1.5. ТР1.

Жорков И.
А-02-23.

$$\int x\sqrt[6]{(3x^2+4)^5}\,dx$$

$$t = 3x^2+4.$$

$$dt = 6x\,dx \Rightarrow x\,dx = \frac{dt}{6}.$$

$$\int t^{\frac{5}{6}}\frac{dt}{6} = \frac{6t^{\frac{11}{6}}}{11\cdot 6} = \frac{t^{\frac{11}{6}}}{11}$$

Обратная замена:

$$\frac{t^{\frac{11}{6}}}{11} = \frac{(3x^2+4)^{\frac{11}{6}}}{11} + C.$$

Ответ: $\frac{(3x^2+4)^{\frac{11}{6}}}{11} + C.$

ТР 2.

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2.5.

$$\int \arccos 3x\,dx = \left\{\begin{matrix} t = 3x \\ x = \frac{t}{3} \\ dx = \frac{1}{3}dt \end{matrix}\right\} = \int \frac{\arccos t}{3}\,dt =$$

$$= \left\{\begin{matrix} u = \arccos t \\ u' = \frac{1}{\sqrt{1-t^2}} \\ v' = 1 \\ v = t \end{matrix}\right\} = \frac{1}{3}\left(t \arccos t + \int \frac{t}{\sqrt{1-t^2}}\,dt\right) =$$

$$= \frac{1}{3}\left(t \arccos t - \sqrt{1-t^2}\right) = x \arccos 3x$$

$$- \frac{\sqrt{1-9x^2}}{3}$$

ТР 3 Жарков И

А-02-23

3.5.

$$\int \frac{3x-10}{\sqrt{x^2-8x+7}}\,dx = \int \frac{3x-10}{\sqrt{(x-4)^2-9}}\,dx =$$

$$= \left\{ \begin{matrix} x-4=t \\ x=t+4 \end{matrix} \right\} = \int \frac{3(t+4)-10}{\sqrt{t^2-9}}\,dt =$$

$$= 3\int \frac{t+4}{\sqrt{t^2-9}}\,dt - 10\int \frac{dt}{\sqrt{t^2-9}} \cancel{dt} =$$

$$= 3\int \frac{x\,dx}{\sqrt{(x-4)^2-9}} - 10\int \frac{dt}{\sqrt{t^2-9}} =$$

$$= \frac{3}{2}\int \frac{dx}{\sqrt{(x-4)^2-9}} - 10\int \frac{dt}{\sqrt{t^2-9}} =$$

$$= \frac{3}{2}\ln\left|(x-4)+\sqrt{(x-4)^2-9}\right| - 10\ln\left|t+\sqrt{t^2-9}\right| + C =$$

$$= -8{,}5\ln\left|(x-4)+\sqrt{(x-4)^2-9}\right| + C$$

ТР 4. Жарков И

4.5. А-02-23

$$\int \frac{3x^2-2}{x^3-x}\,dx = \int \frac{3x^3-3x+3x-2}{x^3-x}\,dx =$$

$$= 3\int dx + \int \frac{3x-2}{x^3-x} =$$

$$x^3-x = x(x-1)(x+1)$$

$$\frac{3x-2}{x^3-x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} =$$

$$= \frac{A(x^2-1)+B(x^2+x)+C(x^2-x)}{x(x-1)(x+1)} =$$

$$= \frac{x^2(A+B+C)+x(B-C)+(-A)}{x^3-x}$$

$$\begin{cases} -A = -2 \\ B - C = 3 \\ A+B+C = 0 \end{cases} \qquad \begin{aligned} &A = 2 \\ &B = 3 + C \\ &C = -2{,}5 \\ &B = 0{,}5. \end{aligned}$$

$$\frac{3x-2}{x^3-x} = 2\frac{1}{x} + \frac{1}{2}\frac{1}{x-1} - \frac{5}{2}\frac{1}{x+1}$$

$$= 3\int dx + 2\int\frac{dx}{x} + \frac{1}{2}\int\frac{dx}{x-1} - \frac{5}{2}\int\frac{dx}{x+1} =$$

$$= 3x + 2\ln|x| + \frac{1}{2}\ln|x-1| - \frac{5}{2}\ln|x+1| + C$$

5.6. TP 5.

$$\int \frac{dx}{(x+1)\sqrt{45x^2+66x+25}} = \left| \begin{array}{l} u = \frac{1}{x+1} \\ x = -\frac{u-1}{u} \\ dx = -(x+1)^2 du \end{array} \right| = \int -\frac{du}{\sqrt{4u^2-24u+45}} =$$

$$= -\int \frac{du}{\sqrt{(2u-6)^2+9}} = \left| \begin{array}{l} t = 2u-6 \\ u = \frac{t+6}{2} \\ du = \frac{1}{2} dt \end{array} \right| = -\int \frac{dt}{2\sqrt{t^2+9}} = -\frac{\ln(\sqrt{t^2+9}+t)}{2} =$$

$$= -\frac{\ln(\sqrt{4u^2-24u+45}+2u-6)}{2} = -\frac{\ln((x+1)\sqrt{45x^2+66x+25} + (-6x-4)\sqrt{x^2+2x+1}) - \ln((x+1)\sqrt{x^2+2x+1})}{2} + C$$

