Типовые расчёты по математическому анализу

Степанов Артём
А-02-23

Раздел "Определенные интегралы"

Типовой расчёт №1

№ 1.18

$$\int \frac{(x^3+x)\mathrm{d}x}{x^4+1} = \int \frac{x(x^2+1)\mathrm{d}x}{x^4+1} = \left\{ \begin{array}{c} x^2 = t \\ 2x\mathrm{d}x = \mathrm{d}t \end{array} \right\} = \frac{1}{2}\int \frac{(t+1)\mathrm{d}t}{t^2+1} = \frac{1}{2}\Big(\int \frac{t\mathrm{d}t}{t^2+1} + \int \frac{\mathrm{d}t}{t^2+1}\Big) =$$

$$= \left\{ \begin{array}{c} t^2+1 = k \\ 2t\mathrm{d}t = \mathrm{d}k \end{array} \right\} = \frac{1}{2}\Big(\frac{1}{2}\int \frac{\mathrm{d}k}{k} + \int \frac{\mathrm{d}t}{t^2+1}\Big) = \frac{1}{4}\ln|k| + \frac{1}{2}\arctan t + C = \frac{1}{4}\ln|x^4+1| + \frac{1}{2}\arctan x^2 + C$$

Ответ: $\frac{1}{4}\ln|x^4+1| + \frac{1}{2}\arctan x^2 + C$

Типовой расчёт №2

№ 2.18

$$\int \arctan\frac{1}{x}\mathrm{d}x = \left\{ \begin{array}{c} \mathrm{d}x = \mathrm{d}V \\ x = V \\ \arctan\frac{1}{x} = U \\ \frac{-\mathrm{d}x}{x^2+1} = \mathrm{d}U \end{array} \right\} = x\arctan\frac{1}{x} - \int x\frac{-\mathrm{d}x}{x^2+1} = x\arctan\frac{1}{x} + \int \frac{x\mathrm{d}x}{x^2+1} = \left\{ \begin{array}{c} x^2+1 = t \\ 2x\mathrm{d}x = \mathrm{d}t \end{array} \right\} =$$

$$= x\arctan\frac{1}{x} + \frac{1}{2}\int \frac{\mathrm{d}t}{t} = x\arctan\frac{1}{x} + \frac{1}{2}\ln|t| + C = x\arctan\frac{1}{x} + \frac{1}{2}\ln|x^2+1| + C$$

Ответ: $x\arctan\frac{1}{x} + \frac{1}{2}\ln|x^2+1| + C$

Типовой расчёт №3

№ 3.18

$$\overset{\text{Int}}{\int \frac{9-2x}{\sqrt{-3x^2+48x-45}}\mathrm{d}x} = \frac{2}{\sqrt{3}}\overset{2}{\int \frac{1-x}{\sqrt{-x^2+16x-15}}\mathrm{d}x} + \frac{7}{\sqrt{3}}\overset{1}{\int \frac{\mathrm{d}x}{\sqrt{-x^2+16x-15}}} = \ldots$$

$$1 = \int \frac{\mathrm{d}x}{\sqrt{-x^2+16x-15}} = \int \frac{\mathrm{d}x}{\sqrt{-x^2+16x-15}} = \int \frac{\mathrm{d}x}{\sqrt{49-(x-8)^2}} = \left\{ \begin{array}{c} x-8 = t \\ \mathrm{d}x = \mathrm{d}t \end{array} \right\} =$$

$$= \int \frac{\mathrm{d}t}{\sqrt{49-t^2}} = \arcsin\frac{x-8}{7} + C$$

$$② = \int \frac{1-x}{\sqrt{-x^2+16x-15}}\mathrm{d}x = \int \overset{①}{\frac{\mathrm{d}x}{\sqrt{-x^2+16x-15}}} - \int \overset{③}{\frac{x\mathrm{d}x}{\sqrt{-x^2+16x-15}}} = \ldots$$

$$③ = \int \frac{x\mathrm{d}x}{\sqrt{-x^2+16x-15}} = \int \frac{(x-8)+8}{\sqrt{49-(x-8)^2}}\mathrm{d}x = \int \overset{④}{\frac{x-8}{\sqrt{49-(x-8)^2}}}\mathrm{d}x + 8\int \overset{①}{\frac{\mathrm{d}x}{\sqrt{49-(x-8)^2}}} = \ldots$$

$$④ = \int \frac{x-8}{\sqrt{49-(x-8)^2}}\mathrm{d}x = \left\{ \begin{array}{l} 49-(x-8)^2 = t \\ -2(x-8)\mathrm{d}x = \mathrm{d}t \end{array} \right\} = -\frac{1}{2}\int \frac{\mathrm{d}t}{\sqrt{t}} = -\sqrt{t} + C = -\sqrt{49-(x-8)^2} + C$$

$$③ = ④ + 8 * ① = -\sqrt{49-(x-8)^2} + 8\arcsin\frac{x-8}{7} + C = -\sqrt{-x^2+16x-15} + 8\arcsin\frac{x-8}{7} + C$$

$$② = ① - ③ = \arcsin\frac{x-8}{7} + \sqrt{-x^2+16x-15} - 8\arcsin\frac{x-8}{7} + C = \sqrt{-x^2+16x-15} - 7\arcsin\frac{x-8}{7} + C$$

$$\text{Int} = \frac{2}{\sqrt{3}} * ② + \frac{7}{\sqrt{3}} * ① = \frac{2\sqrt{-x^2+16x-15} - 14\arcsin\frac{x-8}{7} + 7\arcsin\frac{x-8}{7}}{\sqrt{3}} + C =$$

$$= \frac{2\sqrt{-x^2+16x-15} - 7\arcsin\frac{x-8}{7}}{\sqrt{3}} + C$$

$\boxed{\text{Ответ: } \frac{2\sqrt{-x^2+16x-15}-7\arcsin\frac{x-8}{7}}{\sqrt{3}} + C}$

Типовой расчёт №4

№ 4.18

$$\int \overset{\text{Int}}{\frac{x^4+2x-3}{(x+2)(x+3)^2}}\mathrm{d}x = \int \Big(x - 8 + \frac{43x^2+152x+141}{(x+2)(x+3)^2}\Big)\mathrm{d}x = \int x\mathrm{d}x - \int 8\mathrm{d}x + \int \overset{①}{\frac{43x^2+152x+141}{(x+2)(x+3)^2}}\mathrm{d}x = \ldots$$

$$\frac{43x^2+152x+141}{(x+2)(x+3)^2} = \frac{A}{x+2} + \frac{B}{x+3} + \frac{C}{(x+3)^2} \implies A(x+3)^2 + B(x+2)(x+3) + C(x+2) = 43x^2+152x+141 \implies$$

$$\implies x^2(A+B) + x(6A+5B+C) + 9A+6B+2C = 43x^2+152x+141 \implies \begin{cases} A+B = 43 \\ 6A+5B+C = 152 \\ 9A+6B+2C = 141 \end{cases} \begin{cases} A = 9 \\ B = 34 \\ C = -72 \end{cases} \implies$$

$$\implies ① = \int \frac{43x^2+152x+141}{(x+2)(x+3)^2}\mathrm{d}x = \int \frac{9\mathrm{d}x}{x+2} + \int \frac{34\mathrm{d}x}{x+3} - \int \frac{72\mathrm{d}x}{(x+3)^2} = 9\ln|x+2| + 34\ln|x+3| - \frac{72}{x+3} + C$$

$$\text{Int} = \int x\mathrm{d}x - \int 8\mathrm{d}x + 9\ln|x+2| + 34\ln|x+3| - \frac{72}{x+3} + C = \frac{x^2}{2} - 8x + 9\ln|x+2| + 34\ln|x+3| - \frac{72}{x+3} + C$$

$\boxed{\text{Ответ: } \frac{x^2}{2} - 8x + 9\ln|x+2| + 34\ln|x+3| - \frac{72}{x+3} + C}$

Типовой расчёт №5

№ 5.18

$$\int \frac{\mathrm{d}x}{(x-2)\sqrt{9x^2-12x-8}} = \int \frac{\mathrm{d}x}{(x-2)\sqrt{9(x-2)^2+24(x-2)+4}} = \left\{\begin{array}{c} x-2=t \\ \mathrm{d}x=\mathrm{d}t \end{array}\right\} = \int \frac{\mathrm{d}t}{t\sqrt{9t^2+24t+4}}$$

$$= \int \frac{\mathrm{d}t}{t^2\sqrt{9+\frac{24}{t}+\frac{4}{t^2}}} = \left\{\begin{array}{c} \frac{1}{t}=k \\ -\frac{\mathrm{d}x}{t^2}=\mathrm{d}k \end{array}\right\} = -\int \frac{\mathrm{d}k}{\sqrt{9+24k+4k^2}} = -\int \frac{\mathrm{d}k}{\sqrt{(2k+6)^2-27)}} = \left\{\begin{array}{c} 2k+6=l \\ 2\mathrm{d}k=\mathrm{d}l \end{array}\right\} =$$

$$= -\frac{1}{2}\int \frac{\mathrm{d}l}{\sqrt{l^2-27}} = -\frac{1}{2}\ln|l+\sqrt{l^2-27}|+C = -\frac{1}{2}\ln|2k+6+\sqrt{(2k+6)^2-27}|+C =$$

$$= -\frac{1}{2}\ln\left|\frac{2}{t}+6+\sqrt{9+\frac{24}{t}+\frac{4}{t^2}}\right|+C = -\frac{1}{2}\ln\left|\frac{2}{x-2}+6+\sqrt{\frac{9x^2-12x-8}{(x-2)^2}}\right|+C$$

