3.17

Eg 2.1 (in-class)

Find the minimum to the following function in 3 dimensions

$f_0(x) = \frac{40}{x_1 x_2 x_3} + 20 x_1 x_3 + 10 x_1 x_2 + 40 x_2 x_3$ for strictly positive entries

① compute the first iterations of MART

② Assuming MART converges to $\delta^* = (0.4, 0.2, 0.2, 0.2)^T$, what are the optimal values for x^* and the minimal cost?

$f_0(x)$ is a posynomial with

$$\tilde{A} = \begin{bmatrix} -1 & 1 & 1 & 0 \\ -1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad A\delta = b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

① lift $A \rightarrow$ Add $2R_4$ to R_1, $2R_4$ to R_2, $2R_4$ to R_3

$$B = A + \begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 3 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 3 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

We want to solve $B\delta = \tilde{b} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}$

② Apply MART,

$i = (k \bmod (\text{nb of rows in } B)) + 1$

$x_j^{k+1} = x_j^k \left(\frac{b_i}{(Ax^k)_i}\right)^{\frac{A_{i,j}}{m_i}} \quad m_i = \max\{A_{ij} \mid 1 \le j \le J\}$ < (Remark 2.5)

$C = (40, 20, 10, 40)^T$

Set $\delta^0 = c$

$$B\delta^0 = \begin{pmatrix} 1\cdot 40 + 3\cdot 20 + 3\cdot 10 + 2\cdot 40 \\ 1\cdot 40 + 2\cdot 20 + 3\cdot 10 + 3\cdot 40 \\ 1\cdot 40 + 3\cdot 20 + 2\cdot 10 + 3\cdot 40 \\ 40 + 20 + 10 + 40 \end{pmatrix} = \begin{pmatrix} 210 \\ 230 \\ 240 \\ 110 \end{pmatrix}$$

$k=0,\ i=1$

$k+1$: $\delta_1^1 = \delta_1^0 \left(\frac{\tilde{b}_1}{(B\delta)_1}\right)^{\frac{B_{11}}{m_1}} \qquad m = (3, 3, 3, 1)^T$

$\delta_1^1 = 40\left(\frac{2}{210}\right)^{\frac{1}{3}} = 40 \cdot 105^{-\frac{1}{3}}$

$\delta_2^1 = \delta_2^0 \left(\frac{\tilde{b}_1}{(B\delta^0)_1}\right)^{\frac{B_{12}}{m_1}} = 20 \cdot \left(\frac{2}{210}\right)^{\frac{3}{3}} = \frac{20}{105}$

$\delta_3^1 = \delta_3^0 \left(\frac{2}{210}\right)^{\frac{3}{3}} = \frac{10}{105}$

$\delta_4^1 = \delta_4^0 \left(\frac{2}{210}\right)^{\frac{2}{3}} = \frac{10}{105} = 40 \cdot 105^{-\frac{2}{3}}$

$k=1$

$i=2$

$\vdots$

MART converges to $\delta^* = \begin{pmatrix} 0.4 \\ 0.2 \\ 0.2 \\ 0.2 \end{pmatrix}$

$\Leftrightarrow f_0(x^*) = g_0(\delta^*)$

$g_0(\delta) = \prod_{i=1}^{m} \left(\frac{c_i}{\delta_i}\right)^{\delta_i}$

$g_0(\delta^*) = \left(\frac{40}{0.4}\right)^{0.4} \left(\frac{20}{0.2}\right)^{0.2} \left(\frac{10}{0.2}\right)^{0.2} \left(\frac{40}{0.2}\right)^{0.2}$

$= (100)^{0.4} (100)^{0.2} \underbrace{(50)^{0.2} (200)^{0.2}}_{(100^2)^{0.2}}$

$= 100^{0.4+0.2+2\times 0.2} = 100 = f_0(x^*)$: minimal cost

We remember now $\delta_i^* = \frac{u_j(x^*)}{g_0(\delta^*)} = \frac{u_j(x^*)}{f_0(x^*)}$

$$\begin{cases} u_1(x^*) = \frac{40}{x_1^* x_2^* x_3^*} = 100 \cdot 0.4 = 40 \\ u_2(x^*) = 20 \cdot x_1^* x_3^* = 100 \cdot 0.2 = 20 \\ u_3(x^*) = 10 \cdot x_1^* \cdot x_2^* = 100 \cdot 0.2 = 20 \\ u_4(x^*) = 40 \cdot x_2^* x_3^* = 100 \cdot 0.2 = 20 \end{cases}$$

$$\Leftrightarrow \begin{cases} x_2^* x_3^* = \frac{1}{2} \\ x_1^* x_2^* = 2 \rightarrow x_1^* = \frac{2}{x_2^*} \\ x_1^* x_3^* = 1 \\ x_1^* x_2^* x_3^* = 1 \end{cases}$$

