THE NUMBER OF PATHS IN HEXAGONAL CACTI

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A PROJECT SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
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Abstract

Let $G = (V(G), E(G))$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. A vertex v of a graph G is a cut vertex of G if $G - v$ has more components than G. A block is a maximal subgraph B such that B does not have a cut vertex of B. For a subgraph G having a block B as a subgraph, B is an end block if B has exactly one cut vertex of G. A polygonal cactus is a graph consisting of exactly two endblocks and all the blocks are cycle of the same length. A path is a sequence of distinct vertices, any consecutive of which are adjacent. The length of a path is the number of vertices of the path minus one. In this project, the objective is to use a combinatorial tool such as recurrence relation, generating function and use computer programming to count and study the asymptotic behavior of the number of paths of all possible lengths of hexagonal cacti of n blocks. As a result, the total number of paths in Hexagonal Cacti is obtained.

Keywords : Cactus Graphs / Generating Function / Paths / Recurrence Relations

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บทคัดย่อ

กำหนดให้ $G = (V(G), E(G))$ เป็นกราฟที่มีเซตของจุดคือ $V(G)$ และมีเซตของเส้นคือ $E(G)$ จุด v ของ G จะถูกเรียกว่าจุดตัดของ G ถ้า $G - v$ มีจำนวนคอมโพเนนท์มากกว่าของ G นอกจากนี้ บลอค B คื อกราฟย่อยที่ใหญ่ที่สุดที่ B ไม่มีจุดตัดของ B เอง สำหรับกราฟ G ที่มีกราฟย่อย B เป็นบลอค จะกล่าวว่า B เป็นบลอคขอบถ้า B มีจุดตัดของกราฟ G เพียงจุดเดียว กราฟรูปกระบองเพชรหลายเหลี่ยม คือกราฟที่ทุกบลอคเป็นวัฏจักรที่มีความยาวเท่ากันและมีบลอคขอบเพียงสองบลอคขอบเท่านั้น วิถีคือลำดับ ของจุดที่แตกต่างกัน โดยที่สองจุดที่อยู่ติดกันในลำดับจะต้องประชิดกัน ความยาวของวิถีจะมีค่าเท่ากับจำนวน ของจุดของวิถีนั้นลบด้วยหนึ่ง ในงานนี้มีวัตถุประสงค์เพื่อใช้เครื่องมือทางวิชาคอมบินาทอริก เช่น ความสัมพันธ์-เวียนเกิด และฟังก์ชันก่อกำเนิด อีกทั้งใช้โปรแกรมทางคอมพิวเตอร์ในการนับจำนวนวิถีและศึกษาพฤติกรรม เชิงกำกับของจำนวนวิถีทั้งหมดในทุก ๆ ความยาวที่เป็นไปได้ของกราฟรูปกระบองเพชรหกเหลี่ยมที่มีจำนวน n บลอค จากผลการศึกษาได้จำนวนวิถีทั้งหมดของกราฟรูปกระบองเพชรหกเหลี่ยม

คำสำคัญ กราฟรูปกระบองเพชร / ฟังก์ชันก่อกำเนิด / ความสัมพันธ์เวียนเกิด / วิถี

CONTENTS

CHAPTER 1 INTRODUCTION

In this chapter, we provide basic notations of graphs, motivation and objectives of our project.

1.1 Terminology and Motivation

Let $G = (V(G), E(G))$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. The *order* of G is $|V(G)|$. For a vertex $v \in V(G)$, the *neighbor set* of v in G is the set of vertices that are adjacent to v and is denoted by $N_G(v)$. The *degree* of a vertex v in G is the cardinality of its neighbor seet and is denoted by $deg_G(v)$. A *walk* of length ℓ is a sequence $v_1, ..., v_{\ell+1}$ of $\ell + 1$ vertices such that $v_i v_{i+1} \in E(G)$ for $1 \leq i \leq \ell$. A walk $v_1, ..., v_{\ell+1}$ is a path if all $v_1, ..., v_{\ell+1}$ are distinct. A path of length ℓ is denoted by $P_{\ell+1}$. For two vertices u, v of a graph G, the *distance* between u and v in G is the length of a shortest path that joins u and v. A *cycle* of length n is obtained from a path P_n by adding the edge $v_1 v_{n+1}$ and is denoted by C_n. We say that v is a *cut vertex* of G if $G - v$ has more components than G. A maximal subgraph which does not have a cut vertex is called a *block*. A block B is an *end block* if B has exactly one cut vertex of G, otherwise B is an *inner block*. For two blocks B and B', we say that B *joins* B' if $V(B) \cap V(B') \neq \emptyset$.

For positive integers $g \geq 3$ and $n \geq 1$, a *g-gonal cactus* of n rings is a graph containing n blocks whose all the blocks are C_n and every block joins to at most two other blocks. Thus, a g-gonal cactus has exactly two end blocks. A g-gonal cactus is *regular* if the two cut vertices of each inner block are at the same distance for every inner block. In particular, a regular g-gonal cactus is said to be *para* if the two cut vertices that belong to the same inner block are at distance three. A regular g-gonal cactus is said to be *meta* if the two cut vertices that belong to the same inner block are at distance two. Further, a regular n-gonal cactus is said to be *ortho* if the two cut vertices that belong to the same inner block are adjacent.

In 1940, Mayer and Mayer [10] published their classical book of Statistical Mechanics. One among interesting topics in this book was Theory of Condensation which was extended to the cluster and irreducible integrals by Husimi [9] twelve years later in 1952. Interestingly, Uhlenbeck [12] pointed out that Husimi's integrals can be interpreted by graphs whose each edge is in at most one cycle. These graphs are called *Husimi trees*. From then on, Husimi trees have been attracted much attention as they can be applied to explain many of condensation phenomena, for example of the studies see [6, 8, 11]. The Husimi trees were known in graph theory literature as cacti after Harary and Palmer [7] published their classical book on graph enumeration in 1973 which was 20 years since it was first introduced.

For more example of the study in Husimi trees, or cacti from here on, Doslic and Lits [1], Doslic and Maloy [3] and Doslic and Zubac [2] employed recurrence relations and generating functions to establish the number of independent sets and matching in hexagonal cacti. Further, by the concepts of meromorphic functions and the growth of power series coefficients, the authors established the asymptotic behaviors through simple functions of these recurrence relations.

Hence, in this project, we aim to apply combinatorics tool to establish the number of paths of all possible lengths of hexagonal cacti.

1.2 Thesis Objectives

To use counting tools in combinatorics such as recurrence relation, generating function and to use computer programming to count the number of paths of all lengths of para-hexagonal cacti, ortho-hexagonal cacti and meta-hexagonal cacti.

