Ordinal Optimization

⚙️ Use crude model to screen out good enough designs, saves the computing budget

1. Hard Optimization Problems:
   a. parameter optimization:
      convex optimization, MIP……
   b. policy optimization:
      MDP, RL
2. Simulation-based optimization:
   a. time cost:

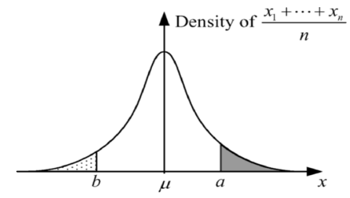
Real system	Performance	Simulation time
Remanufacturing system	Accurately evaluate the average cost of a maintenance strategy by 1000 independent simulations	**30 minutes**
Congestion control and buffer management in a computer network	A simulation of the 1000-second dynamics of a 12000-node-single-bottleneck computer network	**1.5 hours**
Security evaluation and optimization for a large scale electric power grid	A simulation of the 30-second dynamics of a large scale power grid with 5000 buses and 300 generators after a failure event	**2 hours**
Scheduling of a transportation network	A simulation of the 24-hour dynamics of a network with 20 intersections	**2 hours**
Turbine blade design problem	A 3D extrusion simulation with finite element methodology	**7 days**

   b. discrete parameter —> curse of dimensionality
3. Ordinal Optimization:
   a. basic idea:
      i. ordinal comparison
         which better v.s. how much better
      ii. goal softening
         find good enough $(1-\delta)$
   b. math proof:
      i. observed order —> true order converge exponentially
      ii. observed good enough —> truly good enough exponentially

4. Exponential Convergence:

   a. Comparison:

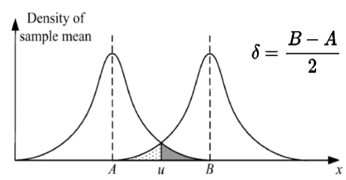
- **Theorem 1** **Assume that design $\boldsymbol{\theta_1}$ and $\boldsymbol{\theta_2}$ have true performance $\boldsymbol{A} < \mathbf{B}$. Denote the performance estimates $\bar{\boldsymbol{J}}(\boldsymbol{X}, \boldsymbol{n}) = \frac{1}{n}\sum_{j=1}^{n} \boldsymbol{x_j}$, where $\boldsymbol{x_j}$ is the sampled performance for the $\boldsymbol{j}$-th replication. There exists a positive $\boldsymbol{\beta}$ such that Prob $\left[\bar{\hat{J}}(\boldsymbol{\theta_1}, \boldsymbol{n}) \geq \bar{\hat{J}}(\boldsymbol{\theta_2}, \boldsymbol{n})\right] = \boldsymbol{O}\left(\boldsymbol{e}^{-\boldsymbol{n\beta}}\right)$**



- Proof of Theorem 1
  - $\text{Prob}\left[\frac{x_1+\cdots+x_n}{n} \geq a\right] = \text{Prob}[x_1 + \cdots + x_n \geq na] \to 0$

    $\text{Prob}\left[\frac{x_1+\cdots+x_n}{n} \leq b\right] = \text{Prob}[x_1 + \cdots + x_n \leq nb] \to 0$

    as $n \to \infty$ (The law of large numbers)



  - A fundamental question is: how fast do these two probabilities decrease?
  - $\text{Prob}[\,\bar{\hat{J}}(\theta_1, n) > \bar{\hat{J}}(\theta_2, n)] \leq \text{Prob}[\bar{\hat{J}}(\theta_1, n) > A + \delta] + \text{Prob}[\bar{\hat{J}}(\theta_2, n) < B - \delta]$
  - Then show that, for every constant $a > \mu$, there exists a positive β such that

$$\text{Prob}[x_1 + \cdots + x_n \geq na] \leq e^{-n\beta}$$

   using Chernoff bound and large number theorem

   b. Selection:

- **Theorem 2** **The alignment probability $\mathbf{Prob}[|\boldsymbol{S} \cap \boldsymbol{G}| \geq \mathbf{1}]$ is bounded from below by the function $\mathbf{1} - \mathbf{e}^{-\frac{gs}{N}}$, that is, $\mathbf{Prob}[|\boldsymbol{S} \cap \boldsymbol{G}| \geq \mathbf{1}] \geq \mathbf{1} - \mathbf{e}^{-\frac{gs}{N}}$**
- Proof of Theorem 2 (for blind pick)
  - $\text{Prob}[|S \cap G| = 0]$ is given by

$$\frac{\binom{N-g}{s}}{\binom{N}{s}}$$

  - Thus, the alignment probability $\text{Prob}[|S \cap G| \geq 1]$ is given by

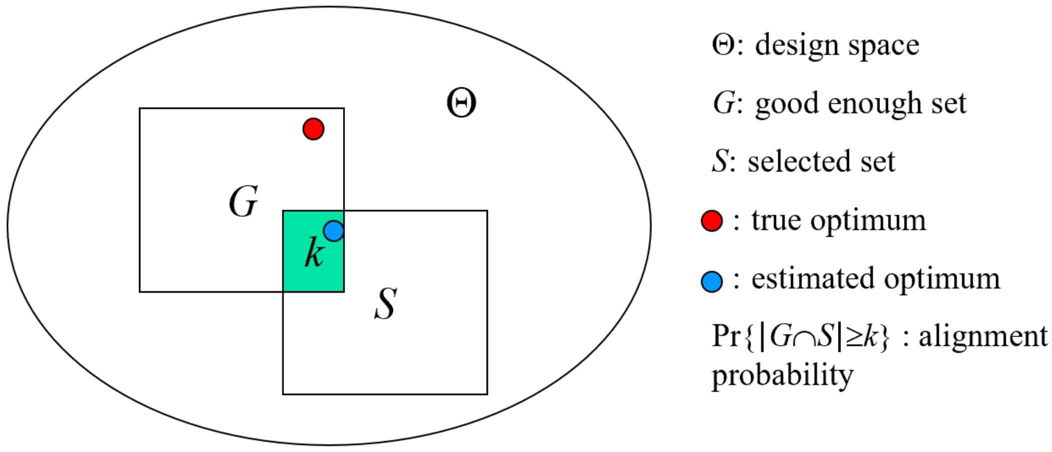
$$\text{Prob}[|S \cap G| \geq 1] = 1 - \text{Prob}[|S \cap G| = 0] = 1 - \frac{\binom{N-g}{s}}{\binom{N}{s}}$$

$$= 1 - \frac{\frac{(N-g)!}{s!(N-g-s)!}}{\frac{N!}{s!(N-s)!}} = 1 - \frac{(N-g)(N-g-1)\cdots(N-g-s+1)}{(N)(N-1)\cdots(N-s+1)}$$

  - Since $\frac{N-g-i}{N-i} \leq \frac{N-g}{N} = 1 - \frac{g}{N}$ for all $i = 0,1 \ldots, S-1$, we have

$$\text{Prob}[|S \cap G| \geq 1] \geq 1 - \left(1 - \frac{g}{N}\right)^{S}$$

5. Application Procedure:



a. random sample

b. user define g and k (intersaction)

c. evaluate the design using crude model

d. estimate the noise level

e. calculate the value of s, let:

$$Pr(G \cap S \geq k) \geq 0.95$$

f. select top s designs

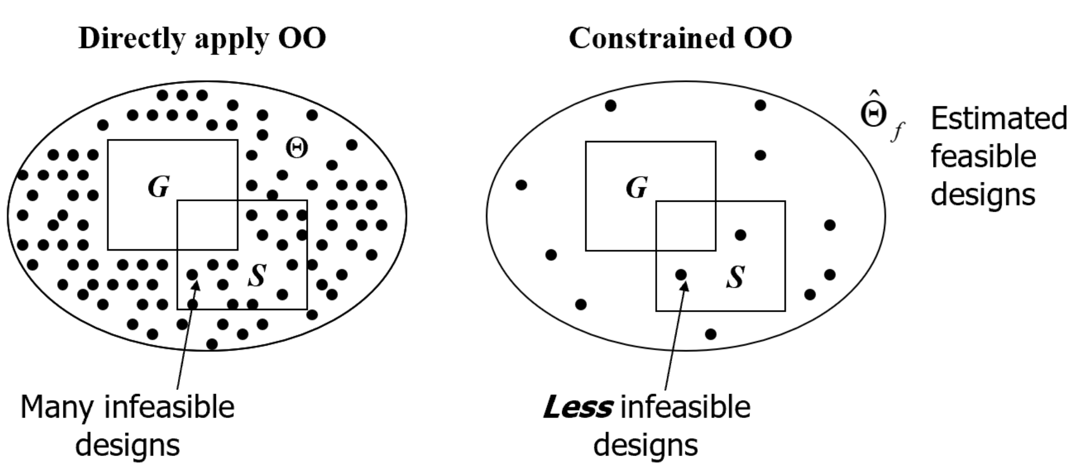
6. Selection Rules:

   a. Blind Picking

   b. Horse Racing (better strategy)

7. Constrained OO:

   a. find feasible designs:



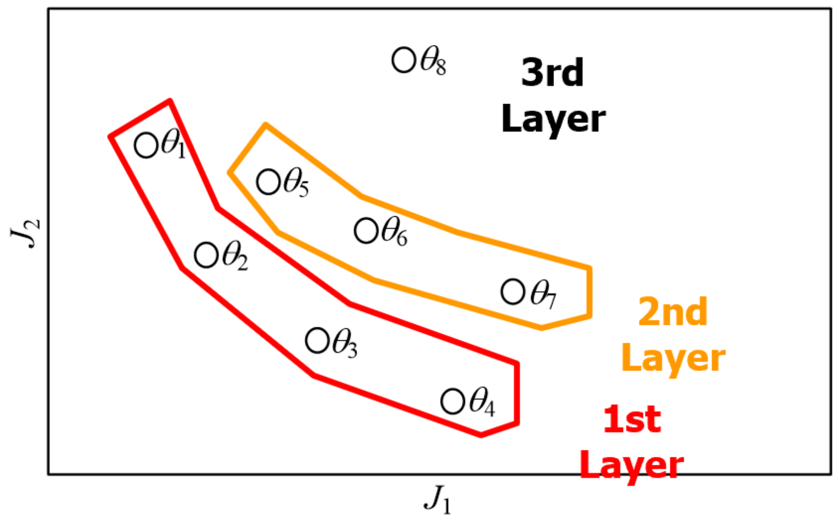
   b. apply OO method

8. Vector OO:

   a. multi-objective

$J_1, J_2 \ldots J_n$

b. different layers:



- same layer: can't make comparison
- 1st layer: Pareto Frontier
- smaller index layer —> better
- good enough set: first n layer

9. Applications:

   a. Remanufacturing systems

   b. Witsenhausen Problem:

> ⚙️ Witsenhausen问题是一个经典的控制理论问题，由荷兰数学家Hendrik Witsenhausen在1968年提出。问题的背景是考虑一个由两个部分组成的系统，其中一个部分是随机的，另一个部分是可控的。问题的目标是设计一个控制器，使得系统的性能最优化。
>
> 具体而言，Witsenhausen问题涉及到一个信道，其中一个输入值通过信道转换成一个输出值。这个信道具有非线性的性质，并且有两个不同的噪声源干扰输入值和输出值。问题的目标是设计一个控制器，使得在给定的控制代价下，系统的平均误差最小化。
>
> Witsenhausen问题是一个非常困难的问题，因为它具有非凸性和非线性的性质，而且没有简单的解析解。近年来，人们通过使用数值方法和优化算法等技术，来逐步解决这个问题，但是仍然存在许多未解决的方面和挑战。

   c. OO for RL:

   - selection of initial parameters
   - online: selection among different policies
   - offline: selection among action space