$\rightarrow \frac{2}{x_2^*} x_2^* \cdot x_3^* = 1 \Rightarrow x_3^* = \frac{1}{2}$

$\checkmark\ x_1^* x_3^* = 2 \cdot \frac{1}{2} = 1$

$\Rightarrow x_2^* = 1$

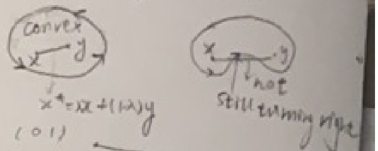
$\Rightarrow x_1^* = 2$

The final minimize of $f_0(x)$ is $x^* = \begin{pmatrix} 2 \\ 1 \\ \frac{1}{2} \end{pmatrix}$

Def 2.1 (Convex sets)

Let $C \subseteq \mathbb{R}^d$, We say that C is convex if for any two points $x, y \in C$, for any $\lambda \in [0,1]$,

$\lambda x + (1-\lambda) y \in C$.

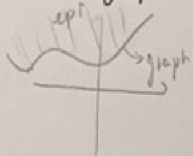


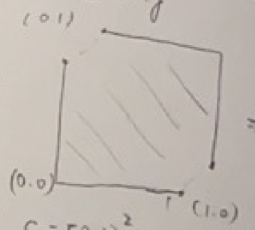
Def 2.2 (Graph-Epigraph)

Let $f : \mathrm{dom}(f) \to \mathbb{R}$

- The graph of f is the set of points $\{(x, f(x)), x \in \mathrm{dom}(f)\} \subset \mathbb{R}^{d+1}$
- The epigraph of f is the set of points above the graph:

$$\mathrm{epi}(f) = \{(x,y) \mid x \in \mathrm{dom}(f),\ y \geq f(x)\}$$





$C = [0,1]^2 - \{(x,0), x \geq 0.75\}$
$- \{(x,1), x \leq 0.25\}$
$- \{(0,y), y \geq 0.75\}$
$- \{(1,y), y \leq 0.25\}$

Def 2.3 (Convex function)

Let $f: \mathrm{dom}(f) \to \mathbb{R}$. We say that f is convex if:

① $\mathrm{dom}(f)$ is convex, and

② for all $x, y \in \mathrm{dom}(f)$, for all $\lambda \in [0,1]$, we have

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$$

Eg 2.1

Let $f(x) = a^T x + c$ for a given vector $a \in \mathbb{R}^d$ and $c \in \mathbb{R}$. f is convex.

Take $x, y \in \mathbb{R}^d$, $\lambda \in [0,1]$

$$\begin{aligned} f(\lambda x + (1-\lambda) y) &= a^T(\lambda x + (1-\lambda) y) + c \\ &= a^T(\lambda x) + a^T((1-\lambda) y) + c \\ &= \lambda a^T x + (1-\lambda) a^T y + c \\ &\overset{1 = 1-\lambda+\lambda}{=} \lambda a^T x + (1-\lambda) a^T y + \lambda c + (1-\lambda) c \\ &\overset{\text{special}}{=} \lambda f(x) + (1-\lambda) f(y) \end{aligned}$$

equality instead of inequality

Eg 2.2

Let $Q \in \mathbb{R}^{d \times d}$ be a positive definite matrix and define $f(x) = x^T Q x$. f is a convex function

$\forall x, y \in \mathbb{R}^d$, $\lambda \in [0,1]$: $f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$

$\lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y)$

$\Leftrightarrow \lambda x^T Q x + (1-\lambda) y^T Q y \geq (\lambda x + (1-\lambda) y)^T Q (\lambda x + (1-\lambda) y)$

