

Notes for Math 597 – Real Analysis

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Chapter 1

Abstract Measure

1.1 σ -Algebra

Definition 1.1. Let X be a set. A collection \mathcal{M} of subsets of X is called a σ -algebra on X if

- $\emptyset \in \mathcal{M}$.
- \mathcal{M} is closed under *complements*: $E \in \mathcal{M} \implies E^c \in \mathcal{M}$.
- \mathcal{M} is closed under *countable unions*: $E_1, E_2, \dots \in \mathcal{M} \implies \bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$.

SIMPLE PROPERTIES:

- $X = \emptyset^c \in \mathcal{M}$.
- $\bigcap_{i=1}^{\infty} E_i = (\bigcup_{i=1}^{\infty} E_i^c)^c \in \mathcal{M}$. It is closed under countable intersections.
- $\bigcup_{i=1}^N E_i = E_1 \cup \dots \cup E_N \cup \emptyset \cup \dots$. It is closed under finite unions (similarly, intersections).
- $E \setminus F = E \cap F^c \in \mathcal{M}$, $E \Delta F = (E \cap F^c) \cup (F \cap E^c) \in \mathcal{M}$.

Example 1.2. (a) $\mathcal{A} = \mathcal{P}(X)$ power algebra.

(b) $\mathcal{A} = \{\emptyset, X\}$ trivial algebra.

(c) Let $B \subset X, B \neq \emptyset, B \neq X$. $\mathcal{A} = \{\emptyset, B, B^c, X\}$.

Lemma 1.3. (An intersection of σ -algebras is a σ -algebra) Let $\mathcal{A}_{\alpha}, \alpha \in I$, be a family a σ -algebras of X . Then $\bigcap_{\alpha \in I} \mathcal{A}_{\alpha}$ is a σ -algebra. (I can be uncountable.)

Proof. DIY

■

Definition 1.4. For $\mathcal{E} \subset \mathcal{P}(X)$ (not necessarily a σ -algebra), let $\langle \mathcal{E} \rangle$ be the intersection of all σ -algebras on X that contains \mathcal{E} . Call it the σ -algebra generated by \mathcal{E} .

- $\langle \mathcal{E} \rangle$ is the *smallest* σ -algebra containing \mathcal{E} and is *unique*.
- $\{\emptyset, B, B^c, X\} = \langle \{B\} \rangle = \langle \{B^c\} \rangle = \langle \{\emptyset, B\} \rangle$.

The above definition gives us (potentially) lots of examples of σ -algebra on a set X

Lemma 1.5. (a) Suppose $\mathcal{E} \subset \mathcal{P}(X)$, \mathcal{A} is a σ -algebra on X . $\mathcal{E} \in \mathcal{A} \implies \langle \mathcal{E} \rangle \in \mathcal{A}$.

(b) $E \subset F \subset \mathcal{P}(X) \implies \langle \mathcal{E} \rangle \subset \langle \mathcal{F} \rangle$.

Proof. ■

Definition 1.6. For a topological space X , the *Borel σ -algebra* $\mathcal{B}(X)$ is the σ -algebra generated by the collection of open sets.

Example 1.7. ($X = \mathbb{R}$) $\mathcal{B}(\mathbb{R})$ contains the following collections:

$$\begin{aligned}\mathcal{E}_1 &= \{(a, b) \mid a < b\}, & \mathcal{E}_2 &= \{[a, b] \mid a < b\}, \\ \mathcal{E}_3 &= \{(a, b] \mid a < b\}, & \mathcal{E}_4 &= \{[a, b) \mid a < b\}, \\ \mathcal{E}_5 &= \{(a, \infty) \mid a \in \mathbb{R}\}, & \mathcal{E}_6 &= \{[a, \infty) \mid a \in \mathbb{R}\}, \\ \mathcal{E}_7 &= \{(-\infty, a) \mid a \in \mathbb{R}\}, & \mathcal{E}_8 &= \{(-\infty, a] \mid a < b\}.\end{aligned}$$

Proposition 1.8. $\mathcal{B}(\mathbb{R}) = \langle \mathcal{E}_i \rangle$ for each $i = 1, \dots, 8$.

Proof. Use 1.5. ■

Definition 1.9. (X, \mathcal{A}) is called a measurable space.

1.2 Measures

Definition 1.10. A measure on (X, \mathcal{A}) is a function $\mu : \mathcal{A} \rightarrow [0, \infty]$ s.t.

- (a) $\mu(\emptyset) = 0$
- (b) (countable additive) For $A_1, A_2, \dots \in \mathcal{A}$ disjoint we have

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

(X, \mathcal{A}, μ) is then called a measure space.

Example 1.11. (a) For any (X, \mathcal{A}) , $\mu(A) = \#A$ counting measure.

(b) For any (X, \mathcal{A}) , let $x_0 \in X$. The Dirac measure at x_0 is

$$\mu(A) = \begin{cases} 1 & x_0 \in A, \\ 0 & x_0 \notin A. \end{cases}$$

(c) For $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, let $a_1, a_2, \dots \in [0, \infty)$. $\mu(A) = \sum_{i \in A} a_i$ is a measure.

(X, \mathcal{A}) measurable space

(X, \mathcal{A}, μ) measure space

$\mu : \mathcal{A} \rightarrow [0, \infty]$ s.t. $\mu(\emptyset) = 0$, countable additivity.

NOTE: $A, B \in \mathcal{A}, A \subset B$, then $\mu(B \setminus A) + \mu(A) = \mu(B) \implies \mu(B \setminus A) = \mu(B) - \mu(A)$ if $\mu(A) < \infty$.

Theorem 1.13. Suppose (X, \mathcal{A}, μ) a measure space. Then

(a) (monotonicity)

$$A, B \in \mathcal{A}, A \subset B \implies \mu(A) \leq \mu(B).$$

(b) (countable subadditivity)

$$A_1, A_2, \dots \in \mathcal{A}, \implies \mu\left(\bigcup_i A_i\right) \leq \sum_i \mu(A_i).$$

(c) (continuity from below/(MCT) from sets)

$$A_1, A_2, \dots \in \mathcal{A}, A_1 \subset A_2 \subset A_3 \subset \dots \implies \mu\left(\bigcup_i A_i\right) = \lim_{n \rightarrow \infty} \mu(A_n).$$

(d) (continuity from above)

$$A_1, A_2, \dots \in \mathcal{A}, A_1 \supset A_2 \supset A_3 \supset \dots, \mu(A_1) < \infty \implies \mu\left(\bigcap_i A_i\right) = \lim_{n \rightarrow \infty} \mu(A_n).$$

Proof. (a), (b), DIY.

For (c), let $B_1 = A_1, B_i = A_i \setminus A_{i-1}, i \geq 2. B_i \in \mathcal{A}$ and are disjoint.

$$\begin{aligned} \bigcup_i^\infty A_i &= \bigcup_i^\infty B_i \\ \implies \mu\left(\bigcup_i^\infty A_i\right) &= \mu\left(\bigcup_i^\infty B_i\right) = \sum_i^\infty \mu(B_i) = \lim_{n \rightarrow \infty} \sum_i^n \mu(B_i) = \lim_{n \rightarrow \infty} \mu(A_n). \end{aligned}$$

For (d), let $E_i = A_1 \setminus A_i$. Hence $E_i \in \mathcal{A}, E_1 \subset E_2 \subset \dots$. We have

$$\bigcup_i^\infty E_i = \bigcup_i^\infty (A_1 \setminus A_i) = A_1 \setminus \left(\bigcap_1^\infty A_i\right) \implies \bigcap_1^\infty A_i = A_1 \setminus \left(\bigcup_1^\infty E_i\right).$$

Hence

$$\mu\left(\bigcap_1^\infty A_i\right) = \mu(A_1) - \mu\left(\bigcup_1^\infty E_i\right) = \mu(A_1) - \lim_{n \rightarrow \infty} \mu(E_n) = \mu(A_1) - \lim_{n \rightarrow \infty} \mu(A_1) - \mu(A_n).$$

■

NOTE: the condition that $\mu(A_1) < \infty$ cannot be dropped.

For example, in $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \text{counting measure})$, let $A_n = \{n, n+1, n+2\}, A_1 \supset A_2 \supset A_3 \supset \dots$. We have $\bigcap_1^\infty A_i = \emptyset \implies \mu\left(\bigcap_1^\infty A_i\right) = 0$.

Definition 1.14. For (X, \mathcal{A}, μ) measure space,

- $A \subset X$ is a μ -null set if $A \in \mathcal{A}, \mu(A) = 0$.
- $A \subset X$ is a μ -subnull set if $\exists B, \mu$ -null set $A \subset B$.
- (X, \mathcal{A}, μ) is a complete measure space if every μ -subnull set is \mathcal{A} -measurable.

Definition 1.15. (X, \mathcal{A}, μ) measure space. A statement $P(x), x \in X$ holds μ -almost everywhere (a.e.) if the set $\{x \in X \mid P(x) \text{ does not hold}\}$ is μ -null.

Definition 1.16. (X, \mathcal{A}, μ) measure space.

- μ is a *finite measure* is $\mu(X) < \infty$.
- μ is a *σ -finite measure* if $X = \bigcup_1^\infty X_n, X_n \in \mathcal{A}, \mu(X_n) < \infty$.

HW: every measure space can be "completed."

1.3 Outer Measures

Definition 1.17. An *outer measure* on X is $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$ s.t.

- $\mu^*(\emptyset) = 0$
- (monotonicity) $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$.
- (countable subadditivity)

$$\forall A_1, A_2, \dots \in X, \mu^* \left(\bigcup_i^\infty A_i \right) \leq \sum_i^\infty \mu^*(A_i).$$

Example 1.18. For $A \subset \mathbb{R}$,

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^\infty (b_i - a_i) \mid \bigcup_1^\infty (a_i, b_i) \supset A \right\}.$$

is an outer measure due to the next proposition.

Proposition 1.19. (1.19) Let $\mathcal{E} \in \mathcal{P}(X)$ s.t. $\emptyset, X \in \mathcal{E}$. Let $\rho : \mathcal{E} \rightarrow [0, \infty]$ s.t. $\rho(\emptyset) = 0$. Then

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^\infty \rho(E_i) \mid E_i \in \mathcal{E}, \forall i \in \mathbb{N}, \bigcup_1^\infty E_i \supset A \right\}$$

is an outer measure on X .

Proof. (a) μ^* is well-defined (inf is taken over non-empty set.)

(b) $\mu^*(\emptyset) = 0$

(c) $A \subset B \implies \mu^*(A) \leq \mu^*(B)$.

We check the countable subadditivity.

Let $A_1, A_2, \dots \subset X$. If one of $\mu^*(A_i) = \infty$, then the result holds. Suppose $\mu^*(A_n) < \infty, \forall n \in \mathbb{N}$.

"Give your self a room of epsilon":

Fix $\varepsilon > 0$. We will show

$$\mu^* \left(\bigcup_1^\infty A_n \right) \leq \sum_1^\infty \mu^*(A_i) + \varepsilon.$$

For each $n \in \mathbb{N}, \exists E_{n,1}, E_{n,2}, \dots \in \mathcal{E}$ s.t.

$$\bigcup_{k=1}^\infty E_{n,k} \supset A_n \quad \text{and} \quad \mu^*(A_n) + \frac{\varepsilon}{2^n} \geq \sum_{k=1}^\infty \rho(E_{n,k}).$$

Then,

$$\bigcup_1^\infty A_n \subset \bigcup_{n=1}^\infty \bigcup_{k=1}^\infty E_{n,k} = \bigcup_{(n,k) \in \mathbb{N}^2} E_{n,k}.$$

RECALL: Tonelli's thm for series. If $a_{ij} \in [0, \infty]$, $\forall i, j \in \mathbb{N}$, then

$$\sum_{(i,j) \in \mathbb{N}^2} a_{ij} = \sum_{i=1}^\infty \sum_{j=1}^\infty a_{ij} = \sum_{j=1}^\infty \sum_{i=1}^\infty a_{ij}.$$

Hence

$$\mu^* \left(\bigcup_{n=1}^\infty A_n \right) \leq \sum_{n=1}^\infty \rho(E_{k,n}) = \sum_{n=1}^\infty \sum_{k=1}^\infty \rho(E_{k,n}) \leq \sum_{n=1}^\infty \left(\mu^*(A_n) + \frac{\varepsilon}{2^n} \right) = \sum_{n=1}^\infty \mu^*(A_n) + \varepsilon.$$

We have shown countable subadditivity. ■

Outer measure is very close to a measure. Here the textbooks diverge.

[Tao11] introduces Lebesgue measure on \mathbb{R} using topological qualities of subsets of \mathbb{R} .

[Fol99] introduces abstract method by Carathéodory and Kolmogorov.

Definition 1.20. Let μ^* be an outer measure on X . We say $A \subset X$ is Carathéodory measurable with respect to μ^* if $\forall E \subset X, \mu^*(E) = \mu^*(E \setminus A) + \mu^*(E \cap A)$.

Lemma 1.21. Let μ^* be an outer measure on X . Suppose B_1, B_2, \dots, B_N are disjoint C -measurable sets. Then,

$$\forall E \subset X, \mu^* \left(E \cap \left(\bigcup_1^N B_i \right) \right) = \sum_{i=1}^n \mu^*(E \cap B_i)$$

Proof.

$$\mu^* \left(E \cap \left(\bigcup_1^N B_i \right) \right) = \mu^*(E \cap B_1) + \mu^* \left(E \cap \left(\bigcup_2^N B_i \right) \right)$$

because B_1 is C -measurable. Then, iterate. ■

Improved version:

B_1, B_2, \dots C -measurable and disjoint $\implies \mu^*(E \cap \bigcup_1^\infty B_n) = \sum_1^\infty \mu^*(E \cap B_n), \forall E \subset X$.

Proof.

$$\begin{aligned} \sum_1^\infty \mu^*(E \cap B_n) &\geq \mu^*\left(E \cap \bigcup_1^\infty B_n\right) \\ &\geq \mu^*\left(E \cap \bigcup_1^N B_n\right) = \sum_1^N \mu^*(E \cap B_n). \end{aligned}$$

Take $N \rightarrow \infty$ or note that $N \in \mathbb{N}$ is arbitrary we get the result. ■

First big theorem:

Theorem 1.22 (Carathéodory extension theorem). *Let μ^* be an outer measure on X . Let \mathcal{A} be the collection of C -measurable sets with respect to μ^* . Then*

- (a) \mathcal{A} is a σ -algebra on X .
- (b) $\mu = \mu^*|_{\mathcal{A}}$ is a measure on (X, \mathcal{A}) .
- (c) (X, \mathcal{A}, μ) is a complete measure space.

Proof. (a) (1) $\emptyset \in \mathcal{A}$.

(2) \mathcal{A} is closed under complements.

(3) To show \mathcal{A} closed under countable unions.

- (finite union)

CLAIM $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$.

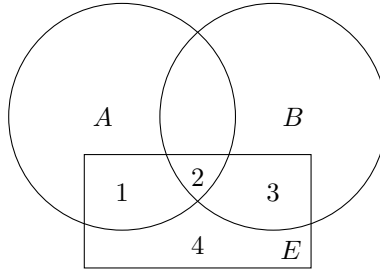


Figure 1.1: Venn diagram of A, B, E

Fix arbitrary $E \subset X$. We need to show

$$\mu^*(E) = \mu^*(E \cap (A \cup B)) + \mu^*(E \setminus (A \cup B)).$$

i.e.

$$\mu^*(1 \cup 2 \cup 3 \cup 4) = \mu^*(1 \cup 2 \cup 3) + \mu^*(4)$$

Since A is C -measurable, we have

$$\mu^*(1 \cup 2 \cup 3 \cup 4) = \mu^*(1 \cup 2) + \mu^*(3 \cup 4)$$

$$\mu^*(1 \cup 2 \cup 3) = \mu^*(1 \cup 2) + \mu^*(3)$$

Similarly since B is C -measurable, we have

$$\mu^*(3 \cup 4) = \mu^*(3) + \mu^*(4)$$

Hence

$$\begin{aligned} \mu^*(1 \cup 2 \cup 3 \cup 4) &= \mu^*(1 \cup 2) + \mu^*(3 \cup 4) \\ &= \mu^*(1 \cup 2 \cup 3) - \mu^*(3) + \mu^*(3) + \mu^*(4) \\ &= \mu^*(1 \cup 2 \cup 3) + \mu^*(4). \end{aligned}$$

- (countable disjoint unions)

Let $A_1, A_2, \dots \in \mathcal{A}$ and *disjoint*.

Fix $E \subset X$ arbitrary. Since μ^* is countably subadditive,

$$\mu^*(E) \leq \mu^*\left(E \cap \bigcup_1^\infty A_n\right) + \mu^*\left(E \setminus \bigcup_1^\infty A_n\right)$$

Fix $n \in \mathbb{N}$.

$$\begin{aligned} &\Rightarrow \bigcup_1^N A_n \in \mathcal{A} \\ &\Rightarrow \mu^*(E) = \mu^*\left(E \cap \bigcup_1^N A_n\right) + \mu^*\left(E \setminus \bigcup_1^N A_n\right) \\ &\geq \sum_1^N \mu^*(E \cap A_n) + \mu^*\left(E \setminus \bigcup_1^\infty A_n\right) \text{ by lemma.} \end{aligned}$$

Take $n \rightarrow \infty$.

- (countable unions)

Let $A_1, A_2, \dots \in \mathcal{A}$. Take $E_1 = A_1, E_n = A_n \setminus \left(\bigcup_1^{n-1} A_i\right)$ for $n \geq 2$. Then $\bigcup A_n = \bigcup E_n$ and E_n 's are disjoint.

(b) Firstly we have $\mu(\emptyset) = \mu^*(\emptyset) = 0$.

Countable additivity of μ^* on \mathcal{A} follows from the improved lemma with $E = X$.

(c) HW. ■

1.4 Hahn-Kolmogorov Theorem

RECALL 1.19 Let $\mathcal{E} \subset \mathcal{P}(X)$ s.t. $\emptyset, X \in \mathcal{E}$. Let $\rho : \mathcal{E} \rightarrow [0, \infty]$ s.t. $\rho(\emptyset) = 0$

$$(\mathcal{E}, \rho) \xrightarrow{1.19} (\mathcal{P}(X), \mu^*) \xrightarrow{\text{C-theorem}} (A, \mu)$$

QUESTION $\mathcal{E} \subset \mathcal{A}$ and $\mu|_{\mathcal{E}} = \rho$? No!