Ответ: $-\frac{1}{2}\ln\left|\frac{2}{x-2}+6+\sqrt{\frac{9x^2-12x-8}{(x-2)^2}}\right|+C$

Типовой расчёт №6

№ 6.18

$$\int \sqrt{\frac{5-3x}{3x-2}}\mathrm{d}x = \left\{\begin{array}{c} \sqrt{\frac{5-3x}{3x-2}}=t \implies x=\frac{5+2t^2}{3+3t^2} \\ \mathrm{d}x=-\frac{2t\mathrm{d}t}{(1+t^2)^2} \end{array}\right\} = \int \frac{-2t^2\mathrm{d}t}{(1+t^2)^2} = -2\int \frac{1+t^2-1}{(1+t^2)^2}\mathrm{d}t =$$

$$= -2\left(\int \frac{\mathrm{d}t}{1+t^2} - \int \frac{\mathrm{d}t}{(1+t^2)^2}\right) = -2\left(\int \frac{\mathrm{d}t}{1+t^2} - \frac{t}{2(t^2+1)} - \frac{1}{2}\int \frac{\mathrm{d}t}{1+t^2}\right) =$$

$$= -2(\arctan t - \frac{t}{2t^2+2} - \frac{1}{2}\arctan t)+C = \frac{\sqrt{5-3x}\sqrt{3x-2}}{3} - \arctan\sqrt{\frac{5-3x}{3x-2}}+C$$

Ответ: $\frac{\sqrt{5-3x}\sqrt{3x-2}}{3} - \arctan\sqrt{\frac{5-3x}{3x-2}}+C$

Типовой расчёт №8

№ 8.18

$$\int \sqrt{-2x^2-7x-2}\,\mathrm{d}x = \int \sqrt{-2\left(x^2+\frac{7}{2}x+1\right)}\mathrm{d}x = \sqrt{2}\int \sqrt{-\left(\left(x+\frac{7}{4}\right)^2-\frac{33}{16}\right)}\mathrm{d}x = \left\{\begin{array}{c} x+\frac{7}{4}=t \\ \mathrm{d}x=\mathrm{d}t \end{array}\right\} =$$

$$= \sqrt{2}\int \sqrt{\frac{33}{16}-t^2}\,\mathrm{d}t = \left\{\begin{array}{c} t=\frac{\sqrt{33}}{4}\sin k \implies k=\arcsin\frac{4t}{\sqrt{33}} \\ \mathrm{d}t=\frac{\sqrt{33}}{4}\cos k\,\mathrm{d}k \end{array}\right\} = \sqrt{2}\int \frac{33}{16}\cos^2 k\,\mathrm{d}k =$$

$$\frac{33\sqrt{2}}{32}\left(\int \cos 2k\,\mathrm{d}k + \int \mathrm{d}k\right) = \frac{33\sqrt{2}}{32}\left(\frac{1}{2}\sin 2k+k\right)+C = \frac{33\sqrt{2}}{32}\left(\frac{16t\sqrt{\frac{33}{16}-t^2}}{33}+\arcsin\frac{4t}{\sqrt{33}}\right)+C =$$

$$= \frac{33\sqrt{2}}{32}\left(16\left(x+\frac{7}{4}\right)\frac{\sqrt{-2x^2-7x-2}}{33\sqrt{2}}+\arcsin\frac{4x+7}{\sqrt{33}}\right)+C = \frac{(4x+7)\sqrt{-2x^2-7x-2}}{16}+\frac{33\sqrt{2}}{32}\arcsin\frac{4x+7}{\sqrt{33}}+C$$

Ответ: $\frac{(4x+7)\sqrt{-2x^2-7x-2}}{16}+\frac{33\sqrt{2}}{32}\arcsin\frac{4x+7}{\sqrt{33}}+C$

Типовой расчёт №9

№ 9.18

$$\int \frac{\sin^3 2x}{\cos^7 2x}\mathrm{d}x = \int \frac{(1-\cos^2 2x)\sin 2x}{\cos^7 2x}\mathrm{d}x = \left\{ \begin{array}{c} \cos 2x = t \\ -2\sin 2x\,\mathrm{d}x = \mathrm{d}t \end{array} \right\} = -\frac{1}{2}\int \frac{(1-t^2)\mathrm{d}t}{t^7} =$$

$$-\frac{1}{2}\left(\int \frac{\mathrm{d}t}{t^7} - \int \frac{(\mathrm{d}t}{t^5}\right) = \frac{1}{12t^6} - \frac{1}{8t^4} + C = \frac{1}{12\cos^6 2x} - \frac{1}{8\cos^4 2x} + C$$

Ответ: $\frac{1}{12\cos^6 2x} - \frac{1}{8\cos^4 2x} + C$

Типовой расчёт №10

№ 10.18

$$\int \frac{\mathrm{d}x}{4\sin^2 x + 2\cos^2 x - 3} = \int \frac{\mathrm{d}x}{\sin^2 x - \cos^2 x} = -\int \frac{\mathrm{d}x}{\cos 2x} = \left\{ \begin{array}{c} 2x = t \\ 2\mathrm{d}x = \mathrm{d}t \end{array} \right\} = -\frac{1}{2}\int \frac{\mathrm{d}t}{\cos t} =$$

$$= -\frac{1}{2}\ln\left|\tan\left(\frac{t}{2} + \frac{\pi}{4}\right)\right| + C = -\frac{1}{2}\ln\left|\tan\left(x + \frac{\pi}{4}\right)\right| + C$$

Ответ: $-\frac{1}{2}\ln\left|\tan\left(x + \frac{\pi}{4}\right)\right| + C$

Типовой расчёт №11

№ 11.18

$$\int\limits_4^5 x^3\sqrt{x^2-16}\mathrm{d}x = \left\{ \begin{array}{c} x^2 = t \\ 2x\,\mathrm{d}x = \mathrm{d}t \end{array} \right\} = \frac{1}{2}\int\limits_{16}^{25} t\sqrt{t-16}\mathrm{d}t = \left\{ \begin{array}{c} t-16 = k \\ \mathrm{d}t = \mathrm{d}k \end{array} \right\} = \frac{1}{2}\int\limits_0^9 (k+16)\sqrt{k}\mathrm{d}t =$$

$$= \frac{1}{2}\left(\int\limits_0^9 k^{\frac{3}{2}}\mathrm{d}t + \int\limits_0^9 16\sqrt{k}\mathrm{d}t\right) = \frac{1}{2}\left(\frac{2k^{\frac{5}{2}}}{5}\Big|_0^9 + 16\frac{2k^{\frac{3}{2}}}{3}\Big|_0^9\right) = \frac{963}{5}$$

Ответ: $\int\limits_4^5 x^3\sqrt{x^2-16}\mathrm{d}x = \frac{963}{5}$

Типовой расчёт №12

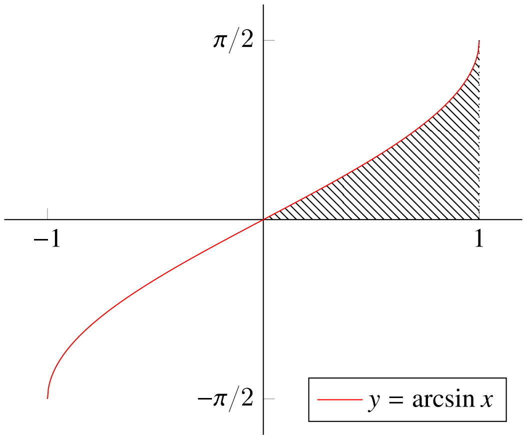
№ 12.18

$$\int\limits_0^{\frac{\pi}{3}} \frac{x\mathrm{d}x}{\cos^2 x} = \left\{ \begin{array}{c} U = \tan x \\ \mathrm{d}U = \frac{\mathrm{d}x}{\cos^2 x} \\ V = x \\ \mathrm{d}V = \mathrm{d}x \end{array} \right\} = x\tan x\Big|_0^{\frac{\pi}{3}} - \int\limits_0^{\frac{\pi}{3}} \frac{\sin x\mathrm{d}x}{\cos x} = x\tan x\Big|_0^{\frac{\pi}{3}} + \ln|\cos x|\Big|_0^{\frac{\pi}{3}} = \frac{\pi\sqrt{3}}{3} + \ln\frac{1}{2} = \frac{\pi\sqrt{3}}{3} - \ln 2$$

Ответ: $\int\limits_0^{\frac{\pi}{3}} \frac{x\mathrm{d}x}{\cos^2 x} = \frac{\pi\sqrt{3}}{3} - \ln 2$