1.3 Scopes of Thesis

This thesis focuses on counting only the number of paths of all possible lengths in para-hexagonal cacti, ortho-hexagonal cacti and meta-hexagonal cacti.

1.4 Working schedule

Activities	Working time									
	Semester 1/2022					Semester 2/2022				
	Aug.	Sep.	Oct.	Nov.	Dec.	Jan.	Feb.	Mar.	Apr.	May.
Topic searching										
Discuss with project advisor										
Literature review										
Work planning										
Counting the number of paths in Para-Hexagonal Cacti with Combinatorial tools and computer programming										
Counting the number of paths in Otho-Hexagonal Cacti with Combinatorial tools and computer programming										
Counting the number of paths in Meta-Hexagonal Cacti with Combinatorial tools and computer programming										
Conclusion and Writing Thesis										

CHAPTER 2 PRELIMINARIES

In this chapter, we state all the results or mathematical concepts that are used in establishing our theorems. We begin with the concept of recurrence relations and generating functions which are the main tools to count the number of paths of Hexagonal cacti.

A *recurrence relation* is a relationship that expresses a term of a sequence a_n in a formula of its previous terms.

Example 2.1. For a natural number $n \geq 1$, we let $a_1 = 1, a_2 = 1, a_3 = 2$ and, when $n \geq 4$, we let

$$a_n = 2 + a_{n-1} + a_{n-3}.$$

That is, the term a_n can be found by the summation of 2 and previous terms a_{n-1} and a_{n-3} with 2.

According to the above formula, the values a_3, a_2 and a_1 must be given and they are called *initial condition.* One might observe that although the formula of two recurrence relations are the same, if their initial conditions are different, the two relations are also different.

For a sequence,

$$a_0, a_1, ...$$

the *ordinary generating function*, or shortly *generating function*, associated with this sequence is the function f whose value at x is

$$f(x) = \sum_{i \geq 0} a_i x^i.$$

Example 2.2. The function f which is defnined to be

$$f(x) = 1 + x + x^2 + \cdots$$

is the generating function of the sequence $1, 1, 1, ...$

It can be observed that generating functions are in the form of power series which need to specify domains and ranges. However, in combinatorial applications, there is no need to concern the convergences of these power series as we focus on the coefficient of an individual term rather than the value of the infinite sum. Hence, the domains and ranges of (bivariate) generating functions are omitted when we do the counting task and we may define them later when the generating functions are considered in the manner of complex analysis. Indeed, if we focus on the coefficient of the term N, the summation will end up at N instead of ∞ but we compromise the notation to be infinite sum for convenience.

We also collect the well-known theorems without proof to study asymptotic behaviors of recurrence relation obtained from generating functions.

Theorem 2.3. *Let f(z)* $= \sum a_n z^n$ *be analytic in some region containing the origin, let a singularity of f(z) of smallest modulus be at a point* $z_0 \neq 0$*,and let* $\epsilon > 0$ *be given. Then there exists N such that for all* $n > N$ *we have*

$$|a_n| < (\frac{1}{|z_0| + \epsilon})^n.$$

Further, for infinitely many n we have

$$|a_n| > (\frac{1}{|z_0| - \epsilon})^n.$$

Theorem 2.4. *If f is analytic throughout the annular region*

$$D : r_1 < |z - z_0| < r_2.$$

Then f(z) can be expressed as a series

$$f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n + \sum_{n=-\infty}^{-1} a_n (z - z_0)^n$$

when $a_n = \dfrac{1}{2\pi i} \oint_c \dfrac{f(z)}{(z - z_0)^{n+1}} dz \ (n = 0, \pm 1, \pm 2, ...)$.

CHAPTER 3 THE NUMBER OF PATHS OF GIVEN LENGTHS OF REGULAR POLYGONAL CACTI

In this chapter, we give all our results concerning the formulae of the numbers of paths of given length of regular polygonal cacti.

For integers $g \geq 3$ and $n \geq 1$, we let $G(n, g)$ be a regular g-gonal cactus of n rings. Since $G(n, g)$ is regular and all the blocks of $G(n, q)$ are C_g, every inner block of $G(n, g)$ has two cut vertices at the same distance, q says. By the minimality of q, we have that

$$q \leq g - q.$$

Therefore, we can let $G(n, g, q)$ be a regular g-gonal cactus of n rings which the two cut vertices of each inner block are at distance q. Further, for an integer $\ell \geq 0$, we let

$p^g_{\ell,q}(n)$ be the number of paths of length ℓ of $G(n, g, q)$.

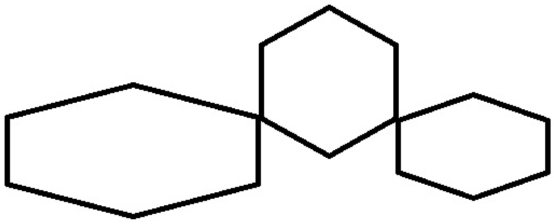

Figure 3.1 A hexagonal cactus of 3 hexagons.

Now, we are ready to establish formulae of counting $p^g_{\ell,q}(n)$.

Theorem 3.1. *If $\ell \geq g$ and $q < g - q$, then*

$$p^g_{\ell,q}(n) = 4\sum_{j=r}^{n}(n-j+1)\sum_{i=1}^{g-1}\sum_{t=1}^{g-1}\binom{j-2}{\frac{\ell-t-i-qj+2q}{g-2q}}$$

where $r = \max\left\{0, \left\lceil \frac{\ell-2g+2}{g-q} \right\rceil\right\} + 2.$

Proof. Because $\ell \geq g$, it follows that every path of length ℓ needs to lie on at least

$$r = \max\left\{0, \left\lceil \frac{\ell-2g+2}{g-q} \right\rceil\right\} + 2$$

rings and lie on at most n rings. For $r \leq j \leq n$, we let j be the number of rings that a path of length lies on.

For the sake of convenient, we count when the path lies on the Ring 1 to Ring j. For integers i, t and a path P of length ℓ, we let i be the number of edges of P in Ring 1 and t be the number of edges of P in Ring j. Since P lies on more than one rings, we have that $1 \leq t \leq g-1$ and $1 \leq i \leq g-1$. Further, we let

x_q be the number of Rings $2, ..., j-1$ that P passes with q edges and

x_{g-q} be the number of Rings $2, ..., j-1$ that P passes with $g-q$ edges.

