ASSIGNMENT 2 ON "MATHS OF QUANTUM COMPUTING"		
Student's Code	AIMS African Institute for Mathematical Sciences CAMEROON	Deadline
rXRB31w8		**29.01.22, 6:00 am pm**
January 29, 2022		Ac. Year: 2021 - 2022
		Lecturer(s): "**Wolfgang Sherer**"

Exercise 1

1. Using the **NOT-gate** $\mathtt{X}$, the Hadamard gate $\mathtt{H}$ and the **C.NOT** get $\Lambda^1(\mathtt{X})$ build a circuit

$$U : {}^{\mathrm{q}}\mathcal{H}^{\otimes 2} \to {}^{\mathrm{q}}\mathcal{H}^{\otimes 2}. \tag{1}$$

$$U|00\rangle = |\Psi^+\rangle. \tag{2}$$

- **Step 1**
  We apply the gate $\mathtt{X}$ on the second qubit and the identity to the first qubit. We have:

$$\begin{aligned}(\mathbb{I}\otimes\mathtt{X})|00\rangle &= \mathbb{I}|0\rangle\otimes\mathtt{X}|0\rangle,\\ &= |0\rangle\otimes|1\rangle.\end{aligned} \tag{3}$$

- **Step 2**
  We apply the Hadamard operator $\mathtt{H}\otimes 1$ on the result of the first step to obtain :

$$\begin{aligned}\mathtt{H}\otimes\mathbb{I}|01\rangle &= \mathtt{H}|0\rangle\otimes|1\rangle,\\ &= \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes|1\rangle,\ \text{since } \mathtt{H}|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle),\\ &= \frac{1}{\sqrt{2}}(|01\rangle+|11\rangle).\end{aligned} \tag{4}$$

- **Step 3**
  We apply the **CNOT-gate**. The operator $\Lambda^1(\mathtt{X})$ is a 2-qubit gate such that when it is applied, the second qubit is flipped when the first qubit is the state 1, and remains the same when the first qubit is in the state 0.
  So we have that:

$$\begin{aligned}\Lambda^1(\mathtt{X})\frac{1}{\sqrt{2}}(|01\rangle+|11\rangle) &= \frac{1}{\sqrt{2}}\left(\Lambda^1(\mathtt{X})|01\rangle+\Lambda^1(\mathtt{X})|11\rangle\right)\\ &= \frac{1}{\sqrt{2}}|01\rangle+|10\rangle\\ &= |\Psi^+\rangle.\end{aligned} \tag{5}$$

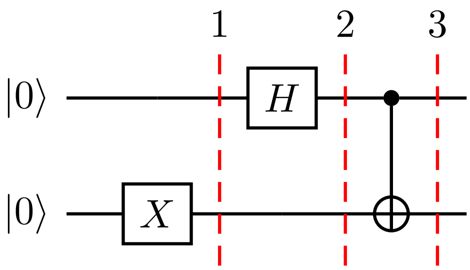

Figure 1: Quantum circuit U

2. We give the matrix U in the computational basis:

$$U = \Lambda^1(\mathtt{X})\,(\mathtt{H}\otimes\mathbb{I})\,(\mathbb{I}\otimes\mathtt{X})\,. \tag{6}$$

We have:

$$\mathbb{I}\otimes\mathtt{X} = \begin{pmatrix} 1.\mathtt{X} & 0\mathtt{X} \\ 0.\mathtt{X} & 1.\mathtt{X} \end{pmatrix} = \begin{pmatrix} 1.\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} & 0.\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \\ 0.\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} & 1.\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{7}$$

Also,

$$\begin{aligned} \mathtt{H}\otimes\mathbb{I} &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\otimes\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1.\mathbb{I} & 1.\mathbb{I} \\ 1.\mathbb{I} & -1.\mathbb{I} \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1.\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 1.\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ 1.\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & -1.\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \end{aligned} \tag{8}$$

And,

$$\begin{aligned} \Lambda^1(\mathtt{X}) &= |0\rangle\langle 0|\otimes\mathbb{I} + |1\rangle\langle 1|\otimes\mathtt{X}, \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\otimes\mathbb{I} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\otimes\mathtt{X}, \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \end{aligned}$$

Now given that:

$$U = \Lambda^1(X)\,(\mathtt{H} \otimes \mathbb{I})\,(\mathbb{I} \otimes \mathtt{X})\,. \tag{9}$$

Then we have that:

$$\begin{aligned} U &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}, \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}. \end{aligned} \tag{10}$$

Therefore the matrix U in the computational basis is given by:

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}. \tag{11}$$

3. See Python code.

4. See Python code

5. Show that the result is indeed $|\Psi^+\rangle$.
   The state vector obtained obtained while programming on qiskit is

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}. \tag{12}$$

$|\Psi^+\rangle$ is given by:

$$\begin{aligned} |\Psi^+\rangle &= \frac{1}{\sqrt{2}} \left(|01\rangle + |10\rangle\right) \\ &= \frac{1}{\sqrt{2}} \left(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle\right) \\ &= \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \end{aligned} \tag{13}$$

$$
\begin{aligned}
|\Psi^+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} 1.\begin{pmatrix}0\\1\end{pmatrix} \\ 0.\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0.\begin{pmatrix}1\\0\end{pmatrix} \\ 1.\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\left(\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right) \\
&= \begin{pmatrix}0\\ \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\\ 0\end{pmatrix} = |\Psi^+\rangle.
\end{aligned} \tag{14}
$$

Hence, the result obtained is indeed $|\Psi^+\rangle$.