6.5. TP6.

$$\int \cos\sqrt{\frac{1-3x}{3x+2}}\,\frac{\sqrt{3x+2}}{\sqrt{1-3x}\,(3x+2)^2}\,dx = \left| t = 1-3x \right| = \int -\frac{\cos\left(\sqrt{\frac{t}{3-t}}\right)}{3\cdot\sqrt{t}\,(3-t)^{\frac{3}{2}}}\,dt =$$

$$= \left| u = \frac{\sqrt{t}}{\sqrt{3-t}} \right| = \int -\frac{1}{9}\cos(u)\,du = \int -\frac{1}{9}\sin(u) = -\frac{1}{9}\sin\left(\frac{1}{\sqrt{3x+2}}\right)+C$$

8.5. TP8.

$$\int \sqrt{-3x^2+13x+1}\,dx = \int \sqrt{\frac{181}{12}-\left(\sqrt{3}\,x-\frac{13}{2\sqrt{3}}\right)^2}\,dx = \left| \begin{array}{l} u=\sqrt{3}x-\frac{13}{2\sqrt{3}} \\ x=\frac{2\sqrt{3}u+13}{6} \\ dx=\frac{1}{\sqrt{3}}du \end{array} \right| =$$

$$= \int \frac{\sqrt{\frac{181}{12}-u^2}}{\sqrt{3}}\,du = \int \frac{\sqrt{181-12u^2}}{6}\,du = \left| \begin{array}{l} v=\arcsin\left(\frac{2\sqrt{3}u}{\sqrt{181}}\right) \\ du=\frac{\sqrt{181}\cos(v)}{2\sqrt{3}}\,dv \end{array} \right| =$$

$$= \frac{1}{6}\int \frac{181\cos^2(v)}{2\sqrt{3}}\,dv = \frac{181}{4\cdot 3\sqrt{3}}\int \frac{\cos(2v)+1}{2}\,dv = \frac{181\sin(2v)}{16\cdot 3\sqrt{3}} + \frac{181v}{8\cdot 3\sqrt{3}} =$$

$$= \frac{181\arcsin\left(\frac{6x-13}{\sqrt{181}}\right)}{8\cdot 3\sqrt{3}} + \frac{x\sqrt{-36x^2+156x+12}}{4\sqrt{3}} - \frac{13\sqrt{-36x^2+156x+12}}{8\cdot 3\sqrt{3}}.$$

9.5. TP 9.

$$\int \sin^6 \frac{x}{2}\, dx = \left| \begin{matrix} u = \frac{x}{2} \\ dx = 2du \end{matrix} \right| = \int 2\sin^6(u)\, du = 2\int \frac{(1-\cos(2u))^3}{8}\, du =$$

$$= \left| \begin{matrix} v = 2u \\ du = \frac{1}{2} dv \end{matrix} \right| = \frac{1}{4}\int \frac{(1-\cos(v))^3}{2}\, dv = \frac{3\sin(2v)}{32} + \frac{\sin^3(v)}{24} - \frac{\sin(v)}{2} + \frac{5v}{16} =$$

$$= \frac{3\sin(4u)}{32} + \frac{\sin^3(2u)}{24} - \frac{\sin(2u)}{2} + \frac{5u}{8} = \frac{3\sin(2x)}{32} + \frac{\sin^3(x)}{24} -$$

$$- \frac{\sin(x)}{2} + \frac{5x}{16}$$

11.5. TP 11.

$$\int_{\frac{1}{\sqrt{2}}}^{1} \frac{dx}{x\sqrt{4x^2-1}} = \operatorname{arctg}\left(\sqrt{4x^2-1}\right)\Big|_{\frac{1}{\sqrt{2}}}^{1} = \frac{\pi}{12}.$$

TP 12.