Thus,

$$qx_q + (g-q)x_{g-q} = \ell - t - i \tag{3.1}$$

$$x_q + x_{g-q} = j - 2 \tag{3.2}$$

which can be solved that

$$x_{g-q} = \frac{\ell - t - i - qj + 2q}{g - 2q}.$$

So, there are $\binom{j-2}{\frac{\ell-t-i-qj+2q}{g-2q}}$ possibilities of selecting Rings $2, ..., n$ so that the path P passes with q edges. Further, there are 2 possibilities of the head and 2 possibilities for the tail of P to lie on Rings 1 and j, respectively. Thus, for each $r \leq j \leq n$, there are $4\binom{j-2}{\frac{\ell-t-i-qj+2q}{g-2q}}$ paths of length ℓ that lie on Ring 1 to Ring j. Since the path of length ℓ crawling across j rings can lie from Rings 1 to j until from Rings $n-j+1$ to n, it follows that the number of paths of length ℓ in $G(n, g, q)$ is

$$p^g_{\ell,q}(n) = 4\sum_{j=r}^{n}(n-j+1)\sum_{i=1}^{g-1}\sum_{t=1}^{g-1}\binom{j-2}{\frac{\ell-t-i-qj+2q}{g-2q}}.$$

This proves Theorem 3.1. □

When $\ell \leq g-1$, it is possible that a path of length ℓ lies on one ring. There are g of these such paths, resulting in there are ng more paths in this case. Thus, by Theorem 3.1, we have that

Theorem 3.2. *If $\ell \leq g-1$ and $q < g-q$, then*

$$p^g_{\ell}(n) = ng + 4\sum_{j=r}^{n}(n-j+1)\sum_{i=1}^{g-1}\sum_{t=1}^{g-1}\binom{j-2}{\frac{\ell-t-i-qj+2q}{g-2q}}$$

where $r = \max\left\{0, \left\lceil\frac{\ell-2g+2}{g-q}\right\rceil\right\} + 2.$

Programming code

CHAPTER 4 THE NUMBER OF ALL PATHS OF HEXAGONAL CACTI

4.1 Generating Functions and Recurrence Relations of the Number of all Paths of Hexagonal Cacti

Let $H(n)$ be a hexagonal cactus of n rings and let $D(n)$ be the total number of paths in $H(n)$. It is worth noting that, when $n = 0$, we let $D(0) = 0$ as there is no graph, resulting in there is no path of any length. We may name all the hexagons of $H(n)$ consecutively by Hexagons 1 to n. That is Hexagons 1 and n are the two end blocks of $H(n)$ while Hexagons $2, ..., n-1$ are the inner hexagons.

Further, we let $\overline{D}(n-1)$ be the total number of paths of lengths at least one in $H(n)$ whose one end vertex is at the cut vertex of Hexagon n and the other end vertex is in Hexagon i for some $1 \leq i \leq n-1$.

Clearly, there are totally 35 paths of lengths 0 to 5 in Hexagon n (excluding the common vertex of Hexagons $n-1$ and n which is counted in $D(n-1)$) and there are $10\overline{D}(n-1)$ paths that cross Hexagons $n-1$ and Hexagon n. Thus, for $n \geq 2$, we have that

$$D(n) = 35 + D(n-1) + 10\overline{D}(n-1). \tag{4.1}$$

Next, we will count $\overline{D}(n)$. It can be checked that there are 10 paths of length at least one start from the common vertex, x say, of Hexagons n and $n+1$ and are in only Hexagon n. For the paths starting from x and end in Hexagons $1, ..., n-1$, there are two possibilities to craw across Hexagon n. Thus, we have that

$$\begin{aligned}
\overline{D}(n) &= 10 + 2\overline{D}(n-1) \\
&= 10 + 2(10 + 2\overline{D}(n-2)) \\
&= 10 + 2(10 + 2(10 + 2\overline{D}(n-3)) \\
&= 10 + 2 \cdot 10 + 2 \cdot 2 \cdot 10 + 2 \cdot 2 \cdot 2\overline{D}(n-3) \\
&= 10 + 2 \cdot 10 + 2 \cdot 2 \cdot 10 + 2 \cdot 2 \cdot 2(10 + 2\overline{D}(n-4)) \\
&= 10 + 2 \cdot 10 + 2 \cdot 2 \cdot 10 + 2 \cdot 2 \cdot 2 \cdot 10 + 2^4\overline{D}(n-4) \\
&= 10(1 + 2 + 2^2 + 2^3) + 2^4\overline{D}(n-4).
\end{aligned}$$

By repeating $n-1$ time, we have

$$\begin{aligned}
\overline{D}(n) &= 10(1 + 2 + 2^2 + ... + 2^{n-2}) + 2^{n-1}\overline{D}(1) \\
&= 10(1 + 2 + 2^2 + ... + 2^{n-1}) \\
&= 10(2^n - 1).
\end{aligned}$$

Thus, we have

$$\overline{D}(n-1) = 10(2^{n-1} - 1).$$

By (4.1), we have that

$$D(n) = 35 + D(n-1) + 10(10(2^{n-1} - 1))$$

$$= D(n-1) + 100 \cdot 2^{n-1} - 65$$
$$= D(n-1) + 50 \cdot 2^{n} - 65.$$

Therefore,

$$D(n) = D(n-1) + 50 \cdot 2^{n} - 65. \tag{4.2}$$

which can be repeatedly substitute an obtain

$$D(n) = 100 \cdot 2^{n} - 65n - 99. \tag{4.3}$$

For $n \geq 2$, we may demonstrate how to find $D(2)$ and $D(3)$ as follows.

$$D(2) = 100 \cdot 2^{2} - 65(2) - 99 = 171$$
$$D(3) = 100 \cdot 2^{3} - 65(3) - 99 = 506$$

In the following, we may find generating function of $D(n)$ from (4.2) in which the generating function will be used to establish asymptotic behavior of $D(n)$.

We let $D(x)$ be the generating fuction of $D(n)$. That is:

$$D(x) = \sum_{n\geq 0} D(n)x^{n}.$$

For $n \geq 2$, we multiply x^n to ((4.2)) and sum all the value n. We have that

$$\sum_{n\geq 2} D(n)x^{n} = \sum_{n\geq 2} D(n-1)x^{n} + 50\sum_{n\geq 2} 2^{n}x^{n} - 65\sum_{n\geq 2} x^{n} \tag{4.4}$$

We consider L.H.S. of (4.4). Thus,

$$\begin{aligned}\sum_{n\geq 2} D(n)x^{n} &= \sum_{n\geq 1} D(n)x^{n} - D(1)x^{1} - D(0)x^{0}\\ &= \sum_{n\geq 0} D(n)x^{n} - 36x - D(0)\\ &= D(x) - 36x\end{aligned}$$

We next consider R.H.S. of (4.4). Thus,

$$\sum_{n\geq 2} D(n-1)x^{n} = x\sum_{n\geq 1} D(n)x^{n}$$

$$= xD(x)$$

$$\begin{aligned} 50\sum_{n\geq 2} 2^n x^n &= 50(\sum_{n\geq 0}(2x)^n - (2x)^1 - 1) \\ &= (1 + 2x + (2x)^2 + ...) - 100x - 50 \\ &= \frac{50}{1-2x} - 100x - 50 \end{aligned}$$

and

$$\begin{aligned} -65\sum_{n\geq 2} x^n &= -65(\sum_{n\geq 0} x^n - x^1 - x^0) \\ &= -65\sum_{n\geq 0} x^n - x - 1 \\ &= -65(1 + x + x^2 + ...) + 65x + 65 \\ &= \frac{-65}{1-x} + 65x + 65 \end{aligned}$$

