Notes for Math 597 – Real Analysis

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Contents

Office hour is Mon 12:30 - 1:30, Tue 12:30 - 1:30 in person EH 5838, Th 1 - 2 online.

Chapter 1

Abstract Measure

1.1 σ-Algebra

Definition 1.1. Let X be a set. A collection $\mathcal{M}$ of subsets of X is called a σ-algebra on X if

- $\emptyset \in \mathcal{M}$.
- $\mathcal{M}$ is closed under complements: $E \in \mathcal{M} \implies E^c \in \mathcal{M}$.
- $\mathcal{M}$ is closed under countable unions: $E_1, E_2, \ldots \in \mathcal{M} \implies \bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$.

SIMPLE PROPERTIES:

- $X = \emptyset^c \in \mathcal{M}$.
- $\bigcap_{i=1}^{\infty} E_i = \left(\bigcup_{i=1}^{n} E_i^c\right)^c \in \mathcal{M}$. It is closed under countable intersections.
- $\bigcup_{i=1}^{N} E_i = E_i \cup \ldots \cup E_n \cup \emptyset \cup \ldots$. It is closed under finite unions (similarly, intersections). sigma
- $E \setminus F = E \cap F^c \in \mathcal{M}, E \triangle F = (E \cap F^c) \cup (F \cap E^c) \in \mathcal{M}$.

Example 1.2. (a) $\mathcal{A} = \mathcal{P}(X)$ power algebra.

(b) $\mathcal{A} = \{\emptyset, X\}$ trivial algebra.

(c) Let $B \subset X, B \neq \emptyset, B \neq X. \mathcal{A} = \{\emptyset, B, B^c, X\}$.

Lemma 1.3. *(An intersection of σ-algebras is a σ-algebra) Let $\mathcal{A}_\alpha, \alpha \in I$, be a family a σ-algebras of X. Then $\bigcap_{\alpha \in I} A_\alpha$ is a σ-algebra. (I can be uncountable.)*

Proof. DIY ■

Definition 1.4. For $\mathcal{E} \subset \mathcal{P}(X)$ (not necessarily a σ-algebra), let $\langle \mathcal{E} \rangle$ be the intersection of all σ-algebras on X that contains $\mathcal{E}$. Call it the σ-algebra generated by $\mathcal{E}$.

- $\langle \mathcal{E} \rangle$ is the smallest σ-algebra containing $\mathcal{E}$ and is unique.
- $\{\emptyset, B, B^c, X\} = \langle \{B\} \rangle = \langle \{B^c\} \rangle = \langle \{\emptyset, B\} \rangle$.

The above definition gives us (potentially) lots of examples of σ-algebra on a set X

Lemma 1.5. *(a) Suppose $\mathcal{E} \subset \mathcal{P}(X), \mathcal{A}$ a σ-algebra on X. $\mathcal{E} \in \mathcal{A} \implies \langle \mathcal{E} \rangle \in \mathcal{A}$.*

(b) $E \subset F \subset \mathcal{P}(X) \implies \langle \mathcal{E} \rangle \subset \langle \mathcal{F} \rangle$.

Proof. ∎

Definition 1.6. For a topological space X, the Borel σ-algebra $\mathcal{B}(X)$ is the σ-algebra generated by the collection of open sets.

Example 1.7. $(X = \mathbb{R})$ $\mathcal{B}(\mathbb{R})$ contains the following collections

$$\mathcal{E}_1 = \{(a,b) \mid a < b\}, \quad \mathcal{E}_2 = \{[a,b] \mid a < b\},$$

$$\mathcal{E}_3 = \{(a,b] \mid a < b\}, \quad \mathcal{E}_4 = \{[a,b) \mid a < b\},$$

$$\mathcal{E}_5 = \{(a,\infty) \mid a \in \mathbb{R}\}, \quad \mathcal{E}_6 = \{[a,\infty) \mid a \in \mathbb{R}\},$$

$$\mathcal{E}_7 = \{(-\infty,a) \mid a \in \mathbb{R}\}, \quad \mathcal{E}_8 = \{(-\infty,a] \mid a < b\}$$

Proposition 1.8. $\mathcal{B}(\mathbb{R}) = \langle \mathcal{E}_i \rangle$ *for each* $i = 1, \ldots, 8$.

Proof. Use 1.5. ∎

Definition 1.9. $(X, \mathcal{A})$ is called a measurable space.

1.2 Measures

Definition 1.10. A measure on $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \to [0, \infty]$ $s.t.$

(a) $\mu(\emptyset) = 0$

(b) (countable additive) For $A_1, A_2, \ldots \in \mathcal{A}$ disjoint we have

$$\mu\left(\bigcup_1^\infty A_i\right) = \sum_{i=1}^\infty \mu(A_i).$$

$(X, \mathcal{A}, \mu)$ is then called a measure space.

Example 1.11. (a) For any $(X, \mathcal{A})$, $\mu(A) = \#A$ counting measure.

(b) For any $(X, \mathcal{A})$, let $x_0 \in X$. The Dirac measure at x_0 is

$$\mu(A) = \begin{cases} 1 & x_0 \in A, \\ 0 & x_0 \notin A. \end{cases}$$

(c) For $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, let $a_1, a_2, \ldots \in [0, \infty)$. $\mu(A) = \sum_{i \in A} a_i$ is a measure.

$(X, \mathcal{A})$ measurable space

$(X, \mathcal{A}, \mu)$ measure space

$\mu : \mathcal{A} \to [0, \infty]$ $s.t.$ $\mu(\emptyset) = 0$, countable additivity.

<u>NOTE</u>: $A, B \in \mathcal{A}, A \subset B$, then $\mu(B \setminus A) + \mu(A) = \mu(B) \implies \mu(B \setminus A) = \mu(B) - \mu(A)$ if $\mu(A) < \infty$.

Theorem 1.13. *$(X, \mathcal{A}, \mu)$ measure space*

(a) (monotonicity)
$$A, B \in \mathcal{A}, A \subset B \implies \mu(A) \leq \mu(B).$$

(b) (countable subadditivity)

$$A_1, A_2, \ldots, \in \mathcal{A}, \implies \mu\left(\bigcup_i^\infty A_i\right) \leq \sum_i^\infty \mu(A_i).$$

(c) (continuity from below/(MCT) from sets)

$$A_1, A_2, \ldots \in \mathcal{A}, A_1 \subset A_2 \subset A_3 \subset \ldots \implies \mu\left(\bigcup_i^\infty A_i\right) = \lim_{n\to\infty} \mu(A_n).$$

(d) (continuity from above)

$$A_1, A_2, \ldots \in \mathcal{A}, A_1 \supset A_2 \supset A_3 \supset \ldots, \mu(A_1) < \infty \implies \mu\left(\bigcap_i^\infty A_i\right) = \lim_{n\to\infty} \mu(A_n).$$

Proof. (a), (b), DIY.

