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Bankruptcy Games

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Abstract: Bankruptcy problems are considered from a game theoretic point of view. Solution concepts from cooperative game theory are studied for bankruptcy games. A necessary and sufficient condition for a division rule for bankruptcy problems to be a game theoretic rule is given. A new division rule which is an adjustment of the proportional rule is given. This rule coincides with the τ-value for bankruptcy games. Properties of the new rule are treated and a set of characterizing properties is given.

Zusammenfassung: In dieser Arbeit werden Bankrottprobleme von spieltheoretischer Warte aus behandelt; insbesondere werden Lösungskonzepte der kooperativen Spieltheorie für „Bankrottspiele" untersucht. Eine notwendige und hinreichende Bedingung wird angegeben dafür, Daß eine Aufteilungsregel für Bankrottprobleme spieltheoretischer Natur ist. Ferner wird eine neue Aufteilungsregel angegeben, welche eine passende Modellierung der Proportionalitätsregel ist. Diese Regel fällt mit dem τ-Wert für Bankrottspiele zusammen. Schließlich werden Eigenschaften dieser neuen Regel untersucht und eine Axiomatisierung angegeben.

Key words: bankruptcy problem, convex game, game theoretic division rule, τ-value.

1 Introduction

Bankruptcy problems arising from the Talmud are studied by O'Neill [6], Aumann and Maschler [1] and Young [11, 12]. The situation is as follows: a number of individuals advances claims on a certain resource whose total value is insufficient to meet all of the claims. We can think of a man dying and leaving behind an estate which is worth less than the sum of his debts. The question is how the estate should be divided among

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the creditors. O'Neill [6] discusses several division methods in such situations. He proposes the method of recursive completion which is in fact the Shapley-value of the cooperative game corresponding to the bankruptcy problem. Aumann and Maschler [1] study three examples of bankruptcy from the Babylonian Talmud and extend the solution given there to all bankruptcy problems. They prove that this extension is the nucleolus of the corresponding bankruptcy games.

Another way of dividing the estate is the proportional method which divides the estate proportionally to the claims. This method is widely used. However, contrary to the two methods mentioned above it is not a game theoretic method. Especially, it is not invariant under strategic equivalence. Consider for example the following two bankruptcy problems. In both the same two claimants figure. In the first problem the estate equals 60 and claimant 1 claims 60, claimant 2 claims 40. In the second problem the estate equals 80 and claimant 1 claims 80, claimant 2 claims 40. The proportional method divides the estates in portions 36, 24 and $53^1/_3$, $26^2/_3$ respectively while the two other methods mentioned both result in divisions of 40, 20 and 60, 20 respectively. We see that these two methods assign the extra 20 which enters in the second problem and which seems to come from an extra loan from the first claimant to the deceased, to the first claimant. Hence the fact that the second problem arises from the first by adding 20 to the estate and to the claim of claimant 1 is reflected in the outcomes of these two methods while the proportional method does not reflect this fact.

In this paper we suggest a method which arises from the proportional method by making some modifications in order to achieve strategic equivalence and to obtain a game theoretical method. This new method is called the adjusted proportional method.

The paper is organized as follows. In Section 2 formal definitions and notations are given. In that section we also give some examples of division rules and we introduce the new one. In Section 3 bankruptcy games are defined and we prove that they are convex games. In Section 4 we consider some solution concepts from cooperative game theory and we study them for bankruptcy games. An interesting result here is that the core and the core cover of bankruptcy games coincide. In this section it is also proved that the adjusted proportional division rule corresponds to the τ-value, Tijs [8], of the corresponding game and a necessary and sufficient condition is given for a division rule to be a game theoretic rule. Finally, in Section 5 properties of the new rule are given. A subset of these properties characterizes the adjusted proportional rule axiomatically.

2 Bankruptcy Problems

A *bankruptcy problem* is an ordered pair $(E; \underset{\sim}{d}) \in \mathbb{R} \times \mathbb{R}^n$, where $0 \leqslant d_1 \leqslant d_2 \leqslant \ldots \leqslant d_n$ and $0 \leqslant E \leqslant d_1 + \ldots + d_n =: D$.

E is the estate which has to be divided among n claimants. We denote the set of claimants by $N = \{1, 2, \ldots, n\}$. Claimant i advances a claim of d_i on E. The problem is now how to divide E among the claimants. A *division rule* is a function f that assigns to every bankruptcy problem $(E; \underline{d})$ for every positive number of claimants a solution $f(E; \underline{d}) = (f_1(E; \underline{d}), \ldots, f_n(E; \underline{d}))$ such that

(i) $f_i(E; \underline{d}) \geqslant 0$ for every $i \in N$ (*individual rationality*)

(ii) $\sum_{i=1}^{n} f_i(E; \underline{d}) = E$ (*efficiency*)

If $E = 0$ it is obvious that by (i) and (ii) no claimant receives anything so in the following we will assume $E > 0$.

Some examples of division rules are the *proportional division* rule which assigns to claimant i the amount $E \cdot d_i \cdot D^{-1}$, the *constrained equal award* or CEA-rule which assigns to claimant i $\alpha \wedge d_i$, that is, the minimum of α and d_i where $\alpha \in [0, d_n]$ is uniquely determined by the efficiency property. Other division rules are the *constrained equal loss* or CEL-rule which assigns to claimant i the maximum of $d_i - \beta$ and 0 where $\beta \in [0, d_n]$ is uniquely determined by the efficiency property, the *recursive completion* rule described by O'Neill [6] and the *CG-consistent* rule given by Aumann and Maschler [1].

