Notes for Math 597 – Real Analysis

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Contents

Office hour is Mon 12:30 - 1:30, Tue 12:30 - 1:30 in person EH 5838, Th 1 - 2 online.

Chapter 1

Abstract Measure

1.1 σ-Algebra

Definition 1.1. Let X be a set. A collection $\mathcal{M}$ of subsets of X is called a σ-algebra on X if

- $\emptyset \in \mathcal{M}$.
- $\mathcal{M}$ is closed under *complements*: $E \in \mathcal{M} \implies E^c \in \mathcal{M}$.
- $\mathcal{M}$ is closed under *countable unions*: $E_1, E_2, \ldots \in \mathcal{M} \implies \bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$.

<u>SIMPLE PROPERTIES:</u>

- $X = \emptyset^c \in \mathcal{M}$.
- $\bigcap_{i=1}^{\infty} E_i = \left(\bigcup_{i=1}^{n} E_i^c\right)^c \in \mathcal{M}$. It is closed under countable intersections.
- $\bigcup_{i=1}^{N} E_i = E_i \cup \ldots \cup E_n \cup \emptyset \cup \ldots$. It is closed under finite unions (similarly, intersections). sigma
- $E \setminus F = E \cap F^c \in \mathcal{M}, E \triangle F = (E \cap F^c) \cup (F \cap E^c) \in \mathcal{M}$.

Example 1.2. (a) $\mathcal{A} = \mathcal{P}(X)$ power algebra.

(b) $\mathcal{A} = \{\emptyset, X\}$ trivial algebra.

(c) Let $B \subset X, B \neq \emptyset, B \neq X. \mathcal{A} = \{\emptyset, B, B^c, X\}$.

Lemma 1.3. *(An intersection of σ-algebras is a σ-algebra) Let $\mathcal{A}_\alpha, \alpha \in I$, be a family a σ-algebras of X. Then $\bigcap_{\alpha \in I} A_\alpha$ is a σ-algebra. (I can be uncountable.)*

Proof. DIY ■

Definition 1.4. For $\mathcal{E} \subset \mathcal{P}(X)$ (not necessarily a σ-algebra), let $\langle \mathcal{E} \rangle$ be the intersection of all σ-algebras on X that contains $\mathcal{E}$. Call it the σ-algebra generated by $\mathcal{E}$.

- $\langle \mathcal{E} \rangle$ is the *smallest* σ-algebra containing $\mathcal{E}$ and is *unique*.
- $\{\emptyset, B, B^c, X\} = \langle \{B\} \rangle = \langle \{B^c\} \rangle = \langle \{\emptyset, B\} \rangle$.

The above definition gives us (potentially) lots of examples of σ-algebra on a set X

Lemma 1.5. *(a) Suppose $\mathcal{E} \subset \mathcal{P}(X), \mathcal{A}$ is a σ-algebra on X. $\mathcal{E} \in \mathcal{A} \implies \langle \mathcal{E} \rangle \in \mathcal{A}$.*

(b) $E \subset F \subset \mathcal{P}(X) \implies \langle \mathcal{E} \rangle \subset \langle \mathcal{F} \rangle$.

Proof. ∎

Definition 1.6. For a topological space X, *the Borel σ-algebra* $\mathcal{B}(X)$ is the σ-algebra generated by the collection of open sets.

Example 1.7. $(X = \mathbb{R})$ $\mathcal{B}(\mathbb{R})$ contains the following collections:

$$\begin{aligned}
\mathcal{E}_1 &= \{(a,b) \mid a < b\}, & \mathcal{E}_2 &= \{[a,b] \mid a < b\}, \\
\mathcal{E}_3 &= \{(a,b] \mid a < b\}, & \mathcal{E}_4 &= \{[a,b) \mid a < b\}, \\
\mathcal{E}_5 &= \{(a,\infty) \mid a \in \mathbb{R}\}, & \mathcal{E}_6 &= \{[a,\infty) \mid a \in \mathbb{R}\}, \\
\mathcal{E}_7 &= \{(-\infty,a) \mid a \in \mathbb{R}\}, & \mathcal{E}_8 &= \{(-\infty,a] \mid a < b\}.
\end{aligned}$$

Proposition 1.8. *$\mathcal{B}(\mathbb{R}) = \langle \mathcal{E}_i \rangle$ for each $i = 1, \dots, 8$.*

Proof. Use 1.5. ∎

Definition 1.9. $(X, \mathcal{A})$ is called a measurable space.

1.2 Measures

Definition 1.10. A measure on $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \to [0, \infty]$ $s.t.$

(a) $\mu(\emptyset) = 0$

(b) (countable additive) For $A_1, A_2, \ldots \in \mathcal{A}$ disjoint we have

$$\mu\left(\bigcup_1^\infty A_i\right) = \sum_{i=1}^\infty \mu(A_i).$$

$(X, \mathcal{A}, \mu)$ is then called a measure space.

Example 1.11. (a) For any $(X, \mathcal{A}), \mu(A) = \#A$ counting measure.

(b) For any $(X, \mathcal{A})$, let $x_0 \in X$. The Dirac measure at x_0 is

$$\mu(A) = \begin{cases} 1 & x_0 \in A, \\ 0 & x_0 \notin A. \end{cases}$$

(c) For $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, let $a_1, a_2, \ldots \in [0, \infty)$. $\mu(A) = \sum_{i \in A} a_i$ is a measure.

$(X, \mathcal{A})$ measurable space

$(X, \mathcal{A}, \mu)$ measure space

$\mu : \mathcal{A} \to [0, \infty]$ $s.t.$ $\mu(\emptyset) = 0$, countable additivity.

<u>Note</u>: $A, B \in \mathcal{A}, A \subset B$, then $\mu(B \setminus A) + \mu(A) = \mu(B) \implies \mu(B \setminus A) = \mu(B) - \mu(A)$ if $\mu(A) < \infty$.

Theorem 1.13. *Suppose $(X, \mathcal{A}, \mu)$ a measure space. Then*

(a) (monotonicity)

$$A, B \in \mathcal{A}, A \subset B \implies \mu(A) \leq \mu(B).$$

(b) (countable subadditivity)

$$A_1, A_2, \ldots, \in \mathcal{A}, \implies \mu\left(\bigcup_i^\infty A_i\right) \leq \sum_i^\infty \mu(A_i).$$

(c) (continuity from below/(MCT) from sets)

$$A_1, A_2, \ldots \in \mathcal{A}, A_1 \subset A_2 \subset A_3 \subset \ldots \implies \mu\left(\bigcup_i^\infty A_i\right) = \lim_{n\to\infty} \mu(A_n).$$

(d) (continuity from above)

$$A_1, A_2, \ldots \in \mathcal{A}, A_1 \supset A_2 \supset A_3 \supset \ldots, \mu(A_1) < \infty \implies \mu\left(\bigcap_i^\infty A_i\right) = \lim_{n\to\infty} \mu(A_n).$$

Proof. (a), (b), DIY.

For (c), let $B_1 = A_1, B_i = A_i \setminus A_{i-1}, i \geq 2. B_i \in \mathcal{A}$ and are disjoint.

$$\begin{aligned} &\bigcup_i^\infty A_i = \bigcup_i^\infty B_i \\ \implies &\mu\left(\bigcup_i^\infty A_i\right) = \mu\left(\bigcup_i^\infty B_i\right) = \sum_i^\infty \mu(B_i) = \lim_{n\to\infty} \sum_i^n \mu(B_i) = \lim_{n\to\infty} \mu(A_n). \end{aligned}$$

For (d), let $E_i = A_1 \setminus A_i$. Hence $E_i \in \mathcal{A}, E_1 \subset E_2 \subset \dots$ We have

$$\bigcup_i^\infty E_i = \bigcup_i^\infty (A_1 \setminus A_i) = A_1 \setminus \left(\bigcap_1^\infty A_i\right) \implies \bigcap_1^\infty A_i = A_1 \setminus \left(\bigcup_1^\infty E_i\right).$$

Hence

$$\mu\left(\bigcap_1^\infty A_i\right) = \mu(A_1) - \mu\left(\bigcup_1^\infty E_i\right) = \mu(A_1) - \lim_{n\to\infty} \mu(E_n) = \mu(A_1) - \lim_{n\to\infty} \mu(A_1) - \mu(A_n).$$

■

<u>NOTE:</u> the condition that $\mu(A_1) < \infty$ cannot be dropped.
For example, in $(\mathbb{N}, \mathcal{P}(N)$, counting measure), let $A_n = \{n, n+1, n+2\}, A_1 \supset A_2 \supset A_3 \supset \dots$ We have $\bigcap_1^\infty = \emptyset \implies \mu\left(\bigcap_1^\infty A_i\right) = 0$.

Definition 1.14. For $(X, \mathcal{A}, \mu)$ measure space,

- $A \subset X$ is a μ-null set if $A \in \mathcal{A}, \mu(A) = 0$.
- $A \subset X$ is a μ-subnull set if $\exists B, \mu$-null set $A \subset B$.
- $(X, \mathcal{A}, \mu)$ is a complete measure space if every μ-subnull set is $\mathcal{A}$-measurable.

Definition 1.15. $(X, \mathcal{A}, \mu)$ measure space. A statement $P(x), x \in X$ holds μ-almost everywhere (a.e.) if the set $\{x \in X \mid P(x)\text{does not hold}\}$ is μ-null.

Definition 1.16. $(X, \mathcal{A}, \mu)$ measure space.

- μ is a *finite measure* is $\mu(X) < \infty$.
- μ is a *σ-finite measure* if $X = \bigcup_1^\infty X_n, X_n \in \mathcal{A}, \mu(X_n) < \infty$.

HW: every measure space can be "completed."

1.3 Outer Measures

Definition 1.17. An *outer measure* on X is $\mu^* : \mathcal{P}(X) \to [0, \infty]$ $s.t.$

- $\mu^*(\emptyset) = 0$
- (monotonicity) $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$.
- (countable subadditivity)

$$\forall A_1, A_2, \ldots \in X, \mu^* \left(\bigcup_i^\infty A_i \right) \leq \sum_i^\infty \mu^*(A_i).$$

Example 1.18. For $A \subset \mathbb{R}$,

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^\infty (b_i - a_i) \,\middle|\, \bigcup_1^\infty (a_i, b_i) \supset A \right\}.$$

is an outer measure due to the next proposition.

Proposition 1.19. *(1.19) Let $\mathcal{E} \in \mathcal{P}(X)$ s.t. $\emptyset, X \in \mathcal{E}$. Let $\rho : \mathcal{E} \to [0, \infty]$ s.t. $\rho(\emptyset) = 0$. Then*

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^\infty \rho(E_i) \,\middle|\, E_i \in \mathcal{E}, \forall i \in N, \bigcup_1^\infty E_i \supset A \right\}$$

is an outer measure on X.

Proof. (a) μ^* is well-defined (inf is taken over non-empty set.)

(b) $\mu^*(\emptyset) = 0$

(c) $A \subset B \implies \mu^*(A) \leq \mu^*(B)$.

We check the countable subadditivity.

Let $A_1, A_2, \ldots \subset X$. If one of $\mu^*(A_i) = \infty$, then the result holds. Suppose $\mu^*(A_n) < \infty, \forall n \in \mathbb{N}$.

"Give your self a room of epsilon":

Fix $\varepsilon > 0$. We will show

$$\mu^* \left(\bigcup_1^\infty A_n \right) \leq \sum_1^\infty \mu^*(A_i) + \varepsilon.$$

For each $n \in \mathbb{N}, \exists E_{n,1}, E_{n,2}, \ldots \in \mathcal{E}$ s.t.

$$\bigcup_{k=1}^\infty E_{n,k} \supset A_n \quad \text{and} \quad \mu^*(A_n) + \frac{\varepsilon}{2^n} \geq \sum_{k=1}^\infty \rho(E_{n,k}).$$

Then,

$$\bigcup_1^\infty A_n \subset \bigcup_{n=1}^\infty \bigcup_{k=1}^\infty E_{n,k} = \bigcup_{(n,k)\in\mathbb{N}^2} E_{n,k}.$$

<u>RECALL:</u> Tonelli's thm for series. If $a_{ij} \in [0,\infty], \forall i,j \in \mathbb{N}$, then

$$\sum_{(i,j)\in\mathbb{N}^2} a_{ij} = \sum_{i=1}^\infty \sum_{j=1^\infty} a_{ij} = \sum_{j=1}^\infty \sum_{i=1}^\infty a_i j.$$

Hence

$$\mu^*\left(\bigcup_{n=1}^\infty A_n\right) \le \sum_{n=1}^\infty \rho(E_{k,n}) = \sum_{n=1}^\infty \sum_{k=1}^\infty \rho(E_{k,n}) \le \sum_{n=1}^\infty \left(\mu^*(A_n) + \frac{\varepsilon}{2^n}\right) = \sum_{n=1}^\infty \mu^*(A_n) + \varepsilon.$$

We have shown countable subadditivity. ∎

Outer measure is very close to a measure. Here the textbooks diverge.

[Tao11] introduces Lebesgue measure on $\mathbb{R}$ using topological qualities of subsets of $\mathbb{R}$. [Fol99] introduces abstract method by Carathéodory and Kolmogorov.

Definition 1.20. Let μ^* be an outer measure on X. We say $A \subset X$ is Carathéodory measurable with respect to μ^* if $\forall E \subset X, \mu^*(E) = \mu^*(E \setminus A) + \mu^*(E \cap A)$.

Lemma 1.21. *Let μ^* be an outer measure on X. Suppose $B_1, B_2, \ldots, B_N$ are disjoint C-measurable sets. Then,*

$$\forall E \subset X, \mu^*\left(E \cap \left(\bigcup_1^N B_i\right)\right) = \sum_{i=1}^n \mu^*(E \cap B_i)$$

Proof.

$$\mu^*\left(E \cap \left(\bigcup_1^N B_i\right)\right) = \mu^*(E \cap B_1) + \mu^*\left(E \cap \left(\bigcup_2^N B_i\right)\right)$$

because B_1 is C-measurable. Then, iterate. ∎

Improved version:

$B_1, B_2, \ldots$ C-measurable and *disjoint* $\implies \mu^*\left(E \cap \bigcup_1^\infty B_n\right) = \sum_1^\infty \mu^*\left(E \cap B_n\right), \forall E \subset X$.

Proof.

$$\begin{aligned}\sum_1^\infty \mu^*(E\cap B_n) &\geq \mu^*\left(E\cap\bigcup_1^\infty B_n\right)\\ &\geq \mu^*\left(E\cap\bigcup_1^N B_n\right) = \sum_1^N \mu^*(E\cap B_n.)\end{aligned}$$

Take $N\to\infty$ or note that $N\in\mathbb{N}$ is arbitrary we get the result. ■

First big theorem:

Theorem 1.22 (Carathéodory extension theorem)**.** *Let μ^* be an outer measure on X. Let $\mathcal{A}$ be the collection of C-measurable sets with respect to μ^*. Then*

(a) $\mathcal{A}$ us a σ-algebra on X.

(b) $\mu=\mu^*|_{\mathcal{A}}$ is a measure on $(X,\mathcal{A})$.

(c) $(X,\mathcal{A},\mu)$ is a complete measure space.

Proof. (a) (1) $\emptyset\in\mathcal{A}$.

(2) $\mathcal{A}$ is closed under complements.

(3) To show $\mathcal{A}$ closed under countable unions.

- (finite union)
  CLAIM $A,B\in\mathcal{A}\implies A\cup B\in\mathcal{A}$.

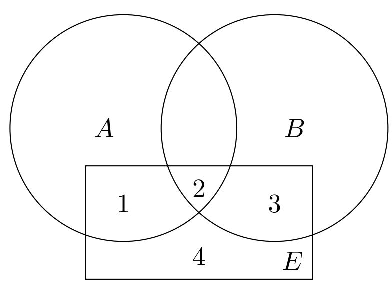

Figure 1.1: Venn diagram of A,B,E

Fix arbitrary $E\subset X$. We need to show

$$\mu^*(E)=\mu^*(E\cap(A\cup B))+\mu^*(E\setminus(A\cup B)).$$

i.e.

$$\mu^*(1\cup 2\cup 3\cup 4)=\mu^*(1\cup 2\cup 3)+\mu^*(4)$$

Since A is C-measurable, we have

$$\mu^*(1 \cup 2 \cup 3 \cup 4) = \mu^*(1 \cup 2) + \mu^*(3 \cup 4)$$

$$\mu^*(1 \cup 2 \cup 3) = \mu^*(1 \cup 2) + \mu^*(3)$$

Similarly since B is C-measurable, we have

$$\mu^*(3 \cup 4) = \mu^*(3) + \mu^*(4)$$

Hence

$$\begin{aligned}
\mu^*(1 \cup 2 \cup 3 \cup 4) &= \mu^*(1 \cup 2) + \mu^*(3 \cup 4) \\
&= \mu^*(1 \cup 2 \cup 3) - \mu^*(3) + \mu^*(3) + \mu^*(4) \\
&= \mu^*(1 \cup 2 \cup 3) + \mu^*(4).
\end{aligned}$$

- (countable disjoint unions)
  Let $A_1, A_2, \ldots \in \mathcal{A}$ and *disjoint*.

  Fix $E \subset X$ arbitrary. Since μ^* is countably subadditive,

  $$\mu^*(E) \leq \mu^*\left(E \cap \bigcup_1^\infty\right) + \mu^*\left(E \setminus \bigcup_1^\infty A_n\right)$$

  Fix $n \in \mathbb{N}$.

  $$\begin{aligned}
  &\implies \bigcup_1^N A_n \in \mathcal{A} \\
  &\implies \mu^*(E) = \mu^*\left(E \cap \bigcup_1^N\right) + \mu^*\left(E \setminus \bigcup_1^N A_n\right) \\
  &\geq \sum_1^N \mu^*(E \cap A_n) + \mu^*\left(E \setminus \bigcup_1^\infty A_n\right) \text{ by lemma.}
  \end{aligned}$$

  Take $n \to \infty$.

- (countable unions)
  Let $A_1, A_2, \ldots \in \mathcal{A}$. Take $E_1 = A_1, E_n = A_n \setminus \left(\bigcup_1^{n-1} A_i\right)$ for $n \geq 2$. Then $\bigcup A_n = \bigcup E_n$ and E_n's are disjoint.

(b) Firstly we have $\mu(\emptyset) = \mu^*(\emptyset) = 0$.

Countable additvity of μ^* on $\mathcal{A}$ follows from the improved lemma with $E = X$.

(c) HW. ■

1.4 Hahn-Kolmogorov Theorem

<u>RECALL</u> 1.19 Let $\mathcal{E} \subset \mathcal{P}(X)$ $s.t.$ $\emptyset, X \in \mathcal{E}$. Let $\rho : \mathcal{E} \to [0, \infty]$ $s.t.$ $\rho(\emptyset) = 0$

$$(\mathcal{E}, \rho) \xrightarrow[1.19]{} (\mathcal{P}(X), \mu^*) \xrightarrow[\text{C-theorem}]{} (A, \mu)$$

<u>QUESTION</u> $\mathcal{E} \subset \mathcal{A}$ and $\mu|_{\mathcal{E}} = \rho$? No!

Definition 1.23. Let $\mathcal{A}_0$ be an algebra on X. We say $\mu_0 : \mathcal{A}_0 \to [0, \infty]$ is a *pre-measure* if

(a) $\mu_0(\emptyset) = 0$.

(b) (finite additivity)

$$\mu_0 \left(\bigcup_1^N A_i 1 \right) = \sum_1^N \mu_0(A_i) \text{ if } A_1, \ldots, A_N \in \mathcal{A}_0 \text{ are disjoint.}$$

(c) (countable additivity within the algebra) If $A \in \mathcal{A}_0$ and

$$A = \bigcup_1^\infty A_n, A_n \in \mathcal{A}_0 \text{ and are disjoint, then } \mu_0(A) = \sum_1^\infty \mu_0(A_n)$$

<u>NOTATION:</u> Folland uses $\mathcal{M}$ for σ-algebra and $\mathcal{A}$ for algebra. (Jinho) uses $\mathcal{A}$ for σ-algebra and $\mathcal{A}_0$ for alegbra.

