CHAPTER 5

Infinite-horizon models of bargaining with one-sided incomplete information

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5.1 Introduction

Bargaining occurs whenever two or more parties can share a surplus if an agreement can be reached on how the surplus should be shared, with a status-quo point that will prevail in the event of disagreement. Until recently, bargaining has been analyzed using the cooperative approach, which typically consists of specifying a set of axioms that the bargaining outcome should satisfy, and then proving that a solution satisfying these axioms exists and is unique. More recently, a second approach has emerged, which relies on the theory of noncooperative games. The typical paper of this type specifies a particular extensive form for the bargaining process, and solves for the noncooperative equilibria. Thus, the noncooperative approach replaces the axioms of the cooperative approach with the need to specify a particular extensive form.

Although this chapter is based on the noncooperative approach, which we believe has considerable power, we should point out that the reliance of the noncooperative approach on particular extensive forms poses two problems. First, because the results depend on the extensive form, one needs to argue that the chosen specification is reasonable – that it is a good approximation to the extensive forms actually played. Second, even if one particular extensive form were used in almost all bargaining, the analysis is incomplete because it has not, at least to-date, begun to address the question of why that extensive form is used. This chapter will consider

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the first point of extending the class of bargaining games for which we have solutions. The second and harder problem, we will leave unresolved.

Fudenberg and Tirole (1983) analyzed the simplest model of noncooperative bargaining that captures bargaining's two key aspects: Bargaining involves a succession of steps, and the bargainers do not know the value to others of reaching an agreement. Their model had only two periods, and only two possible valuations for each player. In each period, one player (the "seller") makes an offer, which the other player (the "buyer") can either accept or reject. Each player is impatient and prefers an agreement today to the same agreement tomorrow. The simplicity of the model permitted a complete characterization of the equilibria. Several common perceptions about the effects of parameter changes on bargaining outcomes were found to be suspect.

However, finite-horizon models are inevitably contrived: Why should negotiations be constrained to end after a fixed number of periods? Moreover, the specification of two-point distributions for the valuations of the bargainers is special. Finally, the assumption that the seller makes all the offers can also be questioned. The present chapter investigates the effect of relaxing the first two assumptions, and discusses relaxing the third, in the case of one-sided incomplete information. The seller's valuation is common knowledge, and only the buyer's valuation is private information.

We find that, as long as the seller makes all of the offers, the conclusions of Fudenberg and Tirole for the one-sided case are essentially unchanged by allowing an infinite bargaining horizon and general distributions: An equilibrium exists and is essentially unique, and the offers decline over time. Although many infinite-horizon games have multiple equilibria, this uniqueness result should not be surprising, since (1) if an agreement occurs, it occurs in finite time; and (2) the seller's offers convey no information because the seller's valuation is common knowledge. "Supergame"-type "punishment strategies" are not equilibria in bargaining games, because a bargainer cannot be punished for accepting the "wrong" offer. Once an offer is accepted, the game ends. The fact that offers decrease over time is similarly intuitive. The seller becomes increasingly pessimistic as each offer is refused. However, neither uniqueness nor decreasing offers holds with two-sided incomplete information, as Fudenberg and Tirole demonstrated in a two-period model. With two-sided incomplete information, the buyer's beliefs about the seller depend on the seller's offers. In particular, we must specify what the buyer infers from an offer to which the equilibrium strategies assign zero probability. In such circumstances, Bayes' rule is inapplicable, and many different inferences can be specified. This leeway in choosing the buyer's "conjectures" generates many equilibria.

The conclusions of noncooperative models of bargaining depend not only on the extensive form chosen but also, of course, on the specification of the payoffs. In particular, models of sequential bargaining assume some sort of impatience on the part of the players. Although most work has modeled these costs as arising from discounting future payoffs, a few studies have modeled impatience as fixed per-period bargaining costs. We examine the fixed per-period cost specification, and explain why that specification may lead to implausible equilibria.

The chapter is organized in the following manner. Section 5.2 reviews some previous work on infinite-horizon bargaining with incomplete information. Section 5.3 proves that if the seller makes all of the offers, an equilibrium exists, and is unique if it is common knowledge that the buyer's valuation strictly exceeds the seller's. This section also investigates the existence of differentiable equilibria. Section 5.4 discusses the case in which the buyer and the seller alternate making offers, and Section 5.5 discusses the specification of the costs of bargaining. Sections 6 and 7 offer some brief thoughts about the choice of the extensive form and the specification of uncertainty.

5.2 Infinite-horizon bargaining under incomplete information: The state of the art

Here, we review briefly the models of Cramton (1983*a*), Sobel and Takahashi (1983), and Rubinstein (1985). (Perry (1982*a*) will be discussed in Section 5.5.) Very schematically, we can distinguish the following steps involved in building these models.

Specification of an extensive form. Cramton (1983*a*) and Sobel and Takahashi (1983) assume that the seller makes all of the offers, at the rate of one per period. Bargaining stops only when the buyer accepts the current offer, then trade takes place at the agreed-upon price. Rubinstein (1985), on the other hand, assumes that the traders take turns making offers. These two representations have a number of features in common. First, the extensive form is given from the outside. As indicated earlier, we have little to say about this assumption. Second, traders are not allowed to bargain with other traders; or, equivalently, bargaining with a given trader is not affected by the potential of bargaining with another trader. Actually, in the three contributions mentioned, traders will never quit the bargaining process. Not only are they prevented from bargaining with other parties, but their costs of bargaining take the form of discounting, and so they have no incentive to stop bargaining with their (unique) partner.

Specification of the payoff structure. We just mentioned that in the three models, the cost of disagreement comes from discounting. Let δ_B and δ_S denote the buyer's and the seller's discount factors, respectively. Typically, if the buyer has valuation b for the object and the seller has valuation or production cost s, agreement at price p at time t yields utilities $\delta_B^t(b-p)$ to the buyer and $\delta_S^t(p-s)$ to the seller (Cramton). This framework is rich enough to include two interesting cases: (1) the production cost is already incurred (the seller owns the object before bargaining, that is, $s=0$ (Sobel and Takahashi)); and (2) the traders bargain on how to divide a pie of a given size (Rubinstein). However, it does not formalize the cases in which bargaining may stop because of disagreement; for example, if $V_S(t,s)$ denotes time-t valuation of a seller with cost s when he quits the bargaining process at time $(t-1)$ to start bargaining with someone else at time t, the seller's payoff is $\delta_S^t V_S(t,s)$.

Specification of the prior information structure. Sobel and Takahashi assume that the asymmetric information concerns the buyer's valuation, which is known only to the buyer. All the rest is common knowledge. Rubinstein assumes instead that one of the traders' discount factors is unknown. Cramton considers two-sided incomplete information: Both the buyer and the seller have incomplete information about the other party's valuation (or production cost).

Solution. The three papers look for special types of equilibria instead of characterizing the equilibrium set. We give only a very brief description of the restrictions used because these are clearly detailed by the authors and they differ greatly. Sobel and Takahashi look for an equilibrium that is the limit of finite-horizon equilibria.

To this purpose, they compute explicitly a sequence of finite-horizon equilibria in a simple case, and derive a limit. Rubinstein imposes some monotonicity conditions on off-the-equilibrium-path conjectures; he also rules out mixed strategies despite using a two-point distribution for the private information. And Cramton looks for equilibria in which the seller at some point of time reveals his information so that the bargaining game becomes a one-sided incomplete-information game, for which he takes the Sobel–Takahashi solution.