For (c), let $B_1 = A_1, B_i = A_i \setminus A_{i-1}, i \geq 2. B_i \in \mathcal{A}$ and are disjoint.

$$\bigcup_i^\infty A_i = \bigcup_i^\infty B_i$$
$$\implies \mu\left(\bigcup_i^\infty A_i\right) = \mu\left(\bigcup_i^\infty B_i\right) = \sum_i^\infty \mu(B_i) = \lim_{n\to\infty} \sum_i^n \mu(B_i) = \lim_{n\to\infty} \mu(A_n).$$

For (d), let $E_i = A_1 \setminus A_i$. Hence $E_i \in \mathcal{A}, E_1 \subset E_2 \subset \dots$ We have

$$\bigcup_i^\infty E_i = \bigcup_i^\infty (A_1 \setminus A_i) = A_1 \setminus \left(\bigcap_1^\infty A_i\right) \implies \bigcap_1^\infty A_i = A_1 \setminus \left(\bigcup_1^\infty E_i\right).$$

Hence

$$\mu\left(\bigcap_1^\infty A_i\right) = \mu(A_1) - \mu\left(\bigcup_1^\infty E_i\right) = \mu(A_1) - \lim_{n\to\infty} \mu(E_n) = \mu(A_1) - \lim_{n\to\infty} \mu(A_1) - \mu(A_n).$$

■

<u>NOTE:</u> the condition that $\mu(A_1) < \infty$ cannot be dropped.
For example, in $(\mathbb{N}, \mathcal{P}(N)$, counting measure), let $A_n = \{n, n+1, n+2\}, A_1 \supset A_2 \supset A_3 \supset \dots$ We have $\bigcap_1^\infty = \emptyset \implies \mu\left(\bigcap_1^\infty A_i\right) = 0$.

Definition 1.14. For $(X, \mathcal{A}, \mu)$ measure space,

- $A \subset X$ is a μ-null set if $A \in \mathcal{A}, \mu(A) = 0$.
- $A \subset X$ is a μ-subnull set if $\exists B$, μ-null set $A \subset B$.
- $(X, \mathcal{A}, \mu)$ is a complete measure space if every μ-subnull set is $\mathcal{A}$-measurable.

Definition 1.15. $(X, \mathcal{A}, \mu)$ measure space. A statement $P(x), x \in X$ holds μ-almost everywhere (a.e.) if the set $\{x \in X \mid P(x)$does not hold$\}$ is μ-null.

Definition 1.16. $(X, \mathcal{A}, \mu)$ measure space.

- μ is a <u>finite measure</u> is $\mu(X) < \infty$.
- μ is a <u>σ-finite measure</u> if $X = \bigcup_1^\infty X_n, X_n \in \mathcal{A}, \mu(X_n) < \infty$.

HW: every measure space can be "completed."

1.3 Outer Measures

Definition 1.17. An <u>outer measure</u> on X is $\mu^* : \mathcal{P}(X) \to [0, \infty]$ $s.t.$

- $\mu^*(\emptyset) = 0$
- (monotonicity) $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$.
- (countable subadditivity)

$$\forall A_1, A_2, \ldots \in X, \mu^*\left(\bigcup_i^\infty A_i\right) \leq \sum_i^\infty \mu^*(A_i).$$

Example 1.18. For $A \subset \mathbb{R}$,

$$\mu^*(A) = \inf\left\{\sum_{i=1}^\infty (b_i - a_i) \,\middle|\, \bigcup_1^\infty (a_i, b_i) \supset A\right\}.$$

is an outer measure due to the next proposition.

Proposition 1.19. *(1.19) Let $\mathcal{E} \in \mathcal{P}(X)$ s.t. $\emptyset, X \in \mathcal{E}$. Let $\rho : \mathcal{E} \to [0, \infty]$ s.t. $\rho(\emptyset) = 0$. Then*

$$\mu^*(A) = \inf\left\{\sum_{i=1}^\infty \rho(E_i) \,\middle|\, E_i \in \mathcal{E}, \forall i \in N, \bigcup_1^\infty E_i \supset A\right\}$$

is an outer measure on X.

Proof. (a) μ^* is well-defined (inf is taken over non-empty set.)

(b) $\mu^*(\emptyset) = 0$

(c) $A \subset B \implies \mu^*(A) \leq \mu^*(B)$.

We check the countable subadditivity.

Let $A_1, A_2, \ldots \subset X$. If one of $\mu^*(A_i) = \infty$, then the result holds. Suppose $\mu^*(A_n) < \infty, \forall n \in \mathbb{N}$.

"Give your self a room of epsilon":

Fix $\varepsilon > 0$. We will show

$$\mu^*\left(\bigcup_1^\infty A_n\right) \leq \sum_1^\infty \mu^*(A_i) + \varepsilon.$$

For each $n \in \mathbb{N}, \exists E_{n,1}, E_{n,2}, \ldots \in \mathcal{E}$ *s.t.*

$$\bigcup_{k=1}^\infty E_{n,k} \supset A_n \quad \text{and} \quad \mu^*(A_n) + \frac{\varepsilon}{2^n} \geq \sum_{k=1}^\infty \rho(E_{n,k}).$$

Then,

$$\bigcup_1^\infty A_n \subset \bigcup_{n=1}^\infty \bigcup_{k=1}^\infty E_{n,k} = \bigcup_{(n,k)\in\mathbb{N}^2} E_{n,k}.$$

<u>RECALL:</u> Tonelli's thm for series. If $a_{ij} \in [0,\infty], \forall i,j \in \mathbb{N}$, then

$$\sum_{(i,j)\in\mathbb{N}^2} a_{ij} = \sum_{i=1}^\infty \sum_{j=1^\infty} a_{ij} = \sum_{j=1}^\infty \sum_{i=1}^\infty a_i j.$$

Hence

$$\mu^*\left(\bigcup_{n=1}^\infty A_n\right) \le \sum_{n=1}^\infty \rho(E_{k,n}) = \sum_{n=1}^\infty \sum_{k=1}^\infty \rho(E_{k,n}) \le \sum_{n=1}^\infty \left(\mu^*(A_n) + \frac{\varepsilon}{2^n}\right) = \sum_{n=1}^\infty \mu^*(A_n) + \varepsilon.$$

We have shown countable subadditivity. ■

Outer measure is very close to a measure. Here the textbooks diverge.

Tao: introduce Lebesgue measure on $\mathbb{R}$ using topological qualities of subsets of $\mathbb{R}$.
Folland: introduce abstract method by Carathéodory and Kolmogorov.

Definition 1.20. Let μ^* be an outer measure on X. We say $A \subset X$ is Carathéodory measurable with respect to μ^* if $\forall E \subset X, \mu^*(E) = \mu^*(E \setminus A) + \mu^*(E \cap A)$.

Lemma 1.21. *Let μ^* be an outer measure on X. Suppose $B_1, B_2, \ldots, B_N$ are disjoint C-measurable sets. Then,*

$$\forall E \subset X, \mu^*\left(E \cap \left(\bigcup_1^N B_i\right)\right) = \sum_{i=1}^n \mu^*(E \cap B_i)$$

Proof.

$$\mu^*\left(E \cap \left(\bigcup_1^N B_i\right)\right) = \mu^*(E \cap B_1) + \mu^*\left(E \cap \left(\bigcup_2^N B_i\right)\right)$$

because B_1 is C-measurable. Then, iterate. ■

Improved version:

$B_1, B_2, \ldots$ C-measurable and <u>disjoint</u> $\implies \mu^*\left(E \cap \bigcup_1^\infty B_n\right) = \sum_1^\infty \mu^*\left(E \cap B_n\right), \forall E \subset X$.

