Nanomagnetism and Spintronics

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Lecture 1

Basic magnetostatics

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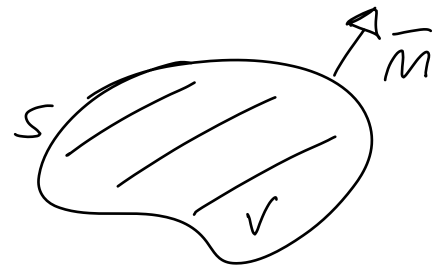
Maxwell equations in the absence of any current

$$\begin{cases} \bar{\nabla} \times \bar{H} = 0 \\ \bar{\nabla} \cdot \bar{B} = 0 \end{cases} \qquad \bar{H} = -\bar{\nabla} U$$



$$\bar{B} = \mu_0 (\bar{H}_M + \bar{M})$$

$$\bar{\nabla} \cdot \bar{B} = 0 \rightarrow \bar{\nabla} \cdot \bar{H}_M = -\bar{\nabla} \cdot \bar{M}$$

$$\bar{\nabla} \cdot (-\bar{\nabla} U) = -\bar{\nabla} \cdot \bar{M}$$

$$\nabla^2 U = \bar{\nabla} \cdot \bar{M}$$

INSIDE A MAGNETIC BODY

$$\nabla^2 U_{in} = \bar{\nabla} \cdot \bar{M} \quad \text{(POISSON)}$$

OUTSIDE $(\bar{M} = 0)$

$$\nabla^2 U_{out} = 0 \quad \text{(LAPLACE)}$$

BOUNDARY CONDITIONS ON THE SURFACE

$$\begin{cases} U_{in} = U_{out} \\ \dfrac{\partial U_{in}}{\partial m} - \dfrac{\partial U_{out}}{\partial m} = \bar{M} \cdot \bar{m} \end{cases}$$

The solution is UNIQUE!

U MUST BE REGULAR AT INFINITY: $\begin{cases} |r U| \\ |r^2 \nabla U| \end{cases}$ bounded as $r \rightarrow \infty$

Exercise: Demonstrate that the boundary conditions $H_{\parallel in} = H_{\parallel out}$ and $B_{\perp in} = B_{\perp out}$ are equivalent to the following condition on the scalar potential U

$$\begin{cases} U_{in} = U_{out} \\ \dfrac{\partial U_{in}}{\partial n} - \dfrac{\partial U_{out}}{\partial n} = \bar{M}\cdot\bar{n} \end{cases}$$

$$\int_S (\bar{H}+\bar{M})\cdot\bar{n}\, dS = 0$$

Proof:

$B_{in\perp} = B_{out\perp}$ (1)

$M_\perp + H_{in\perp} = H_{out}$

$$\bar{n}\cdot\bar{M} - \frac{\partial U_{in}}{\partial n} = -\frac{\partial U_{out}}{\partial n}$$

$$\frac{\partial U_{in}}{\partial n} - \frac{\partial U_{out}}{\partial n} = \bar{M}\cdot\bar{n}$$

$H_{in\parallel} = H_{out\parallel}$ (2)

This is automatically satisfied as $\bar{H} = -\bar{\nabla} U$, so that $\bar{\nabla}\times\bar{H} = 0$ from which (2) can be obtained.

However, we should avoid any divergence of $\bar{H} = -\bar{\nabla} U$ at the interface $\Rightarrow$ U must be continuous.