5.3 Seller makes the offers

We now consider a model in which the seller makes all of the offers and has incomplete information about the buyer's valuation. The seller has production cost $s = 0$ (the object has already been produced and the seller is not allowed to bargain with any other buyer). He has discount factor δ_S, which is common knowledge. The buyer's valuation, b, is known only to him. The seller has a smooth prior cumulative-distribution function $F(b)$, with bounded density $f(b)$, with $0 < \underline{f} \leq f(b) \leq \bar{f}$, concentrated on the interval $[\underline{b},\bar{b}]$, where $\underline{b} \geq 0$. The buyer's discount factor, δ_B, is common knowledge. A *perfect Bayesian equilibrium* is a history-contingent sequence of the seller's offers (p_t), of the buyer's acceptances or refusals of the offers, and of updated beliefs about the buyer's valuation satisfying the usual consistency conditions (i.e., the actions must be optimal given the beliefs, and the beliefs must be derived from the actions by Bayes' rule).

The general case

We will show that an equilibrium exists and that it is unique if $\underline{b}$ *strictly* exceeds s. We begin with two lemmas that hold in either case.

Lemma 1 (Successive skimming). In equilibrium and at any instant, the seller's posterior about the buyer's valuation is the prior truncated at some value b^e: $F(b)/F(b^e)$ for $b \leq b^e$, 1 for $b \geq b^e$.

Proof. Lemma 1 follows from the fact that for any time τ less than or equal to t, if a buyer with valuation b is willing to accept an offer p_τ, then a buyer with valuation $b' > b$ accepts the offer with probability 1. To prove the latter fact, notice that since b accepts p_τ,

$$b - p_\tau \geq \delta_B V_B(b,H_\tau),$$

where $V_B(b,H_\tau)$ is the time-$(\tau + 1)$ valuation of a buyer with valuation b when the history of the game up to and including τ is H_τ. Let us show that

$$b' - p_\tau > \delta_B V_B(b',H_\tau),$$

so that a buyer with valuation b' accepts p_τ with probability 1. Since from time $(\tau + 1)$ on, buyer b can always adopt the optimal strategy of buyer b', that is, accept exactly when buyer b' accepts, then

$$V_B(b',H_\tau) - V_B(b,H_\tau) \leq \sum_{u=0}^{\infty} \delta_B^u \alpha_{\tau+1+u}(b',H_\tau)(b' - b),$$

where u is the index of time periods and $\alpha_{\tau+1+u}(b',H_\tau)$ is the probability conditional on H_τ that agreement is reached at time $(\tau + 1 + u)$ and the buyer uses buyer b''s optimal strategy from time $(\tau + 1)$ on. Therefore,

$$V_B(b',H_\tau) - V_B(b,H_\tau) \leq b' - b,$$

and the conclusion follows by a simple computation.

Lemma 1 implies that the seller's posterior at any instant can be characterized by a unique number, the buyer's highest possible valuation b^e. By abuse of terminology, we will call b^e the posterior.

Lemma 2. The seller never (i.e., in no subgame) charges a price below $\underline{b}$.

Proof. We know that the seller's equilibrium valuation must be nonnegative, and that expected equilibrium surplus cannot exceed $\bar{b}$, so that the expectation over all possible types of the buyer's equilibrium valuation cannot exceed $\bar{b}$. Moreover, following the proof of lemma 1, we can show that the buyer's equilibrium valuation is nondecreasing and has modulus of continuity no greater than 1; that is, if $b' > b$, then

$$V_B(b') \leq V_B(b) + b' - b$$

(because the buyer of type b can always play as though he were type b'). Since the buyer's equilibrium valuation is nondecreasing and does not exceed $\bar{b}$ in expected value, it must be that $V_B(\underline{b}) \leq \bar{b}$, and so $V(\bar{b}) \leq 2\bar{b} - \underline{b}$. This implies that all buyers accept any price below $(\underline{b} - \bar{b})$, and therefore the seller would never charge such prices. Knowing that the lowest possible price is $(\underline{b} - \bar{b})$, all buyers accept prices such that $\underline{b} - p \geq \delta_B[\underline{b} - (\underline{b} - \bar{b})]$, or $p \leq \underline{b} - \delta_B\bar{b}$. Proceeding as before, this implies that for every positive n, all prices below $\underline{b} - \delta_B^n\bar{b}$ are accepted by all buyers, and thus the seller never charges less than $\underline{b}$.

Now we specialize to the case $\underline{b} > 0$. The next lemma shows that if the posterior is sufficiently low the seller charges $\underline{b}$, and uses this fact to establish that the rate at which the seller's posterior decreases is uniformly bounded below over all subgames.

Lemma 3. If $\underline{b} > 0$, there exists N^* such that in all equilibria with probability 1, an offer is accepted in or before period $(N^* + 1)$.

Proof. First, we show that there exists a b^* such that if the seller's posterior is below b^*, he charges $\underline{b}$. We do this by demonstrating that such a b^* exists if the buyer plays myopically and accepts all prices less than his valuation. If a seller chooses to jump down to $\underline{b}$ against a myopic buyer, he will do so against a nonmyopic one, since nonmyopic buyers are less likely to accept prices above $\underline{b}$ but just as likely to accept $\underline{b}$ (because no lower price is ever charged).

Thus, we consider the maximization problem of a seller facing a myopic buyer, when the seller's posterior is b^e. The seller's return to charging price p is at most

$$M(p) = \left[(F(b^e) - F(p))p + \delta_S \int_{\underline{b}}^{p} sf(s)\, ds \right] \Big/ F(b^e).$$

Taking the derivative with respect to p, we have

$$M'(p) \propto F(b^e) - F(p) + pf(p)(\delta_S - 1).$$

As $f(p)$ is bounded below, for b^e sufficiently near b, $M'(p)$ is negative for all p between b^e and $\underline{b}$. Quantity $M(p)$ overstates the "continuation" payoff if p is refused, and so when b^e is sufficiently small, a seller with posterior b^e would charge $\underline{b}$ if the buyer was myopic, and a fortiori would do so against nonmyopic buyers. This establishes the existence of the desired b^*.

Next, we show that there exists N^* such that all equilibria end in $(N^* + 1)$ periods. We do this by showing that in N^* periods, the seller's posterior drops below b^*. We claim that there are constants k and w such that for all initial beliefs $b^e > b^*$, the seller's posterior is no higher than $\max\{\underline{b}, b^e - w\}$ after k additional periods. Assume not – then

$$V_S \leq \bar{b} \left[\frac{\bar{f} w}{F(b^*)} \right] + \delta_S^k \bar{b},$$

where V_S is the seller's valuation and the term in brackets is an upper bound on the probability that an offer is accepted in the first k periods. But for w sufficiently small and k sufficiently large, the right-hand side of this equation is less than $\underline{b}$. Thus, we can define N^* as

$$\left[\left[\frac{k(\bar{b} - b^*)}{w} \right]_{\text{int}} + 1 \right],$$

and all equilibria must end in $(N^* + 1)$ periods.

The proof of lemma 3 makes clear the importance of our assumption that $\underline{b} > 0$. With $\underline{b} > 0$, the potential surplus the seller might hope to extract eventually becomes insignificant compared to the "sure thing" of $\underline{b}$, and thus when the posterior is less than b^*, the seller settles for $\underline{b}$. The second part of the lemma in turn relies crucially on the first: Without the "termination condition" at b^*, the rate at which the seller's posterior fell would not be uniformly bounded below, but would instead decrease with the seller's posterior. When $\underline{b}$ is zero, the equilibria will clearly not be of bounded length, because the seller will never charge zero. This explains why we prove uniqueness only for $\underline{b} > 0$; existence will not be a problem.

Now, we can characterize the unique equilibrium when $\underline{b} > 0$. Let $\beta(p_t, H_{t-1})$ be the least (inf) value of any buyer to buy in period t. An equilibrium is called "weak-Markov" if $\beta(p_t, H_{t-1})$ depends only on p_t (which implies that $V_S(b, H_{t-1})$ depends only on b). Let $\sigma(b^e, H_{t-1})$ be the seller's probability distribution over prices in period t. An equilibrium will be called "strong-Markov" if it is weak-Markov and in addition σ depends only on b^e. In a strong-Markov equilibrium, players' actions depend solely on the "relevant" part of the history, namely, the seller's beliefs and the current offer.