In the following we will denote by $\hat{E}$ the vector in $\mathbb{R}^n$ with every coordinate equal to E. By $\hat{E} \wedge \underline{d}$ we will denote the vector $(E \wedge d_1, \ldots, E \wedge d_n)$. For every $S \subset N$ we denote the sum $\sum_{i \in S} d_i$ by $d(S)$. We define the *minimal right* m_i of claimant i to be the amount that is not claimed by any of the others, so[3]

$$m_i := (E - d(N - i))_+ := \max\{E - d(N - i), 0\}.$$

The division rule we are introducing starts by paying every claimant his minimal right. In Section 3 we will show that this can be done because $m(N) := \sum_{i \in N} m_i \leqslant E$. Now the amount which is left, that is $E' = E - m(N)$, has to be divided. Because claimant i has already received m_i his claim is lowered and becomes $d_i' = d_i - m_i$. Note that $d_i' = d_i - (E - d(N-i))_+ \geqslant d_i - (D - d(N-i))_+ = d_i - d_i = 0$. Further it is considered irrational to clain more than there is available. So every claim which is greater than $E - m(N)$ is reduced till it becomes equal to $E - m(N)$. In this way we get a new claim vector $d'' := \hat{E}' \wedge \underline{d}' \geqslant 0$. Now E' is divided proportional to the new claims. The division rule described here we call the *adjusted proportional* rule, or shorter, the AP-rule.

3 We write $N - \{i\}$ instead of $N-\{i\}$. Also $N - 0 := N$.

A bankruptcy problem for which the minimal rights of all the claimants are equal to zero we call *zero-normalized.* So for a zero-normalized bankruptcy problem $(E; \underline{d})$ we have $E \leqslant d(N-i)$ for every $i \in N$. Following O'Neill [6] we call a bankruptcy problem $(E; \underline{d})$ with $d_i \leqslant E$ for every $i \in N$ a *simple claims* bankruptcy problem. For a zero-normalized simple claims bankruptcy problem the AP-rule reduces to the proportional division rule.

In the following we will denote the outcome of the AP-rule for a bankruptcy problem $(E; d)$ by $t(E; d)$.

So formally,

$$t(E; \underline{d}) = \begin{cases} \underline{m} & \text{if } E' = 0 \\ \underline{m} + \underline{d}''(\sum_{i \in N} d_i'')^{-1} E' & \text{otherwise,} \end{cases}$$

where $\underline{m}$ and $\underline{d}''$ are as defined above. Note that $\sum_{i \in N} d_i' = D - m(N) \geqslant E - m(n) = E'$ and so from $\sum_{i \in N} d_i'' = 0$ it follows that $E' = 0$ because $\sum_{i \in N} d_i'' = 0$ implies $\sum_{i \in N} d_i' = 0$ or $E' = 0$. We can also give more explicit formulas for the AP-rule. Before doing this we will show that it is sufficient to know the rule for bankruptcy problems $(E; \underline{d})$ with $E \leqslant {}^1\!/\!_2 D$.

This is the case because the AP-rule satisfies a property, introduced by Aumann and Maschler [1], called *self-duality.* A division rule f is said to be self-dual if

$$f(E; \underline{d}) = \underline{d} - f(D - E; \underline{d})$$

So if we use a self-dual rule to decide how much of his claim every claimant has to hand in, we get the same outcome as in the case that we apply the rule directly to divide the estate. In Section 4 we will prove that the AP-rule is self-dual, so if we have a bankruptcy problem $(E; \underline{d})$ with $E \geqslant {}^1\!/\!_2 D$ we can use our formulas for the case with $E \leqslant {}^1\!/\!_2 D$ to calculate $t(D - E; \underline{d})$ and then substract this from $\underline{d}$ to get $t(E; \underline{d})$. Let $E \leqslant {}^1\!/\!_2 D$. We distinguish the case (1) $d_n \leqslant {}^1\!/\!_2 D$ and (2) $d_n \geqslant {}^1\!/\!_2 D$. Let $I_0 := [0, d_1]$, $I_1 := [d_1, d_2], \ldots, I_{n-1} := [d_{n-1}, d_n \wedge d(N-n)]$, $I_n := [d_n \wedge d(N-n), \ {}^1\!/\!_2 D]$. In Case 1 we have $I_{n-1} = [d_{n-1}, d_n]$, $I_n = [d_n, \ {}^1\!/\!_2 D]$. In Case 2 we have $I_{n-1} = [d_{n-1}, d(N-n)]$, $I_n = [d(N-n), \ {}^1\!/\!_2 D]$.

Straightforward calculations show that the formulas are:

Case 1

$$t_i(E;d)=\begin{cases} E\cdot d_i(\sum\limits_{k\leqslant j} d_k+(n-j)E)^{-1} & \text{for } E\in I_j \text{ with } i\leqslant j \\ E^2(\sum\limits_{k\leqslant j} d_k+(n-j)E)^{-1} & \text{for } E\in I_j \text{ with } i>j \end{cases}$$

Case 2

$$t_i(E;d)=\begin{cases} \text{the same as in case (1)} & \text{for } E\in I_j \text{ with } j\leqslant n-1 \\ {}^1\!/_2\, d_i & \text{for } E\in I_n,\ i\neq n \\ {}^1\!/_2\, d_n+E-{}^1\!/_2\, D & \text{for } E\in I_n,\ i=n. \end{cases}$$

From the formulas we see that $t_i(E;d)$ is a continuous function of E on $[0, {}^1\!/_2 D]$ and hence with the self-duality property it is a continuous function of E on the whole interval $[0, D]$.