Solution of the differential equation for the scalar potential

Magnetic materials

$$\bar{B} = \mu_0(\bar{H} + \bar{M})$$

$$\bar{H} = -\nabla U$$

$$\bar{\nabla} \cdot \bar{B} = 0$$

$$\boxed{\nabla^2 U = \bar{\nabla} \cdot \bar{M}}$$

The same equations have the same solutions:

$$U(\bar{r}) = \frac{1}{4\pi}\left\{ \int_{Vol} \frac{-\bar{\nabla}\cdot\bar{M}}{|\bar{r}-\bar{r}'|} d\bar{r}' + \int_{Surf} \frac{\bar{M}\cdot\bar{u}}{|\bar{r}-\bar{r}'|} dS \right\}$$

By analogy:

$-\bar{\nabla}\cdot\bar{M}$: VOLUME MAGNETIC CHARGES

$\bar{M}\cdot\bar{u}$: SURFACE MAGNETIC CHARGES

Dielectrics

$$\bar{D} = \varepsilon_0 \bar{E} + \bar{P}$$

$$\bar{E} = -\bar{\nabla} V$$

$$\bar{\nabla} \cdot \bar{E} = \rho/\varepsilon_0$$

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

$$\rho = \rho_{free} + \rho_{pol} = \rho_{free} - \bar{\nabla}\cdot\bar{P}$$

if $\rho_{free} = 0$ (ONLY POL. CHARGES)

$$\boxed{\nabla^2 V = \frac{\bar{\nabla}\cdot\bar{P}}{\varepsilon_0}}$$

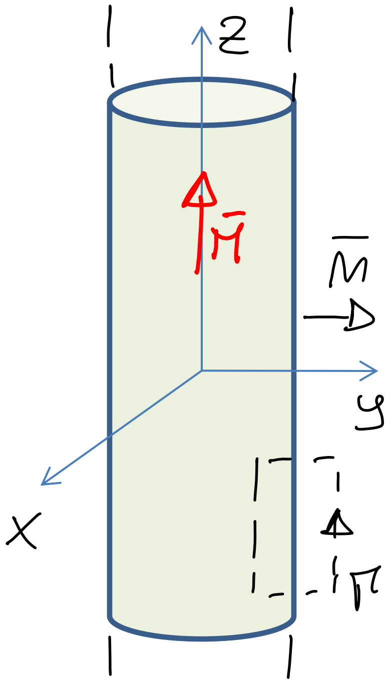
The solution for V is well-known:

$$V = \frac{1}{4\pi\varepsilon_0}\left\{ \int_{Vol} \frac{-\bar{\nabla}\cdot\bar{P}}{|\bar{r}-\bar{r}'|} d\bar{r}' + \int_{Surf.} \frac{\bar{P}\cdot\bar{u}}{|\bar{r}-\bar{r}'|} dS \right\}$$

$$\rho_{pol} = -\bar{\nabla}\cdot\bar{P}$$

$$\sigma_{pol} = \bar{P}\cdot\bar{u}$$

Example 1: infinite circular cylinder uniformly magnetized along z



$$\bar{\nabla}\cdot\bar{M} = 0 \Rightarrow \nabla^2 U = 0 \quad \text{everywhere}$$

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\right]U = 0$$

Boundary conditions: $\bar{M}\cdot\bar{m} = 0$

$$\frac{\partial U_{in}}{\partial r} - \frac{\partial U_{out}}{\partial r} = 0$$

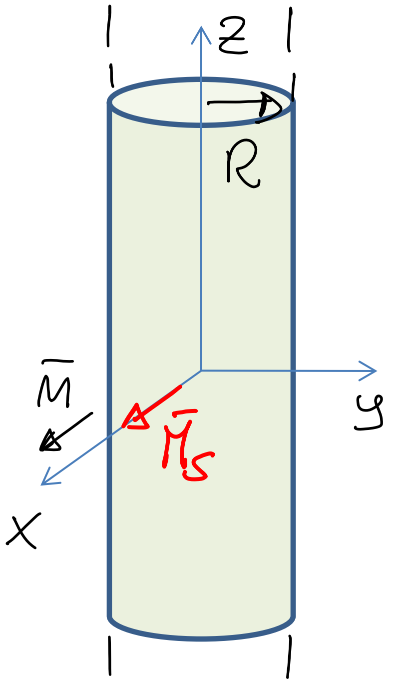
$$\left.\begin{array}{l} H_{in_r} = H_{out_r} \\ H_{in_z} = H_{out_z} \\ \oint_\Gamma \bar{H}\cdot d\bar{l} = 0 \end{array}\right\} \Rightarrow \bar{H}_{in} = \bar{H}_{out} = 0$$