Типовой расчёт №13

№ 13.18



$$S = \int_0^1 (\arcsin x - 0)\mathrm{d}x = \int_0^1 \arcsin x\mathrm{d}x = \left\{\begin{matrix} U = \arcsin x \\ \mathrm{d}U = \frac{\mathrm{d}x}{\sqrt{1-x^2}} \\ V = x \\ \mathrm{d}V = \mathrm{d}x \end{matrix}\right\} = x\arcsin x\Big|_0^1 - \int_0^1 \frac{x\mathrm{d}x}{\sqrt{1-x^2}} =$$

$$= x\arcsin x\Big|_0^1 + \frac{1}{2}\int_0^1 \frac{\mathrm{d}(1-x^2)}{\sqrt{1-x^2}} = x\arcsin x\Big|_0^1 + \sqrt{1-x^2}\Big|_0^1 = \frac{\pi}{2} - 1$$

Ответ: $S = \frac{\pi}{2} - 1$

Типовой расчёт №14

№ 14.18

$$\rho^2 = 3\cos\left(\varphi - \frac{\pi}{3}\right) \implies \rho = \pm\sqrt{3\cos\left(\varphi - \frac{\pi}{3}\right)}$$

$$\cos\left(\varphi - \frac{\pi}{3}\right) \geq 0 \implies -\frac{\pi}{2} \leq \varphi - \frac{\pi}{3} \leq \frac{\pi}{2} \implies -\frac{\pi}{6} \leq \varphi \leq \frac{5\pi}{6}$$

$$S = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{5\pi}{6}} 3\cos\left(\varphi - \frac{\pi}{3}\right)\mathrm{d}\varphi = \frac{3}{2}\int_{-\frac{\pi}{6}}^{\frac{5\pi}{6}} \cos\left(\varphi - \frac{\pi}{3}\right)\mathrm{d}\left(\varphi - \frac{\pi}{3}\right) = \frac{3}{2}\sin\left(\varphi - \frac{\pi}{3}\right)\Big|_{-\frac{\pi}{6}}^{\frac{5\pi}{6}} = 3$$

Ответ: $S = 3$

Типовой расчёт №15

№ 15.18

$$\rho = e^{\varphi};\ \ 0 \leq \varphi \leq \pi$$

$$l = \int_0^{\pi} \sqrt{(e^{\varphi})^2 + ((e^{\varphi})')^2}\mathrm{d}\varphi = \sqrt{2}\int_0^{\pi} \sqrt{e^{2\varphi}}\mathrm{d}\varphi = \sqrt{2}(e^{\pi} - e^0) = e^{\pi}\sqrt{2} - \sqrt{2}$$

Ответ: $l = e^{\pi}\sqrt{2} - \sqrt{2}$

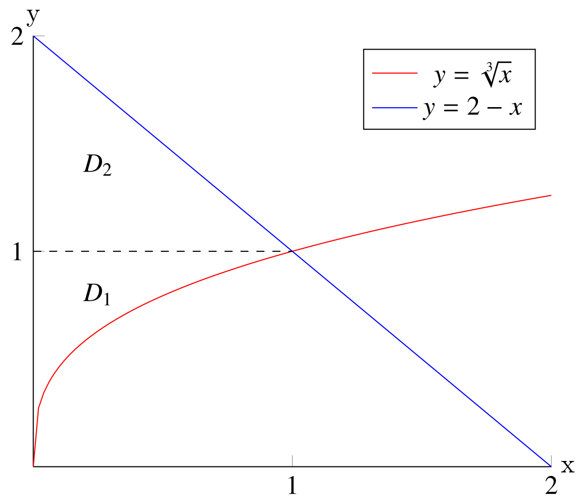
Раздел "Кратные интегралы"

Типовой расчёт №1

№ 1.18

$$\int_0^1 \mathrm{d}y \int_0^{y^3} f\,\mathrm{d}x + \int_1^2 \mathrm{d}y \int_0^{2-y} f\,\mathrm{d}x$$

$$D_1 = \left\{ \begin{array}{c} 0 \le y \le 1 \\ 0 \le x \le y^3 \end{array} \right\} \quad D_2 = \left\{ \begin{array}{c} 1 \le y \le 2 \\ 0 \le x \le 2-y \end{array} \right\}$$



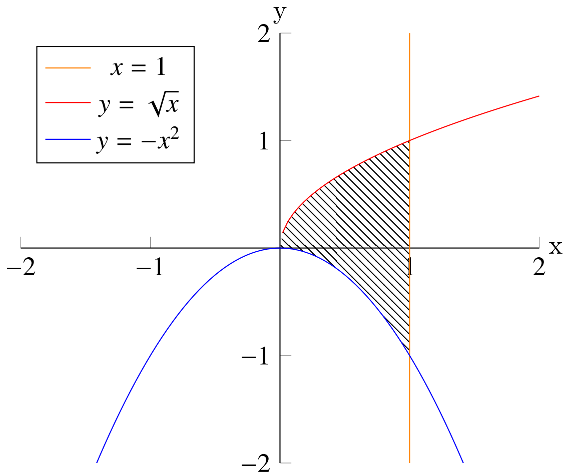
$$\int_0^1 \mathrm{d}y \int_0^{y^3} f\,\mathrm{d}x + \int_1^2 \mathrm{d}y \int_0^{2-y} f\,\mathrm{d}x = \int_0^1 \mathrm{d}x \int_{\sqrt[3]{x}}^1 f\,\mathrm{d}y + \int_0^1 \mathrm{d}x \int_1^{2-x} f\,\mathrm{d}y = \int_0^1 \mathrm{d}x \int_{\sqrt[3]{x}}^{2-x} f\,\mathrm{d}y$$

$$\boxed{\text{Ответ: } \int_0^1 \mathrm{d}x \int_{\sqrt[3]{x}}^{2-x} f\,\mathrm{d}y}$$

Типовой расчёт №2

№ 2.18

$$\iint_D (6xy + 24x^3y^3)\mathrm{d}x\mathrm{d}y \qquad D = \left\{ \begin{array}{c} x = 1 \\ y = \sqrt{x} \\ y = -x^2 \end{array} \right\}$$



$$\iint\limits_D (6xy + 24x^3y^3)\mathrm{d}x\mathrm{d}y = \int\limits_0^1 \mathrm{d}x \int\limits_{-x^2}^{\sqrt{x}} (6xy + 24x^3y^3)\mathrm{d}y = \int\limits_0^1 \left(\frac{6xy^2}{2} + \frac{24x^3y^4}{4}\right)\Bigg|_{-x^2}^{\sqrt{x}} \mathrm{d}x =$$

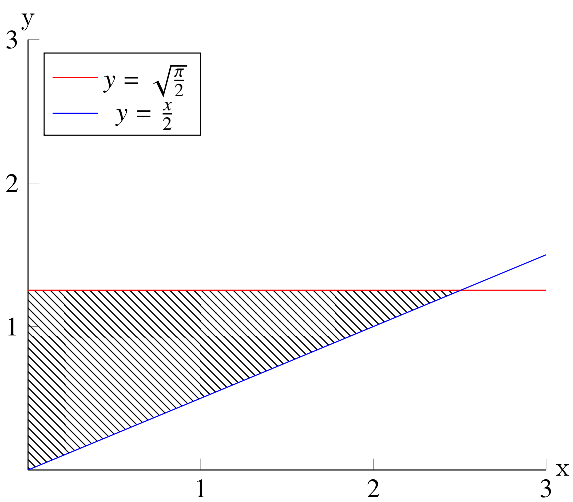
$$\int\limits_0^1 \left(3x^2 + 6x^5 - 3x^5 - 6x^{11}\right)\mathrm{d}x = \left(\frac{3x^3}{3} + \frac{3x^6}{6} - \frac{6x^{12}}{12}\right)\Bigg|_0^1 = 1 + \frac{1}{2} - \frac{1}{2} = 1$$

Ответ: 1

Типовой расчёт №3

№ 3.18

$$\iint\limits_D (y^2\cos(2xy))\mathrm{d}x\mathrm{d}y \qquad D = \left\{ \begin{array}{c} x = 0 \\ y = \sqrt{\frac{\pi}{2}} \\ y = \frac{x}{2} \end{array} \right\}$$



$$\iint\limits_D (y^2\cos(2xy))\mathrm{d}x\mathrm{d}y = \int\limits_0^{\sqrt{\frac{\pi}{2}}} \mathrm{d}y \int\limits_0^{2y} (y^2\cos(2xy))\mathrm{d}x = \int\limits_0^{\sqrt{\frac{\pi}{2}}} \frac{y^2\sin(2xy)}{2y}\Bigg|_0^{2y} \mathrm{d}y = \int\limits_0^{\sqrt{\frac{\pi}{2}}} \frac{y}{2}\sin(4y^2)\,\mathrm{d}y =$$

$$\frac{1}{4}\int\limits_0^{\sqrt{\frac{\pi}{2}}} \sin(4y^2)\,\mathrm{d}(y^2) = -\frac{1}{16}\cos(4y^2)\Big|_0^{\sqrt{\frac{\pi}{2}}} = \frac{1}{16}(\cos(0)-\cos(2\pi)) = 0$$