Strong-Markov equilibria do not necessarily exist, as was discovered by Fudenberg and Tirole (1983) in a two-period model with a discrete distribution over the buyer's valuation, and by Kreps and Wilson (1982*a*) in their treatment of the chain-store paradox. The same forces lead to nonexistence here. Strong-Markov equilibria fail to exist in general, because it may be necessary for the probability of acceptance, $\beta(p)$, to be constant over some interval. The seller's posterior will be the same after any offer in such an interval is refused, but in order for $\beta(p)$ to be constant, the seller's next price will have to depend on the current one. As this discussion suggests, a necessary and sufficient condition for a weak-Markov equilibrium to be strong-Markov is that β be strictly increasing.

We will show that if $\underline{b} > 0$, the unique equilibrium is weak-Markov. The weak-Markov property is unsurprising given that the game ends in finite time and that the seller's offers convey no information. When $\underline{b} = 0$, bargaining can continue indefinitely and we have not been able to show that equilibria must be weak-Markov.

Our proof of uniqueness proceeds inductively. We start by solving what we will call the "one-period" game, in which we impose the constraint that the seller charge $\underline{b}$. Recall from the proof of lemma 3 that if the seller's posterior is sufficiently low (less than b^*), then this constraint is not binding because the seller chooses $\underline{b}$ when he is sufficiently pessimistic. In fact, there exists b^2 that is the largest value of b^e such that the seller charges $\underline{b}$ when his posterior falls below b^2. We then proceed to "work backward" on both the number of "periods remaining" and the seller's posterior simultaneously. Let p_1 be the highest price that buyer b^2 will accept if he expects the price to be $\underline{b}$ next period. In the "two-period" game, the seller is constrained not to charge prices above p_1, and thus the game indeed ends in two periods. Then, we solve for the seller's optimal action in the two-period game. The key to the proof is that if $b^e \leq b^2$, the seller will choose to charge $\underline{b}$ in the two-period game, and indeed in <u>any</u> equilibrium the seller must charge $\underline{b}$ when $b^e \leq b^2$. We then proceed to the "three-period" game, and so on. Because we know that all equilibria end with probability 1 by time $(N^* + 1)$, we need only work backward $(N^* + 1)$ steps. At that point, we will have worked out unique strategies such that (1) the buyer's decisions are best responses to the strategy of the seller, (2) the seller's strategy is an optimal response under the constraint that the seller's first-period price be less than p^{N^*}, and (3) the game ends by $(N^* + 1)$. This immediately implies that at most one equilibrium exists. We claim that it also establishes existence. The only way that the computed strategies could fail to be an equilibrium would be if the first-period constraint on the seller's action were binding. Holding the buyer's strategy fixed, let us consider the seller's optimization. The seller's choice set is compact (in the product topology) and his expected payoff is continuous; therefore, an optimal choice exists. The argument of lemma 3 shows that the seller's optimal choice must terminate the game by $(N^* + 1)$, and so the first-period constraint cannot bind.

After this lengthy overview, we now state and prove our main result. The statement is only generic, since the seller may have several optimal first-period offers.

Proposition 1. If $\underline{b} > 0$, an equilibrium exists and is generically unique. The equilibrium is weak-Markov; it is strong-Markov if and only if the buyer's reservation function $\beta(p)$ is strictly increasing.

Proof. See Appendix 1.

We now assume that there is "enough concavity" in the problem that the seller's optimal action at each instant is unique, in order to give a simpler proof of uniqueness. Moreover, we can show that the equilibrium is (strong-)Markov. The single-valuedness assumption permits us to use a simple dominance argument to show that when the seller's posterior is below b^n, his price is low enough that next period his posterior will be below b^{n-1}.

To state the single-valuedness assumption, (S), we need the following notation:

$$\beta^2(p) \equiv \frac{p - \delta_B \underline{b}}{1 - \delta_B}$$

$$W_S^2(b^e) \equiv \max_p \{[F(b^e) - F(\beta^2(p))]p + \delta_S F(\beta^2(p))\underline{b}\},$$

where $\beta^2(p)$ is the value of the buyer who is indifferent between paying p now or paying $\underline{b}$ next period, and $W_S^2(b^e)$ is the seller's maximal payoff when he is constrained to change $\underline{b}$ next period multiplied by the probability that the seller's posterior is below b^e. In other words, we work with "unconditional probabilities" rather than conditional ones. This is a renormalization and does not change the seller's behavior.

Let $\sigma^2(b^e)$ denote the arg max, which we assume to be unique. Note that σ^2 increases with b^e, and let b^2 be uniquely defined by

$$b^2 = \max\{b^e \le \bar{b} \mid W_S^2(b^e) = F(b^e)\underline{b}\}.$$

Quantity b^2 is the highest posterior for which the seller would choose to charge $\underline{b}$ now, given that $\underline{b}$ will be charged next period.

Let $\beta^n(p)$ be such that

$$\beta^n(p) - p = \delta_B[\beta^n(p) - \sigma^{n-1}(\beta^n(p))].$$

$\beta^n(p)$ is well defined and unique if σ^{n-1} is an increasing function. Consider

$$W_S^n(b^e) \equiv \max_p\{[F(b^e) - F(\beta^n(p))]p + \delta_S W_S^{n-1}(\beta^n(p))\}.$$

Let $\sigma^n(b^e)$ denote the arg max, which we assume to be unique. This is assumption (S).

Assumption (S). For all n, $\sigma^n(b^e)$ is single valued.

We have verified that assumption (S) is satisfied for a uniform distribution. The assumption is quite strong; we use it only to be able to provide a simpler proof of our result.

Under (S), $\sigma^n(b^e)$ is an increasing function of b^e. Then, b^n is uniquely defined by

$$b^n = \max\{b^e \le \bar{b} \mid W_S^n(b^e) = W_S^{n-1}(b^e)\}.$$

Proposition 1′. Under (S), the equilibrium is generically unique and is (strong-)Markov.

Proof. Lemma 3 proved that there exists b^* close to $\underline{b}$ such that if the posterior b^e belongs to $[\underline{b},b^*]$, the seller charges $\underline{b}$ whatever the history. We now proceed by upward induction on b^e.

Lemma 4. If $b_t^e \in [\underline{b},b^2)$, then $\sigma_t(H_t) = \underline{b}$.

Proof. Choose ϵ_1 sufficiently small such that for every $b \in [b^*,b^2]$,

$$(F(b+\epsilon_1) - F(b))(b+\epsilon_1) + \delta_S F(b)\underline{b} < F(b+\epsilon_1)\underline{b}$$

and

$$b^* + \epsilon_1 < b^2.$$

We claim that if at time t, for some history, b_t^e belongs to $(b^*,b^*+\epsilon_1]$, the seller charges $\underline{b}$. He can guarantee himself $\underline{b}$ by offering $\underline{b}$.

Assume that b^e_{t+1} belongs to $(\underline{b},b^*]$. Then, $p_{t+1} = \underline{b}$, and the buyer accepts p_t if and only if his valuation exceeds $\beta^2(p_t)$. Since the seller will change to b next period and $b^e \leq b^2$, the seller charges b this period from the definition of b^2. More generally, the seller will never offer a price leading to $b^e_{t+\tau}$ in $(\underline{b},b^*]$. Alternatively, by offering prices leading to posteriors in $(b^*,b^* + \epsilon_1]$, he obtains at most

$$(F(b^* + \epsilon_1) - F(b^*))(b^* + \epsilon_1) + \delta_S F(b^*)\underline{b} < F(b^* + \epsilon_1)\underline{b}.$$

Therefore, for any history at time t such that $b^e_t \in (\underline{b},b^* + \epsilon_1]$, $p_t = \underline{b}$, $\beta^t = \beta^2$, and $W_S(b^e_t) = W^1_S(b^e_t)$. The same is true by induction for any $b^e_t \in [\underline{b},b^2)$.

