Game Theory

AHdoc

Elemental Knowledges About Game Theory

- (1) Take-Away Game （with removing one, two or there)
- (2) Combinatorial Game
  - The game ends in a finite number of moves no matter how it is played
- (3) P(previous)-position & N(next)-position
  - How to find the P & N-position for combinatorial games
- (4) Nim Problem

- (1) Graph Games
- (2) The SG Function
- (3) Sum of Combinatorial Games
- (4) Coin Turning Games
- (5) Green Hackenbush

Graph Games

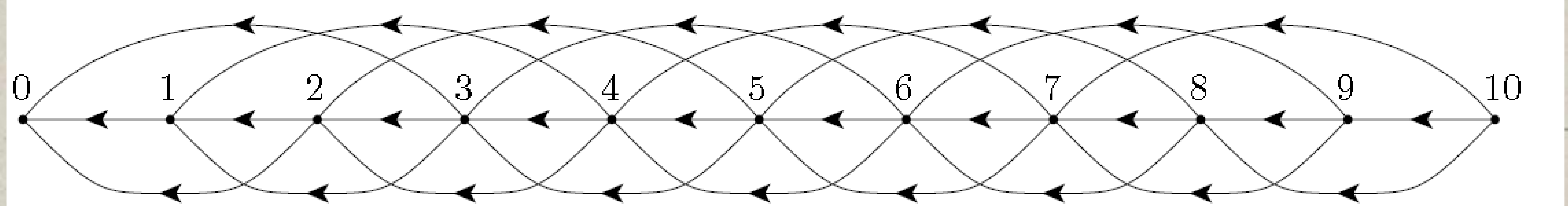
- Definition : A directed graph, G, is a pair (X,F) where X is a nonempty set of vertices (positions) and F is a function that gives for each $x \in X$ a subset of X, $F(x) \subset X$.
- For a given $x \in X$, F(x) represents the positions to which a player may move from x (called the followers of x). If F(x) is empty, x is called a terminal position.

Graph Games

- A two-person win-lose game may be played on such a graph $G = (X,F)$ by stipulating a starting position x0 $\in X$ and using the following rules:
  - (1) Player I moves first, starting at x0.
  - (2) Players alternate moves.
  - (3) At position x, the player whose turn it is to move chooses a position $y \in F(x)$.
  - (4) The player who is confronted with a terminal position at his turn, and thus cannot move, loses.

Graph Game - Subtraction Set

- The subtraction game with subtraction set
- $S = \{1, 2, 3\}$



The Sprague Grundy Function

- Definition. The SG function of a graph, (X,F), is a function, g, defined on X and taking non-nagetive integer values, s.t.

