Basics of Complex Numbers

A complex number is represented as $Z = a + ib$ where a and b are real numbers and

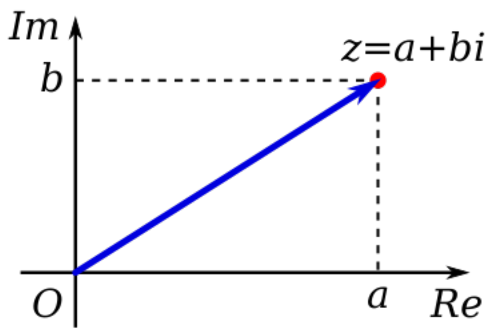
$$Re(Z) = a$$

$$Im(Z) = b$$

$$i = \sqrt{-1}$$

$$i^2 = -1$$

A complex number can be represented on an argand plane with the X-axis as the real part of the complex number and the Y-part as the imaginary part of the complex number



Algebra of Complex numbers

Two complex numbers $Z_1 = a_1 + ib_1$ and $Z_2 = a_2 + ib_2$ can be added and subtracted by seperately adding/subtracting their real and imaginary parts.
Multiplication of two complex numbers,

$$Z_1 \cdot Z_2 = a_1a_2 + ia_1b_2 + ib_1a_2 + i^2b_1b_2 = a_1a_2 + i(a_1b_2 + b_1a_2) - b_1b_2$$

Division of two complex numbers,
Mulitiplying both numerator and denominator by the **conjugate** of the denominator inorder to obtain a real denominator and a complex numerator. For example,

$$\frac{4+3i}{1+2i} = \frac{(4+3i)(1-2i)}{(1+2i)(1-2i)} = \frac{4-8i+3i-6i^2}{1-4i^2} = \frac{12+5i}{5} = \frac{12}{5} + i$$

Complex Conjugate and absolute value

For a complex number $Z = a + ib$, the complex conjugate is defined as $\overline{Z} = a - ib$

$$Z\overline{Z} = (a+ib)(a-ib) = a^2 - iab + iab + b^2 = a^2 + b^2 \tag{1}$$

As per the definition of the absolute value of a complex number

$$|Z| = \sqrt{a^2 + b^2} \tag{2}$$

By (1) and (2), we have

$$Z\overline{Z} = |Z|^2$$

Polar form and Euler's form

Representing the complex number in the form of a $|Z|$ and argument ϕ

$$Z = |Z|(\cos\phi + i\sin\phi)$$

is the Polar representation of a complex number

$$e^{i\phi} = \cos\phi + i\sin\phi$$

Therefore,

$$Z = |Z|e^{i\phi}$$

is the Euler form of the complex number A lot of cool stuff related to complex number on this playlist. Not required for economic aspects though.

Eigen Vectors and Values

$S \in R^{n\times n}$
$A \in R^{n\times n}$
$S^{-1}AS = D$ diagonal matrix, tranformation matrix
Is there a basis transformation?
$\lambda \in R$ is called an eigenvalue of A if there exists a vector $v \neq \phi$ such that

$$A \cdot v = \lambda \cdot v$$

Any such vector v in this case would be called an eigen vector of A for the eigen value λ **Gaussian Reformulation:** Determination of the eigenvalues and eigenvectors.
λ is an eigenvalue of A if there $v \neq \phi$ with $A \cdot v = \lambda \cdot v$

$$\Leftrightarrow A \cdot v - \lambda \cdot v = 0$$

$$\Leftrightarrow (A - \lambda E_n) \cdot v = 0$$

$$\Leftrightarrow v \in \ker(A \cdot \lambda E_n)$$

$$\Leftrightarrow (A - \lambda E_n) \text{ is not invertible}$$

$$\Leftrightarrow \det(A - \lambda E_n) = 0$$

The **characteristic polynomial** $p_A(\lambda)$ is defined as the determinant $\det(A - \lambda E_n)$

$$p_A(\lambda) = \det(A - \lambda E_n)$$

and the kernel $\ker(A - \lambda E_n)$ is called the **eigenspace of A for the eigenvalue** λ and written as

$$eig_A(\lambda) = \ker(A - \lambda E_n)$$

The idea to solve each and every problem here is quite simple,

1. First of all, determine the **characteristic polynomial** [Diagonalized matrix S is the matrix with the eigenspaces of the respective eigenvalues]
2. Find the roots of the characteristic polynomial, these are the eigenvalues of A
3. For each eigenvalue of A, determine the eigenspace by solving the linear equation system $(A - \lambda E_n) \cdot v = 0$

Diagonalization using complex numbers

characterizing which matrices can be transformed into diagonal form, ie possess an eigenvector basis.
A matrix $A \in R^{n\times n}$ is called **diagonalizable** if there is a basis of R^n that consists of eigenvectors of A
NOTE: For each eigenvalue $\rightarrow$ there is at least one eigenvector ie. $\dim(eig_A(\lambda)) \geq 1$ for each eigenvalue λ
$\Rightarrow$ if A possesses n distinct eigenvalue, then there are at least n eigenvectors

$$p_A(\lambda) = (\lambda - \lambda*)^k \cdot g(\lambda) \text{ and } g(\lambda*) \neq 0$$