$\boxed{\text{Ответ: } 0}$

Типовой расчёт №4

№ 4.18

$$\iiint\limits_V (2x^2z\,\mathrm{sh}\,(2xyz))\mathrm{d}x\mathrm{d}y\mathrm{d}z \qquad D = \left\{\begin{array}{ccc} x=2 & y=\frac{1}{2} & z=\frac{1}{2} \\ x=0 & y=0 & z=0 \end{array}\right\}$$

$$\iiint\limits_V (2x^2z\,\mathrm{sh}\,(2xyz))\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int\limits_0^2 \mathrm{d}x\int\limits_0^{\frac{1}{2}}\mathrm{d}z\int\limits_0^{\frac{1}{2}}(2x^2z\,\mathrm{sh}\,(2xyz))\mathrm{d}y = \int\limits_0^2\mathrm{d}x\int\limits_0^{\frac{1}{2}}\frac{2x^2z\,\mathrm{ch}\,(2xyz)}{2xz}\Big|_0^{\frac{1}{2}}\mathrm{d}z =$$

$$= \int\limits_0^2\mathrm{d}x\int\limits_0^{\frac{1}{2}}(x\,\mathrm{ch}\,(xz)-x)\mathrm{d}z = \int\limits_0^2\left(\frac{x\,\mathrm{sh}\,(xz)}{x}-xz\right)\Big|_0^{\frac{1}{2}}\mathrm{d}x = \int\limits_0^2\left(sh\left(\frac{x}{2}\right)-\frac{x}{2}\right)\mathrm{d}x =$$

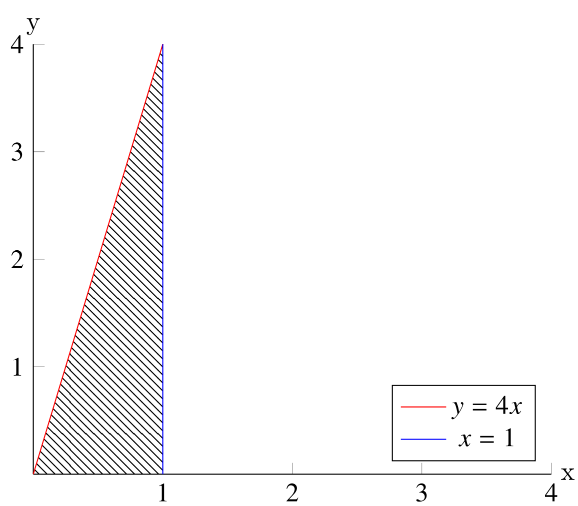
$$= \left(2\,\mathrm{ch}\left(\frac{x}{2}\right)-\frac{x^2}{4}\right)\Big|_0^2 = 2\,\mathrm{ch}\,1-3$$

$\boxed{\text{Ответ: } 2\,\mathrm{ch}\,1-3}$

Типовой расчёт №5

№ 5.18

$$\iiint\limits_V (9+18z)\mathrm{d}x\mathrm{d}y\mathrm{d}z \qquad D = \left\{\begin{array}{cc} y=0 & y=4x \\ \multicolumn{2}{c}{x=1} \\ z=0 & z=\sqrt{xy} \end{array}\right\}$$



$$\iiint\limits_V (9+18z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int\limits_0^1\mathrm{d}x\int\limits_0^{4x}\mathrm{d}y\int\limits_0^{\sqrt{xy}}(9+18z)\mathrm{d}z = \int\limits_0^1\mathrm{d}x\int\limits_0^{4x}(9\sqrt{xy}+9xy)\mathrm{d}y = \int\limits_0^1\left(\frac{6xy^{\frac{3}{2}}}{x}+\frac{9xy^2}{2}\right)\Big|_0^{4x}\mathrm{d}x =$$

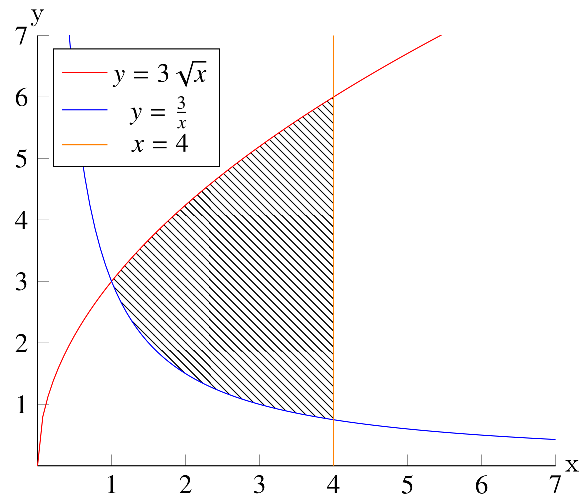
$$= \int_0^1 \left(48x^2 + 72x^3\right) \mathrm{d}x = \left(\frac{48x^3}{3} + \frac{72x^4}{4}\right)\Bigg|_0^1 = 16 + 18 = 34$$

$\boxed{\text{Ответ: } 34}$

Типовой расчёт №6

№ 6.18

$$\left\{ y = 3\sqrt{x}; \qquad y = \frac{3}{x}; \qquad x = 4 \right\}$$



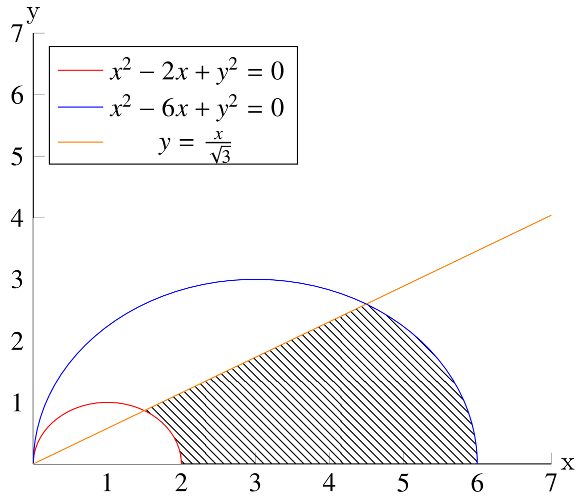
$$S = \iint\limits_D \mathrm{d}x\mathrm{d}y = \int_1^4 \mathrm{d}x \int_{\frac{3}{x}}^{3\sqrt{x}} \mathrm{d}y = \int_1^4 \left(3\sqrt{x} - \frac{3}{x}\right) \mathrm{d}x = \left(2x^{\frac{3}{2}} - 3\ln x\right)\Bigg|_0^4 = 14 - 3\ln 4$$

$\boxed{\text{Ответ: } S = 14 - 3\ln 4}$

Типовой расчёт №7

№ 7.18

$$\left\{ x^2 - 2x + y^2 = 0; \qquad x^2 - 6x + y^2 = 0; \qquad y = 0; \qquad y = \frac{x}{\sqrt{3}} \right\}$$



$$y = 0 \implies \varphi = 0 \qquad y = \frac{x}{\sqrt{3}} \implies \varphi = \frac{\pi}{6}$$

$$x^2 - 2x + y^2 = 0 \implies x^2 + y^2 = 2x \implies \rho^2 = 2\rho\cos\varphi \implies \rho = 2\cos\varphi$$
$$x^2 - 6x + y^2 = 0 \implies x^2 + y^2 = 6x \implies \rho^2 = 6\rho\cos\varphi \implies \rho = 6\cos\varphi$$

$$S = \int\limits_0^{\frac{\pi}{6}} d\varphi \int\limits_{2\cos\varphi}^{6\cos\varphi} \rho\, d\rho = \int\limits_0^{\frac{\pi}{6}} \left(18\cos^2\varphi - 2\cos^2\varphi\right) d\varphi = \int\limits_0^{\frac{\pi}{6}} 16\cos^2\varphi\, d\varphi = \int\limits_0^{\frac{\pi}{6}} \frac{16\cos(2\varphi) + 16}{2} d\varphi =$$

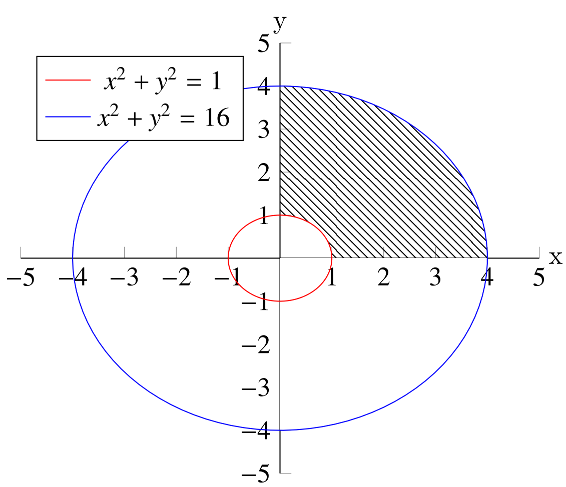
$$= \int\limits_0^{\frac{\pi}{6}} (16\cos(2\varphi) + 16)\, d(2\varphi) = (4\sin(2\varphi) + 8\varphi)\Big|_0^{\frac{\pi}{6}} = 2\sqrt{3} + \frac{4\pi}{3}$$