Example 1.24. $\mathcal{A}_0$ finite disjoint unions of $(a, b]$.

$$\mu_0 \left(\bigcup_1^\infty (a_i, b_i] \right) = \sum_1^\infty (b_i - a_i) \text{ or } b_i^n - a_i^n, e^{b_i} - e^{a_i}, \text{ etc.}$$

Lemma 1.25.

- *(a) + (c) $\implies$ (b).*
- *μ_0 is monotone.*

Theorem 1.26 (Hahn-Kolmogorov Theorem)**.** *Let μ_0 be a pre-measure on algebra $\mathcal{A}_0$ on X. Let μ^* be the outer measure induced by $(\mathcal{A}_0, \mu_0)$ in 1.19. Let $\mathcal{A}$ and μ be the Carathéodory σ-algebra and measure for μ^* $\implies$ $(\mathcal{A}, \mu)$ extends $(\mathcal{A}_0, \mu_0)$ $i.e.$ $\mathcal{A} \supset \mathcal{A}_0, \mu|_{\mathcal{A}_0} = \mu_0$.*

Proof. (a) ($\mathcal{A} \supset \mathcal{A}_0$) Let $A \in \mathcal{A}_0$.

Question: $A \in \mathcal{A}$? i.e. is A C-measurable? i.e. $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c), \forall E \subset$

X.

Fix $E \subset X$.

- (countable) subadditivity of $\mu^* \implies \mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c)$.
- If $\mu^*(E) = \infty$ then $\mu^*(E) = \infty \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$.
- If $\mu^*(E) < \infty$.

  Fix $\varepsilon > 0$. By the definition of $\mu^*, \exists B_1, B_2, \ldots \in \mathcal{A}_0\ s.t.\ \bigcup_1^\infty B_n \supset E$ and

$$\mu^*(E) + \varepsilon \geq \sum_1^\infty \mu_0(B_n) = \sum_1^\infty \left(\mu_0(B_n \cap A) + \mu_0\left(B_n \cap A^c\right)\right).$$

  Note that

$$\bigcup_1^\infty (B_n \cap A) \supset E \cap A, \quad \bigcup_1^\infty (B_n \cap A^c) \supset E \cap A^c \implies \geq$$

(b) Let $A \in \mathcal{A}_0$. We want to show that $\mu(A) = \mu_0(A)$.

By definition, $\mu(A) = \mu^*(A)$.

- Let $B_i = \begin{cases} A & i = 1, \\ \emptyset & i = 2 \end{cases} \in \mathcal{A}_0$ and $\bigcup_1^\infty B_i \supset A$.

  Hence $\mu^*(A) \leq \sum_1^\infty \mu_0(B_i) = \mu_0(A)$.
- Let $B_i \in \mathcal{A}_0, \bigcup_1^\infty B_i \supset A$ an arbitrary collection of sets.

  Let $C_1 = A \cap B_1, C_i = A \cap B_i \setminus \left(\bigcup_{j=1}^{i-1} B_j\right)$. Then $A = \bigcup_1^\infty$ is a disjoint countable union. By countable additivitiy we have

$$\mu_0(A) = \sum_1^\infty \mu_0(C_i) \implies \mu_0(A) \leq \sum_1^\infty \mu_0(B_i).$$

Hence we have $\mu_0(A) = \mu^*(A) = \mu(A)$. We have completed our proof. ■

Definition 1.27. Such $(\mathcal{A}, \mu)$ is called the *Hahn-Kolmogorov extension* of $(\mathcal{A}_0, \mu_0)$, and is also called the *Carathéodory σ-algebra* for $(\mathcal{A}_0, \mu_0)$.

Theorem 1.28 (uniqueness of HK extension)**.** *Let $\mathcal{A}_0$ be an algebra on X, μ_0 be a pre-measure on $\mathcal{A}_0$, $(\mathcal{A}, \mu)$ be the Hahn-Kolmogorov extension of $(\mathcal{A}_0, \mu_0)$. And let $(\mathcal{A}', \mu')$ be another extension of $(\mathcal{A}_0, \mu_0)$.*

*If μ_0 is σ-*finite*, then $\mu \mid_{\mathcal{A} \cap \mathcal{A}'} = \mu' \mid_{\mathcal{A} \cap \mathcal{A}'}$.*

NOTE σ-finite means

$$\forall X, X = \bigcup_1^\infty X_n, X_n \in \mathcal{A}_0, \mu_0(X_n) < \infty.$$

Corollary 1.29. *Let μ_0 be a pre-measure on algebra $\mathcal{A}_0$ on X. Suppose μ_0 is σ-finite, then $\exists!$ measure μ on $\langle \mathcal{A}_0 \rangle$ that extends $\mathcal{A}_0$. Furthermore,*

(a) the completion of $(X, \langle \mathcal{A}_0 \rangle, \mu)$ is the HK extension of $(\mathcal{A}_0, \mu_0)$.

(b)

$$\mu(A) = \inf \left\{ \sum_{i=1}^\infty \mu_0(B_i) \mid B_i \subset A_0, \forall i \in \mathbb{N}, \bigcup_1^\infty B_i \supset A \right\}, \forall A \in \overline{\langle \mathcal{A}_0 \rangle}.$$

Proof of 1.28. Let $A \in \mathcal{A} \cap \mathcal{A}'$. We need to show $\mu(A) = \mu^*(A) = \mu'(A)$.

- $\mu^*(A) \geq \mu'(A)$ (HW)
- $\mu(A) \leq \mu'(A)$:
  (i) Assume $\mu(A) < \infty$. Fix $\varepsilon > 0$. Then $\exists B_i \in \mathcal{A}_0, \forall i \in \mathbb{N}, \bigcup_1^\infty B_i \supset A$ $s.t.$

  $$\mu(A) + \varepsilon = \mu^*(A) + \varepsilon \geq \sum_1^\infty \mu_0(B_i) = \sum_1^\infty \mu(B_i) \geq \mu\left(\bigcup_1^\infty B_i\right) = \mu(B)$$

  Hence $\mu(B \setminus A) = \mu(B) - \mu(A) \leq \varepsilon$.

  On the other hand,

  $$\mu(B) = \lim_{N\to\infty} \mu\left(\bigcup_1^N B_i\right) = \lim_{N\to\infty} \mu'\left(\bigcup_1^N B_i\right) = \mu'(B)$$

  by continuity of measure from below.

  $$\mu(A) \leq \mu(B) = \mu'(B) = \mu'(A) + \mu'(B \setminus A) \leq \mu'(A) = \varepsilon.$$

  (ii) Assume $\mu(A) = \infty$.

  Since μ_0 is σ-finite, $X = \bigcup_1^\infty X_n, X_n \in \mathcal{A}_0, \mu_0(X_0) < \infty$. Replacing X_n by $X_1 \cup \ldots \cup X_n$, we may assume $X_1 \subset X_2 \subset \ldots$.

  $$\forall n \in N, \mu(A \cap X_n) < \infty \implies \mu(A \cap X_n) \leq \mu'(A \cap X_n).$$

Hence
$$\mu(A) = \lim_{N\to\infty} \mu(A \cap X_n) \le \lim_{N\to\infty} \mu'(A \cap X_n) = \mu'(A). \qquad \blacksquare$$

1.5 Borel Measures on $\mathbb{R}$

Definition 1.30. $F : \mathbb{R} \to \mathbb{R}$ is an *increasing* function if $F(x) \le F(y)$ for $x < y$. $F : \mathbb{R} \to \mathbb{R}$ is increasing and right-continuous $\implies$ F is distribution function.

Example 1.31.

- $F(x) = \begin{cases} 1, & x \ge 0 \\ 0, & x < 0. \end{cases}$
- $\mathbb{Q} = \{r_1, r_2, \dots\}, F_n(x) = \begin{cases} 1 & x \ge r_n \\ 0 & x < r_n \end{cases}$. $F(x) = \displaystyle\sum_{n=1}^{\infty} \frac{F_n(x)}{2^n}$ is a distribution function.

<u>NOTE</u> If F is increasing, $F(\infty) := \lim_{x\to\infty} F(x), F(-\infty) := \lim_{x\to-\infty} F(x)$ exists in $[-\infty, \infty]$.

In probability theory, cumulative distribution function (CDF) is a distribution function with $F(\infty) = 1$ and $F(-\infty) = 0$.

There are distributions [Fol99, Ch.9], but these are different from *distribution functions*.

Definition 1.32. Suppose X a topological space. μ on $(X, \mathcal{B}(X))$ is called *locally finite* is $\mu(K) < \infty$ for any compact set $K \subset X$.

Lemma 1.33. *Let μ be a locally finite Borel measure on $\mathbb{R}$ $\implies$*

$$F_\mu(x) = \begin{cases} \mu((0, x]), & x > 0 \\ 0, & x = 0 \\ -\mu((x, 0]), & x < 0 \end{cases} \text{ is a distribution function.}$$

Proof. DIY. Use continuity of measure. $\blacksquare$

Definition 1.34. h-intervals are $\emptyset, (a, b], (a, \infty), (-\infty, b], (\infty, \infty)$.

Lemma 1.35. *Let $\mathcal{H}$ be the collections of finite disjoint unions of h-intervals. Then $\mathcal{H}$ is an algebra on $\mathbb{R}$.*

Proof. DIY. $\blacksquare$

Proposition 1.36 (Distribution function defines a pre-measure)**.** *Let $F : \mathbb{R} \to \mathbb{R}$ be a distribution function. For an h-interval I, define*

$$\ell(I) = \ell_F(I) = \begin{cases} 0, & I = \emptyset \\ F(b) - F(a), & I = (a,b] \\ F(\infty) - F(a), & I = (a,\infty) \\ F(b) - F(\infty), & I = (-\infty, b] \\ F(\infty) - F(-\infty), & I = (-\infty,\infty). \end{cases}$$

Define $\mu_0 = \mu_{0,F} : \mathcal{H} \to [0,\infty]$ by

$$\mu_0(A) := \sum_{k=1}^{N} \ell(I_k) \quad \text{if } A = \bigcup_{k=1}^{N} I_k, \text{ finite disjoint union of } h\text{-intervals.}$$

Then μ_0 is a pre-measure.

Proof. (a) μ_0 is well-defined.

(b) μ_0 is finite additive.

(c) μ_0 is countable additive within $\mathcal{H}$.

Suppose $A \in \mathcal{H}$ and $A = \bigcup_1^\infty A_i$ a disjoint union, $A_i \in \mathcal{H}$. It is enough to consider the case $A = I, A_k = I_k$ all h-intervals. (Why?)

Focus on the case $I = (a,b]$: (HW: check other cases)
We have

$$(a,b] = \bigcup_1^\infty (a_n, b_n], \text{ a disjoint union.}$$

Check

$$F(b) - F(a) \stackrel{?}{=} \sum_1^\infty (F(b_n) - F(a_n))$$

$(a,b] \supset \bigcup_1^N (a_n, b_n] \implies F(b) - F(a) \geq \sum_1^N F(b_n) - F(a_n), \forall N \in \mathbb{N}$. (Arranging them in decreasing order) Take $N \to \infty$ we have

$$F(b) - F(a) \geq \sum_1^\infty (F(b_n) - F(a_n)).$$

Since F is right-continuous, $\exists a' > a$ $s.t.$ $F(a') - F(a) < \varepsilon$. For each $n \in \mathbb{N}$, $\exists b_n' >$

b_n $s.t.$ $F(b_n') - F(b_n) < \frac{\varepsilon}{2^n}$.

$$\begin{aligned}
&\implies [a', b] \subset \bigcup_1^\infty (a_n, b_n') \\
&\implies \exists N \in \mathbb{N}\ s.t.\ [a', b] \subset \bigcup_1^n (a_n, b_n') \\
&\implies F(b) - F(a') \leq \sum_1^N F(b_n') - F(a_n) \\
&\implies F(b) - F(a) \leq F(b) - F(a') + \varepsilon \leq \sum_1^\infty (F(b_n') - F(a_n)) + \varepsilon \\
&\quad \leq \sum_1^\infty \left(F(b_n) - F(a_n) + \frac{\varepsilon}{2^n}\right) + \varepsilon
\end{aligned}$$ ■

Once we have this pre-measure, HK theorem allows us to extended it to a measure.

Theorem 1.37 (Locally finite Borel measures on $\mathbb{R}$)**.**

(a) $F : \mathbb{R} \to \mathbb{R}$ *is a distribution function* $\implies$ $\exists!$ *locally finite Borel measure* μ_F *on* $\mathbb{R}$ *satisfying* $\mu_F((a, b]) = F(b) - F(a), \forall a, b, a < b$.

(b) Suppose $F, G : \mathbb{R} \to \mathbb{R}$ *are distribution functions. Then,* $\mu_F = \mu_G$ *on* $\mathcal{B}(\mathbb{R})$ *if and only if* $F - G$ *is a constant function.*

Proof. HW ■

1.6 Lebesgue-Stieltjes Measures on $\mathbb{R}$

F distribution function $\implies$ μ_F on Carathéodory σ-algebra $\mathcal{A}_{\mu_F}$.
Actually $(\mathcal{A}_{\mu_F}, \mu_F) = (\mathcal{B}(\mathbb{R}), \mu_F)$ (HW3).

Definition 1.38.

- μ_F on $\mathcal{A}_{\mu_F}$ is called the Lebesgue-Stieltjes measure corresponding to F.
- Special case: $F(x) = x \implies$ Lebesgue measure $(\mathcal{B}, m)$.

Example 1.39.

(a) $\mu_F((a, b]) = F(b) - F(a)$. F is right-continuous and increasing $\implies$ $F(x_-) \leq F(x) = F(x_+)$.

(HW) $\mu_F(\{a\}) = F(a) - F(a_-), \mu_F([a, b]) = F(b) - F(a_-), \mu_F((a, b)) = F(b_-) - F(a)$.

(b)

$$F(x) = \begin{cases} 1 & x \leq 0 \\ 0 & x < 0 \end{cases} \implies \mu_F(\{0\}) = 1, \mu_F(\mathbb{R}) = 1, \mu_F(\mathbb{R} \setminus \{0\}) = 0.$$

μ_F is the Dirac measure at 0.

(c)

$$\mathbb{Q} = \{r_1, r_2, \ldots\},\ F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n},\ F_n(x) = \begin{cases} 1 & x \leq r_n \\ 0 & x < r_n \end{cases}$$
$$\implies \mu_F(\{v\}) > 0, \forall v \in \mathbb{Q},\ \mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0.$$

(d) If F is continuous at a, $\mu_F(\{a\}) = 0$.

(e) $F(x) = x \implies m((a, b])) = m((a, b)) = m([a, b]) = b - a.$

(f) $F(x) = e^x, \implies \mu_F((a, b]) = \mu_F((a, b)) = e^b - e^a$

(a), (b) are examples of discrete measure.

Example 1.40 (Middle thirds Cantor set $\mathcal{C} = \bigcup_{n=1}^{\infty} K_n$**).**

$\mathcal{C}$ is uncountable set with $m(\mathcal{C}) = 0$.

$$x \in \mathcal{C} \implies x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}, a_n \in \{0, 2\}.$$

We are interested in the Cantor function F.

Example 1.41. Cantor function F is continuous and increasing. This defines the Cantor measure $\mu_F(\mathbb{R} \setminus \mathcal{C}) = 0, \mu_F(\mathcal{C}) = 1, \mu_F(\{a\}) = 0$. Compare with Lebesgue measure $m(\mathbb{R} \setminus \mathcal{C}) = \infty > 0, \mu(\mathcal{C}) = 0, m(\{a\}) = 0$.

1.7 Regularity Properties of Lebesgue-Stieltjes Measures

Lemma 1.42. *μ is Lebesgue-Stieltjes measure on $\mathbb{R}$ $\implies$*

$$\begin{aligned} \mu(A) &= \inf \left\{ \sum_1^{\infty} \mu((a_i, b_i]) \,\middle|\, \bigcup_1^{\infty} (a_i, b_i] \supset A \right\} \\ &= \inf \left\{ \sum_1^{\infty} \mu((a_i, b_i)) \,\middle|\, \bigcup_1^{\infty} (a_i, b_i) \supset A \right\} \end{aligned}$$

Proof. Using the continuity of measure. ■

Theorem 1.43. *μ is a Lebesgue-Stieltjes measure. Then $\forall A \in \mathcal{A}_\mu$,*

(a) (outer regularity)
$$\mu(A) = \inf\{\mu(O) \mid \textit{open } O \supset A\}.$$

(b) (inner regularity)
$$\mu(A) = \sup\{\mu(K) \mid \textit{compact } K \subset A\}.$$

Proof. (a) Followed from 1.42.

(b) Let $s = \sup\{\dots\}$. Monotonicity $\implies \mu(A) \geq s$.

- (A bounded) $\overline{A} \in \mathcal{B}(\mathbb{R}) \subset \mathcal{A}_\mu, \overline{A}$ bounded $\implies \mu(\overline{A}) < \infty$.

  Fix $\varepsilon > 0$. By 1, $\exists$ open $O \supset \overline{A} \setminus A, \mu(O) - \mu(\overline{A} \setminus A) = \mu(O \setminus (\overline{A} \setminus A)) \leq \varepsilon$.

  Let $K = \underbrace{A \setminus O}_{K \subset A} = \underbrace{\overline{A} \setminus O}_{\text{compact}}$. Show that $\mu(K) \geq \mu(A) - \varepsilon$.

- (A unbounded but $\mu(A) < \infty$) We have
$$A = \bigcup_1^\infty A_n,\ A_n = A \cap [-n, n],\ A_1 \subset A_2 \subset \dots$$
  Hence
$$\lim_{n\to\infty} \mu(A_n) = \mu(A) < \infty.$$

- ($\mu(A) = \infty$)
$$\lim_{n\to\infty} \mu(A_n) = \mu(A) = \infty.$$
  Fix $L > 0$. $\exists N\ s.t.\ \mu(A_N) \geq L$. ■

Definition 1.44. Suppose X a topological space.

A G_σ*-set* is $G = \bigcap_1^\infty O_i$, O_i open. An F_σ*-set* is $F = \bigcup_1^\infty F_i$, F_i closed.

Theorem 1.45. *Suppose μ a LS measure. Then the following statements are equivalent:*

(a) $A \in \mathcal{A}_\mu$.

(b) $A = G \setminus M$, G is a G_σ-set, and M is μ-null.

(c) $A = F \cup N$, F is an F_σ-set, and N is μ-null.

Proof. (b) $\implies$ (a) and (c) $\implies$ (a) are clear.

- (a) $\implies$ (c)
  (i) Assume $\mu(A) < \infty$. By inner regularity,
  $$\forall n \in \mathbb{N}, \exists \text{ compact } K_n \subset A \ s.t.\ \mu(K_n) + \frac{1}{n} \geq \mu(A).$$
  Let $F = \bigcup_1^\infty K_n$. Then $N = A \setminus F$ is μ-null.
  (ii) Assume $\mu(A) = \infty$. We construct
  $$A = \bigcup_{k \in \mathbb{Z}} A_k, A_k = A \cap (k, k+1].$$
  By (i), $\forall k \in \mathbb{Z}, A_k = F_k \cup N_k$. Hence
  $$A = \underbrace{\left(\bigcup_k F_k\right)}_{F\sigma} \cup \underbrace{\left(\bigcup_k N_k\right)}_{\mu\text{-null}}.$$
- (a) $\implies$ (b)
  $$A^c = F \cup N, A = F^c \cup N^c = F^c \setminus N.$$ ■

Proposition 1.46. *Suppose μ a LS measure, $A \in \mathcal{A}_\mu$, $\mu(A) < \infty$. Then*
$$\forall \varepsilon > 0, \exists I = \bigcup_1^{N=N(\varepsilon)} I_i, \text{ disjoint open intervals } s.t.\ \mu(A \triangle I) \leq \varepsilon.$$

Proof. DIY - use outer regularity. ■

Properties of Lebesgue measure

Theorem 1.47.
$$A \in \mathcal{L} \implies A + s \in \mathcal{L}, rA \in \mathcal{L}, \forall r, s \in \mathbb{R}.$$
In addition, $m(A + r) = m(A)$ *and* $m(rA) = rm(A)$.