$$g(x) = \mathrm{mex}\{g(y) : y \in F(x)\}$$

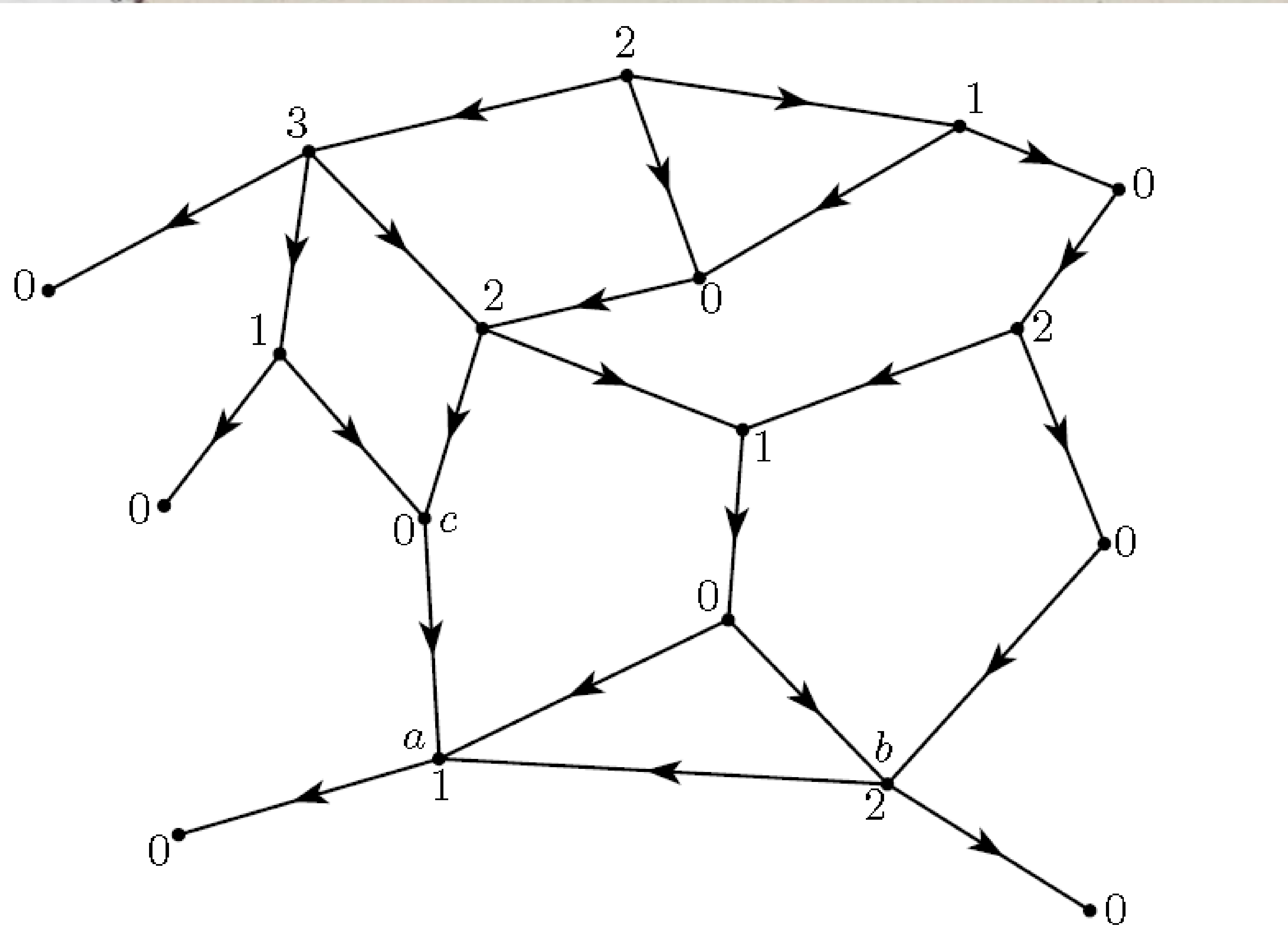
- g(x) = 0 for terminals

SG in Nim Games

- One pile (n chips)
  - $sg(0) = 0$
  - $sg(1) = 1$, $sg(2) = 2$, .. $sg(n) = n$

- Two piles (n chips, m chips)
  - $sg(0,0) = 0$
  - $sg(1,0) = sg(0,1) = 1$
  - $sg(1,1) = mex\{1,1\} = 0$
  - $sg(1,2) = mex\{2,0,1\} = 3 = sg(2,1)$
  - $sg(2,2) = mex\{3,2\} = 0$

SG in Normal Graph



SG in At-Least-Half

- Consider the one-pile game with the rule that you must remove at least half of the counters. The only terminal position is zero.

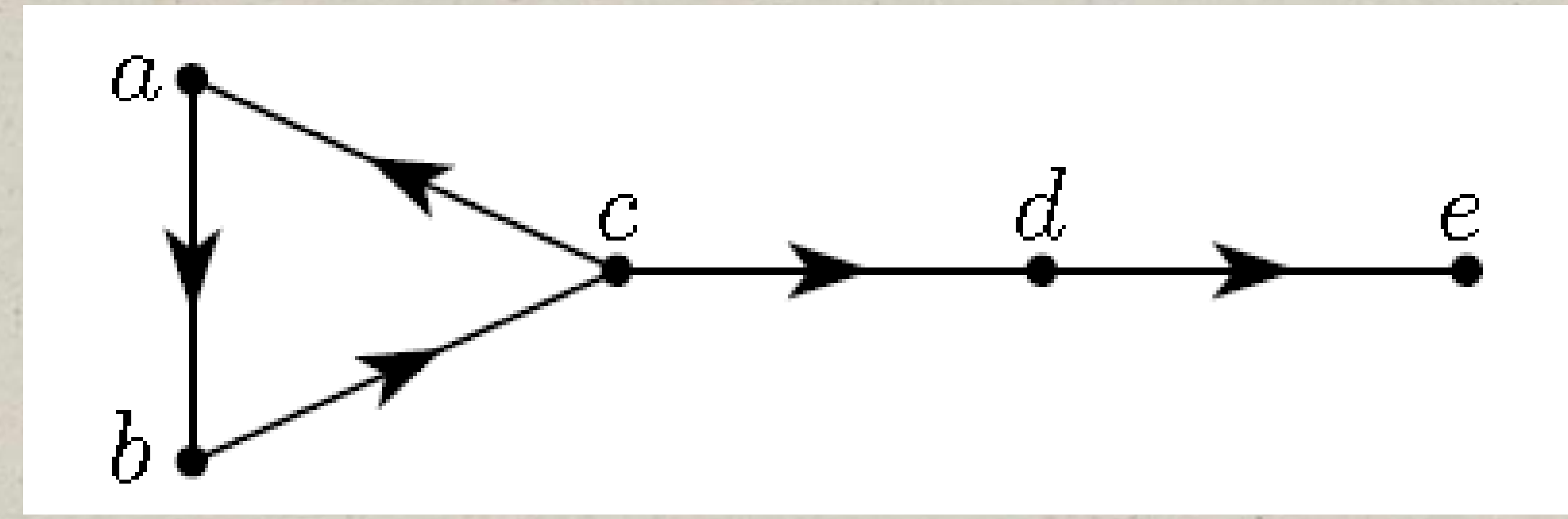
x	0	1	2	3	4	5	6	7	8	9	10	11	12	...
$g(x)$	0	1	2	2	3	3	3	3	4	4	4	4	4	...