Lemma 5. If $b^e_t \in (b^2,b^3)$, then $\sigma_t(H_t) = \sigma^2(b^e_t)$.

Proof. Assume that $b^e_t \in (b^2,b^3)$, and define $\epsilon_2 > 0$ sufficiently small that for every $b \in (b^2,b^3]$,

$$(F(b + \epsilon_2) - F(b))(b + \epsilon_2) + \delta_S W^2_S(b + \epsilon_2) < W^2_S(b + \epsilon_2)$$

and

$$b^2 + \epsilon_2 < b^3.$$

We claim that if at time t, for some history, b^e_t belongs to $(b^2,b^2 + \epsilon_2]$, the seller charges $\sigma^2(b^e_t)$. The seller can guarantee himself $W^2_S(b^e_t)$, as buyers with valuation exceeding $\beta^2(p_t)$ accept p_t since they will never face a better offer than $\underline{b}$. Can the seller do better? If he charges p_t such that $b^e_{t+1} < b^2$, then only buyers with valuations exceeding $\beta^2(p_t)$ accept the offer since they expect $\underline{b}$ at time $(t + 1)$.

More generally, if $p_{t+\tau}$ is accepted by buyers with a valuation less than b^2, the seller obtains at most $W^2_S(b^e_{t+\tau})$. Therefore, an upper bound on what he obtains when his offer leads to a posterior $b^e_{t+1} \geq b^2$ is $(F(b^e_t) - F(b^2))b^e_t + \delta_S W^2_S(b^e_t)$, and hence the seller will not make an offer such that $b^e_{t+1} \geq b^2$. We conclude that if $b^e_t \in (b^2,b^2 + \epsilon_2]$, $p_t = \sigma^2(b^e_t)$, $b^{t+1}_e = \beta^2(p_t)$, and $W_S(b^e_t) = W^2_S(b^e_t)$ on the equilibrium path. The same reasoning applies for $b^e_t \in (b^2 + \epsilon_2,\ b^2 + 2\epsilon_2]$, and so on, until $b^e_t = b^3$.

Let us now choose ϵ_3 such that for every $b \in (b^3,b^4]$,

$$(F(b + \epsilon_3) - F(b))(b + \epsilon_3) + \delta_S W^3_S(b + \epsilon_3) < W^3_S(b + \epsilon_3)$$

and

$$b^3 + \epsilon_3 < b^4.$$

The proof that the seller charges $\sigma^3(b^e_t)$ when $b^e_t \in (b^3,b^4]$ is the same as the previous one. That the seller can guarantee himself $W^3_S(b^e_t)$ is slightly more complicated to demonstrate. It suffices to show that when the seller charges $p_t = \sigma^3(b_t^e)$, a buyer with valuation $\beta^3(p_t)$ accepts it. Imagine that this buyer refuses. Then, $b_{t+1}^e > \beta^3(p_t)$, which implies that $p_{t+1} \geq \sigma^2(b_{t+1}^e) > \sigma^2(\beta^3(p_t))$. Hence, buyer $\beta^3(p_t)$ will not buy at time $(t+1)$, since $\beta^3(p_t) - p_t < \delta_B(\beta^3(p_t) - p_{t+1})$ would contradict the definition of β^3. Similarly, he would not buy later on, and hence he accepts p_t now.

The rest of the proof is by induction on n. Lemma 3 guarantees that this induction takes at most $(N^* + 1)$ steps. Finally, the equilibrium is (strong)-Markov since, by construction, p_t depends only on the posterior b_t^e.

We would prefer not to invoke the restriction that $\underline{b} > 0 = s$. One might expect that the buyer's valuation could sometimes be less than s and that such buyers would not enter the bargaining game, but any buyer whose valuation exceeds s would enter, and thus effectively $\underline{b} = s$. For this case, we can prove that an equilibrium exists by considering a sequence of games with $\underline{b}^n \to s$, showing that there is a limit point of the associated equilibria, and further that this limit is an equilibrium. With $\underline{b} = s$, the seller will never choose to offer price $\underline{b}$, and so bargaining can continue indefinitely. This lack of an "endpoint" has prevented us from establishing uniqueness for this case.

Proposition 2. When $\underline{b} = 0$, a weak-Markov equilibrium exists.

Proof. See Appendix 2.

Smooth-Markov equilibria

Another approach to solving infinite-horizon bargaining games is to assume that a smooth, (strong)-Markov equilibrium exists, and to try to compute it from the differential equation resulting from the first-order conditions for the seller's maximization.

Let $W_S(b^e)$ be the seller's valuation when his posterior is b^e, multiplied by $F(b^e)$. Define

$$J(p,b^e,\beta(\cdot),W_S(\cdot)) = [F(b^e) - F(\beta(p))]p + \delta_S W_S(\beta(p)).$$

Then, $\sigma(b^e)$ must be an arg max of J, and $W_S(b^e)$ the maximized value. As in our previous discussion, we see that σ is strictly increasing if β is strictly increasing. When β has "flat spots," the induced σ will not be strictly increasing and a smooth-Markov equilibrium need not exist.

Differentiating J with respect to b^e and using the envelope theorem, we find that

$$\frac{dW_S}{db^e} = f(b^e)\sigma(b^e).$$

Maximizing J with respect to p, we then find the first-order condition

$$F(b^e) - F[\beta(\sigma(b^e))] - \beta'(\sigma(b))f[\beta(\sigma(b^e))]\{\sigma(b^e) - \delta_S\sigma[\beta(\sigma(b^e))]\} = 0. \quad \textbf{(5.1)}$$

In case the second-order condition is also satisfied, (5.1) and its associated $\beta(p)$ characterize a Markov equilibrium. One such instance occurs when $F(b) = (b/\bar{b})^m$ for $0 \leq b \leq \bar{b}$ and $m > 0$. In this case, $F(\beta)/F(b)$ has the same functional form as F, and we can find a smooth-Markov equilibrium, with the linear form $\beta(b) = \beta p$ and $\sigma(b) = \sigma b$. It can be verified that the second-order condition corresponding to (5.1) is satisfied, and the constants σ and β may then be computed to be the unique solution of

$$(\beta\sigma)^{-m} + \delta_S m(\beta\sigma) = 1 + m$$

$$\beta = \frac{1 - \delta_B(\beta\sigma)}{1 - \delta_B}, \quad \textbf{(5.2)}$$

from which it follows that $\beta > 1$ and $\beta\sigma < 1$. This is the solution obtained as a limit of finite-horizon games by Sobel and Takahashi (1983), which was known to be an equilibrium from Fudenberg and Levine (1983). We have just provided a simpler derivation.

We now comment on a number of features of equilibrium in this model. First, in all cases $\sigma(\cdot)$ is nondecreasing so that equilibrium involves gradual *concessions*. How general a result this is remains to be seen. Fudenberg and Tirole (1983) show that in a finite-horizon model with two-sided incomplete information, prices may rise over time. Whether this can occur in infinite-horizon models is as yet unknown but seems likely.

