Notes for Math 566 – Algebraic Combinatorics

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Contents

Office hours: Tu, Fr 1:00 - 2:20 pm

Chapter 1

Graph and Trees

1.1 Linear Algebra Preliminaries

Let M be a $p \times p$ matrix with entries in $\mathbb{C}$. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$ are defined by

$$\det(t\,\mathrm{id} - M) = \prod_{i=1}^{p}(t - \lambda_i).$$

Taking coefficients of t^{p-1} on both sides we obtain

$$\operatorname{tr} M = \sum_k \lambda_k. \tag{1.1.1}$$

Lemma 1.1.1. *Let $f(t) \in \mathbb{C}[t]$. Then $f(M)$ have eigenvalues $f(\lambda_1), \ldots, f(\lambda_p)$.*

Proof. If M is diagonalizable, then the statement is clear: $f(M)$ has the same eigenvectors as M, with eigenvalues $f(\lambda_k)$. Then use a continuity argument. (Diagonalizable matrices are dense.) Alternative proof: use Jordan's normal form. ■

Combining (**??**) with the lemma, we have

$$\operatorname{tr} M^\ell = \sum_k \lambda_k^\ell. \tag{1.1.2}$$

PROBLEM: [A solution is given in Stanley's textbook.] Let $\alpha_1, \ldots, \alpha_r$ and $\beta_1, \ldots, \beta_r$ be nonzero complex numbers such that for *all* positive integer ℓ we have

$$\alpha_1^\ell + \ldots + \alpha_r^\ell = \beta_1^\ell + \ldots + \beta_r^\ell.$$

Show that this implies that $r = s$, and that α's are a permutation of β's.

In the majority of forthcoming applications, M is symmetric and real. Then it is diagonalizable, with real eigenvalues $\lambda_1, \ldots, \lambda_p$.

1.2 Counting Walk

Let G be a graph on the vertex set $\{1, \ldots, p\}$. (We allow loops and multiple edges.) Let $M = A(G)$ be its adjacency matrix.

OBSERVATION The number of walks of length ℓ from i to j is equal to $(M^\ell)_{ij}$.

In general, counting walks requires knowing the matrix M (equivalently, knowing both the eigenvalues λ_k and the corresponding eigenvectors). On the other hand, some enumerative information can be extracted from the eigenvalues alone:

Proposition 1.2.1. *The number of marked closed walks of length ℓ is equal to $\sum_{k=1}^p \lambda_k^\ell$.*

Here "marked" means that the starting location is fixed, as is a particular instance of passing through it, in case we do it several times.

Proof. By the last observation, the number of marked closed walks of length ℓ is equal to $\operatorname{tr} M^\ell$, which equals to $\sum_{k=1}^p \lambda_k^\ell$ by (**??**). ■

Example 1.2.1. Let $G = K_p$, the complete graph on p vertices. Let J denote the $p \times p$ matrix all of whose entries are 1. Let I denote the $p \times p$ identity matrix. Then $A(G) = J - I$. Obviously $\operatorname{rk} J = 1$ and $\operatorname{tr} J = p$. Hence the eigenvalues of J are $0, \ldots, 0, p$, and the eigenvalues of $A(G) = J - I$ are $-1, \ldots, -1, p - 1$.

Corollary 1.2.1. *There are $(p-1)^\ell + (-1)^\ell (p-1)$ marked closed walks of length ℓ in K_p.*

NOTE This is the number of $(\ell+1)$-letter words in a p-letter alphabet in which no two consecutive letters are identical, and which begin and end by the same letter.

PROBLEM Show that the number of walks of length ℓ between two distinct vertices in K_p differs by 1 from the number of closed walks of length ℓ starting at a given vertex.

1.3 Eigenvalues of Adjacency Matrices

RECALL

$$\# \text{ of marked closed walks of length } \ell = \sum_{i=1}^{p} \lambda_i^\ell.$$

It can be used backwards: using counted walks to compute eigenvalues.