Proof.

$$\sum_1^\infty \mu^*(E \cap B_n) \geq \mu^*\left(E \cap \bigcup_1^\infty B_n\right)$$
$$\geq \mu^*\left(E \cap \bigcup_1^N B_n\right) = \sum_1^N \mu^*(E \cap B_n.)$$

Take $N \to \infty$ or note that $N \in \mathbb{N}$ is arbitrary we get the result. ■

First big theorem:

Theorem 1.22 (Carathéodory extension theorem)**.** *Let μ^* be an outer measure on X. Let $\mathcal{A}$ be the collection of C-measurable sets with respect to μ^*. Then*

(a) $\mathcal{A}$ us a σ-algebra on X.

(b) $\mu = \mu^*|_{\mathcal{A}}$ is a measure on $(X, \mathcal{A})$.

(c) $(X, \mathcal{A}, \mu)$ is a complete measure space.

Proof. (a) (1) $\emptyset \in \mathcal{A}$.

(2) $\mathcal{A}$ is closed under complements.

(3) To show $\mathcal{A}$ closed under countable unions.

- (finite union)
  CLAIM $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$.

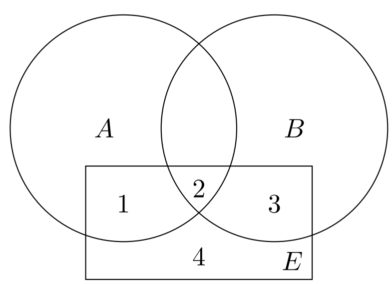

Figure 1.1: Venn diagram of A, B, E

Fix arbitrary $E \subset X$. We need to show

$$\mu^*(E) = \mu^*(E \cap (A \cup B)) + \mu^*(E \setminus (A \cup B)).$$

i.e.

$$\mu^*(1 \cup 2 \cup 3 \cup 4) = \mu^*(1 \cup 2 \cup 3) + \mu^*(4)$$

Since A is C-measurable, we have

$$\mu^*(1 \cup 2 \cup 3 \cup 4) = \mu^*(1 \cup 2) + \mu^*(3 \cup 4)$$

$$\mu^*(1 \cup 2 \cup 3) = \mu^*(1 \cup 2) + \mu^*(3)$$

Similarly since B is C-measurable, we have

$$\mu^*(3 \cup 4) = \mu^*(3) + \mu^*(4)$$

Hence

$$\begin{aligned}\mu^*(1 \cup 2 \cup 3 \cup 4) &= \mu^*(1 \cup 2) + \mu^*(3 \cup 4) \\ &= \mu^*(1 \cup 2 \cup 3) - \mu^*(3) + \mu^*(3) + \mu^*(4) \\ &= \mu^*(1 \cup 2 \cup 3) + \mu^*(4).\end{aligned}$$

- (countable disjoint unions)
  Let $A_1, A_2, \ldots \in \mathcal{A}$ and $\underline{\text{disjoint}}$.

  Fix $E \subset X$ arbitrary. Since μ^* is countably subadditive,

  $$\mu^*(E) \leq \mu^*\left(E \cap \bigcup_1^\infty\right) + \mu^*\left(E \setminus \bigcup_1^\infty A_n\right)$$

  Fix $n \in \mathbb{N}$.

  $$\begin{aligned}&\implies \bigcup_1^N A_n \in \mathcal{A} \\ &\implies \mu^*(E) = \mu^*\left(E \cap \bigcup_1^N\right) + \mu^*\left(E \setminus \bigcup_1^N A_n\right) \\ &\geq \sum_1^N \mu^*(E \cap A_n) + \mu^*\left(E \setminus \bigcup_1^\infty A_n\right) \text{ by lemma.}\end{aligned}$$

  Take $n \to \infty$.

- (countable unions)
  Let $A_1, A_2, \ldots \in \mathcal{A}$. Take $E_1 = A_1, E_n = A_n \setminus \left(\bigcup_1^{n-1} A_i\right)$ for $n \geq 2$. Then $\bigcup A_n = \bigcup E_n$ and E_n's are disjoint.

(b) Firstly we have $\mu(\emptyset) = \mu^*(\emptyset) = 0$.

Countable additvity of μ^* on $\mathcal{A}$ follows from the improved lemma with $E = X$.

(c) HW. ■

1.4 Hahn-Kolmogorov Theorem

RECALL 1.19 Let $\mathcal{E} \subset \mathcal{P}(X)$ $s.t.$ $\emptyset, X \in \mathcal{E}$. Let $\rho : \mathcal{E} \to [0, \infty]$ $s.t.$ $\rho(\emptyset) = 0$

$$(\mathcal{E}, \rho) \xrightarrow[1.19]{} (\mathcal{P}(X), \mu^*) \xrightarrow[\text{C-theorem}]{} (A, \mu)$$

QUESTION $\mathcal{E} \subset \mathcal{A}$ and $\mu|_{\mathcal{E}} = \rho$? No!

Definition 1.23. Let $\mathcal{A}_0$ be an algebra on X. We say $\mu_0 : \mathcal{A}_0 \to [0, \infty]$ is a pre-measure if

(a) $\mu_0(\emptyset) = 0$.

(b) (finite additivity)

$$\mu_0\left(\bigcup_1^N A_i 1\right) = \sum_1^N \mu_0(A_i) \text{ if } A_1, \ldots, A_N \in \mathcal{A}_0 \text{ are disjoint.}$$

(c) (countable additivity within the algebra) If $A \in \mathcal{A}_0$ and

$$A = \bigcup_1^\infty A_n, A_n \in \mathcal{A}_0 \text{ and are disjoint, then } \mu_0(A) = \sum_1^\infty \mu_0(A_n)$$

NOTATION: Folland uses $\mathcal{M}$ for σ-algebra and $\mathcal{A}$ for algebra. (Jinho) uses $\mathcal{A}$ for σ-algebra and $\mathcal{A}_0$ for alegbra.

Example 1.24. $\mathcal{A}_0$ finite disjoint unions of $(a, b]$.

$$\mu_0\left(\bigcup_1^\infty (a_i, b_i]\right) = \sum_1^\infty (b_i - a_i) \text{ or } b_i^n - a_i^n, e^{b_i} - e^{a_i}, \text{ etc.}$$

Lemma 1.25.
- *(a) + (c) $\implies$ (b).*
- *μ_0 is monotone.*

Theorem 1.26 (Hahn-Kolmogorov Theorem)**.** *Let μ_0 be a pre-measure on algebra $\mathcal{A}_0$ on X. Let μ^* be the outer measure induced by $(\mathcal{A}_0, \mu_0)$ in 1.19. Let $\mathcal{A}$ and μ be the Carathéodory σ-algebra and measure for μ^* $\implies$ $(\mathcal{A}, \mu)$ extends $(\mathcal{A}_0, \mu_0)$ $i.e.$ $\mathcal{A} \supset \mathcal{A}_0, \mu|_{\mathcal{A}_0} = \mu_0$.*

Proof. (a) ($\mathcal{A} \supset \mathcal{A}_0$) Let $A \in \mathcal{A}_0$.

Question: $A \in \mathcal{A}$? i.e. is A C-measurable? i.e. $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c), \forall E \subset$

X.

