

732A96 Advanced Machine Learning
Graphical Models and Hidden Markov Models

Jose M. Peña
IDA, Linköping University, Sweden

Lecture 5: Dynamic Bayesian Networks and Hidden Markov Models

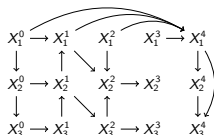
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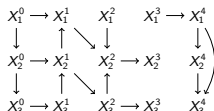
- ▶ Main source
 - ▶ Bishop, C. M. *Pattern Recognition and Machine Learning*. Springer, 2006. Chapter 13.1-13.2.
- ▶ Additional source
 - ▶ Ghahramani, Z. An Introduction to Hidden Markov Models and Bayesian Networks. *International Journal of Pattern Recognition and Artificial Intelligence* 15, 9-42, 2001.

Dynamic Bayesian Networks: Definition

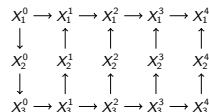
- ▶ To model **sequential data**, e.g. time series data.
- ▶ **Simplification**: Time is discretized in equal width intervals, i.e. $t = 0, 1, \dots$
- ▶ Consider a finite set of discrete random variables $X^t = \{X_1^t, \dots, X_n^t\}$ representing the state at time t of a system described by $V = \{X_1, \dots, X_n\}$.
- ▶ A **dynamic Bayesian network** (DBN) is a BN over $X^{0:T} = \{X^0, \dots, X^T\}$. Thus, it defines $p(X^{0:T})$.



- ▶ **Assumption**: The system is Markovian, i.e. $X^{t+1} \perp_p X^{0:t-1} | X^t$.

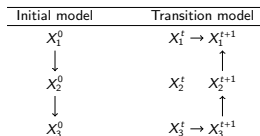


- ▶ **Assumption**: The system is stationary, i.e. $p(X^{t+1} | X^t) = p(X' | X)$.



Dynamic Bayesian Networks

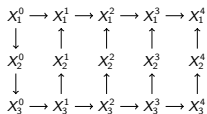
- ▶ Then, a DBN over $X^{0:T}$ can be defined as
 - ▶ a BN over X^0 , and
 - ▶ a BN over $X^t \cup X^{t+1}$ where the nodes in X^t are parentless.



- ▶ The DBN defines

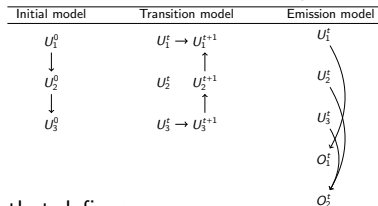
$$p(X^{0:T}) = p(X^0) \prod_{t=0}^{T-1} p(X^{t+1}|X^t) = \left[\prod_{i=1}^n p(X_i^0 | pa_G(X_i^0)) \right] \left[\prod_{t=0}^{T-1} \prod_{i=1}^n p(X_i^{t+1} | pa_G(X_i^{t+1})) \right]$$

- ▶ DBN unrolled for $T = 4$.



Hidden Markov Models: Definition

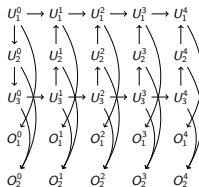
- ▶ To overcome the **Markovian limitation** of DBNs, while keeping sparsity.
- ▶ A **hidden Markov model** (HMM) over $\{Z^{0:T}, X^{0:T}\}$ where $X^{0:T}$ are **observed** and $Z^{0:T}$ are **unobserved** consists of
 - ▶ a DBN over $Z^{0:T}$, and
 - ▶ a BN over $Z^t \cup X^t$ where the nodes in Z^t are parentless.



- ▶ A HMM is a DBN that defines

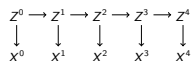
$$p(Z^{0:T}, X^{0:T}) = p(Z^0) \prod_{t=1}^T p(Z^{t+1}|Z^t) \prod_{t=0}^T p(X^t|Z^t)$$

- ▶ HMM unrolled for $T = 4$.



