Notes for Math 597 – Real Analysis

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Contents

Office hour is Mon 12:30 - 1:30, Tue 12:30 - 1:30 in person EH 5838, Th 1 - 2 online.

Chapter 1

Abstract Measure

1.1 σ-Algebra

Definition 1.1. Let X be a set. A collection $\mathcal{M}$ of subsets of X is called a σ-algebra on X if

- $\emptyset \in \mathcal{M}$.

- $\mathcal{M}$ is closed under *complements*: $E \in \mathcal{M} \implies E^c \in \mathcal{M}$.

- $\mathcal{M}$ is closed under *countable unions*: $E_1, E_2, \ldots \in \mathcal{M} \implies \bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$.

SIMPLE PROPERTIES:

- $X = \emptyset^c \in \mathcal{M}$.

- $\bigcap_{i=1}^{\infty} E_i = \left(\bigcup_{i=1}^{n} E_i^c\right)^c \in \mathcal{M}$. It is closed under countable intersections.

- $\bigcup_{i=1}^{N} E_i = E_i \cup \ldots \cup E_n \cup \emptyset \cup \ldots$. It is closed under finite unions (similarly, intersections). sigma

- $E \setminus F = E \cap F^c \in \mathcal{M}, E \triangle F = (E \cap F^c) \cup (F \cap E^c) \in \mathcal{M}$.

Example 1.2. (a) $\mathcal{A} = \mathcal{P}(X)$ power algebra.

(b) $\mathcal{A} = \{\emptyset, X\}$ trivial algebra.

(c) Let $B \subset X, B \neq \emptyset, B \neq X. \mathcal{A} = \{\emptyset, B, B^c, X\}$.

Lemma 1.3. *(An intersection of σ-algebras is a σ-algebra) Let $\mathcal{A}_\alpha, \alpha \in I$, be a family a σ-algebras of X. Then $\bigcap_{\alpha \in I} A_\alpha$ is a σ-algebra. (I can be uncountable.)*

Proof. DIY ■

Definition 1.4. For $\mathcal{E} \subset \mathcal{P}(X)$ (not necessarily a σ-algebra), let $\langle \mathcal{E} \rangle$ be the intersection of all σ-algebras on X that contains $\mathcal{E}$. Call it the σ-algebra generated by $\mathcal{E}$.

- $\langle \mathcal{E} \rangle$ is the *smallest* σ-algebra containing $\mathcal{E}$ and is *unique*.

- $\{\emptyset, B, B^c, X\} = \langle \{B\} \rangle = \langle \{B^c\} \rangle = \langle \{\emptyset, B\} \rangle$.

The above definition gives us (potentially) lots of examples of σ-algebra on a set X

Lemma 1.5. *(a) Suppose $\mathcal{E} \subset \mathcal{P}(X)$, $\mathcal{A}$ is a σ-algebra on X. $\mathcal{E} \in \mathcal{A} \implies \langle \mathcal{E} \rangle \in \mathcal{A}$.*

(b) $E \subset F \subset \mathcal{P}(X) \implies \langle \mathcal{E} \rangle \subset \langle \mathcal{F} \rangle$.

Proof. ∎

Definition 1.6. For a topological space X, *the Borel σ-algebra $\mathcal{B}(X)$* is the σ-algebra generated by the collection of open sets.

Example 1.7. $(X = \mathbb{R})$ $\mathcal{B}(\mathbb{R})$ contains the following collections:

$$\mathcal{E}_1 = \{(a, b) \mid a < b\}, \quad \mathcal{E}_2 = \{[a, b] \mid a < b\},$$
$$\mathcal{E}_3 = \{(a, b] \mid a < b\}, \quad \mathcal{E}_4 = \{[a, b) \mid a < b\},$$
$$\mathcal{E}_5 = \{(a, \infty) \mid a \in \mathbb{R}\}, \quad \mathcal{E}_6 = \{[a, \infty) \mid a \in \mathbb{R}\},$$
$$\mathcal{E}_7 = \{(-\infty, a) \mid a \in \mathbb{R}\}, \quad \mathcal{E}_8 = \{(-\infty, a] \mid a < b\}.$$

Proposition 1.8. $\mathcal{B}(\mathbb{R}) = \langle \mathcal{E}_i \rangle$ *for each $i = 1, \ldots, 8$.*

Proof. Use 1.5. ∎

Definition 1.9. $(X, \mathcal{A})$ is called a measurable space.

1.2 Measures

Definition 1.10. A measure on $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \to [0, \infty]$ *s.t.*

(a) $\mu(\emptyset) = 0$

(b) (countable additive) For $A_1, A_2, \ldots \in \mathcal{A}$ disjoint we have

$$\mu \left(\bigcup_1^\infty A_i \right) = \sum_{i=1}^\infty \mu(A_i).$$

$(X, \mathcal{A}, \mu)$ is then called a measure space.

Example 1.11.   (a) For any $(X, \mathcal{A}), \mu(A) = \#A$ counting measure.

(b) For any $(X, \mathcal{A})$, let $x_0 \in X$. The Dirac measure at x_0 is

$$\mu(A) = \begin{cases} 1 & x_0 \in A, \\ 0 & x_0 \notin A. \end{cases}$$

(c) For $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, let $a_1, a_2, \ldots \in [0, \infty)$. $\mu(A) = \sum_{i \in A} a_i$ is a measure.

$(X, \mathcal{A})$ measurable space

$(X, \mathcal{A}, \mu)$ measure space

$\mu : \mathcal{A} \to [0, \infty]$ s.t. $\mu(\emptyset) = 0$, countable additivity.

<u>NOTE</u>: $A, B \in \mathcal{A}, A \subset B$, then $\mu(B \setminus A) + \mu(A) = \mu(B) \implies \mu(B \setminus A) = \mu(B) - \mu(A)$ if $\mu(A) < \infty$.

Theorem 1.13. *Suppose $(X, \mathcal{A}, \mu)$ a measure space. Then*

(a) (monotonicity)
$$A, B \in \mathcal{A}, A \subset B \implies \mu(A) \leq \mu(B).$$

(b) (countable subadditivity)

$$A_1, A_2, \ldots, \in \mathcal{A}, \implies \mu\left(\bigcup_i^{\infty} A_i\right) \leq \sum_i^{\infty} \mu(A_i).$$

(c) (continuity from below/(MCT) from sets)

$$A_1, A_2, \ldots \in \mathcal{A}, A_1 \subset A_2 \subset A_3 \subset \ldots \implies \mu\left(\bigcup_i^{\infty} A_i\right) = \lim_{n \to \infty} \mu(A_n).$$

(d) (continuity from above)

$$A_1, A_2, \ldots \in \mathcal{A}, A_1 \supset A_2 \supset A_3 \supset \ldots, \mu(A_1) < \infty \implies \mu\left(\bigcap_i^{\infty} A_i\right) = \lim_{n \to \infty} \mu(A_n).$$

Proof. (a), (b), DIY.

For (c), let $B_1 = A_1, B_i = A_i \setminus A_{i-1}, i \geq 2. B_i \in \mathcal{A}$ and are disjoint.

$$\bigcup_i^\infty A_i = \bigcup_i^\infty B_i$$

$$\implies \mu\left(\bigcup_i^\infty A_i\right) = \mu\left(\bigcup_i^\infty B_i\right) = \sum_i^\infty \mu(B_i) = \lim_{n\to\infty} \sum_i^n \mu(B_i) = \lim_{n\to\infty} \mu(A_n).$$

For (d), let $E_i = A_1 \setminus A_i$. Hence $E_i \in \mathcal{A}, E_1 \subset E_2 \subset \ldots$ We have

$$\bigcup_i^\infty E_i = \bigcup_i^\infty (A_1 \setminus A_i) = A_1 \setminus \left(\bigcap_1^\infty A_i\right) \implies \bigcap_1^\infty A_i = A_1 \setminus \left(\bigcup_1^\infty E_i\right).$$

Hence

$$\mu\left(\bigcap_1^\infty A_i\right) = \mu(A_1) - \mu\left(\bigcup_1^\infty E_i\right) = \mu(A_1) - \lim_{n\to\infty} \mu(E_n) = \mu(A_1) - \lim_{n\to\infty} \mu(A_1) - \mu(A_n). \blacksquare$$

<u>NOTE:</u> the condition that $\mu(A_1) < \infty$ cannot be dropped.

For example, in $(\mathbb{N}, \mathcal{P}(N), \text{counting measure})$, let $A_n = \{n, n+1, n+2\}, A_1 \supset A_2 \supset A_3 \supset \ldots$ We have $\bigcap_1^\infty = \emptyset \implies \mu\left(\bigcap_1^\infty A_i\right) = 0$.

Definition 1.14. For $(X, \mathcal{A}, \mu)$ measure space,

- $A \subset X$ is a μ-null set if $A \in \mathcal{A}, \mu(A) = 0$.

- $A \subset X$ is a μ-subnull set if $\exists B, \mu$-null set $A \subset B$.

- $(X, \mathcal{A}, \mu)$ is a complete measure space if every μ-subnull set is $\mathcal{A}$-measurable.

Definition 1.15. $(X, \mathcal{A}, \mu)$ measure space. A statement $P(x), x \in X$ holds μ-almost everywhere (a.e.) if the set $\{x \in X \mid P(x)\text{does not hold}\}$ is μ-null.

Definition 1.16. $(X, \mathcal{A}, \mu)$ measure space.

- μ is a *finite measure* is $\mu(X) < \infty$.

- μ is a *σ-finite measure* if $X = \bigcup_1^\infty X_n, X_n \in \mathcal{A}, \mu(X_n) < \infty$.

HW: every measure space can be "completed."

1.3   Outer Measures

Definition 1.17. An *outer measure* on X is $\mu^* : \mathcal{P}(X) \to [0, \infty]$ *s.t.*

- $\mu^*(\emptyset) = 0$

- (monotonicity) $\mu^*(A) \le \mu^*(B)$ if $A \subset B$.

- (countable subadditivity)

$$\forall A_1, A_2, \ldots \in X, \mu^* \left(\bigcup_i^{\infty} A_i \right) \le \sum_i^{\infty} \mu^*(A_i).$$

Example 1.18. For $A \subset \mathbb{R}$,

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} (b_i - a_i) \ \middle| \ \bigcup_1^{\infty} (a_i, b_i) \supset A \right\}.$$

is an outer measure due to the next proposition.

Proposition 1.19. *(1.19) Let $\mathcal{E} \in \mathcal{P}(X)$ s.t. $\emptyset, X \in \mathcal{E}$. Let $\rho : \mathcal{E} \to [0, \infty]$ s.t. $\rho(\emptyset) = 0$. Then*

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) \ \middle| \ E_i \in \mathcal{E}, \forall i \in N, \bigcup_1^{\infty} E_i \supset A \right\}$$

is an outer measure on X.

Proof.   (a) μ^* is well-defined (inf is taken over non-empty set.)

   (b) $\mu^*(\emptyset) = 0$

   (c) $A \subset B \implies \mu^*(A) \le \mu^*(B)$.

We check the countable subadditivity.

Let $A_1, A_2, \ldots \subset X$. If one of $\mu^*(A_i) = \infty$, then the result holds. Suppose $\mu^*(A_n) < \infty, \forall n \in \mathbb{N}$.

"Give your self a room of epsilon":

Fix $\varepsilon > 0$. We will show

$$\mu^* \left(\bigcup_1^{\infty} A_n \right) \le \sum_1^{\infty} \mu^*(A_i) + \varepsilon.$$

For each $n \in \mathbb{N}, \exists E_{n,1}, E_{n,2}, \ldots \in \mathcal{E}$ s.t.

$$\bigcup_{k=1}^{\infty} E_{n,k} \supset A_n \quad \text{and} \quad \mu^*(A_n) + \frac{\varepsilon}{2^n} \ge \sum_{k=1}^{\infty} \rho(E_{n,k}).$$

Then,

$$\bigcup_1^\infty A_n \subset \bigcup_{n=1}^\infty \bigcup_{k=1}^\infty E_{n,k} = \bigcup_{(n,k)\in\mathbb{N}^2} E_{n,k}.$$

<u>RECALL</u>: Tonelli's thm for series. If $a_{ij} \in [0,\infty], \forall i,j \in \mathbb{N}$, then

$$\sum_{(i,j)\in\mathbb{N}^2} a_{ij} = \sum_{i=1}^\infty \sum_{j=1}^\infty a_{ij} = \sum_{j=1}^\infty \sum_{i=1}^\infty a_i j.$$

Hence

$$\mu^*\left(\bigcup_{n=1}^\infty A_n\right) \le \sum_{n=1}^\infty \rho(E_{k,n}) = \sum_{n=1}^\infty \sum_{k=1}^\infty \rho(E_{k,n}) \le \sum_{n=1}^\infty \left(\mu^*(A_n) + \frac{\varepsilon}{2^n}\right) = \sum_{n=1}^\infty \mu^*(A_n) + \varepsilon.$$

We have shown countable subadditivity.   ■

Outer measure is very close to a measure. Here the textbooks diverge.