Proof. DIY. ■

Example 1.48.

(a) $\mathbb{Q} = \{r_1\}_{i=1}^\infty$, which is dense in $\mathbb{R}$. Let $\varepsilon > 0$ and
$$O = \bigcup_{i=1}^\infty \left(r_i - \frac{\varepsilon}{2^i}, r_i + \frac{\varepsilon}{2^i}\right).$$

O is open and dense in $\mathbb{R}$. We have

$$m(O) \leq \sum_{i=1}^{\infty} \frac{2\varepsilon}{2^i} = 2\varepsilon, \partial O = \overline{O} \setminus O, m(O) = \infty.$$

(b) $\exists$ uncountable set A with $m(A) = 0$.

(c) $\exists A$ with $m(A) > 0$, but A contains no non-empty open interval.

(d) $\exists A \notin \mathcal{L}$ that is Vitali set.

(e) $\exists A \in \mathcal{L} \setminus \mathcal{B}(\mathbb{R})$. We will deal with that later.

Chapter 2

Integration

2.1 Measurable Functions

Definition 2.1. Suppose $(X, \mathcal{A})$, $(Y, \mathcal{B})$ two measurable spaces. $f : X \to Y$ is $(\mathcal{A}, \mathcal{B})$-measurable if

$$\forall B \in \mathcal{B}, f^{-1}(B) \in \mathcal{A}.$$

Lemma 2.2. *Suppose $\mathcal{B} = \langle \mathcal{E} \rangle$. Then*

$$f : X \to Y \textit{ is } (\mathcal{A}, \mathcal{B})\textit{-measurable} \iff \forall E \in \mathcal{E}, f^{-1}(E) \in \mathcal{A}.$$

Proof. $\Longrightarrow$ clear

$\Longleftarrow$ Let $\mathcal{D} = \{E \subset Y \mid f^{-1}(E) \in \mathcal{A}\}$. We have $\mathcal{E} \subset \mathcal{D}$ by assumption. In addition $\mathcal{D}$ is a σ-algebra $\implies \langle \mathcal{E} \rangle \subset \mathcal{D}$. ∎

Definition 2.3. Suppose $(X, \mathcal{A})$ a measurable space.

$$\left.\begin{array}{l} f : X \to \mathbb{R} \\ f : X \to \overline{\mathbb{R}} = [-\infty, \infty] \\ f : X \to \mathbb{C} \end{array}\right\} \text{ is } \mathcal{A}\text{-measurable if } \left\{\begin{array}{l} f \text{ is } (\mathcal{A}, \mathcal{B}(\mathbb{R}))\text{-measurable} \\ f \text{ is } (\mathcal{A}, \mathcal{B}(\overline{\mathbb{R}}))\text{-measurable} \\ \operatorname{Re} f, \operatorname{Im} f : X \to \mathbb{R} \text{ are } \mathcal{A}\text{-measurable.} \end{array}\right.$$

Here $\mathcal{B}(\overline{\mathbb{R}}) = \{E \subset \overline{\mathbb{R}} \mid E \cap R \in \mathcal{B}(\mathbb{R})\}$.

Lemma 2.4. *Suppose $f : X \to \mathbb{R}$. Then the followings are equivalent:*

(a) f *is* $\mathcal{A}$*-measurable*

(b) $\forall a \in \mathbb{R}, f^{-1}((a,\infty)) \in \mathcal{A}$.

(c) $\forall a \in \mathbb{R}, f^{-1}([a,\infty)) \in \mathcal{A}$.

(d) $\forall a \in \mathbb{R}, f^{-1}((-\infty,a)) \in \mathcal{A}$.

(e) $\forall a \in \mathbb{R}, f^{-1}((-\infty,a]) \in \mathcal{A}$.

For $f : X \to \overline{\mathbb{R}}$, change the interval to include $-\infty$ and ∞.

Proof. By 2.2. ■

Example 2.5. $\mathcal{A} = \mathcal{P}(X) \implies$ every function is $\mathcal{A}$ measurable.

$\mathcal{A} = \{\emptyset, X\} \implies$ only $\mathcal{A}$ functions are constant functions.

PROPERTIES Suppose $f, g : X \to \mathbb{R}$, $\mathcal{A}$-measurable functions.

(a) $\phi : \mathbb{R} \to \mathbb{R}$, $\mathcal{B}(\mathbb{R})$ measurable (i.e. Borel measurable) $\implies \phi \circ f : X \to \mathbb{R}$ is $\mathcal{A}$-measurable.

(b) $-f, 3f, f^2, |f|$ are $\mathcal{A}$-measurable, $\frac{1}{f}$ is $\mathcal{A}$-measurable if $f(x) = 0, \forall x \in X$.

(c) $f + g$ is $\mathcal{A}$-measurable

$$(f+g)^{-1}((a,\infty)) = \bigcup_{r\in\mathbb{Q}} \left(f^{-1}((r,\infty)) \cap g^{-1}((a-r,\infty))\right).$$

(d) fg is $\mathcal{A}$-measurable

$$f(x)g(x) = \frac{1}{2}\left((f(x)+g(x))^2 - f(x)^2 - g(x)^2\right).$$

(e) $(f \wedge g)(x) = \min\{f(x), g(x)\}$, $(f \vee g)(x) = \max\{f(x), g(x)\}$ are $\mathcal{A}$-measurable.

(f) $f_n : X \to \overline{\mathbb{R}}$ are a sequence of $\mathcal{A}$-measurable functions $\implies$

$$\sup f_n, \inf f_n, \limsup_{n\to\infty} f_n, \liminf_{n\to\infty} f_n \text{ are } \mathcal{A}\text{-measurable.}$$

(g) If $f(x) = \lim_{n\to\infty} f_n(x)$ converges for every $x \in X$, then f is measurable.

Example 2.6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. Then f is Borel measurable $\implies f$ is Lebesgue measurable. (Preimage of an open set of a continuous function is open.)

Definition 2.7. For $f : X \to \overline{\mathbb{R}}$, let $f^+ = f \vee 0$, $f^- = (-f) \vee 0$.

NOTE $\operatorname{supp} f^+ \cap \operatorname{supp} f^- = \emptyset$. $f(x) = f^+(x) - f^-(x)$. f is $\mathcal{A}$-measurable $\iff f^+, f^-$ measurable.

Definition 2.8. For $E \subset X$, characteristic (indicator) funtion of E

$$\chi_E(x) = 1_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \in E^c. \end{cases}$$

1_E is $\mathcal{A}$-measurable $\iff E \in \mathcal{A}$.

Definition 2.9. Suppose $(X, \mathcal{A})$ a measurable space. A *simple function* $\phi : X \to \mathbb{C}$ that is $\mathcal{A}$-measurable and takes only finitely many values.

$$\phi(X) = \{c_1, \ldots, c_N\}, c_i \neq \pm\infty, E_i = \phi^{-1}(c_i) \in \mathcal{A} \implies \phi = \sum_{i=1}^{N} c_i 1_{E_i}.$$

Theorem 2.10. *Suppose $(X, \mathcal{A})$ a measurable space and $f : X \to [0, \infty]$. Then the followings are equivalent:*

(a) f is $\mathcal{A}$-measurable.

(b) $\exists$ simple functions $0 \leq \phi_1(x) \leq \phi_2(x) \leq \ldots \leq f(x)$ such that

$$\lim_{n \to \infty} \phi_n(x) = f(x),\ \forall x \in X.$$

(f is the pointwise upward limit of simple functions.)

Proof.

- (b) $\implies$ (a) is easy: $f(x) = \sup_{n \in \mathbb{N}} \phi_n(x)$.
- (a) $\implies$ (b): suppose f is $\mathcal{A}$-measurable.

  Fix $n \in \mathbb{N}$. Let $F_n = f^{-1}([2^n, \infty]) \in \mathcal{A}$. For

  $$0 \leq k \leq 2^{2n} - 1,\ E_{n,k} = f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right) \in \mathcal{A}.$$

  Let $\phi_n(x) = \displaystyle\sum_{k=0}^{2^{2n}-1} 1_{E_{n,k}} + 2^n 1_{F_n}$.

  This shows that

  - $0 \leq \phi_1(x) \leq \phi_2(x) \leq \ldots \leq f(x),\ \forall x \in X$.
  - $\forall x \in X \setminus F_n, 0 \leq f(x) - \phi_n(x) \leq \dfrac{1}{2^n}$.

  Since $F_1 \supset F_2 \supset \ldots$ and $\displaystyle\bigcap_1^\infty F_n = f^{-1}(\{\infty\})$, we have

– $x \in f^{-1}([0, \infty)) = X \setminus \left(\bigcap_1^\infty F_n \right) \implies \lim_{n \to \infty} \phi_n(x) = f(x)$.

– $x \in f^{-1}(\{\infty\}) = \bigcap_1^\infty X_n \implies \phi_n(x) \geq 2^n \implies \lim_{n \to \infty} \phi_n(x) = \infty = f(x)$. ■

Corollary 2.11. *If f is bounded on a set $A \subset \mathbb{R}$ (i.e. $\exists L > 0$ s.t. $|f(x)| \leq L, \ \forall x \in A$) then $\phi_n \to f$ uniformly on A.*

Proof. DIY. ■

Corollary 2.12. *$f : X \to \mathbb{C}$, measurable function $\iff$ $\exists$ simple functions $\phi_n : X \to \mathbb{C}$ s.t. $0 \leq |\phi_1| \leq |\phi_2| \leq \ldots \leq |f|$ and ϕ_n converges to f pointwise. (Again, if f is bounded the convergence can be uniform.)*

2.2 Integration of Nonnegative Functions

Definition 2.13. Suppose $(X, \mathcal{A}, \mu)$ a measure space and $\phi = \sum_{i=1}^N c_i 1_{E_i} : X \to [0, \infty]$ a simple function. Let

$$\int \phi = \int \phi \, \mathrm{d}\mu = \int_X \phi \, \mathrm{d}\mu = \sum_1^N c_i \mu(E_i).$$

Proposition 2.14. *Suppose $\phi, \psi \geq 0$ are simple functions. Then,*

- *2.13 is well-defined.*
- $\int c\phi = c \int \phi, c \in [0, \infty)$.
- $\int (\phi + \psi) = \int \phi + \int \psi$.
- $\phi(x) \geq \psi(x), \ \forall x \implies \int \phi \geq \int \psi$.
- $\nu(A) = \int_A \phi \, \mathrm{d}\mu$ *is a measure on* $(X, \mathcal{A})$.

Proof. DIY. ■

Definition 2.15. Suppose $(X, \mathcal{A}, \mu)$, $f : X \to [0, \infty]$ is $\mathcal{A}$-measurable.

Define

$$\int f = \int f \, \mathrm{d}\mu = \sup \left\{ \int \phi \mid 0 \leq \phi \leq f, \phi \text{ simple} \right\}.$$

Proposition 2.16.

- *If f is a simple function then two definitions are the same.*
- $\displaystyle\int cf = c\int f.$
- $f \geq g \geq 0 \implies \displaystyle\int f \geq \int g.$
- $\displaystyle\int f + g = \int f + \int g.$ *(A bit harder to check)*

Theorem 2.17 (Monotone convergence theorem)**.** *Suppose $(X, \mathcal{A}, \mu)$ a measure space and*

- $f : X \to [0, \infty]$ *is $\mathcal{A}$-measurable,* $\forall n \in \mathbb{N}$.
- $0 \leq f_1(x) \leq \ldots$.
- $\displaystyle\lim_{n\to\infty} f_n(x) = f(x).$

Then

$$\int f = \lim_{n\to\infty} \int f_n.$$

Proof. Note that $\lim_{n\to\infty} f_n(x)$ converges $\forall x \in X$ and $\lim_{n\to\infty} f_n(x)$ converges.

- $f_n \leq f \implies \displaystyle\int f_n \leq \int f \implies \lim_{n\to\infty} \int f_n \leq \int f.$
- Fix simple function $0 \leq \phi \leq f$. Enough to show that $\displaystyle\lim_{n\to\infty} \int f_n \geq \int \phi.$

  Now fix $\alpha \in (0, 1)$. Enough to prove that $\displaystyle\lim_{n\to\infty} \int f_n \geq \alpha \int \phi.$

  Let $A_n = \{x \mid f_n(x) \geq \alpha\phi(x)\}$.

  - $A_n \in \mathcal{A}$.
  - $A_1 \subset A_2 \subset \ldots$
  - $\displaystyle\bigcup_{n=1}^{\infty} A_n = X.$ (check!)

  So we have

  $$\int f_n \geq \int f_n 1_{A_n} \geq \int \alpha\phi 1_{A_n} = \alpha\nu(A_n)$$

  where $\nu(A) = \int_A \phi$ is a measure.

  $$\implies \lim_{n\to\infty} \int f_n \geq \lim_{n\to\infty} \nu(A_n) = \alpha\nu(x) = \alpha \int \phi.$$ ■

Corollary 2.18. $f, g \geq 0$ *measurable* $\implies \displaystyle\int f + g = \int f + \int g.$

Proof. $\exists$ simple functions $0 \leq \phi_1 \leq \phi_2 \leq \ldots, \phi_n \to f$ pointwise and $0 \leq \psi_1 \leq \psi_2 \leq \ldots, \psi_n \to g$ pointwise.

By MCT, we have

$$\int (f+g) = \lim_{n\to\infty} \int (\phi_n + \psi_n) = \lim_{n\to\infty} \int \phi_n + \int \psi_n = \int f + \int g. \qquad \blacksquare$$

Corollary 2.19 (Tonelli's theorem for series and integrals)**.** *Given* $s_n \geq 0, \forall n \in \mathbb{N}$ *measurable functions. Then*

$$\int \sum_{n=1}^{\infty} s_n = \sum_{n=1}^{\infty} \int s_n.$$

Proof. Let $f_N = \sum_{n=1}^{N} s_n, 0 \leq f_1 \leq f_2 \leq \ldots$.

$$\lim_{N\to\infty} f_N(x) = \sum_{n=1}^{\infty} s_n(x)$$

By MCT, we have

$$\lim_{N\to\infty} \sum_{1}^{N} s_n = \sum_{1}^{\infty} s_n$$

$\blacksquare$

Theorem 2.20 (Fatou's lemma)**.** *Suppose* $f_n \geq 0$ *measurable. Then*

$$\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n.$$

Recall that

$$\liminf_{n\to\infty} f_n := \lim_{k\to\infty} \inf_{n\geq k} f_n = \sup_{k\in\mathbb{N}} \inf_{n\geq k} f_n,$$

and

$$\lim_{n\to\infty} a_n \text{ exists} \iff \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n.$$

Proof. Let $g_k = \inf_{n\geq k} f_n \implies s_k$ measurable and $0 \leq g_1 \leq g_2 \leq \ldots$. By MCT, we have

$$\int \liminf_{n\to\infty} = \int \lim_{k\to\infty} s_k = \lim_{k\to\infty} \int s_k = \lim_{k\to\infty} \int \inf_{n\geq k} f_n$$

$$\begin{aligned}
&\inf_{n\geq k} f_n \leq f_m, \forall m \geq k \\
\implies &\int \inf_{n\geq k} f_n \leq \int f_m, \forall m \geq k \\
\implies &\int \inf_{n\geq k} f_n \leq \inf_{m\geq k} \int f_m
\end{aligned}$$

■

Example 2.21. Suppose $(\mathbb{R}, \mathcal{L}, m)$

(a) (escape to horizontal infinity) $f_n = 1_{(n,n+1)}$.

We see that $f_n \to 0 = f$ pointwise and $\int f_n = 1, \forall n, \int f = 0$.

(b) (escape to width infinity) $f_n = \frac{1}{n} 1_{(0,n)}$.

(c) (escape to vertical infinity) $f_n = n 1_{(0,1/n)}$.

Lemma 2.22 (Markov's inequality). *$f \geq 0$ is measurable* $\implies$

$$\forall c \in (0,\infty),\ \mu\left(\{x \mid f(x) \geq c\}\right) \leq \frac{1}{c}\int f.$$

Proof. Let $E = \{x \mid f(x) \geq c\}$. Then

$$f(x) \geq c 1_E(x) \implies \int f \geq c \int 1_E = c\mu(E).$$ ■

Proposition 2.23. *Suppose $f \geq 0$ measurable. Then $\int f = 0 \iff f = 0$ almost everywhere (a.e.)*

$$\int f \,\mathrm{d}\mu = \mu(A) = 0,\ A = \{x \mid f(x) > 0\} = f^{-1}((0,\infty])$$

Proof. (a) Assume $f = \phi$ a simple function. We may assume

$$\phi = \sum_{i=1}^{N} c_i 1_{E_i},\ c_i \in (0,\infty),\ E_i\text{'s are disjoint.}$$

$$\begin{aligned}
&\int \phi = \sum_{i=1}^{N} c_i \mu(E_i) = 0 \\
\iff &\mu(E_1) = \ldots = \mu(E_N) = 0 \\
\iff &\mu(A) = 0,\ A = \bigcup_{i=1}^{N} E_i.
\end{aligned}$$

(b) General $f \geq 0$.

(1) Assume $\mu(A) = 0$ (i.e. $f = 0$ a.e.)

Let $0 \leq \phi \leq f, \phi$ is simple.

$$\begin{aligned} &\implies \phi(x) = 0, \ \forall x \in A^c \\ &\implies \phi = 0 \text{ a.e.} \\ &\implies \int \phi = 0 \end{aligned}$$

Then $\int f = 0$ by the definition of $\int f$.

(2) Assume $\inf f = 0$. Let $A_n = f^{-1}\left(\left[\frac{1}{n}, \infty\right]\right)$

$$\begin{aligned} &\implies A_1 \subset A_2 \subset \dots \\ &\quad \bigcup_1^\infty A_n = f^{-1}\left(\bigcup_1^\infty \left[\frac{1}{n}, \infty\right]\right) = f^{-1}((0,\infty)) = A \\ &\quad \mu(A_n) = \mu\left(\left\{x \mid f(x) \geq \frac{1}{n}\right\}\right) \leq n \int f = 0 \\ &\implies \mu(A) = \lim_{n\to\infty} \mu(A_n) = 0 \end{aligned}$$

by the continuity of measure from below. ■

Corollary 2.24. *$f, g \geq 0$ are measurable, $f = g$ a.e. $\implies \int f = \int g$.*

Proof. Let $A = \{x \mid f(x) \geq g(x)\}$. A is measurable (why?). By assumption $\mu(A) = 0$. Hence $f 1_A = 0$ a.e.

$$\begin{aligned} \int f &= \int f(1_A + 1_{A^c}) \\ &= \int f 1_A + \int f 1_{A^c} \\ &= \int f 1_{A^c} \\ &= \int g 1_{A^c} = \int g 1_A + \int g 1_{A^c} = \int g. \end{aligned}$$

■

Corollary 2.25. *$f_n \geq 0$ measurable. Then*

(a)

$$\left.\begin{aligned} 0 \leq f_1 \leq f_2 \leq \ldots \leq f \text{ a.e.} \\ \lim_{n\to\infty} f_n = f \text{ a.e.} \end{aligned}\right\} \implies \lim_{n\to\infty} f_n = \int f.$$

(b)

$$\lim_{n\to\infty} f_n = f \ a.e \implies \int f \le \liminf_{n\to\infty} \int f_n.$$

2.3 Integration of Complex Functions

I was afraid that you are bored.

