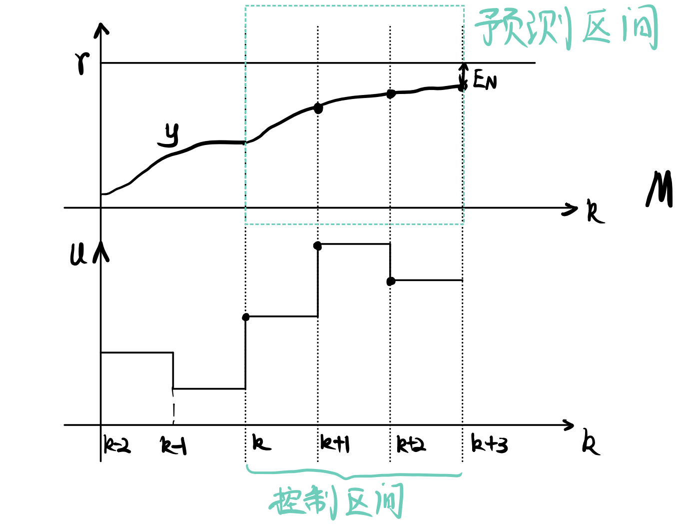
B站 Dr.Can 讲解



假设已获得离散模型:

$$x_{k+1} = Ax_k + Bu_k$$
$$y = Cx + Du$$

MPC:

在 k 时刻:

Step1: 测量/估计系统状态值 x_k

Step2: 基于 $u_k, u_{k+1}, \cdots, u_{k+N-1}$ 做最优化

Step3: 只施加 u_k

思想: 在k时刻,假设未来一段时间的输入信号为 $u_k, u_{k+1}, \cdots, u_{k+N-1}$,根据模型预测系统输出,根据目标函数求解优化问题 得 $u_k, \cdots, u_{k+N-1}$,将 u_k 作为k时刻的系统控制信号.

Step2 常用二次规划来作为优化问题,一般形式: $\min \underbrace{z^TQz}_{\text{二次型}} + \underbrace{C^Tz}_{\text{线性项}}$

$$x_{k+1} = A x_k + B u_k$$

$$x_k = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times1} \quad A_{n\times n} \quad u_k = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{bmatrix}_{p\times1} \quad B_{n\times p}$$

在第k时刻:

$$X_k = \begin{bmatrix} x(k|k) \\ x(k+1|k) \\ \vdots \\ x(k+N|k) \end{bmatrix}_{(N+1)n\times1} \quad U_k = \begin{bmatrix} u(k|k) \\ u(k+1|k) \\ \vdots \\ u(k+N-1|k) \end{bmatrix}_{Np\times1}$$

$x(k+i|k)$: 在k时刻预测的k+i时刻的状态

$u(k+i|k)$: 在k时刻假设的k+i时刻的控制信号

假设 $y=x$,参考信号 $R=0$,误差 $E=y-R=x$.

目标函数:

$$\min J = \left[\sum_{i=0}^{N-1} \underbrace{x^T(k+i|k)Qx(k+i|k)}_{\text{误差}} + \underbrace{u^T(k+i|k)Ru(k+i|k)}_{\text{控制信号}}\right] + x^T(k+N|k)Fx(k+N|k)$$

要使用二次规划(只含一个变量),需消除J中的一个变量(消x,因为我们关心u)

$x(k|k) = x_k$　Step1中测量或估计（状态观测器、卡尔曼滤波等）得到

$$x(k+1|k) = Ax(k|k) + Bu(k|k) = Ax_k + Bu(k|k)$$

$$x(k+2|k) = Ax(k+1|k) + Bu(k+1|k) = A^2x_k + ABu(k|k) + Bu(k+1|k)$$

$$\vdots$$

$$x(k+N|k) = A^Nx_k + A^{N-1}Bu(k|k) + A^{N-2}Bu(k+1|k) + \cdots + ABu(k+N-2|k) + Bu(k+N-1|k)$$

