Solving a Differential Equation

Case 1: Homogeneous DE

we have: $i(t) = 0$ and $k'(t) = \delta k(t)$

Claim: solution is $k(t) = \gamma \cdot e^{\delta t}$
Set of solutions for the homogeneous DE: $\{\gamma \cdot e^{\delta t} : \gamma \in R\}$

Case 2: Inhomogeneous DE we have a general $i(t)$

1. It is sufficient to determine (particular) solution of the Inhomogeneous DE
   $K_n(t) \to$ solution for the homogeneous DE $\Rightarrow K_n'(t) = \delta \cdot K_n(t)$
   $K_p(t) \to$ solution for the Inhomogeneous DE $\Rightarrow K_p'(t) = \delta \cdot K_p(t) + i(t)$
   combining both Equations, we have:

$$\begin{aligned} K'(t) &= K_p'(t) + K_n'(t) \\ &= \delta \cdot K_p(t) + i(t) + \delta \cdot K_n(t) \\ &= \delta \cdot (K_p(t) + K_n(t)) + i(t) \\ &= \delta \cdot K(t) + i(t) \end{aligned}$$

2. Determine one solution for Inhomogeneous DE through variation of constant
   $k(t) = \gamma \cdot e^{\delta t}$
   $k(t) = c(t) \cdot e^{\delta t}$
   substituting this into the DE

$$k'(t) = c'(t) \cdot e^{\lambda t} + c(t) \cdot \delta e^{\delta t} \tag{1}$$

$$k'(t) = k(t) \cdot \lambda + i(t) \tag{2}$$

   from (1) and (2) we have:

$$c'(t) \cdot e^{-\lambda \cdot t} = i(t)$$

   Integrating both sides

$$c(t) = \int i(t) \cdot e^{-\lambda t} dt$$

Example: Solving an Ordinary Differential Equation

$$f'(x) = 1 - x + f(x),\ f(0) = 2$$

1. Homogeneous DE:

$$f'(x) = f(x) \Rightarrow f(x) = \gamma \cdot e^x$$

   check:

$$f'(x) = f(x) = \gamma \cdot e^x - \gamma \cdot e^x = 0$$

2. Variation of constant:

$$f(x) = c(x) \cdot e^x$$

$$f'(x) = c'(x) \cdot e^x + c(x) \cdot e^x$$

$$f'(x) - f(x) = 1 - x$$

Replacing $f'(x)$ and $f(x)$ with the function of $c(x)$ and $c'(x)$

$$c'(x) \cdot e^x + c(x) \cdot e^x - c(x) \cdot e^x = 1 - x$$

$$c'(x) = e^{-x} \cdot (1 - x)$$

$$c(x) = \int e^{-x} \cdot (1 - x) dx$$

$$c(x) = \int e^{-x} dx - \int e^{-x} \cdot x dx$$

$$c(x) = e^{-x} \cdot x$$

Particular solution:

$$f(x) = c(x) \cdot e^x \Rightarrow f(x) = e^{-x} \cdot x \cdot e^x$$

$$f(x) = x \Rightarrow f'(x) = 1$$

3. Set of solutions:

$$f(x) = x + \gamma \cdot e^x$$

4. Initial Value:

$$f(0) = 2$$

$$f(0) = 0 + \gamma = 2 \Rightarrow \gamma = 2$$

$$\rightarrow f(x) = x + 2 \cdot e^x$$

solves the initial value problem

Note: You can also solve the Differential Equation with the help of an integrating factor
Refer to this wikipedia article integrating factor and this text

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System of Differential Equations

A homogeneous system of Linear Differential Equations with constant coefficients is a system of the form:

$$\begin{pmatrix} y_1'(t) \\ \vdots \\ \vdots \\ y_n'(t) \end{pmatrix} = A \cdot \begin{pmatrix} y_1(t) \\ \vdots \\ \vdots \\ y_n(t) \end{pmatrix}, \quad \begin{pmatrix} y_1(0) \\ \vdots \\ \vdots \\ y_n(0) \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \vdots \\ \vdots \\ \gamma_2 \end{pmatrix}$$

$$\text{where A} \in R \text{ and } \begin{pmatrix} \gamma_1 \\ \vdots \\ \vdots \\ \gamma_2 \end{pmatrix} \in R^n$$