Example 1.3.1. Let $G = K_{n,m}$ a complete bipartite graph.

$$\# \text{ of marked closed works of length } \ell = \begin{cases} 0 & \ell = 2k+1 \\ 2n^{\ell/2}m^{\ell/2} & \ell = 2k \end{cases} = (\sqrt{nm})^\ell + (-\sqrt{nm})^\ell$$

$\xRightarrow{\text{Problem}}$ eigenvalues are $\sqrt{nm}, -\sqrt{nm}, 0, \ldots, 0$.

PROBLEM Prove that, for G connected, the $\mathrm{diam}(G) < \#$ of distinct eigenvalues.

Example 1.3.2. $K_p = 1 < 2, K_{n,m} = 2 < 3$.

1.4 Inequalities for the Maximal Eigenvalue

Definition 1.4.1. Suppose G a graph with vertices $= \{1, \ldots, p\}$. Let

$$\lambda_{\max} := \max_i |\lambda_i| = \max \lambda_i.$$

Proposition 1.4.1.

$$\lambda_{\max} \leq \max \deg(G)$$

Proof. For any vector $X = (x_k) \in \mathbb{C}^p$,

$$\max_j |(A(G)X)_j| \leq \max \deg(G) \cdot \max_k |X_k|$$

Now suppose X is an eigenvector of $A(G)$ with eigenvalue λ. Then

$$\max_j |(A(G)X)_j| = |\lambda| \max_k |X_k| \leq \max \deg(G) \cdot \max_k |X_k| \implies |\lambda| \leq \max \deg(G)$$

This holds for all eigenvalue λ_i, which proves our proposition. ∎

ALTERNATE PROOF: by counting closed walks ($\leq \sum \max \deg(G)^\ell$.)

PROBLEM Prove that $\lambda_{\max} \geq$ average degrees of the vertices of G.
HINT for symmetric real matrix M we have $\lambda_{\max} = \max_{|x|=1} x^T M x$.

Corollary 1.4.1. *# of closed walk of length ℓ grows exponentially in ℓ with a rate $\geq$ average degree.*

1.5 Eigenvalue of Block Anti-diagonal Matrices

$$M = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \in \mathbb{R}_{n+m}$$

Lemma 1.5.1. *The non-zero eigenvalues (called "singular values" of B) of M are $\pm\sqrt{\mu_i}$ where μ_i are nonzero eigenvalues of B^TB with multiplicities.*

Note that B^TB is positive definite.

Proof. Let $F_X(t) = \det(t\,\mathrm{id}_p - X)$.

$$\begin{bmatrix} t\,\mathrm{id}_n & -B \\ -B^T & t\,\mathrm{id}_m \end{bmatrix} \begin{bmatrix} \mathrm{id}_n & B \\ 0 & t\,\mathrm{id}_m \end{bmatrix} = \begin{bmatrix} t\,\mathrm{id}_n & 0 \\ -B & -B^TB + t^2\,\mathrm{id}_m \end{bmatrix}$$

$$F_M(t) \cdot t^m = t^n F_{B^TB}(t^2)$$

and the claim follows ■

So now we are equipped to compute the eigenvalue of bipartite graphs.

Example 1.5.1. Suppose $G = K_{n,m}$, B^TB is $m \times m$ matrix with all entries being $n = nJ_m$. So the eigenvalues of $B^TB = nm, 0, 0, \ldots$. So eigenvalues of $A(G)$ is $\sqrt{mn}, -\sqrt{mn}, 0, 0, \ldots$

<u>PROBLEM</u> Let G to be the graph obtained by removing n disjoint edges from $K_{n,n}$. Find the eigenvalue of G.

Example 1.5.2. Let G be a $2n$-cycle. $M_{2n} = A(G) = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$. The $B^TB = 2I_n + M_n$ for an appropriate labeling.