Ответ: $S = 2\sqrt{3} + \frac{4\pi}{3}$

Типовой расчёт №8

№ 8.18

$$D = \left\{ \begin{array}{c} y \geq 0 \quad (y = 0) \\ x \geq 0 \quad (x = 0) \\ x^2 + y^2 = 1 \\ x^2 + y^2 = 16 \end{array} \right\} \qquad \mu = \frac{x + 3y}{x^2 + y^2}$$



$$x^2 + y^2 = 1 \implies \rho = 1 \qquad x^2 + y^2 = 16 \implies \rho = 4$$

$$y \geq 0;\ x \geq 0 \implies 0 \leq \varphi \leq \frac{\pi}{2}$$

$$\mu = \frac{x+3y}{x^2+y^2} = \frac{\cos\varphi + 3\sin\varphi}{\cos^2\varphi + \sin^2\varphi} = \frac{\cos\varphi + 3\sin\varphi}{\rho}$$

$$m = \int_0^{\frac{\pi}{2}} \mathrm{d}\varphi \int_1^4 \frac{\cos\varphi + 3\sin\varphi}{\rho}\rho\,\mathrm{d}\rho = \int_0^{\frac{\pi}{2}} (4\cos\varphi + 12\sin\varphi - \cos\varphi - 3\sin\varphi)\,\mathrm{d}\varphi = \int_0^{\frac{\pi}{2}} (3\cos\varphi + 9\sin\varphi)\,\mathrm{d}\varphi =$$

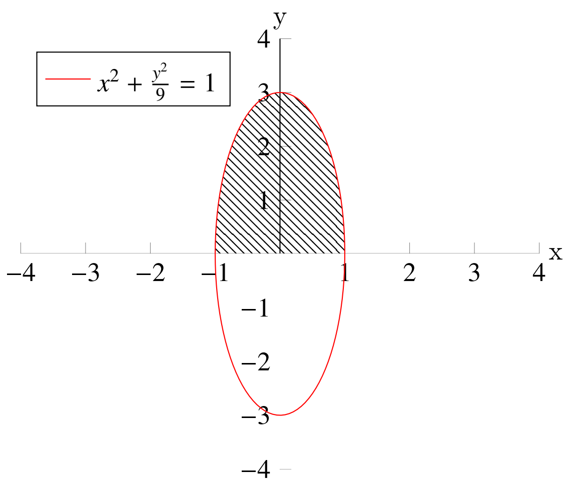
$$= (3\sin\varphi - 9\cos\varphi)\Big|_0^{\frac{\pi}{2}} = 12$$

$\boxed{\text{Ответ: } m = 12}$

Типовой расчёт №9

№ 9.18

$$D = \left\{ \begin{array}{c} x^2 + \frac{y^2}{9} \leq 1 \\ y \geq 0 \end{array} \right\} \qquad \mu = 35x^4y^3$$



$$x = \rho\cos\varphi;\ y = 3\rho\sin\varphi \implies x^2 + \frac{y^2}{9} \leq 1 \implies \rho \leq 1$$

$$\mu = 35x^4y^3 \implies \mu = 945\rho^7\cos^4\varphi\sin^3\varphi$$

$$y \geq 0 \implies 0 \leq \varphi \leq \pi$$

$$m = \int_0^{\pi} \mathrm{d}\varphi \int_0^1 3\rho\, 945\rho^7\cos^4\varphi\sin^3\varphi\,\mathrm{d}\rho = \int_0^{\pi} 315\cos^4\varphi\sin^3\varphi\,\mathrm{d}\varphi = -315\int_0^{\pi}\cos^4\varphi(1-\cos^2\varphi)\,\mathrm{d}(cos\varphi) =$$

$$= -315\left(\frac{\cos^5\varphi}{5} - \frac{\cos^7\varphi}{7}\right)\Big|_0^{\pi} = -315(-\frac{1}{5} + \frac{1}{7} - \frac{1}{5} + \frac{1}{7}) = 126 - 90 = 36$$

$\boxed{\text{Ответ: } m = 36}$

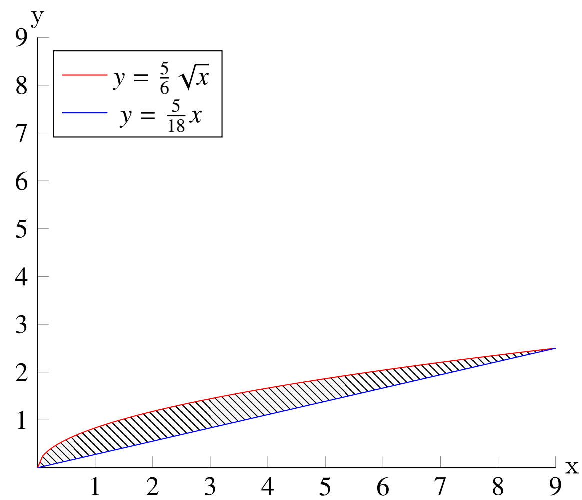
Типовой расчёт №10

№ 10.18

$$V = \left\{ \begin{array}{cc} y = \frac{5}{6}\sqrt{x} & y = \frac{5}{18}x \\ z = 0 & z = \frac{5}{18}(3+\sqrt{x}) \end{array} \right\}$$

$$V = \int\limits_{0}^{9} \mathrm{d}x \int\limits_{\frac{5}{18}x}^{\frac{5}{6}\sqrt{x}} \mathrm{d}y \int\limits_{0}^{\frac{5}{18}(3+\sqrt{x})} \mathrm{d}z = \int\limits_{0}^{9} \mathrm{d}x \int\limits_{\frac{5}{18}x}^{\frac{5}{6}\sqrt{x}} \left(\frac{5}{6} + \frac{5}{18}\sqrt{x}\right) \mathrm{d}y = \int\limits_{0}^{9} \left(\frac{25}{36}\sqrt{x} + \frac{25}{108}x - \frac{25}{108}x - \frac{25}{324}x^{\frac{3}{2}}\right) \mathrm{d}x =$$

$$= \left(\frac{25}{54}x^{\frac{3}{2}} - \frac{5}{162}x^{\frac{5}{2}}\right)\Bigg|_{0}^{9} = 5$$

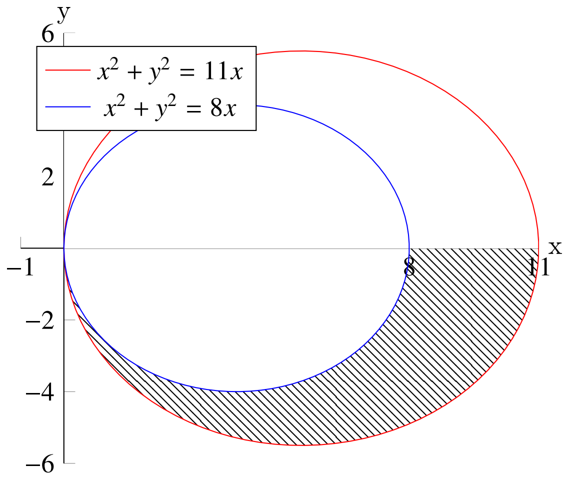


$\boxed{\text{Ответ: } V = 5}$

Типовой расчёт №11

№ 11.18

$$V = \left\{ \begin{array}{cc} x^2 + y^2 = 8x & y \le 0 \\ x^2 + y^2 = 11x & y = 0 \\ z = \sqrt{x^2 + y^2} & z = 0 \end{array} \right\} \implies V = \left\{ \begin{array}{c} \rho = 8\cos\varphi \\ \rho = 11\cos\varphi \\ z = \sqrt{\rho^2} = \rho \\ -\frac{\pi}{2} \le \varphi \le 0 \end{array} \right\}$$



$$V = \int\limits_{-\frac{\pi}{2}}^{0} \mathrm{d}\varphi \int\limits_{8\cos\varphi}^{11\cos\varphi} \mathrm{d}\rho \int\limits_{0}^{\rho} \rho \mathrm{d}z = \int\limits_{-\frac{\pi}{2}}^{0} \mathrm{d}\varphi \int\limits_{8\cos\varphi}^{11\cos\varphi} \rho^2 \mathrm{d}\rho = \int\limits_{-\frac{\pi}{2}}^{0} \left(\frac{1331\cos^3\varphi}{3} - \frac{512\cos^3\varphi}{3}\right) \mathrm{d}\varphi = \int\limits_{-\frac{\pi}{2}}^{0} \frac{819(1-\sin^2\varphi)}{3} \mathrm{d}(\sin\varphi) =$$

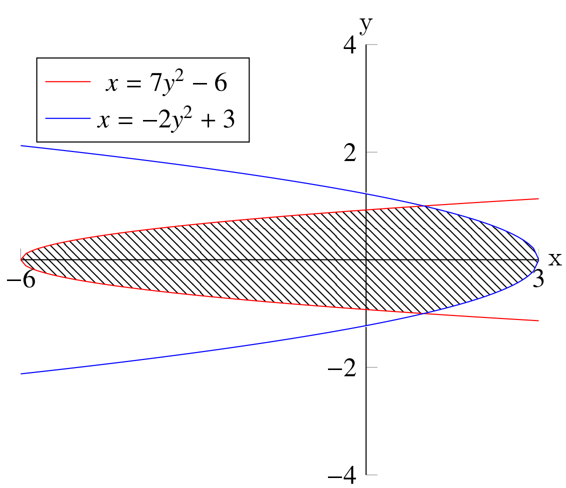
$$= 273\left(\sin\varphi - \frac{\sin^3\varphi}{3}\right)\Bigg|_{-\frac{\pi}{2}}^{0} = 182$$