Fix $E \subset X$.

- (countable) subadditivity of $\mu^* \implies \mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c)$.
- If $\mu^*(E) = \infty$ then $\mu^*(E) = \infty \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$.
- If $\mu^*(E) < \infty$.

  Fix $\varepsilon > 0$. By the definition of $\mu^*, \exists B_1, B_2, \ldots \in \mathcal{A}_0$ $s.t.$ $\bigcup_1^\infty B_n \supset E$ and

$$\mu^*(E) + \varepsilon \geq \sum_1^\infty \mu_0(B_n) = \sum_1^\infty \left(\mu_0(B_n \cap A) + \mu_0\left(B_n \cap A^c\right)\right).$$

  Note that

$$\bigcup_1^\infty (B_n \cap A) \supset E \cap A, \quad \bigcup_1^\infty (B_n \cap A^c) \supset E \cap A^c \implies \geq$$

(b) Let $A \in \mathcal{A}_0$. We want to show that $\mu(A) = \mu_0(A)$.

By definition, $\mu(A) = \mu^*(A)$.

- Let $B_i = \begin{cases} A & i = 1, \\ \emptyset & i = 2 \end{cases} \in \mathcal{A}_0$ and $\bigcup_1^\infty B_i \supset A$.

  Hence $\mu^*(A) \leq \sum_1^\infty \mu_0(B_i) = \mu_0(A)$.
- Let $B_i \in \mathcal{A}_0, \bigcup_1^\infty B_i \supset A$ an arbitrary collection of sets.

  Let $C_1 = A \cap B_1, C_i = A \cap B_i \backslash \left(\bigcup_{j=1}^{i-1} B_j\right)$. Then $A = \bigcup_1^\infty$ is a disjoint countable union. By countable additivitiy we have

$$\mu_0(A) = \sum_1^\infty \mu_0(C_i) \implies \mu_0(A) \leq \sum_1^\infty \mu_0(B_i).$$

Hence we have $\mu_0(A) = \mu^*(A) = \mu(A)$. We have completed our proof. ∎

Definition 1.27. Such $(\mathcal{A}, \mu)$ is called the Hahn-Kolmogorov extension of $(\mathcal{A}_0, \mu_0)$, and is also called the Carathéodory σ-algebra for $(\mathcal{A}_0, \mu_0)$.

Theorem 1.28 (uniqueness of HK extension)**.** *Let $\mathcal{A}_0$ be an algebra on X, μ_0 be a pre-measure on $\mathcal{A}_0$, $(\mathcal{A}, \mu)$ be the Hahn-Kolmogorov extension of $(\mathcal{A}_0, \mu_0)$. And let $(\mathcal{A}', \mu')$ be another extension of $(\mathcal{A}_0, \mu_0)$.*

If μ_0 is σ-finite, then $\mu \mid_{\mathcal{A} \cap \mathcal{A}'} = \mu' \mid_{\mathcal{A} \cap \mathcal{A}'}$.

NOTE σ-finite means

$$\forall X, X = \bigcup_1^\infty X_n, X_n \in \mathcal{A}_0, \mu_0(X_n) < \infty.$$

Corollary 1.29. *Let μ_0 be a pre-measure on algebra $\mathcal{A}_0$ on X. Suppose μ_0 is σ-finite, then $\exists!$ measure μ on $\langle \mathcal{A}_0 \rangle$ that extends $\mathcal{A}_0$. Furthermore,*

(a) the completion of $(X, \langle \mathcal{A}_0 \rangle, \mu)$ is the HK extension of $(\mathcal{A}_0, \mu_0)$.

(b)

$$\mu(A) = \inf \left\{ \sum_{i=1}^\infty \mu_0(B_i) \mid B_i \subset A_0, \forall i \in \mathbb{N}, \bigcup_1^\infty B_i \supset A \right\}, \forall A \in \overline{\langle \mathcal{A}_0 \rangle}.$$

Proof of 1.28. Let $A \in \mathcal{A} \cap \mathcal{A}'$. We need to show $\mu(A) = \mu^*(A) = \mu'(A)$.

- $\mu^*(A) \geq \mu'(A)$ (HW)
- $\mu(A) \leq \mu'(A)$:

  (i) Assume $\mu(A) < \infty$. Fix $\varepsilon > 0$. Then $\exists B_i \in \mathcal{A}_0, \forall i \in \mathbb{N}, \bigcup_1^\infty B_i \supset A \; s.t.$

  $$\mu(A) + \varepsilon = \mu^*(A) + \varepsilon \geq \sum_1^\infty \mu_0(B_i) = \sum_1^\infty \mu(B_i) \geq \mu\left(\bigcup_1^\infty B_i\right) = \mu(B)$$

  Hence $\mu(B \setminus A) = \mu(B) - \mu(A) \leq \varepsilon$.

  On the other hand,

  $$\mu(B) = \lim_{N\to\infty} \mu\left(\bigcup_1^N B_i\right) = \lim_{N\to\infty} \mu'\left(\bigcup_1^N B_i\right) = \mu'(B)$$

  by continuity of measure from below.

  $$\mu(A) \leq \mu(B) = \mu'(B) = \mu'(A) + \mu'(B \setminus A) \leq \mu'(A) = \varepsilon.$$

  (ii) Assume $\mu(A) = \infty$.

  Since μ_0 is σ-finite, $X = \bigcup_1^\infty X_n, X_n \in \mathcal{A}_0, \mu_0(X_0) < \infty$. Replacing X_n by $X_1 \cup \ldots \cup X_n$, we may assume $X_1 \subset X_2 \subset \ldots$.

  $$\forall n \in N, \mu(A \cap X_n) < \infty \implies \mu(A \cap X_n) \leq \mu'(A \cap X_n).$$

Hence
$$\mu(A) = \lim_{N\to\infty} \mu(A\cap X_n) \le \lim_{N\to\infty} \mu'(A\cap X_n) = \mu'(A). \qquad \blacksquare$$

1.5 Borel Measures on $\mathbb{R}$

Definition 1.30. $F : \mathbb{R} \to \mathbb{R}$ is an increasing function if $F(x) \le F(y)$ for $x < y$. $F : \mathbb{R} \to \mathbb{R}$ is increasing and right-continuous $\implies$ F is distribution function.

Example 1.31.

- $$F(x) = \begin{cases} 1, & x \ge 0 \\ 0, & x < 0. \end{cases}$$
- $\mathbb{Q} = \{r_1, r_2, \ldots\}, F_n(x) = \begin{cases} 1 & x \ge r_n \\ 0 & x < r_n. \end{cases}$

  $$F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n} \text{ is a distribution function.}$$

NOTE If F is increasing, $F(\infty) := \lim_{x\to\infty} F(x), F(-\infty) := \lim_{x\to-\infty} F(x)$ exists in $[-\infty, \infty]$.

In probability theory, cumulative distribution function (CDF) is a distribution function with $F(\infty) = 0$ and $F(-\infty) = 0$.

There are distributions [Folland, Ch9], but these are different from distribution functions.

Definition 1.32. Suppose X a topological space. μ on $(X, \mathcal{B}(X))$ is called locally finite is $\mu(K) < \infty$ for any compact set $K \subset X$.