[Tao11] introduces Lebesgue measure on $\mathbb{R}$ using topological qualities of subsets of $\mathbb{R}$. [Fol99] introduces abstract method by Carathéodory and Kolmogorov.

Definition 1.20. Let μ^* be an outer measure on X. We say $A \subset X$ is Carathéodory measurable with respect to μ^* if $\forall E \subset X, \mu^*(E) = \mu^*(E \setminus A) + \mu^*(E \cap A)$.

Lemma 1.21. *Let μ^* be an outer measure on X. Suppose $B_1, B_2, \ldots, B_N$ are disjoint C-measurable sets. Then,*

$$\forall E \subset X, \mu^*\left(E \cap \left(\bigcup_1^N B_i\right)\right) = \sum_{i=1}^n \mu^*(E \cap B_i)$$

Proof.

$$\mu^*\left(E \cap \left(\bigcup_1^N B_i\right)\right) = \mu^*(E \cap B_1) + \mu^*\left(E \cap \left(\bigcup_2^N B_i\right)\right)$$

because B_1 is C-measurable. Then, iterate.   ■

Improved version:

$B_1, B_2, \ldots$ C-measurable and *disjoint* $\implies$ $\mu^*\left(E \cap \bigcup_1^\infty B_n\right) = \sum_1^\infty \mu^*(E \cap B_n), \forall E \subset X$.

Proof.

$$\sum_1^\infty \mu^*(E \cap B_n) \geq \mu^* \left(E \cap \bigcup_1^\infty B_n \right)$$

$$\geq \mu^* \left(E \cap \bigcup_1^N B_n \right) = \sum_1^N \mu^*(E \cap B_n.)$$

Take $N \to \infty$ or note that $N \in \mathbb{N}$ is arbitrary we get the result. ∎

First big theorem:

Theorem 1.22 (Carathéodory extension theorem)**.** *Let μ^* be an outer measure on X. Let $\mathcal{A}$ be the collection of C-measurable sets with respect to μ^*. Then*

(a) *$\mathcal{A}$ us a σ-algebra on X.*

(b) *$\mu = \mu^*|_{\mathcal{A}}$ is a measure on $(X, \mathcal{A})$.*

(c) *$(X, \mathcal{A}, \mu)$ is a complete measure space.*

Proof. (a) (1) $\emptyset \in \mathcal{A}$.

(2) $\mathcal{A}$ is closed under complements.

(3) To show $\mathcal{A}$ closed under countable unions.

- (finite union)
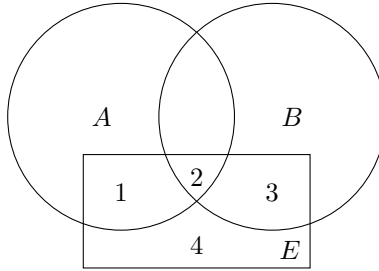 <u>CLAIM</u> $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$.

Figure 1.1: Venn diagram of A, B, E

Fix arbitrary $E \subset X$. We need to show

$$\mu^*(E) = \mu^*(E \cap (A \cup B)) + \mu^*(E \setminus (A \cup B)).$$

i.e.

$$\mu^*(1 \cup 2 \cup 3 \cup 4) = \mu^*(1 \cup 2 \cup 3) + \mu^*(4)$$

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Since A is C-measurable, we have

$$\mu^*(1 \cup 2 \cup 3 \cup 4) = \mu^*(1 \cup 2) + \mu^*(3 \cup 4)$$

$$\mu^*(1 \cup 2 \cup 3) = \mu^*(1 \cup 2) + \mu^*(3)$$

Similarly since B is C-measurable, we have

$$\mu^*(3 \cup 4) = \mu^*(3) + \mu^*(4)$$

Hence

$$\begin{aligned}
\mu^*(1 \cup 2 \cup 3 \cup 4) &= \mu^*(1 \cup 2) + \mu^*(3 \cup 4) \\
&= \mu^*(1 \cup 2 \cup 3) - \mu^*(3) + \mu^*(3) + \mu^*(4) \\
&= \mu^*(1 \cup 2 \cup 3) + \mu^*(4).
\end{aligned}$$

- (countable disjoint unions)

  Let $A_1, A_2, \ldots \in \mathcal{A}$ and *disjoint*.

  Fix $E \subset X$ arbitrary. Since μ^* is countably subadditive,

  $$\mu^*(E) \leq \mu^*\left(E \cap \bigcup_1^\infty \right) + \mu^*\left(E \setminus \bigcup_1^\infty A_n \right)$$

  Fix $n \in \mathbb{N}$.

  $$\implies \bigcup_1^N A_n \in \mathcal{A}$$

  $$\implies \mu^*(E) = \mu^*\left(E \cap \bigcup_1^N \right) + \mu^*\left(E \setminus \bigcup_1^N A_n \right)$$

  $$\geq \sum_1^N \mu^*(E \cap A_n) + \mu^*\left(E \setminus \bigcup_1^\infty A_n \right) \text{ by lemma.}$$

  Take $n \to \infty$.

- (countable unions)

  Let $A_1, A_2, \ldots \in \mathcal{A}$. Take $E_1 = A_1, E_n = A_n \setminus \left(\bigcup_1^{n-1} A_i \right)$ for $n \geq 2$. Then $\bigcup A_n = \bigcup E_n$ and E_n's are disjoint.

(b) Firstly we have $\mu(\emptyset) = \mu^*(\emptyset) = 0$.

Countable additvity of μ^* on $\mathcal{A}$ follows from the improved lemma with $E = X$.

(c) HW.   ∎

1.4   Hahn-Kolmogorov Theorem

<u>RECALL</u> 1.19 Let $\mathcal{E} \subset \mathcal{P}(X)$ s.t. $\emptyset, X \in \mathcal{E}$. Let $\rho : \mathcal{E} \to [0, \infty]$ s.t. $\rho(\emptyset) = 0$

$$(\mathcal{E}, \rho) \xrightarrow[1.19]{} (\mathcal{P}(X), \mu^*) \xrightarrow[\text{C-theorem}]{} (A, \mu)$$

<u>QUESTION</u> $\mathcal{E} \subset \mathcal{A}$ and $\mu|_\mathcal{E} = \rho$? No!

Definition 1.23. Let $\mathcal{A}_0$ be an algebra on X. We say $\mu_0 : \mathcal{A}_0 \to [0, \infty]$ is a *pre-measure* if

(a) $\mu_0(\emptyset) = 0$.

(b) (finite additivity)

$$\mu_0 \left(\bigcup_1^N A_i 1 \right) = \sum_1^N \mu_0(A_i) \text{ if } A_1, \dots, A_N \in \mathcal{A}_0 \text{ are disjoint.}$$

(c) (countable additivity within the algebra) If $A \in \mathcal{A}_0$ and

$$A = \bigcup_1^\infty A_n, A_n \in \mathcal{A}_0 \text{ and are disjoint, then } \mu_0(A) = \sum_1^\infty \mu_0(A_n)$$

<u>NOTATION</u>: Folland uses $\mathcal{M}$ for σ-algebra and $\mathcal{A}$ for algebra. (Jinho) uses $\mathcal{A}$ for σ-algebra and $\mathcal{A}_0$ for alegbra.

Example 1.24. $\mathcal{A}_0$ finite disjoint unions of $(a, b]$.

$$\mu_0 \left(\bigcup_1^\infty (a_i, b_i] \right) = \sum_1^\infty (b_i - a_i) \text{ or } b_i^n - a_i^n, e^{b_i} - e^{a_i}, \text{ etc.}$$

Lemma 1.25.   • *(a) + (c) $\implies$ (b).*

• μ_0 *is monotone.*

Theorem 1.26 (Hahn-Kolmogorov Theorem). *Let μ_0 be a pre-measure on algebra $\mathcal{A}_0$ on X. Let μ^* be the outer measure induced by $(\mathcal{A}_0, \mu_0)$ in 1.19. Let $\mathcal{A}$ and μ be the Carathéodory σ-algebra and measure for $\mu^* \implies (\mathcal{A}, \mu)$ extends $(\mathcal{A}_0, \mu_0)$ i.e. $\mathcal{A} \supset \mathcal{A}_0, \mu|_{\mathcal{A}_0} = \mu_0$.*

Proof.   (a) $(\mathcal{A} \supset \mathcal{A}_0)$ Let $A \in \mathcal{A}_0$.

Question: $A \in \mathcal{A}$? i.e. is A C-measurable? i.e. $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c), \forall E \subset$

X.

Fix $E \subset X$.

- (countable) subadditivity of $\mu^* \implies \mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c)$.

- If $\mu^*(E) = \infty$ then $\mu^*(E) = \infty \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$.

- If $\mu^*(E) < \infty$.

  Fix $\varepsilon > 0$. By the definition of $\mu^*, \exists B_1, B_2, \ldots \in \mathcal{A}_0$ s.t. $\bigcup_1^\infty B_n \supset E$ and

  $$\mu^*(E) + \varepsilon \geq \sum_1^\infty \mu_0(B_n) = \sum_1^\infty \left(\mu_0(B_n \cap A) + \mu_0 \left(B_n \cap A^c \right) \right).$$

  Note that

  $$\bigcup_1^\infty (B_n \cap A) \supset E \cap A, \quad \bigcup_1^\infty (B_n \cap A^c) \supset E \cap A^c \implies \geq$$

(b) Let $A \in \mathcal{A}_0$. We want to show that $\mu(A) = \mu_0(A)$.

By definition, $\mu(A) = \mu^*(A)$.

- Let $B_i = \begin{cases} A & i = 1, \\ \emptyset & i = 2 \end{cases} \in \mathcal{A}_0$ and $\bigcup_1^\infty B_i \supset A$.

  Hence $\mu^*(A) \leq \sum_1^\infty \mu_0(B_i) = \mu_0(A)$.

- Let $B_i \in \mathcal{A}_0, \bigcup_1^\infty B_i \supset A$ an arbitrary collection of sets.

  Let $C_1 = A \cap B_1, C_i = A \cap B_i \setminus \left(\bigcup_{j=1}^{i-1} B_j \right)$. Then $A = \bigcup_1^\infty$ is a disjoint countable union. By countable additivitiy we have

  $$\mu_0(A) = \sum_1^\infty \mu_0(C_i) \implies \mu_0(A) \leq \sum_1^\infty \mu_0(B_i).$$

Hence we have $\mu_0(A) = \mu^*(A) = \mu(A)$. We have completed our proof. ∎

Definition 1.27. Such $(\mathcal{A}, \mu)$ is called the *Hahn-Kolmogorov extension* of $(\mathcal{A}_0, \mu_0)$, and is also called the *Carathéodory σ-algebra* for $(\mathcal{A}_0, \mu_0)$.

Theorem 1.28 (uniqueness of HK extension)**.** *Let $\mathcal{A}_0$ be an algebra on X, μ_0 be a pre-measure on $\mathcal{A}_0$, $(\mathcal{A}, \mu)$ be the Hahn-Kolmogorov extension of $(\mathcal{A}_0, \mu_0)$. And let $(\mathcal{A}', \mu')$ be another extension of $(\mathcal{A}_0, \mu_0)$.*

If μ_0 is σ-finite, then $\mu \mid_{\mathcal{A} \cap \mathcal{A}'} = \mu' \mid_{\mathcal{A} \cap \mathcal{A}'}$.

NOTE σ-finite means

$$\forall X, X = \bigcup_1^\infty X_n, X_n \in \mathcal{A}_0, \mu_0(X_n) < \infty.$$

Corollary 1.29. *Let μ_0 be a pre-measure on algebra $\mathcal{A}_0$ on X. Suppose μ_0 is σ-finite, then $\exists!$ measure μ on $\langle \mathcal{A}_0 \rangle$ that extends $\mathcal{A}_0$. Furthermore,*

   (a) the completion of $(X, \langle \mathcal{A}_0 \rangle, \mu)$ is the HK extension of $(\mathcal{A}_0, \mu_0)$.

   (b)
$$\mu(A) = \inf \left\{ \sum_{i=1}^\infty \mu_0(B_i) \mid B_i \subset A_0, \forall i \in \mathbb{N}, \bigcup_1^\infty B_i \supset A \right\}, \forall A \in \overline{\langle \mathcal{A}_0 \rangle}.$$

Proof of 1.28. Let $A \in \mathcal{A} \cap \mathcal{A}'$. We need to show $\mu(A) = \mu^*(A) = \mu'(A)$.

