A Study On The Entropy Of Wave Function

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$$\int_{-\infty}^{+\infty} |f(x)|^2 \, \mathrm{Ln}\left(|f(x)|^2\right) \mathrm{dx} + \int_{-\infty}^{+\infty} |F(k)|^2 \, \mathrm{Ln}\left(|F(k)|^2\right) \mathrm{dk} = \mathrm{Constant}$$

We use the common definition of Fourier Transform in physics :

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{i k x}\, dx$$

so that it satisfies the Plancherel theorem in the following form :

$$\int_{-\infty}^{+\infty} |F(k)|^2 \, dk = \int_{-\infty}^{+\infty} |f(x)|^2 \, dx$$

In quantum mechanics, if f(x) represents the wave function in the position space, then its fourier transform F(k) represents the wave function in the momentum space.

Meanwhile, $|f(x)|^2$ is the probability that, say, a particle exits at position x (i.e. $|f(x)|^2$ is the probability density function, p.d.f.). Thus, $|f(x)|^2$ must satisfy the normalization condition:

$$\int_{-\infty}^{+\infty} |f(x)|^2 \, dx = 1$$

Due to the Plancherel Theorem , $|F(k)|^2$ satisfies the normalization condition as well (Probably why physicists like to use the definition of Fourier transform with a constant $\frac{1}{\sqrt{2\pi}}$)

Now, if we define the "width" of the p.d.f as :

$$(\Delta x)^2 \equiv \frac{\int_{-\infty}^{\infty} x^2 \, |f(x)|^2 \, \mathrm{dx}}{\int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{dx}} \qquad (\Delta k)^2 \equiv \frac{\int_{-\infty}^{\infty} k^2 \, |F(k)|^2 \, \mathrm{dk}}{\int_{-\infty}^{\infty} |F(k)|^2 \, \mathrm{dk}}$$

then it has been proven in the theory of Fourier Tranform that $(\Delta x)\,(\Delta \mathrm{k}) \geq \frac{1}{2}$.

Heisenberg used the "width" to describe the uncertanty of a wave function, and wrote down the famous principle of uncertainty:

$$(\Delta x)\,(\Delta \mathrm{p}) \geq \frac{\hbar}{2}$$

where p represents momentum, and satisfies the de Broglie relation :

$$p = \hbar\, k$$

where $\hbar$ is the reduced Planck constant：

$$\hbar = h/(2\pi)$$

I got the idea to use the informational entropy from Shannon to describe the uncertainty (the natural logarithm is used here for simplicity of calculation).

$$H \equiv \int_{-\infty}^{+\infty} p(x)\,\mathrm{Log}\left(\frac{1}{p(x)}\right) \mathrm{dx} = -\int_{-\infty}^{+\infty} p(x)\,\mathrm{Log}(p(x))\,\mathrm{dx}$$

Then I propose that the total entropy of $|f(x)|^2$ and $|F(k)|^2$ is a constant. It's actually quite intuitive and consistent with the principle of uncertainty that, if you know more information in one space, you lose information in another space.

I first made some calculation with "Gaussian wave function"

$$f(x) = \sqrt[4]{\frac{2}{\pi}} * \frac{1}{\sqrt{a}} * e^{-\left(\frac{x}{a}\right)^2}$$

where a is a real number that's bigger than 0 and bigger a indicates more uncertainty. The constant is chosen to satisfy the normalization condition.

Its Fourier transform is (simply replace a with $\frac{2}{a}$):

$$F(k) = \sqrt[4]{\frac{2}{\pi}} * \sqrt{\frac{a}{2}} * e^{-\left(\frac{a\,k}{2}\right)^2}$$

The total entropy can be calculated and indeed is independent of a, which is $1+\mathrm{Log}(\pi)$.

If the wave function is multi-dimensional, similarly we obtain:

$$\mathrm{Entropy}_{\mathrm{total}} = n(1 + \mathrm{Log}(\pi))$$

Then I turned to Sech function:

$$f(x) = \sqrt{\frac{1}{2\,a}} * \mathrm{Sech}\left(\frac{x}{a}\right)$$

Its Fourier transform is (simply replace a with $\frac{2}{a\,\pi}$):

$$f(x) = \sqrt{\frac{a\,\pi}{4}} * \mathrm{Sech}\left(\frac{a\,\pi\,k}{2}\right)$$

Again, the total entropy is independent of a. Generally in n dimensions:

$$\mathrm{Entropy}_{\mathrm{total}} = n(4 - \mathrm{Log}(2\,\pi))$$

The total entropy can be calculated easily by hand in these two situations, and in both cases the Fourier transform takes the same form as the original function.

Then I use Mathematica to calculate some different cases. For example:

$$\sqrt{\frac{1}{2\,a}}\ (-a \leq\ x \leq\ a) \longleftrightarrow \frac{\mathrm{Sin}(a\,k)}{\sqrt{a\,\pi}\ k}$$

$$\sqrt{\frac{3}{2\,a}}\ \left(\ 1-\frac{|x|}{a}\ \right)\ (-a \leq\ x \leq\ a) \longleftrightarrow -\sqrt{\frac{3}{4\,\pi\,a^{3}}} * e^{-i\,a\,k} * \left(e^{i\,a\,k}-1\right)^{2}$$

$$\sqrt{\frac{a}{\pi\left(1+(a\,x)^{2}\right)}} \longleftrightarrow \sqrt{\frac{2}{a}}\ \frac{\mathrm{BesselK}\left(0,\ \frac{|k|}{a}\right)}{\pi}$$

The total entropy is calculated numerically for different a (say 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, etc) , and all results are (quite) the same when a changes.

So I got enough courage to make the conjecture that

$$\int_{-\infty}^{+\infty} |f(x)|^{2}\ \mathrm{Ln}\left(|f(x)|^{2}\right)\,\mathrm{dx} + \int_{-\infty}^{+\infty} |F(k)|^{2}\ \mathrm{Ln}\left(|F(k)|^{2}\right)\,\mathrm{dk} = C$$

where C is a constant that only depend on the “type” of f(x), but not its “shape”.