— Jinho Baik on homework

Definition 2.26. $(X, \mathcal{A}, \mu)$ measure space.

- $f : X \to \overline{\mathbb{R}}$ or $f : X \to \mathbb{C}$ measurable functions is called *integrable* if $\int |f| < \infty$. Then

$$\int f = \int f^+ - \int f^- \text{ or } \int f = \int u^+ - \int u^- + i\left(\int v^+ - \int v^-\right).$$

- Suppose $f : X \to \overline{\mathbb{R}}$. Define

$$\int f = \begin{cases} \infty & \int f^+ = \infty, \int f^- < \infty, \\ -\infty & \int f^+ < \infty, \int f^- = \infty. \end{cases}$$

Lemma 2.27. *Suppose $f, g : x \to \overline{\mathbb{R}} \to \mathbb{C}$ integrable. Assume $f(x) + g(x)$ is well-defined $\forall x \in X$. (i.e. $\infty + (-\infty)$, $-\infty + \infty$ do not occur)*

(a) *$f + g$, cf, $c \in \mathbb{C}$ are integrable.*

(b) $\displaystyle\int f + g = \int f + \int g.$

(c) $\displaystyle\left|\int f\right| \le \int |f|$. *(This is essentially triangle inequality.)*

Proof. Check [Fol99, p.53]. ∎

Lemma 2.28. *$(X, \mathcal{A}, \mu)$ measure space and f* integrable *function on X.*

(a) *f is finite a.e. (i.e. $\{x \in X : |f(x)| = \infty\}$ is a null set)*

(b) *The set $\{x \in X : f(x) \neq 0\}$ is σ-finite.*

Proof. HW5Q8. ∎

Proposition 2.29. *Suppose $(X, \mathcal{A}, \mu)$ a measure space.*

(a) If h is integrable on X, then

$$\int_E h = 0, \forall E \in \mathcal{A} \iff \int |h| = 0 \iff h = 0 \text{ a.e.}$$

(b) If f, g are integrable on X then

$$\int_E f = \int_E g, \forall E \in \mathcal{A} \iff f = g \text{ a.e.}$$

Proof. (a) $\int |h| = 0 \iff h = 0$ is shown in 2.23.

$$\int |h| = 0 \implies \left| \int_E h \right| \leq \int_E |h| \leq \int |h| = 0.$$

On the other hand, assume $\int_E h = 0, \forall E \in \mathcal{A}$. $h = u + iv = u^+ - u^- + i(v^+ - v^-)$. Let $B = \{x \mid u^+(x) > 0\}$.

$$0 = \operatorname{Re} \int_B h = \int_B u = \int_B u^+ = \int_B u^+ + \int_{B^c} u^+ = \int u^+ \implies u^+ = 0 \text{ a.e.}$$

Similarly, we get $u^-, v^+, v^- = 0$ a.e..

(b) follows from (a). ∎

Theorem 2.30 (Dominated convergence theorem)**.** *Suppose $(X, \mathcal{A}, \mu)$ a measure space and*

(a) f_n integrable *on X, $\forall n \in \mathbb{N}$.*

(b) $\lim_{n\to\infty} f_n(x) = f(x)$ *a.e. (pointwise)*

(c) $\exists g : X \to [0, \infty]$ *s.t.*

- *g is integrable.*
- *$|f_n(x)| \leq g(x)$ a.e., $\forall n \in \mathbb{N}$.*

Then

$$\lim_{n\to\infty} \int f_n = \int f.$$

Proof. Let F be the countable union of null sets on which (a)-(c) may fail. Modifying the def of f_n, f, g on F we may assume (a)-(c) hold everywhere. (b)+(c) $\implies$ f is integrable.

We consider $\overline{\mathbb{R}}$-valued case only. ($\mathbb{C}$-valued case follows)

$$\begin{aligned}
&g + f_n \geq 0, g - f_n \geq 0 \\
\xrightarrow{\text{Fatou}} &\int g + f \leq \liminf_{n\to\inf} \int g + f_n, \quad \int g - f \leq \liminf_{n\to\inf} \int g - f_n \\
\implies &\int g + \int f \leq \int g + \liminf_{n\to\infty} \int f_n, \quad \int g - \int f \leq \int g - \limsup_{n\to\infty} \int f_n \\
\xrightarrow{\int g<\infty} &\int f \leq \liminf_{n\to\infty} \int f_n, \quad -\int f \leq -\limsup_{n\to\infty} \int f_n. \\
\implies &\int f \leq \liminf_{n\to\infty} \int f_n \leq \limsup_{n\to\infty} \int f_n \leq \int f
\end{aligned}$$

So we should have
$$\int f = \liminf_{n\to\infty} \int f_n = \limsup_{n\to\infty} \int f_n.$$
■

Next we investigate the question:

$$\int \sum_1^\infty f_n \stackrel{?}{=} \sum_1^\infty \int f_n.$$

Tonelli: yes if $f_n \geq 0$. Fubini:

Corollary 2.31 (Fubini's theorem for series and integrals)**.**

$$\left.\begin{array}{c} f_n \textit{ integrable} \\ \sum_1^\infty \int |f_n| < \infty \end{array}\right\} \implies \int \sum_1^\infty f_n = \sum_1^\infty \int f_n.$$

Proof. $G(x) = \sum_1^\infty |f_n(x)| \geq |F_N(x)|, F_N(x) = \sum_1^N f_n(x)$. ■

2.4 L^1 space

Definition 2.32. Suppose V is a vector space over field $\mathbb{R}$ or $\mathbb{C}$. A *seminorm* on V is $\|\cdot\| : V \to [0,\infty)$ *s.t.*

- $\|cv\| = |c|\|v\|, \forall v \in V, \forall c$ scalar
- $\|v + w\| \leq \|v\| + \|w\|$, triangle inequality

A *norm* is a seminorm such that $\|v\| \iff v = 0$.

Lemma 2.33. *A normed vector space is a metric space with metric* $\rho(v, w) = \|v - w\|$.

Proof. (DIY)

- $\rho(v,w)=0 \iff \|v-w\|=0 \iff v-w=0 \iff v=w$.
- $\rho(v,w)=\|v-w\|=\|-1(w-v)\|=|-1|\,\|w-v\|=\rho(w,v)$.
- $\rho(v,w)+\rho(w,z)=\|v-w\|+\|w-z\|\geq\|v-w+w-z\|=\|v-z\|=\rho(v,z)$. ■

Example 2.34. $\mathbb{R}^d$ with $\|x\|_p=\begin{cases}\left(\sum_1^d |x_i|^p\right)^{1/p} & p\in[1,\infty)\\ \max_{1\leq i\leq d}|x_i| & p=\infty\end{cases}$ is a normed vector space.

Unit ball $\{x:\|x\|_p<1\}$.

All $\|\cdot\|_p$ norm induce the same topology i.e. if U is open in p-norm then it is open in p'-norm. This implies that a sequence converging under p-norm also converges under p'-norm.

<u>RECALL</u> f is integrable $\implies \int|f|<\infty$. $f=g$ a.e. $\implies \int f=\int g$.

Definition 2.35. Suppose $(X,\mathcal{A},\mu)$ a measure space.
$f\in L^1(X,\mathcal{A},\mu)=L^1(X,\mu)=L^1(X)=L^1(\mu)$ means f is an integrable function on X.

Lemma 2.36. *$L^1(X,\mathcal{A},\mu)$ is a vector space with seminorm $\|f\|_1=\int|f|$.*

Definition 2.37. Define $f\sim g$ if $f=g$ a.e. $L^1(X,\mathcal{A},\mu)/_\sim=L^1(X,\mathcal{A},\mu)$. " $=$ " is just a notation for convenience!

With new definition we have $L^1(X,\mathcal{A},\mu)$ is a normed vector space. $\rho(f,g)=\int|f-g|$.

Something interesting to discuss is what are the dense subsets of L^1.

Theorem 2.38.

(a) *$\{$ integrable simple functions $\}$ is dense in $L^1(X,\mathcal{A},\mu)$ (with respect to L^1 metric)*

(b) *$(X,\mathcal{A},\mu)=(\mathbb{R},\mathcal{A}_\mu,\mu)$, μ is Lebesgue-Stieltjes measure $\implies$ $\{$ integrable step functions $\}$ is dense in $L^1(X,\mathcal{A},\mu)$*

(c) *$C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R},\mathcal{L},m)$.*

Definition 2.39.

- A step function on $\mathbb{R}$ is $\psi+\sum_1^N c_i 1_{I_i}$, where I_i is an interval.
- $C_c(\mathbb{R})$ is the collection of continuous functions with compact support $\operatorname{supp}(f)=\overline{\{x\in\mathbb{R}\mid f(x)\neq 0\}}$.

Proof. (a) $\exists$ simple functions $0\leq|\phi_1|\leq|\phi_2|\leq\ldots\leq|f|$, $\phi_n\to f$ pointwise $\implies$

$\lim_{n\to\infty} \int |\phi_n - f| = 0$ by DCT. ($|\phi_n - f| \le |\phi_n| + |f| \le 2|f|$)

(b) 1_E approx by $\sum_1^N c_i 1_{I_i}$? Regularity theorem for Lebesgue-Stieltjes measure $\implies$ $\forall \varepsilon' > 0, \exists I = \bigcup_1^N I_i$ *s.t.* $\mu(E \triangle I) < \varepsilon'$.

(c) Suppose $1_{(a,b)}$, $g \in C_c(\mathbb{R})$. $\int |1_{(a,b)} - g| \,\mathrm{d}m \le 1 \cdot \frac{\varepsilon}{2} + 1 \cdot \frac{\varepsilon}{2} = \varepsilon$. ■

2.5 Riemann Integrability

Suppose $P = \{a = t_0 < t_1 < \ldots < t_k = b\}$ a partition of $[a, b]$. Lower Riemann sum of f using P

$$L_P = \sum_{i=1}^{k} \left(\inf_{[t_{i-1}, t_i]} f \right) (t_i - t_{i-1})$$

and upper Riemann sum

$$U_p = \sum_{i=1}^{k} \left(\sup_{[t_{i-1}, t_i]} f \right) (t_i - t_{i-1})$$

Lower Riemann integral of $f = \underline{I} = \sup_P L_P$. Upper Riemann integral of $f = \overline{I} = \inf_P U_P$.

Definition 2.40. A *bounded* function $f : [a, b] \to \mathbb{R}$ is called Riemann (Darboux) integrable if $\underline{I} = \overline{I}$. (If so, $\underline{I} = \overline{I} = \int_a^b f(x) \,\mathrm{d}x$.)

<u>NOTE</u>

- If $P \subset P'$, then $L_P \le L_{P'}, U_{P'} \le U_P$.
- Recall that continuous functions on $[a, b]$ are Riemann integrable on $[a, b]$.

Theorem 2.41. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function.*

(a) If f is Riemann integrable, then f is Lebesgue measurable. (thus Lebesgue integrable) and $\int_a^b f(x) \,\mathrm{d}x = \int_{[a,b]} f \,\mathrm{d}m$.

(b) f is Riemann integrable $\iff$ f is continuous Lebesgue a.e.

Proof. $\exists$partitions $P_1 \subset P_2 \subset P_3 \subset \ldots$ *s.t.* $L_{P_n} \nearrow \underline{I}, U_{P_n} \searrow \overline{I}$.

Define simple (step) functions

$$\phi_n = \sum_{i=1}^{k} \left(\inf_{[t_{i-1}, t_i]} \right) 1_{(t_{i-1}, t_i]}$$

$$\psi_n = \sum_{i=1}^{k} \left(\sup_{[t_{i-1},t_i]} \right) 1_{(t_{i-1},t_i]}$$

Define $\phi = \sup_n \phi_n$, $\psi = \inf_n \psi_n$. Then ϕ, ψ are Lebesgue measurable functions.

<u>NOTE</u>

- $\exists M > 0 \; s.t. \; |\phi_n|, |\psi_n| \le M1[a,b], \forall n \in \mathbb{N}$.
- $\int \phi_n \, \mathrm{d}m = L_{P_n}, \int \psi_n \, \mathrm{d}m = U_{P_n}$.

By DCT, $\underline{I} = \lim\limits_{n\to\infty} \displaystyle\int \phi_n \, \mathrm{d}m = \int \phi \, \mathrm{d}m, \overline{I} = \int \psi \, \mathrm{d}m$.

Thus, f is Riemann integrable $\iff \int \phi = \int \psi \iff \int(\phi - \psi) = 0 \iff \phi = \psi$ Lebesgue a.e.

Recall that $\phi \le f \le \psi, \forall x \in (a,b]$. So $f = \phi$ a.e. Since $(\mathbb{R}, \mathcal{L}, \mu)$ is complete, f is Lebesgue measurable (see HW). The second statement hence follows. ■

2.6 Modes of Convergence

Suppose $f_n, f : X \to \mathbb{C}, S \subset X$.

- $f_n \to f$ *pointwise* on S: $\forall x \in S, \forall \varepsilon > 0, \exists N \in \mathbb{N} \; s.t. \; \forall n \ge \mathbb{N}, |f_n(x) - f(x)| < \varepsilon$.
- $f_n \to f$ *uniformly* on S: $\forall \varepsilon > 0, \exists N \in \mathbb{N} \; s.t. \; \forall x \in X, \forall n \ge \mathbb{N}, |f_n(x) - f(x)| < \varepsilon$.

We can change $\forall \varepsilon > 0$ to $\forall k \in \mathbb{N}$ and bound the distance by $\frac{1}{k}$.

Lemma 2.42. *Let* $B_{n,k} = \{x \in X \mid |f_n(x) - f(x)| < \frac{1}{k}\}$.

(a) $f_n \to f$ *pointwise on* $S \iff S \subset \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} B_{n,k}$.

(b) $f_n \to f$ *uniformly on* $S \iff \exists N_1, N_2, \ldots \in \mathbb{N} \; s.t. \; S \subset \bigcap_{k=1}^{\infty} \bigcap_{n=N_k}^{\infty} B_{n,k}$.

Definition 2.43. Suppose $(X, \mathcal{A}, \mu)$ a measure space.

(a) $f_n \to f$ a.e means $\exists$ null set $E \; s.t. \; f_n \to f$ pointwise on E^c.

(b) $f_n \to f$ in L^1 means $\lim\limits_{n\to\infty} \|f_n - f\| = 0$.

Example 2.44. $(\mathbb{R}, \mathcal{L}, \mu)$. $f = 0$.

(a) $f_n = 1_{(n,n+1)}, f_n = \frac{1}{n} 1_{(0,n)}, f_n = n1_{\left(0,\frac{1}{n}\right)}$. All of $f_n \to f$ pointwise but $\not\to f$ in L^1.

(b) Typewriter functions: $f_n \to f$ in L^1. $f_n \not\to f$ a.e.

Proposition 2.45 (Fast L^1 convergence $\implies$ a.e. convergence)**.** *Suppose* $(x, \mathcal{A}, \mu)$ *measure* *space.* f_n, f *measurable function on* X.

$$\sum_1^\infty \|f_n - f\|_1 < \infty \implies f_n \to f \ a.e.$$

Proof. <u>RECALL</u> Markov's inequality.

Let $E = \bigcup_{k=1}^\infty \bigcap_{N=1}^\infty \bigcup_{n=N}^\infty B_{n,k}^c = \{x \mid f_n(x) \not\to f(x)\}$. By Markov we have

$$\begin{aligned}
&\forall k, \forall N, \mu(B_{n,k}^c) \le k \int |f_n - f| \\
\implies &\forall k, \mu\left(\bigcap_{n=N}^\infty B_{n,k}^c\right) \le \sum_{n=N}^\infty k \|f_n - f\|_1 \to 0 \text{ as } n \to 0 \\
\implies &\forall k, \mu\left(\bigcap_{N=1}^\infty \bigcap_{n=N}^\infty B_{n,k}^c\right) = \lim_{N\to\infty} \mu\left(\bigcap_{n=N}^\infty B_{n,k}^c\right) = 0 \\
\implies &\mu(E) = 0.
\end{aligned}$$

■

Corollary 2.46. $f_n \to f$ *in* $L^1 \implies \exists$*subsequence* $f_{n_j} \to f \ a.e.$

Proof. $\forall j \in \mathbb{N}, \exists n_j \in \mathbb{N} \ s.t. \ \left\|f_{n_j} - f\right\|_1 < \frac{1}{j^2}$. Then $\sum_{j=1}^\infty \left\|f_{n_j} - f\right\|_1 < \infty$. ■

Definition 2.47. f_n, f measurable functions on $(X, \mathcal{A}, \mu)$. $f_n \to f$ *in measure* means

$$\forall \varepsilon > 0, \lim_{n\to\infty} \mu\left(\{x \in X \mid |f_n(x) - f(x)| \ge \varepsilon\}\right) = 0.$$

Example 2.48.

- $f_n = n1_{\left(0, \frac{1}{n}\right)}, f = 0$.

$$\forall \varepsilon > 0, \{x \mid |f_n(x) - f(x)| > \varepsilon\} = \left(0, \frac{1}{n}\right).$$

  (Recall that $f_n \not\to 0$ in L^1.)

- Typewriter function. (Recall that $f_n \not\to 0$ a.e.)

We can easily check that $f_n \to f$ in $L^1 \implies f_n \to f$ in measurable. But the converse is not true.

$f_n \to f$ in measure $\implies \exists f_{n_j} \to f$ a.e. (Check [Fol99])

We have then the following diagram:

$$\begin{tikzcd}
f_n \to f \text{ fast } L^1 \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & f_n \to f \text{ in } L^1 \arrow[r, Rightarrow, shift left] \arrow[d, Rightarrow, shift right, "/" description] & f_n \to f \text{ in measure} \arrow[l, Rightarrow, shift left, "/" description] \arrow[d, Rightarrow] \\
& f_n \to f \text{ a.e.} \arrow[u, Rightarrow, shift right, "/" description] & \exists f_{n_j} \to f \text{ a.e.}
\end{tikzcd}$$

Definition 2.49. f_n, f measurable functions on $(X, \mathcal{A}, \mu)$.

(a) $f_n \to f$ *uniformly a.e* means $\exists$ null set F $s.t.$ $f_n \to f$ uniformly on F^c.

(b) $f_n \to f$ *almost uniformly* means $\forall \varepsilon > 0, \exists F \in \mathcal{A},\ s.t.\ \mu(F) < \varepsilon, f_n \to f$ uniformly on F^c.

Recall 2.42.

Theorem 2.50 (Egoroff)**.** *f_n, f measurable on $(X, \mathcal{A}, \mu)$. Suppose $\mu(X) < \infty$. Then $f_n \to f$ a.e $\iff$ $f_n \to f$ almost uniformly.*

Proof. " $\Longleftarrow$ ": DIY

" $\Longrightarrow$ ": Fix $\varepsilon > 0$.

$f_n \to f$ a.e $\implies \mu\left(\bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c\right) = 0 \implies \forall k, \mu\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{n,k}^c\right) = 0.$

By the continuity of measure from above and since $\mu(X) < \infty$,

$$\forall k, \lim_{N\to\infty} \mu\left(\bigcup_{n=N}^{\infty} B_{n,k}^c\right) = 0 \implies \forall k, \exists N_k \in \mathbb{N}, \mu\left(\bigcup_{n=N_k}^{\infty} B_{n,k}^c\right) < \frac{\varepsilon}{2^k}.$$

Let $F = \bigcup_{k=1}^{\infty} \bigcup_{n=N_k}^{\infty} B_{n,k}^c \implies \mu(F) < \varepsilon, f_n \to F$ uniformly on F^c. ■

Chapter 3

Product Measures

(p.22 - 36, section 1.2 and section 2.5, 2.6 of [Fol99])
The ultimate goal is to prove Fubini's theorem. This is also related to probability in in the sense that a series of events is in product measure.

