ТР математический анализ. Интегралы

ТР. 1.7

$$\int \frac{\sin 2x}{\sqrt{1+\cos 4x}}dx = \int \frac{2\cos x \sin x}{\sqrt{1+\cos 4x}}dx = -\int \frac{d\cos^2 x}{\sqrt{1+\cos 4x}} = -\int \frac{dt}{\sqrt{1+t^2}} =$$

$$= -\ln|t+\sqrt{t^2+1}| = -\ln|\cos^2 x + \sqrt{\cos 4x} + 1| + C$$

$d\cos^2 x = -2\sin x \cos x\, dx$ $\qquad \cos^2 x = t$

$-d\cos^2 x = 2\sin x \cos x\, dx$ $\qquad \cos^4 x = t^2$

Ответ: $-\ln|\cos^2 x + \sqrt{\cos 4x} + 1| + C$

ТР № 2.7

$$\int \arcsin 2x \cdot dx = \left\{\begin{array}{l} u = \arcsin 2x \\ u' = \frac{2}{\sqrt{1-4x^2}} \\ v' = 1 \\ v = x \end{array}\right\} = x\arcsin 2x - \int \frac{2x}{\sqrt{1-4x^2}}dx \ (=)$$

$$\int \frac{2x}{\sqrt{1-4x^2}}dx = \left\{\begin{array}{l} 1-4x^2 = t \\ dt = -8x\,dx \\ -\frac{1}{4}dt = 2x\,dx \end{array}\right\} = -\frac{1}{4}\int \frac{dt}{\sqrt{t}} = -\frac{1}{4}\cdot 2\sqrt{t} = -\frac{1}{2}\sqrt{1-4x^2}$$

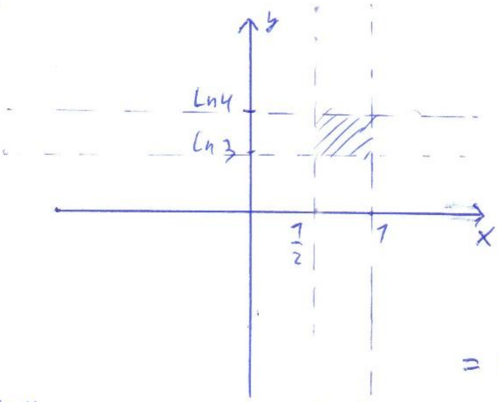
$$(=)\ x\arcsin 2x + \frac{\sqrt{1-4x^2}}{2} + C$$

Ответ: $x\arcsin 2x + \frac{\sqrt{1-4x^2}}{2} + C$

ТР №3

$$\iint\limits_D 4y\, e^{2xy}\, dx\, dy$$

$D:$ $y = \ln 3$
$y = \ln 4$
$x = \frac{1}{2}$
$x = 1$



$$\int_{\ln 3}^{\ln 4} dy \int_{\frac{1}{2}}^{1} 4y^2 e^{2xy}\, dx$$

$$\int_{\frac{1}{2}}^{1} 4y^2 e^{2xy}\, dx = 4y^2\, \frac{e^{2xy}}{2y}\Big|_{\frac{1}{2}}^{1} = 2y(e^{2y} - e^{y}) =$$

$$= 2y\, e^{2y} - 4y\, e^{y}$$

$$\int_{\ln 3}^{\ln 4} 2y\, e^{2y} - 2e^{y} = 2\, \frac{e^{2y}}{2}\Big|_{\ln 3}^{\ln 4} - 2\, e^{y}\Big|_{\ln 3}^{\ln 4} = 16 - 9 - 2(4-3) = 7 - 8 + 6 = 5$$

Ответ: 5

ТР № 4.7

$$\int \frac{x^3-3x}{(x+2)(x+1)^2} \circleddash$$

$$(x+2)(x+1)^2 = (x+2)(x^2+2x+1) = x^3+2x^2+x+2x^2+4x+2 = x^3+4x^2+5x+2$$

$$\frac{x^3-3x}{x^3+4x^2+5x+2} = \frac{P_n(x)}{Q_m(x)},$$ где $n=m \Rightarrow$ дробь неправильная

$$\cancel{\frac{x^3-3x}{x^3+4x^2+5x+2}}$$

$$\begin{array}{r|l} x^3-3x & x^3+4x^2+5x+2 \\ -\;x^3+4x^2+5x+2 & 1 \\ \hline \cancel{-4x^2+5x+2} & \\ -4x^2-8x-2 & \end{array}$$

$$\frac{x^3-3x}{x^3+4x^2+5x+2} = 1 - \frac{4x^2+8x+2}{x^3+4x^2+5x+2}$$

$$\circleddash \int dx - 2\int \frac{2x^2+4x+1}{(x+2)(x+1)^2} \overset{*}{=}$$

$$\frac{2x^2+4x+1}{(x+2)(x+1)^2} = \frac{A}{(x+2)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2} = \frac{Ax^2+2Ax+A+Bx^2+3Bx+2B+Cx+2C}{(x+2)(x+1)^2}$$