It can be shown that when the buyer's and the seller's discount factors converge to 1, the seller's payoff converges to zero. In other words, the seller loses all ability to price discriminate when the time period goes to zero (since then both δ_S and δ_B approach 1). This result was obtained by Sobel and Takahashi and is similar to results of Kreps and Wilson (1982*a*) in the chain-store paradox and of Bulow (1982) and Stokey (1980) in work on durable-goods monopoly. Let us give a rough intuition. The incentive to bargain is due to the destruction of the pie by discounting. By making offers, the seller makes the buyer responsible for destroying the pie if he rejects the offer. The seller uses this leverage to extort the buyer's surplus and, when there is incomplete information, price discriminate. With short time periods, higher-valuation buyers are more willing to free ride on lower-valuation buyers. The seller consequently loses his ability to price discriminate.

Finally, note that if $V_S(b^e) = W_S(b^e)/F(b^e)$,

$$\lim_{m\to\infty} \lim_{\delta_S,\delta_B\to 1} V_S(\bar{b}) = 0 < \lim_{\delta_S,\delta_B\to 1} \lim_{m\to\infty} V_S(\bar{b}) = \bar{b}.$$

Here, $m \to \infty$ means that the seller is nearly certain that the buyer has valuation $\bar{b}$. Thus, in the infinite horizon, it makes a difference what order we pass to the limit.

5.4 Alternating offers

Thus far, we have assumed that the seller makes all of the offers, that is, that the buyer is not allowed to make counter offers but can only accept or reject offers of the seller. This assumption is far from innocuous, especially coupled with our assumption that only the buyer's valuation is private information, which as we suggested seems a good approximation if the seller owns the object before the bargaining starts, and values the object only for its eventual sale. If the seller makes all of the offers and the seller's valuation is known, the offers reveal no information. If the buyer is allowed to make counteroffers, in equilibrium the seller must update his posterior to reflect the information thereby transferred. In particular, we must specify how the seller revises his beliefs if the buyer makes an offer that according to the equilibrium strategies is not made by any type of buyer. Bayes' rule places no restrictions on such inferences, nor does Kreps and Wilson's (1982*b*) more restrictive concept of a sequential equilibrium. This leeway can be used to support a multiplicity of equilibria. If only the seller can make offers, the only zero-probability event that does not terminate the game immediately is if the buyer refuses a price below $\underline{b}$; however, as lemma 2 illustrated, the seller would never charge such a price in *any* equilibrium, and thus what the seller infers from this event is irrelevant. In contrast, when the buyer can make counteroffers, the seller's inferences can change the set of actions that occur in equilibria.

Let us illustrate this point with an example, which has the additional virtue of providing a form of justification for our seller-makes-the-offers specification. Specifically, we will describe an equilibrium in which, although the buyer does make counteroffers, these counteroffers are always rejected by the seller, so that the equilibrium is "observationally equivalent" to one in which the seller makes all of the offers but the time period is twice as long.

Before we present this equilibrium, recall that Rubinstein (1982) proved that for the corresponding complete-information game, there exists a unique equilibrium. The seller always offers $b(1-\delta_B)/(1-\delta_S\delta_B)$, the buyer offers $\delta_S[b(1-\delta_B)/(1-\delta_S\delta_B)]$, and these offers are accepted.

Our example is a "pooling equilibrium," in that all types of the buyer make the same offers, so that the buyer's offer conveys no information. All types of the buyer always offer price zero, which the seller always refuses. Were the buyer to offer a price other than zero, the seller would believe that the buyer has type $K\bar{b}$, where K is some large number (such beliefs are not consistent with the spirit of sequential equilibrium, because they put weight on types outside the initial support of the buyer's valuation. Such beliefs can be understood as resulting from trembles by nature as opposed to the trembles by players which are considered in sequential equilibrium.) The seller's offers are made as with one-sided offers, discussed in Section 5.3, except that the discount factors are δ_S^2 and δ_B^2. The periods in which the buyer makes offers do not count, and play evolves as though the seller made all of the offers and the period length is equal to twice that of the alternating-offers game.

For this to be an equilibrium, it is necessary and sufficient that no type of buyer wish to charge a price other than zero. However, any unexpected price causes the seller to believe that the buyer's valuation is $K\bar{b}$, and thus the seller refuses p unless $p \geq [K\bar{b}(1-\delta_B)/(1-\delta_S\delta_B)]$. Clearly, for K sufficiently large, this will require $p \geq \bar{b}$, which no buyer would offer.

There certainly are many other equilibria. Grossman-Perry (1985) have shown how to embed the one-sided offer equilibrium into the two-sided offer structure using beliefs which assign weight only to types in the interval support of the buyer's calculation.

5.5 Specification of the costs of bargaining

The models that we have discussed so far have modeled the costs of prolonged negotiations as the discounting of future outcomes. This section contrasts that form of the costs with two others; fixed per-period bargaining costs and costs of changing offers.

The assumption of fixed per-period costs is that agreement at price p in period t yields utilities $(b-p-c_Bt)$ and $(p-s-c_St)$, respectively. Fixed per-period costs were included in Rubinstein's (1982) complete-information bargaining model; in equilibrium, the player with lower cost captured the entire surplus. However, as pointed out in Fishburn and Rubinstein (1982), per-period costs are inconsistent with the existence of a "zero agreement," for which the trader has no impatience. Fishburn and Rubinstein show that any preferences at bargaining outcomes that are monotonic, impatient, continuous, and stationary can be represented by discounting if such a zero agreement is possible. Thus, the existence of a zero agreement is of primary importance in choosing a functional form for time preference.

In the absence of a zero agreement, there are outcomes that are inferior to "leaving the game," even when there are known to be gains from trade. Thus, to avoid violating (ex ante) individual rationality, an "exit option" must be included in the specification of the extensive form. With discounting, the exit option is superfluous in a model with only two traders: The value of the outside opportunity is normalized to be zero (and, therefore, even a greatly postponed agreement is preferred to none at all). Rubinstein's (1982) paper did not allow an exit option, so that the lower-cost trader could inflict "infinite damage" on his opponent "relatively" cheaply. This may partially explain his troublesome conclusions in the fixed-cost case. Perry's (1982*a*) model of bargaining with many sellers similarly assumes that the buyer cannot leave the game; and thus its conclusions may be similarly misleading.

The obvious alternative to requiring the players to potentially suffer arbitrarily large bargaining costs is to allow for the possibility of exit, which ensures the (current) reservation value. Although such an option can sensibly be added to bargaining models with complete information, with incomplete information the possibility of exit combined with fixed costs of continuing yields a trivial equilibrium, as was pointed out in Fudenberg and Tirole (1983). The equilibrium is trivial because, when an agent chooses not to exit, he signals that his expected value to continuing, and in particular his valuation, exceeds the sum of the per-period cost and his surplus in the eventual agreement. Consider, for example, the model of Fudenberg and Tirole (1983), with the addition that the buyer decides at the end of the first period whether or not to exit. Let $\underline{b}(c_B)$ denote the type of buyer that is just indifferent between exiting and paying cost c_B to continue. Clearly, the seller will never offer a price below $\underline{b}(c_B)$ in the second period, and so there is no equilibrium in which buyers choose to continue. Perry (1982*b*) analyzes an infinite-horizon, alternating-offers model, and obtains the same result. The only equilibrium in any subgame that begins with the buyer making an exit decision is the trivial one that all buyers leave immediately. Thus, if the seller makes the first move, the equilibrium is simply that of the one-period game, because everyone knows that all valuations of buyer will leave at the end of the first period. If the buyer pays a fee in order to play, the seller will charge a high enough price that the buyer who had been indifferent about staying in will regret having done so. Thus, in the presence of incomplete information, the specification of fixed bargaining costs results in a trivial outcome in which no bargaining in fact occurs. This is highly reminiscent of Diamond's (1971) observation about the effect of fixed search costs, which allowed firms to charge the monopoly price and thus precluded search.