3.1 Product σ-algebra

- Product space $X = \prod_{\alpha \in I} X_\alpha, x = (x_\alpha)_{\alpha \in I}$.
- Coordinate map $\pi_\alpha : X \to X_\alpha$.

Definition 3.1. $(X_\alpha, \mathcal{A}_\alpha)$ measurable space. $\forall \alpha \in I$, the *product σ-algebra* on $X = \prod\limits_{\alpha \in I} X_\alpha$ is

$$\bigotimes_{\alpha \in I} \mathcal{A}_\alpha = \left\langle \bigcup_{\alpha \in I} \pi_\alpha^{-1}\left(\mathcal{A}_\alpha\right) \right\rangle$$

where

$$\pi_\alpha^{-1}\left(A_\alpha\right) = \{\pi_\alpha^{-1}(E) | E \in \mathcal{A}_\alpha\}.$$

<u>Notation</u>

$$I = \{1, \ldots, d\} \implies X = \prod_{i=1}^{d} X_i, x = (x_1, \ldots, x_d), \bigotimes_{i=1}^{d} \mathcal{A}_i = \mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_d.$$

Lemma 3.2. *If I is countable, then*

$$\bigotimes_{\alpha \in I} \mathcal{A}_\alpha = \left\langle \left\{ \prod_{i=1}^{\infty} E_i \mid E_i \in \mathcal{A}_i \right\} \right\rangle$$

Lemma 3.3. *Suppose $\mathcal{A}_\alpha = \langle \mathcal{E}_\alpha \rangle, \forall \alpha \in I$.*

(a) $\pi_\alpha^{-1}(\mathcal{A}_\alpha) = \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle$.

(b) $\displaystyle\bigotimes_\alpha \mathcal{A}_\alpha = \left\langle \bigcup_\alpha \pi_\alpha^{-1}(\mathcal{E}_\alpha) \right\rangle$.

(c) If I is countable, then $\displaystyle\bigotimes_{\alpha \in I} \mathcal{A}_\alpha = \left\langle \left\{ \prod_{i=1}^{\infty} E_i \mid E_i \in \mathcal{E}_i \right\} \right\rangle$.

Proof.

(a)
- $f : Y \to Z$, $\mathcal{B}$ a σ-algebra on $Z \implies f^{-1}(\mathcal{B})$ is a σ-algebra since set union commutes with preimage. Hence $\pi_\alpha^{-1}(\mathcal{A}_\alpha)$ is a σ-algebra on X. Since $\pi_\alpha^{-1}(\mathcal{E}_\alpha) \subset \pi_\alpha^{-1}(\mathcal{A}_\alpha) \implies \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle \subset \pi_\alpha^{-1}(\mathcal{A}_\alpha)$.
- Let $\mathcal{M} = \{B \subset X_\alpha \mid \pi_\alpha^{-1}(B) \in \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle\}$. We show that $\mathcal{A}_\alpha \subset \mathcal{M}$.
  - $\mathcal{M}$ is a σ-algebra. (easy)
  - $\mathcal{E}_\alpha \subset \mathcal{M}$. (by definition)

  So $\mathcal{A}_\alpha = \langle \mathcal{E}_\alpha \rangle \subset \mathcal{M}$. Hence, if $E \in \mathcal{A}_\alpha$, $E \subset \mathcal{M} \implies \pi_\alpha^{-1}(E) \in \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle$ i.e. $\mathcal{A}_\alpha \subset \langle \pi_\alpha^{-1}(\mathcal{E}_\alpha) \rangle$.

(b, c) DIY. ∎

Theorem 3.4. *Suppose $X_1, \ldots, X_d$ metric spaces. Let $X = \prod_1^d X_i$ with product metric $\rho(x, y) = \sum_{i=1}^{d} \rho_i(x, y)$. Then*

(a) $\displaystyle\bigotimes_{i=1}^{d} \mathcal{B}(X_i) \subset \mathcal{B}(X)$.

(b) If, in addition, each X_i has a countable dense subset, then $\displaystyle\bigotimes_{i=1}^{d} \mathcal{B}(X_i) = \mathcal{B}(X)$.

Proof. DIY. ∎

As a consequence, we have $\mathcal{B}(\mathbb{R}^d) = \mathcal{B}(\mathbb{R}) \otimes \ldots \otimes \mathcal{B}(\mathbb{R})$.

Suppose $f = u + iv : X \to \mathbb{C}$. f is measurable $\iff$ $u^{-1}(E) \in \mathcal{A}, v^{-1}(E) \in \mathcal{A}, \forall E \in \mathcal{B}(\mathbb{R}) \iff f^{-1}(F) \in \mathcal{A}, \forall F \in \mathcal{B}(\mathbb{C}) = \mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

p.65. Let's focus on finite product.

You like Minecraft right? It's all rectangles.

Definition 3.5. Suppose X, Y sets.

(a) For a $E \subset X \times Y$, $E_x = \{y \in Y \mid (x, y) \in E\}$ and $E^y = \{x \in X \mid (x, y) \in E\}$.

(b) For $f : X \times Y \to \mathbb{C}$, define $f_x : Y \to \mathbb{C}, f^y : X \to \mathbb{C}$ by $f_x(y) = f(x, y) = f^y(x)$.

(c)

Example 3.6. $(1_E)_x = 1_{E_x}$. $(1_E)^y = 1_{E^y}$.

Proposition 3.7. $(X, \mathcal{A}), (Y, \mathcal{B})$ *measurable spaces.*

(a) $E \in \mathcal{A} \otimes \mathcal{B} \implies E_x \in \mathcal{B}, E^y \in \mathcal{A}, \forall x \in X, y \in Y$.

(b) $f : X \times Y \to \mathbb{C}$ *is* $\mathcal{A} \otimes \mathcal{B}$*-measurable* $\implies$ f_x *is* $\mathcal{B}$*-measurable,* f^y *is* $\mathcal{A}$*-measurable,* $\forall x \in X, y \in Y$.

Proof. (a) Let $\mathcal{F} = \{E \subset X \times Y \mid \text{(a) holds}\}$.

- $\mathcal{F}$ is a σ-algebra (easy)
- $\mathcal{R}_0 := \{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\} \subset \mathcal{F}$ (easy) $\implies \mathcal{A} \otimes \mathcal{B} = \langle \mathcal{R}_0 \rangle \subset \mathcal{F}$

(b) DIY. ■

MIDTERM is up till here.

3.2 Product Measures

Definition 3.8. Suppose $(X, \mathcal{A}), (Y, \mathcal{B})$. A (measurable) rectangle is $R = A \times B, A \in \mathcal{A}, b \in \mathcal{B}$.

Let $\mathcal{R}_0 := \{R = A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$.

$$\mathcal{R} := \left\{ \bigcup_1^N R_i \mid N \in \mathbb{N}, R_1, \dots, R_N \text{ disjoint rectangles} \right\}.$$

Lemma 3.9. $\mathcal{R}$ *is an algebra.* $\langle \mathcal{R}_0 \rangle = \langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B}$.

Theorem 3.10. *Suppose* $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ *measure spaces.*

(a) $\exists$ measure $\mu \times \nu$ on $\mathcal{A} \otimes \mathcal{B}$ satisfying $(\mu \times \nu)(A \otimes B) = \mu(A)\nu(B), \forall A \in \mathcal{A}, B \in \mathcal{B}$.

(b) If μ, ν are σ-finite, then $\mu \times \nu$ is unique.

Proof. (a) Define $\pi : \mathcal{R} \to [0, \infty]$ by $\pi(A \times B) = \mu(A)\nu(B)$ and extend linearly.

CLAIM π is a pre-measure on $\mathcal{R}$.

Enough to check $\pi(A \times B) = \sum_1^\infty \pi(A_n \times B_n)$ if $A \times B = \bigcup_1^\infty (A_n \times B_n)$ disjoint union.

Since $A_n \times B_n$ are disjoint,

$$1_{A \times B}(x, y) = \sum_1^\infty 1_{A_n \times B_n}(x, y),\ 1_A(x)1_B(y) = \sum_1^\infty 1_{A_n}(x)1_{B_n}(y).$$

By Tonelli's theorem for series and integrals, we have

$$\begin{aligned}\mu(A)1_B(y) = \int_x 1_A(x)1_B(y)\ \mathrm{d}\mu(x) \\ = \sum_1^\infty \int_x 1_{A_n}(x)1_{B_n}(y)\ \mathrm{d}\mu(x) = \sum_1^\infty \mu(A_n)1_{B_n}(y).\end{aligned}$$

We then integrate with respect to y to complete the claim.

By HK theorem, $\exists \mu \otimes \nu$ on $\langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B}$ extending π on $\mathcal{R}$.

(b) μ, ν σ-finite $\implies$ π is σ-finite on $\mathcal{R}$ $\implies$ HK uniqueness them applies. ∎

So we have a measure

$$(\mu \times \nu)(E) = \inf \left\{ \sum_1^\infty \mu(A_u)\nu(B_i) \middle| E \subset \bigcup_1^\infty A_i \times B_i, A_i \in \mathcal{A}, B_i \in \mathcal{B} \right\}.$$

Then one questions naturally arises: suppose $f : X \times Y \to \mathbb{C}$,

$$\int_{X \times Y} f\ \mathrm{d}(\mu \times v) \stackrel{?}{=} \int_y \left(\int_x f\ \mathrm{d}\mu \right)\ \mathrm{d}\nu.$$

3.3 Monotone Class Lemma

Definition 3.11. Suppose X is a set, $\mathcal{C} \subset \mathcal{P}(X)$. $\mathcal{C}$ is a *monotone class* on X if

- closed under *countable increasing unions*
  (i.e. $E_n \in \mathcal{C}, E_1 \subset E_2 \subset \dots \implies \bigcup_1^\infty C_i \in \mathcal{C}$.)

- closed under *countable decreasing intersections*
  (i.e. $E_n \in \mathcal{C}, E_1 \supset E_2 \supset \dots \implies \bigcap_1^\infty C_i \in \mathcal{C}$.)

Example 3.12.
- σ-algebra is a monotone class.
- $\bigcap\limits_\alpha \mathcal{C}_\alpha$ is a monotone class $\implies$ if $\mathcal{E} \in \mathcal{P}(X)$, there is unique smallest monotone class containing $\mathcal{E}$.

The importance of this definition shows up in the following theorem:

Theorem 3.13. *Suppose $\mathcal{A}_0$ is an* algebra *on X. Then $\langle \mathcal{A}_0 \rangle$ is the monotone class generated by $\mathcal{A}_0$.*

Proof. Let $\mathcal{A} = \langle \mathcal{A}_0 \rangle$, $\mathcal{C} =$ monotone class generated by $\mathcal{A}_0$.

(a) $\mathcal{A}$ is a σ-algebra $\implies$ $\mathcal{A}$ is a monotone class containing $\mathcal{A}_0 \implies \mathcal{A} \supset \mathcal{C}$.

(b) To show that $\mathcal{C} \supset \mathcal{A}$, we show that $\mathcal{C}$ is a σ-algebra.

  (1) $\emptyset \subset \mathcal{A}_0 \subset \mathcal{C}$.

  (2) Let $\mathcal{C}' = \{E \subset X \mid E^c \subset \mathcal{C}\}$.

  - $\mathcal{C}'$ is a monotone class (easy)
  - $\mathcal{A}_0 \subset \mathcal{C}'$ since $(E \in \mathcal{A}_0 \implies E^c \in \mathcal{A}_0 \subset \mathcal{C})$.

  These two show that $\mathcal{C} \subset \mathcal{C}'$. So $E \in \mathcal{C} \implies E \in \mathcal{C}' \implies E^c \in \mathcal{C}$. So $\mathcal{C}$ is closed under complements.

  (3) For $E \subset X$, let $\mathcal{D}(E) = \{F \in \mathcal{C} \mid E \cup F \in \mathcal{C}\}$.

  - $\mathcal{D}(E) \subset \mathcal{C}$ by definition.
  - $\mathcal{D}(E)$ is a monotone class (easy). $E \cup (\bigcup_1^\infty F_n) = \bigcap_1^\infty (E \cup F_n)$.
  - If $E \in \mathcal{A}_0$, then $\mathcal{A}_0 \subset \mathcal{D}(E)$. $(F \in \mathcal{A}_0 \implies E \cup F \in \mathcal{A}_0 \subset \mathcal{C}$.)

  These show that $\mathcal{C} = \mathcal{D}(E)$ if $E \in \mathcal{A}_0$.

  (4) Let $\mathcal{D} = \{E \in \mathcal{C} \mid \mathcal{D}(E) = \mathcal{C}\} = \{E \in \mathcal{C} \mid E \cup F \in \mathcal{C}, \forall F \in \mathcal{C}\}$.

  - $\mathcal{A}_0 \subset \mathcal{D}$ by (3).
  - $\mathcal{D}$ is a monotone class (easy).
  - $\mathcal{D} \subset \mathcal{C}$ by definition.

  So we conclude that $\mathcal{D} = \mathcal{C}$. Now we have $\mathcal{C}$ is closed under finite unions.

  (5) $\mathcal{C}$ is closed under finite unions and countable increasing unions $\implies$ $\mathcal{C}$ is closed under countable unions. (check) ∎

<u>RECALL</u> $E \in \mathcal{A} \otimes \mathcal{B} \implies E_x \in \mathcal{B}, E^y \in \mathcal{A}, \forall x \in X, y \in Y$. However, the inverse is not necessarily true.

Now comes the main thing:

3.4 Fubini-Tonelli Theorem

Theorem 3.14 (Tonelli for characteristic functions)**.** *Suppose* $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ *are* σ-finite *measure spaces. Suppose* $E \in \mathcal{A} \otimes \mathcal{B}$. *Then*

(a) $\alpha(x) := \nu(E_x) : X \to [0, \infty]$ *is a* $\mathcal{A}$*-measurable function.*

(b) $\beta(y) := \mu(E^y) : Y \to [0, \infty]$ *is a* $\mathcal{B}$*-measurable function.*

(c) $(\mu \times \nu)(E) = \displaystyle\int_X \nu(E_x)\, \mathrm{d}\mu(x) = \int_Y \mu\left(E^y\right)\, \mathrm{d}\nu(y)$.

Proof. (a) Assume μ, ν are finite measures. Let

$$\mathcal{C} = \{E \in \mathcal{A} \otimes \mathcal{B} \mid \text{(a), (b), (c) hold}\}.$$

Enough to prove that $\langle \mathcal{R} \rangle = \mathcal{A} \otimes \mathcal{B} \subset \mathcal{C}$.

Because of monotone class lemma and that $\mathcal{R}$ is a σ-algebra, it is enough to show that $\mathcal{R} \subset \mathcal{C}$ and $\mathcal{C}$ is a monotone class.

- Show that $\mathcal{R} \subset \mathcal{C}$.

$$\alpha(x) = \nu((A \times B)_x) = \begin{cases} \nu(B) & x \in A \\ 0 & x \notin A \end{cases} = \nu(B) 1_A(x).$$

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$$
$$\iff \int_X \nu((A \times B)_x)\, \mathrm{d}\mu(x) = \nu(B)\mu(A)$$

- Show that $\mathcal{C}$ is a monotone class.

(1) Let $E_n \in \mathcal{C}, E_1 \subset E_2 \subset \dots$. Need to show that $E = \bigcup_1^\infty E_n \in \mathcal{C}$.

$$\begin{aligned} & E_n \in \mathcal{C}, E_1 \subset E_2 \subset \dots \\ \implies & E_x = \bigcup_1^\infty (E_n)_x, (E_1)_x \subset (E_2)_x \subset \dots \\ \implies & \alpha(x) = \nu(E_x) = \lim_{n\to\infty} \nu\left((E_n)_x\right), \forall x \in X, \quad \alpha_n(x)\ \mathcal{A}\text{-measurable} \end{aligned}$$

This satisfies (a), (b). For (c), we have

$$\begin{aligned} (\mu\times\nu)(E) &= \lim_{n\to\infty}(\mu\times\nu)(E_n) \\ &= \lim_{n\to\infty}\int_X \nu\left((E_n)_x\right)\, \mathrm{d}\mu(x) \overset{MCT}{=} \int_X \nu(E_x)\, \mathrm{d}\mu(X). \end{aligned}$$

So we have shown countable increasing unions.

(2) Let $F_n \in \mathcal{C}$, $F_1 \supset F_2 \supset \dots$. Need to show that $F \bigcup_1^\infty F_n \in \mathcal{C}$. Using continuity of measure from above instead of below, DCT instead of MCT, we obtained a similar result.

(b) Now assume that μ, ν are σ-finite. Since $X \times Y = \bigcup_1^\infty (X_n \times Y_n)$, where $X_1 \subset X_2 \dots, Y_1 \subset Y_2 \subset \dots$ with $\mu(X_k), \nu(Y_k)$ finite. Apply results from then finite case. (DIY) ■

Theorem 3.15 (Fubini-Tonelli). *Suppose $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are σ-finite measure spaces.*

(a) (Tonelli) If $f : X \times Y \to [0,\infty]$ is $\mathcal{A}\otimes\mathcal{B}$-measurable then

(1) $g(x) := \int_Y f(x,y)\, \mathrm{d}\nu(y) : X \to [0,\infty]$ is a $\mathcal{A}$-measurable function.

(2) $h(y) := \int_X f(x,y)\, \mathrm{d}\mu(x) : Y \to [0,\infty]$ is a $\mathcal{B}$-measurable function.

(3) We have the iterated integral formula

$$\begin{aligned} \int_{X\times Y} f\, \mathrm{d}(\mu\times\nu) &= \int_X \left[\int_Y f(x,y)\, \mathrm{d}\nu(y)\right] \mathrm{d}\mu(x) \\ &= \int_X \left[\int_X f(x,y)\, \mathrm{d}\mu(x)\right] \mathrm{d}\nu(y). \end{aligned}$$

(b) (Fubini) If $f \in L^1(X\times Y, \mu\times\nu)$, then

(1) $f_x \in L^1(Y,\nu)$ for μ-a.e x and $g(x)$ (which is defined μ-a.e) $\in L^1(X,\mu)$.

(2) $f^y \in L^1(X,\mu)$ for ν-a.e y and $h(y)$ (which is defined ν-a.e) $\in L^1(Y,\nu)$.

(3) The iterated integral formula from (a).(3) hold.

Usually we apply Tonelli to $|f|$ to show $f \in L^1(X \times Y, \mu \times \nu)$ and then apply Fubini to evaluate.

Proof. See [Fol99]. ■

3.5 Lebesgue Measure on $\mathbb{R}^d$

Example 3.16 (($\mathbb{R}^2, \mathcal{L} \otimes \mathcal{L}, m \times m$) is not complete)**.** Let $A \in \mathcal{L}, A \neq \emptyset, m(A) = 0$. Let $B \subset [0,1], B \notin \mathcal{L}$ (e.g. Vitali set). Then let $E = A \times B, F = A \times [0,1]$. We can see that $E \subset F$ and $F \in \mathcal{L} \otimes \mathcal{L}, (m \times m)(F) = m(A)m([0,1]) = 0$.

So E is a subnull set but not $\mathcal{L} \otimes \mathcal{L}$-measurable. (otherwise each section of E is measurable, a contradiction.)

Definition 3.17. Let $(\mathbb{R}^d, \mathcal{L}^d, m^d)$ be the *completion* of $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), m \times \ldots \times m)$, which is same(check!) as the *completion* of $(\mathbb{R}^d, \mathcal{L} \otimes \ldots \otimes \mathcal{L}, m \times \ldots \times m)$.

So how do we compute m^d?