SG in Graph with Cycles

- If the graph is allowed to have cycles, new problems arise.

- (1). The SG-function satisfying the conditions of combinatorial games may not exist.
- (2). Even if it does, the simple inductive procedure of the previous sections may not suffice to find it.
- (3). Even if the the Sprague-Grundy function exists and is known, it may not be easy to find a winning strategy.

SG in Graph with Cycles



Sums of Combinatorial Games

- （1）Definition
- （2）SG Theorm

The Sum of Combinatorial Games

- $G_1 = (X_1, F_1)$, $G_2 = (X_2, F_2)$, . . . , $G_n = (X_n, F_n)$

- New graph $G = (X,F)$, called the sum of $G_1, G_2, \ldots, G_n$.
- X is the Cartesian product, $X = X_1 \times \cdots \times X_n$

$$\begin{aligned} F(x) = F(x_1, \ldots, x_n) &= F_1(x_1) \times \{x_2\} \times \cdots \times \{x_n\} \\ &\cup \{x_1\} \times F_2(x_2) \times \cdots \times \{x_n\} \\ &\cup \cdots \\ &\cup \{x_1\} \times \{x_2\} \times \cdots \times F_n(x_n). \end{aligned}$$

The Sprague-Grundy Theorem

- Theorem 2. If g_i is the Sprague-Grundy function of G_i, $i = 1, \ldots, n$, then $G = G_1 + \cdot \cdot \cdot + G_n$ has Sprague-Grundy function
- $g(x_1,\ldots,x_n) = g_1(x_1) \oplus \cdots \oplus g_n(x_n)$

Multi Even if Not All – All if Odd

- Consider the one-pile game.
- Remove
  - –(1) any even number of chips provided it is not the whole pile, or
  - –(2) the whole pile provided it has an odd number of chips.
- There are two terminal positions, zero and two.

x	0	1	2	3	4	5	6	7	8	9	10	11	12	...
$g(x)$	0	1	0	2	1	3	2	4	3	5	4	6	5	...

- Play with 3 piles of sizes 10, 13 and 20.
  - –g(10) = 4, g(13) = 7, g(20) = 9.
- Since 4 xor 7 xor 9 = 10 is not zero, this is an N-position.

Coin Turning Games

- (1) Ending Condition
- (2) Decomposition
- (3) Two Dimensional Coin Turning Games

Coin Turning Games, Ending Condition

- Given a finite number of coins in a row
  - heads or tails
- A move consists of turning over.
  - Always specify that the rightmost coin
  - turned over must go from heads to tails.
- The purpose is to guarantee that the game will end in a finite number of moves no matter how it is played

