Mathematics of Quantum Computing

Group Assignment: Exercise 1

Candidates: GROUP 1
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Question 1 I

Let's determine the matrix representation of U.

- $$U : {}^{q}\mathbb{H}^{\otimes 2} \to {}^{q}\mathbb{H}^{\otimes 2}. \tag{1}$$
  $$U|00\rangle = |\Psi^{+}\rangle \tag{2}$$
- **Step 1** We apply the gate $\mathtt{X}$ on the second qubit and the identity to the first qubit
  $$\begin{aligned}(\mathbb{I} \otimes \mathtt{X})|00\rangle &= \mathbb{I}|0\rangle \otimes \mathtt{X}|0\rangle, \\ &= |0\rangle \otimes |1\rangle.\end{aligned} \tag{3}$$

- **Step 2**
  We apply the Hadamard operator $\mathtt{H} \otimes \mathbb{1}$ on the result of the first step to obtain :

$$
\begin{aligned}
\mathtt{H} \otimes 1|01\rangle &= \mathtt{H}|0\rangle \otimes |1\rangle, \\
&= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |1\rangle, \text{ since } \mathtt{H}|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \\
&= \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle).
\end{aligned}
\tag{4}
$$

Question 1 III

- **Step 3**
  We apply the **CNOT-gate**. The operator $\Lambda^1(\mathtt{X})$ is a 2-qubit gate such that when it is applied, the second qubit is flipped when the first qubit is the state 1, and remains the same when the first qubit is in the state 0.

$$\begin{aligned}
\Lambda^1(\mathtt{X})\frac{1}{\sqrt{2}}(|01\rangle + |11\rangle) &= \frac{1}{\sqrt{2}}\left(\Lambda^1(\mathtt{X})|01\rangle + \Lambda^1(\mathtt{X})|11\rangle\right) \\
&= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\
&= |\Psi^+\rangle.
\end{aligned} \tag{5}$$

Question 1 IV

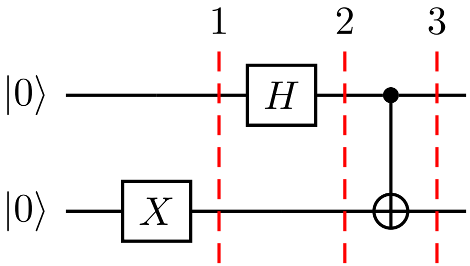

Figure: Quantum circuit of U.

Question 2 I

- Matrix of U in the computational basis:

$$U = \Lambda^1(\mathtt{X})\,(\mathtt{H} \otimes \mathbb{I})\,(\mathbb{I} \otimes \mathtt{X})\,. \tag{6}$$

We have:

$$\mathbb{I}\otimes\mathtt{X} = \begin{pmatrix} 1.\mathtt{X} & 0.\mathtt{X} \\ 0.\mathtt{X} & 1.\mathtt{X} \end{pmatrix} = \begin{pmatrix} 1.\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 0.\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ 0.\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 1.\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{7}$$

Question 2 II

Also,

$$
\begin{aligned}
\mathtt{H} \otimes \mathbb{I} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 1.\mathbb{I} & 1.\mathbb{I} \\ 1.\mathbb{I} & -1.\mathbb{I} \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 1.\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 1.\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ 1.\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & -1.\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}
\end{aligned} \tag{8}
$$

Question 2 III

And,

$$\begin{aligned}
\Lambda^1(\mathtt{X}) &= |0\rangle\langle 0| \otimes \mathbb{I} + |1\rangle\langle 1| \otimes \mathtt{X}, \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbb{I} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \mathtt{X}, \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},
\end{aligned} \tag{9}$$

Question 2 I

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{10}$$

Question 2 I

- Now given that:

$$U = \Lambda^1(\mathtt{X})\left(\mathtt{H} \otimes \mathbb{I}\right)\left(\mathbb{I} \otimes \mathtt{X}\right). \tag{11}$$

Then we have that:

$$\begin{aligned}
U &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix},
\end{aligned} \tag{12}$$

Question 2 I

- $$= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}. \tag{13}$$

Therefore the matrix U in the computational basis is given by:

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}. \tag{14}$$

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Questions 3&4

- See Python code.
- See Python code

Questions 5 I

- Show that the result is indeed $|\Psi^+\rangle$.
  The state vector obtained obtained while programming on qiskit is

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}. \tag{15}$$

$|\Psi^+\rangle$ is given by:

$$\begin{aligned} |\Psi^+\rangle &= \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right) \\ &= \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}\right) \end{aligned} \tag{16}$$

Questions 5 II

$$
\begin{aligned}
&= \frac{1}{\sqrt{2}}\begin{pmatrix} 1.\begin{pmatrix}0\\1\end{pmatrix} \\ 0.\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0.\begin{pmatrix}1\\0\end{pmatrix} \\ 1.\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\left(\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right) \qquad (17) \\
&= \begin{pmatrix}0\\ \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\\ 0\end{pmatrix} = |\Psi^+\rangle.
\end{aligned}
$$

Hence, the result obtained is indeed $|\Psi^+\rangle$.

Thank you for your kind attention