$$\begin{array}{l} \text{при } x^2: \\ \text{при } x^1: \\ \text{при } x^0: \end{array} \begin{cases} A+B=2 \\ 2A+3B=C=4 \\ A+2B+2C=1 \end{cases} \Leftrightarrow \begin{cases} A=2-B \\ 4-2B+3B+C=4 \\ 2-B+2B+2C=1 \end{cases} \Leftrightarrow \begin{cases} A=2-B \\ C=-B \\ B-2B=-1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} A=1 \\ C=-1 \\ B=1 \end{cases}, \quad \frac{2x^2+4x+1}{(x+2)(x+1)^2} = \frac{1}{(x+2)} + \frac{1}{(x+1)} - \frac{1}{(x+2)^2}$$

$$\overset{*}{=} \int dx - 2\int \frac{dx}{x+2} - 2\int \frac{dx}{x+1} + 2\int \frac{dx}{(x+1)^2} = x - 2\ln|x+2| - 2\ln|x+1| +$$

$$+ \frac{2}{(1-2)(x+1)} + C = x - 2\ln\left|\frac{x+2}{x+1}\right| - \frac{2}{(x+1)} + C$$

Ответ: $x - 2\ln\left|\frac{x+2}{x+1}\right| - \frac{2}{(x+1)} + C$

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ТР №6.7

$$\int \frac{3\sqrt{4-x} - 2\sqrt{3x+2}}{(\sqrt{3x+2} + 3\sqrt{4-x})(3x+2)^2}dx = \int \frac{3\sqrt{4-x} + 1\sqrt{3x+2} - 3\sqrt{3x+2}}{(\sqrt{3x+2} + 3\sqrt{4-x})\ (3x+2)^2}dx =$$

$$= \int \left(1 - \frac{3\sqrt{3x+2}}{\sqrt{3x+2} + 3\sqrt{4-x}}\right)\frac{1}{(3x+2)^2}dx = \int \left(1 - \frac{3}{1 + 3\sqrt{\frac{4-x}{3x+2}}}\right)\frac{1}{(3x+2)^2}dx =$$

$$= \left| \begin{array}{ll} \sqrt{\dfrac{4-x}{3x+2}} = t \quad dt^2 = 2t\,dt & \dfrac{-3x-2-12+3x}{(3x+2)^2}dx = 2t\,dt \\ \dfrac{4-x}{3x+2} = t^2 & \dfrac{-14}{(3x+2)^2}dx = 2t\,dt \\ \dfrac{(4-x)'(3x+2) - (4-x)(3x+2)'}{(3x+2)^2}dx = 2t\,dt & \dfrac{dx}{(3x+2)^2} = \dfrac{-t\,dt}{7} \end{array} \right| =$$

$$= \int \left(1 - \frac{3}{3t+1}\right) \cdot \frac{-t\,dt}{7} = -\frac{1}{7}\int\left(1 - \frac{3}{3t+1}\right)\cdot t\,dt = -\frac{1}{7}\int\left(t - \frac{3t}{3t+1}\right)dt =$$

$$= -\frac{1}{7}\int\left(t - 1 + \frac{1}{3t+1}\right)dt = -\frac{1}{7}\int t\,dt + \frac{1}{7}\int dt - \frac{1}{7}\int\frac{dt}{3t+1} = -\frac{t^2}{14} + \frac{t}{7} - \frac{1}{7}\cdot\frac{1}{3}\int\frac{dt}{t+\frac{1}{3}} =$$

$$= -\frac{t^2}{14} + \frac{t}{7} - \frac{1}{21}\ln\left|t + \frac{1}{3}\right| + C = -\frac{4-x}{14(3x+2)} + \frac{1\sqrt{4-x}}{7\sqrt{3x+2}} - \frac{1}{21}\ln\left|\sqrt{\frac{4-x}{3x+2}} + \frac{1}{3}\right| + C$$

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№ 11.7

$$\int_0^{\frac{\pi}{2}} \sin^4\frac{x}{2}\,dx = \int_0^{\frac{\pi}{2}} \frac{(1-\cos x)^2}{4}\,dx = \frac{1}{4}\int_0^{\frac{\pi}{2}} dx - \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos x\,dx + \frac{1}{4}\int_0^{\frac{\pi}{2}} \cos^2 x\,dx = \frac{\pi}{8} - \frac{1}{2}(\sin\frac{\pi}{2} - \underbrace{\sin 0}_{0}) +$$

$$+\frac{1}{4}\int_0^{\frac{\pi}{2}} \frac{1+\cos 2x}{2}\,dx = \frac{\pi}{8} - \frac{1}{2} + \frac{1}{8}\int_0^{\frac{\pi}{2}} dx + \frac{1}{8}\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} \cos 2x\,d2x = \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{16} + \cancel{\frac{1}{16}(\sin\frac{\pi}{2} -}$$

$$\cancel{\sin 0) = \frac{3\pi}{16} - \frac{1}{2} + \frac{1}{16} = \frac{3\pi}{16} - \frac{7}{16}} = \frac{3\pi}{16} - \frac{1}{2} + \frac{1}{16}\int_0^{\pi} \cos u\,du = \frac{3\pi}{16} - \frac{1}{2} + \frac{1}{16}(\underbrace{\sin\pi - \sin 0}_{0}) =$$