Coin Turning Games, Decomposition

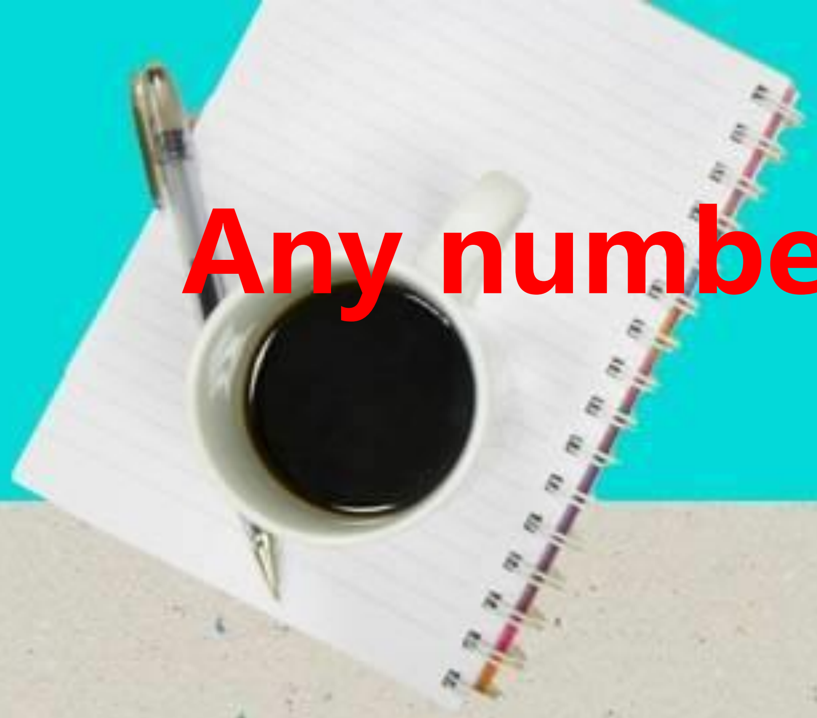
- A position with k heads in positions $x_1, \ldots, x_k$
- is the (disjunctive) **sum of k games** each with exactly one head where for $j = 1, \ldots, k$ the head in game j is at x_j

- Example:
  - THHTTH is the sum of TH, TTH, and TTTTTH.
  - g(THHTTH) = g(TH) xor g(TTH) xor g(TTTTTH)

Subtraction Games

- Recall the simple take-away…

- Some coin, say in position x, must be turned over from heads to tails, and a second coin in one of the positions $x - 1$, $x - 2$, or $x - 3$, must be turned over, except when $x \le 3$, in which case a second coin need not be turned over



Any number of coins may be turned over

- Any number of coins may be turned over but they must be consecutive, and the rightmost coin must be turned from heads to tails.

- g(n) = mex{0, g(n-1), g(n-1) xor g(n-2), …, g(n-1) … xor g(1)}

position x :	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16…
$g(x)$:	1	2	1	4	1	2	1	8	1	2	1	4	1	2	1	16…

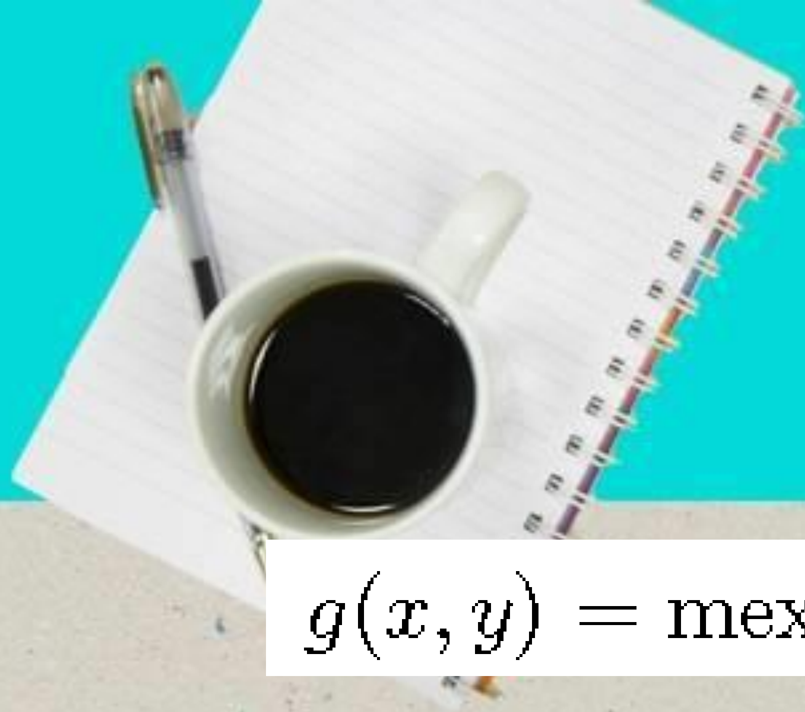
Two-dimensional Coin Turning Games

- We number the coordinates of the array starting at $(0,0)$, with coins at coordinates (x,y) with $x >= 0$ and $y >= 0$

- The southeast coin, (x,y), goes from heads to tails,
- and other coins that are turned over must be in rectangle
  - $\{(a,b)|0<=a<=x,0<=b<=y\}$

Acrostic Twins

- A move is to turn over two coins
- either in the same row or the same column
- with the southeast coin going from heads to tails.