- $\mu^*(A) \geq \mu'(A)$ (HW)

- $\mu(A) \leq \mu'(A)$:

   (i) Assume $\mu(A) < \infty$. Fix $\varepsilon > 0$. Then $\exists B_i \in \mathcal{A}_0, \forall i \in \mathbb{N}, \bigcup_1^\infty B_i \supset A$ s.t.

$$\mu(A) + \varepsilon = \mu^*(A) + \varepsilon \geq \sum_1^\infty \mu_0(B_i) = \sum_1^\infty \mu(B_i) \geq \mu \left(\bigcup_1^\infty B_i \right) = \mu(B)$$

   Hence $\mu(B \setminus A) = \mu(B) - \mu(A) \leq \varepsilon$.

   On the other hand,

$$\mu(B) = \lim_{N \to \infty} \mu \left(\bigcup_1^N B_i \right) = \lim_{N \to \infty} \mu' \left(\bigcup_1^N B_i \right) = \mu'(B)$$

   by continuity of measure from below.

$$\mu(A) \leq \mu(B) = \mu'(B) = \mu'(A) + \mu'(B \setminus A) \leq \mu'(A) = \varepsilon.$$

   (ii) Assume $\mu(A) = \infty$.

   Since μ_0 is σ-finite, $X = \bigcup_1^\infty X_n, X_n \in \mathcal{A}_0, \mu_0(X_0) < \infty$. Replacing X_n by $X_1 \cup \ldots \cup X_n$, we may assume $X_1 \subset X_2 \subset \ldots$.

$$\forall n \in N, \mu(A \cap X_n) < \infty \implies \mu(A \cap X_n) \leq \mu'(A \cap X_n).$$

Hence

$$\mu(A) = \lim_{N \to \infty} \mu(A \cap X_n) \leq \lim_{N \to \infty} \mu'(A \cap X_n) = \mu'(A). \qquad \blacksquare$$

1.5 Borel Measures on $\mathbb{R}$

Definition 1.30. $F : \mathbb{R} \to \mathbb{R}$ is an *increasing* function if $F(x) \leq F(y)$ for $x < y$. $F : \mathbb{R} \to \mathbb{R}$ is increasing and right-continuous $\implies F$ is distribution function.

Example 1.31.

- $F(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0. \end{cases}$

- $\mathbb{Q} = \{r_1, r_2, \ldots\}, F_n(x) = \begin{cases} 1 & x \geq r_n \\ 0 & x < r_n \end{cases}. \ F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n}$ is a distribution function.

<u>NOTE</u> If F is increasing, $F(\infty) := \lim_{x \to \infty} F(x), F(-\infty) := \lim_{x \to -\infty} F(x)$ exists in $[-\infty, \infty]$.

In probability theory, cumulative distribution function (CDF) is a distribution function with $F(\infty) = 1$ and $F(-\infty) = 0$.

There are distributions [Fol99, Ch.9], but these are different from *distribution functions*.

Definition 1.32. Suppose X a topological space. μ on $(X, \mathcal{B}(X))$ is called *locally finite* is $\mu(K) < \infty$ for any compact set $K \subset X$.

Lemma 1.33. *Let μ be a locally finite Borel measure on $\mathbb{R} \implies$*

$$F_\mu(x) = \begin{cases} \mu((0, x]), & x > 0 \\ 0, & x = 0 \ \textit{is a distribution function.} \\ -\mu((x, 0]), & x < 0 \end{cases}$$

Proof. DIY. Use continuity of measure. $\blacksquare$

Definition 1.34. h-intervals are $\emptyset, (a, b], (a, \infty), (-\infty, b], (\infty, \infty)$.

Lemma 1.35. *Let $\mathcal{H}$ be the collections of finite disjoint unions of h-intervals. Then $\mathcal{H}$ is an algebra on $\mathbb{R}$.*

Proof. DIY. $\blacksquare$

Proposition 1.36 (Distribution function defines a pre-measure). *Let $F : \mathbb{R} \to \mathbb{R}$ be a distribution function. For an h-interval I, define*

$$\ell(I) = \ell_F(I) = \begin{cases} 0, & I = \emptyset \\ F(b) - F(a), & I = (a, b] \\ F(\infty) - F(a), & I = (a, \infty) \\ F(b) - F(\infty), & I = (-\infty, b] \\ F(\infty) - F(-\infty), & I = (-\infty, \infty). \end{cases}$$

Define $\mu_0 = \mu_{0,F} : \mathcal{H} \to [0, \infty]$ by

$$\mu_0(A) := \sum_{k=1}^{N} \ell(I_k) \quad \text{if } A = \bigcup_{k=1}^{N} I_k, \text{ finite disjoint union of } h\text{-intervals.}$$

Then μ_0 is a pre-measure.

Proof. (a) μ_0 is well-defined.

(b) μ_0 is finite additive.

(c) μ_0 is countable additive within $\mathcal{H}$.

Suppose $A \in \mathcal{H}$ and $A = \bigcup_1^\infty A_i$ a disjoint union, $A_i \in \mathcal{H}$. It is enough to consider the case $A = I$, $A_k = I_k$ all h-intervals. (Why?)

Focus on the case $I = (a, b]$: (HW: check other cases)
We have

$$(a, b] = \bigcup_1^\infty (a_n, b_n], \text{ a disjoint union.}$$

Check

$$F(b) - F(a) \stackrel{?}{=} \sum_1^\infty (F(b_n) - F(a_n))$$

$(a, b] \supset \bigcup_1^N (a_n, b_n] \implies F(b) - F(a) \geq \sum_1^N F(b_n) - F(a_n), \forall N \in \mathbb{N}$. (Arranging them in decreasing order) Take $N \to \infty$ we have

$$F(b) - F(a) \geq \sum_1^\infty (F(b_n) - F(a_n)).$$

Since F is right-continuous, $\exists a' > a$ s.t. $F(a') - F(a) < \varepsilon$. For each $n \in \mathbb{N}$, $\exists b_n' >$

b_n s.t. $F(b'_n) - F(b_n) < \frac{\varepsilon}{2^n}$.

$$\implies [a', b] \subset \bigcup_1^\infty (a_n, b'_n)$$

$$\implies \exists N \in \mathbb{N} \text{ s.t. } [a', b] \subset \bigcup_1^n (a_n, b'_n)$$

$$\implies F(b) - F(a') \leq \sum_1^N F(b'_n) - F(a_n)$$

$$\implies F(b) - F(a) \leq F(b) - F(a') + \varepsilon \leq \sum_1^\infty (F(b'_n) - F(a_n)) + \varepsilon$$

$$\leq \sum_1^\infty \left(F(b_n) - F(a_n) + \frac{\varepsilon}{2^n} \right) + \varepsilon \qquad \blacksquare$$

Once we have this pre-measure, HK theorem allows us to extended it to a measure.

Theorem 1.37 (Locally finite Borel measures on $\mathbb{R}$)**.**

(a) $F : \mathbb{R} \to \mathbb{R}$ *is a distribution function* $\implies$ $\exists!$ *locally finite Borel measure μ_F on $\mathbb{R}$ satisfying* $\mu_F((a, b]) = F(b) - F(a), \forall a, b, a < b$.

(b) *Suppose $F, G : \mathbb{R} \to \mathbb{R}$ are distribution functions. Then, $\mu_F = \mu_G$ on $\mathcal{B}(\mathbb{R})$ if and only if $F - G$ is a constant function.*

Proof. HW $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\blacksquare$

1.6   Lebesgue-Stieltjes Measures on $\mathbb{R}$

F distribution function $\implies \mu_F$ on Carathéodory σ-algebra $\mathcal{A}_{\mu_F}$.
Actually $(\mathcal{A}_{\mu_F}, \mu_F) = (\mathcal{B}(\mathbb{R}), \mu_F)$ (HW3).

Definition 1.38.   • μ_F on $\mathcal{A}_{\mu_F}$ is called the Lebesgue-Stieltjes measure corresponding to F.

  • Special case: $F(x) = x \implies$ Lebesgue measure $(\mathcal{B}, m)$.

Example 1.39.

(a) $\mu_F((a, b]) = F(b) - F(a)$. F is right-continuous and increasing $\implies$ $F(x_-) \leq F(x) = F(x_+)$.

  (HW) $\mu_F(\{a\}) = F(a) - F(a_-), \mu_F([a, b]) = F(b) - F(a_-), \mu_F((a, b)) = F(b_-) - F(a)$.

(b)
$$F(x) = \begin{cases} 1 & x \leq 0 \\ 0 & x < 0 \end{cases} \implies \mu_F(\{0\}) = 1, \mu_F(\mathbb{R}) = 1, \mu_F(\mathbb{R} \setminus \{0\}) = 0.$$

μ_F is the Dirac measure at 0.

(c)
$$\mathbb{Q} = \{r_1, r_2, \ldots\}, \ F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n}, \ F_n(x) = \begin{cases} 1 & x \leq r_n \\ 0 & x < r_n \end{cases}$$
$$\implies \mu_F(\{v\}) > 0, \forall v \in \mathbb{Q}, \ \mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0.$$

(d) If F is continuous at a, $\mu_F(\{a\}) = 0$.

(e) $F(x) = x \implies m((a,b])) = m((a,b)) = m([a,b]) = b - a$.

(f) $F(x) = e^x, \implies \mu_F((a,b]) = \mu_F((a,b)) = e^b - e^a$

(a), (b) are examples of discrete measure.

Example 1.40 (Middle thirds Cantor set $\mathcal{C} = \bigcup_{n=1}^{\infty} K_n$)**.**

$\mathcal{C}$ is uncountable set with $m(\mathcal{C}) = 0$.

$$x \in \mathcal{C} \implies x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}, a_n \in \{0, 2\}.$$

We are interested in the Cantor function F.

Example 1.41. Cantor function F is continuous and increasing. This defines the Cantor measure $\mu_F(\mathbb{R} \setminus \mathcal{C}) = 0, \mu_F(\mathcal{C}) = 1, \mu_F(\{a\}) = 0$. Compare with Lebesgue measure $m(\mathbb{R} \setminus \mathcal{C}) = \infty > 0, \mu(\mathcal{C}) = 0, m(\{a\}) = 0$.

1.7   Regularity Properties of Lebesgue-Stieltjes Measures

Lemma 1.42. μ *is Lebesgue-Stieltjes measure on* $\mathbb{R} \implies$

$$\mu(A) = \inf \left\{ \sum_1^{\infty} \mu((a_i, b_i]) \ \middle| \ \bigcup_1^{\infty}(a_i, b_i] \supset A \right\}$$
$$= \inf \left\{ \sum_1^{\infty} \mu((a_i, b_i)) \ \middle| \ \bigcup_1^{\infty}(a_i, b_i) \supset A \right\}$$

Proof. Using the continuity of measure. ∎

Theorem 1.43. *μ is a Lebesgue-Stieltjes measure. Then $\forall A \in \mathcal{A}_\mu$,*

  (a) (outer regularity)
$$\mu(A) = \inf\{\mu(O) \mid \text{open } O \supset A\}.$$

  (b) (inner regularity)
$$\mu(A) = \sup\{\mu(K) \mid \text{compact } K \subset A\}.$$

Proof.   (a) Followed from 1.42.

  (b) Let $s = \sup\{\ldots\}$. Monotonicity $\implies \mu(A) \geq s$.

- (A bounded) $\overline{A} \in \mathcal{B}(\mathbb{R}) \subset \mathcal{A}_\mu, \overline{A}$ bounded $\implies \mu(\overline{A}) < \infty$.

  Fix $\varepsilon > 0$. By 1, $\exists$ open $O \supset \overline{A} \setminus A, \mu(O) - \mu(\overline{A} \setminus A) = \mu(O \setminus (\overline{A} \setminus A)) \leq \varepsilon$.

  Let $K = \underbrace{A \setminus O}_{K \subset A} = \underbrace{\overline{A} \setminus O}_{\text{compact}}$. Show that $\mu(K) \geq \mu(A) - \varepsilon$.

- (A unbounded but $\mu(A) < \infty$) We have

$$A = \bigcup_1^\infty A_n, \ A_n = A \cap [-n, n], \ A_1 \subset A_2 \subset \ldots$$

  Hence
$$\lim_{n \to \infty} \mu(A_n) = \mu(A) < \infty.$$

- ($\mu(A) = \infty$)
$$\lim_{n \to \infty} \mu(A_n) = \mu(A) = \infty.$$

  Fix $L > 0$. $\exists N$ s.t. $\mu(A_N) \geq L$. ∎

Definition 1.44. Suppose X a topological space.

A G_σ-set is $G = \bigcup_1^\infty O_i$, O_i open. An F_σ-set is $F = \bigcup_1^\infty F_i$, F_i closed.

Theorem 1.45. *Suppose μ a LS measure. Then the following statements are equivalent:*

  (a) $A \in \mathcal{A}_\mu$.

  (b) $A = G \setminus M$, G is a G_σ-set, and M is μ-null.