Hidden Markov Models: Learning

- ▶ The structure is typically fixed to



- ▶ Consider a sample with a single observation over $X^{0:T}$.
- ▶ Parameter learning: EM algorithm.
- ▶ Cardinality of Z^t ? BIC score to select among a set of plausible values.

Hidden Markov Models: Learning

- Recall that maximizing the log likelihood function over $x^{0:T}$ is inefficient (no closed form solution) and ineffective (multimodal).
- Consider instead maximizing its expectation

$$\begin{aligned} E_{z^{0:T}} [\log p(z^{0:T}, x^{0:T})] &= \sum_{z^{0:T}} p(z^{0:T} | x^{0:T}) \log p(z^{0:T}, x^{0:T}) \\ &= \sum_{z^{0:T}} p(z^{0:T} | x^{0:T}) [\log \theta_{z^0} + \sum_{t=1}^T \log \theta_{z^{t+1}|z^t} + \sum_{t=1}^T \log \theta_{x^t|z^t}] \\ &= \sum_{z^0} p(z^0 | x^{0:T}) \log \theta_{z^0} + \sum_{t=1}^T \sum_{z^t} \sum_{z^{t+1}} p(z^t, z^{t+1} | x^{0:T}) \log \theta_{z^{t+1}|z^t} + \sum_{t=1}^T \sum_{z^t} p(z^t | x^{0:T}) \log \theta_{x^t|z^t} \end{aligned}$$

- Then

- $\theta_{z^0}^{ML} = \frac{p(z^0 | x^{0:T})}{\sum_{z^0} p(z^0 | x^{0:T})}$
- $\theta_{z^{t+1}|z^t}^{ML} = \frac{\sum_{t=1}^T p(z^t, z^{t+1} | x^{0:T})}{\sum_{t=1}^T \sum_{z^{t+1}} p(z^t, z^{t+1} | x^{0:T})}$
- $\theta_{x^t|z^t}^{ML} = \frac{\sum_{t=1}^T p(z^t | x^{0:T}) 1_{\{x^t \in x^{0:T}\}}}{\sum_{t=1}^T p(z^t | x^{0:T})}$

- Note that computing $p(z^0 | x^{0:T})$, $p(z^t, z^{t+1} | x^{0:T})$ and $p(z^t | x^{0:T})$ requires inference: Forward-backward algorithm.

Hidden Markov Models: Forward-Backward Algorithm

$$\begin{aligned}
 p(Z^t | x^{0:T}) &= \frac{p(x^{0:T} | Z^t) p(Z^t)}{p(x^{0:T})} \\
 &= \frac{p(x^{0:t} | Z^t) p(Z^t) p(x^{t+1:T} | Z^t)}{p(x^{0:T})} \text{ by } x^{0:t} \perp_p x^{t+1:T} | Z^t \\
 &= \frac{p(x^{0:t}, Z^t) p(x^{t+1:T} | Z^t)}{p(x^{0:T})} = \frac{\alpha(Z^t) \beta(Z^t)}{\sum_{z^t} \alpha(z^t) \beta(z^t)}
 \end{aligned}$$

$$\begin{aligned}
 p(Z^t, Z^{t+1} | x^{0:T}) &= \frac{p(x^{0:T} | Z^t, Z^{t+1}) p(Z^t, Z^{t+1})}{p(x^{0:T})} \\
 &= \frac{p(x^{0:t} | Z^t) p(x^{t+1} | Z^{t+1}) p(x^{t+2:T} | Z^{t+1}) p(Z^{t+1} | Z^t) p(Z^t)}{p(x^{0:T})} \\
 &\text{by } \begin{aligned} &x^{0:t} \perp_p x^{t+1:T} | Z^t \cup Z^{t+1} \\ &x^{0:t} \perp_p Z^{t+1} | Z^t \\ &x^{t+1:T} \perp_p Z^t | Z^{t+1} \\ &x^{t+1} \perp_p x^{t+2:T} | Z^{t+1} \end{aligned} \\
 &= \frac{\alpha(Z^t) \beta(Z^{t+1}) p(x^{t+1} | Z^{t+1}) p(Z^{t+1} | Z^t)}{\sum_{z^t} \sum_{z^{t+1}} \alpha(z^t) \beta(z^{t+1}) p(x^{t+1} | z^{t+1}) p(z^{t+1} | z^t)}
 \end{aligned}$$