A *rectangle* in $\mathbb{R}^d$ is $R = \prod_{i=1}^{d} E_i$, $E_i \in \mathcal{B}(\mathbb{R})$. Then

$$m^d(E) = \inf \left\{ \sum_1^\infty m^d R_k \;\middle|\; E \subset \bigcup_1^\infty R_k, R_k \text{ rectangle} \right\}.$$

Theorem 3.18. *Let $E \in \mathcal{L}^d$.*

(a) $m^d(E) = \inf \{ m^d(O) \mid \textit{open } O \supset E \} = \sup \{ m^d(K) \mid \textit{compact } K \subset E \}$.

(b) $E = \underbrace{A_1}_{F\sigma} \cup \underbrace{N_1}_{\textit{null}} = \underbrace{A_2}_{G\sigma} \setminus \underbrace{N_2}_{\textit{null}}$.

(c) If $m^d(E) < \infty, \forall \varepsilon > 0, \exists R_1, \ldots, R_m$ rectangles whose sides are intervals *such that* $m^d\left(E \triangle \left(\bigcup_1^m R_i\right)\right) < \varepsilon$.

Proof. Similar to $d = 1$ case. ■

Theorem 3.19. *Integrable "step functions" and $C_c(\mathbb{R}^d)$ are dense in $L^1(\mathbb{R}^d, \mathcal{L}^d, m^d)$.*

Theorem 3.20. *Lebesgue measure in $\mathbb{R}^d$ is translation-invariant.*

Theorem 3.21. *"Effect of linear transformations on Lebesgue measure"*

Skip p. 71-81 of [Fol99] except 3.21.

Chapter 4

Differentiation on Euclidean Space

Suppose $f : [a, b] \to \mathbb{R}$. There are two versions of fundamental theorem of Calculus:

- $\displaystyle\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)$.
- $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)dt = f(x)$.

We focus on the second statement, which implies that

$$\lim_{r\to 0^+}\frac{1}{r}\int_x^{x+r} f(t)\,\mathrm{d}t = \lim_{r\to 0^+}\frac{1}{r}\int_{x-r}^{x} f(t)\,\mathrm{d}t$$

Write $f(x) = \dfrac{1}{r}\displaystyle\int_x^{x+r} f(x)\,\mathrm{d}t$, then

$$\lim_{r\to 0^+}\frac{1}{r}\int_x^{x+r} (f(t) - f(x))\,\mathrm{d}t = \lim_{r\to 0^+}\frac{1}{r}\int_{x-r}^{x} (f(t) - f(x))\,\mathrm{d}t.$$

This generalizes well in $\mathbb{R}^d$:

$$f : \mathbb{R}^d \to \mathbb{R}, \quad \lim_{r\to 0^+}\frac{1}{v(B(x,r))}\int_{B(x,r)} f(t) - f(x)\,\mathrm{d}t = 0.$$

QUESTION to what extent does this hold?

Start from [Fol99, 3.4].

4.1 Hardy-Littlewood Maximal Function

Suppose an open ball in $\mathbb{R}^d$, $B = B(a, r)$. Denote $cB = B(a, cr), c > 0$.

Lemma 4.1 (Vitali-type covering lemma)**.** *Let $B_1, \ldots, B_k$ be a finite collection of open balls in R^d. Then $\exists$ a sub-collection $B'_1, \ldots, B'_m$ of* disjoint *open balls such that*

$$\bigcup_1^m (3B'_j) \supset \bigcup_1^k B_i.$$

Proof. Greedy algorithm. ∎

NOTATION: $\int_E f \, \mathrm{d}m = \int_E f(x) \, \mathrm{d}x$.

Definition 4.2. $f : \mathbb{R}^d \to \mathbb{C}$ is Lebesgue measurable. f is *locally integrable* if

$$\int_K |f| \, \mathrm{d}m < \infty, \forall \text{ compact } K \subset \mathbb{R}^d.$$

We write $f \in L^1_{\text{loc}}(\mathbb{R}^d)$.

Example 4.3. $f(x) = x^2 \in L^1_{\text{loc}}(\mathbb{R}^d)$. (in fact all continuous functions $\in L^1_{\text{loc}}(\mathbb{R}^d)$).

Definition 4.4. For $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, define Hardy-Littlewood maximal function for f

$$Hf(x) = \sup\{A_r(x) \mid r > 0\}, \quad A_r(x) = \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)| \, \mathrm{d}y.$$

Lemma 4.5. *Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$. Then,*

(a) $A_r(x)$ is jointly continuous for $(x, r) \in \mathbb{R}^d \times (0, \infty)$.

(b) $Hf(x)$ is Borel measurable.

Proof.

(a) $(x, r) \to (x^*, r^*) \implies A_r(x) \to A_{r^*}(x^*)$.

Let (x_n, r_n) be any sequence $\to (x^*, r^*)$.

$$A_{r_n}(x_n) \leq \int |f(y)| 1_{B(x_n, r_n)}(y).$$

Apply DCT.

(b) $(Hf)^{-1}((a, \infty)) = \bigcup_{r>0} A_r^{-1}((a, \infty))$ is open.

■

<u>RECALL</u> Markov inequality

$$m\left(\{x \mid |f(x)| \geq c\}\right) \leq \frac{1}{c} \int |f(x)| \, \mathrm{d}x$$

Theorem 4.6 (Hardy-Littlewood maximal inequality)**.** $\exists C_d > 0 \; s.t. \; \forall f \in L^1_{\text{loc}}(\mathbb{R}^d), \forall \alpha > 0$,

$$m\left(\{x \mid Hf(x) > \alpha\}\right) \leq \frac{C_d}{\alpha} \int |f(x)| \, \mathrm{d}x.$$

Proof. Fix $f \in L^1$ and $\alpha > 0$. Let $E\{x \mid (Hf)(x) > \alpha\}$. E is a Borel measurable set. Then

$$x \in E \implies \exists r_x > 0, \; s.t. \; A_{r_x}(x) > \alpha \implies m(B(x, r_x)) < \frac{1}{\alpha} \int_{B(x,r_x)} |f(y)| \, \mathrm{d}y.$$

By inner regularity, we have $m(E) = \sup\{m(K) \mid \text{ compact } K \subset E\}$. Let $K \subset E$ be compact. Then

$$\begin{aligned}
&K \subset \bigcup_{x \in K} B(x, r_x) \\
&\Longrightarrow K \subset \bigcup i = 1^N B_i \\
&\Longrightarrow K \subset \bigcup_{j=1}^{m} (3B'_j), B'_1, \ldots, B'_m \text{ disjoint} \\
&\Longrightarrow m(K) \leq \sum_{j=1}^{n} m(3B'_j) = 3^d \sum_{j=1}^{n} m(B'_j) \\
&\Longrightarrow m(K) \leq \frac{3^d}{\alpha} \sum_{j=1}^{N} \int_{B'_j} |f(y)| \, \mathrm{d}y \\
&\Longrightarrow m(K) \leq \frac{3^d}{\alpha} \int_{\mathbb{R}^d} |f(y)| \, \mathrm{d}y.
\end{aligned}$$

■

4.2 Lebesgue Differentiation Theorem

Theorem 4.7. *Let $f \in L^1(\mathbb{R}^d)$. Then*

$$\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, \mathrm{d}y = 0 \textit{ for a.e } x.$$

Proof. (a) The result is holds for $f \in C_c(\mathbb{R}^d)$ (check!)

(b) Let $f \in L^1(\mathbb{R}^d)$. Fix $\varepsilon > 0$. $\exists g \in C_c(\mathbb{R}^d)$ $s.t.$ $\|f-g\|_1 < \varepsilon$. Then

$$\begin{aligned}&\int_{B(x,r)} |f(y)-f(x)| \,\mathrm{d}y \\ &\le \int_{B(x,r)} |f(y)-g(y)| \,\mathrm{d}y + \int_{B(x,r)} |g(y)-g(x)| \,\mathrm{d}y + \int_{B(x,r)} |g(x)-f(x)| \,\mathrm{d}y.\end{aligned}$$

Let $Q(x) = \limsup_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)-f(x)| \,\mathrm{d}y$. WTS that $m(\{x \mid Q(x) > 0\}) = m\left(\bigcup_{n=1}^\infty \left\{x \mid Q(x) > \frac{1}{n}\right\}\right) = 0$.

Enough to show that $m(E_\alpha) = 0, \forall \alpha > 0, E_\alpha = \{x \mid Q(x) > \alpha\}$.

But $Q(x) \le (H(f-g))(x) + 0 + |g(x)-f(x)|$,

$\{x \mid Q(x) > \alpha \subset \left\{x \mid H(f-g)(x) > \frac{\alpha}{2}\right\} \bigcup \left\{x \mid |g(x)-f(x)| > \frac{\alpha}{2}\right\}$.

$m(\{x \mid Q(x) > \alpha\}) \le \frac{2C_d}{\alpha} \|f-g\|_1 + \frac{2}{\alpha}\|f-g\|_1 \le \frac{2(C_d+1)}{\alpha}\varepsilon$. ■

Corollary 4.8. *This also holds for $f \in L^1_{\text{loc}}(\mathbb{R}^d)$.*

Proof. DIY. ■

Corollary 4.9. *For $f \in L^1_{\text{loc}}(\mathbb{R}^d)$,*

$$\lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y) \,\mathrm{d}y = 0 \textit{ for a.e } x.$$

Proof. DIY. ■

Definition 4.10. Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$. The point $x \in \mathbb{R}^d$ is called a *Lebesgue point* of f if

$$\lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)-f(x)| = 0.$$

$f \in L^1_{\text{loc}}(\mathbb{R}^d) \implies$ a.e point is a Lebesgue point of f.

Definition 4.11. $\{E_r\}_{r>0}$ *shrinks nicely* to x as $r \to 0$ means $E_r \subset B(x,r)$ and $\exists c > 0$ $s.t.$ $cm(B(x,n)) \le m(E_r)$.

Corollary 4.12 (Lebesgue differentiation theorem)**.**

$$\left.\begin{array}{r} E_r \textit{ shrinks nicely to } 0 \\ f \in L^1_{\text{loc}}(\mathbb{R}^d) \\ x \textit{ a Lebesgue point of } f \end{array}\right\} \implies \lim_{r\to 0} \frac{1}{m(E_r)} \int_{E_r+x} |f(y)-f(x)| \,\mathrm{d}y = 0.$$

Proof. DIY. ■

Corollary 4.13. *$f \in L^1_{\text{loc}}(\mathbb{R}^d) \implies F(x) = \int_0^x f(y)\,\mathrm{d}y$ is differentiable and $F'(x) = f(x)$ a.e.*

Rest of [Fol99, Ch.3] will be covered later.

Chapter 5

Normed Vector Spaces

Topological spaces $\supset$ metric spaces $\supset$ normed spaces $\supset$ inner product spaces.

Let's start with metric spaces. [Fol99, 5.1, 6.1, 6.2]

5.1 Metric Spaces and Normed Spaces

Definition 5.1. Suppose Y is a set. A *metric* of Y is $\rho : Y \times Y \to [0, \infty)$ $s.t.$

(a) $\rho(x, y) = \rho(y, x)$

(b) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$

(c) $\rho(x, y) = 0 \iff x = y$.

Example 5.2.

(a) $\mathbb{Q}, \rho(x, y) = |x - y|$.

(b) $\mathbb{R}, \rho(x, y) = |x - y|$.

(c) $\mathbb{R}_+, \rho(x, y) = \left|\ln\left(\frac{y}{x}\right)\right|$.

(d) $\mathbb{R}^d, \rho_1(x, y) = \sum_{i=1}^{d} |x_i - y_i|, \rho_p(x, y) = \left(\sum_{i=1}^{d} |x_i - y_i|^p\right)^{1/p}, \rho_\infty(x, y) = \max_{1 \leq i \leq d} |x_i - y_i|$.

(e) $C([0, 1]), \rho_p(f, g) = \left(\int_0^1 |f - g|^p\right)^{1/p}, \rho_\infty = \max_{x \in [0,1]} |f(x) - g(x)|$.

They are all metric spaces.

Definition 5.3 (Recall 2.32)**.** Suppose V is a vector space over field $\mathbb{R}$ or $\mathbb{C}$. A *seminorm* on V is $\|\cdot\| : V \to [0, \infty)$ $s.t.$

- $\|cv\| = |c|\|v\|, \forall v \in V, \forall c$ scalar
- $\|v + w\| \leq \|v\| + \|w\|$, triangle inequality

A *norm* is a seminorm such that $\|v\| \iff v = 0$.

Norm gives rise to a metric where $\rho(v, w) = \|v - w\|$.

$v_n \to v \iff \lim_{n\to\infty} \|v_n - v\| = 0$.

Example 5.4. (a) $L^1(X, \mathcal{A}, \mu)$

(b) $C([0,1]), \|f\|_1 = \int_0^1 |f(x)|\, \mathrm{d}x, \|f\|_\infty \max_{0\leq x\leq 1} |f(x)|$.

(c) $\mathbb{R}^d, \|x\|_2 = \sqrt{\sum_1^d |x_i|^2}, \|x\|_1 = \sum_1^d |x_i, \|x\|_\infty \max_{1\leq i\leq d} |x_i|$.

5.2 L^p Spaces

Definition 5.5. Suppose $(X, \mathcal{A}, \mu)$ a measure space. f is measurable function. For $0 < p < \infty$, define $\|f\|_p = \left(\int_X |f|^p \,\mathrm{d}\mu\right)^{1/p}$. Define $L^p(X, \mathcal{A}, \mu) = \left\{ f \;\middle|\; \|f\|_p < \infty \right\}$.

Example 5.6.

Definition 5.7. $\ell^p = \ell^p(N) = \{a = (a_1, a_2, \ldots) \mid \|a\|_p = (\sum_1^\infty |a_i|^p)^{1/p} < \infty\}$.

Lemma 5.8. *L^p is a vector space, $\forall p \in (0, \infty)$.*

Proof.

$$\left(\int |cf|^p\right)^{1/p} = |c| \, \|f\|_p \, .$$

Given the following inequality

$$(\alpha + \beta)^p \leq (2 \max(|\alpha|, |\beta|))^p = 2^p \max\left(|\alpha|^p, |\beta|^p\right) \leq 2^p (|\alpha|^p + |\beta|^p)$$

we have

$$\int |f + g|^p \leq 2^p \left(\int (|f|^p + |g|^p)\right) \implies \|f + g\|_p \leq 2 \left(\int (|f|^p + |g|^p)\right)^{1/p} . \qquad \blacksquare$$

But we want to know that whether

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

holds.

Theorem 5.9 (Hölder's Inequality). *Let* $p < \infty, q = \frac{p}{p-1}$ *so* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

Proof.

$$t \leq \frac{t^p}{p} + 1 - \frac{1}{p}, \forall t \geq 0$$

(Take $F(t) = t - \frac{t^p}{p}$)

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}, \forall \alpha, \beta \geq 0 \text{ (Young's inequality)} \tag{5.1}$$

WLOG assume $0 \leq \|f\|_p, \|g\|_q < \infty$. Let $F(x) = \frac{f(x)}{\|f\|_p}, G(x) = \frac{g(x)}{\|g\|_q}$.

$\implies \|F\|_p = 1 = \|G\|_q$.

By (5.1),

$$\int |F(x)G(x)| \leq \int \frac{|F(x)|^p}{p} + \int \frac{|G(x)|^q}{q}$$

$$\frac{\int |f(x)g(x)|}{\|f\|_p \|g\|_q} \leq \frac{1}{p} + \frac{1}{q} = 1.$$

■

Theorem 5.10 (Minkowski's inequality). *Let* $1 \leq p < \infty$. *For* $f, g \in L^p, \|f+g\|_p \leq \|f\|_p + \|g\|_p$.

Proof. $p = 1$ is easy.

Assume $1 < p < \infty$. WLOG assume $\|f+g\|_p \neq 0$. We have

$$\begin{aligned}
\int |f(x)+g(x)|^p &\leq \int |f(x)+g(x)|^{p-1}(|f(x)|+|g(x)|) \\
&\leq \left(\int (|f+g|^{p-1})^q\right)^{1/q} \left(\int |f|^p\right)^{1/p} + \left(\int (|f+g|^{p-1})^q\right)^{1/q} \left(\int |g|^p\right)^{1/p} \\
&\leq \left(\int (|f+g|^{p-1})^q\right)^{1/q} \left[\left(\int |f|^p\right)^{1/p} + \left(\int |g|^p\right)^{1/p}\right] \\
&\leq \left(\int (|f+g|^{p-1})^q\right)^{1/q} \left[\|f\|_p + \|g\|_p\right]
\end{aligned}$$

Since $q(p-1) = p$, divide by $\left(\int (|f+g|^{p-1})^q\right)^{1/q}$ on both sides we have

$$\left(\int |f(x)+g(x)|^p\right)^{1-1/q} \leq \|f\|_p + \|g\|_p . \qquad \blacksquare$$

Hölder: $\|fg\|_1 \leq \|f\|_p \|g\|_q, \frac{1}{p} + \frac{1}{q} = 1$.

Minkowski: $\|f+g\|_p \leq \|f\|_p + \|g\|_p, 1 \leq p < \infty$.

Definition 5.11. For a measurable function f on $(X, \mathcal{A}, \mu)$, let

$$S = \{\alpha \geq 0 \mid \mu(\{x \mid |f(x)| > \alpha\}) = 0\} = \{\alpha \geq 0 \mid f(x) \leq \alpha \text{ a.e}\}.$$

Define $\|f\|_\infty = \begin{cases} \inf S & S \neq \emptyset \\ \infty & S = \emptyset. \end{cases}$. Let $L^\infty(X, \mathcal{A}, \mu) = \{f \mid \|f\|_\infty < \infty\}$.

Example 5.12.

- $(\mathbb{R}, \mathcal{L}, m)$, $f(x) = \frac{1}{x} 1_{(0,\infty)}(x) \neq L^\infty$, $f(x) = x 1_{\mathbb{Q}}(x) + \frac{1}{1+x^2} \in L^\infty$.
- If f is *continuous* on $(\mathbb{R}, \mathcal{L}, m)$, $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$. For $a \in \ell^\infty$, $\|a\|_\infty = \sup_{i\in\mathbb{N}} |a_i|$. ($\ell^\infty = \{a = (a_1, a_2, \ldots) \mid \|a\|_\infty < \infty\} = \{a \mid \exists M \geq 0 \ s.t. \ |a_i| \leq M_i, \forall i\}$)

Lemma 5.13. *(a) For $\alpha \geq \|f\|_\infty, \mu(\{x \mid |f(x)| > \alpha\}) = 0$. For $\alpha < \|f\|_\infty, \mu(\{x \mid |f(x)| > \alpha\}) > 0$.*

(b) $|f(x)| \leq \|f\|_\infty$ a.e.

(c) $f \in L^\infty \iff \exists$ bounded *measurable function g such that $f = g$ a.e.*

Proof. DIY. $\blacksquare$

Theorem 5.14.

(a) $\|fg\|_1 \leq \|f\|_1 \|g\|_\infty$.

(b) $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

(c) $f_n \to f$ in $L^\infty \iff f_n \to f$ uniformly a.e.

Proof. DIY For (c): $\implies$ Let $A_n = \{x \mid |f_n(x) - f(x)| > \|f_n - f\|_\infty\}$. Then $\mu(A_n) = 0$.

Let $A = \bigcup_1^\infty A_n, \mu(A_n) = 0$. $\forall x \in A^c = \bigcap_1^\infty A_n^c, \forall n, |f_n(x) - f(x)| \leq \|f_n - f\|_\infty$. The latter converges to 0 by assumption.

Given $\varepsilon > 0, \exists N \ s.t. \ \|f_n - f\|_\infty < \varepsilon, \forall n \geq N$. So $\forall x \in A^c, \forall n \geq N, |f_n(x) - f(x)| \leq \|f_n - f\|_\infty < \varepsilon$. $\blacksquare$

Proposition 5.15.