$$X_k = \begin{bmatrix} x(k|k) \\ x(k+1|k) \\ \vdots \\ x(k+N|k) \end{bmatrix} = \overbrace{\begin{bmatrix} I_{n\times n} \\ A_{n\times n} \\ A^2 \\ \vdots \\ A^N_{n\times n} \end{bmatrix}}^{M}_{(N+1)n\times n} x_{k\,n\times 1} + \overbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ B & 0 & \cdots & 0 \\ AB & B & \ddots & \\ \vdots & & & \\ A^{N-1}B & A^{N-2}B & & B \end{bmatrix}}^{C}_{(N+1)n\times Np} \begin{bmatrix} u(k|k) \\ u(k+1|k) \\ u(k+2|k) \\ \vdots \\ u(k+N-1|k) \end{bmatrix}_{Np\times 1}$$

0指零矩阵 $0_{n\times p}$

$M_{(N+1)n\times n}$　　$C_{(N+1)n\times Np}$

$$X_k = Mx_k + CU_k$$

$$\min J = \left[\sum_{i=0}^{N-1} \underbrace{x^T(k+i|k)Qx(k+i|k)}_{误差} + \underbrace{u^T(k+i|k)Ru(k+i|k)}_{控制信号}\right] + x^T(k+N|k)Fx(k+N|k)$$

$$= x^T(k|k)Qx(k|k) + x^T(k+1|k)Qx(k+1|k) + \cdots + x^T(k+N-1|k)Qx(k+N-1|k)$$
$$+ x^T(k+N|k)Fx(k+N|k)$$
$$+ u^T(k|k)Ru(k|k) + u^T(k+1|k)Ru(k+1|k) \cdots + u^T(k+N-1|k)Ru(k+N-1|k)$$

$$= \begin{bmatrix} x(k|k) \\ x(k+1|k) \\ \vdots \\ x(k+N-1|k) \\ x(k+N|k) \end{bmatrix}^T \overbrace{\begin{bmatrix} Q & & & & \\ & Q & & & \\ & & \ddots & & \\ & & & Q & \\ & & & & F \end{bmatrix}}^{\bar{Q}} \begin{bmatrix} x(k|k) \\ x(k+1|k) \\ \vdots \\ x(k+N-1|k) \\ x(k+N|k) \end{bmatrix}$$

$$+ \begin{bmatrix} u(k|k) \\ u(k+1|k) \\ \vdots \\ u(k+N-1|k) \end{bmatrix}^T \overbrace{\begin{bmatrix} R & & & \\ & R & & \\ & & \ddots & \\ & & & R \end{bmatrix}}^{\bar{R}} \begin{bmatrix} u(k|k) \\ u(k+1|k) \\ \vdots \\ u(k+N-1|k) \end{bmatrix} = X_k^T\bar{Q}X_k + U_k^T\bar{R}U_k$$

$$\min J = X_k^T \bar{Q} X_k + U_k^T \bar{R} U_k$$

代入 $X_k = M x_k + C U_k$

$$\min J = (x_k^T M^T + U_k^T C^T)\bar{Q}(M x_k + C U_k) + U_k^T \bar{R} U_k$$

$$= x_k^T M^T \bar{Q} M x_k + \underline{x_k^T M^T \bar{Q} C U_k + U_k^T C^T \bar{Q} M x_k} + U_k^T C^T \bar{Q} C U_k + U_k^T \bar{R} U_k$$

J是一个数，等式右边每一项均为一个数，且可证明这两项相等。

$$= x_k^T \underbrace{M^T \bar{Q} M}_{G} x_k + 2 x_k^T \underbrace{M^T \bar{Q} C}_{E} U_k + U_k^T \underbrace{(C^T \bar{Q} C + \bar{R})}_{H} U_k$$

$$= \underbrace{x_k^T G x_k}_{\text{常数项}} + \underbrace{2 x_k^T E U_k}_{\text{线性项}} + \underbrace{U_k^T H U_k}_{\text{二次型}}$$

二次规划

$$M = \begin{bmatrix} I_{n\times n} \\ A_{n\times n} \\ A^2 \\ \vdots \\ A^N_{n\times n} \end{bmatrix}_{(N+1)n\times n} \qquad C = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ B & 0 & \cdots & 0 \\ AB & B & \ddots & \\ \vdots & & & \\ A^{N-1}B & A^{N-2}B & & B \end{bmatrix}_{(N+1)n\times Np}$$