12.5.

$$\int_0^1 (2x+3)4^{2x}dx =$$

$$\int (2x+3)4^{2x}dx = \left| \begin{matrix} u = 2x \\ x = \frac{u}{2} \\ dx = \frac{1}{2}du \end{matrix} \right| = \int \frac{(u+3)4^u}{2}du = \frac{u\cdot 4^u}{2\ln(4)} + \frac{3\cdot 4^u}{2\ln(4)}$$

$$- \frac{4^u}{2\ln^2(4)} = \frac{x4^{2x}}{\ln(4)} + \frac{3\cdot 4^{2x}}{2\ln 4} - \frac{4^{2x}}{2\ln^2 4}$$

$$= \frac{x4^{2x}}{\ln 4} + \frac{3\cdot 4^{2x}}{2\ln 4} - \frac{4^{2x}}{2\ln^2 4}\Big|_0^1 = \frac{77}{2\ln 4} - \frac{15}{2\ln^2 4}$$

13.5. ТР 13.

$y = \ln(x+2),\ x = e-2,\ x = e^4-2,\ y=0.$

$$S = \int_{e-2}^{e^4-2} \ln(x+2)\,dx = \left(x\ln(x+2) + 2\ln(x+2) - x\right)\Big|_{e-2}^{e^4-2} =$$

$$= 3e^4.$$

ТР 14.

14.5.

$\rho^2 = 2\sin 2\varphi \qquad 0 \le \varphi \le \frac{\pi}{2}$

$$S = \frac{1}{2}\int_0^{\frac{\pi}{2}} \rho^2(\varphi)\,d\varphi = \frac{1}{2}\int_0^{\frac{\pi}{2}} 2\sin 2\varphi\,d\varphi = -\cos(2x)\Big|_0^{\frac{\pi}{2}} = 2.$$

Ответ: 2.

ТР 15.

15.5.

$\rho = 2\sin^3\frac{\varphi}{3},\ 0 \le \varphi \le \frac{\pi}{2}.$

$$l = \int_0^{\frac{\pi}{2}} \sqrt{4\sin^6\frac{\varphi}{3} + 4\cos^2\frac{\varphi}{3}\sin^4\frac{\varphi}{3}}\,d\varphi = \left(x - \frac{3\sin\left(\frac{2x}{3}\right)}{2}\right)\Big|_0^{\frac{\pi}{2}} =$$

$$= \frac{\pi}{2} - \frac{3\sqrt{3}}{4}$$

TP 1.

$$\int_{-\sqrt{2}}^{-1} dx \int_{-\sqrt{2-x}}^{0} f\,dy + \int_{-1}^{0} dx \int_{x}^{0} f\,dy \; = $$

$D_1: \begin{cases} -\sqrt{2} \le x \le -1 \\ -\sqrt{2-x} \le y \le 0 \end{cases}$ $\quad D_2: \begin{cases} -1 \le x \le 0 \\ x \le y \le 0 \end{cases}$

$y^2 = -\sqrt{2-x^2} \le 0$

$-y = \sqrt{2-x^2}$

$y^2 = 2-x^2$

$x^2+y^2 = 2$

$-\sqrt{2-x^2} = x$

$2-x^2 = x^2$

$2 = 2x^2$

$x^2 = 1$

$x = \pm 1, \; y = -1$

$D: \begin{cases} -1 \le y \le 0 \\ -\sqrt{2-y^2} \le x \le y \end{cases}$

$$= \int_{-1}^{0} dy \int_{-\sqrt{2-y^2}}^{y} f\,dx$$

Odgovor: $\int_{-1}^{0} dy \int_{-\sqrt{2-y^2}}^{y} f\,dx$

2.5. TP2.

$$\iint\limits_{D} (27x^2y^2+48x^3y^3)\,dx\,dy =$$

$D: x=1;\ y=x^2$
$y=-\sqrt[3]{x}$

$$D: \begin{cases} 0 \le x \le 1 \\ -\sqrt[3]{x} \le y \le x^2 \end{cases}$$

$$= \int_0^1 dx \int_{-\sqrt[3]{x}}^{x^2} (27x^2y^2+48x^3y^3)\,dy = \int_0^1 \left(27x^2\frac{y^3}{3}+48x^3\frac{y^4}{4}\right)\Bigg|_{-\sqrt[3]{x}}^{x^2} dx =$$

$$= \int_0^1 \left(9x^2(x^6+x)+12x^3(x^8-x^{\frac{4}{3}})\right) dx = \int_0^1 \left(9x^8+9x^3+12x^{11}-12x^{\frac{13}{3}}\right) dx =$$

$$= \left(9\frac{x^9}{9}+9\frac{x^4}{4}+12\frac{x^{12}}{12}-12\frac{3x^{\frac{16}{3}}}{16}\right)\Bigg|_0^1 = 1+\frac{9}{4}+1-\frac{36}{16} =$$

$$= 2+\frac{9}{4}-\frac{9}{4} = 2.$$

3.5.