Then k is called the **algebraic multiplicity** of $\lambda*$ and $\dim(eig_A(\lambda*))$ is called the geometric mutliplicity of $\lambda*$

1. Algebraic mulitplicity $\geq$ Geometric mulitplicity
2. For pairwise distinct eigenvalues of A with corresponding eigenvectors- the set $\{v_1, \ldots, v_r\}$ is linearly independent
3. For a matrix to be diagonalizable, the algebraic and the geometric mulitplicity have to be equal for every eigenvalue of A

Proving (2) is pretty simple as you can simply use the principle of induction after solving it for 2 eigenvalues and their corresponding eigenvectors
If roots are complex, we can use the Euler's formula

$$e^{i\phi} = \cos\phi + i\sin\phi$$

what if the matrix cannot be diagonalized? **JORDAN NORMAL FORM** as generalizations and refer for Inhomogeneous Differential Equation system
`https://people.math.harvard.edu/~knill/teaching/math19b_2011/handouts/lecture29.pdf` for a recap

Predator-Prey Model

squirrels-hawks $\rightarrow$ The growth of hawk population depends on availability of squirrels and vice-versa. [Fewer hawks $\rightarrow$ higher the growth rate of squirrels]
Model this through DE system:

$$h'(t) = s(t) - 12 \text{ and } h(0) = 6$$
$$s'(t) = -h(t) \text{ and } s(0) = 20$$

1. get rid of constants: define a new system

$$y_1(t) = h(t) - 10 = -s'(t) \text{ and } y_1(0) = h(0) - 10 = -4$$
$$y_2(t) = s(t) - 12 = h'(t) \text{ and } y_2(0) = s(0) - 12 = 8$$
$$\Rightarrow y_1'(t) = h'(t) = s(t) - 12 = y_2(t)$$
$$\Rightarrow y_2'(t) = -h(t) + 10$$

   new system without constants:

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \mid \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} -4 \\ 8 \end{pmatrix}$$

2. Diagonalize the matrix A [find eigenvalues and the corresponding eigenvectors then form the S matrix]

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

not showing the calculations but the eigenvectors are $+i$, $-i$ and eigenvectors are $\begin{pmatrix}1\\i\end{pmatrix}$ and $\begin{pmatrix}1\\-i\end{pmatrix}$ so the required S matrix is $\begin{pmatrix}1 & 1\\i & -i\end{pmatrix}$

$$S^{-1}AS = \begin{pmatrix}i & 0\\0 & -i\end{pmatrix}$$

3. <u>consider the decoupled system after substitution</u> $y = SZ$

$$Z'(t) = \begin{pmatrix}i & 0\\0 & i\end{pmatrix} Z(t)$$

$\Rightarrow$

$$Z_1'(t) = i \cdot Z_1(t) \Rightarrow Z_1(t) = \gamma_1 \cdot e^{it}$$
$$Z_2'(t) = -i \cdot Z_2(t) \Rightarrow Z_2(t) = \gamma_2 \cdot e^{-it}$$

4. <u>determine y = SZ:</u>

$$y(t) = \begin{pmatrix}1 & 1\\i & -i\end{pmatrix}\begin{pmatrix}\gamma_1 e^{it}\\\gamma_2 e^{-it}\end{pmatrix}$$

$\Rightarrow$

$$y_1(t) = \gamma_1 e^{it} + \gamma_2 e^{-it}$$
$$y_2(t) = \gamma_2 i e^{it} - \gamma_2 i e^{-it}$$

5. <u>Determine γ_1, γ_2 through initial values:</u>

$$y_1(0) = -4 = \gamma_1 \cdot 1 + \gamma_2 \cdot 1$$

$$y_2(0) = 8 = \gamma_1 \cdot i \cdot 1 - \gamma_2 \cdot i \cdot 1$$

$$\rightarrow \left(\begin{array}{cc|c}1 & 1 & -4\\i & -i & 8\end{array}\right) \rightarrow \left(\begin{array}{cc|c}1 & 1 & -4\\0 & -2i & 8+4i\end{array}\right)$$

$$(-2i)\gamma_2 = 8 + 4i$$

$$\Leftrightarrow \gamma_2 = 4i - 2 = -2 + 4i$$

$$\Rightarrow \gamma_1 = -4 - \gamma_2 = -2 - 4i$$

Solutions:

$$y_1(t) = (-2-4i)e^{it} + (-2+4i)e^{-it}$$
$$y_2(t) = (4-2i)e^{it} + (4+2i)e^{-it}$$

6. <u>Use Euler's Formula:</u> replace e^{it} and e^{-it} with $cis\theta$ terms

$$y_1(t) = -4\cos t + 8\sin t \mid h(t) = -4\cos t + 8\sin t + 10$$

$$y_2(t) = 8\cos t + 4\sin t \mid s(t) = 8\cos t + 4\sin t + 12$$