Lemma 1.33. *Let μ be a locally finite Borel measure on $\mathbb{R}$ $\implies$*

$$F_\mu(x) = \begin{cases} \mu((0,x]), & x > 0 \\ 0, & x = 0 \\ -\mu((x,0]), & x < 0 \end{cases} \text{ *is a distribution function.*}$$

Proof. DIY. Use continuity of measure. $\blacksquare$

Definition 1.34. h-intervals are $\emptyset, (a,b], (a,\infty), (-\infty,b], (\infty,\infty)$.

Lemma 1.35. *Let $\mathcal{H}$ be the collections of finite disjoint unions of h-intervals. Then $\mathcal{H}$ is an* *algebra on* $\mathbb{R}$.

Proof. DIY. ■

Proposition 1.36 (Distribution function defines a pre-measure)**.** *Let* $F : \mathbb{R} \to \mathbb{R}$ *be a distribution function. For an* h*-interval* I*, define*

$$\ell(I) = \ell_F(I) = \begin{cases} 0, & I = \emptyset \\ F(b) - F(a), & I = (a, b] \\ F(\infty) - F(a), & I = (a, \infty) \\ F(b) - F(\infty), & I = (-\infty, b] \\ F(\infty) - F(-\infty), & I = (-\infty, \infty). \end{cases}$$

Define $\mu_0 = \mu_{0,F} : \mathcal{H} \to [0, \infty]$ *by*

$$\mu_0(A) := \sum_{k=1}^{N} \ell(I_k) \quad \textit{if } A = \bigcup_{k=1}^{N} I_k, \textit{ finite disjoint union of } h\textit{-intervals.}$$

Then μ_0 *is a pre-measure.*

Proof. (a) μ_0 is well-defined.

(b) μ_0 is finite additive.

(c) μ_0 is countable additive within $\mathcal{H}$.

Suppose $A \in \mathcal{H}$ and $A = \bigcup_1^\infty A_i$ a disjoint union, $A_i \in \mathcal{H}$. It is enough to consider the case $A = I, A_k = I_k$ all h-intervals. (Why?)

Focus on the case $I = (a, b]$: (HW: check other cases)
We have

$$(a, b] = \bigcup_1^\infty (a_n, b_n], \text{ a disjoint union.}$$

Check

$$F(b) - F(a) \stackrel{?}{=} \sum_1^\infty (F(b_n) - F(a_n))$$

$(a, b] \supset \bigcup_1^N (a_n, b_n] \implies F(b) - F(a) \geq \sum_1^N F(b_n) - F(a_n), \forall N \in \mathbb{N}$. (Arranging them in decreasing order) Take $N \to \infty$ we have

$$F(b) - F(a) \geq \sum_1^\infty (F(b_n) - F(a_n)).$$

Since F is right-continuous, $\exists a' > a\ s.t.\ F(a') - F(a) < \varepsilon$. For each $n \in \mathbb{N}$, $\exists b_n' > b_n\ s.t.\ F(b_n') - F(b_n) < \frac{\varepsilon}{2^n}$.

$$\implies [a', b] \subset \bigcup_1^\infty (a_n, b_n')$$

$$\implies \exists N \in \mathbb{N}\ s.t.\ [a', b] \subset \bigcup_1^n (a_n, b_n')$$

$$\implies F(b) - F(a') \leq \sum_1^N F(b_n') - F(a_n)$$

$$\implies F(b) - F(a) \leq F(b) - F(a') + \varepsilon \leq \sum_1^\infty (F(b_n') - F(a_n)) + \varepsilon$$

$$\leq \sum_1^\infty \left(F(b_n) - F(a_n) + \frac{\varepsilon}{2^n}\right) + \varepsilon \qquad \blacksquare$$

Once we have this pre-measure, HK theorem allows us to extended it to a measure.

Theorem 1.37 (Locally finite Borel measures on $\mathbb{R}$)**.**

(a) $F : \mathbb{R} \to \mathbb{R}$ *is a distribution function* $\implies$ $\exists!$ *locally finite Borel measure* μ_F *on* $\mathbb{R}$ *satisfying* $\mu_F((a, b]) = F(b) - F(a), \forall a, b, a < b$.

(b) Suppose $F, G : \mathbb{R} \to \mathbb{R}$ *are distribution functions. Then,* $\mu_F = \mu_G$ *on* $\mathcal{B}(\mathbb{R})$ *if and only if* $F - G$ *is a constant function.*

Proof. HW $\blacksquare$

1.6 Lebesgue-Stieltjes Measures on $\mathbb{R}$

F distribution function $\implies$ μ_F on Carathéodory σ-algebra $\mathcal{A}_{\mu_F}$.
Actually $(\mathcal{A}_{\mu_F}, \mu_F) = (\mathcal{B}(\mathbb{R}), \mu_F)$ (HW3).

Definition 1.38.

- μ_F on $\mathcal{A}_{\mu_F}$ is called the Lebesgue-Stieltjes measure corresponding to F.
- Special case: $F(x) = x \implies$ Lebesgue measure $(\mathcal{B}, m)$.

Example 1.39. (a) $\mu_F((a, b]) = F(b) - F(a)$. F is right-continuous and increasing $\implies$ $F(x_-) \leq F(x) = F(x_+)$.

(HW) $\mu_F(\{a\}) = F(a) - F(a_-), \mu_F([a, b]) = F(b) - F(a_-), \mu_F((a, b)) = F(b_-) - F(a)$.

(b)

$$F(x) = \begin{cases} 1 & x \leq 0 \\ 0 & x < 0 \end{cases} \implies \mu_F(\{0\}) = 1, \mu_F(\mathbb{R}) = 1, \mu_F(\mathbb{R} \setminus \{0\}) = 0.$$

μ_F is the Dirac measure at 0.

(c)

$$\mathbb{Q} = \{r_1, r_2, \ldots\},\ F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{2^n},\ F_n(x) = \begin{cases} 1 & x \leq r_n \\ 0 & x < r_n \end{cases}$$
$$\implies \mu_F(\{v\}) > 0, \forall v \in \mathbb{Q},\ \mu_F(\mathbb{R} \setminus \mathbb{Q}) = 0.$$

(d) If F is continuous at $a, \mu_F(\{a\}) = 0$.

(e) $F(x) = x \implies m((a, b])) = m((a, b)) = m([a, b]) = b - a.$

(f) $F(x) = e^x, \implies \mu_F((a, b]) = \mu_F((a, b)) = e^b - e^a$

(a), (b) are examples of discrete measure.

Example 1.40 (Middle thirds Cantor set $\mathcal{C} = \bigcup_{n=1}^{\infty} K_n$**).**

$\mathcal{C}$ is uncountable set with $m(\mathcal{C}) = 0$.

$$x \in \mathcal{C} \implies x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}, a_n \in \{0, 2\}.$$

We are interested in the Cantor function F.

Example 1.41. Cantor function F is continuous and increasing. This defines the Cantor measure $\mu_F(\mathbb{R} \setminus \mathcal{C}) = 0, \mu_F(\mathcal{C}) = 1, \mu_F(\{a\}) = 0$. Compare with Lebesgue measure $m(\mathbb{R} \setminus \mathcal{C}) = \infty > 0, \mu(\mathcal{C}) = 0, m(\{a\}) = 0$.