(a) *For $1 \le p < \infty$, the collection of simple functions with finite measure support is dense in $L^p(X, \mathcal{A}, \mu)$.*

(b) *For $1 \le p < \infty$, the collection of step functions (by definition they have finite measure support) is dense in $L^p(\mathbb{R}, \mathcal{L}, m)$. So is $C_c(\mathbb{R})$.*

(c) *For $p = \infty$, the collection of simple functions is dense in $L^\infty(X, \mathcal{A}, \mu)$.*

Proof. DIY ■

NOTE: $C_c(\mathbb{R})$ is *not dense* in $L^\infty(\mathbb{R}, \mathcal{L}, m)$.

5.3 Embedding Properties of L^p spaces

Definition 5.16. Two norms $\|\cdot\|, \|\cdot\|'$ on the same spaces V are said to be *equivalent* if

$$\exists c_1, c_2 > 0 \ s.t. \ c_1 \|v\| \le \|v\|' \le c_2 \|v\|, \forall v \in V.$$

So on equivalent norms we have same open sets, same convergence.

Example 5.17.

- For $\mathbb{R}^d$, $\|\cdot\|_p$, $1 \le p \le \infty$ are equivalent.
- For $1 \le p, q \le \infty p \ne q$, $L^p(\mathbb{R}, m)$-norm and $L^q(\mathbb{R}, m)$-norm are *not equivalent*. $L^p(\mathbb{R}, m) \not\subset L^q(\mathbb{R}, m)$, $L^p(\mathbb{R}, m) \not\supset L^q(\mathbb{R}, m)$.

Proposition 5.18. *Suppose $\mu(X) < \infty$, then for any $0 < p < q \le \infty$, $L^q \subseteq L^p$.*

Proof.
- $p = \infty$ is easy.
- Suppose $p < \infty$.

■

Proposition 5.19. *If $0 < p < q \le \infty$ then $\ell^p \subseteq \ell^q$.*

Proposition 5.20. $\forall 0 < p < q < r \le \infty$, $L^p \cap L^r \subset L^q$.

Proof. DIY ■

5.4 Banach Spaces

Theorem 5.21. *Suppose $(V, \|\cdot\|)$ a normed space. Then it is complete $\iff$ Every absolutely convergent series is convergent (i.e. if $\sum_1^\infty \|v_n\| < \infty$ then $\exists s \in V$ s.t. $\sum_1^N v_n \to s$ as $N \to \infty$)*

Proof. $\implies$: DIY. (partial sums form a Cauchy Sequence)

$\impliedby$: Suppose $v_n, n \in \mathbb{N}$ is a Cauchy sequence. $\forall j \in \mathbb{N}, \exists N_j \in \mathbb{N}$ $s.t.$ $\|v_n - v_m\| < \frac{1}{2^j}, \forall n, m \geq N_j$.

WLOG we may assume $N_1 < N_2 < \ldots$. Let $w_1 = v_{N_1}, w_j = v_{N_j} - v_{N_{j-1}}, \forall j \geq 2 \implies \sum_1^\infty \|w_j\| \leq \|v_{N_1}\| + \sum_{j=2}^\infty \frac{1}{2^{j-1}} < \infty \implies \sum_1^k w_j \to \exists s \in V$.

Thus $V_{N_k} \to s$ as $k \to \infty$. v_n is Cauchy $\implies v_n \to s$ as $n \to \infty$. ∎

5.5 Bounded Linear Transformation

Definition 5.22. Suppose $(V, \|\cdot\|), (W, \|\cdot\|')$ two normed spaces. A linear map $T : V \to W$ is said to be a *bounded map* is $\exists c \geq 0$ $s.t.$ $\|T_v\|' \leq C\|v\|, \forall v \in V$.

Proposition 5.23. *Suppose $T : (V, \|\cdot\|) \to (W, \|\cdot\|')$ is a linear map. Then the followings are equivalent:*

(a) T is continuous

(b) T is continuous at 0

(c) T is a bounded map

Proof. (a) $\implies$ (b) is clear.

(b) $\implies$ (c): For $\varepsilon = 1, \exists \delta > 0$ $s.t.$ $\|Tu\|' < \varepsilon = 1$ if $\|u\| < \delta$. Suppose $v \in V, v \neq 0$. Let $u = \frac{\delta}{2\|v\|} v \implies \|u\| = \frac{\delta}{2} < \delta \implies \|Tu\|' < 1 \implies \frac{\delta}{2\|v\|}\|Tv\|' < 1 \implies \|Tu\|' < \frac{2}{\delta}\|v\|$.

(c) $\implies$ (a): Fix $v_0 \in V$. $\|Tv - Tv_0\|' = \|T(v - v_0)\|' \leq C\|v - v_0\|$. ∎

Example 5.24. (a) $T : \ell^1 \to \ell^1, Ta = (a_2, a_3, \ldots)$, $\|Ta\|_1 \leq \|a\|_1$. T is BLT.

(b) $T : (C([-1, 1]), \|\cdot\|_1) \to \mathbb{C}, Tf = f(0)$. This is not continuous.

(c) $T : (C([-1, 1]), \|\cdot\|_\infty) \to \mathbb{C}, Tf = f(0)$ is BLT.

(d) Let A be a $n \times m$ matrix. $T : \mathbb{R}^n \to \mathbb{R}^m, v \mapsto Av$ is BLT.

(e) Let $K(x, y)$ be a continuous function on $[0,1] \times [0,1]$.

$$T : (C_[0,1], \|\cdot\|_\infty) \to (C[0,1], \|\cdot\|_\infty), Tf = \int_0^1 K(x,y) f(y)\,\mathrm{d}y$$

is a BLT.

(f) $T : L^1(\mathbb{R}) \to (C(\mathbb{R}), \|\cdot\|_\infty), (Tf)(t) = \displaystyle\int_{-\infty}^{\infty} e^{-itx} f(x)\,\mathrm{d}x$ (Fourier transform of f)

(g) $T : (C^\infty([0,1]), \|\cdot\|_\infty) \to (C^\infty([0,1]), \|\cdot\|_\infty), (Tf)(x) = f'(x)$ is not bounded.

Definition 5.25. Let $L(V, W) = \{T : V \to W \mid T \text{ is BLT}\}$. For $T \in L(V, W)$, the *operator norm* of T is

$$\begin{aligned}\|T\| &:= \inf\{c \geq 0 \mid \|Tv\|' \leq c\,\|v\|\,, \forall v \in V\} \\ &= \sup\left\{ \frac{\|Tv\|'}{\|v\|} \,\middle|\, v \neq 0, v \in V \right\} \\ &= \sup\left\{\|Tv\|' \mid \|v\| = 1\right\}.\end{aligned}$$

Lemma 5.26. *(a) Above three definitions are equivalent.*

(b) It is indeed a normed space.

Proof. DIY. ■

5.6 Dual of L^p Spaces

Chapter 6

Signed and Complex Measures

[Fol99, Ch.3].

<u>RECALL</u> Suppose $(X, \mathcal{A}, \mu)$ a measure space. $f : X \to [0, \infty]$ measurable. Let $\nu(E) = \int_E f \, \mathrm{d}\mu, E \in \mathcal{A} \implies \nu$ is a measure on $(X, \mathcal{A})$.

6.1 Signed Measures

Definition 6.1. Suppose $(X, \mathcal{A})$ a measurable space. A signed measure is $\nu : \mathcal{A} \to [-\infty, \infty)$ or $\nu : \mathcal{A} \to (-\infty, \infty]$ such that

- $\nu(\emptyset) = 0$.
- $A_1, A_2, \ldots \in \mathcal{A}$, A_i disjoint $\implies \nu\left(\bigcup_1^\infty A_i\right) = \sum_1^\infty \nu(A_i)$ where the series converges absolutely if $\nu\left(\bigcup_1^\infty A_i\right) \in (-\infty, \infty)$.

Example 6.2.

- ν positive measure $\implies \nu$ is a signed measure.
- μ_1, μ_2 positive measures such that either $\nu_1(X) < \infty$ or $\nu_2(X) < \infty \implies \nu = \mu_1 - \mu_2$ a signed measure.
- $f : X \to \bar{\mathbb{R}}$ $s.t.$ $\int_X f^+ \, \mathrm{d}\mu < \infty$ or $\int_X f^- \, \mathrm{d}\mu < \infty \implies \nu(E) = \int_E f \, \mathrm{d}\mu$.

<u>NOTE</u>:

(a) $A \subset B \not\Rightarrow \nu(A) \leq \nu(B)$ since $\nu(B) = \nu(A) + \nu(B \setminus A)$.

(b) $A \subset B, \nu(A) = \infty \implies \nu(B) = \infty$.

Lemma 6.3. *ν is a signed measure on $(X, \mathcal{A})$. Then*

- $E_n \in \mathcal{A}, E_1 \subset E_2 \subset \dots \implies \nu\left(\bigcup_1^\infty E_n\right) = \lim_{n\to\infty} \nu(E_n)$.
- $E_n \in \mathcal{A}, E_1 \supset E_2 \supset \dots, -\infty < \nu(E_1) < \infty \implies \nu\left(\bigcap_1^\infty E_n\right) = \lim_{N\to\infty} \nu(E_n)$.

Definition 6.4. ν is a signed measure on $(X, \mathcal{A})$. Let $E \in \mathcal{A}$. We say

(a) E is *positive* for ν (a positive set for ν) if $\forall F \subset E, F \in \mathcal{A}$, $\nu(F) \geq 0$.

(b) E is *negative* for ν (a negative set for ν) if $\forall F \subset E, F \in \mathcal{A}$, $\nu(F) \leq 0$.

(c) E is *null* for ν (a null set for ν) if $\forall F \subset E, F \in \mathcal{A}$, $\nu(F) = 0$.

NOTE E positive set, $F \subset E \implies \nu(F) \leq \nu(E)$. E negative set, $F \subset E \implies \nu(F) \geq \nu(E)$.

Definition 6.5. Suupose μ, ν are signed measure on $(X, \mathcal{A})$. $\nu \perp \nu$ (singular to each other) means $\exists E, F \in \mathcal{A}\ s.t.\ E \cap F = \emptyset, E \cup F = X$, F is null for μ, E is null for ν.

Example 6.6. For $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$,

(a) Lebesgue measure m

(b) Cantor measure $\mu_C((a, b])$.

(c) Discrete measure $\mu_D = \delta_1 + 2\delta_{-1}$.

For (a), (c), take $E = \mathbb{R} \setminus \{-1, 1\}$, $F = \{-1, 1\}$. For (a), (b), take the cantor set K, $E = \mathbb{R} \setminus K, F = K$.

Lemma 6.7. *ν is a signed measure on $(X, \mathcal{A})$.*

(a) E is positive (for ν) and $G \subset E$ measurable $\implies$ G is positive (for ν).

(b) $E_1, E_2, \dots$ positive sets $\implies \bigcup_1^\infty E_n$ is positive.

Proof. DIY. ■

Lemma 6.8. *ν is a signed measure on $(X, \mathcal{A})$. Suppose $E \in \mathcal{A}$ and $0 < \nu(E) < \infty \implies \exists$ measurable set $A \subset E\ s.t.\ A$ is a positive set (for ν) and $\nu(A) > 0$.*

Proof in [RF10]. If E is a positive set, we are done.

Otherwise, E contains sets of negative measure. Let $n_1 \in \mathbb{N}$ be the smallest such that $\exists E_1 \subset E$ with $\nu(E_1) < -\frac{1}{n_1}$. If $E \setminus E_1$ is a positive set then we are done. Otherwise,

$E \setminus E_1$ contain sets of measure.

Inductively if $E \setminus \bigcup_1^{k_1} E_i$ is not a positive set. Let $n_k \in \mathbb{N}$ be the *smallest* such that $\exists E_k \subset E \setminus \bigcup_1^{k_1} E_i$ with $\nu(E_k) < -\frac{1}{n_k}$.

Note: if $n_k \geq 2, \forall B \subset E \setminus \bigcup_1^{k-1} E_i, \nu(B) \geq -\frac{1}{n_{k-1}}$.

Let $A = E \setminus \bigcup_1^\infty E_k$. Since $E = A \cup \bigcup_1^\infty E_k, \nu(E) = \nu(A) + \sum_1^\infty \nu(E_k) \implies \nu(A) > 0$.

Since $\nu(E), \nu(A)$ are finite, then $\sum_1^\infty \frac{1}{n_k}$ need to be convergent $\implies \lim_{k\to\infty} n_k = \infty$.

Now, if $B \subset A$ then $B \subset E \setminus \bigcup_1^{k-1} E_i$. If $\nu(B) \geq -\frac{1}{n_{k-1}} \implies \nu(B) \geq 0$. Thus A is positive. ■

Theorem 6.9 (The Hahn decomposition theorem)**.** *Suppose ν is a signed measure of $(X, \mathcal{A})$. Then $\exists P, N \in \mathcal{A}$ s.t. $P \cap N = \emptyset, P \cup N = X$, P is positive for ν, and N is negative for ν. If P', N' are another such pair, then $P \triangle P' (= N \triangle N')$ is null for ν.*

Proof. Uniqueness: $P \setminus P' \subset P \cap n' \implies P \setminus P'$ is positive and negative, thus a null set. Same for $P \setminus P'$.

Existence: WLOG assume $\nu : \mathcal{A} \to [-\infty, \infty)$. Let $s = \sup\{\nu(E) \mid E \text{ positive for } \nu\}$. $\exists P_1, P_2, \ldots$ positive sets such that $\lim_{n\to\infty} \nu(P_n) = s$.

Let $P = \bigcup_1^\infty E_n \implies P$ is positive $\implies \begin{cases} s \geq \nu(P) \\ \nu(P) \geq \nu(P_n) \end{cases} \implies \nu(P) = s$. Note that $0 \leq s = \nu(P) < \infty$.

Let $N = X \setminus P$. Is N a negative set?

Suppose not. Then $\exists E \subset N$ *s.t.* $\nu(E) > 0$. Note that $\nu(E) < \infty \implies \exists$ positive set $A \subset EA$ with $\nu(A) > 0$. The P, A are disjoint, $P \cup A$ is a positive set, and $\nu(P \cup A) = \nu(P) + \nu(A) > s$, a contradiction.

So N is a negative set. ■

Theorem 6.10 (Jordan decomposition theorem)**.** *ν signed measure on $(X, \mathcal{A})$. $\exists!$ positive measures ν^+, ν^- on $(X, \mathcal{A})$ s.t. $\nu(E) = \nu^+(E) - \nu^-(E), \forall E \in \mathcal{A}$ and $\nu^+ \perp \nu^-$.*

Proof. $\nu^+(E) = \nu(E \cap P), \nu^-(E) = -\nu(E \cap N)$. DIY. ■

Example 6.11. $(X, \mathcal{A}, \mu), f : X \to \bar{\mathbb{R}}$. Let $\nu(E) = \int_E f \,\mathrm{d}\mu$. $\nu^+ = \int_E f^+ \,\mathrm{d}\mu, \nu^- = \int_E f^- \,\mathrm{d}\mu$.

Definition 6.12. Suppose ν a signed measure on $(X, \mathcal{A})$. *Total variation measure* of ν is $|\nu| = \nu^+ + \nu^-$ (a positive measure on $(X, \mathcal{A})$).

Definition 6.13. $|\nu|(E) = \int_E |f| \,\mathrm{d}\nu$

Lemma 6.14. *(a)* $|\nu(E)| \leq |\nu|(E)$,

(b) E *is a null set for* $\nu \iff E$ *is a null set for* $|\nu|$,

(c) Suppose κ *is another signed measure.* $\kappa \perp \nu \iff \kappa \perp |\nu| \iff \kappa \perp \nu^+$ *and* $\kappa \perp \nu^-$.

Proof. DIY. ∎

Definition 6.15. ν is finite (σ-finite) if $|\nu|$ is a finite (σ-finite) measure. ($\iff \nu^+, \nu^-$ are finite (σ-finite) measures.)

6.2 Absolutely Measurable Spaces

Definition 6.16. μ a positive measure, ν a signed measure on $(X, \mathcal{A})$. $\nu \ll \mu$ (ν is absolutely continuous with respect to μ) $\iff$ $(E \in \mathcal{A}, \mu(E) = 0 \implies \nu(E) = 0) \iff$ all μ-null sets and ν-null sets. (check)

Example 6.17. $(X, \mathcal{A}, \mu), f : X \to \bar{\mathbb{R}}$. $\nu(E) = \int Ef \, \mathrm{d}\mu \implies \nu \ll \mu$.

<u>Notation</u>: $\mathrm{d}\nu = f \, \mathrm{d}\mu$ means ν is the measure defined by $\nu(E) = \int_E f \, \mathrm{d}\mu$.

Lemma 6.18. *μ positive measure, ν signed measure.*

(a) $\nu \ll \mu \iff |\nu| \ll \mu \iff \nu^+ \ll \mu$ *and* $\nu^- \ll \mu$.

(b) $\nu \ll \mu$ *and* $\nu \perp \mu \implies \nu = 0$.

Proof. ∎

Theorem 6.19 (Radon-Nikodym)**.** *ν a σ-finite positive measure, ν a σ-finite signed measure on $(X, \mathcal{A})$. Suppose $\nu \ll \mu$. Then $\exists f : X \to \bar{\mathbb{R}}$ measurable function such that $\nu(E) = \int_E f \, \mathrm{d}\mu$. If g is another such function then $f = g$ a.e.*

Proof. Will follow by proof of Lebesgue-Radon-Nikodym on Monday. ∎

Definition 6.20. Suppose $\nu \ll \mu$. A *Radon-Nikodym derivative* of ν with respect to μ is a function $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} : X \to \bar{\mathbb{R}}$ satisfying $\nu(E) = \int_E \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \, \mathrm{d}\mu, \forall E \in \mathcal{A}$.

<u>Note</u>: 6.19 shows the existence of such functions. If there is another such function g, then $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} = g$ μ-a.e.

<u>Notation</u>:

$$\mathrm{d}\nu = \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \, \mathrm{d}\mu.$$

Example 6.21. $F(x) = e^{2x} : \mathbb{R} \to \mathbb{R}$ is continuous and increasing.

The Lebesgue-Stieltjes measure μ_F on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is the unique locally finite Borel measure satisfying $\mu((a,b]) = e^{2b} - e^{2a}, \forall a < b$.

$$\mu_F(E) \overset{\text{why?}}{=} \int_E 2e^{2x} \, \mathrm{d}x.$$

So $\mu_F \ll m$ and $\frac{\mathrm{d}\mu_F}{\mathrm{d}m} = 2e^{2x}$.

Example 6.22. $F(x) = C(x) : \mathbb{R} \to \mathbb{R}$ the Cantor function. $C'(x) = 0$ Lebesgue a.e.

$$\mu_C(E) \neq \int_E 0 \, \mathrm{d}x.$$

In particular, $c(b) - c(a) \neq \int_a^b c'(x) \, \mathrm{d}x$ even if c is continuous and has derivative a.e. So $\mu_c \not\ll m$. But $\mu_c \perp m$.

6.3 Lebesgue Differentiation Theorem for Regular Borel Measures on $\mathbb{R}^d$

[Fol99, p. 99]

Definition 6.23. A Borel signed measure ν on $\mathbb{R}^d$ is called *regular* if

(a) $|\nu|(\kappa) < \infty, \forall$ compact K.

(b) $|\nu|(E) = \inf\{m(O) \mid \text{ open } O \supset E\}, \forall$ Borel set E.

Example 6.24. LS measure on $\mathbb{R}$ are regular. Lebesgue measure on $\mathbb{R}^d$ is regular (so, the difference of two of them) Note: from (a), ν regular $\implies$ ν is σ-finite,

If $\mathrm{d}\nu = f \, \mathrm{d}m$ regular, then $|\nu|(\kappa) = \int_K |f| \, \mathrm{d}m < \infty$, so $f \in L^1_{\text{loc}}(\mathbb{R}^d)$.

