# Math 494

Yiwei Fu

Jan 27, 2022

Finish the proof on Hilbert's Nullstellenstaz.

**Corollary.** *If $I$ is an ideal of $\mathbb{C}[x_1, \dots, x_n]$ generated by $f_1, \dots, f_k$, and $V$ is the set of all $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$ s.t. $f_i(\alpha) = 0\ \forall i$, then the maximal ideals of $R/I$ are in bijection with $V$.*

$R/I$ is called the "coordinate ring".

*Proof.* Correpondence Theorem $\implies$ maximal ideals of $R/I$ are $\pi(M)$ where $\pi : R \twoheadrightarrow R/I$ and $M$ is a maximal ideal of $R$ containing $I$. (also $M \neq M' \implies \pi(M) \neq \pi(M')$)

An ideal $M$ of $R$ contains $I \iff M$ contains $f_i\ \forall i$.

$M$ is maximal $\iff M = (x_1 - \alpha_1, x_2 - \alpha_2, \dots, x_n - \alpha_n)$. So $f_i \in M \iff f_i(\alpha) = 0$.

So the maximal idals of $R$ containing $I$ are $(x_1 - \alpha_1, \dots, x_n - \alpha_n)$ where $f_i(\alpha) = 0\ \forall i$. ∎

**Lemma** (Zorn's lemma). *If a partially ordered set $S$ in which every chain has a upper bound, then $S$ has at least one maximal element.*

**Corollary.** *If $R =$ ring and $I \neq (1)$ is an ideal of $R$, then $I$ is contained in a maximal ideal.*

*Proof.* Let $S = \{\text{ideals containing } I \text{ which aren't } (1)\}$ partially ordered under containment. If $T$ is a totally ordered subset of $S$ then let $J = \bigcup_{I' \in T} I'$. $J$ is an ideal not containing $(1)$. We can see that $J \in T$ and is an upper bounde of $T$.

By Zorn's lemma we conclude that $S$ contains a maximal element, which is a maximum ideal containing $I$. ∎

**Corollary.** *If a ring $R$ has no maximal ideals then $R$ is the zero ring.*

**Corollary.** *If $f_1, \dots, f_k \in R := \mathbb{C}[x_1, \dots, x_k]$ have no common zeros in $\mathbb{C}^n$, then the ideal*

$(f_1, \ldots, f_k)$ *is* $(1)$ $i.e.$

$$1 = g_1 f_1 + \ldots + g_k f_k, g_1, \quad \ldots, g_k \in \mathbb{C}[x_1, \ldots, x_k]$$

*Proof.* If $(f_1, \ldots, f_k) \neq (1)$ then it is contained in a maximal ideal of $R$ which is $(x_1 - \alpha_1, \ldots, x_n - \alpha_n)$ hwere $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ and $f_i(\alpha) = 0 \; \forall i \implies f_i$'s have a common zero, a contradiction. ■

**Theorem** (Bezout's theorem)**.** *If $f(x, y)$ and $g(x, y)$ are polynomials in $\mathbb{C}[x, y]$ with no (non-constant) common factor. Then they only have finitely many common zeros.*

*In fact*

$$\#\textit{of zeros} \leq (\textit{total deg of } f(x, y)) \cdot (\textit{total deg of } g(x, y)).$$

*Proof.* We have $\mathbb{C}[x, y] = (\mathbb{C}[y])[x] \subseteq (\mathbb{C}(y))[x]$.

The ideal $(f, g)$ in $(\mathbb{C}(y))[x]$ is principal, say it's $(h)$ where $h \in (\mathbb{C}(y))[x]$. If $(h) \neq (1)$ then

$$h = \frac{h_1(x, y)}{u(y)}, h_1 \in \mathbb{C}[x, y], u \in \mathbb{C}[y], u \neq 0.$$

But $u(y)$ is a unit in $\mathbb{C}(y)[x] \implies (h) = (h_1)$.

So we may assume $h \in \mathbb{C}[x, y], (h) \neq (1)$ and $h \mid f, h \mid g$ in $\mathbb{C}(y)[x]$

$$\begin{aligned} &\implies hA = f, hB = g, && A, B \in \mathbb{C}(y)[x] \\ &\implies hA_1 = fu_1, hB_1 = gu_2, && A_1, B_1 \in \mathbb{C}[x, y], u_1, u_2 \in \mathbb{C}[y] \end{aligned}$$

If $g_1 g_2 \in \mathbb{C}^*$ then there is a contradiction. So assume $u_1 \notin \mathbb{C}^*$. Then $u_1$ has a root $\alpha$

$$\begin{aligned} &\implies h(x, \alpha) A_1(x, \alpha) = 0 \text{ in } \mathbb{C}[x] \\ &\implies h(x, \alpha) = 0 \text{ or } A_1(x, \alpha) \\ &\implies y - \alpha \mid h(x, \alpha) \text{ or } y - \alpha \mid A_1(x, \alpha) \end{aligned}$$

■