Table 1

Estate \ Claim	100	200	250	300	400	500
100	$33^1/_3$	40	$45^5/_{11}$	50	60	$66^2/_3$
200	$33^1/_3$	80	$90^{10}/_{11}$	100	120	$166^2/_3$
300	$33^1/_3$	80	$113^7/_{11}$	150	220	$266^2/_3$

In Table 1 $t(E;d)$ is given for some bankruptcy problems with three creditors. The examples of the Talmud given by Aumann and Maschler [1] are included in the table. (They correspond to $E = 100, 200$ and 300 respectively.)

For E equal to 100 and 300 the AP-rule gives the same division as prescribed by the Mishna. For E equal to 200 the division given in the Talmud is (50, 75, 75). For $E = 250$ the extension of the Mishna solution given by Aumann and Maschler divides the estate by assigning to claimant 1 50 and to claimants 2 and 3 both 100.

3 Bankruptcy Games

In this section we study *cooperative games* arising from bankruptcy situations. An *n-person cooperative game in characteristic function form* is an ordered pair $\langle N, v\rangle$ where $N = \{1, 2, \dots, n\}$ is the *set of players* and v is the characteristic function which assigns to every $S \in 2^N$ an element of $\mathbb{R}$ with $v(\emptyset) = 0$. The subsets of N are called *coalitions* and $v(S)$ is regarded as the worth of S for every coalition S. Such a game is called *superadditive* if

$$v(S) + v(T) \leqslant v(S \cup T) \quad \text{for every } S, T \in 2^N \text{ with } S \cap T = \emptyset,$$

and *convex* if

$$v(S) + v(T) \leqslant v(S \cup T) + v(S \cap T) \quad \text{for every } S, T \in 2^N.$$

It follows that every convex game is superadditive.

One of the problems considered in cooperative game theory is how to divide $v(N)$ once the coalition N is formed. A *payoff vector* is a vector $x \in \mathbb{R}^n$ with $\sum_{i \in N} x_i = v(N)$, where x_i represents the payoff to player i and $\sum_{i \in N} x_i = v(N)$. *Solution concepts* associate payoff vectors with games.

For any bankruptcy problem $(E; \underline{d})$ we define a corresponding game as follows. The set of players is equal to the set of claimants. The characteristic functions $v_{E;d}$ is defined to be

$$v_{E;d}(S) = (E - d(N - S))_+.$$

The worth of coalition S in the game $v_{E;d}$ is that amount of the estate which is not claimed by the complement of S. In this sense $v_{E;d}(S)$ is a generalization of the minimal right concept. Note that $v_{E;d}(\{i\}) = m_i$. The definition of $v_{E;d}$ has already been given by O'Neill [6]. In the following we will call the game $v_{E;d}$ the bankruptcy game corresponding to the bankruptcy problem $(E; \underline{d})$.

The following theorem states that bankruptcy games are convex.

Theorem 1: Bankruptcy games are convex games.

Proof: We make use of the following equivalent formulation of the convexity of a game given by Shapley [7].

A game $\langle N, v\rangle$ is convex if for all $S, T \in 2^N, i \in N$ with $T \subset S \subset N-i$

$$v(S \cup \{i\}) - v(S) \geqslant v(T \cup \{i\}) - v(T)$$

Let v be a bankruptcy game arising from a bankruptcy situation $(E; \underline{d})$.

Let $T \subset S \subset N-i$. We have to prove that $v(S \cup \{i\}) + v(T) \geqslant v(T) \cup \{i\}) + v(S)$. Let $A := E - D$. Then

$$v(S \cup \{i\}) + v(T) = \max\{A + d(S) + d_i, 0\} + \max\{A + d(T), 0\}$$

$$= \max\{2A + d(S) + d(T) + d_i, A + d(S) + d_i, A + d(T), 0\} \quad \text{and}$$

$$v(T \cup \{i\}) + v(S) = \max\{A + d(T) + d_i, 0\} + \max\{A + d(S), 0\}$$

$$= \max\{2A + d(S) + d(T) + d_i, A + d(T) + d_i, A + d(S), 0\}$$

From $d(S) \geqslant d(T)$ now follows that $A + d(T) + d_i \leqslant A + d(S) + d_i$, further $A + d(S) \leqslant A + d(S) + d_i$ and hence $v(S \cup \{i\}) + v(T) \geqslant v(T \cup \{i\}) + v(S)$. It follows that v is a convex game.

Now we can show that $m(N) \leqslant E$ as we promised in Section 2. We have $m(N) = \sum_{i \in N} v(\{i\}) \leqslant v(N) = E$ where the inequality follows from the superadditivity of v and the equalities follow from the definitions of $m(N)$ and $v(N)$.

The question arises whether all non-negative convex games are bankruptcy games. The following example shows that this need not be the case.

Example: Let $\langle N, v\rangle$ be the game with $N = \{1, 2, \ldots, n\}$,

$$v(N) = 1, \quad v(N-i) = 1/2 \quad \text{for all } i \in N,$$

$$v(N - \{i, j\}) = 1/2 \quad \text{for all } i, j \in N, i \neq j,$$

$$v(S) = 0 \quad \text{for all other } S \in 2^N. \quad \text{Then } \langle N, v\rangle \text{ is a convex game.}$$

Suppose v is a bankruptcy game corresponding to the bankruptcy situation $(E; \underline{d})$. Then $E = 1$ and $d_i = E - v(N - i) = {}^1/_2$ for all $i \in N$. Then $v(N - \{i, j\}) = (E - d_i - d_j)_+ = 0$ in contradiction with $v(N - \{i, j\}) = {}^1/_2$.