1.7 Regularity Properties of Lebesgue-Stieltjes Measures

Lemma 1.42. *μ is Lebesgue-Stieltjes measure on $\mathbb{R}$ $\implies$*

$$\begin{aligned} \mu(A) &= \inf \left\{ \sum_1^{\infty} \mu((a_i, b_i]) \;\middle|\; \bigcup_1^{\infty} (a_i, b_i] \supset A \right\} \\ &= \inf \left\{ \sum_1^{\infty} \mu((a_i, b_i)) \;\middle|\; \bigcup_1^{\infty} (a_i, b_i) \supset A \right\} \end{aligned}$$

Proof. Using the continuity of measure. ∎

Theorem 1.43. *μ is a Lebesgue-Stieltjes measure. Then $\forall A \in \mathcal{A}_\mu$,*

(a) (outer regularity)

$$\mu(A) = \inf\{\mu(O) \mid \textit{open } O \supset A\}.$$

(b) (inner regularity)

$$\mu(A) = \sup\{\mu(K) \mid \textit{compact } K \subset A\}.$$

Proof. (a) Followed from 1.42.

(b) Let $s = \sup\{\ldots\}$. Monotonicity $\implies \mu(A) \geq s$.

- (A bounded) $\overline{A} \in \mathcal{B}(\mathbb{R}) \subset \mathcal{A}_\mu$, $\overline{A}$ bounded $\implies \mu(\overline{A}) < \infty$.

  Fix $\varepsilon > 0$. By 1, $\exists$ open $O \supset \overline{A} \setminus A$, $\mu(O) - \mu(\overline{A} \setminus A) = \mu(O \setminus (\overline{A} \setminus A)) \leq \varepsilon$.

  Let $K = \underbrace{A \setminus O}_{K \subset A} = \underbrace{\overline{A} \setminus O}_{\text{compact}}$. Show that $\mu(K) \geq \mu(A) - \varepsilon$.

- (A unbounded but $\mu(A) < \infty$) We have

  $$A = \bigcup_1^\infty A_n,\ A_n = A \cap [-n, n],\ A_1 \subset A_2 \subset \ldots$$

  Hence

  $$\lim_{n \to \infty} \mu(A_n) = \mu(A) < \infty.$$

- ($\mu(A) = \infty$)

  $$\lim_{n \to \infty} \mu(A_n) = \mu(A) = \infty.$$

  Fix $L > 0$. $\exists N \ s.t.\ \mu(A_N) \geq L$. ∎

Definition 1.44. Suppose X a topological space.

A <u>$G\sigma$-set</u> is $G = \bigcup_1^\infty O_i$, O_i open. An <u>$F\sigma$-set</u> is $F = \bigcup_1^\infty F_i$, F_i closed.

Theorem 1.45. *Suppose μ a LS measure. Then the following statements are equivalent:*

(a) $A \in \mathcal{A}_\mu$.

(b) $A = G \setminus M$, G is a $G\sigma$-set, and M is μ-null.

(c) $A = F \cup N$, F is a $F\sigma$-set, and N is μ-null.

Proof. (b) $\implies$ (a) and (c) $\implies$ (a) are clear.

- (a) $\implies$ (c)
  (i) Assume $\mu(A) < \infty$. By inner regularity,
  $$\forall n \in \mathbb{N}, \exists \text{ compact } K_n \subset A \; s.t. \; \mu(K_n) + \frac{1}{n} \geq \mu(A).$$
  Let $F = \bigcup_1^\infty K_n$. Then $N = A \setminus F$ is μ-null.
  (ii) Assume $\mu(A) = \infty$. We construct
  $$A = \bigcup_{k \in \mathbb{Z}} A_k, A_k = A \cap (k, k+1].$$
  By (i), $\forall k \in \mathbb{Z}, A_k = F_k \cup N_k$. Hence
  $$A = \underbrace{\left(\bigcup_k F_k\right)}_{F\sigma} \cup \underbrace{\left(\bigcup_k N_k\right)}_{\mu\text{-null}}.$$
- (a) $\implies$ (b)
  $$A^c = F \cup N, A = F^c \cup N^c = F^c \setminus N.$$ ■

Proposition 1.46. *Suppose μ a LS measure, $A \in \mathcal{A}_\mu$, $\mu(A) < \infty$. Then*
$$\forall \varepsilon > 0, \exists I = \bigcup_1^{N=N(\varepsilon)} I_i, \text{ disjoint open intervals } s.t. \; \mu(A \triangle I) \leq \varepsilon.$$

Proof. DIY - use outer regularity. ■

Properties of Lebesgue measure

Theorem 1.47.
$$A \in \mathcal{L} \implies A + s \in \mathcal{L}, rA \in \mathcal{L}, \forall r, s \in \mathbb{R}.$$
In addition, $m(A + r) = m(A)$ and $m(rA) = rm(A)$.

Proof. DIY. ■

Example 1.48. (a) $\mathbb{Q} = \{r_1\}_{i=1}^\infty$, which is dense in $\mathbb{R}$. Let $\varepsilon > 0$ and
$$O = \bigcup_{i=1}^\infty \left(r_i - \frac{\varepsilon}{2^i}, r_i + \frac{\varepsilon}{2^i}\right).$$

O is open and dense in $\mathbb{R}$. We have

$$m(O) \leq \sum_{i=1}^{\infty} \frac{2\varepsilon}{2^i} = 2\varepsilon, \partial O = \overline{O} \setminus O, m(O) = \infty.$$

(b) $\exists$ uncountable set A with $m(A) = 0$.

(c) $\exists A$ with $m(A) > 0$, but A contains no non-empty open interval.

(d) $\exists A \notin \mathcal{L}$ that is Vitali set.

(e) $\exists A \in \mathcal{L} \setminus \mathcal{B}(\mathbb{R})$. We will deal with that later.

Chapter 2

Integration

2.1 Measurable Functions

Definition 2.1. Suppose $(X, \mathcal{A})$, $(Y, \mathcal{B})$ two measurable spaces. $f : X \to Y$ is $(\mathcal{A}, \mathcal{B})$-measurable if

$$\forall B \in \mathcal{B}, f^{-1}(B) \in \mathcal{A}.$$

Lemma 2.2. *Suppose $\mathcal{B} = \langle \mathcal{E} \rangle$. Then*

$$f : X \to Y \textit{ is } (\mathcal{A}, \mathcal{B})\textit{-measurable} \iff \forall E \in \mathcal{E}, f^{-1}(E) \in \mathcal{A}.$$

Proof. $\implies$ clear

$\impliedby$ Let $\mathcal{D} = \{E \subset Y \mid f^{-1}(E) \in \mathcal{A}\}$. We have $\mathcal{E} \subset \mathcal{D}$ by assumption. In addition $\mathcal{D}$ is a σ-algebra $\implies \langle \mathcal{E} \rangle \subset \mathcal{D}$. ■

Definition 2.3. Suppose $(X, \mathcal{A})$ a measurable space.

$$\left.\begin{array}{l} f : X \to \mathbb{R} \\ f : X \to \overline{\mathbb{R}} = [-\infty, \infty] \\ f : X \to \mathbb{C} \end{array}\right\} \text{ is } \mathcal{A}\text{-measurable if } \left\{\begin{array}{l} f \text{ is } (\mathcal{A}, \mathcal{B}(\mathbb{R}))\text{-measurable} \\ f \text{ is } (\mathcal{A}, \mathcal{B}(\overline{\mathbb{R}}))\text{-measurable} \\ \Re f, \Im f : X \to \mathbb{R} \text{are } \mathcal{A}\text{-measurable.} \end{array}\right.$$

Here $\mathcal{B}(\overline{\mathbb{R}}) = \{E \subset \overline{\mathbb{R}} \mid E \cap R \in \mathcal{B}(\mathbb{R})\}$.