Definition 1.23. Let \mathcal{A}_0 be an algebra on X . We say $\mu_0 : \mathcal{A}_0 \rightarrow [0, \infty]$ is a *pre-measure* if

- (a) $\mu_0(\emptyset) = 0$.
- (b) (finite additivity)

$$\mu_0 \left(\bigcup_1^N A_i \right) = \sum_1^N \mu_0(A_i) \text{ if } A_1, \dots, A_N \in \mathcal{A}_0 \text{ are disjoint.}$$

- (c) (countable additivity within the algebra) If $A \in \mathcal{A}_0$ and

$$A = \bigcup_1^\infty A_n, A_n \in \mathcal{A}_0 \text{ and are disjoint, then } \mu_0(A) = \sum_1^\infty \mu_0(A_n)$$

NOTATION: Folland uses \mathcal{M} for σ -algebra and \mathcal{A} for algebra. (Jinho) uses \mathcal{A} for σ -algebra and \mathcal{A}_0 for algebra.

Example 1.24. \mathcal{A}_0 finite disjoint unions of $(a, b]$.

$$\mu_0 \left(\bigcup_1^\infty (a_i, b_i] \right) = \sum_1^\infty (b_i - a_i) \text{ or } b_i^n - a_i^n, e^{b_i} - e^{a_i}, \text{ etc.}$$

Lemma 1.25. • $(a) + (c) \implies (b)$.

- μ_0 is monotone.

Theorem 1.26 (Hahn-Kolmogorov Theorem). Let μ_0 be a pre-measure on algebra \mathcal{A}_0 on X . Let μ^* be the outer measure induced by (\mathcal{A}_0, μ_0) in 1.19. Let \mathcal{A} and μ be the Carathéodory σ -algebra and measure for $\mu^* \implies (\mathcal{A}, \mu)$ extends (\mathcal{A}_0, μ_0) i.e. $\mathcal{A} \supset \mathcal{A}_0, \mu|_{\mathcal{A}_0} = \mu_0$.

Proof. (a) $(\mathcal{A} \supset \mathcal{A}_0)$ Let $A \in \mathcal{A}_0$.

Question: $A \in \mathcal{A}$? i.e. is A C -measurable? i.e. $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c), \forall E \subset X$

X .

Fix $E \subset X$.

- (countable) subadditivity of $\mu^* \implies \mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c)$.
- If $\mu^*(E) = \infty$ then $\mu^*(E) = \infty \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$.
- If $\mu^*(E) < \infty$.

Fix $\varepsilon > 0$. By the definition of μ^* , $\exists B_1, B_2, \dots \in \mathcal{A}_0$ s.t. $\bigcup_1^\infty B_n \supset E$ and

$$\mu^*(E) + \varepsilon \geq \sum_1^\infty \mu_0(B_n) = \sum_1^\infty (\mu_0(B_n \cap A) + \mu_0(B_n \cap A^c)).$$

Note that

$$\bigcup_1^\infty (B_n \cap A) \supset E \cap A, \quad \bigcup_1^\infty (B_n \cap A^c) \supset E \cap A^c \implies \geq$$

(b) Let $A \in \mathcal{A}_0$. We want to show that $\mu(A) = \mu_0(A)$.

By definition, $\mu(A) = \mu^*(A)$.

- Let $B_i = \begin{cases} A & i = 1, \\ \emptyset & i = 2 \end{cases} \in \mathcal{A}_0$ and $\bigcup_1^\infty B_i \supset A$.

Hence $\mu^*(A) \leq \sum_1^\infty \mu_0(B_i) = \mu_0(A)$.

- Let $B_i \in \mathcal{A}_0, \bigcup_1^\infty B_i \supset A$ an arbitrary collection of sets.

Let $C_1 = A \cap B_1, C_i = A \cap B_i \setminus \left(\bigcup_{j=1}^{i-1} B_j \right)$. Then $A = \bigcup_1^\infty C_i$ is a disjoint countable union. By countable additivity we have

$$\mu_0(A) = \sum_1^\infty \mu_0(C_i) \implies \mu_0(A) \leq \sum_1^\infty \mu_0(B_i).$$

Hence we have $\mu_0(A) = \mu^*(A) = \mu(A)$. We have completed our proof. ■

Definition 1.27. Such (\mathcal{A}, μ) is called the *Hahn-Kolmogorov extension* of (\mathcal{A}_0, μ_0) , and is also called the *Carathéodory σ -algebra* for (\mathcal{A}_0, μ_0) .

Theorem 1.28 (uniqueness of HK extension). *Let \mathcal{A}_0 be an algebra on X , μ_0 be a pre-measure on \mathcal{A}_0 , (\mathcal{A}, μ) be the Hahn-Kolmogorov extension of (\mathcal{A}_0, μ_0) . And let (\mathcal{A}', μ') be another extension of (\mathcal{A}_0, μ_0) .*

If μ_0 is σ -finite, then $\mu|_{\mathcal{A} \cap \mathcal{A}'} = \mu'|_{\mathcal{A} \cap \mathcal{A}'}$.

NOTE σ -finite means

$$\forall X, X = \bigcup_1^\infty X_n, X_n \in \mathcal{A}_0, \mu_0(X_n) < \infty.$$

Corollary 1.29. Let μ_0 be a pre-measure on algebra \mathcal{A}_0 on X . Suppose μ_0 is σ -finite, then $\exists!$ measure μ on $\langle \mathcal{A}_0 \rangle$ that extends \mathcal{A}_0 . Furthermore,

(a) the completion of $(X, \langle \mathcal{A}_0 \rangle, \mu)$ is the HK extension of (\mathcal{A}_0, μ_0) .

(b)

$$\mu(A) = \inf \left\{ \sum_{i=1}^\infty \mu_0(B_i) \mid B_i \subset A_0, \forall i \in \mathbb{N}, \bigcup_1^\infty B_i \supset A \right\}, \forall A \in \overline{\langle \mathcal{A}_0 \rangle}.$$

Proof of 1.28. Let $A \in \mathcal{A} \cap \mathcal{A}'$. We need to show $\mu(A) = \mu^*(A) = \mu'(A)$.

- $\mu^*(A) \geq \mu'(A)$ (HW)

- $\mu(A) \leq \mu'(A)$:

(i) Assume $\mu(A) < \infty$. Fix $\varepsilon > 0$. Then $\exists B_i \in \mathcal{A}_0, \forall i \in \mathbb{N}, \bigcup_1^\infty B_i \supset A$ s.t.

$$\mu(A) + \varepsilon = \mu^*(A) + \varepsilon \geq \sum_1^\infty \mu_0(B_i) = \sum_1^\infty \mu(B_i) \geq \mu\left(\bigcup_1^\infty B_i\right) = \mu(B)$$

Hence $\mu(B \setminus A) = \mu(B) - \mu(A) \leq \varepsilon$.

On the other hand,

$$\mu(B) = \lim_{N \rightarrow \infty} \mu\left(\bigcup_1^N B_i\right) = \lim_{N \rightarrow \infty} \mu'\left(\bigcup_1^N B_i\right) = \mu'(B)$$

by continuity of measure from below.

$$\mu(A) \leq \mu(B) = \mu'(B) = \mu'(A) + \mu'(B \setminus A) \leq \mu'(A) = \varepsilon.$$

(ii) Assume $\mu(A) = \infty$.

Since μ_0 is σ -finite, $X = \bigcup_1^\infty X_n, X_n \in \mathcal{A}_0, \mu_0(X_n) < \infty$. Replacing X_n by $X_1 \cup \dots \cup X_n$, we may assume $X_1 \subset X_2 \subset \dots$

$$\forall n \in \mathbb{N}, \mu(A \cap X_n) < \infty \implies \mu(A \cap X_n) \leq \mu'(A \cap X_n).$$

Hence

$$\mu(A) = \lim_{N \rightarrow \infty} \mu(A \cap X_N) \leq \lim_{N \rightarrow \infty} \mu'(A \cap X_N) = \mu'(A). \quad \blacksquare$$

1.5 Borel Measures on \mathbb{R}

Definition 1.30. $F : \mathbb{R} \rightarrow \mathbb{R}$ is an *increasing* function if $F(x) \leq F(y)$ for $x < y$. $F : \mathbb{R} \rightarrow \mathbb{R}$ is increasing and right-continuous $\implies F$ is distribution function.

Example 1.31.

- $F(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0. \end{cases}$
- $\mathbb{Q} = \{r_1, r_2, \dots\}$, $F_n(x) = \begin{cases} 1 & x \geq r_n \\ 0 & x < r_n \end{cases}$. $F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n}$ is a distribution function.

NOTE If F is increasing, $F(\infty) := \lim_{x \rightarrow \infty} F(x)$, $F(-\infty) := \lim_{x \rightarrow -\infty} F(x)$ exists in $[-\infty, \infty]$.

In probability theory, cumulative distribution function (CDF) is a distribution function with $F(\infty) = 1$ and $F(-\infty) = 0$.

There are distributions [Fol99, Ch.9], but these are different from *distribution functions*.

Definition 1.32. Suppose X a topological space. μ on $(X, \mathcal{B}(X))$ is called *locally finite* is $\mu(K) < \infty$ for any compact set $K \subset X$.

Lemma 1.33. Let μ be a locally finite Borel measure on $\mathbb{R} \implies$

$$F_\mu(x) = \begin{cases} \mu((0, x]), & x > 0 \\ 0, & x = 0 \\ -\mu((x, 0]), & x < 0 \end{cases} \text{ is a distribution function.}$$

Proof. DIY. Use continuity of measure. ■

Definition 1.34. h -intervals are $\emptyset, (a, b], (a, \infty), (-\infty, b], (-\infty, \infty)$.

Lemma 1.35. Let \mathcal{H} be the collections of finite disjoint unions of h -intervals. Then \mathcal{H} is an algebra on \mathbb{R} .

Proof. DIY. ■

Proposition 1.36 (Distribution function defines a pre-measure). *Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be a distribution function. For an h -interval I , define*

$$\ell(I) = \ell_F(I) = \begin{cases} 0, & I = \emptyset \\ F(b) - F(a), & I = (a, b] \\ F(\infty) - F(a), & I = (a, \infty) \\ F(b) - F(-\infty), & I = (-\infty, b] \\ F(\infty) - F(-\infty), & I = (-\infty, \infty). \end{cases}$$

Define $\mu_0 = \mu_{0,F} : \mathcal{H} \rightarrow [0, \infty]$ by

$$\mu_0(A) := \sum_{k=1}^N \ell(I_k) \quad \text{if } A = \bigcup_{k=1}^N I_k, \text{ finite disjoint union of } h\text{-intervals.}$$

Then μ_0 is a pre-measure.

Proof. (a) μ_0 is well-defined.

(b) μ_0 is finite additive.

(c) μ_0 is countable additive within \mathcal{H} .

Suppose $A \in \mathcal{H}$ and $A = \bigcup_1^\infty A_i$ a disjoint union, $A_i \in \mathcal{H}$. It is enough to consider the case $A = I, A_k = I_k$ all h -intervals. (Why?)

Focus on the case $I = (a, b]$: (HW: check other cases)

We have

$$(a, b] = \bigcup_1^\infty (a_n, b_n], \text{ a disjoint union.}$$

Check

$$F(b) - F(a) \stackrel{?}{=} \sum_1^\infty (F(b_n) - F(a_n))$$

$(a, b] \supset \bigcup_1^N (a_n, b_n] \implies F(b) - F(a) \geq \sum_1^N (F(b_n) - F(a_n)), \forall N \in \mathbb{N}$. (Arranging them in decreasing order) Take $N \rightarrow \infty$ we have

$$F(b) - F(a) \geq \sum_1^\infty (F(b_n) - F(a_n)).$$

Since F is right-continuous, $\exists a' > a$ s.t. $F(a') - F(a) < \varepsilon$. For each $n \in \mathbb{N}$, $\exists b'_n >$

$$b_n \text{ s.t. } F(b'_n) - F(b_n) < \frac{\varepsilon}{2^n}.$$

$$\implies [a', b] \subset \bigcup_1^\infty (a_n, b'_n)$$

$$\implies \exists N \in \mathbb{N} \text{ s.t. } [a', b] \subset \bigcup_1^N (a_n, b'_n)$$

$$\implies F(b) - F(a') \leq \sum_1^N F(b'_n) - F(a_n)$$

$$\implies F(b) - F(a) \leq F(b) - F(a') + \varepsilon \leq \sum_1^\infty (F(b'_n) - F(a_n)) + \varepsilon$$

$$\leq \sum_1^\infty \left(F(b_n) - F(a_n) + \frac{\varepsilon}{2^n} \right) + \varepsilon \quad \blacksquare$$

Once we have this pre-measure, HK theorem allows us to extend it to a measure.

Theorem 1.37 (Locally finite Borel measures on \mathbb{R}).

- (a) $F : \mathbb{R} \rightarrow \mathbb{R}$ is a distribution function $\implies \exists!$ locally finite Borel measure μ_F on \mathbb{R} satisfying $\mu_F((a, b]) = F(b) - F(a), \forall a, b, a < b$.
- (b) Suppose $F, G : \mathbb{R} \rightarrow \mathbb{R}$ are distribution functions. Then, $\mu_F = \mu_G$ on $\mathcal{B}(\mathbb{R})$ if and only if $F - G$ is a constant function.

Proof. HW ■

1.6 Lebesgue-Stieltjes Measures on \mathbb{R}

F distribution function $\implies \mu_F$ on Carathéodory σ -algebra \mathcal{A}_{μ_F} .

Actually $(\mathcal{A}_{\mu_F}, \mu_F) = (\mathcal{B}(\mathbb{R}), \mu_F)$ (HW3).

Definition 1.38. • μ_F on \mathcal{A}_{μ_F} is called the Lebesgue-Stieltjes measure corresponding to F .

- Special case: $F(x) = x \implies$ Lebesgue measure (\mathcal{B}, m) .

Example 1.39.

- (a) $\mu_F((a, b]) = F(b) - F(a)$. F is right-continuous and increasing $\implies F(x_-) \leq F(x) = F(x_+)$.

$$\text{(HW)} \quad \mu_F(\{a\}) = F(a) - F(a_-), \mu_F([a, b]) = F(b) - F(a_-), \mu_F((a, b)) = F(b_-) - F(a).$$

(b)

$$F(x) = \begin{cases} 1 & x \leq 0 \\ 0 & x < 0 \end{cases} \implies \mu_F(\{0\}) = 1, \mu_F(\mathbb{R}) = 1, \mu_F(\mathbb{R} \setminus \{0\}) = 0.$$

μ_F is the Dirac measure at 0.

(c)

$$\mathbb{Q} = \{r_1, r_2, \dots\}, F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n}, F_n(x) = \begin{cases} 1 & x \leq r_n \\ 0 & x < r_n \end{cases} \\ \implies \mu_F(\{v\}) > 0, \forall v \in \mathbb{Q}, \mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0.$$

(d) If F is continuous at a , $\mu_F(\{a\}) = 0$.(e) $F(x) = x \implies m((a, b]) = m((a, b)) = m([a, b]) = b - a$.(f) $F(x) = e^x, \implies \mu_F((a, b]) = \mu_F((a, b)) = e^b - e^a$

(a), (b) are examples of discrete measure.

Example 1.40 (Middle thirds Cantor set $\mathcal{C} = \bigcup_{n=1}^{\infty} K_n$). \mathcal{C} is uncountable set with $m(\mathcal{C}) = 0$.

$$x \in \mathcal{C} \implies x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}, a_n \in \{0, 2\}.$$

We are interested in the Cantor function F .

Example 1.41. Cantor function F is continuous and increasing. This defines the Cantor measure $\mu_F(\mathbb{R} \setminus \mathcal{C}) = 0, \mu_F(\mathcal{C}) = 1, \mu_F(\{a\}) = 0$. Compare with Lebesgue measure $m(\mathbb{R} \setminus \mathcal{C}) = \infty > 0, \mu(\mathcal{C}) = 0, m(\{a\}) = 0$.

1.7 Regularity Properties of Lebesgue-Stieltjes Measures

Lemma 1.42. μ is Lebesgue-Stieltjes measure on $\mathbb{R} \implies$

$$\begin{aligned} \mu(A) &= \inf \left\{ \sum_1^{\infty} \mu((a_i, b_i]) \mid \bigcup_1^{\infty} (a_i, b_i] \supset A \right\} \\ &= \inf \left\{ \sum_1^{\infty} \mu((a_i, b_i)) \mid \bigcup_1^{\infty} (a_i, b_i) \supset A \right\} \end{aligned}$$

Proof. Using the continuity of measure. ■

Theorem 1.43. μ is a Lebesgue-Stieltjes measure. Then $\forall A \in \mathcal{A}_\mu$,

(a) (outer regularity)

$$\mu(A) = \inf\{\mu(O) \mid \text{open } O \supset A\}.$$

(b) (inner regularity)

$$\mu(A) = \sup\{\mu(K) \mid \text{compact } K \subset A\}.$$

Proof. (a) Followed from 1.42.

(b) Let $s = \sup\{\dots\}$. Monotonicity $\implies \mu(A) \geq s$.

- (A bounded) $\bar{A} \in \mathcal{B}(\mathbb{R}) \subset \mathcal{A}_\mu$, \bar{A} bounded $\implies \mu(\bar{A}) < \infty$.

Fix $\varepsilon > 0$. By 1, \exists open $O \supset \bar{A} \setminus A$, $\mu(O) - \mu(\bar{A} \setminus A) = \mu(O \setminus (\bar{A} \setminus A)) \leq \varepsilon$.

Let $K = \underbrace{A \setminus O}_{K \subset A} = \underbrace{\bar{A} \setminus O}_{\text{compact}}$. Show that $\mu(K) \geq \mu(A) - \varepsilon$.