Plug in (4.4), we have $D(x) - 36x = xD(x) + \frac{50}{1-2x} - 100x - 50 + \frac{-65}{1-x} + 65x + 65$ which can be solved that

$$D(x) = \frac{36x + 27x^2 + 2x^3}{1 - 4x + 5x^2 - 2x^3} \tag{4.5}$$

It is worth noting that the generating function of (4.5) can be solved to obtain recurrence relation of $D(n)$ as follows. Recall that $D(x) = \sum_{n\geq 0} D(n)x^n$ and $D(0) = 0$. Thus,

$$\begin{aligned} 36 + 27x + 2x^2 &= (1 - 4x + 5x^2 - 2x^3)\sum_{n\geq 0} D(n)x^n \\ &= \sum_{n\geq 0} D(n)x^n - 4\sum_{n\geq 0} D(n)x^{n+1} + 5\sum_{n\geq 0} D(n)x^{n+2} - 2\sum_{n\geq 0} D(n)x^{n+3} \\ &= (D(1)x + D(2)x^2 + \sum_{n\geq 3} D(n)x^n) - 4(D(1)x^2 + \sum_{n\geq 3} D(n-1)x^n) \\ &+ 5\sum_{n\geq 3} D(n-2)x^n - 2\sum_{n\geq 4} D(n-3)x^n \\ &= 36 + 27x + 2x^2 \\ &+ \sum_{n\geq 3}(D(n) - 4D(n-1) + 5D(n-2) - 2D(n-3))x^n \end{aligned}$$

Since L.H.S of the equation is a polynomial of degree 3, the coefficients of x^n when $n \geq 3$ of R.H.G. are all zero. That is,

$$D(n) = 4D(n-1) - 5D(n-2) + 2D(n-3)$$

for all $n \geq 4$ where $D(1) = 36, D(2) = 171$ and $D(3) = 506$, establishing the recurrence relation of $D(n)$.

4.2 Asymptotic Behavior of The Number of all Paths

From $D(x) = \dfrac{36x + 27x^2 + 2x^3}{1 - 4x + 5x^2 - 2x^3}$ we're going to factorization that denominator $1 - 4x + 5x^2 - 2x^3 = -(x-1)^2(x-0.5)$ then we get $x_0 = \dfrac{1}{2}, x_1 = 1, x_2 = 1$
Laurent's series at $\dfrac{1}{2}$ is $f(x) = \displaystyle\sum_{n=0}^{\infty} c_n(x-x_0)^n + \sum_{n=1}^{\infty} a_n(x-x_0)^{-n}$
and $c_n = \dfrac{1}{2\pi i}\displaystyle\int_c \frac{f(x)}{(x-x_0)^{n+1}}\,dx; n = 0, \pm1, \pm2, ...$

Normal part

$$\begin{aligned}
c_n &= \frac{1}{2\pi i}\int_c \frac{\frac{36x+27x^2+2x^3}{1-4x+5x^2-2x^3}}{(x-\frac{1}{2})^{n+1}}\,dx; n = 1, 2, ...\\
&= \frac{1}{2\pi i}\int_c \frac{36x + 27x^2 + 2x^3}{(-2x^2 + 4x - 2)(x-\frac{1}{2})(x-\frac{1}{2})^{n+1}}\,dx\\
&= \frac{1}{2\pi i}\int_c \frac{36x + 27x^2 + 2x^3}{(-2x^2 + 4x - 2)(x-\frac{1}{2})^{n+2}}\,dx
\end{aligned}$$

Consider of $c_n = \dfrac{1}{2\pi i}\displaystyle\int_c \frac{36x + 27x^2 + 2x^3}{(-2x^2 + 4x - 2)(x-\frac{1}{2})^{n+2}}\,dx$

Generalized Cauchy Int Formula of $x_0 = \dfrac{1}{2}$

$$\begin{aligned}
g^{(n)}(x_0) &= \frac{n!}{2\pi i}\int_c \frac{g(x)}{(x-x_0)^{n+1}}\,dx\\
&= \frac{1}{2\pi i}\int_c \frac{g(x)}{(x-x_0)^{n+1}}\,dx
\end{aligned}$$

$\therefore g(x) = \dfrac{36 + 27x + 2x^2}{(-2x^2 + 4x - 2)}; n = n+1$ and $n+1 = (n+1)+1 = n+2$

$$c_n = \frac{1}{2\pi i}\int_c \frac{36x+27x^2+2x^3}{(-2x^2+4x-2)(x-\frac{1}{2})^{n+2}}\,dx = \frac{1}{n+1}!g^{n+1}(\frac{1}{2})$$

$$\begin{aligned} while \qquad g(x) &= \frac{36x+27x^2+2x^3}{(-2x^2+4x-2)} \\ g'(x) &= -\frac{x^3-3x^2-45x-18}{(x-1)^3} \\ g''(x) &= -\frac{96x+99}{(x-1)^4} \\ g'''(x) &= \frac{288x-300}{(x-1)^5} \\ g''''(x) &= -\frac{1152x+1212}{(x-1)^6} \end{aligned}$$

Main part

$$\begin{aligned} c_n &= \frac{1}{2\pi i}\int_c \frac{\frac{36x+27x^2+2x^3}{1-4x+5x^2-2x^3}}{(x-\frac{1}{2})^{n+1}}\,dx; n=-1,-2,... \\ &= \frac{1}{2\pi i}\int_c \frac{(36x+27x^2+2x^3)(x-\frac{1}{2})^{-n-1}}{(-2x^2+4x-2)}\,dx \\ &= \frac{1}{2\pi i}\int_c \frac{(36x+27x^2+2x^3)(x-\frac{1}{2})^{-n-2}}{(-2x^2+4x-2)}\,dx \\ a_n &= \frac{1}{2\pi i}\int_c \frac{(36x+27x^2+2x^3)(x-\frac{1}{2})^{n-2}}{(-2x^2+4x-2)}\,dx; n=1,2,... \end{aligned}$$