  (c) $A = F \cup N$, F is an F_σ-set, and N is μ-null.

Proof. (b) $\implies$ (a) and (c) $\implies$ (a) are clear.

- (a) $\implies$ (c)

  (i) Assume $\mu(A) < \infty$. By inner regularity,

  $$\forall n \in \mathbb{N}, \exists \text{ compact } K_n \subset A \text{ s.t. } \mu(K_n) + \frac{1}{n} \geq \mu(A).$$

  Let $F = \bigcup_1^\infty K_n$. Then $N = A \setminus F$ is μ-null.

  (ii) Assume $\mu(A) = \infty$. We construct

  $$A = \bigcup_{k \in \mathbb{Z}} A_k, A_k = A \cap (k, k+1].$$

  By (i), $\forall k \in \mathbb{Z}, A_k = F_k \cup N_k$. Hence

  $$A = \underbrace{\left(\bigcup_k F_k \right)}_{F\sigma} \cup \underbrace{\left(\bigcup_k N_k \right)}_{\mu\text{-null}}.$$

- (a) $\implies$ (b)
  $$A^c = F \cup N, A = F^c \cup N^c = F^c \setminus N. \qquad \blacksquare$$

Proposition 1.46. *Suppose μ a LS measure, $A \in \mathcal{A}_\mu$, $\mu(A) < \infty$. Then*

$$\forall \varepsilon > 0, \exists I = \bigcup_1^{N = N(\varepsilon)} I_i, \text{ disjoint open intervals s.t. } \mu(A \triangle I) \leq \varepsilon.$$

Proof. DIY - use outer regularity. $\qquad \blacksquare$

Properties of Lebesgue measure

Theorem 1.47.
$$A \in \mathcal{L} \implies A + s \in \mathcal{L}, rA \in \mathcal{L}, \forall r, s \in \mathbb{R}.$$

In addition, $m(A + r) = m(A)$ and $m(rA) = rm(A)$.

Proof. DIY. $\qquad \blacksquare$

Example 1.48.

(a) $\mathbb{Q} = \{r_1\}_{i=1}^\infty$, which is dense in $\mathbb{R}$. Let $\varepsilon > 0$ and

$$O = \bigcup_{i=1}^\infty \left(r_i - \frac{\varepsilon}{2^i}, r_i + \frac{\varepsilon}{2^i} \right).$$

O is open and dense in $\mathbb{R}$. We have

$$m(O) \leq \sum_{i=1}^{\infty} \frac{2\varepsilon}{2^i} = 2\varepsilon, \partial O = \overline{O} \setminus O, m(O) = \infty.$$

(b) $\exists$ uncountable set A with $m(A) = 0$.

(c) $\exists A$ with $m(A) > 0$, but A contains no non-empty open interval.

(d) $\exists A \notin \mathcal{L}$ that is Vitali set.

(e) $\exists A \in \mathcal{L} \setminus \mathcal{B}(\mathbb{R})$. We will deal with that later.

Chapter 2

Integration

2.1 Measurable Functions

Definition 2.1. Suppose $(X, \mathcal{A})$, $(Y, \mathcal{B})$ two measurable spaces. $f : X \to Y$ is $(\mathcal{A}, \mathcal{B})$-measurable if

$$\forall B \in \mathcal{B}, f^{-1}(B) \in \mathcal{A}.$$

Lemma 2.2. *Suppose $\mathcal{B} = \langle \mathcal{E} \rangle$. Then*

$$f : X \to Y \text{ is } (\mathcal{A}, \mathcal{B})\text{-measurable} \iff \forall E \in \mathcal{E}, f^{-1}(E) \in \mathcal{A}.$$

Proof. $\implies$ clear

$\impliedby$ Let $\mathcal{D} = \{E \subset Y \mid f^{-1}(E) \in \mathcal{A}\}$. We have $\mathcal{E} \subset \mathcal{D}$ by assumption. In addition $\mathcal{D}$ is a σ-algebra $\implies \langle \mathcal{E} \rangle \subset \mathcal{D}$. $\blacksquare$

Definition 2.3. Suppose $(X, \mathcal{A})$ a measurable space.

$$\left. \begin{array}{l} f : X \to \mathbb{R} \\ f : X \to \overline{\mathbb{R}} = [-\infty, \infty] \\ f : X \to \mathbb{C} \end{array} \right\} \text{ is } \mathcal{A}\text{-measurable if } \left\{ \begin{array}{l} f \text{ is } (\mathcal{A}, \mathcal{B}(\mathbb{R}))\text{-measurable} \\ f \text{ is } (\mathcal{A}, \mathcal{B}(\overline{\mathbb{R}}))\text{-measurable} \\ \operatorname{Re} f, \operatorname{Im} f : X \to \mathbb{R} \text{ are } \mathcal{A}\text{-measurable.} \end{array} \right.$$

Here $\mathcal{B}(\overline{\mathbb{R}}) = \{E \subset \overline{\mathbb{R}} \mid E \cap R \in \mathcal{B}(\mathbb{R})\}$.

Lemma 2.4. *Suppose $f : X \to \mathbb{R}$. Then the followings are equivalent:*

(a) *f is $\mathcal{A}$-measurable*

(b) $\forall a \in \mathbb{R}, f^{-1}((a, \infty)) \in \mathcal{A}.$

(c) $\forall a \in \mathbb{R}, f^{-1}([a, \infty)) \in \mathcal{A}.$

(d) $\forall a \in \mathbb{R}, f^{-1}((-\infty, a)) \in \mathcal{A}.$

(e) $\forall a \in \mathbb{R}, f^{-1}((-\infty, a]) \in \mathcal{A}.$

For $f : X \to \overline{\mathbb{R}}$, change the interval to include $-\infty$ and ∞.

Proof. By 2.2.   ■

Example 2.5. $\mathcal{A} = \mathcal{P}(X) \implies$ every function is $\mathcal{A}$ measurable.

$\mathcal{A} = \{\emptyset, X\} \implies$ only $\mathcal{A}$ functions are constant functions.

<u>PROPERTIES</u> Suppose $f, g : X \to \mathbb{R}$, $\mathcal{A}$-measurable functions.

(a) $\phi : \mathbb{R} \to \mathbb{R}$, $\mathcal{B}(\mathbb{R})$ measurable (i.e. Borel measurable) $\implies \phi \circ f : X \to \mathbb{R}$ is $\mathcal{A}$-measurable.

(b) $-f, 3f, f^2, |f|$ are $\mathcal{A}$-measurable, $\frac{1}{f}$ is $\mathcal{A}$-measurable if $f(x) = 0, \forall x \in X$.

(c) $f + g$ is $\mathcal{A}$-measurable

$$(f + g)^{-1}((a, \infty)) = \bigcup_{r \in \mathbb{Q}} \left(f^{-1}((r, \infty)) \cap g^{-1}((a - r, \infty)) \right).$$

(d) fg is $\mathcal{A}$-measurable

$$f(x)g(x) = \frac{1}{2} \left((f(x) + g(x))^2 - f(x)^2 - g(x)^2 \right).$$

(e) $(f \wedge g)(x) = \min\{f(x), g(x)\}$, $(f \vee g)(x) = \max\{f(x), g(x)\}$ are $\mathcal{A}$-measurable.

(f) $f_n : X \to \overline{\mathbb{R}}$ are a sequence of $\mathcal{A}$-measurable functions $\implies$

$$\sup f_n, \inf f_n, \limsup_{n \to \infty} f_n, \liminf_{n \to \infty} f_n \text{ are } \mathcal{A}\text{-measurable.}$$

(g) If $f(x) = \lim_{n \to \infty} f_n(x)$ converges for every $x \in X$, then f is measurable.

Example 2.6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. Then f is Borel measurable $\implies f$ is Lebesgue measurable. (Preimage of an open set of a continuous function is open.)

Definition 2.7. For $f : X \to \overline{\mathbb{R}}$, let $f^+ = f \vee 0$, $f^- = (-f) \vee 0$.

<u>NOTE</u> $\operatorname{supp} f^+ \cap \operatorname{supp} f^- = \emptyset$. $f(x) = f^+(x) - f^-(x)$. f is $\mathcal{A}$-measurable $\iff f^+, f^-$ measurable.

Definition 2.8. For $E \subset X$, characteristic (indicator) funtion of E

$$\chi_E(x) = 1_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \in E^c. \end{cases}$$

1_E is $\mathcal{A}$-measurable $\iff E \in \mathcal{A}$.

Definition 2.9. Suppose $(X, \mathcal{A})$ a measurable space. A *simple function* $\phi : X \to \mathbb{C}$ that is $\mathcal{A}$-measurable and takes only finitely many values.

$$\phi(X) = \{c_1, \ldots, c_N\}, c_i \neq \pm\infty, E_i = \phi^{-1}(c_i) \in \mathcal{A} \implies \phi = \sum_{i=1}^{N} c_i 1_{E_i}.$$

Theorem 2.10. *Suppose $(X, \mathcal{A})$ a measurable space and $f : X \to [0, \infty]$. Then the followings are equivalent:*

(a) *f is $\mathcal{A}$-measurable.*

(b) *$\exists$ simple functions $0 \leq \phi_1(x) \leq \phi_2(x) \leq \ldots \leq f(x)$ such that*

$$\lim_{n \to \infty} \phi_n(x) = f(x), \ \forall x \in X.$$

(f is the pointwise upward limit of simple functions.)

Proof.   • (b) $\implies$ (a) is easy: $f(x) = \sup_{n \in \mathbb{N}} \phi_n(x)$.

• (a) $\implies$ (b): suppose f is $\mathcal{A}$-measurable.

Fix $n \in \mathbb{N}$. Let $F_n = f^{-1}([2^n, \infty]) \in \mathcal{A}$. For

$$0 \leq k \leq 2^{2n} - 1, \ E_{n,k} = f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right) \in \mathcal{A}.$$

Let $\phi_n(x) = \displaystyle\sum_{k=0}^{2^{2n}-1} 1_{E_{n,k}} + 2^n 1_{F_n}$.

This shows that

– $0 \leq \phi_1(x) \leq \phi_2(x) \leq \ldots \leq f(x), \ \forall x \in X$.

– $\forall x \in X \setminus F_n, 0 \leq f(x) - \phi_n(x) \leq \dfrac{1}{2^n}$.

Since $F_1 \supset F_2 \supset \ldots$ and $\displaystyle\bigcap_{1}^{\infty} F_n = f^{-1}(\{\infty\})$, we have

$$- \quad x \in f^{-1}([0, \infty)) = X \setminus \left(\bigcap_1^\infty F_n \right) \implies \lim_{n \to \infty} \phi_n(x) = f(x).$$

$$- \quad x \in f^{-1}(\{\infty\}) = \bigcap_1^\infty X_n \implies \phi_n(x) \geq 2^n \implies \lim_{n \to \infty} \phi_n(x) = \infty = f(x). \qquad \blacksquare$$

Corollary 2.11. *If f is bounded on a set $A \subset \mathbb{R}$ (i.e. $\exists L > 0$ s.t. $|f(x)| \leq L$, $\forall x \in A$) then $\phi_n \to f$ uniformly on A.*

Proof. DIY. $\qquad \blacksquare$

Corollary 2.12. $f : X \to \mathbb{C}$, *measurable function* $\iff$ $\exists$ *simple functions* $\phi_n : X \to \mathbb{C}$ *s.t.* $0 \leq |\phi_1| \leq |\phi_2| \leq \ldots \leq |f|$ *and ϕ_n converges to f pointwise. (Again, if f is bounded the convergence can be uniform.)*

2.2   Integration of Nonnegative Functions

Definition 2.13. Suppose $(X, \mathcal{A}, \mu)$ a measure space and $\phi = \sum_{i=1}^N c_i 1_{E_i} : X \to [0, \infty]$ a simple function. Let

$$\int \phi = \int \phi \, \mathrm{d}\mu = \int_X \phi \, \mathrm{d}\mu = \sum_1^N c_i \mu(E_i).$$

Proposition 2.14. *Suppose $\phi, \psi \geq 0$ are simple functions. Then,*

- *2.13 is well-defined.*

- $\displaystyle\int c\phi = c \int \phi, c \in [0, \infty).$

- $\displaystyle\int (\phi + \psi) = \int \phi + \int \psi.$

- $\phi(x) \geq \psi(x)$, $\forall x \implies \displaystyle\int \phi \geq \int \psi.$

- $\nu(A) = \displaystyle\int_A \phi \, \mathrm{d}\mu$ *is a measure on* $(X, \mathcal{A})$.

Proof. DIY. $\qquad \blacksquare$

Definition 2.15. Suppose $(X, \mathcal{A}, \mu)$, $f : X \to [0, \infty]$ is $\mathcal{A}$-measurable.