- (A unbounded but $\mu(A) < \infty$) We have

$$A = \bigcup_{n=1}^{\infty} A_n, \quad A_n = A \cap [-n, n], \quad A_1 \subset A_2 \subset \dots$$

Hence

$$\lim_{n \rightarrow \infty} \mu(A_n) = \mu(A) < \infty.$$

- ($\mu(A) = \infty$)

$$\lim_{n \rightarrow \infty} \mu(A_n) = \mu(A) = \infty.$$

Fix $L > 0$. $\exists N$ s.t. $\mu(A_N) \geq L$. ■

Definition 1.44. Suppose X a topological space.

A G_σ -set is $G = \bigcap_{i=1}^{\infty} O_i$, O_i open. An F_σ -set is $F = \bigcup_{i=1}^{\infty} F_i$, F_i closed.

Theorem 1.45. Suppose μ a LS measure. Then the following statements are equivalent:

- (a) $A \in \mathcal{A}_\mu$.
- (b) $A = G \setminus M$, G is a G_σ -set, and M is μ -null.
- (c) $A = F \cup N$, F is an F_σ -set, and N is μ -null.

Proof. (b) \implies (a) and (c) \implies (a) are clear.

- (a) \implies (c)

(i) Assume $\mu(A) < \infty$. By inner regularity,

$$\forall n \in \mathbb{N}, \exists \text{ compact } K_n \subset A \text{ s.t. } \mu(K_n) + \frac{1}{n} \geq \mu(A).$$

Let $F = \bigcup_1^\infty K_n$. Then $N = A \setminus F$ is μ -null.

(ii) Assume $\mu(A) = \infty$. We construct

$$A = \bigcup_{k \in \mathbb{Z}} A_k, A_k = A \cap (k, k+1].$$

By (i), $\forall k \in \mathbb{Z}, A_k = F_k \cup N_k$. Hence

$$A = \underbrace{\left(\bigcup_k F_k \right)}_{F_\sigma} \cup \underbrace{\left(\bigcup_k N_k \right)}_{\mu\text{-null}}.$$

- (a) \implies (b)

$$A^c = F \cup N, A = F^c \cup N^c = F^c \setminus N. \quad \blacksquare$$

Proposition 1.46. Suppose μ a LS measure, $A \in \mathcal{A}_\mu$, $\mu(A) < \infty$. Then

$$\forall \varepsilon > 0, \exists I = \bigcup_1^{N=N(\varepsilon)} I_i, \text{ disjoint open intervals s.t. } \mu(A \Delta I) \leq \varepsilon.$$

Proof. DIY - use outer regularity. ■

Properties of Lebesgue measure

Theorem 1.47.

$$A \in \mathcal{L} \implies A + s \in \mathcal{L}, rA \in \mathcal{L}, \forall r, s \in \mathbb{R}.$$

In addition, $m(A + r) = m(A)$ and $m(rA) = rm(A)$.

Proof. DIY. ■

Example 1.48.

(a) $\mathbb{Q} = \{r_1\}_{i=1}^\infty$, which is dense in \mathbb{R} . Let $\varepsilon > 0$ and

$$O = \bigcup_{i=1}^\infty \left(r_i - \frac{\varepsilon}{2^i}, r_i + \frac{\varepsilon}{2^i} \right).$$

O is open and dense in \mathbb{R} . We have

$$m(O) \leq \sum_{i=1}^{\infty} \frac{2\varepsilon}{2^i} = 2\varepsilon, \partial O = \overline{O} \setminus O, m(O) = \infty.$$

- (b) \exists uncountable set A with $m(A) = 0$.
- (c) $\exists A$ with $m(A) > 0$, but A contains no non-empty open interval.
- (d) $\exists A \notin \mathcal{L}$ that is Vitali set.
- (e) $\exists A \in \mathcal{L} \setminus \mathcal{B}(\mathbb{R})$. We will deal with that later.

Chapter 2

Integration

2.1 Measurable Functions

Definition 2.1. Suppose (X, \mathcal{A}) , (Y, \mathcal{B}) two measurable spaces. $f : X \rightarrow Y$ is $(\mathcal{A}, \mathcal{B})$ -measurable if

$$\forall B \in \mathcal{B}, f^{-1}(B) \in \mathcal{A}.$$

Lemma 2.2. Suppose $\mathcal{B} = \langle \mathcal{E} \rangle$. Then

$$f : X \rightarrow Y \text{ is } (\mathcal{A}, \mathcal{B})\text{-measurable} \iff \forall E \in \mathcal{E}, f^{-1}(E) \in \mathcal{A}.$$

Proof. \implies clear

\Leftarrow Let $\mathcal{D} = \{E \subset Y \mid f^{-1}(E) \in \mathcal{A}\}$. We have $\mathcal{E} \subset \mathcal{D}$ by assumption. In addition \mathcal{D} is a σ -algebra $\implies \langle \mathcal{E} \rangle \subset \mathcal{D}$. ■

Definition 2.3. Suppose (X, \mathcal{A}) a measurable space.

$$\left. \begin{array}{l} f : X \rightarrow \mathbb{R} \\ f : X \rightarrow \overline{\mathbb{R}} = [-\infty, \infty] \\ f : X \rightarrow \mathbb{C} \end{array} \right\} \text{ is } \mathcal{A}\text{-measurable if } \left\{ \begin{array}{l} f \text{ is } (\mathcal{A}, \mathcal{B}(\mathbb{R}))\text{-measurable} \\ f \text{ is } (\mathcal{A}, \mathcal{B}(\overline{\mathbb{R}}))\text{-measurable} \\ \operatorname{Re} f, \operatorname{Im} f : X \rightarrow \mathbb{R} \text{ are } \mathcal{A}\text{-measurable.} \end{array} \right.$$

Here $\mathcal{B}(\overline{\mathbb{R}}) = \{E \subset \overline{\mathbb{R}} \mid E \cap \mathbb{R} \in \mathcal{B}(\mathbb{R})\}$.

Lemma 2.4. Suppose $f : X \rightarrow \mathbb{R}$. Then the followings are equivalent:

(a) f is \mathcal{A} -measurable

- (b) $\forall a \in \mathbb{R}, f^{-1}((a, \infty)) \in \mathcal{A}$.
- (c) $\forall a \in \mathbb{R}, f^{-1}([a, \infty)) \in \mathcal{A}$.
- (d) $\forall a \in \mathbb{R}, f^{-1}((-\infty, a)) \in \mathcal{A}$.
- (e) $\forall a \in \mathbb{R}, f^{-1}((-\infty, a]) \in \mathcal{A}$.

For $f : X \rightarrow \overline{\mathbb{R}}$, change the interval to include $-\infty$ and ∞ .

Proof. By 2.2. ■

Example 2.5. $\mathcal{A} = \mathcal{P}(X) \implies$ every function is \mathcal{A} measurable.

$\mathcal{A} = \{\emptyset, X\} \implies$ only \mathcal{A} functions are constant functions.

PROPERTIES Suppose $f, g : X \rightarrow \mathbb{R}$, \mathcal{A} -measurable functions.

- (a) $\phi : \mathbb{R} \rightarrow \mathbb{R}$, $\mathcal{B}(\mathbb{R})$ measurable (i.e. Borel measurable) $\implies \phi \circ f : X \rightarrow \mathbb{R}$ is \mathcal{A} -measurable.
- (b) $-f, 3f, f^2, |f|$ are \mathcal{A} -measurable, $\frac{1}{f}$ is \mathcal{A} -measurable if $f(x) \neq 0, \forall x \in X$.
- (c) $f + g$ is \mathcal{A} -measurable

$$(f + g)^{-1}((a, \infty)) = \bigcup_{r \in \mathbb{Q}} (f^{-1}((r, \infty)) \cap g^{-1}((a - r, \infty))).$$

- (d) fg is \mathcal{A} -measurable

$$f(x)g(x) = \frac{1}{2} ((f(x) + g(x))^2 - f(x)^2 - g(x)^2).$$

- (e) $(f \wedge g)(x) = \min\{f(x), g(x)\}$, $(f \vee g)(x) = \max\{f(x), g(x)\}$ are \mathcal{A} -measurable.
- (f) $f_n : X \rightarrow \overline{\mathbb{R}}$ are a sequence of \mathcal{A} -measurable functions \implies

$$\sup f_n, \inf f_n, \limsup_{n \rightarrow \infty} f_n, \liminf_{n \rightarrow \infty} f_n \text{ are } \mathcal{A}\text{-measurable.}$$

- (g) If $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ converges for every $x \in X$, then f is measurable.

Example 2.6. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous. Then f is Borel measurable $\implies f$ is Lebesgue measurable. (Preimage of an open set of a continuous function is open.)

Definition 2.7. For $f : X \rightarrow \overline{\mathbb{R}}$, let $f^+ = f \vee 0$, $f^- = (-f) \vee 0$.

NOTE $\text{supp } f^+ \cap \text{supp } f^- = \emptyset$. $f(x) = f^+(x) - f^-(x)$. f is \mathcal{A} -measurable $\iff f^+, f^-$ measurable.

Definition 2.8. For $E \subset X$, characteristic (indicator) function of E

$$\chi_E(x) = 1_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \in E^c. \end{cases}$$

1_E is \mathcal{A} -measurable $\iff E \in \mathcal{A}$.

Definition 2.9. Suppose (X, \mathcal{A}) a measurable space. A *simple function* $\phi : X \rightarrow \mathbb{C}$ that is \mathcal{A} -measurable and takes only finitely many values.

$$\phi(X) = \{c_1, \dots, c_N\}, c_i \neq \pm\infty, E_i = \phi^{-1}(c_i) \in \mathcal{A} \implies \phi = \sum_{i=1}^N c_i 1_{E_i}.$$

Theorem 2.10. Suppose (X, \mathcal{A}) a measurable space and $f : X \rightarrow [0, \infty]$. Then the followings are equivalent:

- (a) f is \mathcal{A} -measurable.
- (b) \exists simple functions $0 \leq \phi_1(x) \leq \phi_2(x) \leq \dots \leq f(x)$ such that

$$\lim_{n \rightarrow \infty} \phi_n(x) = f(x), \forall x \in X.$$

(f is the pointwise upward limit of simple functions.)

Proof. • (b) \implies (a) is easy: $f(x) = \sup_{n \in \mathbb{N}} \phi_n(x)$.

- (a) \implies (b): suppose f is \mathcal{A} -measurable.

Fix $n \in \mathbb{N}$. Let $F_n = f^{-1}([2^n, \infty]) \in \mathcal{A}$. For

$$0 \leq k \leq 2^{2n} - 1, E_{n,k} = f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right) \in \mathcal{A}.$$

$$\text{Let } \phi_n(x) = \sum_{k=0}^{2^{2n}-1} 1_{E_{n,k}} + 2^n 1_{F_n}.$$

This shows that

- $0 \leq \phi_1(x) \leq \phi_2(x) \leq \dots \leq f(x), \forall x \in X$.
- $\forall x \in X \setminus F_n, 0 \leq f(x) - \phi_n(x) \leq \frac{1}{2^n}$.

Since $F_1 \supset F_2 \supset \dots$ and $\bigcap_{n=1}^{\infty} F_n = f^{-1}(\{\infty\})$, we have

$$\begin{aligned}
- & x \in f^{-1}([0, \infty)) = X \setminus \left(\bigcap_1^\infty F_n \right) \implies \lim_{n \rightarrow \infty} \phi_n(x) = f(x). \\
- & x \in f^{-1}(\{\infty\}) = \bigcap_1^\infty X_n \implies \phi_n(x) \geq 2^n \implies \lim_{n \rightarrow \infty} \phi_n(x) = \infty = f(x). \quad \blacksquare
\end{aligned}$$

Corollary 2.11. If f is bounded on a set $A \subset \mathbb{R}$ (i.e. $\exists L > 0$ s.t. $|f(x)| \leq L, \forall x \in A$) then $\phi_n \rightarrow f$ uniformly on A .

Proof. DIY. ■

Corollary 2.12. $f : X \rightarrow \mathbb{C}$, measurable function $\iff \exists$ simple functions $\phi_n : X \rightarrow \mathbb{C}$ s.t. $0 \leq |\phi_1| \leq |\phi_2| \leq \dots \leq |f|$ and ϕ_n converges to f pointwise. (Again, if f is bounded the convergence can be uniform.)

2.2 Integration of Nonnegative Functions

Definition 2.13. Suppose (X, \mathcal{A}, μ) a measure space and $\phi = \sum_{i=1}^N c_i 1_{E_i} : X \rightarrow [0, \infty]$ a simple function. Let

$$\int \phi = \int \phi \, d\mu = \int_X \phi \, d\mu = \sum_1^N c_i \mu(E_i).$$

Proposition 2.14. Suppose $\phi, \psi \geq 0$ are simple functions. Then,

- 2.13 is well-defined.
- $\int c\phi = c \int \phi, c \in [0, \infty)$.
- $\int (\phi + \psi) = \int \phi + \int \psi$.
- $\phi(x) \geq \psi(x), \forall x \implies \int \phi \geq \int \psi$.
- $\nu(A) = \int_A \phi \, d\mu$ is a measure on (X, \mathcal{A}) .

Proof. DIY. ■

Definition 2.15. Suppose $(X, \mathcal{A}, \mu), f : X \rightarrow [0, \infty]$ is \mathcal{A} -measurable.

Define

$$\int f = \int f \, d\mu = \sup \left\{ \int \phi \mid 0 \leq \phi \leq f, \phi \text{ simple} \right\}.$$

Proposition 2.16.

- If f is a simple function then two definitions are the same.
- $\int cf = c \int f$.
- $f \geq g \geq 0 \implies \int f \geq \int g$.
- $\int f + g = \int f + \int g$. (A bit harder to check)

Theorem 2.17 (Monotone convergence theorem). Suppose (X, \mathcal{A}, μ) a measure space and

- $f : X \rightarrow [0, \infty]$ is \mathcal{A} -measurable, $\forall n \in \mathbb{N}$.
- $0 \leq f_1(x) \leq \dots$
- $\lim_{n \rightarrow \infty} f_n(x) = f(x)$.

Then

$$\int f = \lim_{n \rightarrow \infty} \int f_n.$$

Proof. Note that $\lim_{n \rightarrow \infty} f_n(x)$ converges $\forall x \in X$ and $\lim_{n \rightarrow \infty} f_n(x)$ converges.

- $f_n \leq f \implies \int f_n \leq \int f \implies \lim_{n \rightarrow \infty} \int f_n \leq \int f$.
- Fix simple function $0 \leq \phi \leq f$. Enough to show that $\lim_{n \rightarrow \infty} \int f_n \geq \int \phi$.

Now fix $\alpha \in (0, 1)$. Enough to prove that $\lim_{n \rightarrow \infty} \int f_n \geq \alpha \int \phi$.

Let $A_n = \{x \mid f_n(x) \geq \alpha \phi(x)\}$.

- $A_n \in \mathcal{A}$.
- $A_1 \subset A_2 \subset \dots$
- $\bigcup_{n=1}^{\infty} A_n = X$. (check!)

So we have

$$\int f_n \geq \int f_n 1_{A_n} \geq \int \alpha \phi 1_{A_n} = \alpha \nu(A_n)$$

where $\nu(A) = \int_A \phi$ is a measure.

$$\implies \lim_{n \rightarrow \infty} \int f_n \geq \lim_{n \rightarrow \infty} \nu(A_n) = \alpha \nu(X) = \alpha \int \phi. \quad \blacksquare$$

Corollary 2.18. $f, g \geq 0$ measurable $\implies \int f + g = \int f + \int g$.

Proof. \exists simple functions $0 \leq \phi_1 \leq \phi_2 \leq \dots, \phi_n \rightarrow f$ pointwise and $0 \leq \psi_1 \leq \psi_2 \leq \dots, \psi_n \rightarrow g$ pointwise.

By MCT, we have

$$\int (f + g) = \lim_{n \rightarrow \infty} \int (\phi_n + \psi_n) = \lim_{n \rightarrow \infty} \int \phi_n + \int \psi_n = \int f + \int g. \quad \blacksquare$$

Corollary 2.19 (Tonelli's theorem for series and integrals). *Given $s_n \geq 0, \forall n \in \mathbb{N}$ measurable functions. Then*

$$\int \sum_{n=1}^{\infty} s_n = \sum_{n=1}^{\infty} \int s_n.$$

Proof. Let $f_N = \sum_{n=1}^N s_n, 0 \leq f_1 \leq f_2 \leq \dots$

$$\lim_{N \rightarrow \infty} f_N(x) = \sum_{n=1}^{\infty} s_n(x)$$

By MCT, we have

$$\lim_{N \rightarrow \infty} \sum_{n=1}^N s_n = \sum_{n=1}^{\infty} s_n \quad \blacksquare$$

Theorem 2.20 (Fatou's lemma). *Suppose $f_n \geq 0$ measurable. Then*

$$\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n.$$

Recall that

$$\liminf_{n \rightarrow \infty} f_n := \lim_{k \rightarrow \infty} \inf_{n \geq k} f_n = \sup_{k \in \mathbb{N}} \inf_{n \geq k} f_n,$$

and

$$\lim_{n \rightarrow \infty} a_n \text{ exists} \iff \limsup_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n.$$

Proof. Let $g_k = \inf_{n \geq k} f_n \implies s_k$ measurable and $0 \leq g_1 \leq g_2 \leq \dots$. By MCT, we have

$$\int \liminf_{n \rightarrow \infty} f_n = \int \lim_{k \rightarrow \infty} g_k = \lim_{k \rightarrow \infty} \int g_k = \lim_{k \rightarrow \infty} \int \inf_{n \geq k} f_n$$

$$\begin{aligned}
& \inf_{n \geq k} f_n \leq f_m, \forall m \geq k \\
\implies & \int \inf_{n \geq k} f_n \leq \int f_m, \forall m \geq k \\
\implies & \int \inf_{n \geq k} f_n \leq \inf_{m \geq k} \int f_m
\end{aligned}$$

■

Example 2.21. Suppose $(\mathbb{R}, \mathcal{L}, m)$

(a) (escape to horizontal infinity) $f_n = 1_{(n, n+1)}$.

We see that $f_n \rightarrow 0 = f$ pointwise and $\int f_n = 1, \forall n, \int f = 0$.

(b) (escape to width infinity) $f_n = \frac{1}{n} 1_{(0, n)}$.