ТР3.

$$\iint_D y \sin xy \, dx\,dy \qquad D: y = \frac{\pi}{2};\ y = \pi;\ x = 1;\ x = 2.$$

$$\int_{\frac{\pi}{2}}^{\pi} y\,dy \int_1^2 \sin xy \, dx = \int_{\frac{\pi}{2}}^{\pi} y\,dy\, \frac{1}{y}\left(-\sin xy\right)\Big|_1^2 = \int_{\frac{\pi}{2}}^{\pi} dy\left(-\cos 2y + \cos y\right) =$$

$$= \int_{\frac{\pi}{2}}^{\pi} \cos y\,dy - \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \cos 2y\, d2y = \left(\sin y - \frac{1}{2}\sin 2y\right)\Big|_{\frac{\pi}{2}}^{\pi} = \sin\pi - \frac{1}{2}\sin 2\pi -$$

$$- \sin\frac{\pi}{2} + \frac{1}{2}\sin\pi = -1$$

5.5. TP 5.

$$\iiint\limits_V (1+2x^3)\,dx\,dy\,dz = \qquad V: \begin{cases} y=9x,\ y=0,\ x=1 \\ z=\sqrt{xy},\ z=0. \end{cases}$$

$$V: \begin{cases} 0 \le x \le 1 \\ 0 \le y \le 9x \\ 0 \le z \le \sqrt{xy} \end{cases}$$

$$= \int_0^1 dx \int_0^{9x} dy \int_0^{\sqrt{xy}} (1+2x^3)\,dz = \int_0^1 (1+2x^3)\,dx \int_0^{9x} dy\,(z)\Big|_0^{\sqrt{xy}} =$$

$$= \int_0^1 (1+2x^3)\,d\int_0^{9x} \sqrt{x}\sqrt{y}\,dy = \int_0^1 (\sqrt{x}+2x^3\sqrt{x})\,dx\, \frac{2y^{\frac{3}{2}}}{3}\Big|_0^{9x} =$$

$$= 18\int_0^1 (\sqrt{x}+2x^3\sqrt{x})\,x\sqrt{x}\,dx = 18\int_0^1 (x^2+2x^5)\,dx = 18\left(\frac{x^3}{3}+2\frac{x^6}{6}\right)\Big|_0^1 =$$

$$= 12.$$

6.5. ТР6.

$y=\frac{3}{x};\ y=8e^{x};\ y=3;\ y=8.$ $\qquad D:\begin{cases}3\le y\le 8\\ \ln\frac{y}{8}\le x\le -\frac{3}{y}\end{cases}$

$$S=\iint\limits_D dx\,dy=\int_3^8 dy\int_{\ln\frac{y}{8}}^{\frac{3}{y}}dx=\int_3^8\left(\frac{3}{y}-\ln\frac{y}{8}\right)dy=3\ln y\Big|_3^8-$$

$$-\int_3^8\ln\frac{y}{8}\,dy=\left|\begin{matrix}u=\ln\frac{y}{8}\\ dv=dy\\ du=\frac{dy}{y}\\ v=y\end{matrix}\right|=3(\ln 8-\ln 3)-\left(y\ln\frac{y}{8}\Big|_3^8-\int_3^8 y\frac{dy}{y}\right)=$$

$$=3\ln\frac{8}{3}-\left(8\ln 1-3\ln\frac{3}{8}-y\Big|_3^8\right)=3\left(\ln\frac{8}{3}+\ln\frac{3}{8}\right)+5=5.$$

8.5. ТР8.

$x=2,\ y=0,\ y^2=2x\ (y\ge 0)$ $\qquad \mu=\frac{7x^2}{8}+2y;$ $\quad D:\begin{cases}0\le x\le 2\\ 0\le y\le\sqrt{2x}\end{cases}$

$$M=\iint\limits_D \mu\,dx\,dy=\int_0^2 dx\int_0^{\sqrt{2x}}\left(\frac{7}{8}x^2+2y\right)dy=\int_0^2\left(\frac{7}{8}x^2y+y^2\right)\Big|_0^{\sqrt{2x}}dx=$$

$$=\int_0^2\left(\frac{7}{8}x^2\sqrt{2x}+2x\right)dx=\int_0^2\left(\frac{7}{8}\sqrt{2}\,x^{\frac{5}{2}}+2x\right)dx=\frac{7}{8}\sqrt{2}\,\frac{2x^{\frac{7}{2}}}{7}\Big|_0^2+$$

$$+x^2\Big|_0^2=2\frac{\sqrt{2}}{8}\cdot 2^{\frac{7}{2}}+4=8.$$

Ответ: 8.

10.5. ТР10.