$Q_{n\times n}$ $\quad R_{p\times p}$ $\quad F_{n\times n}$

$$\bar{Q} = \underbrace{\begin{bmatrix} Q & & & & \\ & Q & & & \\ & & \ddots & & \\ & & & Q & \\ & & & & F \end{bmatrix}}_{(N+1)n\times(N+1)n} \Big\} N\text{项} \qquad \bar{R} = \underbrace{\begin{bmatrix} R & & & \\ & R & & \\ & & \ddots & \\ & & & R \end{bmatrix}}_{Np\times Np} \Big\} N\text{项}$$

$G = M^T \bar{Q} M \qquad [n\times n]$

$E = M^T \bar{Q} C \qquad [n\times Np]$

$H = C^T \bar{Q} C + \bar{R} \qquad [Np\times Np]$

一般情况，$y=Cx$，参考信号 $r\neq 0$，误差 $E=y-r$

目标函数

$$\min J = \left[\sum_{i=0}^{N-1} E^T(k+i|k)\,Q\,E(k+i|k) + U^T(k+i|k)\,R\,U(k+i|k)\right] + E^T(k+N|k)\,F\,E(k+N|k)$$

$$\begin{aligned}
E^TQE &= (y-r)^TQ(y-r)\\
&= (y^T-r^T)Q(y-r)\\
&= (x^TC^T-r^T)Q(Cx-r)\\
&= x^TC^TQCx - x^TC^TQr - r^TQCx + r^TQr\\
&= x^TC^TQCx - 2x^TC^TQr + \underbrace{\cancel{r^TQr}}_{\text{与优化无关}}
\end{aligned}$$

$$\begin{aligned}
\left[\sum_{i=0}^{N-1} E^T(k+i|k)\,Q\,E(k+i|k)\right] + E^T(k+N|k)\,F\,E(k+N|k) = {} & \left[x^T(k|k)C^TQCx(k|k) - 2x^T(k|k)C^TQr\right] + \\
& \left[x^T(k+1|k)C^TQCx(k+1|k) - 2x^T(k+1|k)C^TQr\right] + \\
& \vdots \\
& \left[x^T(k+N-1|k)C^TQCx(k+N-1|k) - 2x^T(k+N-1|k)C^TQr\right] + \\
& \left[x^T(k+N|k)C^TFCx(k+N|k) - 2x^T(k+N|k)C^TFr\right]
\end{aligned}$$

$$= \begin{bmatrix} x(k|k)\\ x(k+1|k)\\ x(k+2|k)\\ \vdots\\ x(k+N|k)\end{bmatrix}^T \underbrace{\begin{bmatrix} C^TQC & & & \\ & C^TQC & & \\ & & C^TQC & \\ & & & \ddots & \\ & & & & C^TFC\end{bmatrix}}_{\bar{Q}_1} \begin{bmatrix} x(k|k)\\ x(k+1|k)\\ x(k+2|k)\\ \vdots\\ x(k+N|k)\end{bmatrix} -$$

$$2\begin{bmatrix} x(k|k)\\ x(k+1|k)\\ x(k+2|k)\\ \vdots\\ x(k+N|k)\end{bmatrix}^T \underbrace{\begin{bmatrix} C^TQr\\ C^TQr\\ C^TQr\\ \vdots\\ C^TFr\end{bmatrix}}_{\bar{Q}_2} = X_k^T\bar{Q}_1\underbrace{X_k}_{(N+1)n\times 1} - 2X_k^T\bar{Q}_2$$