The solution is $\left\{\begin{array}{l} \bar{B} = \mu_0 \bar{M} \\ \bar{H}_M = 0 \end{array}\right.$

$U = 0$

$E_M = 0$

Example 2: infinite circular cylinder uniformly magnetized along x



$$\bar{\nabla}\cdot\bar{M} = 0 \Rightarrow \nabla^2 U = 0 \text{ everywhere}$$

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\right]U = 0$$

Boundary conditions: $\bar{M}\cdot\bar{n} =$

$$\frac{\partial U_{in}}{\partial r} - \frac{\partial U_{out}}{\partial r} = M_s \cos\phi$$

The solution is:

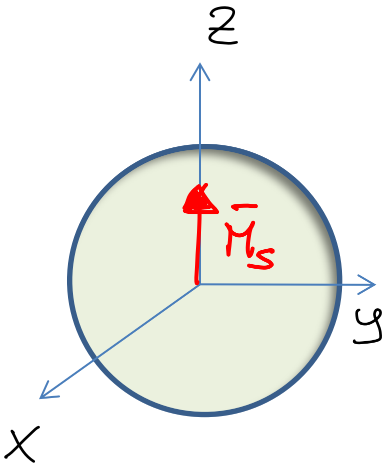
$$U = \frac{M_s}{2}\cos\phi \cdot \begin{cases} \rho & \text{if } \rho < R \\ R^2/\rho & \text{if } \rho > R \end{cases}$$

$$H_{x_{in}} = -\frac{M_s}{2} \qquad H_{y_{in}} = H_{z_{in}} = 0$$

$$E_M = \frac{1}{L}\int -\frac{\mu_0}{2}\bar{H}_M\cdot\bar{M}\,d\tau = \frac{\pi}{4}R^2 M_s^2 \mu_0$$

per unit length

Example 3: sphere uniformly magnetized along z



$$\bar{\nabla}\cdot\bar{M} = 0 \Rightarrow \nabla^2 U = 0 \text{ everywhere}$$

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]U = 0$$

Boundary conditions: $\bar{n}\cdot\bar{M} = M_s\cos\theta$

$$\frac{\partial U_{in}}{\partial r} - \frac{\partial U_{out}}{\partial r} = M_s\cos\theta$$

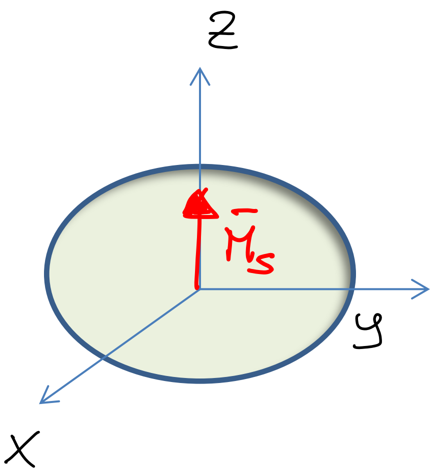
The solution is:

$$U = \frac{M_s}{3}\cos\theta\cdot\begin{cases} r & \text{if } r < R \\ \frac{R^3}{r^2} & \text{if } r > R \end{cases}$$

$$H_{x_{in}} = H_{y_{in}} = 0 \qquad H_{z_{in}} = -\frac{M_s}{3}$$

$$E_M = -\frac{\mu_0}{2}\int_V \bar{M}\cdot\bar{H}\,dV = +\frac{\mu_0}{2}\,\frac{4}{3}\pi R^3\,\frac{M_s^2}{3} = \frac{2}{9}\mu_0\pi R^3 M_s^2$$

Ellipsoid uniformly magnetized



THEOREM 1: if and only if the surface of a FM body is of a second degree, the internal field is uniform when M is unif.