$y = 20\sqrt{2y}$, $x = 5\sqrt{2y}$, $z = 0$, $z + y = \frac{1}{2}$

$$V: \begin{cases} 0 \le y \le \frac{1}{2} \\ 5\sqrt{2y} \le x \le 20\sqrt{2y} \\ 0 \le z \le \frac{1}{2} - y \end{cases}$$

$$V = \iiint\limits_V dx\,dy\,dz = \int_0^{\frac{1}{2}} dy \int_{5\sqrt{2y}}^{20\sqrt{2y}} dx \int_0^{\frac{1}{2}-y} dz = \int_0^{\frac{1}{2}} dy \int_{5\sqrt{2y}}^{20\sqrt{2y}} \left(\frac{1}{2} - y\right) dx =$$

$$= \int_0^{\frac{1}{2}} \left(\frac{1}{2} - y\right)\left(20\sqrt{2y} - 5\sqrt{2y}\right) dy = 15\sqrt{2}\left(\frac{1}{3}\left(\left(\frac{1}{2}\right)^{\frac{3}{2}}\right) - \frac{2}{5}\left(\frac{1}{2}\right)^{\frac{5}{2}}\right) =$$

$$= \frac{15}{2} \cdot \frac{5-3}{15} = 1.$$

Ответ: $V = 1$.

12.5. ТР 12.

$y = -6x^2+8;\ y=2;\ z = x-x^2-y^2-1;\ z = x-x^2-y^2-5.$

$2 = -6x^2+8$

$x = \pm 1$

при $y=0$

$x^2 = \frac{8}{6} = \frac{2}{\sqrt{3}}$

$z = -(x^2-x)-y^2-1 = -(x-\frac{1}{2})^2-y^2-\frac{3}{4}$

$z = x-x^2-y^2-5$

$z = -(x-\frac{1}{2})^2-y^2-4\frac{3}{4}$

$\cancel{-6x^2+8x = x^2-y^2-4.}$

$$\int_{-1}^{1} dx \int_{2}^{-6x^2+8x+x^2-y^2-4} dy \int_{x-x^2-y^2-5} dz = \int_{-1}^{1} dx \int_{2}^{-6x^2+6} dy\,(x-x^2-y^2-1-x+x^2+y^2+5) =$$

$$= \int_{-1}^{1} 4\,dx \int_{2}^{-6x^2+8} dy = \int_{-1}^{1} 4(-6x^2+6)\,dy = 24\left(-\frac{1}{3}x^3\Big|_{-1}^{1}+2\right) = 32.$$

Ответ: 32.

.5:

ТР 1.

$u = xz^2 - \sqrt{x^3 y}$; $S: x^2 - y^2 - 3z + 12 = 0$ $M(2; 2; 4)$

$\bar{N} = \left\{ \frac{\partial F}{\partial x}; \frac{\partial F}{\partial y}; \frac{\partial F}{\partial z} \right\}$

$\frac{\partial F}{\partial x} = 2x$; $\frac{\partial F}{\partial y} = -2y$; $\frac{\partial F}{\partial z} = -3$. В т. M: $\frac{\partial F}{\partial x} = 4$; $\frac{\partial F}{\partial y} = -4$; $\frac{\partial F}{\partial z} = -3$

$\bar{N} = \{4; -4; -3\}$. $|N| = \sqrt{41}$

$\vec{n} = \frac{\bar{N}}{|\bar{N}|} = \left\{ \frac{4}{\sqrt{41}}; -\frac{4}{\sqrt{41}}; \frac{-3}{\sqrt{41}} \right\}$

$\frac{\partial u}{\partial x} = z^2 - \frac{3}{2}\sqrt{x y}$; $\frac{\partial u}{\partial y} = -\frac{\sqrt{x^3}}{2\sqrt{y}}$; $\frac{\partial u}{\partial z} = 2xz$.

В т. M: $\frac{\partial u}{\partial x} = 13$; $\frac{\partial u}{\partial y} = -1$; $\frac{\partial u}{\partial z} = 16$.

$\frac{\partial u}{\partial u_1} = \frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\cos\beta + \frac{\partial u}{\partial z}\cos\gamma$

$\cos\alpha = \frac{-4}{\sqrt{41}}$; $\cos\beta = \frac{4}{\sqrt{41}}$; $\cos\gamma = \frac{3}{\sqrt{41}}$

$\frac{\partial u}{\partial u_1} = 13\left(\frac{-4}{\sqrt{41}}\right) - 1\frac{4}{\sqrt{41}} + 16\frac{3}{\sqrt{41}} = \frac{-8}{\sqrt{41}}$

Ответ: $\frac{-8}{\sqrt{41}}$

.5. TP2.