$\min J = X_k^T\bar{Q}_1X_k - 2X_k^T\bar{Q}_2 + U_k^T\bar{R}U_k$　　代入 $X_k = Mx_k + DU_k$

$$\begin{aligned}
&= (x_k^TM^T + U_k^TD^T)\bar{Q}_1(Mx_k + DU_k) - 2(x_k^TM^T + U_k^TD^T)\bar{Q}_2 + U_k^T\bar{R}U_k\\
&= \cancel{x_k^TM^T\bar{Q}_1Mx_k} + x_k^TM^T\bar{Q}_1DU_k + U_k^TD^T\bar{Q}_1Mx_k + U_k^TD^T\bar{Q}_1DU_k\\
&\quad \cancel{-2x_k^TM^T\bar{Q}_2} - 2U_k^TD^T\bar{Q}_2 + U_k^T\bar{R}U_k\\
&= \underbrace{2U_k^T\underbrace{(D^T\bar{Q}_1Mx_k - D^T\bar{Q}_2)}_{E,\ N_p\times 1}}_{\text{线性项}} + \underbrace{U_k^T\underbrace{(D^T\bar{Q}D + \bar{R})}_{H}U_k}_{\text{二次项}}
\end{aligned}$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad}_{\text{二次规划}}$$

$$M^T = \begin{bmatrix} I_{n\times n} \\ A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}_{(N+1)n \times n} \qquad D = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ B & 0 & \cdots & 0 \\ AB & B & \ddots & \\ \vdots & & & \\ A^{N-1}B & A^{N-2}B & & B \end{bmatrix}_{(N+1)n \times Np}$$

$A_{n\times n}$

$B_{n\times p}$

$$\bar{Q}_1 = \begin{bmatrix} C^TQC & & & \\ & C^TQC & & \\ & & C^TQC & \\ & & & \ddots \\ & & & & C^TFC \end{bmatrix}_{(N+1)n \times (N+1)n} \qquad \bar{Q}_2 = \begin{bmatrix} C^TQr \\ C^TQr \\ C^TQr \\ \vdots \\ C^TFr \end{bmatrix}_{(N+1)n \times 1}$$

$y_{m\times 1} = C_{m\times n} x_{n\times 1}$

$Q_{m\times m}$ $\quad r_{m\times 1}$

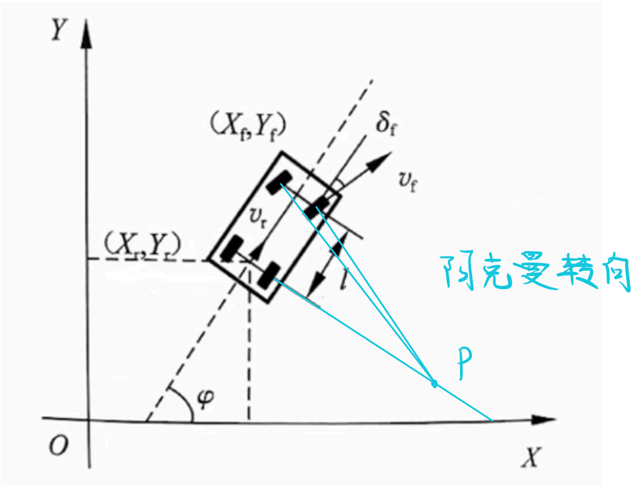
$R_{p\times p}$

$F_{m\times m}$

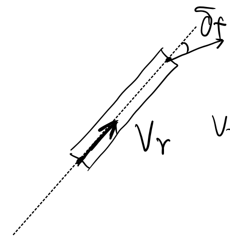
$$\bar{R} = \begin{bmatrix} R & & \\ & R & \\ & & \ddots \\ & & & R \end{bmatrix}_{Np \times Np}$$

B站 模型预测控制 MPC

车辆运动学建模



抽象为两轮模型



$V_f \cos\delta_f = V_r$

$$\dot{X} = V_r \cos\varphi$$
$$\dot{Y} = V_r \sin\varphi$$
$$\dot{\varphi} = \frac{V_f \sin\delta_f}{l} = \frac{V_r \tan\delta_f}{l}$$

状态量 $\xi = \begin{bmatrix} X \\ Y \\ \varphi \end{bmatrix}$ 　控制量 $u = \begin{bmatrix} V \\ \delta \end{bmatrix}$ 　X, Y 为车辆后轴中心坐标，V 后轴速度 δ 前轮偏角

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} \cos\varphi \\ \sin\varphi \\ \frac{\tan\delta}{l} \end{bmatrix} V$$