Without proof we state that all non-negative 3-person zero-normalized convex games (that is convex games $\langle N, v\rangle$ with $v(i) = 0$ for all $i \in N$) are bankruptcy games.

4 Game Theoretic Division Rules

A well known solution concept in game theory is the core introduced by Gillies [4]. For a cooperative game $\langle N, v\rangle$ the core $C(v)$ is defined as follows

$$C(v) := \{x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geqslant v(S) \text{ for all } S \in 2^N\}.$$

If $v(N)$ is divided according to a payoff vector in the core no coalition can do better by working on its own. However, the core of a game can be empty. Shapley [7] has proved that the core of a convex game is not empty, hence bankruptcy games have non-empty cores.

For a game $\langle N, v\rangle$ the vector M^v with coordinates $M_i^v = v(N) - v(N - i)$ is called the *upper vector* of v.

M_i^v is the maximal payoff player i can except to get; if he asks for more the others will do better by throwing him out of the grand coalition N. Now we look at a coalition $S \ni i$ and calculate what is left for player i if the other members of S get their maximal payoff. That is

$$R^v(S, i) = v(S) - \sum_{j \in S - i} M_j^v.$$

The minimal payoff that player i will consent to get is

$$\mu_i^v = \max_{S \in i} R^v(S, i),$$

because he can ensure himself this payoff by offering the members of a coalition S, for which the maximum is achieved, their maximal payoff and remaining with μ_i^v.

The vector μ^v is called the *lower* vector of v. Note that $\mu_i^v \leqslant v(i)$ for all $i \in N$. Tijs and Lipperts [10] have defined the *core cover* to be

$$\mathrm{CC}(v) = \{x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = v(N), \mu^v \leqslant x \leqslant M^v\}.$$

They proved that in general $C(v) \subset \mathrm{CC}(v)$ and gave some classes of games for which equality holds. Although bankruptcy games do not belong to those classes in general, a consequence of the next theorem is that equality holds for them as well.

Theorem 2: Let $v_{E;d}$ be a bankruptcy game and let v be a game such that $0 \leqslant v \leqslant v_{E;d}$ and $v(S) = v_{E;d}(S)$ if $|S| \in \{0, n-1, n\}$. Then $C(v) = \mathrm{CC}(v)$.

Proof: Note that $x \in C(v_{E;d})$ implies $x \in C(v)$ hence $C(v) \neq \emptyset$. Take an $x \in \mathrm{CC}(v)$. Then

$$\begin{aligned}
\sum_{i \in S} x_i = \sum_{i \in N} x_i - \sum_{i \in N-S} x_i &\geqslant v(N) - \sum_{i \in N-S} (v(N) - v(N-i)) \\
&= v_{E;d}(N) - \sum_{i \in N-S} (v_{E;d}(N) - v_{E;d}(N-i)) \\
&= E - \sum_{i \in N-S} (E - (E-d_i)_+) \\
&\geqslant E - d(N-S)
\end{aligned}$$

Further $\sum_{i \in S} x_i \geqslant \sum_{i \in S} \mu_i^v \geqslant \sum_{i \in S} v(\{i\}) \geqslant 0$. So $x(S) \geqslant (E - d(N-S))_+ = v_{E;d}(S)$ and we have $x \in C(v_{E;d})$ which implies $x \in C(v)$. Since we already know that $C(v) \subset \mathrm{CC}(v)$ it follows that $C(v) = \mathrm{CC}(v)$.

Let $(E;\underline{d})$ be a bankruptcy problem and $\langle N, v\rangle$ the corresponding bankruptcy game. Then $M_i^v = v(N) - v(N-i) = E - (E-d_i)_+ = E \wedge d_i$. Driessen and Tijs [3] have proved that for a convex game the i-th coordinate of the lower vector is equal to the worth of the coalition $\{i\}$ in that game. Hence $\mu_i^v = v(\{i\}) = m_i$ and $\mathrm{CC}(v) = \{x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = v(N), m \leqslant x \leqslant \hat{E} \wedge \underline{d}\}$. It follows that the extreme points of $C(v) = \mathrm{CC}(v)$ are of the form $((m_i)_{i \in S}, x_k, (E \wedge d_j)_{j \in N-S \cup k})$ with $m_k \leqslant x_k \leqslant E \wedge d_k$ and with coordinates summing up to $v(N)$.

Tijs [8] has defined the *τ-value* for games with non-empty core over in the following way

$$\tau^v := \lambda \underline{M}^v + (1-\lambda)\underline{\mu}^v = \underline{\mu}^v + \lambda(\underline{M}^v - \underline{\mu}^v)$$

where λ is uniquely determined by the fact that $\sum_{i \in N} \tau_i^v = v(N)$ (efficiency). The following theorem states that for a bankruptcy problem the AP-rule gives a division which is equal to the τ-value of the corresponding bankruptcy game.

Theorem 3: For a bankruptcy problem the AP-rule yields the τ-value of the corresponding bankruptcy game.

Proof: Let $(E; \underline{d})$ be a bankruptcy problem and v the corresponding bankruptcy game. Then $\tau^v = \underline{\mu}^v + \lambda(\underline{M}^v - \underline{\mu}^v) = \underline{v} + \lambda((\hat{E} \wedge \underline{d}) - \underline{v})$ and $t(E; D) = \underline{v} + \gamma((\underline{d} - \underline{v}) \wedge \hat{E}')$ where $\underline{v} = (v(1), v(2), \ldots, v(n))$ and $E' = E - \sum_{j \in N} v(j)$. We will show that $(\hat{E} \wedge \underline{d}) - \underline{v} = (\underline{d} - \underline{v}) \wedge \hat{E}'$.