Consider of $a_n = \frac{1}{2\pi i}\int_c \frac{(36x+27x^2+2x^3)(x-\frac{1}{2})^{n-2}}{(-2x^2+4x-2)}\,dx; n=1,2,...$

$$\begin{aligned} (n=1) \qquad a_n &= \frac{1}{2\pi i}\int_c \frac{(36x+27x^2+2x^3)(x-\frac{1}{2})^{1-2}}{(-2x^2+4x-2)}\,dx \\ &= \frac{1}{2\pi i}\int_c \frac{(36x+27x^2+2x^3)}{(-2x^2+4x-2)(x-\frac{1}{2})^1}\,dx \\ &= 0!g(\frac{1}{2}) \\ &= 1(\frac{(36(\frac{1}{2})+27(\frac{1}{2})^2+2(\frac{1}{2})^3)}{(-2(\frac{1}{2})^2+4(\frac{1}{2})-2)}) \\ &= -50 \\ (n=2) \qquad a_n &= \frac{1}{2\pi i}\int_c \frac{(36x+27x^2+2x^3)(x-\frac{1}{2})^{2-2}}{(-2x^2+4x-2)}\,dx \\ &= \frac{1}{2\pi i}\int_c \frac{(36x+27x^2+2x^3)}{(-2x^2+4x-2)(x-\frac{1}{2})^0}\,dx \\ &= 0 \qquad (Cauchy\ integral\ Theorem) \\ (n=3) \qquad a_n &= \frac{1}{2\pi i}\int_c \frac{(36x+27x^2+2x^3)(x-\frac{1}{2})^{3-2}}{(-2x^2+4x-2)}\,dx \end{aligned}$$

$$= 0 \qquad (Cauchy\ integral\ Theorem)$$

$$\therefore \sum_{n=1}^{\infty} a_n(x-x_0)^{-n} = \frac{a_n}{(x-x_0)^n} = \frac{-50}{(x-(\frac{1}{2}))} = \frac{-50}{x-0.5}$$

Analytic and asymptotic methods

$$\begin{aligned}
PP(f;z_0) &= \sum_{j=1}^{r} \frac{a_{-j}}{(z-z_0)^j} \\
&= \sum_{j=1}^{r} \frac{(-1)^j a_{-j}}{z_0^j(1-(z/z_0))^j} \\
&= \sum_{j=1}^{r} \frac{(-1)^j a_{-j}}{z_0^j} \sum_{n\geq 0} \binom{n+j-1}{n} (z/z_0)^n \\
&= \sum_{n\geq 0} z^n \sum_{j=1}^{r} \frac{(-1)^j a_{-j}}{z_0^{n+j}} \binom{n+j-1}{n}
\end{aligned}$$

$$\begin{aligned}
PP(f;z_0) &= \frac{-50}{(x-(0.5))} \\
&= \frac{(-1)(-50)}{(0.5)(1-(x/0.5))} \\
&= \sum_{n\geq 0} (\frac{50}{0.5}) \frac{1}{0.5^n} x^n \\
&= \sum_{n\geq 0} (\frac{50}{0.5^{n+1}}) x^n \\
&= \sum_{n\geq 0} 100 \cdot 2^n x^n
\end{aligned}$$

$\therefore D_n \approx 100 \cdot 2^n$

CHAPTER 5 THE NUMBER OF PATHS OF GIVEN LENGTHS OF PARA-HEXAGONAL CACTI

In this chapter, we show all our current results concerning the formula of the numbers of paths of length at most 7 of para-hexagonal cacti.

Observation 5.1. *For $0 \leq \ell \leq 3n+4$ and $1 \leq k \leq n-1$ such that $k = \lfloor \frac{\ell}{3} \rfloor$, we let G be a para-hexagonal cactus of n hexagons and let $\overline{p}_\ell(n)$ be the number of paths of length ℓ in G having one end vertex at the vertex of distance three from the cut vertex of the first hexagon. Then,*

$$\overline{p}_\ell(n) = \begin{cases} 1 & \textit{if } \ell = 0, \\ 2 & \textit{if } \ell \in \{1, 2\}, \\ 2^k & \textit{if } \ell = 3k, \\ 3 \cdot 2^k & \textit{if } \ell \in \{3k+1, 3k+2\}, \\ 2^n & \textit{if } \ell \in \{3n+2, 3n+1, 3n\}, \\ 0 & \textit{if } \ell \in \{3n+3, 3n+4\}. \end{cases}$$

Next we will establish the formula of $p_\ell(n)$, the number of paths of length ℓ in para-hexagonal cacti.

Lemma 5.2. *Let G a para-hexagonal cactus of n hexagon and let $p_\ell(n)$ be the number of paths of length ℓ in G. Then,*

$$p_\ell(n) = \begin{cases} 5n+1 & \text{if } \ell = 0, \\ 6n & \text{if } \ell = 1, \\ 10n-4 & \text{if } \ell = 2, \\ 14n-8 & \text{if } \ell = 3, \\ 18n-12 & \text{if } \ell = 4, \end{cases}$$

Proof. Clearly, when $\ell = 0$, $p_0(n)$ is the number of vertices of G which is $5n+1$. When $\ell = 1$, $p_1(n)$ is the number of edges of G which is $6n$. When $\ell = 2$, we count $p_2(n)$ via the combination of th neighbours of each vertex choose two. There are $4n+2$ vertices of degree two and there are $n-1$ vertices of degree four. Thus, $p_2(n) = (4n+2)\binom{2}{2} + (n-1)\binom{4}{2} = 10n-4$. This completes the proofs when $0 \le \ell \le 2$. We distinguish 3 cases according to the value ℓ.

Case 1: $\ell = 3$.

When $\ell = 3$, there are six of P_3 that are completely in the n^{th} hexagon. Further, there are $2\overline{p}_2(n-1)$ of paths P_3 that have one edge in the n^{th} hexagon, there are $2\overline{p}_1(n-1)$ of paths P_3 that have two edges in the n^{th} hexagon and there are $p_3(n-1)$ that do not have any edge in the n^{th} hexagon. Thus

$$p_3(n) = 6 + p_3(n-1) + 2\overline{p}_2(n-1) + 2\overline{p}_1(n-1) = 14 + p_3(n-1).$$

By applying this equation $n-1$ times, we obtain

$$p_3(n) = 14(n-1) + p_3(1) = 14n-8.$$

This proves Case 1.

Case 2: $\ell = 4$.

When $\ell = 4$, there are six of P_4 that are completely in the n^{th} hexagon. Further, there are $2\overline{p}_3(n-1)$ of paths P_4 that have two edge in the n^{th} hexagon, there are $2\overline{p}_2(n-1)$ of paths P_4 that have two edge in the n^{th} hexagon, there are $2\overline{p}_1(n-1)$ of paths P_4 that have two edges in the n^{th} hexagon and there are $p_4(n-1)$ that do not have any edge in the n^{th} hexagon. Thus

$$p_4(n) = 6 + p_4(n-1) + 2\overline{p}_3(n-1) + 2\overline{p}_2(n-1) + 2\overline{p}_1(n-1) = 18 + p_4(n-1).$$

By applying this equation $n-1$ times, we obtain

$$p_4(n) = 18(n-1) + p_4(1) = 18n - 12.$$

This proves Case 2.