Acrostic Twins

$$g(x,y) = \text{mex}\{g(x,b), g(a,y) : 0 \le b < y, 0 \le a < x\}.$$

	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	0	3	2	5	4	7	6	9	8
2	2	3	0	1	6	7	4	5	10	11
3	3	2	1	0	7	6	5	4	11	10
4	4	5	6	7	0	1	2	3	12	13
5	5	4	7	6	1	0	3	2	13	12
6	6	7	4	5	2	3	0	1	14	15
7	7	6	5	4	3	2	1	0	15	14
8	8	9	10	11	12	13	14	15	0	1
9	9	8	11	10	13	12	15	14	1	0

For Acrostic Twins, $g(x,y) = x \oplus y$.

Turning Corners

- A move consists of turning over four distinct coins at the corners of a rectangle, i.e.
  - (a, b), (a, y), (x, b) and (x, y)
- where $0 \leq a < x$ and $0 \leq b < y$, the coin at (x, y) going from heads to tails.

Turning Corners

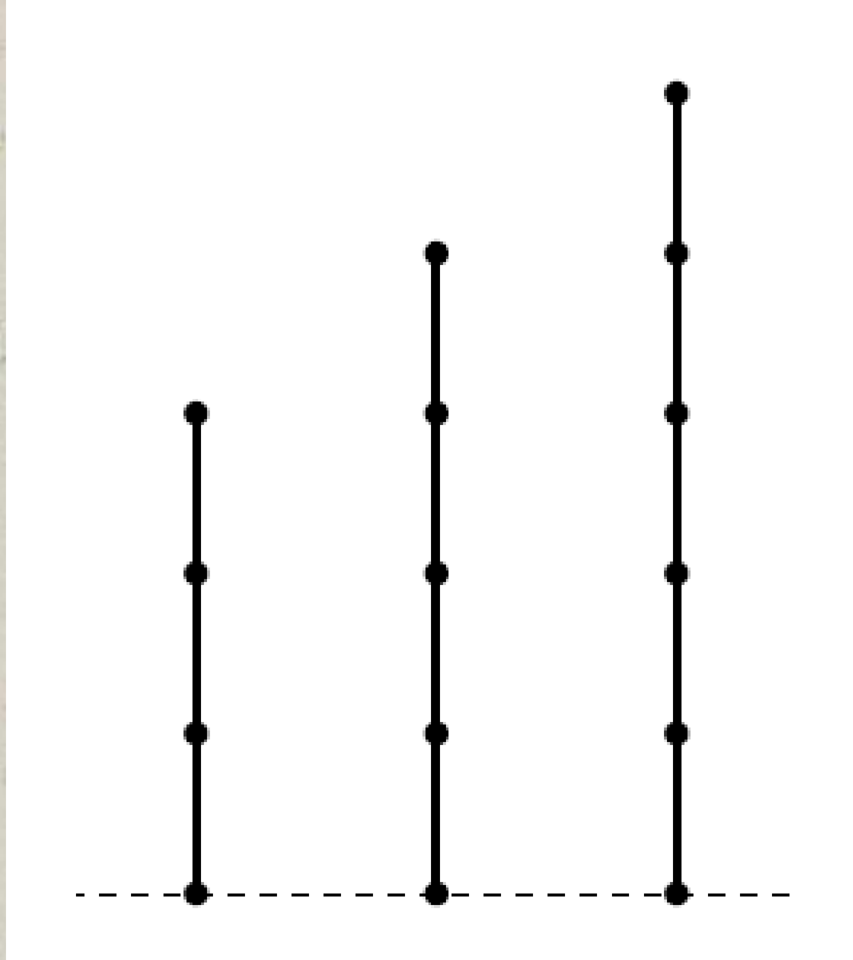
$$g(x,y) = \operatorname{mex}\{g(x,b) \oplus g(a,y) \oplus g(a,b) : 0 \le a < x, 0 \le b < y\}.$$