Suppose $d_i \leqslant E$. Because of the superadditivity of v we have then $E - d_i = v(N-i) \geqslant \sum_{j \in N-i} v(j)$ which leads to $E - \sum_{j \in N} v(j) \geqslant d_i - v(i)$ and hence $(E \wedge d_i) - v(i) = (d_i - v(i)) \wedge E'$.

Suppose $d_i > E$, then $E - d_i < 0 = \sum_{j \in N-i} v(j)$ which leads to $E - \sum_{j \in N} v(j) < d_i - v(i)$ and hence $(E \wedge d_i) - v(i) = (d_i - v(i)) \wedge E'$. So $(\hat{E} \wedge \underline{d}) - \underline{v} = (\underline{d} - \underline{v}) \wedge \hat{E}'$, because of the efficiency of both τ^v and $t(E; \underline{d})$ we have that $\lambda = \gamma$ and hence $\tau^v = t(E; \underline{d})$.

Now we will prove the self-duality of the AP-rule as was promised in Section 2.

Theorem 4: The AP-rule is self-dual.

Proof: Let $(E; \underline{d})$ be a bankruptcy problem and $\langle N, v \rangle$ the corresponding bankruptcy game. We have to prove that $t(E; \underline{d}) = \underline{d} - t(D - E; \underline{d})$. We will do this by proving that $\tau^v = \underline{d} - \tau^w$ where $\langle N, w \rangle$ is the bankruptcy game corresponding to $(D - E; \underline{d})$. We have

$$\tau^v = \underline{v} + \lambda(\underline{M}^v - \underline{v}) \quad \text{and} \quad \tau^w = \underline{w} + \gamma(\underline{M}^w - \underline{w}).$$

It is easily verified that $w(S) = d(S) - E + v(N - S)$ for every $S \in 2^N$.

From this it follows that $w(i) = d_i - M_i^v$ and $M_i^w = d_i - v(i)$ for every $i \in N$. Hence $\tau^w = \underline{d} - \underline{M}^v + \gamma(\underline{M}^v - \underline{v})$. From the definition it follows that $M_i^v \geqslant v(i)$ for all $i \in N$. Hence $M^v(N) = \sum_{i \in N} v(i)$ implies $v(i) = M_i^v$ for all $i \in N$ and then it follows immediately that $\tau^v = \underline{d} - \tau^w$. So in the following we will suppose that $M^v(N) \neq \sum_{i \in N} v(i)$. From $\sum_{i \in N} \tau_i^w = D - \sum_{i \in N} \tau_i^v$ it follows then that $\gamma = 1 - \lambda$ which leads to $\underline{d} - \tau^w = \underline{M}^v - (1-\lambda)(\underline{M}^v - \underline{v}) = v + \lambda(\underline{M}^v - \underline{v}) = \tau^v$ and the proof is completed.

With the aid of the self-duality property we can give an alternative way of computing the AP-rule. Let $(E;\underline{d})$ be a bankruptcy problem with $E \leqslant {}^1/_2 D$ then $t(E;\underline{d}) = \lambda\underline{d}^*$ where λ is determined by efficiency and

$$
\begin{aligned}
d_i^* &= d_i && \text{if } d_i \leqslant E \\
d_i^* &= E && \text{if } E \leqslant d_i \leqslant D - E \\
d_i^* &= d_i - D + 2E && \text{if } d_i \geqslant D - E
\end{aligned}
$$

The self-duality property now enables us to compute $t(E;\underline{d})$ for problems with $E \geqslant {}^1/_2 D$ as well.

Let us call a division rule f for bankruptcy problems a *game theoretic division rule* if it is possible to construct a solution concept F for cooperative games such that $f(E;\underline{d}) = F(v_{E;d})$ for all bankruptcy problems $(E;\underline{d})$. We just saw that the AP-rule is a game theoretic division rule. O'Neill [6] has shown that the recursive completion rule assigns to a bankruptcy problem the Shapley-value of the corresponding bankruptcy game, hence it is also a game theoretic division rule. The CG-consistent rule is also a game theoretic division rule because CG$(E;\underline{d})$ equals the nucleolus of $v_{E;d}$, cf. Aumann and Maschler [1].

Because the bankruptcy games corresponding to the problems $(E;\underline{d})$ and $(E;\hat{E} \wedge \underline{d})$ are the same a necessary condition for a division rule f to be a game theoretic division rule is that $f(E;\underline{d}) = f(E;\hat{E} \wedge \underline{d})$. The following theorem states that this is also a sufficient condition.

Theorem 5: A division rule f for bankruptcy problems is a game theoretic division rule if and only if $f(E;\underline{d}) = f(E;\hat{E} \wedge \underline{d})$.

Proof: We already saw the only if part. Suppose f is a division rule with $f(E;\underline{d}) = f(E;\hat{E} \wedge \underline{d})$.

We define a solution concept F for cooperative games as follows. For every cooperative game $\langle N, v\rangle$ $F(v) := f(v(N); \underline{K}^v)$ where $\underline{K}^v$ is the vector with i-th coordinate $K_i^v = M_i^v + n^{-1}(v(N) - \sum_{i \in N} M_i^v)_+$.