$\boxed{\text{Ответ: } V = 182}$

Типовой расчёт №12

№ 12.18

$$V = \left\{ \begin{array}{c} x = 7y^2 - 6 \\ x = -2y^2 + 3 \\ z = 3 + 5x^2 - 8y^2 \\ z = -2 + 5x^2 - 8y^2 \end{array} \right\}$$



$$V = \int\limits_{-1}^{1} \mathrm{d}y \int\limits_{7y^2-6}^{-2y^2+3} \mathrm{d}x \int\limits_{-2+5x^2-8y^2}^{3+5x^2-8y^2} \mathrm{d}z = \int\limits_{-1}^{1} \mathrm{d}y \int\limits_{7y^2-6}^{-2y^2+3} 5\mathrm{d}x = \int\limits_{-1}^{1} \left(45 - 45y^2\right) \mathrm{d}y = \left(45y - 15y^3\right)\Big|_{-1}^{1} = 60$$

$\boxed{\text{Ответ: } V = 60}$

Типовой расчёт №13

№ 13.18

$$V = \left\{ \begin{array}{c} z = 3\dfrac{\sqrt{x^2+y^2}}{2} \\ z = \frac{5}{2} - x^2 - y^2 \end{array} \right\} \implies V = \left\{ \begin{array}{c} z = \frac{3}{2}\rho \\ z = \frac{5}{2} - \rho^2 \\ x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{array} \right\} \qquad \frac{3}{2}\rho = \frac{5}{2} - \rho^2 \implies \rho = 1$$

$$V = \int\limits_0^{2\pi} \mathrm{d}\varphi \int\limits_0^1 \rho \mathrm{d}\rho \int\limits_{\frac{3}{2}\rho}^{\frac{5}{2}-\rho^2} \mathrm{d}z = \int\limits_0^{2\pi} \mathrm{d}\varphi \int\limits_0^1 \left(\frac{5}{2}\rho - \rho^3 - \frac{3}{2}\rho^2\right) \mathrm{d}\rho = \int\limits_0^{2\pi} \left(\frac{5}{4}\rho^2 - \frac{\rho^4}{4} - \frac{\rho^3}{2}\right)\Bigg|_0^1 \mathrm{d}\varphi = \frac{1}{2}\int\limits_0^{2\pi} \mathrm{d}\varphi = \pi$$

$\boxed{\text{Ответ: } V = \pi}$

Типовой расчёт №14

№ 14.18

$$V = \left\{ \begin{array}{c} z = 26((x-1)^2 + y^2) \\ z = 50 - 52x \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} z = 26\rho^2 - 52\rho\cos\varphi + 24 \\ z = 50 - 52\rho\cos\varphi \\ x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{array} \right\} \quad 26\rho^2 - 52\rho\cos\varphi + 24 = 50 - 52\rho\cos\varphi \implies \rho = 1$$

$$V = \int\limits_0^{2\pi} \mathrm{d}\varphi \int\limits_0^1 \rho \mathrm{d}\rho \int\limits_{26\rho^2 - 52\rho\cos\varphi + 24}^{50-52\rho\cos\varphi} \mathrm{d}z = 26\int\limits_0^{2\pi} \mathrm{d}\varphi \int\limits_0^1 \left(\rho - \rho^3\right) \mathrm{d}\rho = 26\int\limits_0^{2\pi} \left(\frac{\rho^2}{2} - \frac{\rho^4}{4}\right)\Bigg|_0^1 \mathrm{d}\varphi = \frac{26}{4}\int\limits_0^{2\pi} \mathrm{d}\varphi = 13\pi$$

$\boxed{\text{Ответ: } V = 13\pi}$

Типовой расчёт №15

№ 15.18

$$V = \left\{ \begin{array}{c} 36 \le x^+ y^2 + z^2 \le 144 \\ -\sqrt{\frac{x^2+y^2}{3}} \le z \le -\sqrt{\frac{x^2+y^2}{15}} \\ 0 \le y \le -\sqrt{3}x \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} 6 \le \rho \le 12 \\ \frac{2\pi}{3} \le \theta \le \arctan\sqrt{\frac{-1}{15}} \\ 0 \le \varphi \le -\frac{\pi}{3} \end{array} \right\}$$

$$V = \int\limits_{\frac{2\pi}{3}}^{\arctan\sqrt{\frac{-1}{15}}} \mathrm{d}\theta \int\limits_6^{12} \mathrm{d}\rho \int\limits_0^{-\frac{\pi}{3}} \rho^2 \sin\theta \mathrm{d}\varphi = \int\limits_{\frac{2\pi}{3}}^{\arctan\sqrt{\frac{-1}{15}}} \mathrm{d}\theta \int\limits_6^{12} -\frac{\pi}{3}\rho^2 \sin\theta \mathrm{d}\rho = \int\limits_{\frac{2\pi}{3}}^{\arctan\sqrt{\frac{-1}{15}}} \left(-\frac{\pi}{9}\rho^3 \sin\theta\right)\Bigg|_6^{12} \mathrm{d}\theta =$$

$$= \pi \int\limits_{\frac{2\pi}{3}}^{\arctan\sqrt{\frac{-1}{15}}} (24\sin\theta - 192\sin\theta)\,\mathrm{d}\theta = -168\pi\,(-\cos\theta)\Bigg|_{\frac{2\pi}{3}}^{\arctan\sqrt{\frac{-1}{15}}} = 168\pi\left(\frac{-\sqrt{\frac{1}{15}}}{\sqrt{1+\frac{1}{15}}} - \cos\frac{2\pi}{3}\right) = 42\pi$$

$\boxed{\text{Ответ: } V = 42\pi}$

Типовой расчёт №16

№ 16.18

$$V=\left\{\begin{array}{c} x^2+y^2=1 \\ x^2+y^2=z \\ x=0;\ y=0;\ z=0;(x\geq 0;\ y\geq 0)\end{array}\right\}\Leftrightarrow\left\{\begin{array}{c}\rho=1\\ \rho^2=z\\ 0\leq\varphi\leq\frac{\pi}{2}\end{array}\right\}\quad \mu=10y=10\rho\sin\varphi$$

$$m=\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\int_0^1\rho\mathrm{d}\rho\int_0^{\rho^2}10\rho\sin\varphi\mathrm{d}z=\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\int_0^1 10\rho^4\sin\varphi\mathrm{d}\rho=\int_0^{\frac{\pi}{2}}2\sin\varphi\mathrm{d}\varphi=2\left(-cos\varphi\right)\Big|_0^{\frac{\pi}{2}}=2$$

$\boxed{\text{Ответ: } m=2}$

Раздел "Векторный анализ"

Типовой расчёт №1

№ 1.18

$$U=y\ln(1+x^2)-\arctan z\qquad \vec{I}=2\vec{i}-3\vec{j}-2\vec{k}\qquad M(0;1;1)$$

$$\left.\begin{array}{l}\frac{\partial U}{\partial x}=\frac{2xy}{1+x^2}\implies\frac{\partial U}{\partial x}(\mathrm{M})=0\\ \frac{\partial U}{\partial y}=\ln(1+x^2)\implies\frac{\partial U}{\partial y}(\mathrm{M})=0\\ \frac{\partial U}{\partial z}=\frac{-1}{1+z^2}\implies\frac{\partial U}{\partial z}(\mathrm{M})=-\frac{1}{2}\end{array}\right\}\implies\overrightarrow{\mathrm{grad}U}(M)=-\frac{1}{2}\vec{k}$$

$$\vec{I_0}=\frac{2\vec{i}-3\vec{j}-2\vec{k}}{\sqrt{4+9+4}}=\frac{2}{\sqrt{17}}\vec{i}-\frac{3}{\sqrt{17}}\vec{j}-\frac{2}{\sqrt{17}}\vec{k}$$

$$\frac{\partial U}{\partial I}(\mathrm{M})=(\overrightarrow{\mathrm{grad}U}(M),\vec{I_0})=0*\frac{2}{\sqrt{17}}-0*\frac{3}{\sqrt{17}}+\frac{1}{2}*\frac{2}{\sqrt{17}}=\frac{1}{\sqrt{17}}$$

$\boxed{\text{Ответ: } \frac{\partial U}{\partial I}(\mathrm{M})=\frac{1}{\sqrt{17}}}$

