A Ring Theory Proof that Infinitely Many Primes Exist

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1 Introduction

Definition 1. *Let $\mathbb{Z}$ be the ring of integers. Let $\mathbf{P}$ denote the set of all primes.*

Theorem 1. $\mathbf{P}$ *is infinite.*

Proof. Let $p_i \in \mathbf{P}$ denote a prime number. Let (p_i) denote the principal ideal generated by prime number p_i. Every integer except $\{-1, 1\}$ has a prime factor. Thus, every integer $x \in \mathbb{Z} \setminus \{-1, 1\}$ belongs to (p_i) for some prime p_i.

Consider the ideal (p_i). It clearly does not contain 1, so $1 + (p_i)$ is a coset of this ideal. This gives us the following relation

$$1 + (p_i) \subset \mathbb{Z} \setminus (p_i). \tag{1}$$

Running this over all the primes P gives us

$$\bigcap_{p_i \in \mathbf{P}} 1 + (p_i) \subset \bigcap_{p_i \in \mathbf{P}} \mathbb{Z} \setminus (p_i) \tag{2}$$

Since every integer other than $\{1, -1\}$ lies in one of the ideals (p_i), by running p_i over $\mathbf{P}$, we obtain,

$$\bigcap_{p_i \in \mathbf{P}} \mathbb{Z} \setminus (p_i) = \{1, -1\}. \tag{3}$$

The equality sign is due to the fact that $\{1, -1\}$ are the only integers that don't have a prime factor. From equations 2 and 3, we obtain,

$$\bigcap_{p_i \in \mathbf{P}} 1 + (p_i) \subset \{1, -1\}. \tag{4}$$

Now, let us suppose that $\mathbf{P}$ is finite. In that case, let the product of all primes be $m = \prod p_i$. Let it generate a principal ideal (m).

$$(m) = \bigcap_{p_i \in \mathbf{P}} (p_i) \tag{5}$$

For an arbitrary ideal (p_i), we have

$$1 + (m) \subset 1 + (p_i). \tag{6}$$

Run p_i over $\mathbf{P}$, we obtain

$$1 + (m) \subset \bigcap_{p_i \in \mathbf{P}} 1 + (p_i) \subset \{1, -1\} \tag{7}$$

This is a contradiction. $\square$

2 The Argument, in a different language

Our proof is presented in the dialect of Ring Theory, but the idea of it is not local to this linguistic specification. Our contradiction is based on that $\{-1, 1\}$ should contain a coset of the ideal (m), – but it does not. Similarly, Furstenberg's proof, written in the language of topology, drives the contradiction that $\{1, -1\}$ should be open, – but it is not. Conrad, in his exposition, translates the proof into the language of Arithmetic Progressions, landing the contradition on that $\{1, -1\}$ should contain an AP (i.e. $1 + m\mathbb{Z}$), – but it does not.

All of these are variants of the same mathematical idea.

Similarly, "the countries that got tea via China through the Silk Road (land) referred to it in various forms of the word 'cha.' On the other hand, the countries that traded with China via sea – through the Min Nan port called it in different forms of 'te'."

References

[1] K. Conrad, "The 'Topological' Proof Of The Infinitude Of Prime," `https://kconrad.math.uconn.edu/blurbs/`

[2] H. Furstenberg, "On the Infinitude of Primes," Amer. Math. Monthly 62 (1955), 353.

[3] India in Pixels, "Cha, if by the Land. Tea, if by the Sea." The Authors' Personal Facebook Feed. (2020) `https://www.reddit.com/r/tea/comments/g49o1o/the_countries_that_got_tea_via_china_through_the/?utm_source=amp&utm_medium=&utm_content=tp_title`