Т.Р. 1. КМ-4. Хохлов П.

А-07-23

1.21.

Найти производную скал. поля $U(x,y,z)$ в точке M по напр. вектора $\bar{l}$

$u = \sin(x+2y) + \sqrt{xyz}$

$\bar{l} = 4\vec{i} + 3\vec{j}$

$M = \left(\frac{\pi}{2}, \frac{3}{2}\pi, 3\right)$

Решение: $u'_x = \cos(x+2y) + \frac{1}{2}\cdot\frac{yz}{\sqrt{xyz}} = \cos\left(\frac{\pi}{2}+3\pi\right) + \frac{1}{2}\cdot\frac{9\pi}{2\cdot\sqrt{\frac{9\pi^2}{4}}} = \cos\frac{7\pi}{2} + \frac{3}{2} = \frac{3}{2}$

$u'_y = 2\cdot\cos(x+2y) + \frac{1}{2}\cdot\frac{xz}{\sqrt{xyz}} = 2\cos\frac{7\pi}{2} + \frac{3\pi}{4\cdot\sqrt{\frac{9\pi^2}{4}}} = \frac{1}{2}$

$u'_z = \frac{xy}{2\sqrt{xyz}} = \frac{\frac{3}{4}\pi^2}{2\cdot\sqrt{\frac{9\pi^2}{4}}} = \frac{\pi}{4}$

$\overline{\text{grad}}\, u = \frac{3}{2}\vec{i} + \frac{1}{2}\vec{j} + \frac{\pi}{4}\cdot\vec{k}$

$\bar{l} = \{4; 3; 0\}$

$|s| = |l| = \sqrt{16+9+0} = \sqrt{25} = 5$

$\frac{du}{ds} = \frac{(\text{grad}\,u, \bar{s})}{|s|} = \frac{\frac{3}{2}\cdot 4 + \frac{1}{2}\cdot 3 + \frac{\pi}{4}\cdot 0}{5} = \frac{6+\frac{3}{2}}{5} = \frac{15}{10} =$

$= 1,5$

Ответ: 1,5

2.21. Найти угол между градиентами скалярных полей $U(x, y, z)$ и $\vartheta(x, y, z)$ в точке M

$$U = \frac{z^2}{x^2 \cdot y^2}\ ;\quad \vartheta = \frac{3x^2}{\sqrt{2}} - \frac{y^2}{\sqrt{2}} + \sqrt{2} \cdot z^2$$

$$M = \left(\frac{2}{3};\ 2;\ \sqrt{\frac{2}{3}}\right)$$

Част. произв: $\frac{\partial \vartheta}{\partial x} = \frac{6x}{\sqrt{2}} = 3x \cdot \sqrt{2} = 3 \cdot \frac{2}{3} \cdot \sqrt{2} = 2\sqrt{2}$

$$\frac{\partial \vartheta}{\partial y} = -\frac{2y}{\sqrt{2}} = -2\sqrt{2}$$

$$\frac{\partial \vartheta}{\partial z} = 2\sqrt{2} \cdot z = 2\sqrt{\frac{2 \cdot 2}{3}} = \frac{4 \cdot \sqrt{3}}{3}$$

$$\frac{\partial U}{\partial x} = \frac{z^2}{y^2} \cdot \frac{(-2)}{x^3} = \frac{-2 \cdot \frac{2}{3}}{4 \cdot \left(\frac{2}{3}\right)^2} = -\frac{9}{8}$$

$$\frac{\partial U}{\partial y} = \frac{z^2}{x^2} \cdot \frac{(-2)}{y^3} = \frac{-2 \cdot \frac{2}{3}}{\frac{4}{9} \cdot 8} = -\frac{3}{8}$$

$$\frac{\partial U}{\partial z} = \frac{2z}{x^2 y^2} = \frac{2\sqrt{\frac{2}{3}}}{\frac{4}{9} \cdot 4} = \frac{9}{8} \cdot \sqrt{\frac{2}{3}} = \frac{3}{8} \cdot \sqrt{6}$$