For an ellipsoid: $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1$

$$\bar{H}_{in} = -N \cdot \bar{M}$$

└ DEMAGNETIZING TENSOR

THEOREM 2: the trace of N is $= 1$

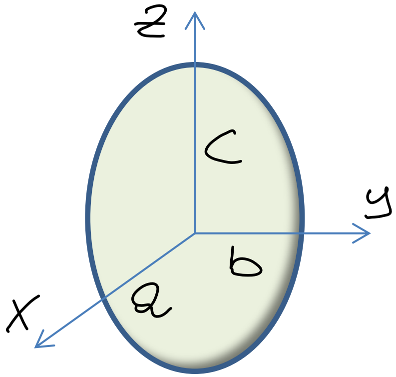
For a sphere: $N = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$

For a cylinder (infinite along z): $N = \begin{pmatrix} 1/2 & & \\ & 1/2 & \\ & & 0 \end{pmatrix}$

$$E_M = \frac{\mu_0}{2} V \left(N_x M_x^2 + N_y M_y^2 + N_z M_z^2\right)$$

SHAPE ANISOTROPY ENERGY

PROLATE SPHEROID



$$c > a = b \Rightarrow N_x = N_y > N_z$$

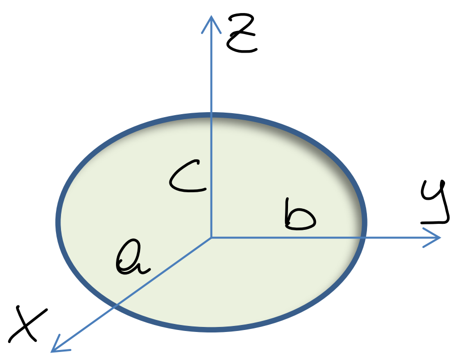
$$E_M = \frac{\mu_0}{2}V\left[N_x\left(M_x^2+M_y^2\right)+N_z M_z^2\right] =$$

$$= \frac{\mu_0}{2}V\left(N_z-N_x\right)M_z^2 + \underbrace{\frac{\mu_0}{2}V N_x\left(M_x^2+M_y^2+M_z^2\right)}_{\text{COST}}$$

$$= \frac{\mu_0}{2}V\underbrace{\left(N_z-N_x\right)}_{-}M_s^2\cos^2\theta$$

UNIAXIAL ANISOTROPY

OBLATE SPHEROID

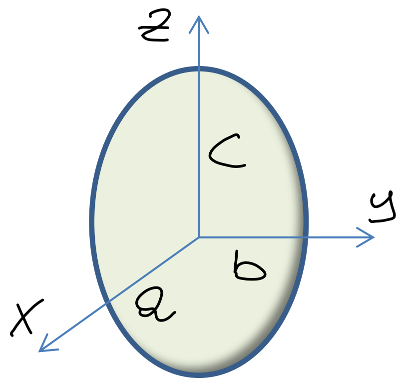


$$c < a = b \Rightarrow N_x = N_y < N_z$$

$$E_M = \frac{\mu_0}{2}V\underbrace{\left(N_z-N_x\right)}_{+}M_s^2\cos^2\theta$$

PLANE ANISOTROPY

Demagnetizing tensor components along the three axis for a prolate spheroid



$$a = b$$

$$p = \frac{c}{a} (> 1), \qquad \xi = \frac{\sqrt{p^2 - 1}}{p}$$

$$N_z = \frac{1}{p^2 - 1}\left[\frac{1}{2\xi} ln\left(\frac{1+\xi}{1-\xi}\right) - 1\right]; \quad N_x = N_y = \frac{1 - N_z}{2}$$

Osborn , J. A., Demagnetizing factors of the general ellipsoid, Phys. Rev., 67, 351-7 (1945)

Demagnetizing factors for non-ellipsoidal shapes

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Demagnetizing factors for rectangular ferromagnetic prisms