Lemma 2.4. *Suppose $f : X \to \mathbb{R}$. Then the followings are equivalent:*

(a) f is $\mathcal{A}$-measurable

(b) $\forall a \in \mathbb{R}, f^{-1}((a,\infty)) \in \mathcal{A}$.

(c) $\forall a \in \mathbb{R}, f^{-1}([a,\infty)) \in \mathcal{A}$.

(d) $\forall a \in \mathbb{R}, f^{-1}((-\infty,a)) \in \mathcal{A}$.

(e) $\forall a \in \mathbb{R}, f^{-1}((-\infty,a]) \in \mathcal{A}$.

For $f : X \to \overline{\mathbb{R}}$, change the interval to include $-\infty$ and ∞.

Proof. By 2.2. ∎

Example 2.5. $\mathcal{A} = \mathcal{P}(X) \implies$ every function is $\mathcal{A}$ measurable.

$\mathcal{A} = \{\emptyset, X\} \implies$ only $\mathcal{A}$ functions are constant functions.

<u>Properties</u> Suppose $f, g : X \to \mathbb{R}$, $\mathcal{A}$-measurable functions.

(a) $\phi : \mathbb{R} \to \mathbb{R}$, $\mathcal{B}(\mathbb{R})$ measurable (i.e. Borel measurable) $\implies \phi \circ f : X \to \mathbb{R}$ is $\mathcal{A}$-measurable.

(b) $-f, 3f, f^2, |f|$ are $\mathcal{A}$-measurable, $\frac{1}{f}$ is $\mathcal{A}$-measurable if $f(x) = 0, \forall x \in X$.

(c) $f + g$ is $\mathcal{A}$-measurable

$$(f+g)^{-1}((a,\infty)) = \bigcup_{r \in \mathbb{Q}} \left(f^{-1}((r,\infty)) \cap g^{-1}((a-r,\infty))\right).$$

(d) fg is $\mathcal{A}$-measurable

$$f(x)g(x) = \frac{1}{2}\left((f(x)+g(x))^2 - f(x)^2 - g(x)^2\right).$$

(e) $(f \wedge g)(x) = \min\{f(x), g(x)\}$, $(f \vee g)(x) = \max\{f(x), g(x)\}$ are $\mathcal{A}$-measurable.

(f) $f_n : X \to \overline{\mathbb{R}}$ are a sequence of $\mathcal{A}$-measurable functions $\implies$

$$\sup f_n, \inf f_n, \limsup_{n\to\infty} f_n, \liminf_{n\to\infty} f_n \text{ are } \mathcal{A}\text{-measurable.}$$

(g) If $f(x) = \lim_{n\to\infty} f_n(x)$ converges for every $x \in X$, then f is measurable.

Example 2.6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. Then f is Borel measurable $\implies f$ is Lebesgue measurable. (Preimage of an open set of a continuous function is open.)

Definition 2.7. For $f : X \to \overline{\mathbb{R}}$, let $f^+ = f \vee 0$, $f^- = (-f) \vee 0$.

<u>Note</u> $\operatorname{supp} f^+ \cap \operatorname{supp} f^- = \emptyset$. $f(x) = f^+(x) - f^-(x)$. f is $\mathcal{A}$-measurable $\iff f^+, f^-$ measurable.

Definition 2.8. For $E \subset X$, characteristic (indicator) funtion of E

$$\chi_E(x) = 1_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \in E^c. \end{cases}$$

1_E is $\mathcal{A}$-measurable $\iff E \in \mathcal{A}$.

Definition 2.9. Suppose $(X, \mathcal{A})$ a measurable space. A simple function $\phi : X \to \mathbb{C}$ that is $\mathcal{A}$-measurable and takes only finitely many values.

$$\phi(X) = \{c_1, \ldots, c_N\}, c_i \neq \pm\infty, E_i = \phi^{-1}(c_i) \in \mathcal{A} \implies \phi = \sum_{i=1}^{N} c_i 1_{E_i}.$$

Theorem 2.10. *Suppose $(X, \mathcal{A})$ a measurable space and $f : X \to [0, \infty]$. Then the followings are equivalent:*

(a) f is $\mathcal{A}$-measurable.

(b) $\exists$ simple functions $0 \leq \phi_1(x) \leq \phi_2(x) \leq \ldots \leq f(x)$ such that

$$\lim_{n\to\infty} \phi_n(x) = f(x),\ \forall x \in X.$$

(f is the pointwise upward limit of simple functions.)

Proof.

- (b) $\implies$ (a) is easy: $f(x) = \sup_{n\in\mathbb{N}} \phi_n(x)$.
- (a) $\implies$ (b): suppose f is $\mathcal{A}$-measurable.

  Fix $n \in \mathbb{N}$. Let $F_n = f^{-1}([2^n, \infty]) \in \mathcal{A}$. For

  $$0 \leq k \leq 2^{2n} - 1,\ E_{n,k} = f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right) \in \mathcal{A}.$$

  Let $\phi_n(x) = \sum_{k=0}^{2^{2n}-1} 1_{E_{n,k}} + 2^n 1_{F_n}$.

  We have

  $$0 \leq \phi_1(x) \leq \phi_2(x) \leq \ldots \leq f(x)$$

  and

  $$\forall x \in X, 0 \leq f(x) - \phi_n(x) \leq \frac{1}{2^n}.$$

■

Corollary 2.11. *If f is bounded on a set $A \subset \mathbb{R}$ (i.e. $\exists n L > 0$ $s.t.$ $|f(x)| \leq L,\ \forall x \in A$) then $\phi_n \to f$ uniformly on A.*

Proof. DIY. ■

Corollary 2.12. *$f : X \to \mathbb{C}$, measurable function $\iff$ $\exists$ simple functions $\phi_n : X \to \mathbb{C}$ $s.t.$*

2.2 Integration of Nonnegative Functions

Definition 2.13. Suppose $(X, \mathcal{A}, \mu)$ a measure space and $\phi = \sum_{i=1}^{N} c_i 1_{E_i} : X \to [0, \infty]$ a simple function. Let

$$\int \phi = \int \phi d\mu = \int_X \phi d\mu = \sum_1^N c_i \mu(E_i).$$

Proposition 2.14. *Suppose $\phi, \psi \geq 0$ are simple functions. Then,*

- *2.13 is well-defined.*
- $\displaystyle\int c\phi = c \int \phi, c \in [0, \infty).$
- $\displaystyle\int (\phi + \psi) = \int \phi + \int \psi.$
- $\displaystyle\phi(x) \geq \psi(x),\ \forall x \implies \int \phi \geq \int \psi.$
- $\displaystyle\nu(A) = \int_A \phi d\mu$ *is a measure on $(X, \mathcal{A})$.*

Proof. DIY. ■

Definition 2.15. Suppose $(X, \mathcal{A}, \mu)$, $f : X \to [0, \infty]$ is $\mathcal{A}$-measurable.