Define

$$\int f = \int f \, \mathrm{d}\mu = \sup \left\{ \int \phi \mid 0 \leq \phi \leq f, \phi \text{ simple} \right\}.$$

Proposition 2.16.

- *If f is a simple function then two definitions are the same.*

- $\displaystyle \int cf = c \int f.$

- $\displaystyle f \geq g \geq 0 \implies \int f \geq \int g.$

- $\displaystyle \int f + g = \int f + \int g.$ *(A bit harder to check)*

Theorem 2.17 (Monotone convergence theorem). *Suppose $(X, \mathcal{A}, \mu)$ a measure space and*

- $f : X \to [0, \infty]$ *is $\mathcal{A}$-measurable, $\forall n \in \mathbb{N}$.*

- $0 \leq f_1(x) \leq \dots.$

- $\displaystyle \lim_{n \to \infty} f_n(x) = f(x).$

Then
$$\int f = \lim_{n \to \infty} \int f_n.$$

Proof. Note that $\lim_{n \to \infty} f_n(x)$ converges $\forall x \in X$ and $\lim_{n \to \infty} f_n(x)$ converges.

- $\displaystyle f_n \leq f \implies \int f_n \leq \int f \implies \lim_{n \to \infty} \int f_n \leq \int f.$

- Fix simple function $0 \leq \phi \leq f$. Enough to show that $\displaystyle \lim_{n \to \infty} \int f_n \geq \int \phi.$

  Now fix $\alpha \in (0, 1)$. Enough to prove that $\displaystyle \lim_{n \to \infty} \int f_n \geq \alpha \int \phi.$

  Let $A_n = \{x \mid f_n(x) \geq \alpha \phi(x)\}$.

  - $A_n \in \mathcal{A}$.

  - $A_1 \subset A_2 \subset \dots$

  - $\displaystyle \bigcup_{n=1}^{\infty} A_n = X.$ (check!)

  So we have
  $$\int f_n \geq \int f_n 1_{A_n} \geq \int \alpha \phi 1_{A_n} = \alpha \nu(A_n)$$

  where $\nu(A) = \int_A \phi$ is a measure.

  $$\implies \lim_{n \to \infty} \int f_n \geq \lim_{n \to \infty} \nu(A_n) = \alpha \nu(x) = \alpha \int \phi. \qquad \blacksquare$$

Corollary 2.18. $\displaystyle f, g \geq 0 \text{ measurable} \implies \int f + g = \int f + \int g.$

Proof. $\exists$ simple functions $0 \le \phi_1 \le \phi_2 \le \ldots, \phi_n \to f$ pointwise and $0 \le \psi_1 \le \psi_2 \le \ldots, \psi_n \to g$ pointwise.

By MCT, we have

$$\int (f + g) = \lim_{n \to \infty} \int (\phi_n + \psi_n) = \lim_{n \to \infty} \int \phi_n + \int \psi_n = \int f + \int g. \qquad \blacksquare$$

Corollary 2.19 (Tonelli's theorem for series and integrals)**.** *Given* $s_n \ge 0, \forall n \in \mathbb{N}$ *measurable functions. Then*

$$\int \sum_{n=1}^{\infty} s_n = \sum_{n=1}^{\infty} \int s_n.$$

Proof. Let $f_N = \sum_{n=1}^{N} s_n, 0 \le f_1 \le f_2 \le \ldots.$

$$\lim_{N \to \infty} f_N(x) = \sum_{n=1}^{\infty} s_n(x)$$

By MCT, we have

$$\lim_{N \to \infty} \sum_{1}^{N} s_n = \sum_{1}^{\infty} s_n$$

$\blacksquare$

Theorem 2.20 (Fatou's lemma)**.** *Suppose* $f_n \ge 0$ *measurable. Then*

$$\int \liminf_{n \to \infty} f_n \le \liminf_{n \to \infty} \int f_n.$$

Recall that

$$\liminf_{n \to \infty} f_n := \lim_{k \to \infty} \inf_{n \ge k} f_n = \sup_{k \in \mathbb{N}} \inf_{n \ge k} f_n,$$

and

$$\lim_{n \to \infty} a_n \text{ exists} \iff \limsup_{n \to \infty} a_n = \liminf_{n \to \infty} a_n.$$

Proof. Let $g_k = \inf_{n \ge k} f_n \implies s_k$ measurable and $0 \le g_1 \le g_2 \le \ldots.$ By MCT, we have

$$\int \liminf_{n \to \infty} = \int \lim_{k \to \infty} s_k = \lim_{k \to \infty} \int s_k = \lim_{k \to \infty} \int \inf_{n \ge k} f_n$$

$$\inf_{n \geq k} f_n \leq f_m, \forall m \geq k$$

$$\implies \int \inf_{n \geq k} f_n \leq \int f_m, \forall m \geq k$$

$$\implies \int \inf_{n \geq k} f_n \leq \inf_{m \geq k} \int f_m$$

$\blacksquare$

Example 2.21. Suppose $(\mathbb{R}, \mathcal{L}, m)$

(a) (escape to horizontal infinity) $f_n = 1_{(n,n+1)}$.

 We see that $f_n \to 0 = f$ pointwise and $\int f_n = 1, \forall n, \int f = 0$.

(b) (escape to width infinity) $f_n = \frac{1}{n} 1_{(0,n)}$.

(c) (escape to vertical infinity) $f_n = n 1_{(0,1/n)}$.

Lemma 2.22 (Markov's inequality). *$f \geq 0$ is measurable $\implies$*

$$\forall c \in (0, \infty), \ \mu\left(\{x \mid f(x) \geq c\}\right) \leq \frac{1}{c} \int f.$$

Proof. Let $E = \{x \mid f(x) \geq c\}$. Then

$$f(x) \geq c 1_E(x) \implies \int f \geq c \int 1_E = c\mu(E). \qquad \blacksquare$$

Proposition 2.23. *Suppose $f \geq 0$ measurable. Then $\int f = 0 \iff f = 0$ almost everywhere (a.e.)*

$$\int f \, \mathrm{d}\mu = \mu(A) = 0, \ A = \{x \mid f(x) > 0\} = f^{-1}((0, \infty])$$

Proof. (a) Assume $f = \phi$ a simple function. We may assume

$$\phi = \sum_{i=1}^{N} c_i 1_{E_i}, \ c_i \in (0, \infty), \ E_i\text{'s are disjoint.}$$

$$\int \phi = \sum_{i=1}^{N} c_i \mu(E_i) = 0$$

$$\iff \mu(E_1) = \ldots = \mu(E_N) = 0$$

$$\iff \mu(A) = 0, \ A = \bigcup_{i=1}^{N} E_i.$$

(b) General $f \geq 0$.

(1) Assume $\mu(A) = 0$ (i.e. $f = 0$ a.e.)

Let $0 \leq \phi \leq f, \phi$ is simple.

$$\implies \phi(x) = 0, \ \forall x \in A^c$$
$$\implies \phi = 0 \text{ a.e.}$$
$$\implies \int \phi = 0$$

Then $\int f = 0$ by the definition of $\int f$.

(2) Assume $\inf f = 0$. Let $A_n = f^{-1}\left(\left[\frac{1}{n}, \infty\right]\right)$

$$\implies A_1 \subset A_2 \subset \dots$$
$$\bigcup_1^\infty A_n = f^{-1}\left(\bigcup_1^\infty \left[\frac{1}{n}, \infty\right]\right) = f^{-1}((0, \infty)) = A$$
$$\mu(A_n) = \mu\left(\left\{x \mid f(x) \geq \frac{1}{n}\right\}\right) \leq n \int f = 0$$
$$\implies \mu(A) = \lim_{n \to \infty} \mu(A_n) = 0$$

by the continuity of measure from below. ∎

Corollary 2.24. $f, g \geq 0$ *are measurable,* $f = g$ *a.e.* $\implies \int f = \int g$.

Proof. Let $A = \{x \mid f(x) \geq g(x)\}$. A is measurable (why?). By assumption $\mu(A) = 0$. Hence $f1_A = 0$ a.e.

$$\int f = \int f(1_A + 1_{A^c})$$
$$= \int f1_A + \int f1_{A^c}$$
$$= \int f1_{A^c}$$
$$= \int g1_{A^c} = \int g1_A + \int g1_{A^c} = \int g. \qquad \blacksquare$$

Corollary 2.25. $f_n \geq 0$ *measurable. Then*

(a)

$$\left.\begin{array}{c} 0 \leq f_1 \leq f_2 \leq \dots \leq f \text{ a.e.} \\ \lim_{n \to \infty} f_n = f \text{ a.e.} \end{array}\right\} \implies \lim_{n \to \infty} f_n = \int f.$$

(b)

$$\lim_{n \to \infty} f_n = f \ a.e \implies \int f \leq \liminf_{n \to \infty} \int f_n.$$

2.3   Integration of Complex Functions

I was afraid that you are bored.

— Jinho Baik on homework

Definition 2.26. $(X, \mathcal{A}, \mu)$ measure space.

- $f : X \to \overline{\mathbb{R}}$ or $f : X \to \mathbb{C}$ measurable functions is called *integrable* if $\int |f| < \infty$. Then

$$\int f = \int f^+ - \int f^- \text{ or } \int f = \int u^+ - \int u^- + i \left(\int v^+ - \int v^- \right).$$

- Suppose $f : X \to \overline{\mathbb{R}}$. Define

$$\int f = \begin{cases} \infty & \int f^+ = \infty, \int f^- < \infty, \\ -\infty & \int f^+ < \infty, \int f^- = \infty. \end{cases}$$

Lemma 2.27. *Suppose* $f, g : x \to \overline{\mathbb{R}} \to \mathbb{C}$ *integrable. Assume* $f(x) + g(x)$ *is well-defined* $\forall x \in X$. *(i.e.* $\infty + (-\infty), -\infty + \infty$ *do not occur)*

(a) $f + g$, cf, $c \in \mathbb{C}$ *are integrable.*

(b) $\int f + g = \int f + \int g.$

(c) $\left| \int f \right| \leq \int |f|$. *(This is essentially triangle inequality.)*

Proof. Check [Fol99, p.53]. ∎

Lemma 2.28. $(X, \mathcal{A}, \mu)$ *measure space and* f *integrable function on* X.

(a) f *is finite a.e. (i.e.* $\{x \in X : |f(x)| = \infty\}$ *is a null set)*

(b) *The set* $\{x \in X : f(x) \neq 0\}$ *is* σ-*finite.*

Proof. HW5Q8. ∎

Proposition 2.29. *Suppose* $(X, \mathcal{A}, \mu)$ *a measure space.*

(a) *If h is integrable on X, then*

$$\int_E h = 0, \forall E \in \mathcal{A} \iff \int |h| = 0 \iff h = 0 \text{ a.e.}$$

(b) *If f, g are integrable on X then*

$$\int_E f = \int_E g, \forall E \in \mathcal{A} \iff f = g \text{ a.e.}$$

Proof.   (a) $\int |h| = 0 \iff h = 0$ is shown in 2.23.

$$\int |h| = 0 \implies \left| \int_E h \right| \leq \int_E |h| \leq \int |h| = 0.$$

On the other hand, assume $\int_E h = 0, \forall E \in \mathcal{A}$. $h = u + iv = u^+ - u^- + i(v^+ - v^-)$. Let $B = \{x \mid u^+(x) > 0\}$.

$$0 = \operatorname{Re} \int_B h = \int_B u = \int_B u^+ = \int_B u^+ + \int_{B^c} u^+ = \int u^+ \implies u^+ = 0 \text{ a.e.}$$

Similarly, we get $u^-, v^+, v^- = 0$ a.e..

(b) follows from (a).   ∎

Theorem 2.30 (Dominated convergence theorem). *Suppose $(X, \mathcal{A}, \mu)$ a measure space and*

(a) *f_n integrable on X, $\forall n \in \mathbb{N}$.*

(b) *$\displaystyle\lim_{n \to \infty} f_n(x) = f(x)$ a.e. (pointwise)*

(c) *$\exists g : X \to [0, \infty]$ s.t.*

- *g is integrable.*

- *$|f_n(x)| \leq g(x)$ a.e., $\forall n \in \mathbb{N}$.*

Then

$$\lim_{n \to \infty} \int f_n = \int f.$$

Proof. Let F be the countable union of null sets on which (a)-(c) may fail. Modifying the def of f_n, f, g on F we may assume (a)-(c) hold everywhere. (b)+(c) $\implies f$ is integrable.