Lemma 5.3. *If $p_5(n)$ is the number of paths of length six of a para-hexagonal cactus of n hexagons, then*

$$p_5(n) = \begin{cases} 6 & \text{if } n = 1, \\ 28 & \text{if } n = 2, \\ 30n - 32 & \text{if } n \geq 3. \end{cases}$$

Proof. There are six of P_5 that are completely in the n^{th} hexagon. Thus, $p_5(1) = 6$. It is routine to check that $p_5(2) = 28$ and $p_5(3) = 58$. Thus, we may assume that $n \geq 3$.

It can be observed that every path of length five has $0, 1, 2, 3$ or 4 edges in the n^{th} hexagon. There are $p_5(n-1)$ paths of length five which has no edge in the n^{th} hexagon, there are $2\overline{p}_4(n-1)$ paths of length five which has one edge in the n^{th} hexagon, there are $2\overline{p}_3(n-1)$ paths of length five which has two edges in the n^{th} hexagon, there are $2\overline{p}_2(n-1)$ paths of length five which has three edges in the n^{th} hexagon, there are $2\overline{p}_1(n-1)$ paths of length five which has four edges in the n^{th} hexagon. Thus,

$$p_5(n) = 6 + p_5(n-1) + 2(\overline{p}_4(n-1) + \overline{p}_3(n-1) + \overline{p}_2(n-1) + \overline{p}_1(n-1))$$

By Observation 5.1, we have that

$$p_5(n) = 6 + p_5(n-1) + 2(6 + 2 + 2 + 2) = p_5(n-1) + 30.$$

By applying this equation $n-3$ times, we obtain

$$p_5(n) = 30(n-3) + p_5(3) = 30n - 32.$$

This completes the proof. $\square$

Lemma 5.4. *If $p_6(n)$ is the number of paths of length six of a para-hexagonal cactus of n hexagons, then*

$$p_6(n) = \begin{cases} 0 & \textit{if } n = 1, \\ 20 & \textit{if } n = 2, \\ 36n - 52 & \textit{if } n \geq 3. \end{cases}$$

Proof. There is no path of length six in a hexagon. Thus, $p_6(1) = 0$. It is routine to check that $p_6(2) = 20$ and $p_6(3) = 56$. Thus, we may assume that $n \geq 3$.

It can be observed that every path of length six has $0, 1, 2, 3, 4$ or 5 edges in the n^{th} hexagon. There are $p_6(n-1)$ paths of length six which has no edge in the n^{th} hexagon, there are $2\overline{p}_5(n-1)$ paths of length six which has one edge in the n^{th} hexagon, there are $2\overline{p}_4(n-1)$ paths of length six which has two edges in the n^{th} hexagon, there are $2\overline{p}_3(n-1)$ paths of length six which has three edges in the n^{th} hexagon, there are $2\overline{p}_2(n-1)$ paths of length six which has four edges in the n^{th} hexagon, there are $2\overline{p}_1(n-1)$ paths of length six which has five edges in the n^{th} hexagon. Thus,

$$p_6(n) = p_6(n-1) + 2(\overline{p}_5(n-1) + \overline{p}_4(n-1) + \overline{p}_3(n-1) + \overline{p}_2(n-1) + \overline{p}_1(n-1))$$

By Observation 5.1, we have that

$$p_6(n) = p_6(n-1) + 2(6 + 6 + 2 + 2 + 2) = p_6(n-1) + 36.$$

By applying this equation $n-3$ times, we obtain

$$p_6(n) = 36(n-3) + p_6(3) = 36n - 52.$$

This completes the proof. $\square$

Lemma 5.5. *If $p_7(n)$ is the number of path of length seven of a para-hexagonal cactus of n hexagons, then*

$$p_7(n) = \begin{cases} 0 & \text{if } n = 1, \\ 16 & \text{if } n = 2, \\ 40n - 64 & \text{if } n \geq 3. \end{cases}$$

Proof. There is no path of length seven in a hexagon. Thus, $p_7(1) = 0$. It is routine to check that $p_7(2) = 16$ and $p_7(3) = 56$. Thus, we may assume that $n \geq 4$.

It can be observed that every path of length seven has $0, 1, 2, 3, 4, 5$ or 6 edges in the n^{th} hexagon. There are $p_7(n-1)$ paths of length seven which has no edge in the n^{th} hexagon, there are $2\overline{p}_6(n-1)$ paths of length seven which has one edge in the n^{th} hexagon, there are $2\overline{p}_5(n-1)$ paths of length seven which has two edges in the n^{th} hexagon, there are $2\overline{p}_4(n-1)$ paths of length seven which has three edges in the n^{th} hexagon, there are $2\overline{p}_3(n-1)$ paths of length seven which has four edges in the n^{th} hexagon, there are $2\overline{p}_2(n-1)$ paths of length seven which has five edges in the n^{th} hexagon. Thus,

$$p_7(n) = p_7(n-1) + 2(\overline{p}_6(n-1) + \overline{p}_5(n-1) + \overline{p}_4(n-1) + \overline{p}_3(n-1) + \overline{p}_2(n-1))$$

By Observation 5.1, we have that

$$p_7(n) = p_7(n-1) + 2(4 + 6 + 6 + 2 + 2) = p_7(n-1) + 40.$$

By applying this equation $n-3$ times, we obtain

$$p_7(n) = 40(n-3) + p_7(3) = 40n - 64.$$

This completes the proof. $\square$

In the following, for a given length $\ell \geq 8$ of a path, the smallest n such that a para-hexagonal cactus of n hexagons that can contain P_ℓ is $\lfloor \frac{\ell-2}{3} \rfloor$ (this is not true when $\ell \leq 7$) and the smallest number of n such that we may count the number of P_ℓ in general case is $\lceil \frac{\ell+2}{3} \rceil$. We are ready to establish the number of paths P_ℓ for all $\ell \geq 8$ as follows:

Theorem 5.6. *For integers $\ell \geq 8$ and $n \geq \lfloor \frac{\ell-2}{3} \rfloor$, we let $p_\ell(n)$ be the number of path of length ℓ of a para-hexagonal cactus of n hexagons. If $\ell = 3k$, then*

$$p_\ell(n) = \begin{cases} 2^k & \text{if } n = k-1, \\ 7 \cdot 2^k & \text{if } n = k, \\ 20 \cdot (n-k-1) \cdot 2^{k-1} + 17 \cdot 2^k & \text{if } n \geq \lceil \frac{\ell+2}{3} \rceil. \end{cases}$$

Proof. We first consider the case when $n = k-1$. Thus there are only two possibilities which are (i) the first hexagon contains 5 edges of P_{3k} and the $(k-1)^{th}$ hexagon contains 4 edges of P_{3k} and (ii) the first hexagon contains 4 edges of P_{3k} and the $(k-1)^{th}$ hexagon contains 5 edges of P_{3k}. In each case, there are 2^{k-1} of P_{3k}. Thus, $p_{3k}(k-1) = 2^k$.