Then for a bankruptcy game $v_{E;d}$ we have $F(v_{E;d}) = f(E; \hat{E} \wedge \underline{d}) = f(E; \underline{d})$ and the proof is completed.

It is immediate that the proportional rule is not a game theoretic rule because in general it assigns different outcomes to $(E; \underline{d})$ and $(E; \hat{E} \wedge \underline{d})$. For the same reason the CEL-rule is not a game theoretic rule. Because this is not so immediate as in the case of the proportional rule we will illustrate it with an example. Let $(E; \underline{d})$ be the 3 claimants bankruptcy problem with $E = 100$, $d_1 = 40$, $d_2 = 50$, $d_3 = 110$, then CEL $(E; \underline{d}) = (6^2/_3, 16^2/_3, 76^2/_3)$ but CEL $(E; \hat{E} \wedge \underline{d}) = (10, 20, 70)$.

We recall that $\mathrm{CEA}_i(E; \underline{d}) = d_i \wedge \alpha$ where $\alpha \in [0, d_n]$ is determined by efficiency. It follows that $\alpha \leqslant E$ and hence $\mathrm{CEA}_i(E; \hat{E} \wedge \underline{d}) = E \wedge d_i \wedge \alpha = d_i \wedge \alpha = \mathrm{CEA}_i(E; \underline{d})$ for every $i \in N$ and we see that the CEA-rule is a game theoretic rule.

We have seen that the core of a bankruptcy game is not empty. The question arises under what conditions a division rule for bankruptcy problems will give an outcome which is always in the core of the corresponding bankruptcy game. Theorem 5 below states that this is the case whenever the division rule is *reasonable*, i.e. it does not assign more than his claim to any claimant. Assigning more than d_i to claimant i would seem highly unreasonable as there is not even enough to fulfil everybody's claims.

Theorem 6: Let f be a division rule for bankruptcy problems. Then $f(E; \underline{d}) \in C(v_{E;d})$ if and only if $f_i(E; \underline{d}) \leqslant d_i$ for all $i \in N$.

Proof: From $C(v_{E;d}) = CC(v_{E;d}) = \{x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = v(N), \underline{m} \leqslant x \leqslant \hat{E} \wedge \underline{d}\}$ the only if part follows immediately. Further, efficiency and $f_i(E; \underline{d}) \leqslant d_i$ for all $i \in N$ imply that for every $i \in N$ $f_i(E; \underline{d}) = E - \sum_{j \in N-i} f_j(E; \underline{d}) \geqslant E - d(N-i)$. Individual rationality implies that $f_i(E; \underline{d}) \geqslant 0$ for all $i \in N$, so $f_i(E; \underline{d}) \geqslant m_i$ for all $i \in N$ and $f(E; \underline{d}) \in C(v_{E;d})$.

All the division rules treated in this paper fulfil the reasonability condition so they all assign outcomes to bankruptcy problems which are in the core of the corresponding bankruptcy games.

5 Properties of the AP-Rule

In this section we will study some properties of the AP-rule and we will give a set of properties which characterizes the AP-rule axiomatically.

The first property that we want to introduce is the *minimal right* property which states that it doesn't matter whether a division rule is applied directly to a bankruptcy problem or the minimal rights are handed out to the claimants first and the rule is applied then. Formally, a division rule f satisfies the minimial right property if $f(E;\underset{\sim}{d}) = \underset{\sim}{m} + f(E - m(N), \underset{\sim}{d} - \underset{\sim}{m})$. From the way we introduced the AP-rule it follows that it satisfies the minimal right property. In fact, all game theoretic rules which are invanant under strategic equivalence also satisfy the minimal right property. A solution concept F is said to be invariant under strategic equivalence if $F(kv + a) = kF(v) + a$ for all games v, $k \in \mathbb{R}_+$ and $a \in \mathbb{R}^n$, where $(kv + a)(S) := kv(S) + \sum_{i \in S} a_i$. It follows that the recursive completion rule and the rule of Aumann and Maschler satisfy the minimal right property as well.

The second property is a well known property called *symmetry.*

A division rule f is called symmetric if $d_i = d_j$ implies $f_i(E;\underset{\sim}{d}) = f_j(E;\underset{\sim}{d})$. This property is a weak form of the anonymity property which requires that f is a symmetrical function of $d_1, \ldots, d_n$. Cf. Moulin [5]. From the definition it follows that the AP-rule satisfies the anonymity property and hence the symmetry property.

Suppose now that one of the claimants dies leaving behind parts of his claim to different heirs. These heirs become new claimants, each one claiming the part of the original claim he received. Together their claims sum up to the original claim. Our third property states that if the bankruptcy problem is a zero-normalized simple claims problem nothing changes for the other claimants. This property is called the *additivity of claims* property. A division rule f is said to satisfy the additivity of claims property if for every zero-normalized simple claims bankruptcy problem $(E;\underset{\sim}{d}) = (E;(d_1, d_2, \ldots, d_n))$ which changes by splitting up d_i in $d_{i,1}, d_{i,2}, \ldots, d_{i,k}$ into a bankruptcy problem

$$(E;\underset{\sim}{d}') = (E;(d_1, \ldots, d_{i-1}, d_{i,1}, d_{i,2}, \ldots, d_{i,k}, \ldots, d_n))$$

we have

$$f_j(E;\underset{\sim}{d}) = f_j(E;\underset{\sim}{d}') \quad \text{for every } j \in N - i.$$

This last property can also be found in O'Neill [6] where it is defined for all simple claims problems and where it is called strategy-proofness. Together with four other axioms it is used there to characterize the proportional division rule. In the context of cost allocation it has also been defined and used by Banker [2] to characterize the proportional cost allocation rule. One of the axioms O'Neill uses is an axiom requiring continuity of the division rule in at least one point for all coordinates while Banker characterizes the rule in the case that it allocates cost in rational numbers only and remarks that an axiom requiring continuity is needed in order to relax this assumption. In the proof of the next theorem the proportional division rule is characterized axiomatically by the last two properties, efficiency and individual rationality without the use of a continuity axiom. The theorem states that the AP-rule is characterized by the three properties introduced in this section plus the fact that it is a game theoretic rule.