$u = \frac{x^2}{yz^2}$; $V = \frac{x^3}{2} + 6y^3 + 3\sqrt{6}\, z^3$; $M(\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}})$

1) $\cos\alpha = \frac{(\text{grad} V \cdot \text{grad} u)}{|\text{grad} V| |\text{grad} u|}$

2) $\text{grad} V = \frac{3x^2}{2}\vec{i} + 18y^2\vec{j} + 9\sqrt{6}\vec{k}$

$\text{grad} V_M = 3\vec{i} + 9\vec{j} + 3\sqrt{6}\vec{k}$. $|\text{grad} V| = 12$.

$\text{grad} u = \frac{2x}{yz^2}\vec{i} - \frac{x^2}{z^2y^2}\vec{j} - \frac{2x^2}{yz^3}\vec{k}$

$\text{grad} u_M = 12\vec{i} - 12\vec{j} - 4\sqrt{2}(\sqrt{3})^3\vec{k}$; $|\text{grad} u| = \sqrt{1152}$

3) $\cos\alpha = \frac{36 - 108 - 3\sqrt{6} \cdot 4\sqrt{2}(\sqrt{3})^3}{12\sqrt{1152}} = -\frac{1}{\sqrt{2}}$

4) $\alpha = \arccos(-\frac{1}{\sqrt{2}}) = 135°$.

Ответ: 135°.

3.5. ТР3.

$a = x\vec{i} + 4y\vec{j}$

$a_x = x \quad a_y = 4y \quad a_z = 0$

$\frac{dx}{x} = \frac{dy}{4y} = \frac{dz}{z} \qquad dz = 0$

$\int \frac{dx}{x} = \int \frac{dy}{4y} \Rightarrow \ln|x| = \frac{1}{4}\ln|y|$

$4\ln|x| = \ln|y|$

$\ln x^4 = \ln y$

$cx^4 = y$

Ответ: $y = cx^4$.

1.5. ТР4.

$a = xi + yj + xyzk$; $S: x^2+y^2=1$

$P_1: z=0$; $P_2: z=5$

$\varphi(x,y,z) = x^2+y^2-1$

$\varphi'_x = 2x$; $\varphi'_y = 2y$; $\varphi'_z = 0$.

$\bar{N} = (x;y;0)$; $|\bar{N}| = \sqrt{x^2+y^2+0^2} = 1$

$\iint\limits_{\sigma} P\cos(n;x)\,d\sigma$

$\bar{n} = \left(\frac{x}{\sqrt{x^2+y^2}}; \frac{y}{\sqrt{x^2+y^2}}; 0\right) = (x;y;0)$

$(\bar{a};\bar{n}) = \frac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2} = 1$

$\Pi = \iint\limits_{\sigma}(\bar{a};\bar{n})\,d\sigma = \iint\limits_{\sigma} d\sigma = |\sigma| = 2\pi RH = 2\pi\cdot 1\cdot 5 = 10\pi.$

Ответ: 10π.

5.5. ТРЗ.

$a = 2x\bar{i} + zy\vec{j}$; P: $x+y+z=1$; $z = 1-x-y$

при $z=0$

$N = \{1;1;1\}$ $y=1-x$; $|\bar{N}| = \sqrt{3}$; $n = \{\frac{2}{\sqrt{3}}; \frac{1}{\sqrt{3}}; 0\}$.

$$\iint\limits_S \frac{4x+z}{\sqrt{3}}\sqrt{3}\,dS = \int_0^1 dx \int_0^{1-x} \frac{4x+1-x-y}{\sqrt{3}}\sqrt{3}\,dy = \int_0^1 dx \int_0^{1-x} (3x-y+1)\,dy$$

$$= \int_0^1 dx \int_0^{1-x} (3x-y+1)\,dy = \int_0^1 \left(3x(1-x) - \frac{(1-x)^2}{2} + 1 - x\right) dx =$$

$$= \int_0^1 3x\,dx - 3x^2\,dx - \frac{1}{2}\,dx + x\,dx - \frac{x^2}{2}\,dx + dx - x\,dx =$$

$$= \frac{3}{2} - \frac{7}{6} + \frac{1}{2} = \frac{12-7}{6} = \frac{5}{6}$$

Ответ: $\frac{5}{6}$.