$\dot{\xi} = f(\xi, u)$ 　非线性模型，无法写为 $\dot{x} = Ax + Bu$ 的形式

线性化：

$\dot{\xi}_r = f(\xi_r, u_r)$，r 指参考量

将 $\dot{\xi} = f(\xi, u)$ 在参考轨迹点处泰勒展开且忽略高阶项

$$\dot{\xi} = f(\xi_r, u_r) + \left.\frac{\partial f(\xi, u)}{\partial \xi}\right|_{\substack{\xi = \xi_r \\ u = u_r}} (\xi - \xi_r) + \left.\frac{\partial f(\xi, u)}{\partial u}\right|_{\substack{\xi = \xi_r \\ u = u_r}} (u - u_r)$$

令 $\tilde{\xi} = \xi - \xi_r$，即误差

$$\dot{\tilde{\xi}} = \frac{\partial f}{\partial \xi}(\xi - \xi_r) + \frac{\partial f}{\partial u}(u - u_r)$$
$$= A\tilde{\xi} + B\tilde{u}$$

$\frac{\partial f}{\partial \xi}$，$\frac{\partial f}{\partial u}$ 均为雅可比矩阵

$$\frac{\partial f}{\partial \xi} = \begin{bmatrix} 0 & 0 & -V\sin\varphi_r \\ 0 & 0 & V\cos\varphi_r \\ 0 & 0 & 0 \end{bmatrix} \qquad \frac{\partial f}{\partial u} = \begin{bmatrix} \cos\varphi_r & 0 \\ \sin\varphi_r & 0 \\ \frac{\tan\delta}{l} & \frac{V}{l\cos^2\delta} \end{bmatrix}$$

$$\tilde{\xi} = \begin{bmatrix} X - X_r \\ Y - Y_r \\ \varphi - \varphi_r \end{bmatrix} \qquad \tilde{u} = u - u_r = \begin{bmatrix} v \\ \delta \end{bmatrix}$$

离散化（前向欧拉法）：

$$\dot{\tilde{\xi}} = \frac{\tilde{\xi}(k+1) - \tilde{\xi}(k)}{T} = A\tilde{\xi}(k) + B\tilde{u}(k)$$

$$\tilde{\xi}(k+1) = (I + TA)\tilde{\xi}(k) + TB\tilde{u}(k)$$

$$= \tilde{A}\tilde{\xi}(k) + \tilde{B}\tilde{u}(k) = \tilde{A}\tilde{\xi}(k) + \tilde{B}u(k)$$

$$\tilde{A} = \begin{bmatrix} 1 & 0 & -TV\sin\varphi_r \\ 0 & 1 & TV\cos\varphi_r \\ 0 & 0 & 1 \end{bmatrix} \qquad \tilde{B} = \begin{bmatrix} T\cos\varphi_r & 0 \\ T\sin\varphi_r & 0 \\ T\frac{\tan\delta}{L} & \frac{TV}{L\cos^2\delta} \end{bmatrix}$$

线性时变

预测

$$\tilde{\xi}(k+1) = \tilde{A}\tilde{\xi}(k) + \tilde{B}u(k)$$

$$\tilde{\xi}(k+2) = \tilde{A}^2\tilde{\xi}(k) + \tilde{A}\tilde{B}u(k) + \tilde{B}u(k+1)$$

$$\tilde{\xi}(k+3) = \tilde{A}^3\tilde{\xi}(k) + \tilde{A}^2\tilde{B}u(k) + \tilde{A}\tilde{B}u(k+1) + \tilde{B}u(k+2)$$

$$\vdots$$

$$\tilde{\xi}(k+N) = \tilde{A}^N\tilde{\xi}(k) + \tilde{A}^{N-1}\tilde{B}u(k) + \tilde{A}^{N-2}\tilde{B}u(k+1) + \cdots + \tilde{A}\tilde{B}u(k+N-2) + \tilde{B}u(k+N-1)$$