Градиенты: $\operatorname{grad} \vartheta = \frac{\partial \vartheta}{\partial x}\vec{i} + \frac{\partial \vartheta}{\partial y}\vec{j} + \frac{\partial \vartheta}{\partial z} \cdot \vec{k} = 2\sqrt{2}\,\vec{j} + 2\sqrt{2}\,\vec{j} + \frac{4}{\sqrt{3}}$

$$\operatorname{grad} U = -\frac{9}{8}\vec{j} - \frac{3}{8}\vec{j} + \frac{3}{8}\sqrt{6}\,\vec{k}$$

$$(\overline{\text{grad}\, v}, \overline{\text{grad}\, u}) = -\frac{9\sqrt{2}}{4} + \frac{3\sqrt{2}}{4} + \frac{3}{2}\sqrt{2} = 0$$

$\Downarrow$

$$\cos\alpha = \frac{(\overrightarrow{\text{grad}\, v}, \overrightarrow{\text{grad}\, u})}{|\overline{\text{grad}\, v}| \cdot |\overline{\text{grad}\, u}|} = 0 \Rightarrow \alpha = 90^\circ = \frac{\pi}{2}$$

Ответ: $\frac{\pi}{2}$

Т.Р 7.21. Найти поток через поверхность S

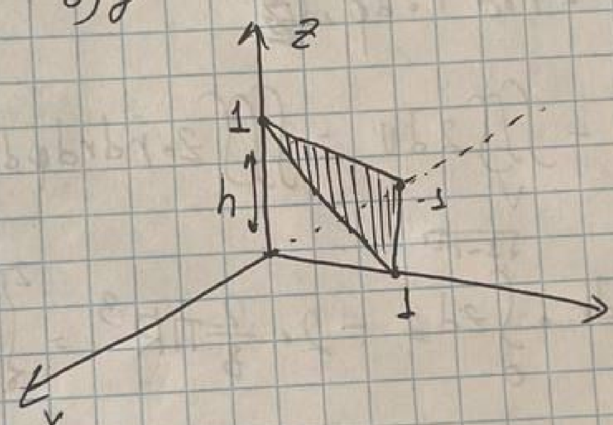
$$a = (2yz - x)\vec{i} + (xz + 2y)\vec{j} + (x^2 + z)\vec{k}$$

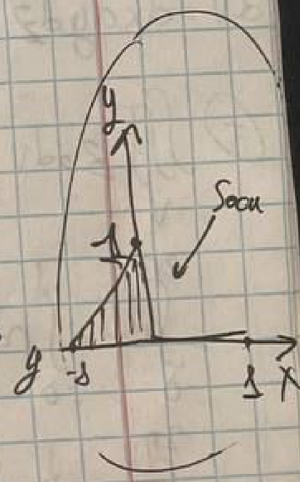
S: $y - x + z = 1$

$x = 0$

$y = 0$

$z = 0$





$$\text{div}\,\vec{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} = -1 + 2 + 1 = 2$$

Используя ТОГ

$$П = \iiint_V \text{div}\,\bar{a} \cdot dV = 2\iiint_V dV = 2|V| = 2 \cdot \frac{1}{2} \cdot 1 \cdot \frac{1}{3} = \frac{1}{3}$$

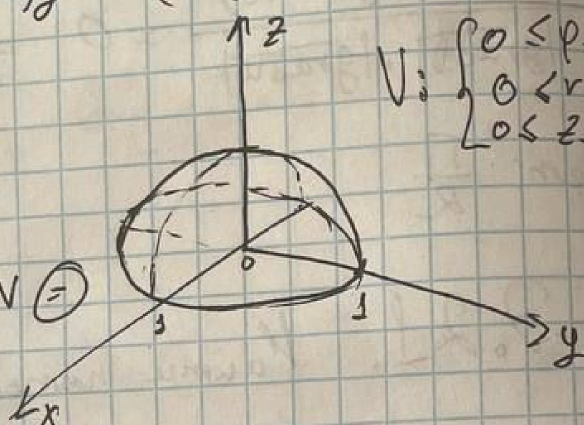
Ответ: $\frac{1}{3}$

9.21 Найти поток через S (внешн. нормаль)

$a = (zx + y)\vec{i} - (2y - x)\vec{j} - (x^2+y^2)\vec{k}$

$S: \begin{cases} x^2+y^2+z^2 = 1 \\ z = 0 \ (z \geq 0) \end{cases}$

$V: \begin{cases} 0 \leq \varphi \leq 2\pi \\ 0 < r \leq 1 \\ 0 \leq z \leq \sqrt{1-r^2} \end{cases}$