(c) (escape to vertical infinity) $f_n = n 1_{(0, 1/n)}$.

Lemma 2.22 (Markov's inequality). $f \geq 0$ is measurable \implies

$$\forall c \in (0, \infty), \mu(\{x \mid f(x) \geq c\}) \leq \frac{1}{c} \int f.$$

Proof. Let $E = \{x \mid f(x) \geq c\}$. Then

$$f(x) \geq c 1_E(x) \implies \int f \geq c \int 1_E = c \mu(E).$$

■

Proposition 2.23. Suppose $f \geq 0$ measurable. Then $\int f = 0 \iff f = 0$ almost everywhere (a.e.)

$$\int f \, d\mu = \mu(A) = 0, \quad A = \{x \mid f(x) > 0\} = f^{-1}((0, \infty])$$

Proof. (a) Assume $f = \phi$ a simple function. We may assume

$$\phi = \sum_{i=1}^N c_i 1_{E_i}, \quad c_i \in (0, \infty), \quad E_i \text{'s are disjoint.}$$

$$\begin{aligned}
& \int \phi = \sum_{i=1}^N c_i \mu(E_i) = 0 \\
& \iff \mu(E_1) = \dots = \mu(E_N) = 0 \\
& \iff \mu(A) = 0, \quad A = \bigcup_{i=1}^N E_i.
\end{aligned}$$

(b) General $f \geq 0$.

(1) Assume $\mu(A) = 0$ (i.e. $f = 0$ a.e.)

Let $0 \leq \phi \leq f$, ϕ is simple.

$$\implies \phi(x) = 0, \forall x \in A^c$$

$$\implies \phi = 0 \text{ a.e.}$$

$$\implies \int \phi = 0$$

Then $\int f = 0$ by the definition of $\int f$.

(2) Assume $\inf f = 0$. Let $A_n = f^{-1}([\frac{1}{n}, \infty])$

$$\implies A_1 \subset A_2 \subset \dots$$

$$\bigcup_1^\infty A_n = f^{-1}\left(\bigcup_1^\infty \left[\frac{1}{n}, \infty\right]\right) = f^{-1}((0, \infty)) = A$$

$$\mu(A_n) = \mu\left(\left\{x \mid f(x) \geq \frac{1}{n}\right\}\right) \leq n \int f = 0$$

$$\implies \mu(A) = \lim_{n \rightarrow \infty} \mu(A_n) = 0$$

by the continuity of measure from below. ■

Corollary 2.24. $f, g \geq 0$ are measurable, $f = g$ a.e. $\implies \int f = \int g$.

Proof. Let $A = \{x \mid f(x) \geq g(x)\}$. A is measurable (why?). By assumption $\mu(A) = 0$. Hence $f1_A = 0$ a.e.

$$\begin{aligned} \int f &= \int f(1_A + 1_{A^c}) \\ &= \int f1_A + \int f1_{A^c} \\ &= \int f1_{A^c} \\ &= \int g1_{A^c} = \int g1_A + \int g1_{A^c} = \int g. \end{aligned} \quad \blacksquare$$

Corollary 2.25. $f_n \geq 0$ measurable. Then

(a)

$$\left. \begin{array}{l} 0 \leq f_1 \leq f_2 \leq \dots \leq f \text{ a.e.} \\ \lim_{n \rightarrow \infty} f_n = f \text{ a.e.} \end{array} \right\} \implies \lim_{n \rightarrow \infty} \int f_n = \int f.$$

(b)

$$\lim_{n \rightarrow \infty} f_n = f \text{ a.e.} \implies \int f \leq \liminf_{n \rightarrow \infty} \int f_n.$$

2.3 Integration of Complex Functions

I was afraid that you are bored.

— Jinho Baik on homework

Definition 2.26. (X, \mathcal{A}, μ) measure space.

- $f : X \rightarrow \overline{\mathbb{R}}$ or $f : X \rightarrow \mathbb{C}$ measurable functions is called *integrable* if $\int |f| < \infty$.
Then

$$\int f = \int f^+ - \int f^- \text{ or } \int f = \int u^+ - \int u^- + i \left(\int v^+ - \int v^- \right).$$

- Suppose $f : X \rightarrow \overline{\mathbb{R}}$. Define

$$\int f = \begin{cases} \infty & \int f^+ = \infty, \int f^- < \infty, \\ -\infty & \int f^+ < \infty, \int f^- = \infty. \end{cases}$$

Lemma 2.27. Suppose $f, g : x \rightarrow \overline{\mathbb{R}} \rightarrow \mathbb{C}$ integrable. Assume $f(x) + g(x)$ is well-defined $\forall x \in X$. (i.e. $\infty + (-\infty)$, $-\infty + \infty$ do not occur)

- (a) $f + g, cf, c \in \mathbb{C}$ are integrable.
- (b) $\int f + g = \int f + \int g$.
- (c) $\left| \int f \right| \leq \int |f|$. (This is essentially triangle inequality.)

Proof. Check [Fol99, p.53]. ■

Lemma 2.28. (X, \mathcal{A}, μ) measure space and f integrable function on X .

- (a) f is finite a.e. (i.e. $\{x \in X : |f(x)| = \infty\}$ is a null set)
- (b) The set $\{x \in X : f(x) \neq 0\}$ is σ -finite.

Proof. HW5Q8. ■

Proposition 2.29. Suppose (X, \mathcal{A}, μ) a measure space.

(a) If h is integrable on X , then

$$\int_E h = 0, \forall E \in \mathcal{A} \iff \int |h| = 0 \iff h = 0 \text{ a.e.}$$

(b) If f, g are integrable on X then

$$\int_E f = \int_E g, \forall E \in \mathcal{A} \iff f = g \text{ a.e.}$$

Proof. (a) $\int |h| = 0 \iff h = 0$ is shown in 2.23.

$$\int |h| = 0 \implies \left| \int_E h \right| \leq \int_E |h| \leq \int |h| = 0.$$

On the other hand, assume $\int_E h = 0, \forall E \in \mathcal{A}$. $h = u + iv = u^+ - u^- + i(v^+ - v^-)$.
Let $B = \{x \mid u^+(x) > 0\}$.

$$0 = \operatorname{Re} \int_B h = \int_B u = \int_B u^+ = \int_B u^+ + \int_{B^c} u^+ = \int u^+ \implies u^+ = 0 \text{ a.e.}$$

Similarly, we get $u^-, v^+, v^- = 0$ a.e..

(b) follows from (a). ■

Theorem 2.30 (Dominated convergence theorem). Suppose (X, \mathcal{A}, μ) a measure space and

- (a) f_n integrable on $X, \forall n \in \mathbb{N}$.
- (b) $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ a.e. (pointwise)
- (c) $\exists g : X \rightarrow [0, \infty]$ s.t.
 - g is integrable.
 - $|f_n(x)| \leq g(x)$ a.e., $\forall n \in \mathbb{N}$.

Then

$$\lim_{n \rightarrow \infty} \int f_n = \int f.$$

Proof. Let F be the countable union of null sets on which (a)-(c) may fail. Modifying the def of f_n, f, g on F we may assume (a)-(c) hold everywhere. (b)+(c) $\implies f$ is integrable.

We consider $\overline{\mathbb{R}}$ -valued case only. (\mathbb{C} -valued case follows)

$$\begin{aligned}
 & g + f_n \geq 0, g - f_n \geq 0 \\
 & \xRightarrow{\text{Fatou}} \int g + f \leq \liminf_{n \rightarrow \infty} \int g + f_n, \quad \int g - f \leq \liminf_{n \rightarrow \infty} \int g - f_n \\
 & \implies \int g + \int f \leq \int g + \liminf_{n \rightarrow \infty} \int f_n, \quad \int g - \int f \leq \int g - \limsup_{n \rightarrow \infty} \int f_n \\
 & \xRightarrow{\int g < \infty} \int f \leq \liminf_{n \rightarrow \infty} \int f_n, \quad -\int f \leq -\limsup_{n \rightarrow \infty} \int f_n \\
 & \implies \int f \leq \liminf_{n \rightarrow \infty} \int f_n \leq \limsup_{n \rightarrow \infty} \int f_n \leq \int f
 \end{aligned}$$

So we should have

$$\int f = \liminf_{n \rightarrow \infty} \int f_n = \limsup_{n \rightarrow \infty} \int f_n. \quad \blacksquare$$

Next we investigate the question:

$$\int \sum_1^\infty f_n \stackrel{?}{=} \sum_1^\infty \int f_n.$$

Tonelli: yes if $f_n \geq 0$. Fubini:

Corollary 2.31 (Fubini's theorem for series and integrals).

$$\left. \sum_1^\infty \int |f_n| < \infty \right\} \implies \int \sum_1^\infty f_n = \sum_1^\infty \int f_n.$$

$$\text{Proof. } G(x) = \sum_1^\infty |f_n(x)| \geq |F_N(x)|, F_N(x) = \sum_1^N f_n(x). \quad \blacksquare$$

2.4 L^1 space

Definition 2.32. Suppose V is a vector space over field \mathbb{R} or \mathbb{C} . A *seminorm* on V is $\|\cdot\| : V \rightarrow [0, \infty)$ s.t.

- $\|cv\| = |c|\|v\|, \forall v \in V, \forall c \text{ scalar}$
- $\|v + w\| \leq \|v\| + \|w\|$, triangle inequality

A *norm* is a seminorm such that $\|v\| \iff v = 0$.

Lemma 2.33. A normed vector space is a metric space with metric $\rho(v, w) = \|v - w\|$.

Proof. (DIY)

- $\rho(v, w) = 0 \iff \|v - w\| = 0 \iff v - w = 0 \iff v = w.$
- $\rho(v, w) = \|v - w\| = \|-1(w - v)\| = |-1| \|w - v\| = \rho(w, v).$
- $\rho(v, w) + \rho(w, z) = \|v - w\| + \|w - z\| \geq \|v - w + w - z\| = \|v - z\| = \rho(v, z). \quad \blacksquare$

Example 2.34. \mathbb{R}^d with $\|x\|_p = \begin{cases} \left(\sum_1^d |x_i|^p \right)^{1/p} & p \in [1, \infty) \\ \max_{1 \leq i \leq d} |x_i| & p = \infty \end{cases}$ is a normed vector space.

Unit ball $\{x : \|x\|_p < 1\}.$

All $\|\cdot\|_p$ norm induce the same topology i.e. if U is open in p -norm then it is open in p' -norm. This implies that a sequence converging under p -norm also converges under p' -norm.

RECALL f is integrable $\implies \int |f| < \infty. f = g$ a.e. $\implies \int f = \int g.$

Definition 2.35. Suppose (X, \mathcal{A}, μ) a measure space.

$f \in L^1(X, \mathcal{A}, \mu) = L^1(X, \mu) = L^1(X) = L^1(\mu)$ means f is an integrable function on X .

Lemma 2.36. $L^1(X, \mathcal{A}, \mu)$ is a vector space with seminorm $\|f\|_1 = \int |f|.$

Definition 2.37. Define $f \sim g$ if $f = g$ a.e. $L^1(X, \mathcal{A}, \mu)/\sim = L^1(X, \mathcal{A}, \mu).$ “=” is just a notation for convenience!

With new definition we have $L^1(X, \mathcal{A}, \mu)$ is a normed vector space. $\rho(f, g) = \int |f - g|.$

Something interesting to discuss is what are the dense subsets of L^1 .

Theorem 2.38.

- (a) $\{ \text{integrable simple functions} \}$ is dense in $L^1(X, \mathcal{A}, \mu)$ (with respect to L^1 metric)
- (b) $(X, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{A}_\mu, \mu), \mu$ is Lebesgue-Stieltjes measure $\implies \{ \text{integrable step functions} \}$ is dense in $L^1(X, \mathcal{A}, \mu)$
- (c) $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R}, \mathcal{L}, m).$

Definition 2.39.

- A step function on \mathbb{R} is $\psi + \sum_1^N c_i 1_{I_i},$ where I_i is an interval.
- $C_c(\mathbb{R})$ is the collection of continuous functions with compact support $\text{supp}(f) = \overline{\{x \in \mathbb{R} \mid f(x) \neq 0\}}.$

Proof. (a) \exists simple functions $0 \leq |\phi_1| \leq |\phi_2| \leq \dots \leq |f|, \phi_n \rightarrow f$ pointwise \implies

$$\lim_{n \rightarrow \infty} \int |\phi_n - f| = 0 \text{ by DCT. } (|\phi_n - f| \leq |\phi_n| + |f| \leq 2|f|)$$

(b) 1_E approx by $\sum_1^N c_i 1_{I_i}$? Regularity theorem for Lebesgue-Stieltjes measure $\implies \forall \varepsilon' > 0, \exists I = \bigcup_1^N I_i$ s.t. $\mu(E \Delta I) < \varepsilon'$.

(c) Suppose $1_{(a,b)}, g \in C_c(\mathbb{R})$. $\int |1_{(a,b)} - g| dm \leq 1 \cdot \frac{\varepsilon}{2} + 1 \cdot \frac{\varepsilon}{2} = \varepsilon$. ■

2.5 Riemann Integrability

Suppose $P = \{a = t_0 < t_1 < \dots < t_k = b\}$ a partition of $[a, b]$. Lower Riemann sum of f using P

$$L_P = \sum_{i=1}^k \left(\inf_{[t_{i-1}, t_i]} f \right) (t_i - t_{i-1})$$

and upper Riemann sum

$$U_P = \sum_{i=1}^k \left(\sup_{[t_{i-1}, t_i]} f \right) (t_i - t_{i-1})$$

Lower Riemann integral of $f = \underline{I} = \sup_P L_P$. Upper Riemann integral of $f = \bar{I} = \inf_P U_P$.

Definition 2.40. A bounded function $f : [a, b] \rightarrow \mathbb{R}$ is called Riemann (Darboux) integrable if $\underline{I} = \bar{I}$. (If so, $\underline{I} = \bar{I} = \int_a^b f(x) dx$.)

NOTE

- If $P \subset P'$, then $L_P \leq L_{P'}, U_{P'} \leq U_P$.
- Recall that continuous functions on $[a, b]$ are Riemann integrable on $[a, b]$.

Theorem 2.41. Let $f : [a, b] \rightarrow \mathbb{R}$ be a bounded function.

(a) If f is Riemann integrable, then f is Lebesgue measurable. (thus Lebesgue integrable) and

$$\int_a^b f(x) dx = \int_{[a,b]} f dm.$$

(b) f is Riemann integrable $\iff f$ is continuous Lebesgue a.e.

Proof. \exists partitions $P_1 \subset P_2 \subset P_3 \subset \dots$ s.t. $L_{P_n} \nearrow \underline{I}, U_{P_n} \searrow \bar{I}$.

Define simple (step) functions

$$\phi_n = \sum_{i=1}^k \left(\inf_{[t_{i-1}, t_i]} f \right) 1_{(t_{i-1}, t_i]}$$

$$\psi_n = \sum_{i=1}^k \left(\sup_{[t_{i-1}, t_i]} \right) 1_{(t_{i-1}, t_i]}$$

Define $\phi = \sup_n \phi_n$, $\psi = \inf_n \psi_n$. Then ϕ, ψ are Lebesgue measurable functions.

NOTE

- $\exists M > 0$ s.t. $|\phi_n|, |\psi_n| \leq M 1_{[a, b]}, \forall n \in \mathbb{N}$.
- $\int \phi_n \, dm = L_{P_n}, \int \psi_n \, dm = U_{P_n}$.

By DCT, $\underline{I} = \lim_{n \rightarrow \infty} \int \phi_n \, dm = \int \phi \, dm, \bar{I} = \int \psi \, dm$.

Thus, f is Riemann integrable $\iff \int \phi = \int \psi \iff \int (\phi - \psi) = 0 \iff \phi = \psi$ Lebesgue a.e.

Recall that $\phi \leq f \leq \psi, \forall x \in (a, b]$. So $f = \phi$ a.e. Since $(\mathbb{R}, \mathcal{L}, \mu)$ is complete, f is Lebesgue measurable (see HW). The second statement hence follows. ■

2.6 Modes of Convergence

Suppose $f_n, f : X \rightarrow \mathbb{C}, S \subset X$.

- $f_n \rightarrow f$ pointwise on S : $\forall x \in S, \forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N, |f_n(x) - f(x)| < \varepsilon$.
- $f_n \rightarrow f$ uniformly on S : $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall x \in S, \forall n \geq N, |f_n(x) - f(x)| < \varepsilon$.

We can change $\forall \varepsilon > 0$ to $\forall k \in \mathbb{N}$ and bound the distance by $\frac{1}{k}$.

Lemma 2.42. Let $B_{n,k} = \{x \in X \mid |f_n(x) - f(x)| < \frac{1}{k}\}$.

$$(a) \ f_n \rightarrow f \text{ pointwise on } S \iff S \subset \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} B_{n,k}.$$

$$(b) \ f_n \rightarrow f \text{ uniformly on } S \iff \exists N_1, N_2, \dots \in \mathbb{N} \text{ s.t. } S \subset \bigcap_{k=1}^{\infty} \bigcap_{n=N_k}^{\infty} B_{n,k}.$$

Definition 2.43. Suppose (X, \mathcal{A}, μ) a measure space.

- (a) $f_n \rightarrow f$ a.e means \exists null set E s.t. $f_n \rightarrow f$ pointwise on E^c .
- (b) $f_n \rightarrow f$ in L^1 means $\lim_{n \rightarrow \infty} \|f_n - f\| = 0$.

Example 2.44. $(\mathbb{R}, \mathcal{L}, \mu)$. $f = 0$.

- (a) $f_n = 1_{(n, n+1)}, f_n = \frac{1}{n} 1_{(0, n)}, f_n = n 1_{(0, \frac{1}{n})}$. All of $f_n \rightarrow f$ pointwise but $\nrightarrow f$ in L^1 .
- (b) Typewriter functions: $f_n \rightarrow f$ in L^1 . $f_n \nrightarrow f$ a.e.

Proposition 2.45 (Fast L^1 convergence \implies a.e. convergence). Suppose (x, \mathcal{A}, μ) measure

space. f_n, f measurable function on X .

$$\sum_1^\infty \|f_n - f\|_1 < \infty \implies f_n \rightarrow f \text{ a.e.}$$

Proof. RECALL Markov's inequality.