$\Leftrightarrow \lambda x^T Q x + (1-\lambda) y^T Q y \geq \lambda^2 x^T Q x + (1-\lambda)^2 y^T Q y + 2\lambda(1-\lambda) x^T Q y$

$\Leftrightarrow (\lambda - \lambda^2) x^T Q x + ((1-\lambda) - (1-\lambda)^2) y^T Q y - 2\lambda(1-\lambda) x^T Q y \geq 0$

$\Leftrightarrow \lambda(1-\lambda) x^T Q x + (1-\lambda)\lambda y^T Q y - 2\lambda(1-\lambda) x^T Q y \geq 0$

Assuming $\lambda \neq 1, \lambda \neq 0$. $\Leftrightarrow x^T Q x + y^T Q y - 2x^T Q y \geq 0$

$\Leftrightarrow (x-y)^T Q (x-y) \geq 0$ which is true because $Q > 0$

Let's look at $\lambda = 0, \lambda = 1$. then either $x^T Q x \geq x^T Q x$

or $y^T Q y \geq y^T Q y$ } both are true.

Proposition 2.2

Let $f: \mathrm{dom}(f) \to \mathbb{R}$. f is convex $\Leftrightarrow$ $\mathrm{epi}(f)$ is convex.

$\Rightarrow$ Assume f is convex, Remember $(x, y) \in \mathrm{epi}(f) \Leftrightarrow y \geq f(x)$

Consider $\underline{(x_1, y_1), (x_2, y_2) \in \mathrm{epi}(f)}$. $\lambda \in [0, 1]$

We know that $f(x_1) \leq y_1$, $f(x_2) \leq y_2$

Convexity of f emplies: $f(\underbrace{\lambda x_1 + (1-\lambda) x_2}_{\tilde{x}}) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$

$$\leq \underbrace{\lambda y_1 + (1-\lambda) y_2}_{\tilde{y}}$$

$$f(\tilde{x}) \leq \tilde{y} \Rightarrow (\tilde{x}, \tilde{y}) \in \mathrm{epi}(f)$$

$$\underline{\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2)} = (\lambda x_1 + (1-\lambda) x_2, \lambda y_1 + (1-\lambda) y_2)$$

$$= \underline{(\tilde{x}, \tilde{y})} \in \mathrm{epi}(f)$$

$$\Rightarrow \mathrm{epi}(f) \text{ is convex.}$$

$\Leftarrow$ Assume $\mathrm{epi}(f)$ is convex.

Consider $x_1, x_2 \in \mathrm{dom}(f)$ and $\lambda \in [0, 1]$

By definition of the epigraph $(x_1, f(x_1)), (x_2, f(x_2)) \in \mathrm{epi}(f)$

But $\mathrm{epi}(f)$ is convex

$$\Rightarrow \lambda(x_1, f(x_1)) + (1-\lambda)(x_2, f(x_2)) \in \mathrm{epi}(f)$$

$$\Rightarrow (\underbrace{\lambda x_1 + (1-\lambda) x_2}_{f(\sim)}, \underbrace{\lambda f(x_1) + (1-\lambda) f(x_2)}_{\leq y}) \in \mathrm{epi}(f)$$

By definition,

$$f(\lambda x_1 + (1-\lambda) x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2) \Rightarrow f \text{ is convex}$$

Proposition 2.3 (Jensen's inequality)

Let $f: \mathrm{dom}(f) \to \mathbb{R}$ be a convex function. Let $x_1, \cdots, x_n$ be n points in $\mathrm{dom}(f) \subset \mathbb{R}^d$ and let $\lambda_1 \cdots, \lambda_n$ be n nonnegative numbers such that $\sum \lambda_i = 1$. Then

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i f(x_i)$$

Proposition 2.4

Let f be a convex function on the open set $\mathrm{dom}(f)$. Then f is continuous.

Def 2.4 (Lipshitz continuity)

A function $f: \mathrm{dom}(f) \to \mathbb{R}$ is said to be <u>Lipschitz continuous</u> with Lipshitz constant L (sometimes expressed as L-Lipschitz or, if context is clear, simply f is Lipschitz) if $\|f(x) - f(y)\| \leq L\|x - y\|$, $\forall x, y \in \mathrm{dom}(f)$.