{Z}_{>0} \times \mathbb{Z}_{>0})$  the poset of *all* Young diagrams ordered by inclusion.

**Definition 4.7.2.** Denote  $f^\lambda := \#\{\text{standard Young diagram of size } \lambda\}$ .

**Example 4.7.1.**

**Theorem 4.7.1** (J.S. Frame, G. de B. Robinson, R.M. Thrall, 1954). For  $|\lambda| = n$  we have

$$f^\lambda = \frac{n!}{\prod_{x \in \lambda} h_\lambda(x)}.$$

**Example 4.7.2.**

**Theorem 4.7.2** (R.M. Thrall, 1952). The number of standard shifted tableaux of shape  $\lambda^*$  is equal to

$$\frac{n!}{\prod_{x \in \lambda^*} h_{\lambda^*}(x)},$$

where  $h_{\lambda^*}(x)$  is the size of "shifted hook".

**Example 4.7.3.**

*Proof of the Hooklength formula.* [C. Greene, A. Nijenhuis, H. Wilf 1979]

Suppose  $\lambda$  Young diagram with  $|\lambda| = n$ . Set  $H(\lambda) = \prod_{x \in \lambda} h_\lambda(x)$ . Here  $x$  is a unit box.

Want to show that  $f^\lambda = \frac{n!}{H(\lambda)}$ .

$$\text{del}(\lambda) = \{\text{corner boxes of } \lambda\}.$$

We have a recurrence relation

$$f^\lambda = \sum_{v \in \text{del}(\lambda)} f^{\lambda - \{v\}}.$$

It suffices to prove that

$$\begin{aligned} \frac{n!}{H(\lambda)} &= \sum_{v \in \text{del}(\lambda)} \frac{(n-1)!}{H(\lambda - \{v\})} \\ \iff n &= \sum_{v \in \text{del}(\lambda)} \frac{H(\lambda)}{H(\lambda - \{v\})}. \end{aligned}$$

Idea: construct a Markov chain on  $\lambda$  with certain properties:

- Start at a box  $u \in \lambda$ .
- Terminate (with probability 1) when it reaches  $v \in \text{del}(\lambda)$ .

Let  $\mathbb{P}(u, v) = \mathbb{P}(\text{end in } v \text{ when starting at } u)$ .  $\sum_{v \in \text{del}(\lambda)} \mathbb{P}(u, v) = 1$ .

- $\sum_{u \in \lambda} \mathbb{P}(u, v) = \frac{H(\lambda)}{H(\lambda - \{v\})}$ .

If such Markov chain exists, we get

$$n = \sum_{u \in \lambda} \sum_{v \in \text{del}(\lambda)} \mathbb{P}(u, v) = \sum_{v \in \text{del}(\lambda)} \sum_{u \in \lambda} \mathbb{P}(u, v) = \sum_{v \in \text{del}(\lambda)} \frac{H(\lambda)}{H(\lambda - \{v\})}.$$

We propose the following Markov chain defined by Algorithm 3. We claim that this walk

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**Algorithm 3** The Hook Walk

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1: function WALK(Young diagram  $\lambda$ , box  $u \in \lambda$ )
2:    $v \leftarrow u$ 
3:   while  $v \notin \text{del}(\lambda)$  do                                 $\triangleright$  Continue until  $v$  is a corner book
4:     Choose  $v' \in \text{hook}(v) \setminus \{v\}$  uniformly at random
5:      $v \leftarrow v'$ 
6:   end while
7:   return  $v$ 
8: end function

```

---

satisfies the above conditions.

We introduce some lemmas:

**Definition 4.7.3** (Lattice path). Suppose  $r, s \in \mathbb{Z}_{>0}$ .

$$\text{LP}(r, s) := \{\text{lattice paths from } (1, 1) \text{ to } (r+1, s+1)\}.$$

**Lemma 4.7.1.** Suppose  $a_1, \dots, a_r \in \mathbb{R}_{>0}, b_1, \dots, b_r \in \mathbb{R}_{>0}$  with  $a_{r+1} = 0$  and  $b_{r+1} = 0$ . Then

$$\sum_{\pi \in \text{LP}(r, s)} \prod_{\substack{(i, j) \in \pi \\ (i, j) \neq (r+1, s+1)}} \frac{1}{a_i + b_j} = \frac{1}{\prod_{i=1}^r a_i \prod_{j=1}^s b_j}. \quad (4.7.1)$$

**Example 4.7.4.** Take  $r = 1, s = 2$ . The lattice would look like

$$\begin{array}{ccc} \frac{1}{a_1 + b_1} & \frac{1}{a_1 + b_2} & \frac{1}{a_1} \\ \frac{1}{b_1} & \frac{1}{b_2} & \end{array}$$

$$\frac{1}{a_1 + b_1} \left( \frac{1}{b_1 b_2} + \frac{1}{a_1 + b_2} \left( \frac{1}{a_1} + \frac{1}{b_2} \right) \right) = \frac{1}{a_1 + b_1} \left( \frac{1}{b_1 b_2} + \frac{1}{a_1 b_2} \right) = \frac{1}{b_1 a_1 b_1}.$$

*Proof.* Double induction on  $r, s$ .

**Base case:** easy.

**Induction assumption:**

**Induction step:** LHS of (4.7.1) equals to

$$\frac{1}{a_1 + b_1} \left( \frac{b_1 + a_1}{\prod a_i \prod b_j} \right),$$

which equals to RHS of (4.7.1). ■

Now let  $\text{LP}'(r, s)$  denote the sequences that start at  $(1, 1)$ , end at  $(r + 1, s + 1)$ , where each step is either horizontal or vertical with any length.

**Lemma 4.7.2 (Lemma').** Suppose  $a_1, \dots, a_r \in \mathbb{R}_{>0}, b_1, \dots, b_r \in \mathbb{R}_{>0}$  with  $a_{r+1} = 0$  and  $b_{r+1} = 0$ .

$$\sum_{\pi \in \text{LP}'(r, s)} \prod_{\substack{(i, j) \in \pi \\ (i, j) \neq (r+1, s+1)}} \frac{1}{a_i + b_j} = \prod_{i=1}^r \left( 1 + \frac{1}{a_i} \right) \prod_{j=1}^s \left( 1 + \frac{1}{b_j} \right). \quad (4.7.2)$$

*Proof.* Follows from Lemma 4.7.1. (cheat) ■

In the case of hooklength formula,  $1 + \frac{1}{a_i} = 1 + \frac{1}{h_\lambda(x) - 1} = \frac{h_\lambda(x)}{h_\lambda(x) - 1}$ , so the RHS of (4.7.2) becomes  $\frac{H(\lambda)}{H(\lambda - \{v\})}$ . ■