We consider $\overline{\mathbb{R}}$-valued case only. ($\mathbb{C}$-valued case follows)

$$g + f_n \geq 0, g - f_n \geq 0$$

$$\xRightarrow{\text{Fatou}} \int g + f \leq \liminf_{n \to \inf} \int g + f_n, \quad \int g - f \leq \liminf_{n \to \inf} \int g - f_n$$

$$\implies \int g + \int f \leq \int g + \liminf_{n \to \infty} \int f_n, \quad \int g - \int f \leq \int g - \limsup_{n \to \infty} \int f_n$$

$$\xRightarrow{\int g < \infty} \int f \leq \liminf_{n \to \infty} \int f_n, \quad -\int f \leq -\limsup_{n \to \infty} \int f_n.$$

$$\implies \int f \leq \liminf_{n \to \infty} \int f_n \leq \limsup_{n \to \infty} \int f_n \leq \int f$$

So we should have

$$\int f = \liminf_{n \to \infty} \int f_n = \limsup_{n \to \infty} \int f_n. \qquad \blacksquare$$

Next we investigate the question:

$$\int \sum_1^\infty f_n \stackrel{?}{=} \sum_1^\infty \int f_n.$$

Tonelli: yes if $f_n \geq 0$. Fubini:

Corollary 2.31 (Fubini's theorem for series and integrals)**.**

$$\left. \begin{array}{c} f_n \text{ integrable} \\ \sum_1^\infty \int |f_n| < \infty \end{array} \right\} \implies \int \sum_1^\infty f_n = \sum_1^\infty \int f_n.$$

Proof. $G(x) = \sum_1^\infty |f_n(x)| \geq |F_N(x)|, F_N(x) = \sum_1^N f_n(x).$ $\qquad \blacksquare$

2.4 L^1 space

Definition 2.32. Suppose V is a vector space over field $\mathbb{R}$ or $\mathbb{C}$. A *seminorm* on V is $\| \cdot \| : V \to [0, \infty)$ *s.t.*

- $\|cv\| = |c| \|v\|, \forall v \in V, \forall c$ scalar

- $\|v + w\| \leq \|v\| + \|w\|$, triangle inequality

A *norm* is a seminorm such that $\|v\| \iff v = 0$.

Lemma 2.33. *A normed vector space is a metric space with metric $\rho(v, w) = \|v - w\|$.*

Proof. (DIY)

- $\rho(v, w) = 0 \iff \|v - w\| = 0 \iff v - w = 0 \iff v = w.$

- $\rho(v, w) = \|v - w\| = \|-1(w - v)\| = |-1| \, \|w - v\| = \rho(w, v).$

- $\rho(v, w) + \rho(w, z) = \|v - w\| + \|w - z\| \geq \|v - w + w - z\| = \|v - z\| = \rho(v, z).$ ∎

Example 2.34. $\mathbb{R}^d$ with $\|x\|_p = \begin{cases} \left(\displaystyle\sum_1^d |x_i|^p \right)^{1/p} & p \in [1, \infty) \\ \displaystyle\max_{1 \leq i \leq d} |x_i| & p = \infty \end{cases}$ is a normed vector space.

Unit ball $\{x : \|x\|_p < 1\}$.

All $\| \cdot \|_p$ norm induce the same topology i.e. if U is open in p-norm then it is open in p'-norm. This implies that a sequence converging under p-norm also converges under p'-norm.

<u>RECALL</u> f is integrable $\implies \int |f| < \infty$. $f = g$ a.e. $\implies \int f = \int g$.

Definition 2.35. Suppose $(X, \mathcal{A}, \mu)$ a measure space.
$f \in L^1(X, \mathcal{A}, \mu) = L^1(X, \mu) = L^1(X) = L^1(\mu)$ means f is an integrable function on X.

Lemma 2.36. $L^1(X, \mathcal{A}, \mu)$ *is a vector space with seminorm* $\|f\|_1 = \displaystyle\int |f|$.

Definition 2.37. Define $f \sim g$ if $f = g$ a.e. $L^1(X, \mathcal{A}, \mu)/_\sim = L^1(X, \mathcal{A}, \mu)$. " $=$ " is just a notation for convenience!

With new definition we have $L^1(X, \mathcal{A}, \mu)$ is a normed vector space. $\rho(f, g) = \displaystyle\int |f - g|$.

Something interesting to discuss is what are the dense subsets of L^1.

Theorem 2.38.

(a) { integrable simple functions } is dense in $L^1(X, \mathcal{A}, \mu)$ *(with respect to L^1 metric)*

(b) $(X, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{A}_\mu, \mu)$, μ *is Lebesgue-Stieltjes measure* $\implies$ { integrable step functions } *is dense in* $L^1(X, \mathcal{A}, \mu)$

(c) $C_c(\mathbb{R})$ *is dense in* $L^1(\mathbb{R}, \mathcal{L}, m)$.

Definition 2.39.

- A step function on $\mathbb{R}$ is $\psi + \sum_1^N c_i 1_{I_i}$, where I_i is an interval.

- $C_c(\mathbb{R})$ is the collection of continuous functions with compact support $\text{supp}(f) = \overline{\{x \in \mathbb{R} \mid f(x) \neq 0\}}$.

Proof. (a) $\exists$ simple functions $0 \leq |\phi_1| \leq |\phi_2| \leq \ldots \leq |f|$, $\phi_n \to f$ pointwise $\implies$

$$\lim_{n \to \infty} \int |\phi_n - f| = 0 \text{ by DCT. } (|\phi_n - f| \leq |\phi_n| + |f| \leq 2|f|)$$

(b) 1_E approx by $\sum_1^N c_i 1_{I_i}$? Regularity theorem for Lebesgue-Stieltjes measure $\implies$ $\forall \varepsilon' > 0, \exists I = \bigcup_1^N I_i$ s.t. $\mu(E \triangle I) < \varepsilon'$.

(c) Suppose $1_{(a,b)}, g \in C_c(\mathbb{R})$. $\int |1_{(a,b)} - g| \, \mathrm{d}m \leq 1 \cdot \dfrac{\varepsilon}{2} + 1 \cdot \dfrac{\varepsilon}{2} = \varepsilon.$ ∎

2.5 Riemann Integrability

Suppose $P = \{a = t_0 < t_1 < \ldots < t_k = b\}$ a partition of $[a, b]$. Lower Riemann sum of f using P

$$L_P = \sum_{i=1}^k \left(\inf_{[t_{i-1}, t_i]} f \right) (t_i - t_{i-1})$$

and upper Riemann sum

$$U_p = \sum_{i=1}^k \left(\sup_{[t_{i-1}, t_i]} f \right) (t_i - t_{i-1})$$

Lower Riemann integral of $f = \underline{I} = \sup_P L_P$. Upper Riemann integral of $f = \overline{I} = \inf_P U_P$.

Definition 2.40. A *bounded* function $f : [a, b] \to \mathbb{R}$ is called Riemann (Darboux) integrable if $\underline{I} = \overline{I}$. (If so, $\underline{I} = \overline{I} = \int_a^b f(x) \, \mathrm{d}x$.)

<u>NOTE</u>

- If $P \subset P'$, then $L_P \leq L_{P'}, U_{P'} \leq U_P$.

- Recall that continuous functions on $[a, b]$ are Riemann integrable on $[a, b]$.

Theorem 2.41. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function.*

(a) *If f is Riemann integrable, then f is Lebesgue measurable. (thus Lebesgue integrable) and*
$$\int_a^b f(x) \, \mathrm{d}x = \int_{[a,b]} f \, \mathrm{d}m.$$

(b) *f is Riemann integrable $\iff$ f is continuous Lebesgue a.e.*

Proof. $\exists$partitions $P_1 \subset P_2 \subset P_3 \subset \ldots$ s.t. $L_{P_n} \nearrow \underline{I}, U_{P_n} \searrow \overline{I}$.

Define simple (step) functions

$$\phi_n = \sum_{i=1}^k \left(\inf_{[t_{i-1}, t_i]} \right) 1_{(t_{i-1}, t_i]}$$

$$\psi_n = \sum_{i=1}^{k} \left(\sup_{[t_{i-1}, t_i]} \right) 1_{(t_{i-1}, t_i]}$$

Define $\phi = \sup_n \phi_n$, $\psi = \inf_n \psi_n$. Then ϕ, ψ are Lebesgue measurable functions.

NOTE

- $\exists M > 0$ s.t. $|\phi_n|, |\psi_n| \leq M 1[a,b], \forall n \in \mathbb{N}$.

- $\int \phi_n \, dm = L_{P_n}, \int \psi_n \, dm = U_{P_n}$.

By DCT, $\underline{I} = \lim_{n \to \infty} \int \phi_n \, dm = \int \phi \, dm, \overline{I} = \int \psi \, dm$.

Thus, f is Riemann integrable $\iff \int \phi = \int \psi \iff \int (\phi - \psi) = 0 \iff \phi = \psi$ Lebesgue a.e.

Recall that $\phi \leq f \leq \psi, \forall x \in (a, b]$. So $f = \phi$ a.e. Since $(\mathbb{R}, \mathcal{L}, \mu)$ is complete, f is Lebesgue measurable (see HW). The second statement hence follows. ∎

2.6   Modes of Convergence

Suppose $f_n, f : X \to \mathbb{C}, S \subset X$.

- $f_n \to f$ *pointwise* on S: $\forall x \in S, \forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \geq \mathbb{N}, |f_n(x) - f(x)| < \varepsilon$.

- $f_n \to f$ *uniformly* on S: $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall x \in X, \forall n \geq \mathbb{N}, |f_n(x) - f(x)| < \varepsilon$.

We can change $\forall \varepsilon > 0$ to $\forall k \in \mathbb{N}$ and bound the distance by $\frac{1}{k}$.

Lemma 2.42. *Let* $B_{n,k} = \{x \in X \mid |f_n(x) - f(x)| < \frac{1}{k}\}$.

(a) $f_n \to f$ *pointwise on* $S \iff S \subset \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} B_{n,k}$.

(b) $f_n \to f$ *uniformly on* $S \iff \exists N_1, N_2, \ldots \in \mathbb{N}$ *s.t.* $S \subset \bigcap_{k=1}^{\infty} \bigcap_{n=N_k}^{\infty} B_{n,k}$.

Definition 2.43. Suppose $(X, \mathcal{A}, \mu)$ a measure space.

(a) $f_n \to f$ a.e means $\exists$ null set E s.t. $f_n \to f$ pointwise on E^c.

(b) $f_n \to f$ in L^1 means $\lim_{n \to \infty} \|f_n - f\| = 0$.

Example 2.44. $(\mathbb{R}, \mathcal{L}, \mu)$. $f = 0$.

(a) $f_n = 1_{(n, n+1)}, f_n = \frac{1}{n} 1_{(0, n)}, f_n = n 1_{(0, \frac{1}{n})}$. All of $f_n \to f$ pointwise but $\nrightarrow f$ in L^1.

(b) Typewriter functions: $f_n \to f$ in L^1. $f_n \nrightarrow f$ a.e.

Proposition 2.45 (Fast L^1 convergence $\implies$ a.e. convergence)**.** *Suppose* $(x, \mathcal{A}, \mu)$ *measure*

space. f_n, f measurable function on X.

$$\sum_{1}^{\infty} \|f_n - f\|_1 < \infty \implies f_n \to f \text{ a.e.}$$

Proof. <u>RECALL</u> Markov's inequality.

Let $E = \bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c = \{x \mid f_n(x) \not\to f(x)\}$. By Markov we have

$$\forall k, \forall N, \mu(B_{n,k}^c) \le k \int |f_n - f|$$

$$\implies \forall k, \mu\left(\bigcap_{n=N}^{\infty} B_{n,k}^c\right) \le \sum_{n=N}^{\infty} k \|f_n - f\|_1 \to 0 \text{ as } n \to 0$$

$$\implies \forall k, \mu\left(\bigcap_{N=1}^{\infty} \bigcap_{n=N}^{\infty} B_{n,k}^c\right) = \lim_{N \to \infty} \mu\left(\bigcap_{n=N}^{\infty} B_{n,k}^c\right) = 0$$

$$\implies \mu(E) = 0. \qquad \blacksquare$$

Corollary 2.46. $f_n \to f$ *in* $L^1 \implies \exists \text{subsequence } f_{n_j} \to f$ *a.e.*

Proof. $\forall j \in \mathbb{N}, \exists n_j \in \mathbb{N} \text{ s.t. } \|f_{n_j} - f\|_1 < \frac{1}{j^2}$. Then $\sum_{j=1}^{\infty} \|f_{n_j} - f\|_1 < \infty$. $\qquad \blacksquare$

Definition 2.47. f_n, f measurable functions on $(X, \mathcal{A}, \mu)$. $f_n \to f$ *in measure* means

$$\forall \varepsilon > 0, \lim_{n \to \infty} \mu\left(\{x \in X \mid |f_n(x) - f(x)| \ge \varepsilon\}\right) = 0.$$

Example 2.48.   • $f_n = n \mathbb{1}_{(0, \frac{1}{n})}, f = 0$.

$$\forall \varepsilon > 0, \{x \mid |f_n(x) - f(x)| > \varepsilon\} = \left(0, \frac{1}{n}\right).$$

(Recall that $f_n \not\to 0$ in L^1.)