Fixed bargaining costs are formally similar to entry fees to participate; yet entry fees have been shown to be optimal in the theory of optimal auctions (Maskin and Riley (1980)). The difference is that in auction theory, unlike bargaining, the seller is allowed to precommit to future actions, and thus to "promise" a nonnegative expected return to those who choose to pay the fee. Note that one way out of the dilemma in bargaining may be to modify the extensive form to allow such payments, so that the seller can pay the continuation fee.

The second alternative specification of bargaining costs that we wish to discuss is one in which it is costly to change offers. Such costs were introduced to the bargaining literature by Crawford (1981), who assumed that having made initial demands, bargainers could "back down" at a cost. More recently, Anderson (1983) studied repeated games with costs of adjustment. Although costs of adjustment may seem artificial and ad hoc in the context of bargaining between individuals, they are perhaps more plausible if the bargainers are agents for others, as in union–management negotiations.

These are the main alternatives to the discounting formulation that we have employed. Still other formulations may emerge with the continued development of the empirical literature on sequential bargaining.

5.6 Why should we study sequential processes? The thorny question of the extensive form

Here, we offer a few thoughts on the nature of the extensive form. It should be clear that these thoughts are incomplete. Their only purpose is to raise some questions we deem important for bargaining theory.

Myerson and Satterthwaite (1983) have studied the optimal negotiation mechanisms between a buyer and a seller. This work has been extended to situations with multiple buyers and sellers (double auctions) by Wilson (1982). According to the revelation principle, the optimal negotiation is a revelation game in which the buyer(s) and the seller(s) announce their characteristics simultaneously. Therefore, it seems that one could as well restrict attention to static revelation games and never be interested in sequential bargaining. A number of considerations actually go against this first intuition.

For one thing, real-world bargaining is almost always sequential. Myerson–Satterthwaite-type revelation games are not played. Thus, it seems that there is scope for sequential bargaining theory. Students of bargaining theory cannot content themselves with this proof-of-the-pudding argument. One must ask why such revelation games are not played, and when the Myerson–Satterthwaite model is internally consistent.

Imagine that two parties meet and want to discuss freely the possibility of a trade that might be advantageous. Their using the Myerson–Satterthwaite mechanism requires two fundamental assumptions: (1) the traders agree to bargain this way, and (2) the traders can commit themselves to not ever reopen the bargaining process in case of disagreement. It is immediately evident that these two conditions are likely not to be satisfied in real-world conditions, for the following reasons.

1. Most of the time, the traders have at least some of their private information before meeting. Depending on his information, a trader may want to use a bargaining mechanism that differs from the revelation game. One could object to this reasoning by noticing that, because the revelation game is the most efficient game, there could be transfers inducing the traders to play that game. However, this neglects the fact that choosing a bargaining mechanism itself conveys information and changes the outcome of the subsequent bargaining game. In particular, accepting the revelation game is not neutral: It says something about the trader. We are aware that we are raising a deep question without bringing any element of answer.
2. It is well known that any bargaining mechanism under asymmetric information and individual rationality constraints implies inefficiency. Traders may quit without realizing gains from trade. This is especially characteristic of the Myerson–Satterthwaite mechanism. Thus, there is an incentive to renegotiate later. This point is addressed in greater detail in Cramton (1983*b*).

What, then, is left of the Myerson–Satterthwaite analysis? We think that this mechanism is of interest for two reasons:

1. From a normative point of view, it gives a lower bound on the inefficiency associated with voluntary bargaining.
2. From a positive point of view, it may be applied to some special cases. Imagine, for example, that the parties meet when they have symmetric information. They know that later on they will acquire private information (value of a project, its cost), and that they will have to make a decision (production) on this basis. In this case, they decide to bargain according to the Myerson–Satterthwaite mechanism if they have a means of enforcing the absence of renegotiation in case of disagreement. One could think of reputation as an incentive not to renegotiate.

5.7 Specification of the information structure

The literature on sequential bargaining has up to now assumed that the random variables on which there is asymmetric information are uncorrelated. This may be a reasonable assumption in a number of cases. For example, the seller's production cost and the buyer's valuation for the object can be assumed to be independant. Similarly, costs of bargaining are likely to be uncorrelated between them and with the previous variables. However, there are two channels through which a trader can learn about his own valuation.

1. *Correlated values.* Imagine, for example, that the seller owns a used car, and knows its quality. His willingness to sell the car depends on this quality. In addition, the buyer's willingness to buy the car would depend on this parameter if he knew it. In this case, the valuations are correlated and the buyer is eager to learn about the seller's information not only to discover his bargaining power but also to assess the value of a deal.
2. *Learning from an unknown distribution.* Imagine that there are several sellers, whose production costs are drawn from a probability distribution that is unknown to the buyer. Imagine further that the buyer can switch sellers. When the buyer bargains with a given seller, he learns not only the specific characteristics of this seller (and therefore about his bargaining power), but also about the other sellers' characteristics. Therefore, the buyer learns about his expected profit if he switches to another seller. Even though the buyer and the seller's characteristics may be uncorrelated, the buyer learns about more than his bargaining power. Another possibility leading to the same effect is the correlation of production costs between sellers. Indeed, the case of independent draws from a distribution that is unknown to the buyer is processed like that of correlated draws by the buyer.

An interesting example that combines items (1) and (2) can be found in the work of Ordover and Rubinstein (1983) on litigation. In their paper, one of the bargaining parties knows who will win if the dispute is resolved in court, that is, if disagreement occurs. On the one hand, the parties are interested in their valuations after disagreement, and they can learn something about them before disagreement as outlined in item (2). On the other hand, the valuations after disagreement are correlated.

Whereas in the independent-draws model, the only purpose of learning is to discover one's bargaining power, when draws are correlated between traders the parties learn about their positions *after* bargaining with the current partner whether there is agreement or disagreement. Consequently, during the bargaining process the parties must take into account two kinds of "curses":

1. *The celebrated winner's curse in case of agreement.* For example, the fact that the seller of the used car accepts the buyer's offer may be a bad signal about the quality of the car.
2. *The "bargaining curse."* The seller's making a low offer may not be good news to the buyer if the seller knows the quality of the car. In the unknown-distribution framework, the seller's making a high offer may signal that the production costs of the other potential sellers are likely to

be high as well. On the other hand, such learning may not be a curse, but good news. For instance, in the used car example the seller's turning down the buyer's offer may signal a high quality. Such transmission of information can easily be embodied in bargaining models. Although we will not pursue this topic here, it is clear that some new insight can be gained from it.

5.8 Conclusion

As we stated in Section 5.3, the outcome of bargaining with one-sided information is fairly easy to characterize if the player whose valuation is known makes all of the offers. In this case, the price must decrease over time, and things are generally "well behaved." With alternating offers, however, there are multiple equilibria, which are qualitatively very dissimilar. Thus, the problem of the choice of extensive form is fairly severe, even when only one-sided incomplete information is being considered. If both player's valuations are private information, the situation is even more complex. We fear that in this case, few generalizations will be possible, and that even for convenient specifications of the functional form of the distributions over the valuations, the problem of characterizing the equilibria will be quite difficult. Cramton (1983*a*) is a start in this direction.

Throughout this paper, because the bargaining costs took the form of discounting and players had no other opportunities to trade, players had no incentive to stop bargaining. If traders have alternative bargaining partners, we would expect them to switch to a new partner whenever they become sufficiently pessimistic about the valuation of the party with whom they are currently negotiating. Thus, the length of bargaining between any pair of traders could be endogeneously determined by the outside opportunities. Shaked and Sutton (1984) have modeled bargaining with several sellers under complete information. Because the sellers are known to have the same valuation in equilibrium, traders never quit bargaining without an agreement if there exist gains from trade. Thus, the Shaked–Sutton model again predicts that traders will never stop negotiating. In a forthcoming paper, we analyze bargaining with many traders and incomplete information to study the effect of outside opportunities on equilibrium prices and on the length of negotiations.