6.5. TP 6.

$a = 7x\,i + 9\pi y\,j + k$; $P: x + \frac{y}{3} + z = 1$

$\bar N = (1; \frac{1}{3}; 1)$; $|\bar N| = \sqrt{1 + \frac{1}{9} + 1} = \sqrt{\frac{19}{9}} = \frac{\sqrt{19}}{3}$

$\bar n = \frac{\bar N}{|\bar N|} = \left(\frac{3}{\sqrt{19}}; \frac{1}{\sqrt{19}}; \frac{3}{\sqrt{19}}\right)$

$(\bar a; \vec n) = \frac{21x + 9\pi y + 3}{\sqrt{19}}$; $\Pi = \iint\limits_{\sigma} (\bar a; \bar n)\, d\sigma$

$x + \frac{y}{3} = 1$

$y = (1-x)3$; $x = 1$; $y = 3$; $z = 1$.

$$\begin{cases} 0 \le x \le 1 \\ 0 \le y \le (1-x)3 \end{cases}$$

$d\sigma = \sqrt{1 + (z'_x)^2 + (z'_y)^2} = \frac{\sqrt{19}}{3}$; $z = 1 - x - \frac{y}{3}$

$z'_x = -1$; $z'_y = -\frac{1}{3}$

$$\Pi = \int_0^1 dx \int_0^{(1-x)3} \left(\frac{21x + 9\pi y + 3}{\sqrt{19}} \cdot \frac{\sqrt{19}}{3}\right) dy = \int_0^1 dx \int_0^{(1-x)3} (7x + 3\pi y + 1)\, dy =$$

$$= \int_0^1 \left(21x(1-x) + \frac{9\pi(1-x)^2}{2} + (1-x)3\right) dx = \int_0^1 \left(18x - 21x^2 + \frac{9\pi}{2} - \frac{18\pi}{2}x\right.$$

$$\left. - \frac{9\pi x^2}{2} + 3\right) dx = 5 + \frac{9}{2}\pi$$

Ответ: $5 + \frac{9}{2}\pi$.

7.5. ТР 7.

$a = (e^{-z} - x)\bar{i} + (xz + 3y)\bar{j} + (z + x^2)\bar{k}$

$S: 2x + y + z = 2,\ x = 0;\ y = 0;\ z = 0;\ \frac{x}{1} + \frac{y}{2} + \frac{z}{2} = 1$

$\operatorname{div} \bar{a} = -1 + 3 + 1 = 3$

$$\Pi = \iiint_V 3\,dV = 3\int_0^1 dx \int_{0}^{2-2x} dy \int_0^{2-2x-y} dz = 3\int_0^1 dx \int_0^{2-2x} (2-2x-y)\,dy =$$

$$= 3\int_0^1 \left(4 - 4x - 2x(2-2x) - \frac{(2-2x)^2}{2}\right) dx = 3\left(2\frac{x^3}{3} - 4\frac{x^2}{2} + 2x\right)\Big|_0^1 =$$

$= 2.$

Ответ: 2.

8.5. ТР8.

$a = (z+y)i + yj - xk$ $\qquad S: \begin{cases} x^2+z^2=2y \\ y=2 \end{cases}$

$div\,\bar{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$

$div\,\bar{a} = 1$

$\Pi = \iiint\limits_V div\,\bar{a}\,dV = \iiint\limits_V dV.$ $\qquad V: \begin{cases} 0 \le \varphi \le 2\pi \\ 0 \le \rho \le 2 \\ \frac{\rho^2}{2} \le z \le 2 \end{cases}$

$V = \iiint dx\,dy\,dz$

$V = \int\limits_0^{2\pi} d\varphi \int\limits_0^2 \rho\,d\rho \int\limits_{\rho^2/2}^2 dz = 2\pi \int\limits_0^2 \left(2\rho - \frac{\rho^3}{3}\right) d\rho =$

$= 2\pi\left(\rho^2 - \frac{\rho^4}{8}\right)\Big|_0^2 = 2\pi(4-2) = 4\pi.$

Ответ: 4π.

9.5. ТРУ.

$a = xz\,i + z\,j + y\,k$ $\qquad S: \begin{cases} x^2+y^2 = 1-z \\ z=0 \end{cases}$

$\operatorname{div} a = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$

$a_x = xz \qquad \frac{\partial a_x}{\partial x} = z$

$a_y = z \qquad \frac{\partial a_y}{\partial y} = 0$

$a_z = y \qquad \frac{\partial a_z}{\partial z} = 0$

$\operatorname{div} \bar{a} = z + 0 + 0 = z$

$\Pi = \iiint\limits_V \operatorname{div} \bar{a}\, dV$ $\qquad V: \begin{cases} 0 \le \varphi \le 2\pi \\ 0 \le \rho \le 1 \\ 0 \le z \le 1-\rho^2 \end{cases}$

$\Pi = \iiint\limits_V z\,dV = \int_0^{2\pi} d\varphi \int_0^1 \rho\,d\rho \int_0^{1-\rho^2} z\,dz = \int_0^{2\pi} d\varphi \int_0^1 \rho\,d\rho \cdot \frac{z^2}{2}\Big|_0^{1-\rho^2} =$

$= \int_0^{2\pi} d\varphi \int_0^1 \frac{\rho(1-2\rho^2+\rho^4)}{2}\,d\rho = \frac{1}{2}\int_0^{2\pi} d\varphi \left(\frac{\rho^2}{2} - \frac{2\rho^4}{4} + \frac{\rho^6}{6}\right)\Big|_0^1 =$

$= \frac{1}{2}\cdot\frac{1}{6}\int_0^{2\pi} d\varphi = \frac{1}{12}\cdot 2\pi = \frac{\pi}{6}$

Ответ: $\frac{\pi}{6}$.