• Typewriter function. (Recall that $f_n \not\to 0$ a.e.)

We can easily check that $f_n \to f$ in $L^1 \implies f_n \to f$ in measurable. But the converse is not true.

$f_n \to f$ in measure $\implies \exists f_{n_j} \to f$ a.e. (Check [Fol99])

We have then the following diagram:

$$f_n \to f \text{ fast } L^1 \implies f_n \to f \text{ in } L^1 \overset{\longrightarrow}{\underset{\longleftarrow\!\!\!/}{}} f_n \to f \text{ in measure}$$

$$f_n \to f \text{ a.e.} \qquad\qquad \exists f_{n_j} \to f \text{ a.e.}$$

Definition 2.49. f_n, f measurable functions on $(X, \mathcal{A}, \mu)$.

(a) $f_n \to f$ *uniformly a.e* means $\exists$ null set F *s.t.* $f_n \to f$ uniformly on F^c.

(b) $f_n \to f$ *almost uniformly* means $\forall \varepsilon > 0, \exists F \in \mathcal{A},$ *s.t.* $\mu(F) < \varepsilon, f_n \to f$ uniformly on F^c.

Recall 2.42.

Theorem 2.50 (Egoroff). *f_n, f measurable on $(X, \mathcal{A}, \mu)$. Suppose $\mu(X) < \infty$. Then $f_n \to f$ a.e $\iff f_n \to f$ almost uniformly.*

Proof. " $\impliedby$ ": DIY

" $\implies$ ": Fix $\varepsilon > 0$.

$$f_n \to f \text{ a.e} \implies \mu\left(\bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c\right) = 0 \implies \forall k, \mu\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c\right) = 0.$$

By the continuity of measure from above and since $\mu(X) < \infty$,

$$\forall k, \lim_{N \to \infty} \mu\left(\bigcup_{n=N}^{\infty} B_{n,k}^c\right) = 0 \implies \forall k, \exists N_k \in \mathbb{N}, \mu\left(\bigcup_{n=N_k}^{\infty} B_{n,k}^c\right) < \frac{\varepsilon}{2^k}.$$

Let $F = \bigcup_{k=1}^{\infty} \bigcup_{n=N_k}^{\infty} B_{n,k}^c \implies \mu(F) < \varepsilon, f_n \to F$ uniformly on F^c. ∎

Chapter 3

Product Measures

(p.22 - 36, section 1.2 and section 2.5, 2.6 of [Fol99])
The ultimate goal is to prove Fubini's theorem. This is also related to probability in in the sense that a series of events is in product measure.

3.1 Product σ-algebra

- Product space $X = \prod_{\alpha \in I} X_\alpha, x = (x_\alpha)_{\alpha \in I}$.

- Coordinate map $\pi_\alpha : X \to X_\alpha$.

Definition 3.1. $(X_\alpha, \mathcal{A}_\alpha)$ measurable space. $\forall \alpha \in I$, the *product σ-algebra* on $X = \prod_{\alpha \in I} X_\alpha$ is

$$\bigotimes_{\alpha \in I} \mathcal{A}_\alpha = \left\langle \bigcup_{\alpha \in I} \pi_\alpha^{-1}(\mathcal{A}_\alpha) \right\rangle$$

where

$$\pi_\alpha^{-1}(A_\alpha) = \{\pi_\alpha^{-1}(E) | E \in \mathcal{A}_\alpha\}.$$

<u>NOTATION</u>

$$I = \{1, \ldots, d\} \implies X = \prod_{i=1}^d X_i, x = (x_1, \ldots, x_d), \bigotimes_{i=1}^d \mathcal{A}_i = \mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_d.$$

Lemma 3.2. *If I is countable, then*

$$\bigotimes_{\alpha \in I} \mathcal{A}_\alpha = \left\langle \left\{ \prod_{i=1}^\infty E_i \mid E_i \in \mathcal{A}_i \right\} \right\rangle$$

Lemma 3.3. *Suppose $\mathcal{A}_\alpha = \langle \mathcal{E}_\alpha \rangle, \forall \alpha \in I$.*

(a) $\pi_\alpha^{-1}(\mathcal{A}_\alpha) = \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle$.

(b) $\bigotimes_\alpha \mathcal{A}_\alpha = \left\langle \bigcup_\alpha \pi_\alpha^{-1}(\mathcal{E}_\alpha) \right\rangle$.

(c) *If I is countable, then* $\bigotimes_{\alpha \in I} \mathcal{A}_\alpha = \left\langle \left\{ \prod_{i=1}^\infty E_i \mid E_i \in \mathcal{E}_i \right\} \right\rangle$.

Proof.

(a)   • $f : Y \to Z$, $\mathcal{B}$ a σ-algebra on $Z \implies f^{-1}(\mathcal{B})$ is a σ-algebra since set union commutes with preimage. Hence $\pi_\alpha^{-1}(\mathcal{A}_\alpha)$ is a σ-algebra on X. Since $\pi_\alpha^{-1}(\mathcal{E}_\alpha) \subset \pi_\alpha^{-1}(\mathcal{A}_\alpha) \implies \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle \subset \pi_\alpha^{-1}(\mathcal{A}_\alpha)$.

     • Let $\mathcal{M} = \{ B \subset X_\alpha \mid \pi_\alpha^{-1}(B) \in \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle \}$. We show that $\mathcal{A}_\alpha \subset \mathcal{M}$.

       – $\mathcal{M}$ is a σ-algebra. (easy)

       – $\mathcal{E}_\alpha \subset \mathcal{M}$. (by definition)

     So $\mathcal{A}_\alpha = \langle \mathcal{E}_\alpha \rangle \subset \mathcal{M}$. Hence, if $E \in \mathcal{A}_\alpha$, $E \subset \mathcal{M} \implies \pi_\alpha^{-1}(E) \in \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle$ i.e. $\mathcal{A}_\alpha \subset \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle$.

(b, c) DIY.   ∎

Theorem 3.4. *Suppose $X_1, \ldots, X_d$ metric spaces. Let $X = \prod_1^d X_i$ with product metric $\rho(x, y) = \sum_{i=1}^d \rho_i(x, y)$. Then*

(a) $\bigotimes_{i=1}^d \mathcal{B}(X_i) \subset \mathcal{B}(X)$.

(b) *If, in addition, each X_i has a countable dense subset, then* $\bigotimes_{i=1}^d \mathcal{B}(X_i) = \mathcal{B}(X)$.

Proof. DIY.   ∎

As a consequence, we have $\mathcal{B}(\mathbb{R}^d) = \mathcal{B}(\mathbb{R}) \otimes \ldots \otimes \mathcal{B}(\mathbb{R})$.

Suppose $f = u + iv : X \to \mathbb{C}$. f is measurable $\iff$ $u^{-1}(E) \in \mathcal{A}, v^{-1}(E) \in \mathcal{A}, \forall E \in \mathcal{B}(\mathbb{R})$ $\iff$ $f^{-1}(F) \in \mathcal{A}, \forall F \in \mathcal{B}(\mathbb{C}) = \mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

p.65. Let's focus on finite product.

You like Minecraft right? It's all rectangles.

Definition 3.5. Suppose X, Y sets.

(a) For a $E \subset X \times Y$, $E_x = \{y \in Y \mid (x, y) \in E\}$ and $E^y = \{x \in X \mid (x, y) \in E\}$.

(b) For $f : X \times Y \to \mathbb{C}$, define $f_x : Y \to \mathbb{C}, f^y : X \to \mathbb{C}$ by $f_x(y) = f(x, y) = f^y(x)$.

(c)

Example 3.6. $(1_E)_x = 1_{E_x}$. $(1_E)^y = 1_{E^y}$.

Proposition 3.7. $(X, \mathcal{A}), (Y, \mathcal{B})$ *measurable spaces.*

(a) $E \in \mathcal{A} \otimes \mathcal{B} \implies E_x \in \mathcal{B}, E^y \in \mathcal{A}, \forall x \in X, y \in Y$.

(b) $f : X \times Y \to \mathbb{C}$ *is* $\mathcal{A} \otimes \mathcal{B}$*-measurable* $\implies$ f_x *is* $\mathcal{B}$*-measurable,* f^y *is* $\mathcal{A}$*-measurable,* $\forall x \in X, y \in Y$.

Proof.   (a) Let $\mathcal{F} = \{E \subset X \times Y \mid$ (a) holds$\}$.

- $\mathcal{F}$ is a σ-algebra (easy)

- $\mathcal{R}_0 := \{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\} \subset \mathcal{F}$ (easy) $\implies \mathcal{A} \otimes \mathcal{B} = \langle \mathcal{R}_0 \rangle \subset \mathcal{F}$

(b) DIY.   ∎

MIDTERM is up till here.

3.2   Product Measures

Definition 3.8. Suppose $(X, \mathcal{A}), (Y, \mathcal{B})$. A (measurable) rectangle is $R = A \times B, A \in \mathcal{A}, b \in \mathcal{B}$.

Let $\mathcal{R}_0 := \{R = A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$.

$$\mathcal{R} := \left\{ \bigcup_1^N R_i \mid N \in \mathbb{N}, R_1, \dots, R_N \text{ disjoint rectangles} \right\}.$$

Lemma 3.9. $\mathcal{R}$ *is an algebra.* $\langle \mathcal{R}_0 \rangle = \langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B}$.

Theorem 3.10. *Suppose* $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ *measure spaces.*

(a) *$\exists$ measure $\mu \times \nu$ on $\mathcal{A} \otimes \mathcal{B}$ satisfying $(\mu \times \nu)(A \otimes B) = \mu(A)\nu(B), \forall A \in \mathcal{A}, B \in \mathcal{B}$.*

(b) *If μ, ν are σ-finite, then $\mu \times \nu$ is unique.*

Proof.   (a) Define $\pi : \mathcal{R} \to [0, \infty]$ by $\pi(A \times B) = \mu(A)\nu(B)$ and extend linearly.

<u>CLAIM</u> π is a pre-measure on $\mathcal{R}$.

Enough to check $\pi(A \times B) = \sum_1^\infty \pi(A_n \times B_n)$ if $A \times B = \bigcup_1^\infty (A_n \times B_n)$ disjoint union.

Since $A_n \times B_n$ are disjoint,

$$1_{A \times B}(x, y) = \sum_1^\infty 1_{A_n \times B_n}(x, y), \ 1_A(x)1_B(y) = \sum_1^\infty 1_{A_n}(x)1_{B_n}(y).$$

By Tonelli's theorem for series and integrals, we have

$$\mu(A)1_B(y) = \int_x 1_A(x)1_B(y) \, \mathrm{d}\mu(x)$$
$$= \sum_1^\infty \int_x 1_{A_n}(x)1_{B_n}(y) \, \mathrm{d}\mu(x) = \sum_1^\infty \mu(A_n)1_{B_n}(y).$$

We then integrate with respect to y to complete the claim.

By HK theorem, $\exists \mu \otimes \nu$ on $\langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B}$ extending π on $\mathcal{R}$.

(b) μ, ν σ-finite $\implies \pi$ is σ-finite on $\mathcal{R} \implies$ HK uniqueness them applies.   ∎

So we have a measure

$$(\mu \times \nu)(E) = \inf \left\{ \sum_1^\infty \mu(A_u)\nu(B_i) \middle| E \subset \bigcup_1^\infty A_i \times B_i, A_i \in \mathcal{A}, B_i \in \mathcal{B} \right\}.$$

Then one questions naturally arises: suppose $f : X \times Y \to \mathbb{C}$,

$$\int_{X \times Y} f \, \mathrm{d}(\mu \times v) \overset{?}{=} \int_y \left(\int_x f \, \mathrm{d}\mu \right) \mathrm{d}\nu.$$

3.3   Monotone Class Lemma

Definition 3.11. Suppose X is a set, $\mathcal{C} \subset \mathcal{P}(X)$. $\mathcal{C}$ is a *monotone class* on X if

- closed under *countable increasing unions*
  (i.e. $E_n \in \mathcal{C}, E_1 \subset E_2 \subset \ldots \implies \bigcup_1^\infty C_i \in \mathcal{C}$.)

- closed under *countable decreasing intersections*
  (i.e. $E_n \in \mathcal{C}, E_1 \supset E_2 \supset \ldots \implies \bigcap_1^\infty C_i \in \mathcal{C}$.)

Example 3.12.   • σ-algebra is a monotone class.

- $\bigcap\limits_\alpha \mathcal{C}_\alpha$ is a monotone class $\implies$ if $\mathcal{E} \in \mathcal{P}(X)$, there is unique smallest monotone class containing $\mathcal{E}$.