$\Pi = \iiint\limits_V \operatorname{div}\vec{a} \cdot dV = \iiint\limits_V (z-2)\,dV =$

$\operatorname{div}\vec{a} = z - 2 + 0 = (z-2)$

$dV = dx\,dy\,dz = r \cdot dr \cdot d\varphi \cdot dz$

$= \iiint\limits_V z \cdot dV - \iiint\limits_V 2\,dV = \iiint z \cdot r\,dr\,d\varphi\,dz - 2 \cdot |V| =$

$= \int_0^{2\pi} d\varphi \cdot \int_0^1 r\,dr \cdot \int_0^{\sqrt{1-r^2}} z\,dz - 2 \cdot \frac{2}{3}\pi R^3 = \int_0^{2\pi} d\varphi \cdot \int_0^1 r\,dr \cdot \frac{z^2}{2}\Big|_0^{\sqrt{1-r^2}} - \frac{4}{3}\pi$

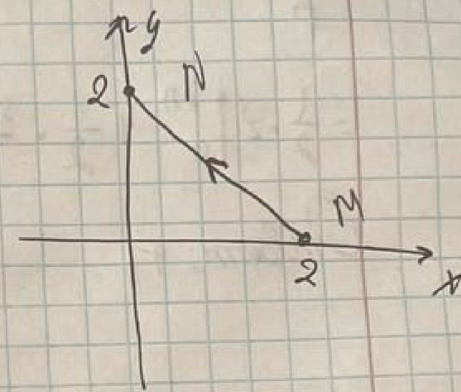
$= \int_0^{2\pi} d\varphi \cdot \int_0^1 r\,dr \cdot \frac{1-r^2}{2} - \frac{4}{3}\pi = \frac{1}{2}\int_0^{2\pi} d\varphi \cdot \left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Big|_0^1 - \frac{4}{3}\pi =$

$= \frac{1}{2}\int_0^{2\pi} d\varphi \cdot \frac{1}{4} - \frac{4}{3}\pi = \frac{2\pi}{2} \cdot \frac{1}{4} - \frac{4}{3}\pi = \frac{\pi}{4} - \frac{4}{3}\pi = -\frac{13}{12}\pi$

10.21. Найти работу $\vec{F}$ при перемещении вдоль L от M до N

$F = (x^2+y^2)\vec{i} + y^2\vec{j}$

L: MN || M(2;0) N(0;2)



ур-е: $y = -x+2 = 2-x$

$dy = -dx$ Измен. $x \to$ от 2 до 0

Работа $F = A = \int\limits_L (\vec{F}, d\vec{s}) = \int\limits_L (x^2+y^2)dx + y^2 \cdot dy =$

$$= \int_2^0 \left(x^2 + (2-x)^2\right)dx - (2-x)^2 dx = \int_2^0 x^2 dx + \cancel{(2-x)^2 dx - (2-x)^2}$$

$$= \int_2^0 x^2 dx = \frac{x^3}{3}\Big|_2^0 = 0 - \frac{8}{3} = -\frac{8}{3}$$

Ответ: $-\frac{8}{3}$

11.21. Найти циркуляцию $\vec{a}$ вдоль контура Г

$\vec{a} = xz\vec{i} + x\vec{j} + z^2\vec{k}$ $0 \le t \le 2\pi$

Циркуляция поля $Ц = \oint\limits_Г (\vec{a}, d\vec{s}) =$

$Г: \begin{cases} x = \cos t \\ y = \sin t \\ z = \sin t \end{cases}$

$$= \int\limits_Г xz\,dx + x\,dy + z^2 dz = \int \cancel{\cos t \cdot \sin t \cdot (-\sin t)dt} +$$

$$+ \cos t \cdot \cos t\,dt + \cancel{\sin^2 t \cdot \cos t\,dt} =$$

$dx = x'dt = -\sin t\,dt$

$dy = dz = \cos t\,dt$

$$= \int_{\Gamma} \cos^2 t \, dt = \int_0^{2\pi} \cos^2 t \, dt = \int_0^{2\pi} \frac{1+\cos 2t}{2} dt =$$

$$= \frac{1}{2} \cdot t \Big|_0^{2\pi} + \frac{1}{2} \cdot \frac{\sin 2t}{2} \Big|_0^{2\pi} = \pi - 0 = \pi$$

Ответ: π