Theorem 7: The AP-rule is the unique game theoretic division rule for bankruptcy problems which satisfies (1) the minimal right property, (2) the symmetry property, and (3) the additivity of claims property.

Proof: We already saw that the AP-rule is a game theoretic rule and that it satisfies the properties (1) and (2). That it also satisfies the third property follows from the remark in Section 2 that for a zero-normalized simple claims bankruptcy problem the AP-rule reduces to the proportional division rule. Let f be a game theoretic division rule for bankruptcy problems which satisfies these three properties. Let $(E;\underline{d})$ be a zero-normalized simple claims bankruptcy problem. In the following we will denote $f_j(E;\underline{d})$ by f_j for all $j \in N$. For an $i \in N$ with $d_i > 0$ let $k \in \mathbb{N}$ be such that $d_i \cdot k^{-1} \leqslant d_j$ whenever $d_j > 0$. We consider the bankruptcy problem where claimant i is replaced by k claimants all with the same claim equal to $d_i \cdot k^{-1}$. By property (2) and (3) and efficiency these new claimants receive the amount $f_i \cdot k^{-1}$. Let $j \in N$, we consider now the bankruptcy problem arising from the last problem by replacing claimant j by $q_k := [d_j \cdot d_i^{-1} \cdot k]$ claimants with claim $d_i \cdot k^{-1}$ and one claimant with claim $r_{j,k} = d_j - q_k \cdot d_i \cdot k^{-1}$. Here $[x]$ stands for the largest integer smaller than or equal to x. So $d_j \cdot d_i^{-1} \cdot k - 1 < q_k \leqslant d_j \cdot d_i^{-1} \cdot k$. Because of symmetry all the claimants with claim $d_i \cdot k^{-1}$ receive $f_i \cdot k^{-1}$. Let $f_{j,k}$ denote the amount the claimant $r_{j,k}$ receives. We have $r_{j,k} < d_i \cdot k^{-1}$. Because of this inequality, property (3) and individual rationality it follows that $0 \leqslant f_{j,k} \leqslant f_i \cdot k^{-1}$. Because of property (3) we also have $f_j = q_k \cdot f_i \cdot k^{-1} + f_{j,k}$ and hence $d_j \cdot d_i^{-1} - k^{-1} \leqslant f_j \cdot f_i^{-1} \leqslant d_j \cdot d_i^{-1} + k^{-1}$. Because we can go through the above procedure for every k which is big enough we get $f_j \cdot f_i^{-1} = d_j \cdot d_i^{-1}$. From this it follows that the division rule f is the same as the proportional division rule for zero-normalized simple claims bankruptcy problems and hence f is equal to the AP-rule for such problems. Let $(E;\underline{d})$ be a general bankruptcy problem. Then $f(E;\underline{d}) = \underline{m} + f(E - m(N); \underline{d} - \underline{m}) = \underline{m} + f(E - m(N); \hat{E}' \wedge (\underline{d} - \underline{m})$, where $\hat{E}'$ is the vector in $\mathbb{R}^n$ with all coordinates equal to $E - m(N)$ and where the first equality follows from property (1) and the second from the fact that f is a game theoretic division rule. The problem $(E - m(N); \hat{E}' \wedge (\underline{d} - \underline{m})$ is a zero-normalized simple claims bankruptcy problems so $\underline{m} + f(E - m(N); \hat{E}' \wedge (\underline{d} - \underline{m})) = \underline{m} + t(E - m(N); \hat{E}' \wedge (\underline{d} - \underline{m})) = \underline{m} + t(E - m(N); \underline{d} - \underline{m}) = t(E; \underline{d})$. Here the second and third equalities follow from the fact that the AP-rule is a game theoretic division rule and from the minimal right property respectively. We see that $f(E; \underline{d}) = t(E; \underline{d})$ for all bankruptcy problems $(E; \underline{d})$ and hence the AP-rule is the unique game theoretic division rule for bankruptcy problems which satisfies (1), (2) and (3).

A property usually required for a division rule is the *monotonicity* property. This property states that if the amount to be distributed becomes greater this may not be disadvantageous for any claimant. A division rule f is said to satisfy the monotonicity property if

$$E' \geqslant E \text{ implies } f_i(E'; \underline{d}) \geqslant f_i(E; \underline{d}) \quad \text{for all } i \in N. \tag{*}$$

We will show that the AP-rule is montonic. Because the AP-rule is self-dual it suffices to show that (*) holds for $\frac{1}{2} D \geqslant E' \geqslant E$. We will do this by calculating the partial derivatives $\frac{\partial t_i(E; \underline{d})}{\partial E}$ on the open intervals $I_0^o, I_j^o, \ldots, I_n^o$ with $I_0, I_1, \ldots, I_n$ as defined in Section 2. From the formulas given there we derive:

Case 1: $d_n \leqslant \frac{1}{2} D$. Then

$$\frac{\partial t_i(E; \underline{d})}{\partial E} = \begin{cases} (d_i \sum\limits_{k \leqslant j} d_k \cdot (\sum\limits_{k \leqslant j} d_k + (n-j)E)^{-2} & \text{for } E \in I_j \text{ with } i \leqslant j \\ \\ (2E \sum\limits_{k \leqslant j} d_k + (n-j)E^2) \cdot (\sum\limits_{k \leqslant j} d_k + (n-j)E)^{-2} & \text{for } E \in I_j \text{ with } i > j \end{cases}$$

Case 2: $d_n \geqslant \frac{1}{2} D$. Then

$$\frac{\partial t_i(E; \underline{d})}{\partial E} = \begin{cases} \text{the same as in case (1)} & \text{for } E \in I_j,\ j \leqslant n-1 \\ 0 \quad \text{for } E \in I_n,\ i \neq n \\ 1 \quad \text{for } E \in I_n,\ i = n \end{cases}$$

We see that $t_i(E; \underline{d})$ is not differentiable in the points $d_1, d_2, \ldots, d_{n-1}, d_n \wedge d(N-n)$.

From the fact that $\frac{\partial t_i(E;\underline{d})}{\partial E} \geqslant 0$ on the open intervals it follows that $t_i(E;\underline{d})$ is monotonic on these intervals. From the fact that $t_i(E;\underline{d})$ is a continuous function of E (Section 2) it follows that $t_i(E;\underline{d})$ is monotonic on $[0, {}^1/_2 D]$ and from the self-duality follows monotonicity on $[0, D]$.

Another property the AP-rule satisfies is the homogeneity property. A division rule f is homogeneous if $f(\lambda E; \lambda \underline{d}) = \lambda f(E;\underline{d})$ for all $\lambda > 0$ and bankruptcy problems $(E;\underline{d})$. The meaning of this property is that if the estates and the claims are measured in another monetary unit the division of the estate doesn't really change, the amounts appointed to every claimant are merely expressed in the new monetary unit too.

Aumann and Maschler [1] introduced a property they call *order-preservingness.* We will show that the AP-rule satisfies this property. A division rule f is said to be order preserving if

$$0 \leqslant f_1(E;\underline{d}) \leqslant \ldots \leqslant f_n(E;\underline{d}) \quad \text{and} \quad 0 \leqslant d_1 - f_1(E;\underline{d}) \leqslant \ldots \leqslant d_n - f_n(E;\underline{d})$$

for every bankruptcy problem $(E;\underline{d})$. A division rule which is order preserving appoints to a person with a higher claim an award and a loss that is not less than that appointed to a person with a lower claim. As we saw in Section 4 $t(E;\underline{d}) = \underline{m} + \lambda(\underline{M} - \underline{m}) = \lambda \underline{M} + (1-\lambda)\underline{m}$ with $0 \leqslant \lambda \leqslant 1$. We recall that $m_i = (E - d(N-i))_+$ and $M_i = d_i \wedge E$. From this it follows that $0 \leqslant m_1 \leqslant m_2 \ldots \leqslant m_n$ and $0 \leqslant M_1 \leqslant M_2 \leqslant \ldots \leqslant M_n$. With the above expression for $t(E;\underline{d})$ it follows that $t_1(E;\underline{d}) \leqslant \ldots \leqslant t_n(E;\underline{d})$. Further we have $d_i - t_i(E;\underline{d}) = t_i(D-E;\underline{d})$ and hence $0 \leqslant d_1 - t_1(E;\underline{d}) \leqslant \ldots \leqslant d_n - t_n(E;\underline{d})$. So the AP-rule is order-preserving.

The proportional rule satisfies all the properties mentioned in this section except the minimal right property. This is why the proportional rule is not invariant under the adding of constants to individual debts as the example in the introduction illustrates.

We also saw that the proportional rule is not a game theoretic rule. The adjusted proportional rule shows that after some modifications we can still obtain the idea of proportional division in a game theoretical context.

References

1. Aumann, RJ, Maschler M (1985) Game theoretic analysis of a bankruptcy problem from the Talmud. J Economic Theory 36:195–213
2. Banker RD (1981) Equity considerations in traditional full cost allocation practices: an axiomatic perspective. In: Moriarity S (ed) Joint cost allocations. University of Oklahama, pp 110–130

3. Driessen TSH, Tijs SH (1985) The τ-value, the core and semiconvex games. Int J Game Theory 14:229–248
4. Gillies DB (1953) Some theorems on n-person games. PhD Thesis, Princeton University Press
5. Moulin H (1985) Equal or proportional division of a surplus, and other methods. Working paper, Virginia polytechnic institute and state university
6. O'Neill B (1982) A problem of rights arbitration from the Talmud. Math Soc Sci 2:345–371
7. Shapley LS (1971) Cores of convex games. Int J Game Theory 1:11–26
8. Tijs SH (1981) Bounds for the core and τ-value. In Moeschlin O, Pallaschke D (eds) Game theory and mathematical economics. North-Holland Publ Cie, pp 123–132
9. Tijs SH, Driessen TSH (1986) Extensions of solutions concepts by means of multiplicative ϵ-tax games. Math Soc Sci 12:9–20
10. Tijs SH, Lipperts FAS (1982) The hypercube and the core cover of n-person cooperative games. Cahiers Centre D'Études Rech Opér 24:27–37
11. Young HP Taxation and bankruptcy. Mimeo, University of Maryland
12. Young HP Taxation, bankruptcy and estate allocation, Mimeo. University of Maryland

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