Let $E = \bigcup_{k=1}^\infty \bigcap_{N=1}^\infty \bigcup_{n=N}^\infty B_{n,k}^c = \{x \mid f_n(x) \not\rightarrow f(x)\}$. By Markov we have

$$\begin{aligned} \forall k, \forall N, \mu(B_{n,k}^c) &\leq k \int |f_n - f| \\ \implies \forall k, \mu\left(\bigcap_{n=N}^\infty B_{n,k}^c\right) &\leq \sum_{n=N}^\infty k \|f_n - f\|_1 \rightarrow 0 \text{ as } n \rightarrow \infty \\ \implies \forall k, \mu\left(\bigcap_{N=1}^\infty \bigcap_{n=N}^\infty B_{n,k}^c\right) &= \lim_{N \rightarrow \infty} \mu\left(\bigcap_{n=N}^\infty B_{n,k}^c\right) = 0 \\ \implies \mu(E) &= 0. \end{aligned}$$

■

Corollary 2.46. $f_n \rightarrow f$ in $L^1 \implies \exists \text{subsequence } f_{n_j} \rightarrow f \text{ a.e.}$

Proof. $\forall j \in \mathbb{N}, \exists n_j \in \mathbb{N} \text{ s.t. } \|f_{n_j} - f\|_1 < \frac{1}{j^2}$. Then $\sum_{j=1}^\infty \|f_{n_j} - f\|_1 < \infty$.

■

Definition 2.47. f_n, f measurable functions on (X, \mathcal{A}, μ) . $f_n \rightarrow f$ in measure means

$$\forall \varepsilon > 0, \lim_{n \rightarrow \infty} \mu(\{x \in X \mid |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

Example 2.48. • $f_n = n1_{(0, \frac{1}{n})}, f = 0$.

$$\forall \varepsilon > 0, \{x \mid |f_n(x) - f(x)| > \varepsilon\} = \left(0, \frac{1}{n}\right).$$

(Recall that $f_n \not\rightarrow 0$ in L^1 .)

- Typewriter function. (Recall that $f_n \not\rightarrow 0$ a.e.)

We can easily check that $f_n \rightarrow f$ in $L^1 \implies f_n \rightarrow f$ in measure. But the converse is not true.

$f_n \rightarrow f$ in measure $\implies \exists f_{n_j} \rightarrow f$ a.e. (Check [Fol99])

We have then the following diagram:

$$\begin{array}{ccccc}
 f_n \rightarrow f \text{ fast } L^1 & \implies & f_n \rightarrow f \text{ in } L^1 & \xRightarrow{\quad} & f_n \rightarrow f \text{ in measure} \\
 & \searrow & \Downarrow & & \Downarrow \\
 & & f_n \rightarrow f \text{ a.e.} & & \exists f_{n_j} \rightarrow f \text{ a.e.}
 \end{array}$$

Definition 2.49. f_n, f measurable functions on (X, \mathcal{A}, μ) .

- (a) $f_n \rightarrow f$ *uniformly a.e* means \exists null set F s.t. $f_n \rightarrow f$ uniformly on F^c .
- (b) $f_n \rightarrow f$ *almost uniformly* means $\forall \varepsilon > 0, \exists F \in \mathcal{A}$, s.t. $\mu(F) < \varepsilon$, $f_n \rightarrow f$ uniformly on F^c .

Recall 2.42.

Theorem 2.50 (Egoroff). f_n, f measurable on (X, \mathcal{A}, μ) . Suppose $\mu(X) < \infty$. Then $f_n \rightarrow f$ a.e $\iff f_n \rightarrow f$ almost uniformly.

Proof. " \Leftarrow ": DIY

" \Rightarrow ": Fix $\varepsilon > 0$.

$$f_n \rightarrow f \text{ a.e} \implies \mu \left(\bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c \right) = 0 \implies \forall k, \mu \left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c \right) = 0.$$

By the continuity of measure from above and since $\mu(X) < \infty$,

$$\forall k, \lim_{N \rightarrow \infty} \mu \left(\bigcup_{n=N}^{\infty} B_{n,k}^c \right) = 0 \implies \forall k, \exists N_k \in \mathbb{N}, \mu \left(\bigcup_{n=N_k}^{\infty} B_{n,k}^c \right) < \frac{\varepsilon}{2^k}.$$

$$\text{Let } F = \bigcup_{k=1}^{\infty} \bigcup_{n=N_k}^{\infty} B_{n,k}^c \implies \mu(F) < \varepsilon, f_n \rightarrow f \text{ uniformly on } F^c. \quad \blacksquare$$

Chapter 3

Product Measures

(p.22 - 36, section 1.2 and section 2.5, 2.6 of [Fol99])

The ultimate goal is to prove Fubini's theorem. This is also related to probability in the sense that a series of events is in product measure.

3.1 Product σ -algebra

- Product space $X = \prod_{\alpha \in I} X_{\alpha}, x = (x_{\alpha})_{\alpha \in I}$.
- Coordinate map $\pi_{\alpha} : X \rightarrow X_{\alpha}$.

Definition 3.1. $(X_{\alpha}, \mathcal{A}_{\alpha})$ measurable space. $\forall \alpha \in I$, the *product σ -algebra* on $X = \prod_{\alpha \in I} X_{\alpha}$ is

$$\bigotimes_{\alpha \in I} \mathcal{A}_{\alpha} = \left\langle \bigcup_{\alpha \in I} \pi_{\alpha}^{-1}(\mathcal{A}_{\alpha}) \right\rangle$$

where

$$\pi_{\alpha}^{-1}(A_{\alpha}) = \{\pi_{\alpha}^{-1}(E) | E \in \mathcal{A}_{\alpha}\}.$$

NOTATION

$$I = \{1, \dots, d\} \implies X = \prod_{i=1}^d X_i, x = (x_1, \dots, x_d), \bigotimes_{i=1}^d \mathcal{A}_i = \mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_d.$$

Lemma 3.2. *If I is countable, then*

$$\bigotimes_{\alpha \in I} \mathcal{A}_\alpha = \left\langle \left\{ \prod_{i=1}^{\infty} E_i \mid E_i \in \mathcal{A}_i \right\} \right\rangle$$

Lemma 3.3. *Suppose $\mathcal{A}_\alpha = \langle \mathcal{E}_\alpha \rangle, \forall \alpha \in I$.*

- (a) $\pi_\alpha^{-1}(\mathcal{A}_\alpha) = \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle$.
- (b) $\bigotimes_{\alpha} \mathcal{A}_\alpha = \left\langle \bigcup_{\alpha} \pi_\alpha^{-1}(\mathcal{E}_\alpha) \right\rangle$.
- (c) *If I is countable, then* $\bigotimes_{\alpha \in I} \mathcal{A}_\alpha = \left\langle \left\{ \prod_{i=1}^{\infty} E_i \mid E_i \in \mathcal{E}_i \right\} \right\rangle$.

Proof.

- (a) • $f : Y \rightarrow Z, \mathcal{B}$ a σ -algebra on $Z \implies f^{-1}(\mathcal{B})$ is a σ -algebra since set union commutes with preimage. Hence $\pi_\alpha^{-1}(\mathcal{A}_\alpha)$ is a σ -algebra on X . Since $\pi_\alpha^{-1}(\mathcal{E}_\alpha) \subset \pi_\alpha^{-1}(\mathcal{A}_\alpha) \implies \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle \subset \pi_\alpha^{-1}(\mathcal{A}_\alpha)$.
 - Let $\mathcal{M} = \{B \subset X_\alpha \mid \pi_\alpha^{-1}(B) \in \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle\}$. We show that $\mathcal{A}_\alpha \subset \mathcal{M}$.
 - \mathcal{M} is a σ -algebra. (easy)
 - $\mathcal{E}_\alpha \subset \mathcal{M}$. (by definition)
- So $\mathcal{A}_\alpha = \langle \mathcal{E}_\alpha \rangle \subset \mathcal{M}$. Hence, if $E \in \mathcal{A}_\alpha, E \subset \mathcal{M} \implies \pi_\alpha^{-1}(E) \in \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle$ i.e. $\mathcal{A}_\alpha \subset \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle$.

(b, c) DIY. ■

Theorem 3.4. *Suppose X_1, \dots, X_d metric spaces. Let $X = \prod_1^d X_i$ with product metric $\rho(x, y) =$*

$$\sum_{i=1}^d \rho_i(x, y). \text{ Then}$$

$$(a) \bigotimes_{i=1}^d \mathcal{B}(X_i) \subset \mathcal{B}(X).$$

$$(b) \text{ If, in addition, each } X_i \text{ has a countable dense subset, then } \bigotimes_{i=1}^d \mathcal{B}(X_i) = \mathcal{B}(X).$$

Proof. DIY. ■

As a consequence, we have $\mathcal{B}(\mathbb{R}^d) = \mathcal{B}(\mathbb{R}) \otimes \dots \otimes \mathcal{B}(\mathbb{R})$.

Suppose $f = u + iv : X \rightarrow \mathbb{C}$. f is measurable $\iff u^{-1}(E) \in \mathcal{A}, v^{-1}(E) \in \mathcal{A}, \forall E \in \mathcal{B}(\mathbb{R}) \iff f^{-1}(F) \in \mathcal{A}, \forall F \in \mathcal{B}(\mathbb{C}) = \mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

p.65. Let's focus on finite product.

You like Minecraft right? It's all rectangles.

Definition 3.5. Suppose X, Y sets.

- (a) For a $E \subset X \times Y$, $E_x = \{y \in Y \mid (x, y) \in E\}$ and $E^y = \{x \in X \mid (x, y) \in E\}$.
- (b) For $f : X \times Y \rightarrow \mathbb{C}$, define $f_x : Y \rightarrow \mathbb{C}$, $f^y : X \rightarrow \mathbb{C}$ by $f_x(y) = f(x, y) = f^y(x)$.
- (c)

Example 3.6. $(1_E)_x = 1_{E_x}$. $(1_E)^y = 1_{E^y}$.

Proposition 3.7. $(X, \mathcal{A}), (Y, \mathcal{B})$ measurable spaces.

- (a) $E \in \mathcal{A} \otimes \mathcal{B} \implies E_x \in \mathcal{B}, E^y \in \mathcal{A}, \forall x \in X, y \in Y$.
- (b) $f : X \times Y \rightarrow \mathbb{C}$ is $\mathcal{A} \otimes \mathcal{B}$ -measurable $\implies f_x$ is \mathcal{B} -measurable, f^y is \mathcal{A} -measurable, $\forall x \in X, y \in Y$.

Proof. (a) Let $\mathcal{F} = \{E \subset X \times Y \mid \text{(a) holds}\}$.

- \mathcal{F} is a σ -algebra (easy)
- $\mathcal{R}_0 := \{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\} \subset \mathcal{F}$ (easy) $\implies \mathcal{A} \otimes \mathcal{B} = \langle \mathcal{R}_0 \rangle \subset \mathcal{F}$

(b) DIY. ■

MIDTERM is up till here.

3.2 Product Measures

Definition 3.8. Suppose $(X, \mathcal{A}), (Y, \mathcal{B})$. A (measurable) rectangle is $R = A \times B, A \in \mathcal{A}, B \in \mathcal{B}$.

Let $\mathcal{R}_0 := \{R = A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$.

$$\mathcal{R} := \left\{ \bigcup_1^N R_i \mid N \in \mathbb{N}, R_1, \dots, R_N \text{ disjoint rectangles} \right\}.$$

Lemma 3.9. \mathcal{R} is an algebra. $\langle \mathcal{R}_0 \rangle = \langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B}$.

Theorem 3.10. Suppose $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ measure spaces.

(a) \exists measure $\mu \times \nu$ on $\mathcal{A} \otimes \mathcal{B}$ satisfying $(\mu \times \nu)(A \otimes B) = \mu(A)\nu(B), \forall A \in \mathcal{A}, B \in \mathcal{B}$.

(b) If μ, ν are σ -finite, then $\mu \times \nu$ is unique.

Proof. (a) Define $\pi : \mathcal{R} \rightarrow [0, \infty]$ by $\pi(A \times B) = \mu(A)\nu(B)$ and extend linearly.

CLAIM π is a pre-measure on \mathcal{R} .

Enough to check $\pi(A \times B) = \sum_1^\infty \pi(A_n \times B_n)$ if $A \times B = \bigcup_1^\infty (A_n \times B_n)$ disjoint union.

Since $A_n \times B_n$ are disjoint,

$$1_{A \times B}(x, y) = \sum_1^\infty 1_{A_n \times B_n}(x, y), \quad 1_A(x)1_B(y) = \sum_1^\infty 1_{A_n}(x)1_{B_n}(y).$$

By Tonelli's theorem for series and integrals, we have

$$\begin{aligned} \mu(A)1_B(y) &= \int_x 1_A(x)1_B(y) \, d\mu(x) \\ &= \sum_1^\infty \int_x 1_{A_n}(x)1_{B_n}(y) \, d\mu(x) = \sum_1^\infty \mu(A_n)1_{B_n}(y). \end{aligned}$$

We then integrate with respect to y to complete the claim.

By HK theorem, $\exists \mu \otimes \nu$ on $\langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B}$ extending π on \mathcal{R} .

(b) μ, ν σ -finite $\implies \pi$ is σ -finite on $\mathcal{R} \implies$ HK uniqueness them applies. ■

So we have a measure

$$(\mu \times \nu)(E) = \inf \left\{ \sum_1^\infty \mu(A_i)\nu(B_i) \mid E \subset \bigcup_1^\infty A_i \times B_i, A_i \in \mathcal{A}, B_i \in \mathcal{B} \right\}.$$

Then one questions naturally arises: suppose $f : X \times Y \rightarrow \mathbb{C}$,

$$\int_{X \times Y} f \, d(\mu \times \nu) \stackrel{?}{=} \int_y \left(\int_x f \, d\mu \right) \, d\nu.$$

3.3 Monotone Class Lemma

Definition 3.11. Suppose X is a set, $\mathcal{C} \subset \mathcal{P}(X)$. \mathcal{C} is a *monotone class* on X if

- closed under *countable increasing unions*
(i.e. $E_n \in \mathcal{C}, E_1 \subset E_2 \subset \dots \implies \bigcup_1^\infty E_i \in \mathcal{C}$.)

- closed under *countable decreasing intersections*
(i.e. $E_n \in \mathcal{C}, E_1 \supset E_2 \supset \dots \implies \bigcap_{i=1}^{\infty} E_i \in \mathcal{C}$.)

Example 3.12. • σ -algebra is a monotone class.

- $\bigcap_{\alpha} \mathcal{C}_{\alpha}$ is a monotone class \implies if $\mathcal{E} \in \mathcal{P}(X)$, there is unique smallest monotone class containing \mathcal{E} .

The importance of this definition shows up in the following theorem:

Theorem 3.13. Suppose \mathcal{A}_0 is an algebra on X . Then $\langle \mathcal{A}_0 \rangle$ is the monotone class generated by \mathcal{A}_0 .

Proof. Let $\mathcal{A} = \langle \mathcal{A}_0 \rangle$, \mathcal{C} = monotone class generated by \mathcal{A}_0 .

- (a) \mathcal{A} is a σ -algebra $\implies \mathcal{A}$ is a monotone class containing $\mathcal{A}_0 \implies \mathcal{A} \supset \mathcal{C}$.
 (b) To show that $\mathcal{C} \supset \mathcal{A}$, we show that \mathcal{C} is a σ -algebra.

(1) $\emptyset \subset \mathcal{A}_0 \subset \mathcal{C}$.

(2) Let $\mathcal{C}' = \{E \subset X \mid E^c \in \mathcal{C}\}$.

- \mathcal{C}' is a monotone class (easy)
- $\mathcal{A}_0 \subset \mathcal{C}'$ since $(E \in \mathcal{A}_0 \implies E^c \in \mathcal{A}_0 \subset \mathcal{C})$.

These two show that $\mathcal{C} \subset \mathcal{C}'$. So $E \in \mathcal{C} \implies E \in \mathcal{C}' \implies E^c \in \mathcal{C}$. So \mathcal{C} is closed under complements.

(3) For $E \subset X$, let $\mathcal{D}(E) = \{F \in \mathcal{C} \mid E \cup F \in \mathcal{C}\}$.

- $\mathcal{D}(E) \subset \mathcal{C}$ by definition.
- $\mathcal{D}(E)$ is a monotone class (easy). $E \cup (\bigcup_{i=1}^{\infty} F_n) = \bigcap_{i=1}^{\infty} (E \cup F_n)$.
- If $E \in \mathcal{A}_0$, then $\mathcal{A}_0 \subset \mathcal{D}(E)$. ($F \in \mathcal{A}_0 \implies E \cup F \in \mathcal{A}_0 \subset \mathcal{C}$.)

These show that $\mathcal{C} = \mathcal{D}(E)$ if $E \in \mathcal{A}_0$.

(4) Let $\mathcal{D} = \{E \in \mathcal{C} \mid \mathcal{D}(E) = \mathcal{C}\} = \{E \in \mathcal{C} \mid E \cup F \in \mathcal{C}, \forall F \in \mathcal{C}\}$.

- $\mathcal{A}_0 \subset \mathcal{D}$ by (3).
- \mathcal{D} is a monotone class (easy).
- $\mathcal{D} \subset \mathcal{C}$ by definition.

So we conclude that $\mathcal{D} = \mathcal{C}$. Now we have \mathcal{C} is closed under finite unions.

- (5) \mathcal{C} is closed under finite unions and countable increasing unions $\implies \mathcal{C}$ is closed under countable unions. (check) ■

RECALL $E \in \mathcal{A} \otimes \mathcal{B} \implies E_x \in \mathcal{B}, E^y \in \mathcal{A}, \forall x \in X, y \in Y$. However, the inverse is not necessarily true.

Now comes the main thing:

3.4 Fubini-Tonelli Theorem

Theorem 3.14 (Tonelli for characteristic functions). *Suppose $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ are σ -finite measure spaces. Suppose $E \in \mathcal{A} \otimes \mathcal{B}$. Then*

- (a) $\alpha(x) := \nu(E_x) : X \rightarrow [0, \infty]$ is a \mathcal{A} -measurable function.
- (b) $\beta(y) := \mu(E^y) : Y \rightarrow [0, \infty]$ is a \mathcal{B} -measurable function.
- (c) $(\mu \times \nu)(E) = \int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y)$.