10.5. TP10.

$\bar{F} = x^3\bar{i} - y^3\bar{j}$ $L: x^2+y^2=4;\ x\geq 0;\ y\geq 0;\ M(2;0);\ N(0;2)$

$F_x = x^3;\ F_y = -y^3.$

$$\begin{cases} x = R\cos t = 2\cos t \\ y = 2\sin t \\ 0 \leq t \leq \frac{\pi}{2} \end{cases}$$

$x' = -2\sin t;\ y' = 2\cos t$

$$A = \int_L (F\,d\bar{s}) = \int_0^{\frac{\pi}{2}} (F_x x' + F_y y')\,dt = \int_0^{\frac{\pi}{2}} (x^3 x' - y^3 y')\,dt =$$

$$= -16\int_0^{\frac{\pi}{2}} (\cos^3 t \sin t)\,dt - 16\int_0^{\frac{\pi}{2}} \sin^3 t \cos t\,dt = \left| \begin{matrix} p = \cos t \\ p' = -\sin t \\ dp = -\sin t\,dt \\ p_1 = 1;\ p_2 = 0 \end{matrix} \right| /$$

$$\left| \begin{matrix} u = \sin t \\ u' = \cos t \\ du = \cos t\,dt \\ u_1 = 0,\ u_2 = 1 \end{matrix} \right| = 16\int_1^0 p^3\,dp - 16\int_0^1 u^3\,du = -4 - 4 = -8$$

Ответ: $A = -8$.

11.5. TP 11.

$a = (y-z)i + (z-x)j + (x-y)k \qquad \Gamma: \begin{cases} x = 4\cos t, \; y = 4\sin t \\ z = 1-\cos t \end{cases}$

$$u = \oint_\Gamma a_x dx + a_y dy + a_z dz = \oint_\Gamma ((y-z)x' + (z-x)y' + (x-y)z')dt$$

$$= \int_0^{2\pi} ((4\sin t - 1 + \cos t)(-4\sin t) + (1-\cos t - 4\cos t)4\cos t + (4\cos t -$$

$$- 4\sin t)\sin t)dt = \int_0^{2\pi} (-16\sin^2 t + 4\sin t - 4\sin t\cos t + 4\cos t -$$

$$- 4\cos^2 t + 4\sin t\cos t - 4\sin^2 t)dt = \int_0^{2\pi} (-20\sin^2 t - 20\cos^2 t +$$

$$+ 4\sin t + 4\cos t)dt = \int_0^{2\pi} (-20 + 4\sin t + 4\cos t)dt =$$

$$= \int_0^{2\pi} -20dt + 4\int_0^{2\pi} \sin t dt + 4\int_0^{2\pi} \cos t dt = -20t\Big|_0^{2\pi} - 4\cos t\Big|_0^{2\pi} + 4\sin t\Big|_0^{2\pi} =$$

$$= -20 \cdot 2\pi \cdot 4(\cos 2\pi - \cos 0) + 4(\sin 2\pi - \sin 0) = -40\pi.$$

Ответ: -40π.

12.5. ТР12.

$\bar{a} = (x-y)\bar{i} + x\bar{j} - z\bar{k}$

$\Gamma: \begin{cases} x^2+y^2=1 \\ z=5 \end{cases}$

$a_x = x-y,\ a_y = x,\ a_z = -z.$

$\Gamma: \begin{cases} x = 1\cos t \\ y = 1\sin t \\ z = 5 \end{cases} \quad 0 \le t \le 2\pi$

$$U = \oint_\Gamma (\bar{a}\, d\bar{s}) = \int_0^{2\pi} ((x-y)x' + xy' + z\cdot z')dt =$$

$$= \int_0^{2\pi} ((\cos t - \sin t)(-\sin t) + \cos t \cos t - 5\cdot 0)\, dt =$$

$$= \int_0^{2\pi} (-\cos t \sin t + \sin^2 t + \cos^2 t)\, dt = \int_0^{2\pi} (1 - \tfrac{1}{2}\sin 2t)\, dt =$$

$$= \left(t + \frac{\cos 2t}{4}\right)\Big|_0^{2\pi} = 2\pi + \frac{1}{4} - \frac{1}{4} = 2\pi.$$

Ответ: 2π.