The noncooperative approach to bargaining theory is still in its infancy. Although much remains to be done, substantial progress has been made in the past few years. Solving a wider variety of extensive forms may permit some generalizations to emerge. The problem of the choice of extensive forms by the players remains open.

APPENDIX 1
Proof of Proposition 1

We proceed by induction on n. For each n, we construct p^n, the highest price the seller is allowed to charge. The index n will keep track of the number of periods remaining: When $p < p^n$, the game will be shown to end in at most n periods. For $n = 1$, set

$$p^1 = \underline{b}, \qquad W_S^1(b^e) = F(b^e)\underline{b}, \qquad \beta^2(p) = \frac{p - \delta_B \underline{b}}{1 - \delta_B},$$

$$\sigma^1(b^e) = \bar{\sigma}^1(b^e \mid p) = \underline{b}, \qquad \text{and} \qquad b^1 = \underline{b},$$

Here, $p^1 = \underline{b}$ is the price the seller is required to charge to guarantee that the game ends immediately, and $W_S^1(b^e)$ is the seller's payoff to charging $\underline{b}$ multiplied by the probability that the posterior is below b^e. In other words, we work with "unconditional probabilities" instead of conditional ones; the conditional probability of a sale at $\underline{b}$ is 1, but the unconditional probabilities prove simpler to work with. Note that this renormalization does not affect the seller's optimal behavior. Quantity $\beta^2(p)$ is the reservation value of the buyer who is just indifferent between p in this period and $\underline{b}$ in the next one. Because the seller's offers will be nonincreasing and no less than $\underline{b}$, if the seller charges $\underline{b}$ in this period, buyers must expect $\underline{b}$ in subsequent periods. The term $\beta^{n+1}(p)$ will be the lowest reservation value of a buyer who accepts p when there will be n *subsequent* periods, that is, in the $(n + 1)$-period game. Observe that W_S^1 and β^1 are continuous and nondecreasing. If $p \leq p^1$, the game is over; if $p > p^1$, it (with some probability) lasts at least one more period. Value $\sigma^n(b^e)$ is the correspondence that yields the seller's optimal choices in the n-period game when he is constrained to charge no more than p^n (when $n = 1$, this constraint forces σ^1 to be single valued). Quantity $\bar{\sigma}^1(b^e \mid p)$ is the expected value of the seller's price if the last price was p. And b^1 is a dummy, which will not be used; for larger n, b^n will be a bound on b^e that guarantees that the seller charges no more than p^{n-1} in the next period.

In the n-period game, the seller is constrained to not charge more than p^n, where p^n is chosen such that if p^n is charged, the buyer's reservation value is at most b^n; so the next period's price is below p^{n-1} and, by inductive hypothesis, the game ends in $(n - 1)$ additional periods.

We will now define $W_S^n(b^e)$, $\beta^{n+1}(p)$, p^n, $\sigma^n(b^e)$, and $\bar{\sigma}^n(b^e,p)$ recursively, and prove by induction that

1. W_S^n and β^{n+1} are continuous and nondecreasing, and that β^n is the unique solution of $p \in (1 - \delta_B)\beta^n(p) + \delta_B \hat{\sigma}^n(\beta^n(p))$, where $\hat{\sigma}^n$ is the convexification of σ^n;

2. When a price $p \leq p^n$ is charged, the game lasts n or fewer periods, and so for $p \leq p^n$, $\beta^{n+1}(p) = \beta^n(p)$;
3. When $b^e \leq b^n$, the seller charges a price less than p^{n-1};
4. $\sigma^n(b^e) < b^e$, where σ^n is nonempty, nondecreasing, and has a compact graph;
5. In the $(n+1)$-period game, the buyer with valuation $\beta^{n+1}(p)$ is just indifferent between paying p now and waiting one period, and strictly prefers buying next period to waiting longer;
6. The expected price that the seller charges in period n, $\bar{\sigma}^n(b^e \mid p)$, is uniquely determined, given b^e and the price p charged in previous period;
7. In any equilibrium, the buyer must play according to β^{N^*+1}, and the seller's initial price belongs to $\sigma^{N^*+1}(\bar{b})$.

Having by inductive hypothesis proved claims (1) through (5) for the $(n-1)$-period game, let us extend them to the n-period game. First, we solve for W_S^n and σ^n. Let $c \geq \underline{b}$ be a given constant. Define the maximization problem, denoted $J(p, b^e, \beta, W_S, c)$, as follows:

$$\max_p \{p[F(b^e) - F(\beta(p))] + \delta_S W_s(\beta(p))\}$$

$$\text{subject to } \underline{b} \leq p \leq \min\{b^e, c\}.$$

Since, by inductive hypothesis, β and W_S will be continuous, and the constraint set is compact, the arg max correspondence has a nonempty image and a compact graph. Moreover, the correspondence is nondecreasing, since increasing b^e strictly increases the gradient of the objective function. (The correspondence is strictly increasing whenever the objective is continuously differentiable, but it may be flat if not.)

Let σ denote the arg max correspondence, $\bar{\sigma}$ the expected price charged by the seller, and $\hat{\sigma}$ the correspondence whose image is the convex hull of the image σ. Note that $\hat{\sigma}$ is continuous, is convex valued, and contains $\bar{\sigma}$. Finally, note that $\sigma(\underline{b}) = \underline{b}$, whereas for $b > \underline{b}$, $\sigma(b) < b$.

Now, we can find W_S^n, p^n, and β^{n+1}. Consider first $J(p, b^e, \beta^n, W_S^{n-1}, \bar{b})$. Associated with this are $\sigma^n(b^e, \bar{b})$ and $\hat{\sigma}^n(b^e, \bar{b})$. Define b^n to be the largest value of b^e for which $p^{n-1} \in \hat{\sigma}^n$. The key is that when $b^e \leq b^n$, we can without loss restrict the seller to not charge more than p^{n-1}; that is, $J(p, b^e, \beta^n, W_S^{n-1}, \bar{b}) = J(p, b^e, \beta^n, W_S^{n-1}, p^{n-1})$. However, by inductive hypothesis, when $p \leq p^{n-1}$, the game ends in $(n-1)$ periods, and so $\beta^n(p) = \beta^{n-1}(p)$ and $\beta^{n-1}(p) \leq b^{n-1}$ (from the definition of b^{n-1}), implying that $W_S^{n-1}(\beta^n(p)) = W_S^{n-2}(\beta^{n-1}(p))$. Thus, when $b^e \leq b^n$, the n-period game in fact ends in at most $(n-1)$ periods, and the behavior we previously determined for the $(n-1)$-stage game must still apply. We conclude that for $b^e \leq b^n$, $\sigma^n(b^e, \bar{b}) = \sigma^{n-1}(b^e, \bar{b}) = \sigma^{n-1}(b^e)$. This argument holds only for $n > 2$; for $n = 2$, $p^{n-1} = \underline{b}$, and the result is trivial.

Next, we must define p^n, the bound on the seller's price that ensures that next period's price is less than p^{n-1} and therefore that the game ends in n periods. This situation is complicated slightly by the possible discontinuity of σ. Let $\underline{p}^{n-1}$ be the largest value in $\sigma^{n-1}(b^n)$ less than or equal to p^{n-1}, and define $p^n = (1 - \delta_B)b^n + \delta_B \underline{p}^{n-1}$. We claim that if $p \leq p^n$ in the n-period game, then the next-period price will be less than $\underline{p}^{n-1}$ and the game in fact ends in n periods. Assume not – then the seller's posterior next period, $\beta^n(p)$, must exceed b^n. From the definition of p^n, this implies that $(1 - \delta_B)\beta^n(p) > p^n - \delta_B \underline{p}^{n-1}$. Since $\beta^n(p) < p \leq p^n$, $\sigma^{n-1}(\beta^n(p)) \leq \sigma^{n-1}(p^n) \leq \sigma^{n-1}(b^n) \leq \underline{p}^{n-1}$. Yet, by inductive hypothesis, β^n satisfies $p \in (1 - \delta_B)\beta^n(p) + \delta_B \hat{\sigma}^{n-1}(\beta^n(p))$, and so $(1 - \delta_B)\beta^n(p) \geq p - \delta_B \underline{p}^{n-1}$, which is a contradiction. This means that imposing the constraint $p \leq p^n$ guarantees that the n-period game in fact ends in n periods. Later, we will show that if $p \leq p^n$, the $(n + 1)$-stage game ends in n periods as well.