Типовой расчёт №2

№ 2.18

$$U=\frac{y^2z^3}{x}\qquad V=\frac{1}{x\sqrt{2}}-\frac{2\sqrt{2}}{y}-\frac{3\sqrt{2}}{2z}\qquad M\left(\frac{1}{\sqrt{2}};\ \sqrt{2};\ \frac{\sqrt{3}}{2}\right)$$

$$\left.\begin{array}{l}\frac{\partial U}{\partial x}=\frac{-y^2z^3}{x^2}\implies\frac{\partial U}{\partial x}(\mathrm{M})=-\frac{3\sqrt{3}}{2}\\ \frac{\partial U}{\partial y}=\frac{2yz^3}{x}\implies\frac{\partial U}{\partial y}(\mathrm{M})=\frac{3\sqrt{3}}{2}\\ \frac{\partial U}{\partial z}=\frac{3y^2z^2}{x}\implies\frac{\partial U}{\partial z}(\mathrm{M})=\frac{9\sqrt{3}}{2}\end{array}\right\}\implies\overrightarrow{\mathrm{grad}U}(M)=-\frac{3\sqrt{3}}{2}\vec{i}+\frac{3\sqrt{3}}{2}\vec{j}+\frac{9\sqrt{3}}{2}\vec{k}$$

$$\left.\begin{array}{l}\frac{\partial V}{\partial x}=\frac{-1}{x^2\sqrt{2}}\implies\frac{\partial V}{\partial x}(\mathrm{M})=-\sqrt{2}\\ \frac{\partial V}{\partial y}=\frac{2\sqrt{2}}{y^2}\implies\frac{\partial V}{\partial y}(\mathrm{M})=\sqrt{2}\\ \frac{\partial V}{\partial z}=\frac{3\sqrt{3}}{2z^2}\implies\frac{\partial V}{\partial z}(\mathrm{M})=2\sqrt{3}\end{array}\right\}\implies\overrightarrow{\mathrm{grad}V}(M)=-\sqrt{2}\vec{i}+\sqrt{2}\vec{j}+2\sqrt{3}\vec{k}$$

$$\cos\alpha=\frac{(\overrightarrow{\mathrm{grad}U}(M),\overrightarrow{\mathrm{grad}V}(M))}{|\overrightarrow{\mathrm{grad}U}(M)|\,|\overrightarrow{\mathrm{grad}V}(M)|}=\frac{\frac{3\sqrt{6}}{2}+\frac{3\sqrt{6}}{2}+\frac{18\sqrt{6}}{2}}{\sqrt{\frac{27}{4}+\frac{27}{4}+\frac{162}{4}}\sqrt{2+2+12}}=1\implies\alpha=0$$

$\boxed{\text{Ответ: } \alpha=0}$

Типовой расчёт №3

№ 3.18

$$\vec{a} = x\vec{i} + y\vec{j} \implies a_x = x;\ a_y = y;\ a_z = 0;$$

$$\begin{cases} \frac{\mathrm{d}x}{x} = \frac{\mathrm{d}y}{y} \\ \mathrm{d}z = 0 \end{cases} \implies \begin{cases} \int \frac{\mathrm{d}x}{x} = \int \frac{\mathrm{d}y}{y} \\ z = C_1 \end{cases} \implies \begin{cases} \ln x = \ln y + \ln C_2 \implies x = C_2\, y \\ z = C_1 \end{cases}$$

$\boxed{\text{Ответ: } x = C_2\, y;\ z = C_1}$

Типовой расчёт №4

№ 4.18

$$\vec{a} = (x + xy)\vec{i} + (y - x^2)\vec{j} + (z - 1)\vec{k} \qquad S:\ x^2 + y^2 = z^2\ (z \geq 0) \qquad P:\ z = 3$$

$$F(x, y, z) = x^2 + y^2 - z^2 \qquad \vec{N} = \left\{\frac{\partial F}{\partial x};\ \frac{\partial F}{\partial y};\ \frac{\partial F}{\partial z}\right\} = \{2x;\ 2y;\ -2z\}$$

$$\vec{n_0} = \frac{\vec{N}}{|\vec{N}|} = \frac{\{2x;\ 2y;\ -2z\}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}\{x;\ y;\ -z\}$$

S – площадь боковой поверхности конуса: R = 3, h = 3 $\implies$ образующая L $= \sqrt{R^2 + h^2} = 3\sqrt{2}$

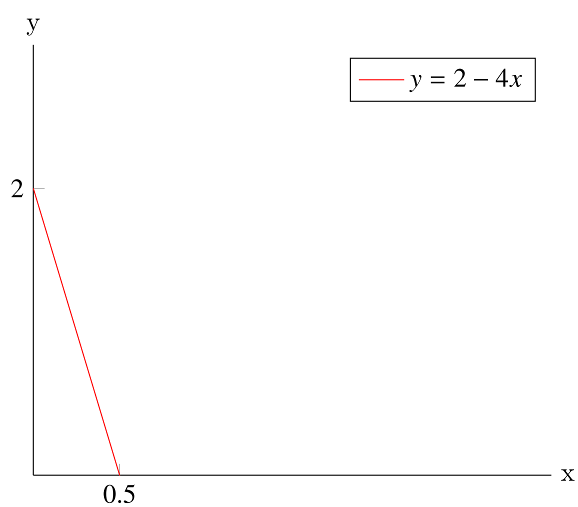
$$\Pi = \iint\limits_S (\vec{a}, \vec{n_0})\,\mathrm{d}S = \iint\limits_S \frac{x^2 + x^2y + y^2 - x^2y - z^2 + z}{\sqrt{x^2 + y^2 + z^2}}\,\mathrm{d}S = \iint\limits_S \frac{\mathrm{d}S}{\sqrt{2}} = \frac{\pi RL}{\sqrt{2}} = 9\pi$$

$\boxed{\text{Ответ: } \Pi = 9\pi}$

Типовой расчёт №5

№ 5.18

$$\vec{a} = 2x\vec{i} + y\vec{j} - 2z\vec{k} \qquad P:\ 2x + \frac{y}{2} + z = 1 \implies z = 1 - 2x - \frac{y}{2}$$



$$F(x, y, z) = 2x + \frac{y}{2} + z - 1 \qquad \vec{N} = \left\{\frac{\partial F}{\partial x};\ \frac{\partial F}{\partial y};\ \frac{\partial F}{\partial z}\right\} = \left\{2;\ \frac{1}{2};\ 1\right\} \qquad |\vec{N}| = \sqrt{4 + \frac{1}{4} + 1} = \frac{\sqrt{21}}{2}$$

$$\vec{n_0} = \frac{\vec{N}}{|\vec{N}|} = \left\{\frac{4}{\sqrt{21}};\ \frac{1}{\sqrt{21}}\ \frac{2}{\sqrt{21}}\right\} \implies \cos\gamma = \frac{2}{\sqrt{21}}$$

$$\Pi = \iint\limits_S (\vec{a}, \vec{n_0})\,\mathrm{d}S = \iint\limits_S \frac{8x + y - 4z}{\sqrt{21}}\,\frac{\mathrm{d}x\mathrm{d}y}{|\cos\gamma|} = \iint\limits_S (4x - 0.5y - 2z)\mathrm{d}x\mathrm{d}y = \iint\limits_S (8x + 1.5y - 2)\mathrm{d}x\mathrm{d}y =$$

$$= \int\limits_0^{0.5} \mathrm{d}x \int\limits_0^{2-4x} (8x + 1.5y - 2)\mathrm{d}y = \int\limits_0^{0.5} \left(16x - 32x^2 + \frac{3}{4}(2-4x)^2 - 4 + 8x\right)\mathrm{d}x = \int\limits_0^{0.5} \left(-20x^2 + 12x - 1\right)\mathrm{d}x =$$

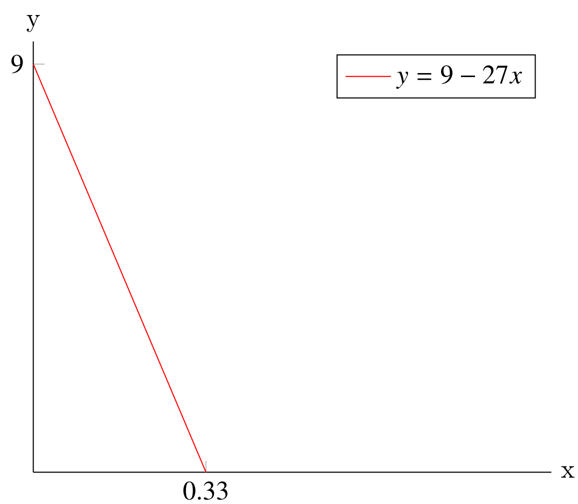
$$= \left(\frac{-20x^3}{3} + 6x^2 - x\right)\Bigg|_0^{0.5} = \frac{1}{6}$$

$\boxed{\text{Ответ: } \Pi = \frac{1}{6}}$

Типовой расчёт №6

№ 6.18

$$\vec{a} = (27\pi - 1)x\vec{i} + (34\pi y + 3)\vec{j} + 20\pi z\vec{k} \qquad P:\ 3x + \frac{y}{9} + z = 1 \implies z = 1 - 3x - \frac{y}{9}$$



$$F(x, y, z) = 3x + \frac{y}{9} + z - 1 \qquad \vec{N} = \left\{\frac{\partial F}{\partial x};\ \frac{\partial F}{\partial y};\ \frac{\partial F}{\partial z}\right\} = \left\{3;\ \frac{1}{9};\ 1\right\} \qquad |\vec{N}| = \sqrt{9 + \frac{1}{81} + 1} = \frac{\sqrt{811}}{9}$$

$$\vec{n_0} = \frac{\vec{N}}{|\vec{N}|} = \left\{\frac{27}{\sqrt{811}};\ \frac{1}{\sqrt{811}}\ \frac{9}{\sqrt{811}}\right\} \implies \cos\gamma = \frac{9}{\sqrt{811}}$$

$$\Pi = \iint\limits_S (\vec{a}, \vec{n_0})\,\mathrm{d}S = \iint\limits_S \frac{((27\pi - 1)27x + 34\pi y + 3 + 180\pi z)}{\sqrt{811}}\,\frac{\mathrm{d}x\mathrm{d}y}{|\cos\gamma|} = \frac{1}{9}\iint\limits_S (189\pi x - 27x + 14\pi y + 180\pi + 3)\mathrm{d}x\mathrm{d}y =$$

$$= \frac{1}{9}\int\limits_0^{\frac{1}{3}} \mathrm{d}x \int\limits_0^{9-27x} (189\pi x - 27x + 14\pi y + 180\pi + 3)\mathrm{d}y = \frac{1}{9}\int\limits_0^{\frac{1}{3}} (9 - 27x)(189\pi x - 27x + 7\pi(9 - 27x) + 180\pi + 3)\mathrm{d}x =$$

$$= \frac{1}{9}\int\limits_0^{\frac{1}{3}} (9 - 27x)(243\pi x - 27x + 3)\mathrm{d}x = \frac{27}{9}\int\limits_0^{\frac{1}{3}} (81\pi - 243\pi x - 12x + 27x^2 + 1)\mathrm{d}x =$$

$$= \frac{27}{9}\left(81\pi x - \frac{243\pi x^2}{2} - 6x^2 + 9x^3 + x\right)\Bigg|_0^{\frac{1}{3}} = \frac{81\pi}{2}$$