The importance of this definition shows up in the following theorem:

Theorem 3.13. *Suppose $\mathcal{A}_0$ is an algebra on X. Then $\langle \mathcal{A}_0 \rangle$ is the monotone class generated by $\mathcal{A}_0$.*

Proof. Let $\mathcal{A} = \langle \mathcal{A}_0 \rangle$, $\mathcal{C} = $ monotone class generated by $\mathcal{A}_0$.

(a) $\mathcal{A}$ is a σ-algebra $\implies$ $\mathcal{A}$ is a monotone class containing $\mathcal{A}_0$ $\implies$ $\mathcal{A} \supset \mathcal{C}$.

(b) To show that $\mathcal{C} \supset \mathcal{A}$, we show that $\mathcal{C}$ is a σ-algebra.

  (1) $\emptyset \subset \mathcal{A}_0 \subset \mathcal{C}$.

  (2) Let $\mathcal{C}' = \{E \subset X \mid E^c \subset \mathcal{C}\}$.

   - $\mathcal{C}'$ is a monotone class (easy)

   - $\mathcal{A}_0 \subset \mathcal{C}'$ since $(E \in \mathcal{A}_0 \implies E^c \in \mathcal{A}_0 \subset \mathcal{C})$.

   These two show that $\mathcal{C} \subset \mathcal{C}'$. So $E \in \mathcal{C} \implies E \in \mathcal{C}' \implies E^c \in \mathcal{C}$. So $\mathcal{C}$ is closed under complements.

  (3) For $E \subset X$, let $\mathcal{D}(E) = \{F \in \mathcal{C} \mid E \cup F \in \mathcal{C}\}$.

   - $\mathcal{D}(E) \subset \mathcal{C}$ by definition.

   - $\mathcal{D}(E)$ is a monotone class (easy). $E \cup (\bigcup_1^\infty F_n) = \bigcap_1^\infty (E \cup F_n)$.

   - If $E \in \mathcal{A}_0$, then $\mathcal{A}_0 \subset \mathcal{D}(E)$. $(F \in \mathcal{A}_0 \implies E \cup F \in \mathcal{A}_0 \subset \mathcal{C}.)$

   These show that $\mathcal{C} = \mathcal{D}(E)$ if $E \in \mathcal{A}_0$.

  (4) Let $\mathcal{D} = \{E \in \mathcal{C} \mid \mathcal{D}(E) = \mathcal{C}\} = \{E \in \mathcal{C} \mid E \cup F \in \mathcal{C}, \forall F \in \mathcal{C}\}$.

   - $\mathcal{A}_0 \subset \mathcal{D}$ by (3).

   - $\mathcal{D}$ is a monotone class (easy).

   - $\mathcal{D} \subset \mathcal{C}$ by definition.

   So we conclude that $\mathcal{D} = \mathcal{C}$. Now we have $\mathcal{C}$ is closed under finite unions.

  (5) $\mathcal{C}$ is closed under finite unions and countable increasing unions $\implies$ $\mathcal{C}$ is closed under countable unions. (check) ∎

<u>RECALL</u> $E \in \mathcal{A} \otimes \mathcal{B} \implies E_x \in \mathcal{B}, E^y \in \mathcal{A}, \forall x \in X, y \in Y$. However, the inverse is not necessarily true.

Now comes the main thing:

3.4   Fubini-Tonelli Theorem

Theorem 3.14 (Tonelli for characteristic functions)**.** *Suppose* $(X, \mathcal{A}, \mu)$, $(Y\mathcal{B}, \nu)$ *are* σ-*finite measure spaces. Suppose* $E \in \mathcal{A} \otimes \mathcal{B}$. *Then*

(a) $\alpha(x) := \nu(E_x) : X \to [0, \infty]$ *is a* $\mathcal{A}$-*measurable function.*

(b) $\beta(y) := \mu(E^y) : Y \to [0, \infty]$ *is a* $\mathcal{B}$-*measurable function.*

(c) $(\mu \times \nu)(E) = \displaystyle\int_X \nu(E_x) \, \mathrm{d}\mu(x) = \int_Y \mu\left(E^y\right) \, \mathrm{d}\nu(y).$

Proof.   (a) Assume μ, ν are finite measures. Let

$$\mathcal{C} = \{E \in \mathcal{A} \otimes \mathcal{B} \mid \text{(a), (b), (c) hold}\} \, .$$

Enough to prove that $\langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B} \subset \mathcal{C}$.

Because of monotone class lemma and that $\mathcal{R}$ is a σ-algebra, it is enough to show that $\mathcal{R} \subset \mathcal{C}$ and $\mathcal{C}$ is a monotone class.

- Show that $\mathcal{R} \subset \mathcal{C}$.

$$\alpha(x) = \nu((A \times B)_x) = \begin{cases} \nu(B) & x \in A \\ 0 & x \notin A \end{cases} = \nu(B) 1_A(x).$$

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$$

$$\iff \int_X \nu((A \times B)_x) \, \mathrm{d}\mu(x) = \nu(B)\mu(A)$$

- Show that $\mathcal{C}$ is a monotone class.

(1) Let $E_n \in \mathcal{C}, E_1 \subset E_2 \subset \dots$. Need to show that $E = \bigcup_1^\infty E_n \in \mathcal{C}$.

$$E_n \in \mathcal{C}, E_1 \subset E_2 \subset \dots$$

$$\implies E_x = \bigcup_1^\infty (E_n)_x, (E_1)_x \subset (E_2)_x \subset \dots$$

$$\implies \alpha(x) = \nu(E_x) = \lim_{n \to \infty} \nu\left((E_n)_x\right), \forall x \in X, \quad \alpha_n(x) \; \mathcal{A}\text{-measurable}$$

This satisfies (a), (b). For (c), we have

$$(\mu \times \nu)(E) = \lim_{n \to \infty} (\mu \times \nu)(E_n)$$
$$= \lim_{n \to \infty} \int_X \nu\left((E_n)_x\right) \, \mathrm{d}\mu(x) \overset{MCT}{=} \int_X \nu(E_x) \, \mathrm{d}\mu(X).$$

So we have shown countable increasing unions.

(2) Let $F_n \in \mathcal{C}$, $F_1 \supset F_2 \supset \dots$. Need to show that $F \bigcup_1^\infty F_n \in \mathcal{C}$. Using continuity of measure from above instead of below, DCT instead of MCT, we obtained a similar result.

(b) Now assume that μ, ν are σ-finite. Since $X \times Y = \bigcup_1^\infty (X_n \times Y_n)$, where $X_1 \subset X_2 \dots, Y_1 \subset Y_2 \subset \dots$ with $\mu(X_k), \nu(Y_k)$ finite. Apply results from then finite case. (DIY) ∎

Theorem 3.15 (Fubini-Tonelli). *Suppose $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are σ-finite measure spaces.*

(a) *(Tonelli) If $f : X \times Y \to [0, \infty]$ is $\mathcal{A} \otimes \mathcal{B}$-measurable then*

(1) *$g(x) := \displaystyle\int_Y f(x, y) \, \mathrm{d}\nu(y) : X \to [0, \infty]$ is a $\mathcal{A}$-measurable function.*

(2) *$h(y) := \displaystyle\int_X f(x, y) \, \mathrm{d}\mu(x) : Y \to [0, \infty]$ is a $\mathcal{B}$-measurable function.*

(3) *We have the iterated integral formula*

$$\int_{X \times Y} f \, \mathrm{d}(\mu \times \nu) = \int_X \left[\int_Y f(x, y) \, \mathrm{d}\nu(y) \right] \, \mathrm{d}\mu(x)$$
$$= \int_X \left[\int_X f(x, y) \, \mathrm{d}\mu(x) \right] \, \mathrm{d}\nu(y).$$

(b) *(Fubini) If $f \in L^1(X \times Y, \mu \times \nu)$, then*

(1) *$f_x \in L^1(Y, \nu)$ for μ-a.e x and $g(x)$ (which is defined μ-a.e) $\in L^1(X, \mu)$.*

(2) *$f^y \in L^1(X, \mu)$ for ν-a.e y and $h(y)$ (which is defined ν-a.e) $\in L^1(Y, \nu)$.*

(3) The iterated integral formula from (a).(3) hold.

Usually we apply Tonelli to $|f|$ to show $f \in L^1(X \times Y, \mu \times \nu)$ and then apply Fubini to evaluate.

Proof. See [Fol99].   ∎

3.5 Lebesgue Measure on $\mathbb{R}^d$

Example 3.16 (($\mathbb{R}^2, \mathcal{L} \otimes \mathcal{L}, m \times m$) is not complete). Let $A \in \mathcal{L}, A \neq \emptyset, m(A) = 0$. Let $B \subset [0,1], B \notin \mathcal{L}$ (e.g. Vitali set). Then let $E = A \times B, F = A \times [0,1]$. We can see that $E \subset F$ and $F \in \mathcal{L} \otimes \mathcal{L}, (m \times m)(F) = m(A)m([0,1]) = 0$.

So E is a subnull set but not $\mathcal{L} \otimes \mathcal{L}$-measurable. (otherwise each section of E is measurable, a contradiction.)

Definition 3.17. Let $(\mathbb{R}^d, \mathcal{L}^d, m^d)$ be the *completion* of $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), m \times \ldots \times m)$, which is same(check!) as the *completion* of $(\mathbb{R}^d, \mathcal{L} \otimes \ldots \otimes \mathcal{L}, m \times \ldots \times m)$.

So how do we compute m^d?

A *rectangle* in $\mathbb{R}^d$ is $R = \prod_{i=1}^d E_i$, $E_i \in \mathcal{B}(\mathbb{R})$. Then

$$m^d(E) = \inf \left\{ \sum_1^\infty m^d R_k \ \bigg| \ E \subset \bigcup_1^\infty R_k, R_k \text{ rectangle} \right\}.$$

Theorem 3.18. *Let $E \in \mathcal{L}^d$.*

(a) $m^d(E) = \inf \left\{ m^d(O) \mid \text{open } O \supset E \right\} = \sup \left\{ m^d(K) \mid \text{compact } K \subset E \right\}.$

(b) $E = \underbrace{A_1}_{F\sigma} \cup \underbrace{N_1}_{null} = \underbrace{A_2}_{G\sigma} \setminus \underbrace{N_2}_{null}.$

(c) *If $m^d(E) < \infty, \forall \varepsilon > 0, \exists R_1, \ldots, R_m$ rectangles whose sides are intervals such that*
$$m^d \left(E \triangle \left(\bigcup_1^m R_i \right) \right) < \varepsilon.$$

Proof. Similar to $d = 1$ case.   ∎

Theorem 3.19. *Integrable "step functions" and $C_c(\mathbb{R}^d)$ are dense in $L^1(\mathbb{R}^d, \mathcal{L}^d, m^d)$.*

Theorem 3.20. *Lebesgue measure in $\mathbb{R}^d$ is translation-invariant.*

Theorem 3.21. *"Effect of linear transformations on Lebesgue measure"*

Skip p. 71-81 of [Fol99] except 3.21.

Chapter 4

Differentiation on Euclidean Space

Suppose $f : [a, b] \to \mathbb{R}$. There are two versions of fundamental theorem of Calculus:

- $\displaystyle \int_a^b f'(x) \, \mathrm{d}x = f(b) - f(a).$

- $\displaystyle \frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t) dt = f(x).$

We focus on the second statement, which implies that

$$\lim_{r \to 0^+} \frac{1}{r} \int_x^{x+r} f(t) \, \mathrm{d}t = \lim_{r \to 0^+} \frac{1}{r} \int_{x-r}^x f(t) \, \mathrm{d}t$$

Write $f(x) = \dfrac{1}{r} \displaystyle\int_x^{x+r} f(x) \, \mathrm{d}t$, then

$$\lim_{r \to 0^+} \frac{1}{r} \int_x^{x+r} (f(t) - f(x)) \, \mathrm{d}t = \lim_{r \to 0^+} \frac{1}{r} \int_{x-r}^x (f(t) - f(x)) \, \mathrm{d}t.$$

This generalizes well in $\mathbb{R}^d$:

$$f : \mathbb{R}^d \to \mathbb{R}, \quad \lim_{r \to 0^+} \frac{1}{v(B(x,r))} \int_{B(x,r)} f(t) - f(x) \, \mathrm{d}t = 0.$$

QUESTION to what extent does this hold?

Start from [Fol99, 3.4].

4.1   Hardy-Littlewood Maximal Function

Suppose an open ball in $\mathbb{R}^d$, $B = B(a, r)$. Denote $cB = B(a, cr), c > 0$.

Lemma 4.1 (Vitali-type covering lemma). *Let $B_1, \ldots, B_k$ be a finite collection of open balls in R^d. Then $\exists$ a sub-collection $B_1', \ldots, B_m'$ of* disjoint *open balls such that*

$$\bigcup_1^m (3B_j') \supset \bigcup_1^k B_i.$$

Proof.   Greedy algorithm.   ■

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