Given p^n, we consider the optimization problem $J(p,b^e,\beta^n,W_S^{n-1},p^n)$. The solution to this problem is $\sigma^n(b^e)$, with convex null $\hat{\sigma}^n(b^e)$. As shown above, we know that for $b^e \leq b^n, W_S^n(b^e) = W_S^{n-1}(b^e)$, and $\sigma^n(b^e,\bar{b}) = \sigma^n(b^e) = \sigma^{n-1}(b^e)$; therefore, behavior below b^n is not changed by increasing the number of periods.

Next, we work backward one period to show that $\beta^{n+1}(p)$ is uniquely defined by the assumed equation. The valuation of the buyer who is just indifferent between paying p in period $(n + 1)$ and waiting must satisfy

$$(\beta^{n+1}(p) - p) \in \delta_B[\beta^{n+1}(p) - \hat{\sigma}^n(\beta^{n+1}(p))]$$

or

$$p \in (1 - \delta_B)\beta^{n+1}(p) + \delta_B \hat{\sigma}^n(\beta^{n+1}(p)). \tag{A.1}$$

The right-hand side of (A.1) is a continuous, convex-valued, strictly increasing correspondence, and thus has a unique inverse function $\beta^{n+1}(p)$, which is nondecreasing and has modulus of continuity smaller than $1/(1 - \delta_B)$.

Note that since $\max\{\hat{\sigma}^n(\beta^{n+1}(p))\} < \beta^{n+1}(p)$, the choice of buyer $\beta^{n+1}(p)$ whether to accept p in period $(n + 1)$ or to wait one period and then buy is unaffected if we replace the anticipated next-period probability distribution over elements of $\sigma^n(\beta^{n+1}(p))$ by its expected value $\bar{\sigma}^n$. Because $\bar{\sigma}^n$ must lie in $\hat{\sigma}^n$, equation (A.1) defining β^{n+1} ensures that the buyer of valuation $\beta^{n+1}(p)$ is indifferent between paying p in period $(n + 1)$ and waiting to face $\bar{\sigma}^n$ next period. If buyer $\beta^{n+1}(p)$ were willing to wait more than one period, then all buyers with lower valuations would strictly prefer to wait, and there would be no sales in period n. This would contradict the behavior that we derived for the n-period game.

Thus, we have verified the inductive hypotheses for $k = n$. Because we know that all equilibria end in at most $(N^* + 1)$ periods, we know that after $(N^* + 1)$ steps, we will have $b^{N^*+1} = \bar{b}$, and the induction is complete. Thus, the first price charged must (1) be an element of $\sigma^{N^*+1}(\bar{b})$ and, moreover, (2) be less than or equal to p^{N^*+1}, and that thereafter equilibrium play is uniquely determined. Thus, if an equilibrium exists, it is unique up to the seller's choice of an initial price (or any probability distribution over $\sigma^{N^*+1}(\bar{b})$). The argument given before the statement of proposition 1 shows that an equilibrium does in fact exist, because given the functions $\beta^n(p)$, the seller will choose to end the game in no more than $(N^* + 1)$ periods.

APPENDIX 2
Proof of Proposition 2

To proceed, we need the following lemma.

Lemma 6. The functions $\beta(p)$ and $W(b^e)$ derived in the proof of proposition 1 are equicontinuous.

Proof. We observed earlier that the modulus of continuity of $\beta(p)$ is no greater than $1/(1 - \delta_B)$. Recall the definition of $W(b^e)$:

$$W(b^e) = \max_p\{[F(b^e) - F(\beta(p))]p + \delta W(\beta(p))\}.$$

Now, consider $W(b_1^e) - W(b_2^e)$, where $b_1^e > b_2^e$. Let p_1 and p_2 be the respective maximizing prices. We claim that

$$W(b_1^e) - W(b_2^e) \leq (F(b_1^e) - F(b_2^e))p_1,$$

because the seller could always choose to offer price p_1 when his beliefs were b_2^e, and so we have used a lower bound on $W(b_2^e)$. However, since the density, $f(b)$, is bounded by $\bar{f}$, we have

$$W(b_1^e) - W(b_2^e) \leq \bar{b}\bar{f}|b_1^e - b_2^e|.$$

Proof of proposition 2. Consider the sequence of games with buyer valuation densities

$$f^n(b) = \begin{cases} \dfrac{f(b)}{1 - F(\underline{b}^n)} & \text{if } b \geq \underline{b}^n, \\ 0 & \text{if } b < \underline{b}^n, \end{cases}$$

where $\underline{b}^n \to 0$ as $n \to \infty$. Each of these games has a unique weak-Markov equilibrium (β^n, W^n, σ^n). Since the family of functions (β^n, W^n) is equicontinuous, it has a uniformly convergent subsequence converging to continuous (nondecreasing) functions (β, W). For notational simplicity, assume that (β^n, W^n) actually converge to (β, W). There are now two distinct concepts of the limiting σ. First, there is σ, which is the arg max correspondence for the seller's optimization problem $J(p,b,\beta,W,\bar{b})$; this is monotonic and piecewise continuous, as always. Furthermore, since each J^n is Lipschitz continuous in β^n and W^n (in the uniform topology), J^n converges uniformly to J, and from the theorem of the maximum, the limit points of σ^n are contained in σ. Thus, at continuity points of σ, σ^n converges to σ.

Second, there is the $\hat{\sigma}$ correspondence, defined uniquely as the solution of

$$\hat{\sigma}(b) = \frac{\beta^{-1}(b) - (1 - \delta_B)b}{\delta_B},$$

where equality is the equality of sets.

Let us show that limit points of $\hat{\sigma}^n$ are in $\hat{\sigma}$. Suppose, in fact, that for some b, $s^n \in \hat{\sigma}^n(b) \rightarrow p$. This is true if and only if $g^n \equiv \delta_B(s^n + (1 - \delta_B)b) \rightarrow \delta_B(p + (1 - \delta_B)b) \equiv g$. From the definition of $\hat{\sigma}$ given previously, $p \in \hat{\sigma}(b)$ if and only if $g \in \beta^{-1}(b)$. Consider the sequence $\beta(g^n)$. Since $b = \beta^n(g^n)$ and the β^n converge uniformly, then $\beta(g^n)$ converges to b. Since $g^n \rightarrow g$ and β is continuous, $\beta(g^n) \rightarrow \beta(g)$, and so $b = \beta(g)$, or $g \in \beta^{-1}(b)$. Thus, we can conclude that at continuity points of $\hat{\sigma}$, $\hat{\sigma}^n$ converges to $\hat{\sigma}$, and since $\hat{\sigma}^n$ is the convex hull of σ^n, that $\hat{\sigma}$ and σ agree wherever they are continuous. Finally, since σ and $\hat{\sigma}$ are monotonic, they are continuous except at countably many points, and thus $\hat{\sigma}$ is the convex hull of σ. Therefore, the optimal seller behavior given β and W (i.e., σ) is consistent with playing the mixed strategies in $\hat{\sigma}$, which in turn induce the desired behavior from the buyer, and we indeed have an equilibrium.

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