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We consider a uniform and homogeneous ferromagnetic particle in the shape of a rectangular prism, and we define the origin of a Cartesian coordinate system at the center of this prism. More specifically, we assume (as in Ref. 5) that the prism extends over the volume $-a \leqslant x \leqslant a$, $-b \leqslant y \leqslant b$ and $-c \leqslant z \leqslant c$, see Fig. 1. If this prism is saturated along z, a surface charge is created on its faces $z = \pm c$. The potential due to this charge can be calculated by well-known integrals on these surfaces, and the magnetic field is the gradient of that potential. It takes another integration of this field over the prism volume to obtain the magnetostatic self-energy, but all these integrations are nearly the same as in Ref. 9. On the whole, the algebra is non-trivial but straightforward in principle. The *magnetometric* demagnetizing factor in the z-direction, D_z, is defined as the factor that makes the magnetostatic self-energy per unit volume equal to $2\pi D_z M_s^2$.

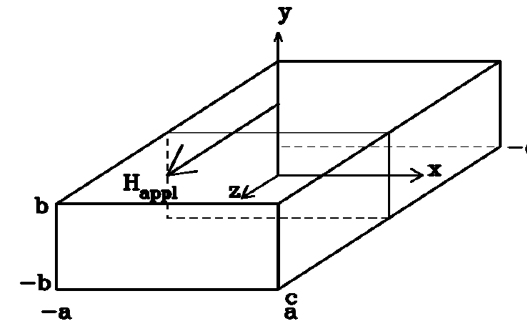

TABLE I. The demagnetizing factor, D_z^s, of a prolate spheroid and the magnetometric demagnetizing factor, D_z^p, of a square prism, for an aspect ratio, p.

p	D_z^s	D_z^p
2.0	0.17356	0.19832
3.0	0.10871	0.14036
4.0	0.075407	0.10845
5.0	0.055821	0.088316
6.0	0.043230	0.074466
7.0	0.034609	0.064363
8.0	0.028421	0.056670
9.0	0.023816	0.050617
10.0	0.020286	0.045731
11.0	0.017515	0.041705

$$\pi D_z = \frac{b^2-c^2}{2bc}\ln\left(\frac{\sqrt{a^2+b^2+c^2}-a}{\sqrt{a^2+b^2+c^2}+a}\right) + \frac{a^2-c^2}{2ac}\ln\left(\frac{\sqrt{a^2+b^2+c^2}-b}{\sqrt{a^2+b^2+c^2}+b}\right) + \frac{b}{2c}\ln\left(\frac{\sqrt{a^2+b^2}+a}{\sqrt{a^2+b^2}-a}\right) + \frac{a}{2c}\ln\left(\frac{\sqrt{a^2+b^2}+b}{\sqrt{a^2+b^2}-b}\right)$$

$$+ \frac{c}{2a}\ln\left(\frac{\sqrt{b^2+c^2}-b}{\sqrt{b^2+c^2}+b}\right) + \frac{c}{2b}\ln\left(\frac{\sqrt{a^2+c^2}-a}{\sqrt{a^2+c^2}+a}\right) + 2\arctan\left(\frac{ab}{c\sqrt{a^2+b^2+c^2}}\right) + \frac{a^3+b^3-2c^3}{3abc}$$

$$+ \frac{a^2+b^2-2c^2}{3abc}\sqrt{a^2+b^2+c^2} + \frac{c}{ab}\left(\sqrt{a^2+c^2}+\sqrt{b^2+c^2}\right) - \frac{(a^2+b^2)^{3/2}+(b^2+c^2)^{3/2}+(c^2+a^2)^{3/2}}{3abc}.$$

The other two demagnetizing factors, D_x and D_y, can be derived from this equation by applying twice the cyclic permutation $c \to a \to b \to c$. It is readily seen that these factors obey the relation

$$D_x + D_y + D_z = 1, \tag{2}$$