Proof. (a) Assume μ, ν are finite measures. Let

$$\mathcal{C} = \{E \in \mathcal{A} \otimes \mathcal{B} \mid \text{(a), (b), (c) hold}\}.$$

Enough to prove that $\langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B} \subset \mathcal{C}$.

Because of monotone class lemma and that \mathcal{R} is a σ -algebra, it is enough to show that $\mathcal{R} \subset \mathcal{C}$ and \mathcal{C} is a monotone class.

- Show that $\mathcal{R} \subset \mathcal{C}$.

$$\alpha(x) = \nu((A \times B)_x) = \begin{cases} \nu(B) & x \in A \\ 0 & x \notin A \end{cases} = \nu(B)1_A(x).$$

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$$

$$\iff \int_X \nu((A \times B)_x) d\mu(x) = \nu(B)\mu(A)$$

- Show that \mathcal{C} is a monotone class.

(1) Let $E_n \in \mathcal{C}, E_1 \subset E_2 \subset \dots$. Need to show that $E = \bigcup_1^\infty E_n \in \mathcal{C}$.

$$\begin{aligned} E_n &\in \mathcal{C}, E_1 \subset E_2 \subset \dots \\ \implies E_x &= \bigcup_1^\infty (E_n)_x, (E_1)_x \subset (E_2)_x \subset \dots \\ \implies \alpha(x) &= \nu(E_x) = \lim_{n \rightarrow \infty} \nu((E_n)_x), \forall x \in X, \quad \alpha_n(x) \text{ } \mathcal{A}\text{-measurable} \end{aligned}$$

This satisfies (a), (b). For (c), we have

$$\begin{aligned} (\mu \times \nu)(E) &= \lim_{n \rightarrow \infty} (\mu \times \nu)(E_n) \\ &= \lim_{n \rightarrow \infty} \int_X \nu((E_n)_x) \, d\mu(x) \stackrel{MCT}{=} \int_X \nu(E_x) \, d\mu(X). \end{aligned}$$

So we have shown countable increasing unions.

(2) Let $F_n \in \mathcal{C}, F_1 \supset F_2 \supset \dots$. Need to show that $F = \bigcap_1^\infty F_n \in \mathcal{C}$. Using continuity of measure from above instead of below, DCT instead of MCT, we obtained a similar result.

(b) Now assume that μ, ν are σ -finite. Since $X \times Y = \bigcup_1^\infty (X_n \times Y_n)$, where $X_1 \subset X_2 \subset \dots, Y_1 \subset Y_2 \subset \dots$ with $\mu(X_k), \nu(Y_k)$ finite. Apply results from then finite case. (DIY) ■

Theorem 3.15 (Fubini-Tonelli). Suppose (X, \mathcal{A}, μ) and (Y, \mathcal{B}, ν) are σ -finite measure spaces.

(a) (Tonelli) If $f : X \times Y \rightarrow [0, \infty]$ is $\mathcal{A} \otimes \mathcal{B}$ -measurable then

- (1) $g(x) := \int_Y f(x, y) \, d\nu(y) : X \rightarrow [0, \infty]$ is a \mathcal{A} -measurable function.
- (2) $h(y) := \int_X f(x, y) \, d\mu(x) : Y \rightarrow [0, \infty]$ is a \mathcal{B} -measurable function.
- (3) We have the iterated integral formula

$$\begin{aligned} \int_{X \times Y} f \, d(\mu \times \nu) &= \int_X \left[\int_Y f(x, y) \, d\nu(y) \right] d\mu(x) \\ &= \int_Y \left[\int_X f(x, y) \, d\mu(x) \right] d\nu(y). \end{aligned}$$

(b) (Fubini) If $f \in L^1(X \times Y, \mu \times \nu)$, then

- (1) $f_x \in L^1(Y, \nu)$ for μ -a.e x and $g(x)$ (which is defined μ -a.e) $\in L^1(X, \mu)$.
- (2) $f_y \in L^1(X, \mu)$ for ν -a.e y and $h(y)$ (which is defined ν -a.e) $\in L^1(Y, \nu)$.

(3) The iterated integral formula from (a).(3) hold.

Usually we apply Tonelli to $|f|$ to show $f \in L^1(X \times Y, \mu \times \nu)$ and then apply Fubini to evaluate.

Proof. See [Fol99]. ■

3.5 Lebesgue Measure on \mathbb{R}^d

Example 3.16 ($(\mathbb{R}^2, \mathcal{L} \otimes \mathcal{L}, m \times m)$ is not complete). Let $A \in \mathcal{L}$, $A \neq \emptyset$, $m(A) = 0$. Let $B \subset [0, 1]$, $B \notin \mathcal{L}$ (e.g. Vitali set). Then let $E = A \times B$, $F = A \times [0, 1]$. We can see that $E \subset F$ and $F \in \mathcal{L} \otimes \mathcal{L}$, $(m \times m)(F) = m(A)m([0, 1]) = 0$.

So E is a subnull set but not $\mathcal{L} \otimes \mathcal{L}$ -measurable. (otherwise each section of E is measurable, a contradiction.)

Definition 3.17. Let $(\mathbb{R}^d, \mathcal{L}^d, m^d)$ be the completion of $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), m \times \dots \times m)$, which is same(check!) as the completion of $(\mathbb{R}^d, \mathcal{L} \otimes \dots \otimes \mathcal{L}, m \times \dots \times m)$.

So how do we compute m^d ?

A rectangle in \mathbb{R}^d is $R = \prod_{i=1}^d E_i$, $E_i \in \mathcal{B}(\mathbb{R})$. Then

$$m^d(E) = \inf \left\{ \sum_1^\infty m^d R_k \mid E \subset \bigcup_1^\infty R_k, R_k \text{ rectangle} \right\}.$$

Theorem 3.18. Let $E \in \mathcal{L}^d$.

- (a) $m^d(E) = \inf \{m^d(O) \mid \text{open } O \supset E\} = \sup \{m^d(K) \mid \text{compact } K \subset E\}$.
- (b) $E = \underbrace{A_1}_{F\sigma} \cup \underbrace{N_1}_{\text{null}} = \underbrace{A_2}_{G\sigma} \setminus \underbrace{N_2}_{\text{null}}$.
- (c) If $m^d(E) < \infty$, $\forall \varepsilon > 0$, $\exists R_1, \dots, R_m$ rectangles whose sides are intervals such that $m^d\left(E \triangle \left(\bigcup_1^m R_i\right)\right) < \varepsilon$.

Proof. Similar to $d = 1$ case. ■

Theorem 3.19. Integrable "step functions" and $C_c(\mathbb{R}^d)$ are dense in $L^1(\mathbb{R}^d, \mathcal{L}^d, m^d)$.

Theorem 3.20. Lebesgue measure in \mathbb{R}^d is translation-invariant.

Theorem 3.21. "Effect of linear transformations on Lebesgue measure"

Skip p. 71-81 of [Fol99] except 3.21.

Chapter 4

Differentiation on Euclidean Space

Suppose $f : [a, b] \rightarrow \mathbb{R}$. There are two versions of fundamental theorem of Calculus:

- $\int_a^b f'(x) \, dx = f(b) - f(a)$.
- $\frac{d}{dx} \int_a^x f(t) \, dt = f(x)$.

We focus on the second statement, which implies that

$$\lim_{r \rightarrow 0^+} \frac{1}{r} \int_x^{x+r} f(t) \, dt = \lim_{r \rightarrow 0^+} \frac{1}{r} \int_{x-r}^x f(t) \, dt$$

Write $f(x) = \frac{1}{r} \int_x^{x+r} f(x) \, dt$, then

$$\lim_{r \rightarrow 0^+} \frac{1}{r} \int_x^{x+r} (f(t) - f(x)) \, dt = \lim_{r \rightarrow 0^+} \frac{1}{r} \int_{x-r}^x (f(t) - f(x)) \, dt.$$

This generalizes well in \mathbb{R}^d :

$$f : \mathbb{R}^d \rightarrow \mathbb{R}, \quad \lim_{r \rightarrow 0^+} \frac{1}{v(B(x, r))} \int_{B(x, r)} f(t) - f(x) \, dt = 0.$$

QUESTION to what extent does this hold?

Start from [Fol99, 3.4].

4.1 Hardy-Littlewood Maximal Function

Suppose an open ball in \mathbb{R}^d , $B = B(a, r)$. Denote $cB = B(a, cr)$, $c > 0$.

Lemma 4.1 (Vitali-type covering lemma). *Let B_1, \dots, B_k be a finite collection of open balls in \mathbb{R}^d . Then \exists a sub-collection B'_1, \dots, B'_m of disjoint open balls such that*

$$\bigcup_{j=1}^m (3B'_j) \supset \bigcup_{i=1}^k B_i.$$

Proof. Greedy algorithm. ■

NOTATION: $\int_E f \, dm = \int_E f(x) \, dx$.

Definition 4.2. $f : \mathbb{R}^d \rightarrow \mathbb{C}$ is Lebesgue measurable. f is *locally integrable* if

$$\int_K |f| \, dm < \infty, \forall \text{ compact } K \subset \mathbb{R}^d.$$

We write $f \in L^1_{\text{loc}}(\mathbb{R}^d)$.

Example 4.3. $f(x) = x^2 \in L^1_{\text{loc}}(\mathbb{R}^d)$. (in fact all continuous functions $\in L^1_{\text{loc}}(\mathbb{R}^d)$).

Definition 4.4. For $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, define Hardy-Littlewood maximal function for f

$$Hf(x) = \sup\{A_r(x) \mid r > 0\}, \quad A_r(x) = \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y)| \, dy.$$

Lemma 4.5. *Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$. Then,*

- (a) $A_r(x)$ is jointly continuous for $(x, r) \in \mathbb{R}^d \times (0, \infty)$.
- (b) $Hf(x)$ is Borel measurable.

Proof.

- (a) $(x, r) \rightarrow (x^*, r^*) \implies A_r(x) \rightarrow A_{r^*}(x^*)$.

Let (x_n, r_n) be any sequence $\rightarrow (x^*, r^*)$.

$$A_{r_n}(x_n) \leq \int |f(y)| 1_{B(x_n, r_n)}(y).$$

Apply DCT.

- (b) $(Hf)^{-1}((a, \infty)) = \bigcup_{r>0} A_r^{-1}((a, \infty))$ is open.

■

RECALL Markov inequality

$$m(\{x \mid |f(x)| \geq c\}) \leq \frac{1}{c} \int |f(x)| \, dx$$

Theorem 4.6 (Hardy-Littlewood maximal inequality). $\exists C_d > 0$ s.t. $\forall f \in L^1_{\text{loc}}(\mathbb{R}^d), \forall \alpha > 0$,

$$m(\{x \mid Hf(x) > \alpha\}) \leq \frac{C_d}{\alpha} \int |f(x)| \, dx.$$

Proof. Fix $f \in L^1$ and $\alpha > 0$. Let $E = \{x \mid (Hf)(x) > \alpha\}$. E is a Borel measurable set. Then

$$x \in E \implies \exists r_x > 0, \text{ s.t. } A_{r_x}(x) > \alpha \implies m(B(x, r_x)) < \frac{1}{\alpha} \int_{B(x, r_x)} |f(y)| \, dy.$$

By inner regularity, we have $m(E) = \sup\{m(K) \mid \text{compact } K \subset E\}$. Let $K \subset E$ be compact. Then

$$\begin{aligned} K &\subset \bigcup_{x \in K} B(x, r_x) \\ \implies K &\subset \bigcup_{i=1}^N B_i \\ \implies K &\subset \bigcup_{j=1}^m (3B'_j), B'_1, \dots, B'_m \text{ disjoint} \\ \implies m(K) &\leq \sum_{j=1}^n m(3B'_j) = 3^d \sum_{j=1}^n m(B'_j) \\ \implies m(K) &\leq \frac{3^d}{\alpha} \sum_{j=1}^N \int_{B'_j} |f(y)| \, dy \\ \implies m(K) &\leq \frac{3^d}{\alpha} \int_{\mathbb{R}^d} |f(y)| \, dy. \end{aligned}$$

■

4.2 Lebesgue Differentiation Theorem

Theorem 4.7. Let $f \in L^1(\mathbb{R}^d)$. Then

$$\lim_{r \rightarrow 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| \, dy = 0 \text{ for a.e. } x.$$

Proof. (a) The result holds for $f \in C_c(\mathbb{R}^d)$ (check!)

(b) Let $f \in L^1(\mathbb{R}^d)$. Fix $\varepsilon > 0$. $\exists g \in C_c(\mathbb{R}^d)$ s.t. $\|f - g\|_1 < \varepsilon$. Then

$$\begin{aligned} & \int_{B(x,r)} |f(y) - f(x)| \, dy \\ & \leq \int_{B(x,r)} |f(y) - g(y)| \, dy + \int_{B(x,r)} |g(y) - g(x)| \, dy + \int_{B(x,r)} |g(x) - f(x)| \, dy. \end{aligned}$$

Let $Q(x) = \limsup_{r \rightarrow 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, dy$. WTS that $m(\{x \mid Q(x) > 0\}) = m(\bigcup_{n=1}^{\infty} \{x \mid Q(x) > \frac{1}{n}\}) = 0$.

Enough to show that $m(E_\alpha) = 0, \forall \alpha > 0, E_\alpha = \{x \mid Q(x) > \alpha\}$.

But $Q(x) \leq (H(f - g))(x) + 0 + |g(x) - f(x)|$,

$$\{x \mid Q(x) > \alpha\} \subset \{x \mid H(f - g)(x) > \frac{\alpha}{2}\} \cup \{x \mid |g(x) - f(x)| > \frac{\alpha}{2}\}.$$

$$m(\{x \mid Q(x) > \alpha\}) \leq \frac{2C_d}{\alpha} \|f - g\|_1 + \frac{2}{\alpha} \|f - g\|_1 \leq \frac{2(C_d+1)}{\alpha} \varepsilon. \quad \blacksquare$$

Corollary 4.8. This also holds for $f \in L^1_{\text{loc}}(\mathbb{R}^d)$.

Proof. DIY. ■

Corollary 4.9. For $f \in L^1_{\text{loc}}(\mathbb{R}^d)$,

$$\lim_{r \rightarrow 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y) \, dy = 0 \text{ for a.e } x.$$

Proof. DIY. ■

Definition 4.10. Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$. The point $x \in \mathbb{R}^d$ is called a *Lebesgue point* of f if

$$\lim_{r \rightarrow 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, dy = 0.$$

$f \in L^1_{\text{loc}}(\mathbb{R}^d) \implies$ a.e point is a Lebesgue point of f .

Definition 4.11. $\{E_r\}_{r>0}$ *shrinks nicely* to x as $r \rightarrow 0$ means $E_r \subset B(x,r)$ and $\exists c > 0$ s.t. $cm(B(x,n)) \leq m(E_r)$.

Corollary 4.12 (Lebesgue differentiation theorem).

$$\left. \begin{array}{l} E_r \text{ shrinks nicely to } 0 \\ f \in L^1_{\text{loc}}(\mathbb{R}^d) \\ x \text{ a Lebesgue point of } f \end{array} \right\} \implies \lim_{r \rightarrow 0} \frac{1}{m(E_r)} \int_{E_r+x} |f(y) - f(x)| \, dy = 0.$$

Proof. DIY. ■

Corollary 4.13. $f \in L^1_{\text{loc}}(\mathbb{R}^d) \implies F(x) = \int_0^x f(y) \, dy$ is differentiable and $F'(x) = f(x)$ a.e.

Rest of [Fol99, Ch.3] will be covered later.

Chapter 5

Normed Vector Spaces

Topological spaces \supset metric spaces \supset normed spaces \supset inner product spaces.

Let's start with metric spaces. [Fol99, 5.1, 6.1, 6.2]

5.1 Metric Spaces and Normed Spaces

Definition 5.1. Suppose Y is a set. A *metric* of Y is $\rho : Y \times Y \rightarrow [0, \infty)$ s.t.

- (a) $\rho(x, y) = \rho(y, x)$
- (b) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$
- (c) $\rho(x, y) = 0 \iff x = y.$

Example 5.2.

- (a) $\mathbb{Q}, \rho(x, y) = |x - y|.$
- (b) $\mathbb{R}, \rho(x, y) = |x - y|.$
- (c) $\mathbb{R}_+, \rho(x, y) = \left| \ln \left(\frac{y}{x} \right) \right|.$
- (d) $\mathbb{R}^d, \rho_1(x, y) = \sum_{i=1}^d |x_i - y_i|, \rho_p(x, y) = \left(\sum_{i=1}^d |x_i - y_i|^p \right)^{1/p}, \rho_\infty(x, y) = \max_{1 \leq i \leq d} |x_i - y_i|.$
- (e) $C([0, 1]), \rho_p(f, g) = \left(\int_0^1 |f - g|^p \right)^{1/p}, \rho_\infty = \max_{x \in [0, 1]} |f(x) - g(x)|.$

They are all metric spaces.

Definition 5.3 (Recall 2.32). Suppose V is a vector space over field \mathbb{R} or \mathbb{C} . A *seminorm*

on V is $\|\cdot\| : V \rightarrow [0, \infty)$ s.t.

- $\|cv\| = |c|\|v\|, \forall v \in V, \forall c \text{ scalar}$
- $\|v + w\| \leq \|v\| + \|w\|$, triangle inequality

A norm is a seminorm such that $\|v\| \iff v = 0$.

Norm gives rise to a metric where $\rho(v, w) = \|v - w\|$.

$v_n \rightarrow v \iff \lim_{n \rightarrow \infty} \|v_n - v\| = 0$.

Example 5.4. (a) $L^1(X, \mathcal{A}, \mu)$

(b) $C([0, 1]), \|f\|_1 = \int_0^1 |f(x)| dx, \|f\|_\infty = \max_{0 \leq x \leq 1} |f(x)|$.

(c) $\mathbb{R}^d, \|x\|_2 = \sqrt{\sum_1^d |x_i|^2}, \|x\|_1 = \sum_1^d |x_i|, \|x\|_\infty = \max_{1 \leq i \leq d} |x_i|$.