Define

$$\int f = \int f d\mu = \sup \left\{ \int \phi \mid 0 \leq \phi \leq f, \phi \text{ simple} \right\}.$$

Proposition 2.16.

- *If f is a simple function then two definitions are the same.*
- $\displaystyle\int cf = c \int f.$
- $\displaystyle f \geq g \geq 0 \implies \int f \geq \int g.$
- $\displaystyle\int f + g = \int f + \int g.$ *(A bit harder to check)*

Theorem 2.17 (Monotone convergence theorem). *Suppose $(X, \mathcal{A}, \mu)$ a measure space and*

- $f : X \to [0, \infty]$ *is $\mathcal{A}$-measurable,* $\forall n \in \mathbb{N}$.
- $0 \le f_1(x) \le \dots$.
- $\lim_{n\to\infty} f_n(x) = f(x)$.
-

Then

$$\int f = \lim_{n\to\infty} \int f_n.$$

Proof. TBC ■

Corollary 2.18. $f, g \ge 0$ *measurable* $\implies \int f + g = \int f + \int g.$

Corollary 2.19 (Tonelli's theorem for series and integrals). *Given $s_n \ge 0, \forall n \in \mathbb{N}$ measurable functions. Then*

$$\int \sum_{n=1}^{\infty} s_n = \sum_{n=1}^{\infty} \int s_n.$$

Proof. Let $f_N = \sum_{n=1}^{N} s_n, 0 \le f_1 \le f_2 \le \dots$.

$$\lim_{N\to\infty} f_N(x) = \sum_{n=1}^{\infty} s_n(x)$$

By MCT, we have

$$\lim_{N\to\infty} \sum_{1}^{N} s_n = \sum_{1}^{\infty} s_n$$

■

Theorem 2.20 (Fotou's lemma). *Suppose $f_n \ge 0$ measurable. Then*

$$\int \liminf_{n\to\infty} f_n \le \liminf_{n\to\infty} \int f_n.$$

Recall that

$$\liminf_{n\to\infty} f_n := \lim_{k\to\infty} \inf_{n\ge k} f_n = \sum_{k\in\mathbb{N}} \inf_{n\ge k} f_n,$$

and

$$\lim_{n\to\infty} a_n \text{ exists} \iff \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n.$$

Proof. Let $g_k = \inf_{n \geq k} f_n \implies s_k$ measurable and $0 \leq g_1 \leq g_2 \leq \ldots$. By MCT, we have

$$\int \liminf_{n \to \infty} = \int \lim_{k \to \infty} s_k = \lim_{k \to \infty} \int s_k = \lim_{k \to \infty} \int \inf_{n \geq k} f_n$$

$$\begin{aligned}
&\inf_{n \geq k} f_n \leq f_m, \forall m \geq k \\
\implies &\int \inf_{n \geq k} f_n \leq \int f_m, \forall m \geq k \\
\implies &\int \inf_{n \geq k} f_n \leq \inf_{m \geq k} \int f_m
\end{aligned}$$

■

Example 2.21. Suppose $(\mathbb{R}, \mathcal{L}, m)$

(a) (escape to horizontal infinity) $f_n = 1_{(n,n+1)}$.

We see that $f_n \to 0 = f$ pointwise and $\int f_n = 1, \forall n, \int f = 0$.

(b) (escape to width infinity) $f_n = \frac{1}{n} 1_{(0,n)}$.

(c) (escape to vertical infinity) $f_n = n 1_{(0,1/n)}$.

Lemma 2.22 (Markov's inequality). *$f \geq 0$ is measurable* $\implies$

$$\forall c \in (0, \infty),\ \mu\left(\{x \mid f(x) \geq c\}\right) \leq \frac{1}{c} \int f.$$

Proof. Let $E = \{x \mid f(x) \geq c\}$. Then

$$f(x) \geq c 1_E(x) \implies \int f \geq c \int 1_E = c\mu(E).$$

■

Proposition 2.23. *Suppose $f \geq 0$ measurable. Then $\int f = 0 \iff f = 0$ almost everywhere (a.e.)*

$$\int f d\mu = \mu(A) = 0,\ A = \{x \mid f(x) > 0\} = f^{-1}((0, \infty])$$

Proof. (a) Assume $f = \phi$ a simple function. We may assume

$$\phi = \sum_{i=1}^{N} c_i 1_{E_i},\ c_i \in (0, \infty),\ E_i\text{'s are disjoint.}$$

$$
\begin{aligned}
&\int \phi = \sum_{i=1}^{N} c_i \mu(E_i) = 0 \\
\iff &\mu(E_1) = \ldots = \mu(E_N) = 0 \\
\iff &\mu(A) = 0,\ A = \bigcup_{i=1}^{N} E_i.
\end{aligned}
$$

(b) General $f \geq 0$.

(1) Assume $\mu(A) = 0$ (i.e. $f = 0$ a.e.)

Let $0 \leq \phi \leq f, \phi$ is simple.

$$
\begin{aligned}
&\implies \phi(x) = 0,\ \forall x \in A^c \\
&\implies \phi = 0 \text{ a.e.} \\
&\implies \int \phi = 0
\end{aligned}
$$

Then $\int f = 0$ by the definition of $\int f$.

(2) Assume $\inf f = 0$. Let $A_n = f^{-1}\left(\left[\frac{1}{n}, \infty\right]\right)$

$$
\begin{aligned}
\implies &A_1 \subset A_2 \subset \ldots \\
&\bigcup_1^\infty A_n = f^{-1}\left(\bigcup_1^\infty \left[\frac{1}{n}, \infty\right]\right) = f^{-1}((0,\infty)) = A \\
&\mu(A_n) = \mu\left(\left\{x \mid f(x) \geq \frac{1}{n}\right\}\right) \leq n \int f = 0 \\
\implies &\mu(A) = \lim_{n\to\infty} \mu(A_n) = 0
\end{aligned}
$$

by the continuity of measure from below. ∎

Corollary 2.24. *$f, g \geq 0$ are measurable, $f = g$ a.e. $\implies \int f = \int g$.*

Proof. Let $A = \{x \mid f(x) \geq g(x)\}$. A is measurable (why?). By assumption $\mu(A) = 0$.

Hence $f1_A = 0$ a.e.

$$\begin{aligned}\int f &= \int f(1_A + 1_{A^c}) \\ &= \int f1_A + \int f1_{A^c} \\ &= \int f1_{A^c} \\ &= \int g1_{A^c} = \int g1_A + \int g1_{A^c} = \int g.\end{aligned}$$ ■

Corollary 2.25. $f_n \geq 0$ *measurable. Then*

(a)

$$\left.\begin{array}{r} 0 \leq f_1 \leq f_2 \leq \ldots \leq f \text{ a.e.} \\ \lim_{n\to\infty} f_n = f \text{ a.e.} \end{array}\right\} \implies \lim_{n\to\infty} f_n = \int f.$$

(b)

$$\lim_{n\to\infty} f_n = f \ a.e \implies \int f \leq \liminf_{n\to\infty} \int f_n.$$