We next consider the case when $n = k$. Thus, all the possibilities are (i) for all $1 \leq i \leq 5$, the first hexagon contains i edges of P_{3k} and the k^{th} hexagon contains $6-i$ edges of P_{3k}, (ii) the first hexagon does not contain any edge of P_{3k} and (iii) the k^{th} hexagon does not contain any edge of P_{3k}. In each of the cases (ii) and (iii), there are $p_{3k}(k-1) = 2^k$ of P_{3k}. Further, for the case (i), it can be checked that there are 2^k of P_{3k}. Thus, $p_{3k}(k) = 7 \cdot 2^k$.

To consider the case when $n \geq \lceil \frac{\ell+2}{3} \rceil$, we first assume that $n = k+1$. In this case, all the possibilities are (i) for all $1 \leq i \leq 2$, the first hexagon contains i edges of P_{3k} and the $(k+1)^{th}$ hexagon contains $3-i$ edges of P_{3k}, (ii) the first hexagon does not contain any edge of P_{3k} and (iii) the $(k+1)^{th}$ hexagon does not contain any edge of P_{3k}. In each of the cases (ii) and (iii), there are $p_{3k}(k) = 2^k$ of P_{3k}. However, it is possible that the cases (ii) and (iii) occur at the same time and this yields the over counting. Thus, there are

$$p_{3k}(k) + p_{3k}(k) - p_{3k}(k-1) = 7 \cdot 2^k + 7 \cdot 2^k - 2^k = 13 \cdot 2^k$$

of P_{3k} in cases (ii) and (iii). For each i of the case (i), it can be checked that there are 2^{k+2} of P_{3k}. Thus,

$$p_{3k}(k+1) = 13 \cdot 2^k + 2^{k+2} = 17 \cdot 2^k.$$

Now, for the general case $n \geq \lceil \frac{\ell+2}{3} \rceil$, it can be observed that every path of length ℓ has $0, 1, 2, 3, 4$ or 5 edges in the n^{th} hexagon. There are $p_\ell(n-1)$ paths of length ℓ which has no edge in the n^{th} hexagon, there are $2\overline{p}_{\ell-1}(n-1)$ paths of length ℓ which has one edge in the n^{th} hexagon, there are $2\overline{p}_{\ell-2}(n-1)$ paths of length ℓ which has two edges in the n^{th} hexagon, there are $2\overline{p}_{\ell-3}(n-1)$ paths of length ℓ which has three edges in the n^{th} hexagon, there are $2\overline{p}_{\ell-4}(n-1)$ paths of length ℓ which has four edges in the n^{th} hexagon and there are $2\overline{p}_{\ell-5}(n-1)$ paths of length ℓ which has five edges in the n^{th} hexagon. Thus,

$$p_\ell(n) = p_\ell(n-1) + 2(\overline{p}_{\ell-1}(n-1) + \overline{p}_{\ell-2}(n-1) + \overline{p}_{\ell-3}(n-1) + \overline{p}_{\ell-4}(n-1) + \overline{p}_{\ell-5}(n-1))$$

By Observation 5.1, we have that

$$p_\ell(n) = p_\ell(n-1) + 20 \cdot 2^{k-1}.$$

By applying this equation $n-k-1$ times, we obtain

$$\begin{aligned} p_\ell(n) &= 20(n-k-1) \cdot 2^{k-1} + p_\ell(k+1) \\ &= 20(n-k-1) \cdot 2^{k-1} + p_{3k}(k+1) \\ &= 20(n-k-1) \cdot 2^{k-1} + 17 \cdot 2^k. \end{aligned}$$

This completes the proof.

$\square$

Theorem 5.7. *For integers $\ell \geq 8$ and $n \geq \lfloor \frac{\ell-2}{3} \rfloor$, we let $p_\ell(n)$ be the number of path of length ℓ of a para-hexagonal cactus of n hexagons. If $\ell = 3k+1$, then*

$$p_\ell(n) = \begin{cases} 2^{k-1} & \text{if } n = k-1, \\ 5 \cdot 2^k & \text{if } n = k, \\ 21 \cdot (n-k-1) \cdot 2^{k-1} + 31 \cdot 2^{k-1} & \text{if } n \geq \lceil \frac{\ell+2}{3} \rceil. \end{cases}$$

Proof. We first consider the case when $n = k-1$. Thus there is only one possibility which is the first hexagon and the $(k-1)^{th}$ hexagon contains 5 edges of P_{3k+1}. There are 2^{k-1} of P_{3k}. Thus, $p_{3k+1}(k-1) = 2^{k-1}$.

We next consider the case when $n = k$. Thus, all the possibilities are (i) for all $2 \leq i \leq 5$, the first hexagon contains i edges of P_{3k+1} and the k^{th} hexagon contains $7-i$ edges of P_{3k+1}, (ii) the first hexagon does not contain any edge of P_{3k+1} and (iii) the k^{th} hexagon does not contain any edge of P_{3k+1}. In each of the cases (ii) and (iii), there are $p_{3k+1}(k-1) = 2^{k-1}$ of P_{3k+1}. Further, for the case (i), it can be checked that there are 2^{k+2} of P_{3k+1}. Thus, $p_{3k+1}(k) = 2^{k+2} + 2^{k-1} + 2^{k-1} = 5 \cdot 2^k$.

To consider the case when $n \geq \lceil \frac{\ell+2}{3} \rceil$, we first assume that $n = k+1$. In this case, all the possibilities are (i) for all $1 \leq i \leq 3$, the first hexagon contains i edges of P_{3k+1} and the $(k+1)^{th}$ hexagon contains $4-i$ edges of P_{3k+1}, (ii) the first hexagon does not contain any edge of P_{3k+1} and (iii) the $(k+1)^{th}$ hexagon does not contain any edge of P_{3k+1}. In each of the cases (ii) and (iii), there are $p_{3k+1}(k) = 2^k$ of P_{3k+1}. However, it is possible that the cases (ii) and (iii) occur at the same time and this yields the over counting. Thus, there are

$$p_{3k+1}(k) + p_{3k+1}(k) - p_{3k+1}(k-1) = 5 \cdot 2^k + 5 \cdot 2^k - 2^{k-1} = 19 \cdot 2^{k-1}.$$

of P_{3k+1} in cases (ii) and (iii). For each i of the case (i), it can be checked that there are 2^{k+1} of P_{3k+1}. Thus,