预测时，将A视为不变，哪怕A是时变的

控制区间（可以比预测区间短，但绝不能比预测区间长）

$$\begin{bmatrix} \tilde{\xi}(k+1) \\ \tilde{\xi}(k+2) \\ \tilde{\xi}(k+3) \\ \vdots \\ \tilde{\xi}(k+N) \end{bmatrix} = \begin{bmatrix} \tilde{A} \\ \tilde{A}^2 \\ \tilde{A}^3 \\ \vdots \\ \tilde{A}^N \end{bmatrix} \tilde{\xi}(k) + \begin{bmatrix} \tilde{B} & 0 & & & \\ \tilde{A}\tilde{B} & \tilde{B} & 0 & & \\ \tilde{A}^2\tilde{B} & \tilde{A}\tilde{B} & \tilde{B} & \ddots & \\ \vdots & \vdots & \vdots & & \ddots \\ \tilde{A}^{N-1}\tilde{B} & \tilde{A}^{N-2}\tilde{B} & \tilde{A}^{N-3}\tilde{B} & \cdots & \tilde{B} \end{bmatrix} \begin{bmatrix} u(k) \\ u(k+1) \\ u(k+2) \\ \vdots \\ u(k+N-1) \end{bmatrix}$$

$$Y_k = \Psi\tilde{\xi}_k + \theta U_k$$

$$\tilde{\xi}_k = \begin{bmatrix} \tilde{X}_k \\ \tilde{Y}_k \\ \tilde{\varphi}_k \end{bmatrix}$$

当预测区间长度Np等于控制区间长度Nc+1时，θ才为下三角矩阵。

优化—二次规划(QP)

二次规划 $\min_{\Delta U, \varepsilon} \frac{1}{2}x^T H x + f^T x$

s.t. $Ax \leq b$

$Aeq\, x = Beq$

$lb \leq x \leq ub$

ε为松弛因子，为了更快得结果

优化目标 $\min \ Y_k^T Q Y_k + U_k^T R U_k$ 并不符合二次规划的形式

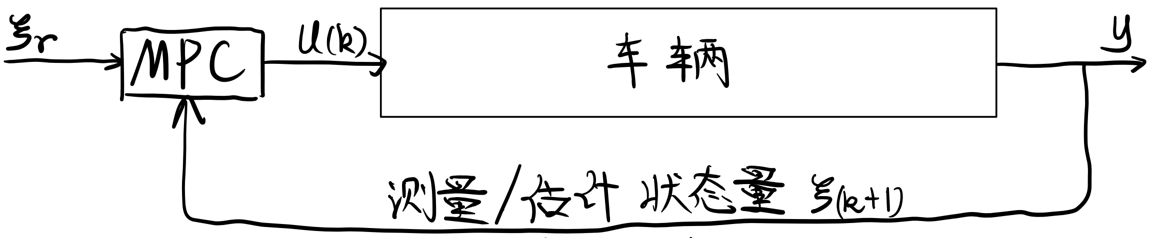
↓ 代入 $Y_k = \underbrace{\Phi \xi_k}_{E} + \theta U_k$

$\min \ 2E^T Q \theta U_k + U_k^T(\theta^T Q \theta + R) U_k$ 二次规划

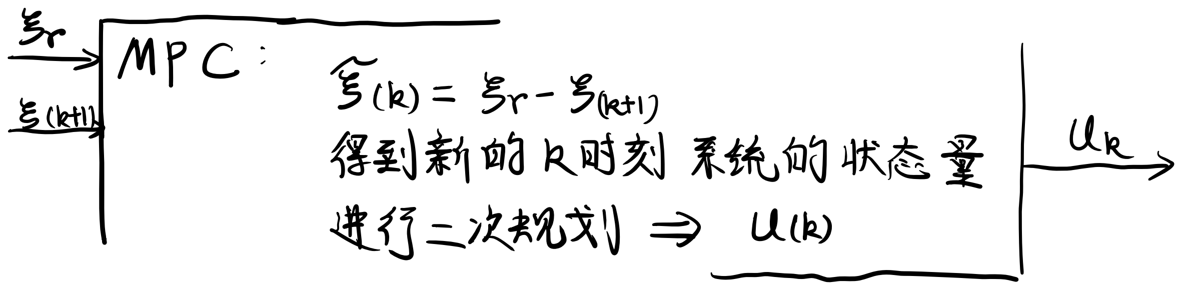
s.t. $lb \leq U_k \leq ub$ 参考 Matlab 中 quadprog 函数.