5.2 L^p Spaces

Definition 5.5. Suppose (X, \mathcal{A}, μ) a measure space. f is measurable function. For $0 < p < \infty$, define $\|f\|_p = \left(\int_X |f|^p d\mu \right)^{1/p}$. Define $L^p(X, \mathcal{A}, \mu) = \left\{ f \mid \|f\|_p < \infty \right\}$.

Example 5.6.

Definition 5.7. $\ell^p = \ell^p(N) = \{a = (a_1, a_2, \dots) \mid \|a\|_p = (\sum_1^\infty |a_i|^p)^{1/p} < \infty\}$.

Lemma 5.8. L^p is a vector space, $\forall p \in (0, \infty)$.

Proof.

$$\left(\int |cf|^p \right)^{1/p} = |c| \|f\|_p.$$

Given the following inequality

$$(\alpha + \beta)^p \leq (2 \max(|\alpha|, |\beta|))^p = 2^p \max(|\alpha|^p, |\beta|^p) \leq 2^p (|\alpha|^p + |\beta|^p)$$

we have

$$\int |f + g|^p \leq 2^p \left(\int (|f|^p + |g|^p) \right) \implies \|f + g\|_p \leq 2 \left(\int (|f|^p + |g|^p) \right)^{1/p}. \quad \blacksquare$$

But we want to know that whether

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

holds.

Theorem 5.9 (Hölder's Inequality). *Let $p < \infty, q = \frac{p}{p-1}$ so $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

Proof.

$$t \leq \frac{t^p}{p} + 1 - \frac{1}{p}, \forall t \geq 0$$

(Take $F(t) = t - \frac{t^p}{p}$)

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}, \forall \alpha, \beta \geq 0 \text{ (Young's inequality)} \quad (5.1)$$

WLOG assume $0 \leq \|f\|_p, \|g\|_q < \infty$. Let $F(x) = \frac{f(x)}{\|f\|_p}, G(x) = \frac{g(x)}{\|g\|_q}$.

$$\implies \|F\|_p = 1 = \|G\|_q.$$

By (5.1),

$$\begin{aligned} \int |F(x)G(x)| &\leq \int \frac{|F(x)|^p}{p} + \int \frac{|G(x)|^q}{q} \\ \frac{\int |f(x)g(x)|}{\|f\|_p \|g\|_q} &\leq \frac{1}{p} + \frac{1}{q} = 1. \end{aligned}$$

■

Theorem 5.10 (Minkowski's inequality). *Let $1 \leq p < \infty$. For $f, g \in L^p, \|f + g\|_p \leq \|f\|_p + \|g\|_p$.*

Proof. $p = 1$ is easy.

Assume $1 < p < \infty$. WLOG assume $\|f + g\|_p \neq 0$. We have

$$\begin{aligned} \int |f(x) + g(x)|^p &\leq \int |f(x) + g(x)|^{p-1} (|f(x)| + |g(x)|) \\ &\leq \left(\int (|f| + |g|^{p-1})^q \right)^{1/q} \left(\int |f|^p \right)^{1/p} + \left(\int (|f| + |g|^{p-1})^q \right)^{1/q} \left(\int |g|^p \right)^{1/p} \\ &\leq \left(\int (|f| + |g|^{p-1})^q \right)^{1/q} \left[\left(\int |f|^p \right)^{1/p} + \left(\int |g|^p \right)^{1/p} \right] \\ &\leq \left(\int (|f| + |g|^{p-1})^q \right)^{1/q} [\|f\|_p + \|g\|_p] \end{aligned}$$

Since $q(p-1) = p$, divide by $(\int (|f+g|^{p-1})^q)^{1/q}$ on both sides we have

$$\left(\int |f(x) + g(x)|^p \right)^{1-1/q} \leq \|f\|_p + \|g\|_p. \quad \blacksquare$$

Hölder: $\|fg\|_1 \leq \|f\|_p \|g\|_q, \frac{1}{p} + \frac{1}{q} = 1$.

Minkowski: $\|f+g\|_p \leq \|f\|_p + \|g\|_p, 1 \leq p < \infty$.

Definition 5.11. For a measurable function f on (X, \mathcal{A}, μ) , let

$$S = \{\alpha \geq 0 \mid \mu(\{x \mid |f(x)| > \alpha\}) = 0\} = \{\alpha \geq 0 \mid f(x) \leq \alpha \text{ a.e.}\}.$$

Define $\|f\|_\infty = \begin{cases} \inf S & S \neq \emptyset \\ \infty & S = \emptyset. \end{cases}$. Let $L^\infty(X, \mathcal{A}, \mu) = \{f \mid \|f\|_\infty < \infty\}$.

Example 5.12.

- $(\mathbb{R}, \mathcal{L}, m), f(x) = \frac{1}{x} 1_{(0, \infty)}(x) \notin L^\infty, f(x) = x 1_{\mathbb{Q}}(x) + \frac{1}{1+x^2} \in L^\infty$.
- If f is continuous on $(\mathbb{R}, \mathcal{L}, m), \|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|$. For $a \in \ell^\infty, \|a\|_\infty = \sup_{i \in \mathbb{N}} |a_i|$.
 $(\ell^\infty = \{a = (a_1, a_2, \dots) \mid \|a\|_\infty < \infty\} = \{a \mid \exists M \geq 0 \text{ s.t. } |a_i| \leq M_i, \forall i\})$

Lemma 5.13. (a) For $\alpha \geq \|f\|_\infty, \mu(\{x \mid |f(x)| > \alpha\}) = 0$. For $\alpha < \|f\|_\infty, \mu(\{x \mid |f(x)| > \alpha\}) > 0$.

(b) $|f(x)| \leq \|f\|_\infty$ a.e.

(c) $f \in L^\infty \iff \exists$ bounded measurable function g such that $f = g$ a.e.

Proof. DIY. ■

Theorem 5.14.

(a) $\|fg\|_1 \leq \|f\|_1 \|g\|_\infty$.

(b) $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

(c) $f_n \rightarrow f$ in $L^\infty \iff f_n \rightarrow f$ uniformly a.e.

Proof. DIY For (c): \implies Let $A_n = \{x \mid |f_n(x) - f(x)| > \|f_n - f\|_\infty\}$. Then $\mu(A_n) = 0$.

Let $A = \bigcup_1^\infty A_n, \mu(A_n) = 0, \forall x \in A^c = \bigcap_1^\infty A_n^c, \forall n, |f_n(x) - f(x)| \leq \|f_n - f\|_\infty$. The latter converges to 0 by assumption.

Given $\varepsilon > 0, \exists N$ s.t. $\|f_n - f\|_\infty < \varepsilon, \forall n \geq N$. So $\forall x \in A^c, \forall n \geq N, |f_n(x) - f(x)| \leq \|f_n - f\|_\infty < \varepsilon$. ■

Proposition 5.15.

- (a) For $1 \leq p < \infty$, the collection of simple functions with finite measure support is dense in $L^p(X, \mathcal{A}, \mu)$.
- (b) For $1 \leq p < \infty$, the collection of step functions (by definition they have finite measure support) is dense in $L^p(\mathbb{R}, \mathcal{L}, m)$. So is $C_c(\mathbb{R})$.
- (c) For $p = \infty$, the collection of simple functions is dense in $L^\infty(X, \mathcal{A}, \mu)$.

Proof. DIY ■

NOTE: $C_c(\mathbb{R})$ is not dense in $L^\infty(\mathbb{R}, \mathcal{L}, m)$.

5.3 Embedding Properties of L^p spaces

Definition 5.16. Two norms $\|\cdot\|, \|\cdot\|'$ on the same spaces V are said to be *equivalent* if

$$\exists c_1, c_2 > 0 \text{ s.t. } c_1 \|v\| \leq \|v\|' \leq c_2 \|v\|, \forall v \in V.$$

So on equivalent norms we have same open sets, same convergence.

Example 5.17.

- For $\mathbb{R}^d, \|\cdot\|_p, 1 \leq p \leq \infty$ are equivalent.
- For $1 \leq p, q \leq \infty, p \neq q$, $L^p(\mathbb{R}, m)$ -norm and $L^q(\mathbb{R}, m)$ -norm are not equivalent.
 $L^p(\mathbb{R}, m) \not\subseteq L^q(\mathbb{R}, m), L^p(\mathbb{R}, m) \not\supseteq L^q(\mathbb{R}, m)$.

Proposition 5.18. Suppose $\mu(X) < \infty$, then for any $0 < p < q \leq \infty, L^q \subseteq L^p$.

Proof. • $p = \infty$ is easy.

- Suppose $p < \infty$. ■

Proposition 5.19. If $0 < p < q \leq \infty$ then $\ell^p \subseteq \ell^q$.

Proposition 5.20. $\forall 0 < p < q < r \leq \infty, L^p \cap L^r \subset L^q$.

Proof. DIY ■

5.4 Banach Spaces

Theorem 5.21. Suppose $(V, \|\cdot\|)$ a normed space. Then it is complete \iff Every absolutely convergent series is convergent (i.e. if $\sum_1^\infty \|v_n\| < \infty$ then $\exists s \in V$ s.t. $\sum_1^N v_n \rightarrow s$ as $N \rightarrow \infty$)

Proof. \implies : DIY. (partial sums form a Cauchy Sequence)

\impliedby : Suppose $v_n, n \in \mathbb{N}$ is a Cauchy sequence. $\forall j \in \mathbb{N}, \exists N_j \in \mathbb{N}$ s.t. $\|v_n - v_m\| < \frac{1}{2^j}, \forall n, m \geq N_j$.

WLOG we may assume $N_1 < N_2 < \dots$. Let $w_1 = v_{N_1}, w_j = v_{N_j} - v_{N_{j-1}}, \forall j \geq 2 \implies \sum_1^\infty \|w_j\| \leq \|v_{N_1}\| + \sum_{j=2}^\infty \frac{1}{2^{j-1}} < \infty \implies \sum_1^k w_j \rightarrow \exists s \in V$.

Thus $v_{N_k} \rightarrow s$ as $k \rightarrow \infty$. v_n is Cauchy $\implies v_n \rightarrow s$ as $n \rightarrow \infty$. ■

5.5 Bounded Linear Transformation

Definition 5.22. Suppose $(V, \|\cdot\|), (W, \|\cdot\|')$ two normed spaces. A linear map $T : V \rightarrow W$ is said to be a *bounded map* if $\exists c \geq 0$ s.t. $\|Tv\|' \leq c\|v\|, \forall v \in V$.

Proposition 5.23. Suppose $T : (V, \|\cdot\|) \rightarrow (W, \|\cdot\|')$ is a linear map. Then the followings are equivalent:

- (a) T is continuous
- (b) T is continuous at 0
- (c) T is a bounded map

Proof. (a) \implies (b) is clear.

(b) \implies (c): For $\varepsilon = 1, \exists \delta > 0$ s.t. $\|Tu\|' < \varepsilon = 1$ if $\|u\| < \delta$. Suppose $v \in V, v \neq 0$. Let $u = \frac{\delta}{2\|v\|}v \implies \|u\| = \frac{\delta}{2} < \delta \implies \|Tu\|' < 1 \implies \frac{\delta}{2\|v\|} \|Tv\|' < 1 \implies \|Tu\|' < \frac{2}{\delta} \|v\|$.

(c) \implies (a): Fix $v_0 \in V$. $\|Tv - Tv_0\|' = \|T(v - v_0)\|' \leq C\|v - v_0\|$. ■

Example 5.24. (a) $T : \ell^1 \rightarrow \ell^1, Ta = (a_2, a_3, \dots), \|Ta\|_1 \leq \|a\|_1$. T is BLT.

(b) $T : (C([-1, 1]), \|\cdot\|_1) \rightarrow \mathbb{C}, Tf = f(0)$. This is not continuous.

(c) $T : (C([-1, 1]), \|\cdot\|_\infty) \rightarrow \mathbb{C}, Tf = f(0)$ is BLT.

(d) Let A be a $n \times m$ matrix. $T : \mathbb{R}^n \rightarrow \mathbb{R}^m, v \mapsto Av$ is BLT.

(e) Let $K(x, y)$ be a continuous function on $[0, 1] \times [0, 1]$.

$$T : (C[0, 1], \|\cdot\|_\infty) \rightarrow (C[0, 1], \|\cdot\|_\infty), Tf = \int_0^1 K(x, y)f(y) \, dy$$

is a BLT.

(f) $T : L^1(\mathbb{R}) \rightarrow (C(\mathbb{R}), \|\cdot\|_\infty), (Tf)(t) = \int_{-\infty}^{\infty} e^{-itx} f(x) \, dx$ (Fourier transform of f)

(g) $T : (C^\infty([0, 1]), \|\cdot\|_\infty) \rightarrow (C^\infty([0, 1]), \|\cdot\|_\infty), (Tf)(x) = f'(x)$ is not bounded.

Definition 5.25. Let $L(V, W) = \{T : V \rightarrow W \mid T \text{ is BLT}\}$. For $T \in L(V, W)$, the *operator norm* of T is

$$\begin{aligned} \|T\| &:= \inf\{c \geq 0 \mid \|Tv\|' \leq c\|v\|, \forall v \in V\} \\ &= \sup\left\{\frac{\|Tv\|'}{\|v\|} \mid v \neq 0, v \in V\right\} \\ &= \sup\{\|Tv\|' \mid \|v\| = 1\}. \end{aligned}$$

Lemma 5.26. (a) Above three definitions are equivalent.

(b) It is indeed a normed space.

Proof. DIY. ■

5.6 Dual of L^p Spaces

Chapter 6

Signed and Complex Measures

[Fol99, Ch.3].

RECALL Suppose (X, \mathcal{A}, μ) a measure space. $f : X \rightarrow [0, \infty]$ measurable. Let $\nu(E) = \int_E f \, d\mu, E \in \mathcal{A} \implies \nu$ is a measure on (X, \mathcal{A}) .

6.1 Signed Measures

Definition 6.1. Suppose (X, \mathcal{A}) a measurable space. A signed measure is $\nu : \mathcal{A} \rightarrow [-\infty, \infty]$ or $\nu : \mathcal{A} \rightarrow (-\infty, \infty]$ such that

- $\nu(\emptyset) = 0$.
- $A_1, A_2, \dots \in \mathcal{A}, A_i$ disjoint $\implies \nu\left(\bigcup_1^\infty A_i\right) = \sum_1^\infty \nu(A_i)$ where the series converges absolutely if $\nu\left(\bigcup_1^\infty A_i\right) \in (-\infty, \infty)$.

Example 6.2.

- ν positive measure $\implies \nu$ is a signed measure.
- μ_1, μ_2 positive measures such that either $\nu_1(X) < \infty$ or $\nu_2(X) < \infty \implies \nu = \mu_1 - \mu_2$ a signed measure.
- $f : X \rightarrow \bar{\mathbb{R}}$ s.t. $\int_X f^+ \, d\mu < \infty$ or $\int_X f^- \, d\mu < \infty \implies \nu(E) = \int_E f \, d\mu$.

NOTE:

- (a) $A \subset B \nRightarrow \nu(A) \leq \nu(B)$ since $\nu(B) = \nu(A) + \nu(B \setminus A)$.

(b) $A \subset B, \nu(A) = \infty \implies \nu(B) = \infty$.

Lemma 6.3. ν is a signed measure on (X, \mathcal{A}) . Then

- $E_n \in \mathcal{A}, E_1 \subset E_2 \subset \dots \implies \nu\left(\bigcup_1^\infty E_n\right) = \lim_{n \rightarrow \infty} \nu(E_n)$.
- $E_n \in \mathcal{A}, E_1 \supset E_2 \supset \dots, -\infty < \nu(E_1) < \infty \implies \nu\left(\bigcap_1^\infty E_n\right) = \lim_{n \rightarrow \infty} \nu(E_n)$.

Definition 6.4. ν is a signed measure on (X, \mathcal{A}) . Let $E \in \mathcal{A}$. We say

- (a) E is *positive* for ν (a positive set for ν) if $\forall F \subset E, F \in \mathcal{A}, \nu(F) \geq 0$.
- (b) E is *negative* for ν (a negative set for ν) if $\forall F \subset E, F \in \mathcal{A}, \nu(F) \leq 0$.
- (c) E is *null* for ν (a null set for ν) if $\forall F \subset E, F \in \mathcal{A}, \nu(F) = 0$.

NOTE E positive set, $F \subset E \implies \nu(F) \leq \nu(E)$. E negative set, $F \subset E \implies \nu(F) \geq \nu(E)$.

Definition 6.5. μ, ν signed measure on (X, \mathcal{A}) .

$\nu \perp \mu$ (singular to each other) means $\exists E, F \in \mathcal{A}$ s.t. $E \cap F = \emptyset, E \cup F = X, F$ is null for μ, E is null for ν .

Example 6.6. For $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$,

- (a) Lebesgue measure m
- (b) Cantor measure $\mu_C((a, b])$.
- (c) Discrete measure $\mu_D = \delta_1 + 2\delta_{-1}$.

For (a), (c), take $E = \mathbb{R} \setminus \{-1, 1\}, F = \{-1, 1\}$. For (a), (b), take the cantor set $K, E = \mathbb{R} \setminus K, F = K$.

Lemma 6.7. ν is a signed measure on (X, \mathcal{A}) .

- (a) E is positive (for ν) and $G \subset E$ measurable $\implies G$ is positive (for ν).
- (b) E_1, E_2, \dots positive sets $\implies \bigcup_1^\infty E_n$ is positive.

Proof. DIY. ■

Lemma 6.8. ν is a signed measure on (X, \mathcal{A}) . Suppose $E \in \mathcal{A}$ and $0 < \nu(E) < \infty \implies \exists$ measurable set $A \subset E$ s.t. A is a positive set (for ν) and $\nu(A) > 0$.

Proof in [RF10]. If E is a positive set, we are done.

Otherwise, E contains sets of negative measure. Let $n_1 \in \mathbb{N}$ be the smallest such that $\exists E_1 \subset E$ with $\nu(E_1) < -\frac{1}{n_1}$. If $E \setminus E_1$ is a positive set then we are done. Otherwise, $E \setminus E_1$ contain sets of measure.