So if the eigenvalue of n-cycle are $\lambda_1, \ldots, \lambda_n$. Then the eigenvalues of $2n$-cycles are $\pm\sqrt{\lambda_i + 2}$.

1.6 Eigenvalues of Circulant Matrices

Definition 1.6.1. A circulant matrix is of the form

$$M = \begin{bmatrix} s_0 & s_1 & s_2 & \dots & s_{p-1} \\ s_{p-1} & s_0 & s_1 & \dots & s_{p-2} \\ \vdots & & & & \\ s_1 & s_2 & s_3 & \dots & s_0 \end{bmatrix}.$$

Lemma 1.6.1. *M has eigenvalues*

$$\lambda_k = \sum_{j=0}^{p-1} s_j e^{\frac{2\pi i}{p} jk}, k = 0, 1, \dots, p-1.$$

Notice that

$$\lambda_k = \sum_{j=0}^{p-1} s_j e^{\frac{2\pi i}{p} jk} = s\left(e^{\frac{2\pi i}{p} k}\right) \quad p\text{-th root of unity.}$$

where

$$s(x) = \sum_{j=0}^{p-1} s_j x^j.$$

Proof. Let

$$T = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}$$

We have that the eigenvalues of T and p-th roots of unity and characteristic polynomial is $t^p - 1$.

Key observation: $M = s(T)$. ∎

Definition 1.6.2. A graph G is circulant if $A(G)$ is circulant, for some choice of vertex labeling.

Corollary 1.6.1. *The eigenvalue of p-cycle are*

$$2\cos\left(\frac{2\pi k}{p}\right), k = 0, 1, \dots, p-1.$$

Proof. By **??**, we have that

$$\lambda_k = e^{\frac{2\pi i}{p}k} + e^{\frac{2\pi i}{p}(p-1)k} = e^{\frac{2\pi ik}{p}} + e^{-\frac{2\pi ik}{p}} = 2\cos\left(\frac{2\pi k}{p}\right).$$ ∎

Remark. This formula is consistent with the formula linking the eigenvalues of a $2n$-cycle and an n-cycle: if $2\cos\alpha = \lambda$, then $2\cos\frac{\alpha}{2} = \pm\sqrt{2+\lambda}$.

PROBLEM Find the eigenvalues of the graph obtains by removing n disjoint edges from K_{2n}.

1.7 Eigenvalues of Cartesian Products

Definition 1.7.1. Suppose G, H are graphs with no loops. Define graph $G \times H$ where

$$V(G \times H) = \{(g,h) : g \in V(G), h \in V(H)\},$$

and we have two kinds of edges:

- $(g,h) - (g',h)$ for $g - g'$
- $(g,h) - (g,h')$ for $h - h'$

Example 1.7.1. 1. Grid graph = path $\times$ path

2. Discrete annulus (cylinder) = cycle $\times$ path
3. Discrete torus = cycle $\times$ cycle
4. n-cube graph

Proposition 1.7.1. *If G has eigenvalues $\lambda_1, \lambda_2, ldots$, H has eigenvalues $\mu_1, \mu_2, \ldots$ Then $G \times H$ has eigenvalues $\lambda_i + \mu_j$ for any pair i, j.*

Proof 1. (Tensor product) V_G, V_H are vector spaces formally spanned by vertices of G, H. Take $u = \sum \alpha_g g \in V_G, v = \sum \beta_h h \in V_H$. We have

$$u \otimes v = \sum_{g,h} \alpha_g \beta_h (g,h) \in V_{G\times H}.$$

The

$$A(G \times H)(u \otimes v) = (A(G)u) \otimes v + u \otimes (A(h)v)$$

Suppose u, v are eigenvectors $i.e.$ $A(G)u = \lambda u, A(H)v = \mu v$. Then we get

$$A(G \times H)(u \otimes v) = \lambda u \otimes v + u \otimes \mu v = (\lambda + \mu)(u \otimes v). \qquad \blacksquare$$