控制:

二次优化 得到 U_k，只取 $u(k)$ 作控制量.



反馈控制



MPC从输入 参考信号以及测量/估计的系统状态量到输出 控制信号，中间的优化过程用到了被控系统的物理模型，所以更复杂，控制效果也更好，对于不同被控系统，都需要重新设计MPC。而PID属于通用控制器，使用过程中只需要考虑如何调整控制器参数，不需要考虑被控系统的物理模型(但还是需要结合模型考虑加入控制器后的系统稳定性)

MPC适用于MIMO系统，而PID更适用于SISO系统；PID只考虑了输出的准度，而MPC不但可以考虑输出的准度，还考虑了输入。

修正:

离散化得到的模型: $\xi(k+1) = \tilde{A}(k)\xi(k) + \tilde{B}(k)u(k)$

有些情况下我们更希望控制量的变化量，即 $\Delta U(k)$

令 $z(k) = \begin{bmatrix} \xi(k) \\ u(k-1) \end{bmatrix}$ $\quad \hat{A} = \begin{bmatrix} \tilde{A}(k) & \tilde{B}(k) \\ 0 & I_p \end{bmatrix}$ $\quad \hat{B} = \begin{bmatrix} \tilde{B}(k) \\ I_p \end{bmatrix}$ $\quad \hat{C} = [C \quad 0]$

$$\begin{cases} z(k+1) = \hat{A} z(k) + \hat{B} \Delta U(k) \\ \eta(k) = \hat{C} z(k) \end{cases}$$

预测:

$$Y(k) = \Psi z(k) + \theta \Delta U(k)$$

优化:

$$J = Y^T_{(k)} Q Y(k) + \Delta U^T_{(k)} R \Delta U(k) + \underbrace{\rho \varepsilon^2}_{\text{松弛项}}$$

$\downarrow$ 代入 $Y(k) = \Psi z(k) + \theta \Delta U(k)$
且令 $E = \Psi z(k)$

$$J = \Delta U^T_{(k)} (\theta^T Q \theta + R) \Delta U(k) + \rho \varepsilon^2 + 2E^T Q \theta \Delta U(k)$$

$$= [\Delta U^T_{(k)} \quad \varepsilon] \begin{bmatrix} \theta^T Q \theta + R & 0 \\ 0 & \rho \end{bmatrix} \begin{bmatrix} \Delta U(k) \\ \varepsilon \end{bmatrix} + [2E^T Q \theta \quad 0] \begin{bmatrix} \Delta U(k) \\ \varepsilon \end{bmatrix}$$

$$= X^T H X + 2 f^T X$$

约束条件处理:

通常我们会约束: $\Delta U_{min} \leq \Delta U(k) \leq \Delta U_{max}$

$U_{min} \leq U(k) \leq U_{max}$

如何转换为二次规划的标准约束形式?

$$\text{s.t.} \quad \begin{aligned} & A x \leq b \\ & A_{eq} x = B_{eq} \\ & lb \leq x \leq ub \end{aligned}$$

$\Delta U_{min} \leq \Delta U(k) \leq \Delta U_{max} \rightarrow lb \leq x \leq ub$

令 $A = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & 1 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 1 & \cdots & \cdots & 1 \end{bmatrix}_{N_c \times N_c} \otimes I_p$ $\qquad \otimes$: 克罗内克积

$$\begin{bmatrix} A & 0 \\ -A & 0 \end{bmatrix} \begin{bmatrix} \Delta U(k) \\ \varepsilon \end{bmatrix} \leq \begin{bmatrix} U_{max} - U(k) \\ -U_{min} + U(k) \end{bmatrix} \rightarrow A x \leq b$$

$$A \Delta U(k) \leq U_{max} - U(k)$$
$$A \Delta U(k) \geq U_{min} - U(k)$$

$$U_{min} \leq U(k) + A\Delta U(k) \leq U_{max}$$

控制:

$$U(k+1) = U(k) + \Delta U(k)$$