$$p_{3k+1}(k+1) = 19 \cdot 2^{k-1} + 3 \cdot 2^{k+1} = 31 \cdot 2^{k-1}.$$

Now, for the general case $n \geq \lceil \frac{\ell+2}{3} \rceil$, it can be observed that every path of length ℓ has $0, 1, 2, 3, 4$ or 5 edges in the n^{th} hexagon. There are $p_\ell(n-1)$ paths of length ℓ which has no edge in the n^{th} hexagon, there are $2\overline{p}_{\ell-1}(n-1)$ paths of length ℓ which has one edge in the n^{th} hexagon, there are $2\overline{p}_{\ell-2}(n-1)$ paths of length ℓ which has two edges in the n^{th} hexagon, there are $2\overline{p}_{\ell-3}(n-1)$ paths of length ℓ which has three edges in the n^{th} hexagon, there are $2\overline{p}_{\ell-4}(n-1)$ paths of length ℓ which has four edges in the n^{th} hexagon and there are $2\overline{p}_{\ell-5}(n-1)$ paths of length ℓ which has five edges in the n^{th} hexagon. Thus,

$$p_\ell(n) = p_\ell(n-1) + 2(\overline{p}_{\ell-1}(n-1) + \overline{p}_{\ell-2}(n-1) + \overline{p}_{\ell-3}(n-1) + \overline{p}_{\ell-4}(n-1) + \overline{p}_{\ell-5}(n-1))$$

By Observation 5.1, we have that

$$p_\ell(n) = p_\ell(n-1) + 21 \cdot 2^{k-1}.$$

By applying this equation $n-k-1$ times, we obtain

$$\begin{aligned} p_\ell(n) &= 21(n-k-1) \cdot 2^{k-1} + p_\ell(k+1) \\ &= 21(n-k-1) \cdot 2^{k-1} + p_{3k+1}(k+1) \\ &= 21(n-k-1) \cdot 2^{k-1} + 31 \cdot 2^{k-1}. \end{aligned}$$

This completes the proof.

$\square$

Theorem 5.8. *For integers $\ell \geq 8$ and $n \geq \lfloor \frac{\ell-2}{3} \rfloor$, we let $p_\ell(n)$ be the number of path of length ℓ of a para-hexagonal cactus of n hexagons. If $\ell = 3k+2$, then*

$$p_\ell(n) = \begin{cases} 0 & \text{if } n = k-1, \\ 3 \cdot 2^k & \text{if } n = k, \\ 30 \cdot (n-k-1) \cdot 2^{k-1} + 14 \cdot 2^k & \text{if } n \geq \lceil \frac{\ell+2}{3} \rceil. \end{cases}$$

Proof. We first consider the case when $n = k-1$. Thus there is only one possibility which is the first hexagon and the $(k-1)^{th}$ hexagon contains 5 edges of P_{3k+2}. There are 2^{k-1} of P_{3k}. Thus, $p_{3k+2}(k-1) = 0$.

We next consider the case when $n = k$. Thus, all the possibilities are (i) for all $3 \leq i \leq 5$, the first hexagon contains i edges of P_{3k+2} and the k^{th} hexagon contains $8 - i$ edges of P_{3k+2}, (ii) the first hexagon does not contain any edge of P_{3k+2} and (iii) the k^{th} hexagon does not contain any edge of P_{3k+2}. In each of the cases (ii) and (iii), there are $p_{3k+2}(k-1) = 0$ of P_{3k+2}. Further, for the case (i), it can be checked that there are $3 \cdot 2^k$ of P_{3k+2}. Thus, $p_{3k+2}(k) = 3 \cdot 2^k + 0 + 0 = 3 \cdot 2^k$.

To consider the case when $n \geq \lceil \frac{\ell+2}{3} \rceil$, we first assume that $n = k+1$. In this case, all the possibilities are (i) for all $1 \leq i \leq 4$, the first hexagon contains i edges of P_{3k+2} and the $(k+1)^{th}$ hexagon contains $5 - i$ edges of P_{3k+2}, (ii) the first hexagon does not contain any edge of P_{3k+2} and (iii) the $(k+1)^{th}$ hexagon does not contain any edge of P_{3k+2}. In each of the cases (ii) and (iii), there are $p_{3k+2}(k) = 0$ of P_{3k+2}. However, it is possible that the cases (ii) and (iii) occur at the same time and this yields the over counting. Thus, there are

$$p_{3k+2}(k) + p_{3k+2}(k) - p_{3k+2}(k-1) = 3 \cdot 2^k + 3 \cdot 2^k - 0 = 6 \cdot 2^k.$$

of P_{3k+2} in cases (ii) and (iii). For each i of the case (i), it can be checked that there are 2^{k+3} of P_{3k+2}. Thus,

$$p_{3k+2}(k+1) = 6 \cdot 2^k + 2^{k+3} = 14 \cdot 2^k.$$

Now, for the general case $n \geq \lceil \frac{\ell+2}{3} \rceil$, it can be observed that every path of length ℓ has $0, 1, 2, 3, 4$ or 5 edges in the n^{th} hexagon. There are $p_\ell(n-1)$ paths of length ℓ which has no edge in the n^{th} hexagon, there are $2\overline{p}_{\ell-1}(n-1)$ paths of length ℓ which has one edge in the n^{th} hexagon, there are $2\overline{p}_{\ell-2}(n-1)$ paths of length ℓ which has two edges in the n^{th} hexagon, there are $2\overline{p}_{\ell-3}(n-1)$ paths of length ℓ which has three edges in the n^{th} hexagon, there are $2\overline{p}_{\ell-4}(n-1)$ paths of length ℓ which has four edges in the n^{th} hexagon and there are $2\overline{p}_{\ell-5}(n-1)$ paths of length ℓ which has five edges in the n^{th} hexagon. Thus,

$$p_\ell(n) = p_\ell(n-1) + 2(\overline{p}_{\ell-1}(n-1) + \overline{p}_{\ell-2}(n-1) + \overline{p}_{\ell-3}(n-1) + \overline{p}_{\ell-4}(n-1) + \overline{p}_{\ell-5}(n-1))$$

By Observation 5.1, we have that

$$p_\ell(n) = p_\ell(n-1) + 30 \cdot 2^{k-1}.$$

By applying this equation $n-k-1$ times, we obtain

$$\begin{aligned}
p_\ell(n) &= 30(n-k-1)\cdot 2^{k-1} + p_\ell(k+1)\\
&= 30(n-k-1)\cdot 2^{k-1} + p_{3k+1}(k+1)\\
&= 30(n-k-1)\cdot 2^{k-1} + 14\cdot 2^{k}.
\end{aligned}$$

This completes the proof. $\square$

CHAPTER 6 CONCLUSION

In this project, we applied the concept of generating functions, which have been used ...

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