The key idea is that if A is a diagonal matrix, then each row is a Differential Equation that is independent of the others, solve each row separately **decoupled system**

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

We need a basis and a corresponding basis transformation S $\in R^{n\times n}$ such that $S^{-1}AS = D$ is diagonal
Then we substitute, y = SZ $\Rightarrow Z = S^{-1}y$
The DE system,

$$y' = A \cdot y$$

becomes

$$S \cdot z = A \cdot S \cdot Z$$

$$Z' = S^{-1} \cdot A \cdot S \cdot Z = D \cdot Z$$

Solve the new decoupled system and transform solution back through $y = SZ$
Example:

$$\gamma'(t) = \frac{5}{2} \cdot \gamma(t) - \phi(t)$$

$$\phi'(t) = \frac{-1}{4} \cdot \gamma(t) + \frac{5}{2}\phi(t)$$

matrix form

$$\begin{pmatrix} \gamma' \\ \phi' \end{pmatrix} = \begin{pmatrix} \frac{5}{2} & -1 \\ \frac{-1}{4} & \frac{5}{2} \end{pmatrix} \begin{pmatrix} \gamma \\ \phi \end{pmatrix}$$

now, determine the eigenvalues and eigenvectors for the matrix

$$p_A(\lambda) = \det(A - \lambda I_2) = \det \begin{pmatrix} \frac{5}{2} - \lambda & -1 \\ \frac{-1}{4} & \frac{5}{2} - \lambda \end{pmatrix} = \left(\frac{5}{2} - \lambda\right)^2 - \frac{1}{4} = \frac{25}{4} + \lambda^2 - 5\lambda - \frac{1}{4} = 6 + \lambda^2 - 5\lambda = (\lambda - 3)(\lambda - 2)$$

The roots of the equation are $\lambda_1 = 3$ and $\lambda_2 = 2$
For the respective eigenspaces, we get

$$eig_A(3) = \ker(A - 3I_2) = \ker \begin{pmatrix} \frac{-1}{2} & -1 \\ \frac{-1}{4} & \frac{-1}{2} \end{pmatrix} = \ker \begin{pmatrix} \frac{1}{2} & -1 \\ 0 & 0 \end{pmatrix} = span \left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$$

$$eig_A(2) = \ker(A - 2I_2) = \ker \begin{pmatrix} \frac{1}{2} & -1 \\ \frac{-1}{4} & \frac{1}{2} \end{pmatrix} = \ker \begin{pmatrix} \frac{1}{2} & -1 \\ 0 & 0 \end{pmatrix} = span \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$$

$$S = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}$$

$$S^{-1} = \frac{-1}{4} \begin{pmatrix} 1 & -2 \\ -1 & -2 \end{pmatrix}$$

$$S^{-1}AS = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} = D$$

Solve the system:

$$Z' = DZ = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

$$Z_1(t) = \gamma_1 e^{3t}$$

$$Z_2(t) = \gamma_2 e^{2t}$$

$$\begin{pmatrix}\gamma\\ \phi\end{pmatrix} = \begin{pmatrix}-2 & 2\\ 1 & 1\end{pmatrix}\begin{pmatrix}\gamma_1 e^{3t}\\ \gamma_2 e^{2t}\end{pmatrix}$$

$$\gamma(t) = -2\gamma_1 e^{3t} + 2\gamma_2 e^{2t}$$

$$\phi(t) = \gamma_1 e^{3t}\gamma_2 e^{3t}$$

we know the initial values: $\gamma(0) = 60$ and $\phi(0) = 60$

$$-2\gamma_1 + 2\gamma_2 = 60$$

$$\gamma_1 + \gamma_2 = 60$$

solving, we get $\gamma_1 = 15$ and $\gamma_2 = 45$

$$\gamma(t) = -30e^{3t} + 90e^{2t}$$

$$\phi(t) = 15e^{3t} + 45e^{2t}$$