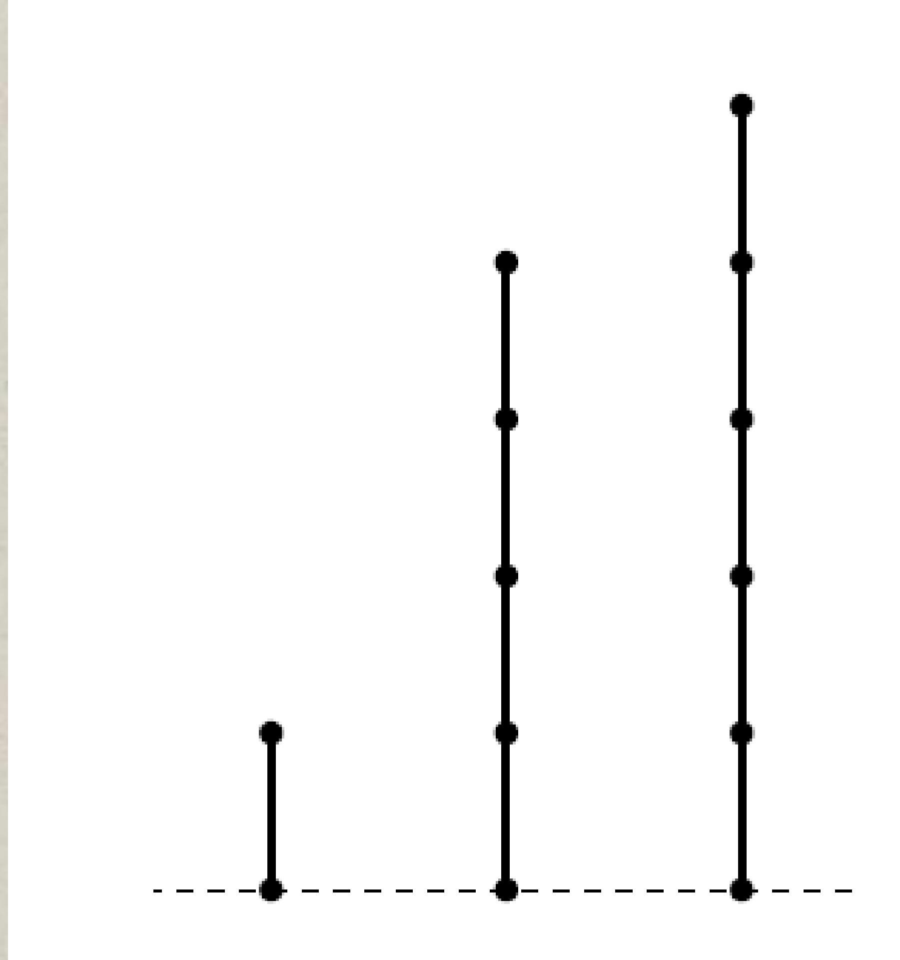
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	0	2	3	1	8	10	11	9	12	14	15	13	4	6	7	5
3	0	3	1	2	12	15	13	14	4	7	5	6	8	11	9	10
4	0	4	8	12	6	2	14	10	11	15	3	7	13	9	5	1
5	0	5	10	15	2	7	8	13	3	6	9	12	1	4	11	14
6	0	6	11	13	14	8	5	3	7	1	12	10	9	15	2	4
7	0	7	9	14	10	13	3	4	15	8	6	1	5	2	12	11
8	0	8	12	4	11	3	7	15	13	5	1	9	6	14	10	2
9	0	9	14	7	15	6	1	8	5	12	11	2	10	3	4	13
10	0	10	15	5	3	9	12	6	1	11	14	4	2	8	13	7
11	0	11	13	6	7	12	10	1	9	2	4	15	14	5	3	8
12	0	12	4	8	13	1	9	5	6	10	2	14	11	7	15	3
13	0	13	6	11	9	4	15	2	14	3	8	5	7	10	1	12
14	0	14	7	9	5	11	2	12	10	4	13	3	15	1	8	6
15	0	15	5	10	1	14	4	11	2	13	7	8	3	12	6	9