Proof 2. (Marked closed walk) Walk in $G \times H \xleftrightarrow{1-1}$ a <u>shuffle</u> of marked closed walks in $G \& H$.

$$\begin{aligned}
&\# \text{ of closed walks of length } \ell \text{ in } G \times H \\
=&\sum_k \binom{\ell}{k} \sum_i \lambda_i^k \sum_j \mu_j^{\ell-k} \\
=&\sum_i \sum_j \sum_k \binom{\ell}{k} \lambda_i^k \mu_j^{\ell-k} \\
=&\sum_{i,j} (\lambda_i + \mu_j)^\ell
\end{aligned}$$

This set of numbers are unique by problem in lecture 1, so they must be the eigenvalues of $G \times H$. $\blacksquare$

<u>PROBLEM</u> Take a 3×3 grid, find the number of marked closed walks of length ℓ.
<u>PROBLEM</u> Direct problem of 8-cycle and K_2.

<u>n-CUBE GRAPH:</u>

$$(K_2)^n = \underbrace{K_2 \times K_2 \times \cdots K_2}_{n \text{ times}}.$$

Example 1.7.2. When $n = 3$, we have a 3-D cube:

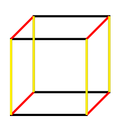


Figure 1.1: Cube graph $K_2 \times K_2 \times K_2$

K_2 has adjacency matrix $A(K_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ with eigenvalues $\pm 1 \implies$ eigenvalues of $(K_2)^n$ are

$$\lambda = \underbrace{\pm 1 \pm 1 \pm \ldots \pm 1}_{n \text{ times}}.$$

Proposition 1.7.2. *The eigenvalues of $(K_2)^n$ are of the form $n - 2k$ where $k = 0, 1, \ldots, n$, each* *with multiplicities* $\binom{n}{k}$ $i.e.$ *the number of marked closed walks of length* ℓ *in the* $n-$*cube graph is*

$$\sum_{k=0}^{n} \binom{n}{k} (n-2k)^{\ell}$$

which is 0 *when* ℓ *is odd.*

1.8 Random Walks

Let G be a regular graph of degree d on p vertices.

Example 1.8.1. $G = (K_2)^n$ is regular with $d = n$.

A simple random walk on G originating at a vertex v is a random walk with equal probabilities for each adjacent vertices.

$$\begin{aligned}\mathbb{P}\left(\text{walk is back at } v \text{ after } \ell \text{ steps}\right) &= \frac{1}{d^{\ell}} \#\{\text{marked closed walks of length } \ell \text{ orginiating from } v\} \\ &= d^{-\ell} p^{-1} \sum_{1}^{p} \lambda_i^{\ell}.\end{aligned}$$

assuming that $\mathrm{Aut}(G)$ acts transitively on vertices.

Notice that an arbitrary regular G does not necessarily have that condition, but the converse is true.

Example 1.8.2. The probability that a simple random walk on $(K_2)^n$ returns to its origin after ℓ steps is

$$\frac{1}{n^{\ell} 2^n} \sum_{k=0}^{n} \binom{n}{k} (n-2k)^{\ell}$$

Chapter 2

Tilings, Spanning Trees, and Electric Networks

2.1 Domino Tilings ("Dimers")

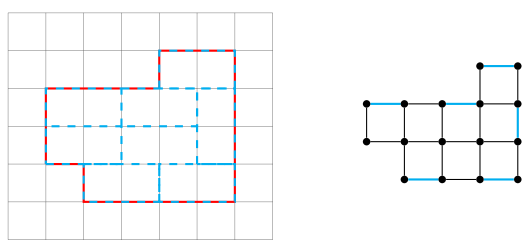
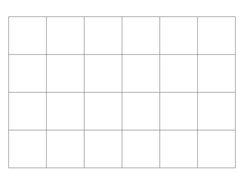


Figure 2.1: An example of domino tiling and perfect matching in its dual graph

A domino tiles decompose part of grids into 1×2 rectangles.