Inductively if $E \setminus \bigcup_1^{k_1} E_i$ is not a positive set. Let $n_k \in \mathbb{N}$ be the *smallest* such that $\exists E_k \subset E \setminus \bigcup_1^{k_1} E_i$ with $\nu(E_k) < -\frac{1}{n_k}$.

Note: if $n_k \geq 2, \forall B \subset E \setminus \bigcup_1^{k-1} E_i, \nu(B) \geq -\frac{1}{n_{k-1}}$.

Let $A = E \setminus \bigcup_1^\infty E_k$. Since $E = A \cup \bigcup_1^\infty E_k, \nu(E) = \nu(A) + \sum_1^\infty \nu(E_k) \implies \nu(A) > 0$.

Since $\nu(E), \nu(A)$ are finite, then $\sum_1^\infty \frac{1}{n_k}$ need to be convergent $\implies \lim_{k \rightarrow \infty} n_k = \infty$.

Now, if $B \subset A$ then $B \subset E \setminus \bigcup_1^{k-1} E_i$. If $\nu(B) \geq -\frac{1}{n_{k-1}} \implies \nu(B) \geq 0$. Thus A is positive. ■

Theorem 6.9 (The Hahn decomposition theorem). *Suppose ν is a signed measure of (X, \mathcal{A}) . Then $\exists P, N \in \mathcal{A}$ s.t. $P \cap N = \emptyset, P \cup N = X$, P is positive for ν , and N is negative for ν . If P', N' are another such pair, then $P \triangle P' (= N \triangle N')$ is null for ν .*

Proof. Uniqueness: $P \setminus P' \subset P \cap N' \implies P \setminus P'$ is positive and negative, thus a null set. Same for $P' \setminus P$.

Existence: WLOG assume $\nu : \mathcal{A} \rightarrow [-\infty, \infty)$. Let $s = \sup\{\nu(E) \mid E \text{ positive for } \nu\}$. $\exists P_1, P_2, \dots$ positive sets such that $\lim_{n \rightarrow \infty} \nu(P_n) = s$.

Let $P = \bigcup_1^\infty P_n \implies P$ is positive $\implies \begin{cases} s \geq \nu(P) \\ \nu(P) \geq \nu(P_n) \end{cases} \implies \nu(P) = s$. Note that $0 \leq s = \nu(P) < \infty$.

Let $N = X \setminus P$. Is N a negative set?

Suppose not. Then $\exists E \subset N$ s.t. $\nu(E) > 0$. Note that $\nu(E) < \infty \implies \exists$ positive set $A \subset EA$ with $\nu(A) > 0$. The P, A are disjoint, $P \cup A$ is a positive set, and $\nu(P \cup A) = \nu(P) + \nu(A) > s$, a contradiction.

So N is a negative set. ■

Theorem 6.10 (Jordan decomposition theorem). *ν signed measure on (X, \mathcal{A}) . $\exists!$ positive measures ν^+, ν^- on (X, \mathcal{A}) s.t. $\nu(E) = \nu^+(E) - \nu^-(E), \forall E \in \mathcal{A}$ and $\nu^+ \perp \nu^-$.*

Proof. $\nu^+(E) = \nu(E \cap P), \nu^-(E) = -\nu(E \cap N)$. DIY. ■

Example 6.11. $(X, \mathcal{A}, \mu), f : X \rightarrow \bar{\mathbb{R}}$. Let $\nu(E) = \int_E f d\mu, \nu^+ = \int_E f^+ d\mu, \nu^- = \int_E f^- d\mu$.

Definition 6.12. Suppose ν a signed measure on (X, \mathcal{A}) . *Total variation measure* of ν is $|\nu| = \nu^+ \nu^-$ (a positive measure on (X, \mathcal{A})).

Definition 6.13. $|\nu|(E) = \int_E |f| d\nu$

Lemma 6.14. (a) $|\nu(E)| \leq |\nu|(E)$,

(b) E is a null set for $\nu \iff E$ is a null set for $|\nu|$,

(c) Suppose κ is another signed measure. $\kappa \perp \nu \iff \kappa \perp |\nu| \iff \kappa \perp \nu^+$ and $\kappa \perp \nu^-$.

Proof. DIY. ■

Definition 6.15. ν is finite (σ -finite) if $|\nu|$ is a finite (σ -finite) measure. ($\iff \nu^+, \nu^-$ are finite (σ -finite) measures.)

6.2 Absolutely Measurable Spaces

Definition 6.16. μ a positive measure, ν a signed measure on (X, \mathcal{A}) . $\nu \ll \mu$ (ν is absolutely continuous with respect to μ) $\iff (E \in \mathcal{A}, \mu(E) = 0 \implies \nu(E) = 0) \iff$ all μ -null sets and ν -null sets. (check)

Example 6.17. $(X, \mathcal{A}, \mu), f : X \rightarrow \bar{\mathbb{R}}. \nu(E) = \int E f d\mu \implies \nu \ll \mu$.

NOTATION: $d\nu = f d\mu$ means ν is the measure defined by $\nu(E) = \int_E f d\mu$.

Lemma 6.18. μ positive measure, ν signed measure.

(a) $\nu \ll \mu \iff |\nu| \ll \mu \iff \nu^+ \ll \mu$ and $\nu^- \ll \mu$.

(b) $\nu \ll \mu$ and $\nu \perp \mu \implies \nu = 0$.

Proof. ■

Theorem 6.19 (Radon-Nikodym). ν a σ -finite positive measure, ν a σ -finite signed measure on (X, \mathcal{A}) . Suppose $\nu \ll \mu$. Then $\exists f : X \rightarrow \bar{\mathbb{R}}$ measurable function such that $\nu(E) = \int_E f d\mu$. If g is another such function then $f = g$ a.e.

Proof. Will follow by proof of Lebesgue-Radon-Nikodym on Monday. ■

Definition 6.20. Suppose $\nu \ll \mu$. A *Radon-Nikodym derivative* of ν with respect to μ is a function $\frac{d\nu}{d\mu} : X \rightarrow \bar{\mathbb{R}}$ satisfying $\nu(E) = \int_E \frac{d\nu}{d\mu} d\mu, \forall E \in \mathcal{A}$.

NOTE: 6.19 shows the existence of such functions. If there is another such function g , then $\frac{d\nu}{d\mu} = g$ μ -a.e.

NOTATION:

$$d\nu = \frac{d\nu}{d\mu} d\mu.$$

Example 6.21. $F(x) = e^{2x} : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and increasing.

The Lebesgue-Stieltjes measure μ_F on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is the unique locally finite Borel measure satisfying $\mu((a, b]) = e^{2b} - e^{2a}, \forall a < b$.

$$\mu_F(E) \stackrel{\text{why?}}{=} \int_E 2e^{2x} dx.$$

So $\mu_F \ll m$ and $\frac{d\mu_F}{dm} = 2e^{2x}$.

Example 6.22. $F(x) = C(x) : \mathbb{R} \rightarrow \mathbb{R}$ the Cantor function. $C'(x) = 0$ Lebesgue a.e.

$$\mu_C(E) \neq \int_E 0 dx.$$

In particular, $c(b) - c(a) \neq \int_a^b c'(x) dx$ even if c is continuous and has derivative a.e. So $\mu_c \not\ll m$. But $\mu_c \perp m$.

6.3 Lebesgue Differentiation Theorem for Regular Borel Measures on \mathbb{R}^d

[Fol99, p. 99]

Definition 6.23. A Borel signed measure ν on \mathbb{R}^d is called *regular* if

- (a) $|\nu|(K) < \infty, \forall$ compact K .
- (b) $|\nu|(E) = \inf\{m(O) \mid \text{open } O \supset E\}, \forall$ Borel set E .

Example 6.24. LS measure on \mathbb{R} are regular. Lebesgue measure on \mathbb{R}^d is regular (so, the difference of two of them) Note: from (a), ν regular $\implies \nu$ is σ -finite,

If $d\nu = f dm$ regular, then $|\nu|(K) = \int_K |f| dm < \infty$, so $f \in L^1_{\text{loc}}(\mathbb{R}^d)$.

Lemma 6.25. If $f \in L^1_{\text{loc}}(\mathbb{R}^d) \iff d\nu = f dm$ is regular

Proof. Read the book. ■

RECALL Lebesgue differentiation theorem

Corollary 6.26. Let ρ be a regular signed Borel measure on \mathbb{R}^d . Suppose $\rho \ll m \implies$ For Lebesgue a.e.- x , $\lim_{r \rightarrow 0} \frac{\rho(E_r)}{m(E_r)} = \frac{d\rho}{dm}(x)$ for every $E_r \rightarrow x$ nicely.

Proposition 6.27. Let λ be a regular positive Borel measure on \mathbb{R}^d . Suppose $\lambda \perp m$. For Lebesgue a.e.- x , $\lim_{r \rightarrow 0} \frac{\lambda(E_1)}{m(E_1)} = 0$ for every $E_r \rightarrow x$ nicely.

Proof. Enough to consider $E_1 = B(x, r)$

$$\left\{ x \mid \limsup_{r \rightarrow 0} \frac{\lambda(E_1)}{m(E_1)} \neq 0 \right\} = \bigcup_{n=1}^{\infty} G_n, G_n = \left\{ x \mid \limsup_{r \rightarrow 0} \frac{\lambda(E_1)}{m(E_1)} > \frac{1}{n} \right\}$$

Enough to show that $m(G_n) = 0, \forall n$.

$\lambda \perp m \implies \mathbb{R}^d = A \cup B$ disjoint. $\lambda(A) = 0, m(B) = 0$, Enough to show $m(G_n \cap A) = 0$.

Fix $\varepsilon > 0$. Since λ is regular, \exists open $O \supset A$ s.t. $\lambda(O) \leq \lambda(A) + \varepsilon = \varepsilon$. $\forall x \in G_n \cap A, \exists r_x > 0$ s.t. $\frac{\lambda(B(x, r_x))}{m(B(x, r_x))} > \frac{1}{n}$ and $B(x, r_x) \subset O$.

Let $K \subset G_n \cap A$, compact. $K \subset \bigcup_{x \in K} B(x, r_x) \implies \exists$ finite subcover $\implies \exists B_1, B_2, \dots, B_N$ disjoint, $K \subset \bigcup_1^N 3B_i$.

$$\implies m(K) \leq 3^d \sum_1^N m(B_i) \leq 3^d n \sum_1^N \lambda(B_i) = 3^d n \lambda\left(\bigcap_1^N B_i\right) \leq 3^d n \lambda(O) \leq 3^d n \varepsilon \implies m(G_n \cap A) \leq 3^d n \varepsilon. \blacksquare$$

Theorem 6.28 (LDT for regular Borel measures). Suppose ν is a regular Borel signed measure on \mathbb{R}^d and $d\nu = d\lambda + f dm, \lambda \perp m \implies$ for Leb a.e. $x, \lim_{r \rightarrow 0} \frac{\nu(E_r)}{m(E_r)} = f(x)$ for every $E_r \rightarrow x$ nicely.

Proof. ν regular $\implies \lambda, f dm$ are regular. \blacksquare

6.4 Monotone Differentiation Theorem

[Fol99, 3.5]

Definition 6.29. For $F : \mathbb{R} \rightarrow \mathbb{R}$ that is increasing, denote $F(x+) = \lim_{y \downarrow x} F(y) = \inf_{y > x} F(y), F(x-) = \lim_{y \uparrow x} F(y) = \sup_{y < x} F(y)$.

Lemma 6.30. F is increasing $\implies D = \{x \mid F \text{ is discontinuous at } x\}$ is countable.

Proof. $x \in D \implies F(x+) > F(x-)$ since $F \nearrow$. For $x, y \in D, x \neq y \implies I_x, I_y$ disjoint. For each $x \in D$, let $I_x = (F(x-), F(x+)) \implies \exists f : D \rightarrow \mathbb{Q}$ is 1-1. I_x is open interval, not empty $\implies D$ is countable. \blacksquare

Theorem 6.31 (Monotone differentiation theorem). Suppose $F \nearrow \implies$

- F is differentiable Lebesgue a.e.
- $G(x) = F(x+)$ is differentiable Lebesgue a.e.

- $G' = F'$ a.e.

Proof. G is increasing, right-continuous on $\mathbb{R} \implies \exists$ Lebesgue-Stieltjes measure μ_G on \mathbb{R} (so, regular).

$$\frac{G(x+h) - G(x)}{h} = \begin{cases} \frac{\mu_G((x, x+h])}{m((x, x+h])} & h > 0, \\ \frac{\mu_G((x+h, x])}{m((x+h, x])} & h < 0 \end{cases}$$

converges for Lebesgue a.e x by LDT. So G' exists a.e.

Let $H(x) = G(x) - F(x) \geq 0$. We have

$$\{x \mid H(x) > 0\} \subset \{x \mid x \text{ is discontinuous at } x\}.$$

So $\{x \mid H(x) > 0\}$ it is countable. Denote the set as $\{x_n\}$.

Let $\mu = \sum_n H(x_n)\delta_{x_n}$. Then

$$\mu((-N, N)) = \sum_{x_n \in (-N, N)} H(x_n) \stackrel{check}{\leq} G(N) - F(-N) < \infty.$$

So μ is a locally finite Borel measure on $\mathbb{R} \implies \mu$ is regular. Hence

$$\left| \frac{H(x+h) - H(x)}{h} \right| \leq \frac{H(x+h) + H(x)}{|h|} \leq 4 \frac{\mu((x-2h, x+2h))}{4|h|} \xrightarrow{\text{LDT}, \mu \perp m} 0$$

for Lebesgue a.e. x .

So H is differentiable a.e and $H' = 0$ a.e. ■

Proposition 6.32. $F \nearrow \implies \int_a^b F'(x) dx \leq F(b) - F(a)$.

Example 6.33.

- $F(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$. $F'(x) = 0$ a.e and $\int_{-1}^1 F'(x) dx = 0 < F(1) - F(-1) = 1$.
- $F(x)$ Cantor function. $F'(x) = 0$ a.e. and $\int_0^1 F'(x) dx = 0 \leq F(1) - F(0) = 1$.

Lemma 6.34. $F \in \text{BV}$ and right-continuous $\implies T_F \in \text{NBV}$.

Proof. $T_F \nearrow$, bounded $\implies T_F \in \text{BV}$, $T_F(-\infty) = 0$. Is T_F right-continuous?

Suppose it is not. $\exists a \in \mathbb{R}$ s.t. $c := T_F(a+) - T_F(a) > 0$. Fix $\varepsilon > 0$. Since $F(x)$ and

$g(x) := T_F(x+)$ are right continuous, $\exists \delta > 0$ s.t.

$$|F(y) - F(a)| < \varepsilon, \quad |g(y) - g(a)| < \varepsilon \quad \forall y \in (a, a + \delta].$$

So $T_F(y) - T_F(a+) \leq T_F(y+) - T_F(a+) < \varepsilon$.

$\exists a = x_0 < x_1 < x_2 < \dots < x_n = a + \delta$ s.t.

$$\begin{aligned} \sum_{i=1}^n |F(x_i) - F(x_{i-1})| &\geq T_F(a + \delta) - T_F(a) - \frac{c}{4} \\ &\geq T_F(a+) - T_F(a) - \frac{c}{4} = \frac{3}{4}c. \end{aligned}$$

This shows that $\sum_{i=2}^n |F(x_i) - F(x_{i-1})| \geq \frac{3}{4}c - \varepsilon$ since

Consider $[a, x_1]$. $\exists a = t_0 < t_1 < \dots < t_k = x_1$ s.t.

$$\sum_{i=1}^k |F(t_i) - F(t_{i-1})| \geq T_F(x_1) - T_F(a) - \frac{c}{4} \geq \frac{3}{4}c.$$

So we can write $[a, a + \delta] = [a, x_1] \cup [x_1, a + \delta]$. So

$$\begin{aligned} \varepsilon + c &\geq T_F(a + \delta) - T_F(a+) + T_F(a+) - T_F(a) \\ &= T_F(a + \delta) - T_F(a) \\ &\geq \sum_{j=1}^k |F(t_j) - F(t_{j-1})| + \sum_{i=2}^n |F(x_i) - F(x_{i-1})| \geq \frac{3}{4}c - \varepsilon + \frac{3}{4}c = \frac{3}{2}c - \varepsilon \\ \implies c &\leq 4\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $c = 0$, a contradiction. ■

Corollary 6.35. $F \in \text{NBV} \iff F = F_1 - F_2, F_1, F_2 \in \text{NBV} \text{ and } \nearrow$.

Proof. $F =$. ■

Theorem 6.36.

(a) μ is a finite signed Borel measure on $\mathbb{R} \implies F(x) := \mu((-\infty, x]) \in \text{NBV}$.

(b) $F \in \text{NBV} \implies \exists!$ finite signed Borel measure μ_F on \mathbb{R} satisfying $\mu((-\infty, x]) = F(x)$.

Proof. (a) $\mu = \mu^+ - \mu^- \implies F = F^+ - F^-, F^\pm(x) = \mu^\pm((-\infty, x])$ is increasing, bounded, right-continuous, and $F^\pm(-\infty) = 0$.

- (b) $F \in \text{NBV} \implies F = F_1 - F_2, F_1, F_2 \in \text{NBV}$ and are increasing. So $\exists \mu_{F_1}, \mu_{F_2}$ Lebesgue-Stieltjes measure. $\mu_F := \mu_{F_1} - \mu_{F_2}$. Uniqueness is left for homework. ■

Proposition 6.37. *Let $F \in \text{NBV}$. Then*

- (a) F is differentiable a.e, $F \in L^1(\mathbb{R}, m)$.
- (b) $d\mu_F = d\lambda + F'dm, \lambda \perp m$.
- (c) $\mu_F \perp m \iff F' = 0$ Lebesgue a.e.
- (d) $\mu_F \ll m \iff \int_{-\infty}^x F'(t) dt = F(x)$.

Proof. Check (a), (b), (c).

- (d) $\mu_F \ll m \iff \lambda = 0 \iff d\mu_F = F'dm \iff \mu_F = \int_E F' dm, \forall E \text{ Borel}$
 $\iff F(x) = \int_{-\infty}^x F'(t) dt, \forall x \in \mathbb{R}$. (by uniqueness) ■

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