Green Hackenbush

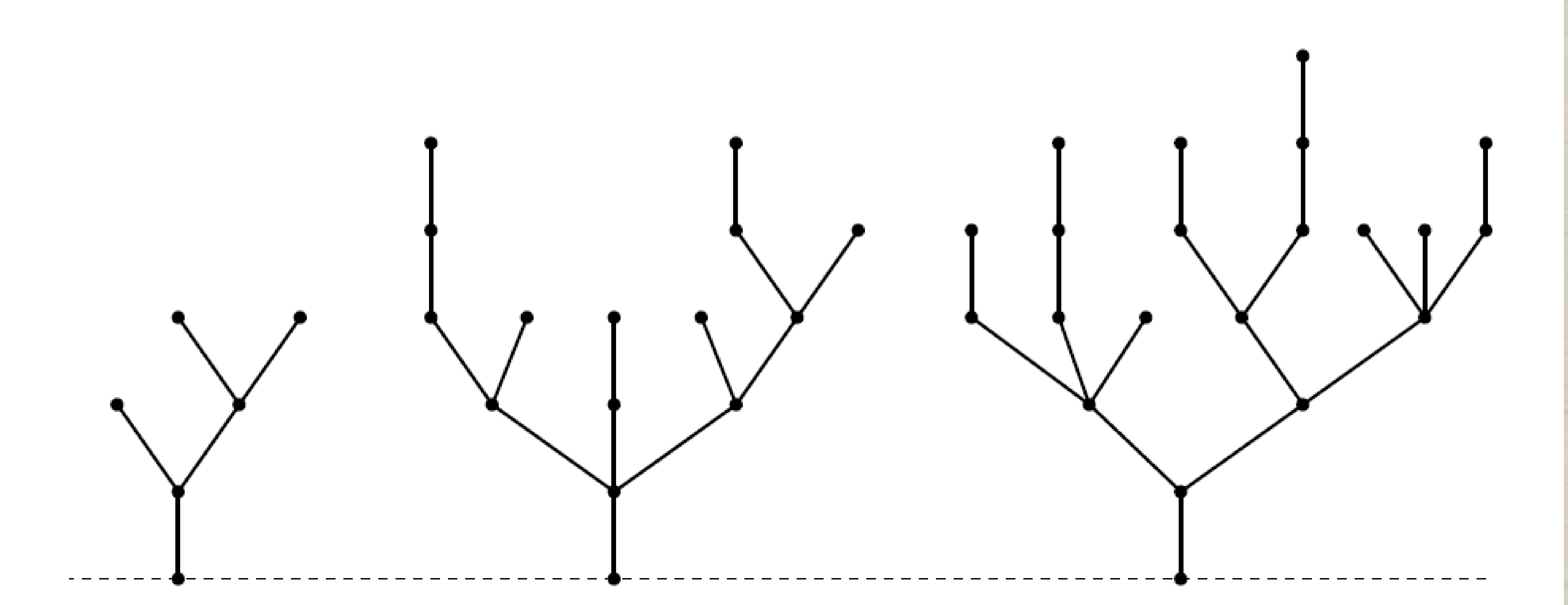
- **(Blue Red Hackenbush)**
- Played by hacking away edges from a rooted graph and removing those pieces of the graph that are no longer connected to the ground.

Green Hackenbush - Bamboo Stalks





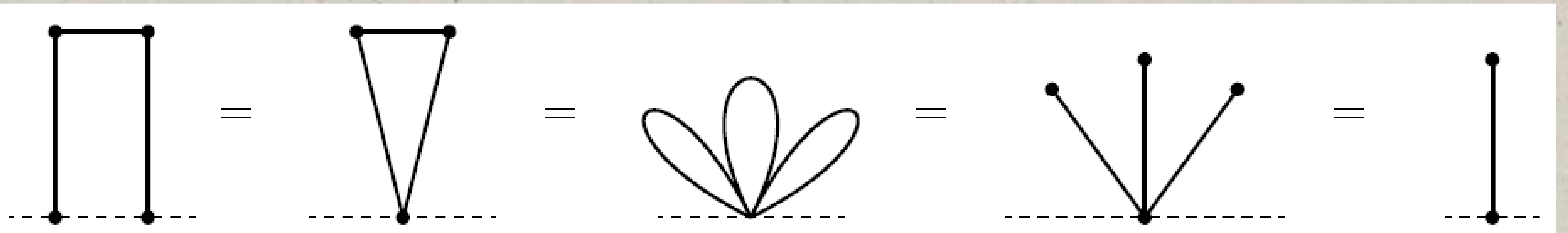
Green Hackenbush on Trees



- **Colon Principle:**
  - **When branches come together at a vertex, one may replace the branches by a non-branching stalk of length equal to their nim sum.**

Green Hackenbush on general rooted graphs

- **The Fusion Principle:**
- The vertices on any circuit may be fused without changing the Sprague-Grundy value of the graph.
- We fuse two neighboring vertices by bringing them together into a single vertex and bending the edge joining them into a loop.