Think of it another way: the "dual graph" where squares are vertices, and there exists an edge between two vertices iff the corresponding squares shares an edge. A tiling is a perfect matching between these vertices.

Special case: $m \times n$ rectangular boards



Without loss of generality, assume that n is even. We denote the answer as $T(m,n)$

The dual graph G is m-chain $\times$ n-chain. Notice that G is bipartite.

$M = A(G)$ has the form $\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$ given appropriate labeling of vertices where B is a square matrix.

<u>CLAIM</u> $T(m,n) =$ the permanent of matrix B.

Permanents do not have nice properties, thus they are hard to calculate. In order to better calculate the permanent of B, let $\tilde{B}$ obtained from B by replacing the 1's by corresponding to <u>vertical</u> tiles by i's where $i^2 = -1$.

Proposition 2.1.1. $T(m,n) = \mathrm{per}(B) = \pm\det(\tilde{B})$.

Lemma 2.1.1 (exercise). *Any two domino tilings of a rectangular board are related to each other via "flips" of the form (two horizontal $\leftrightarrow$ two vertical)*

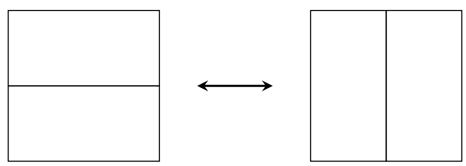


Figure 2.2: Example of a flip

Proof of Prop. This is equivalent to all nonzero terms terms in $\det(\tilde{B})$ are equal and are ± 1. The latter claim follows from the former, since since the all-horizontal tiling contributes ± 1.

Then it is enough to show that the contributions of two tilings that differ by a flip are equal to each other.

It means swapping two diagonal entries, thus change the sign of permutation, but one of them is 1^2 while the other being i^2, so the result does not change. ■

Now we can use some linear algebra to calculate the determinant. Denote $\tilde{M} = \begin{bmatrix} 0 & \tilde{B} \\ \tilde{B}^T & 0 \end{bmatrix}$.
Then $\det(\tilde{M}) = \pm(\det(\tilde{B}))^2 = \pm(T(m,n))^2$.

<u>OBSERVATION</u> We have

$$M = \mathrm{id}_m \otimes A_n + A_m \otimes \mathrm{id}_n,$$

where A_n, A_m are adjacency matrices of chain graphs. Similarly,

$$\tilde{M} = \mathrm{id}_m \otimes A_n + iA_m \otimes \mathrm{id}_n,$$

since $\tilde{M}$ obtained by vertical tile with i's. Hence the eigenvalues of $\tilde{M}$ are $\lambda_i + i\mu_k$.

Now we only need to find the eigenvalues of chain graph. For a n-chain, we have

$$A_n = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

Proposition 2.1.2. *The eigenvalues of A_n are*

$$\lambda_k = 2\cos\left(\frac{k\pi}{n+1}\right) \quad \textit{for } k = 1, \dots, n.$$

Proof. An eigenvector $u = (u_1, \dots, u_n)^T$ of A_n associated with eigenvalue λ satisfies

$$u_{j-1} + u_{j+1} = \lambda u_j, \quad 1 \leq i \leq n$$

with the convention that $u_0 = u_{n+1} = 0$.