Hidden Markov Models: Forward-Backward Algorithm

$$\begin{aligned}
 \alpha(Z^t) &= p(x^t|Z^t)p(Z^t)p(x^{0:t-1}|Z^t) \text{ by } X^{0:t-1} \perp_p X^t|Z^t \\
 &= p(x^t|Z^t)p(x^{0:t-1}, Z^t) = p(x^t|Z^t) \sum_{z^{t-1}} p(x^{0:t-1}, Z^t|z^{t-1})p(z^{t-1}) \\
 &= p(x^t|Z^t) \sum_{z^{t-1}} p(x^{0:t-1}|z^{t-1})p(Z^t|z^{t-1})p(z^{t-1}) \text{ by } X^{0:t-1} \perp_p Z^t|Z^{t-1} \\
 &= p(x^t|Z^t) \sum_{z^{t-1}} p(x^{0:t-1}, z^{t-1})p(Z^t|z^{t-1}) = p(x^t|Z^t) \sum_{z^{t-1}} \alpha(z^{t-1})p(Z^t|z^{t-1})
 \end{aligned}$$

$$\alpha(Z^0) = p(x^0|Z^0)p(Z^0)$$

$$\begin{aligned}
 \beta(Z^t) &= \sum_{z^{t+1}} p(x^{t+1:T}, z^{t+1}|Z^t) = \sum_{z^{t+1}} p(x^{t+1:T}|z^{t+1}, Z^t)p(z^{t+1}|Z^t) \\
 &= \sum_{z^{t+1}} p(x^{t+1:T}|z^{t+1})p(z^{t+1}|Z^t) \text{ by } X^{t+1:T} \perp_p Z^t|Z^{t+1} \\
 &= \sum_{z^{t+1}} p(x^{t+2:T}|z^{t+1})p(x^{t+1}|z^{t+1})p(z^{t+1}|Z^t) \text{ by } X^{t+2:T} \perp_p X^{t+1}|Z^{t+1} \\
 &= \sum_{z^{t+1}} \beta(z^{t+1})p(x^{t+1}|z^{t+1})p(z^{t+1}|Z^t)
 \end{aligned}$$

$$\beta(Z^T) = 1 \text{ by } p(Z^T|x^{0:T}) = \frac{\alpha(Z^T)\beta(Z^T)}{p(x^{0:T})} = p(Z^T|x^{0:T})\beta(Z^T)$$

Hidden Markov Models: Forward-Backward Algorithm

FB algorithm

$$\alpha(Z^0) := p(x^0|Z^0)p(Z^0)$$

For $t = 1, \dots, T$ do

$$\alpha(Z^t) := p(x^t|Z^t) \sum_{z^{t-1}} \alpha(z^{t-1}) p(Z^t|z^{t-1})$$

$$\beta(Z^T) := 1$$

For $t = T, \dots, 0$ do

$$\beta(Z^t) := \sum_{z^{t+1}} \beta(z^{t+1}) p(x^{t+1}|z^{t+1}) p(z^{t+1}|Z^t)$$

Return $\alpha(Z^0), \dots, \alpha(Z^T), \beta(Z^0), \dots, \beta(Z^T)$

- ▶ Unlike the LS algorithm, the FB algorithms consists of two independent steps.
- ▶ **Filtering:** $p(Z^t|x^{0:t}) = \frac{\alpha(Z^t)}{\sum_{z^t} \alpha(z^t)}$.
- ▶ **Smoothing:** $p(Z^t|x^{0:T}) = \frac{\alpha(Z^t)\beta(Z^t)}{\sum_{z^t} \alpha(z^t)\beta(z^t)}$.