Exercises

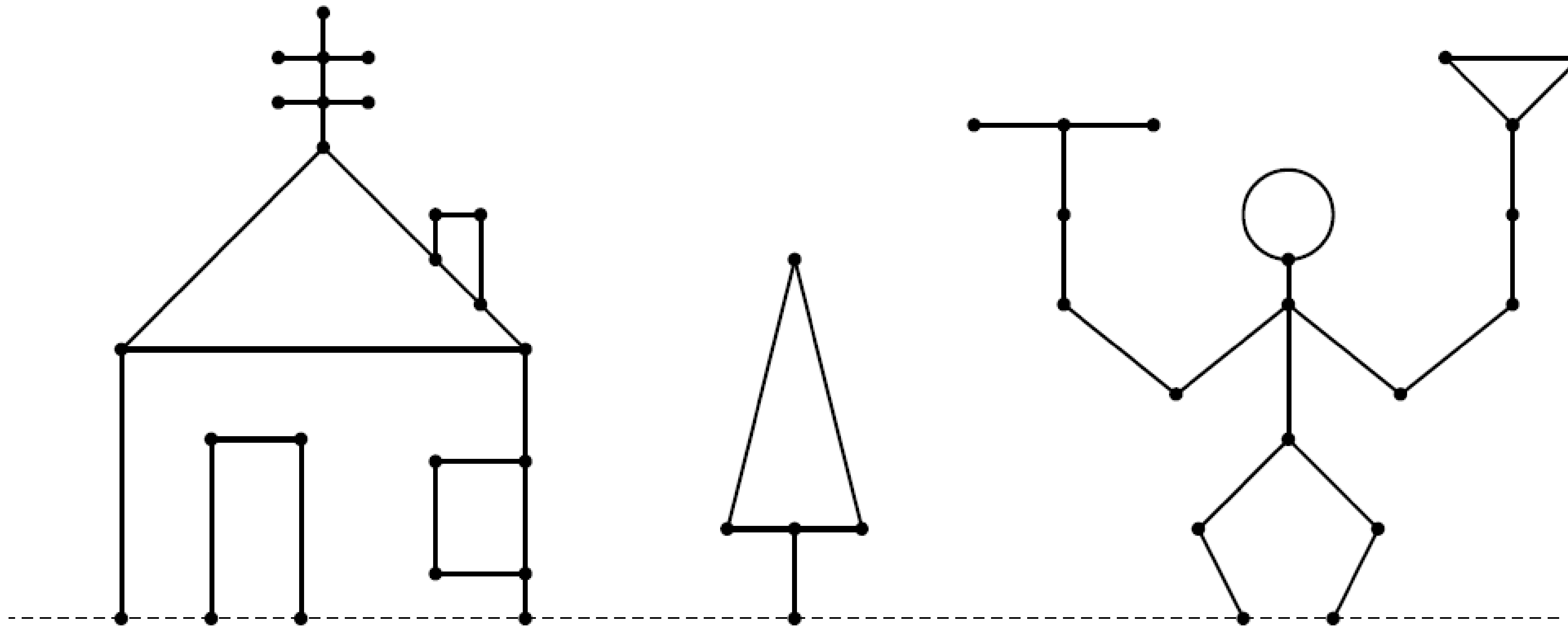
- **http://acm.hust.edu.cn/vjudge/contest/126769**

- hdu 3537, hdu 3389, hdu 3863, hdu 3951, hdu 2149,
- hdu 1850, hdu 2176, hdu 1527, hdu 2177, hdu 1517,
- hdu 2486, hdu 4315, hdu 1538, hdu 3404, hdu 1404,
- hdu 1536, hdu 1729, hdu 1730, hdu 1760, hdu 1848,
- hdu 1849, hdu 1851, hdu 1907, hdu 2873, hdu 2999,
- hdu 3595, hdu 4203, hdu 3590

- poj 1740, poj2484, poj 2234, poj 2975,
- poj 2368, poj 2311, poj 2425, poj 1678,
- poj 2068, poj 2599, poj1704, poj3533

- **More?**

Exercises



Exercises