A divine revelation: recall that

$$sin\alpha + \sin\beta = 2\cos\frac{\beta-\alpha}{2}\sin\frac{\alpha+\beta}{2}.$$

This suggest taking

$$u_j = \sin\left(\frac{\pi k j}{n+1}\right) \quad \text{for } j = 1, \dots, n.$$

with eigenvalue

$$\lambda_k = 2\cos\left(\frac{k\pi}{n+1}\right).$$

■

Example 2.1.1.

$$n = 3, \det(t\,\mathrm{id} - A_3) = t^3 - 2t = t(t-\sqrt{2})(t+\sqrt{2}).$$

So the eigenvalues are

$$\lambda_1 = \sqrt{2} = 2\cos\left(\frac{1\pi}{4}\right), \lambda_2 = 0 = 2\cos\left(\frac{2\pi}{4}\right), \lambda_2 = -\sqrt{2} = 2\cos\left(\frac{3\pi}{4}\right).$$

Now

$$\begin{aligned}
\det \tilde{M} &= \prod_{j=1}^{n}\prod_{k=1}^{m}\left(2\cos\frac{j\pi}{n+1}+i2\cos\frac{k\pi}{m+1}\right)\\
&= \prod_{j=1}^{n/2}\prod_{k=1}^{m}\left(2\cos\frac{j\pi}{n+1}+i2\cos\frac{k\pi}{m+1}\right)\left(2\cos\frac{(n+1-j)\pi}{n+1}+i2\cos\frac{k\pi}{m+1}\right)\\
&= \prod_{j=1}^{n/2}\prod_{k=1}^{m}\left(2\cos\frac{j\pi}{n+1}+i2\cos\frac{k\pi}{m+1}\right)\left(-2\cos\frac{j\pi}{n+1}+i2\cos\frac{k\pi}{m+1}\right)\\
&= \pm\prod_{j=1}^{n/2}\prod_{k=1}^{m}\left(4\cos^2\frac{j\pi}{n+1}+4\cos^2\frac{k\pi}{m+1}\right)
\end{aligned}$$

Theorem 2.1.1 (P.Kasteleyn, M.Fisher, H.N.V.Temperley, 1961)**.** *When m is even,*

$$T(m,n)=\prod_{j=1}^{n/2}\prod_{k=1}^{m/2}\left(4\cos^2\frac{j\pi}{n+1}+4\cos^2\frac{k\pi}{m+1}\right).$$

When m is odd,

$$T(m,n)=\prod_{j=1}^{n/2}2\cos\frac{j\pi}{n+1}\prod_{k=1}^{(m-1)/2}\left(4\cos^2\frac{j\pi}{n+1}+4\cos^2\frac{k\pi}{m+1}\right).$$

Example 2.1.2. For $n=m=8$, we get $T(8,8)=12,988,816=3604^2$.

<u>PROBLEM</u> For any positive integer $a\in\mathbb{Z}_{>0}$, $T(4a,4a)$ is a perfect square, $T(4a-2,4a-2)$ is twice a perfect square.

Asymptotics of $T(n,n)$: reasonable to expect $T(n,n)\sim e^{cn^2}$.

We take the natural log of $T(n,n)$:

$$\begin{aligned}
\frac{\ln T(n,n)}{n^2} &= \frac{1}{n^2}\sum_{k=1}^{n/2}\sum_{j=1}^{n/2}\ln\left(4\cos^2\frac{\pi k}{n+1}+4\cos^2\frac{\pi j}{n+1}\right)\\
&\sim \frac{1}{\pi^2}\sum\sum\left(\frac{\pi}{n+1}\right)^2\ln\left(4\cos^2\frac{\pi k}{n+1}+4\cos^2\frac{\pi j}{n+1}\right)
\end{aligned}$$

Notice that the right hand side is a Riemann sum of the function $\ln(4\cos^2 x+4\cos^2 y)$.

So the sum approaches to

$$\frac{1}{\pi^2}\int_0^{\pi/2}\int_0^{\pi/2}\ln(4\cos^2 x + 4\cos^2 y)dxdy = \frac{K}{\pi}$$

where K is Catalan's constant. As of today, it is not known whether it is irrational, nor transcendental.

So we have $T(n,n) \approx 1.34n^2$.

Another way to define Catalan's constant:

$$K = \beta(2) = \sum_{i=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \dots.$$