Hidden Markov Models: Viterbi Algorithm

- ▶ To compute the most probable configuration for HMMs.

Viterbi algorithm

$$\omega(Z^0) := \log p(Z^0) + \log p(x^0|Z^0)$$

For $t = 0, \dots, T$ do

$$\omega(Z^{t+1}) := \log p(x^{t+1}|Z^{t+1}) + \max_{z^t} [\log p(z^{t+1}|z^t) + \omega(z^t)]$$

$$\psi(Z^{t+1}) := \arg \max_{z^t} [\log p(z^{t+1}|z^t) + \omega(z^t)]$$

For $t = T, \dots, 0$ do

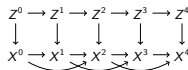
$$u_{\max}^t := \psi(u_{\max}^{t+1})$$

Return $u_{\max}^{0:T}$

- ▶ **Exercise.** Prove that the Viterbi algorithm is correct.

Autoregressive Hidden Markov Models

- ▶ To overcome the poor modeling of long range correlations in HMMs, by allowing $pa_G(X^t) \neq \emptyset$.



- ▶ **Exercise.** Derive the EM algorithm for AR-HMMs with $pa_G(X^t) = \{X^{t-1}\}$. Specifically, derive the recursive expressions for $\alpha(Z^t)$ and $\beta(Z^t)$ to be used in the FB algorithm.
- ▶ **Hint.**

$$\begin{aligned}
 p(Z^t | x^{0:T}) &= \frac{p(x^{0:t-1}, x^{t+1:T} | Z^t, x^t) p(Z^t, x^t)}{p(x^{0:T})} \\
 &= \frac{p(x^{0:t-1} | Z^t, x^t) p(Z^t, x^t) p(x^{t+1:T} | Z^t, x^t)}{p(x^{0:T})} \text{ by } x^{0:t-1} \perp_p x^{t+1:T} | Z^t \cup x^t \\
 &= \frac{p(x^{0:t}, Z^t) p(x^{t+1:T} | Z^t, x^t)}{p(x^{0:T})} = \frac{\alpha(Z^t) \beta(Z^t)}{\sum_{z^t} \alpha(z^t) \beta(z^t)}
 \end{aligned}$$

Autoregressive Hidden Markov Models

► **Hint.**

$$\begin{aligned} p(Z^t, Z^{t+1} | x^{0:T}) &= \frac{p(x^{0:t-1}, x^{t+2:T} | Z^t, Z^{t+1}, x^t, x^{t+1}) p(Z^t, Z^{t+1}, x^t, x^{t+1})}{p(x^{0:T})} \\ &= \frac{p(x^{0:t-1} | Z^t, x^t) p(x^{t+2:T} | Z^{t+1}, x^{t+1}) p(x^{t+1} | Z^{t+1}, x^t) p(Z^{t+1} | Z^t) p(Z^t, x^t)}{p(x^{0:T})} \\ &\quad \text{by } X^{0:t-1} \perp_p X^{t+2:T} | Z^t \cup Z^{t+1} \cup X^t \cup X^{t+1} \\ &\quad X^{0:t-1} \perp_p Z^{t+1} \cup X^{t+1} | Z^t \cup X^t \\ &\quad X^{t+2:T} \perp_p Z^t \cup X^t | Z^{t+1} \cup X^{t+1} \\ &\quad X^{t+1} \perp_p Z^t | Z^{t+1} \cup X^t \\ &\quad Z^{t+1} \perp_p X^t | Z^t \\ &= \frac{\alpha(Z^t) \beta(Z^{t+1}) p(x^{t+1} | Z^{t+1}, x^t) p(Z^{t+1} | Z^t)}{\sum_{z^t} \sum_{z^{t+1}} \alpha(z^t) \beta(z^{t+1}) p(x^{t+1} | z^{t+1}, x^t) p(z^{t+1} | z^t)} \end{aligned}$$

► **Exercise.** Derive the Viterbi algorithm for AR-HMMs.