Lemma 6.25. *If $f \in L^1_{\text{loc}}(\mathbb{R}^d) \iff \mathrm{d}\nu = f \, \mathrm{d}m$ is regular*

Proof. Read the book. ∎

<u>RECALL</u> Lebesgue differentiation theorem

Corollary 6.26. *Let ρ be a* regular *signed Borel measure on $\mathbb{R}^d$. Suppose $\rho \ll m \implies$ For Lebesgue a.e.-x, $\lim_{r\to 0} \frac{\rho(E_r)}{m(E_r)} = \frac{\mathrm{d}\rho}{\mathrm{d}m}(x)$ for every $E_r \to x$ nicely.*

Proposition 6.27. *Let λ be a* regular *positive Borel measure on $\mathbb{R}^d$. Suppose $\lambda \perp m$. For Lebesgue a.e.-x, $\lim_{r\to 0} \frac{\lambda(E_1)}{m(E_1)} = 0$ for every $E_r \to x$ nicely.*

Proof. Enough to consider $E_1 = B(x, r)$

$$\left\{x \mid \limsup_{r\to 0} \frac{\lambda(E_1)}{m(E_1)} \neq 0\right\} = \bigcup_{n=1}^{\infty} G_n, G_n = \left\{x \mid \limsup_{r\to 0} \frac{\lambda(E_1)}{m(E_1)} > \frac{1}{n}\right\}$$

Enough to show that $m(G_n) = 0, \forall n$.

$\lambda \perp m \implies \mathbb{R}^d = A \cup B$ disjoint. $\lambda(A) = 0, m(B) = 0$, Enough to show $m(G_n \cap A) = 0$.

Fix $\varepsilon > 0$. Since λ is regular, $\exists$ open $O \supset A$ $s.t.$ $\lambda(O) \le \lambda(A) + \varepsilon = \varepsilon$. $\forall x \in G_n \cap A, \exists r_x > 0$ $s.t.$ $\frac{\lambda(B(x,r_x))}{m(B(x,r_x))} > \frac{1}{n}$ and $B(x, r_x) \subset O$.

Let $K \subset G_n \cap A$, compact. $K \subset \bigcup_{x\in K} B(x, r_x) \implies \exists$ finite subcover $\implies \exists B_1, B_2, \ldots, E_N$ disjoint, $K \subset \bigcup_1^N 3B_i$.

$\implies m(K) \le 3^d \sum_1^N m(B_i) \le 3^d n \sum_1^N \lambda(B_i) = 3^d n \lambda\left(\bigcap_1^N B_i\right) \le 3^d n \lambda(O) \le 3^d n\varepsilon \implies m(G_n \cap A) \le 3^d n\varepsilon$. ∎

Theorem 6.28 (LDT for regular Borel measures)**.** *Suppose ν is a regular Borel signed meaaure on $\mathbb{R}^d$ and $\mathrm{d}\nu = \mathrm{d}\lambda + f\,\mathrm{d}m$, $\lambda \perp m \implies$ for Leb a.e. x, $\lim_{r\to 0} \frac{\nu(E_r)}{m(E_r)} = f(x)$ for every $E_r \to x$ nicely.*

Proof. ν regular $\implies \lambda, f\,\mathrm{d}m$ are regular. ∎

6.4 Monotone Differentiation Theorem

[Fol99, 3.5]

Definition 6.29. For $F : \mathbb{R} \to \mathbb{R}$ that is increasing, denote $F(x+) = \lim_{y\downarrow x} F(y) = \inf_{y>x} F(y), F(x-) = \lim_{y\uparrow x} F(y) = \sup_{y<x} F(y)$.

Lemma 6.30. *F is increasing $\implies D = \{x \mid F$ is discontinuous at $x\}$ is countable.*

Proof. $x \in D \implies F(x+) > F(x-)$ since $F \nearrow$. For $x, y \in D, x \neq y \implies I_x, I_y$ disjoint. For each $x \in D$, let $I_x = (F(x-), F(x+)) \implies \exists f : D \to \mathbb{Q}$ is 1-1. I_x is open interval, not empty $\implies D$ is countable. ∎

Theorem 6.31 (Monotone differentiation theorem)**.** *Suppose $F \nearrow \implies$*

- *F is differentiable Lebesgue a.e.*
- *$G(x) = F(x+)$ is differentiable Lebesgue a.e.*
- *$G' = F'$ a.e.*

Proof. G is increasing, right-continuous on $\mathbb{R} \implies \exists$ Lebesgue-Stieltjes measure μ_G on $\mathbb{R}$ (so, regular).

$$\frac{G(x+h)-G(x)}{h} = \begin{cases} \dfrac{\mu_G((x,x+h])}{m((x,x+h])} & h>0, \\ \dfrac{\mu_G((x+h,x])}{m((x+h,x])} & h<0 \end{cases}$$

converges for Lebesgue a.e x by LDT. So G' exists a.e.

Let $H(x) = G(x) - F(x) \geq 0$. We have

$$\{x \mid H(x) > 0\} \subset \{x \mid x \text{ is discontinuous at } x\}.$$

So $\{x \mid H(x) > 0\}$ it is countable. Denote the set as $\{x_n\}$.

Let $\mu = \sum_n H(x_n)\delta_{x_n}$. Then

$$\mu((-N,N)) = \sum_{x_n \in (-N,N)} H(x_n) \overset{check}{\leq} G(N) - F(-N) < \infty.$$

So μ is a locally finite Borel measure on $\mathbb{R} \implies \mu$ is regular. Hence

$$\left| \frac{H(x+h)-H(x)}{h} \right| \leq \frac{H(x+h)+H(x)}{|h|} \leq 4\frac{\mu((x-2h,x+2h))}{4|h|} \xrightarrow{\text{LDT},\mu\perp m} 0$$

for Lebesgue a.e. x.

So H is differentiable a.e and $H' = 0$ a.e. ■

Proposition 6.32. $F \nearrow \implies \displaystyle\int_a^b F'(x)\,\mathrm{d}x \leq F(b) - F(a)$.

Example 6.33.

- $F(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$. $F'(x) = 0$ a.e and $\displaystyle\int_{-1}^1 F'(x)\,\mathrm{d}x = 0 < F(1) - F(-1) = 1$.
- $F(x)$ Cantor function. $F'(x) = 0$ a.e. and $\displaystyle\int_0^1 F'(x)\,\mathrm{d}x = 0 \leq F(1) - F(0) = 1$.

6.5 Functions of Bounded Variation

Definition 6.34. For $F : \mathbb{R} \to \mathbb{R}$, the total variation function of F is $T_F : \mathbb{R} \to [0, \infty]$,

$$T_F(x) = \sup \left\{ \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \mid n \in \mathbb{N}, -\infty < x_0 < x_1 < \ldots < x_n = x \right\}.$$

Lemma 6.35. *For $a < b$,*

$$T_F(b) = T_F(a) + \sup \left\{ \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \mid n \in \mathbb{N}, a = x_0 < x_1 < \ldots < x_n = b \right\}$$

Note that T_F is increasing.

Definition 6.36. $F \in \mathrm{BV}$ (F is of bounded variation) means $T_F(\infty) = \lim_{x\to\infty} T_F(x) < \infty$.

$F \in \mathrm{BV}([a, b])$ means $\sup \left\{ \sum_1^N |F(x_i) - F(x_{i-1})| \mid a = x_0 < x_1 < \ldots < x_n = b \right\} < \infty$.

Note that $F \in \mathrm{BV} \implies F$ is bounded.

Example 6.37.

(a) $F(x) = \sin x \notin \mathrm{BV}, \in \mathrm{BV}([a, b])$.

(b) $F(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases} \notin \mathrm{BV}([a, b])$ for $a < 0 < b$.

(c) $F, G \in \mathrm{BV} \implies \alpha F + \beta G \in \mathrm{BV}$.

(d) $F \nearrow$ and bounded $\implies F \in \mathrm{BV}$.

(e) F *Lipschitz* on $[a, b] \implies F \in \mathrm{BV}([a, b])$. (*Lipschitz* $\implies \exists M \geq 0$ *s.t.* $|F(x)-F(y)| \leq M|x - y|, \forall x, y$.)

(f) F differentiable, F' bounded on $[a, b] \implies F \in \mathrm{BV}([a, b])$.

(g) $F(x) = \int_{-\infty}^{x} f(t) \in L^1(\mathbb{R}) \implies F \in \mathrm{BV}$ since

$$\sum_1^N |F(x_i) - F(x_{i-1})| \leq \sum_1^N \int_{x_{i-1}}^{x_i} |f(t)| \,\mathrm{d}t = \int_{x_0}^{x} |f(t)| \,\mathrm{d}t \leq \int_{-\infty}^{\infty} |f(t)| \,\mathrm{d}t < \infty.$$

Definition 6.38. $\mathrm{NBV} = \{G \in \mathrm{BV} \mid G \text{ right-continuous}, G(-\infty) = 0\}$.

Example 6.39.

(a) $F \nearrow$, bounded, right-continuous, $F(-\infty) = 0$.

(b) $F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t$, $f \in L^1(\mathbb{R})$. (Midterm $\implies$ F is uniformly continuous.)

Lemma 6.40. $F \in \mathrm{BV}$ *and right-continuous* $\implies T_F \in \mathrm{NBV}$.

Proof. $T_F \nearrow$, bounded $\implies T_F \in \mathrm{BV}, T_F(-\infty) = 0$. Is T_F right-continuous?

Suppose it is not. $\exists a \in \mathbb{R}$ *s.t.* $c := T_F(a+) - T_F(a) > 0$. Fix $\varepsilon > 0$. Since $F(x)$ and $g(x) := T_F(x+)$ are right continuous, $\exists \delta > 0$ *s.t.*

$$|F(y) - F(a)| < \varepsilon, \quad |g(y) - g(a)| < \varepsilon \quad \forall y \in (a, a+\delta].$$

So $T_F(y) - T_F(a+) \leq T_F(y+) - T_F(a+) < \varepsilon$.

$\exists a = x_0 < x_1 < x_2 < \ldots < x_n = a + \delta$ *s.t.*

$$\begin{aligned}\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| &\geq T_F(a+\delta) - T_F(a) - \frac{c}{4} \\ &\geq T_F(a+) - T_F(a) - \frac{c}{4} = \frac{3}{4}c.\end{aligned}$$

This shows that $\sum_{i=2}^{n} |F(x_i) - F(x_{i-1})| \geq \frac{3}{4}c - \varepsilon$ since

Consider $[a, x_1]$. $\exists a = t_0 < t_1 < \ldots < t_k = x_1$ *s.t.*

$$\sum_{i=1}^{k} |F(t_i) - F(t_{i-1})| \geq T_F(x_1) - T_F(a) - \frac{c}{4} \geq \frac{3}{4}c.$$

So we can write $[a, a+\delta] = [a, x_1] \cup [x_1, a+\delta]$. So

$$\begin{aligned}\varepsilon + c &\geq T_F(a+\delta) - T_F(a+) + T_F(a+) - T_F(a) \\ &= T_F(a+\delta) - T_F(a) \\ &\geq \sum_{j=1}^{k} |F(t_j) - F(t_{j-1})| + \sum_{i=2}^{n} |F(x_i) - F(x_{i-1})| \geq \frac{3}{4}c - \varepsilon + \frac{3}{4}c = \frac{3}{2} - \varepsilon \\ \implies c &\leq 4\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $c = 0$, a contradiction. ∎

Corollary 6.41. $F \in \mathrm{NBV} \iff F = F_1 - F_2, F_1, F_2 \in \mathrm{NBV}$ *and* $\nearrow$.

Proof. ∎

Theorem 6.42.

(a) μ *is a finite* signed *Borel measure on* $\mathbb{R} \implies F(x) := \mu((-\infty, x]) \in \mathrm{NBV}$.

(b) $F \in \mathrm{NBV} \implies \exists!$ *finite* signed *Borel measure* μ_F *on* $\mathbb{R}$ *satisfying* $\mu((-\infty, x]) = F(x)$.

Proof. (a) $\mu = \mu^+ - \mu^- \implies F = F^+ - F^-, F^{\pm}(x) = \mu^{\pm}((-\infty, x])$ is increasing, bounded, right-continuous, and $F^{\pm}(-\infty) = 0$.

(b) $F \in \mathrm{NBV} \implies F = F_1 - F_2, F_1, F_2 \in \mathrm{NBV}$ and are increasing. So $\exists \mu_{F_1}, \mu_{F_2}$ Lebesgue-Stieltjes measure. $\mu_F := \mu_{F_1} - \mu_{F_2}$. Uniqueness is left for homework. ∎

Proposition 6.43. *Let* $F \in \mathrm{NBV}$. *Then*

(a) F *is differentiable a.e,* $F \in L^1(\mathbb{R}, m)$.

(b) $\mathrm{d}\mu_F = \mathrm{d}\lambda + F'\mathrm{d}m, \lambda \perp m$.

(c) $\mu_F \perp m \iff F' = 0$ *Lebesgue a.e.*

(d) $\mu_F \ll m \iff \displaystyle\int_{-\infty}^{x} F'(t)\, \mathrm{d}t = F(x)$.

Proof. Check (a), (b), (c).

(d) $\mu_F \ll m \iff \lambda = 0 \iff \mathrm{d}\mu_F = F'\mathrm{d}m \iff \mu_F = \int_E F'\, \mathrm{d}m, \forall E$ Borel $\iff F(x) = \int_{-\infty}^{x} F'(t)\, \mathrm{d}t, \forall x \in \mathbb{R}$. (by uniqueness) ∎

6.6 Absolutely Continuous Functions

Definition 6.44. $F : \mathbb{R} \to \mathbb{R}$ is absolutely continuous ($F \in \mathrm{AC}$) means $\forall \varepsilon > 0, \exists \delta > 0$ *s.t.* if $(a_1, b_1), \ldots, (a_N, b_N)$ are *disjoint* open intervals satisfying $\sum_{n=1}^{N}(b_n - a_n) < \delta$, then $\displaystyle\sum_{n=1}^{N} |F(b_n) - F(a_n)| < \varepsilon$.

Lemma 6.45. *(a)* $F \in \mathrm{AC} \implies F$ *is uniformly continuous.*

(b) F *is Lipschitz* $\implies F \in \mathrm{AC}$.

(c) $F(x) = \displaystyle\int_{\infty}^{x} f(t)\, \mathrm{d}t, f \in L^1 \implies F \in \mathrm{AC}$.

Proof.

$$\sum_{n=1}^{N} |F(b_n) - F(a_n)| = \sum_{1}^{N} \left| \int_{a_n}^{b_n} f(t)\, \mathrm{d}t \right| \leq \sum_{1}^{N} \int_{a_n}^{b_n} |f(t)|\, \mathrm{d}t = \int_E |f|\, \mathrm{d}m$$

where $E = \bigcup_1^N (a_n, b_n)$. By midterm Q1, If $f \in L^1(X, \mu)$ then $\forall \varepsilon > 0, \exists \delta > 0$ *s.t.* $\mu(E) < \delta \implies \int_E |f| < \varepsilon$. ■

The inverse of (a) is not always true. The Cantor function $C(x)$ is uniformly continuous but $C \notin \mathrm{AC}$.

Proposition 6.46. *Suppose $F \in \mathrm{NBV}$. Then $F \in \mathrm{AC} \iff \mu_F \ll m$.*

Corollary 6.47. $F \in \mathrm{NBV} \cap \mathrm{AC} \iff F(x) = \displaystyle\int_\infty^x f(t)\,\mathrm{d}t$ *for some $f \in L^1(\mathbb{R}, m)$. If this holds, $f = F'$ Lebesgue a.e.*

Lemma 6.48. $F \in \mathrm{AC}([a,b]) \implies F \in \mathrm{NBV}([a,b])$.

Proof. Check. (read the textbook) ■

Theorem 6.49 (Fundamental theorem of Calculus)**.** *For $F : [a,b] \to \mathbb{R}$, TFAE:*

(a) $F \in \mathrm{AC}([a,b])$,

(b) $F(x) - F(a) = \displaystyle\int_a^x f(t)\,\mathrm{d}t$ *for some $f \in L^1([a,b], m)$,*

(c) F is differentiable a.e on $[a,b]$ and $F(x) - F(a) = \int_a^x F'(t)\,\mathrm{d}t$.

Proof of Prop. $\impliedby$: Suppose $\mu_F \ll m$. Then $F(x) = \int_{-\infty}^x F'(t)\,\mathrm{d}t, F' \in L^1 \implies F \in \mathrm{AC}$.

$\implies$: Suppose $F \in \mathrm{AC}$.

Note: since F is continuous, $\mu_F((a,b]) = \lim_{n\to\infty} \mu_F\left((a, b - \frac{1}{n}]\right) = \lim_{h\to\infty} F\left(b - \frac{1}{n}\right) - F(a) = F(b) - F(a)$.

Let E be a Borel set with $m(E) = 0$. Fix $\varepsilon > 0$. Let $\delta > 0$ be the constant from $F \in \mathrm{AC}$. Since m and μ_F are *regular*,

$$\exists \text{ open } U_1 \supset U_2 \supset \ldots \supset E \textit{ s.t. } \lim_{n\to\infty} m(U_n) = m(E) = 0,$$
$$\exists \text{ open } V_1 \supset V_2 \supset \ldots \supset E \textit{ s.t. } \lim_{n\to\infty} \mu_F(V_n) = \mu_F(E).$$

Let $O_n = U_n \cap V_n$. O_n is open and $O_1 \supset O_2 \supset \ldots \supset E$. Then

$$\lim_{n\to\infty} m(O_n) = m(E) = 0, \quad \lim_{n\to\infty} \mu_F(O_n) = \mu_F(E) \text{ (think about it)}.$$

WLOG, we may assume $m(O_1) < \delta$. Each $O_n = \bigcup_{k=1}^\infty (a_k^n, b_K^n)$ disjoint, $\sum_{k=1}^N (b_k^n, a_k^n) \leq$

$m(O_n) \leq m(O_1) \leq \delta \implies$

$$\mu_F\left(\bigcup_{k=1}^N (a_k^n, b_K^n)\right) = \sum_{k=1}^N \mu_F(a_k^n, b_K^n) = \sum_{k=1}^N F(b_k^n) - F(a_k^n).$$

Take the absolute value we have

$$\left|\mu_F\left(\bigcup_{k=1}^N (a_k^n, b_K^n)\right)\right| \leq \sum_{k=1}^N |F(b_k^n) - F(a_k^n)| < \varepsilon.$$

Hence

$$|\mu_F(O_n)| = \lim_{n\to\infty}\left|\mu_F\left(\bigcup_{k=1}^N (a_k^n, b_K^n)\right)\right| \leq \varepsilon \implies |\mu_F(E)| = \lim_{n\to\infty} |\mu_F(O_n)| \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary we conclude that $\mu_F(E) = 0$. ■

Definition 6.50. Suppose μ a finite signed Borel measure on $\mathbb{R}$.

- μ is a *discrete* measure means $\exists$countable set $\{x_n\}$ and $c_n \neq 0$ $s.t.$ $\sum_1^\infty c_n < \infty$ and $\mu = \sum_n c_n \delta_{x_n}$.
- μ is a *continuous* measure means $\mu(\{a\}) = 0, \forall a \in \mathbb{R}$.

Lemma 6.51. *(a) $\mu = \mu_d + \mu_c$ uniquely, where μ_d is a discrete measure and μ_c is a continuous measure.*

(b) μ discrete $\implies$ $\mu \perp m$.

(c) $\mu \ll m \implies \mu$ is continuous.

Corollary 6.52. *Suppose μ is finite signed Borel measure on $\mathbb{R}$. Then μ can be uniquely written as*

$$\mu = \mu_d + \mu_{ac} + \mu_{sc}$$

where $\mu_{ac} \in$ AC and μ_{sc} is singularly continuous (continuous and $\perp$ m).

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