$\boxed{\text{Ответ: } \Pi = \frac{81\pi}{2}}$

Типовой расчёт №7

№ 7.18

$$\vec{a} = (\sqrt{z} + y)\vec{i} + 3x\vec{j} + (3z + 5x)\vec{k} \qquad S : z^2 = 8(x^2 + y^2); \qquad z = 2$$

$$\Pi = \oiiint\limits_V \operatorname{div}\vec{a}\,\mathrm{d}V \qquad \operatorname{div}\vec{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} = 0 + 0 + 3 = 3$$

$$V - \text{конус: } \mathrm{R} = \frac{\sqrt{2}}{2},\ \mathrm{h} = 2 \implies$$

$$\implies \Pi = \oiiint\limits_V 3\mathrm{d}V = 3\pi\frac{1}{3}R^2 h = 3\pi\frac{1}{3}\frac{2}{4}2 = \pi$$

$\boxed{\text{Ответ: } \Pi = \pi}$

Типовой расчёт №8

№ 8.18

$$\vec{a} = z\vec{i} + (3y - x)\vec{j} - z\vec{k} \qquad S : \begin{cases} x^2 + y^2 = 1 \\ z = x^2 + y^2 + 2; \qquad z = 0 \end{cases} \Leftrightarrow \begin{cases} 0 \le \rho \le 1 \\ 0 \le z \le \rho^2 + 2 \\ 0 \le \varphi \le 2\pi \end{cases}$$

$$\operatorname{div}\vec{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} = 0 + 3 - 1 = 2 \implies \Pi = \oiiint\limits_V \operatorname{div}\vec{a}\,\mathrm{d}V = 2\oiiint\limits_V \mathrm{d}V = 2\int\limits_0^{2\pi}\mathrm{d}\varphi\int\limits_0^1 \rho\mathrm{d}\rho\int\limits_0^{\rho^2+2}\mathrm{d}z =$$

$$= 2\int\limits_0^{2\pi}\mathrm{d}\varphi\int\limits_0^1 (\rho^3 + 2\rho)\mathrm{d}\rho = 2\int\limits_0^{2\pi}\frac{5}{4}\mathrm{d}\varphi = \frac{5}{2}\varphi\Big|_0^{2\pi} = 5\pi$$

$\boxed{\text{Ответ: } \Pi = 5\pi}$

Типовой расчёт №9

№ 9.18

$$\vec{a} = xy\vec{i} + yz\vec{j} + xz\vec{k} \qquad S : \begin{cases} x^2 + y^2 = 4 \\ z = 0; \qquad z = 1 \end{cases} \Leftrightarrow \begin{cases} 0 \le \rho \le 2 \\ 0 \le z \le 1 \\ 0 \le \varphi \le 2\pi \end{cases}$$

$$\operatorname{div}\vec{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} = y + z + x = \rho\cos\varphi + \rho\sin\varphi + z \implies \Pi = \oiiint\limits_V \operatorname{div}\vec{a}\,\mathrm{d}V = \oiiint\limits_V (\rho(\cos\varphi + \sin\varphi) + z)\mathrm{d}V =$$

$$= \int\limits_0^{2\pi}\mathrm{d}\varphi\int\limits_0^2 \rho\mathrm{d}\rho\int\limits_0^1 (\rho(\cos\varphi + \sin\varphi) + z)\mathrm{d}z = \int\limits_0^{2\pi}\mathrm{d}\varphi\int\limits_0^2 \left(\rho^2(\cos\varphi + \sin\varphi) + \frac{\rho}{2}\right)\mathrm{d}\rho = \int\limits_0^{2\pi}\left(\frac{8}{3}(\cos\varphi + \sin\varphi) + 1\right)\mathrm{d}\varphi =$$

$$= \left(\frac{8}{3} (\sin\varphi 0 \cos\varphi) + \varphi \right) \Big|_0^{2\pi} = 2\pi$$

$\boxed{\text{Ответ: } \Pi = 2\pi}$

Типовой расчёт №10

№ 10.18

$$\vec{F} = \left(x + y\sqrt{x^2 + y^2} \right) \vec{i} + \left(y - \sqrt{x^2 + y^2} \right) \vec{j} \qquad L : x^2 + y^2 = 16 \implies x = 4\cos\varphi;\ y = 4\sin\varphi$$

$$A = \int\limits_{\cup MN} F \mathrm{d}l = \int\limits_0^{\frac{\pi}{2}} \left(\left(4\cos\varphi + 4\sin\varphi \sqrt{16\cos^2\varphi + 16\sin^2\varphi} \right) (-4\sin\varphi) + \left(4\sin\varphi - \sqrt{16\cos^2\varphi + 16\sin^2\varphi} \right) (4\cos\varphi) \right) \mathrm{d}\varphi =$$

$$= -16 \int\limits_0^{\frac{\pi}{2}} \left(4\sin^2\varphi + \cos\varphi \right) \mathrm{d}\varphi = -16 \left(4 \int\limits_0^{\frac{\pi}{2}} \left(1 - \cos^2\varphi \right) \mathrm{d}\varphi + \int\limits_0^{\frac{\pi}{2}} \cos\varphi \mathrm{d}\varphi \right) =$$

$$= -16 \left(2 \int\limits_0^{\frac{\pi}{2}} \mathrm{d}\varphi - \int\limits_0^{\frac{\pi}{2}} \cos 2\varphi \mathrm{d}(2\varphi) + \int\limits_0^{\frac{\pi}{2}} \cos\varphi \mathrm{d}\varphi \right) = -16 \left(2\varphi - \sin(2\varphi) + \sin\varphi \right) \Big|_0^{\frac{\pi}{2}} = -16\pi - 16$$

$\boxed{\text{Ответ: } A = -16\pi - 16}$

Типовой расчёт №11

№ 11.18

$$\vec{a} = z\vec{i} + x\vec{j} + y\vec{k} \qquad \Gamma : \begin{cases} x = 2\cos t \\ y = 2\sin t \\ z = 0 \end{cases}$$

$$\text{Ц} = \int\limits_0^{2\pi} (z(-2\sin t) + x(2\cos t) + y(0))\, \mathrm{d}t = \int\limits_0^{2\pi} 4\cos^2 t \mathrm{d}t = 2 \int\limits_0^{2\pi} (\cos(2t) + 1) \mathrm{d}t = \int\limits_0^{2\pi} \cos(2t) \mathrm{d}(2t) + 2 \int\limits_0^{2\pi} \mathrm{d}t =$$

$$= (\sin(2t) + 2t) \Big|_0^{2\pi} = 4\pi$$

$\boxed{\text{Ответ: Ц} = 4\pi}$

Типовой расчёт №12

№ 12.18

$$\vec{a} = 2yz\vec{i} + xz\vec{j} - x^2\vec{k} \qquad \Gamma : \begin{cases} x^2 + y^2 + z^2 = 25 \\ x^2 + y^2 = 9 \qquad z > 0 \end{cases} \implies 9 + z^2 = 25 \implies z = 4$$

$$\text{Ц} = \int\limits_0^{2\pi} \left(2yz(-3\sin\varphi) + xz(3\cos\varphi) - x^2(0) \right) \mathrm{d}\varphi = 36 \int\limits_0^{2\pi} \left(\cos^2\varphi - 2\sin^2\varphi \right) \mathrm{d}\varphi = 36 \left(\frac{3}{2} \int\limits_0^{2\pi} (\cos(2\varphi) + 1) \mathrm{d}\varphi - 2 \int\limits_0^{2\pi} \mathrm{d}\varphi \right) =$$

$$= 36\left(\frac{3}{4}\int\limits_0^{2\pi}\cos(2\varphi)\mathrm{d}(2\varphi) + \frac{3}{2}\int\limits_0^{2\pi}\mathrm{d}\varphi - 2\int\limits_0^{2\pi}\mathrm{d}\varphi\right) = \left(\frac{3}{4}\sin(2\varphi) - \frac{1}{2}\varphi\right)\Bigg|_0^{2\pi} = -36\pi \implies |Ц| = 36\pi$$

$\boxed{\text{Ответ: } |Ц| = 36\pi}$