смена пределов интегрирования
$0 \le x \le \frac{\pi}{2} \iff 0 \le 2x \le \pi$

$$= \frac{3\pi}{16} - \frac{1}{2}$$

Ответ: $\frac{3\pi}{16} - \frac{1}{2}$

№ 12.7

$$\int_0^{\frac{\pi}{2}} x\sin^2\frac{x}{2}\,dx = \left|\begin{matrix} \frac{x}{2} = u & dx = 2du \\ x = 2u & 0 \le x \le \frac{\pi}{2} \\ du = \frac{1}{2}dx & 0 \le \frac{x}{2} \le \frac{\pi}{4} \end{matrix}\right| = \int_0^{\frac{\pi}{4}} 4u\sin^2 u\,du = 4\int_0^{\frac{\pi}{4}} u\left(\frac{1-\cos 2u}{2}\right)du =$$

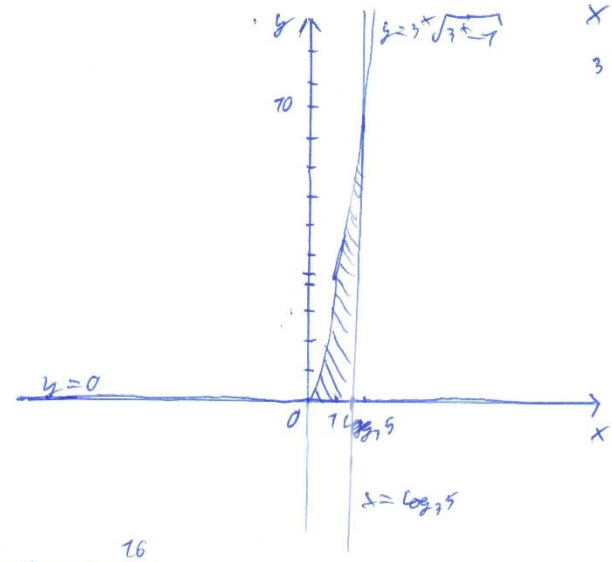
$$= 2\int_0^{\frac{\pi}{4}} u\,du - 2\int_0^{\frac{\pi}{4}} u\cos 2u\,du = 2\,\frac{u^2}{2}\Big|_0^{\frac{\pi}{4}} - 2\left(\frac{u\sin 2u}{2} + \frac{\cos 2u}{4}\right)\Big|_0^{\frac{\pi}{4}} = \frac{\pi^2}{16} - \frac{\pi}{4}\sin\frac{\pi}{2} +$$

$$+ \underbrace{\frac{\cos\frac{\pi}{2}}{2}}_{0} - \underbrace{0\sin 0}_{0} - \frac{\cos 0}{2}) = \frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2}$$

Ответ: $\frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2}$

№ 13.7

$y = 3^x\sqrt{3^x-1}$ $\quad y = 0;\quad x = \log_3 5$



y	0	$3\sqrt{2}$	
x	0	1	$\log_3 5$

$$3^{\log_3 5}\sqrt{3^{\log_3 5}-1} = 5\sqrt{5-1} = 10$$

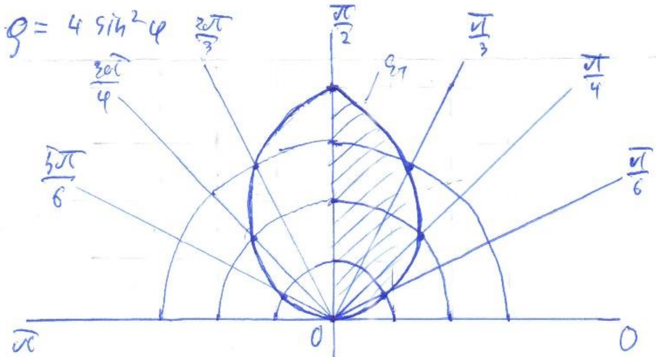
$$\int_0^{\log_3 5} 3^x\sqrt{3^x-1}\,dx = \left| \begin{array}{l} d(3^x-1) = 3^x\ln 3\,dx \\ dx = \dfrac{d(3^x-1)}{3^x\ln 3} \\ 0 \le x \le \log_3 5 \\ 1 \le 3^x \le 5 \\ 0 \le 3^x - 1 \le 4 \\ 3^x-1=u \end{array} \right| =$$

$$= \int_0^4 \frac{\sqrt{u}\,du}{\ln 3} = \frac{1}{\ln 3}\int_0^4 \sqrt{u}\,du = \frac{1}{\ln 3}\,\frac{2}{3}\,u^{\frac{3}{2}}\Big|_0^4 = \frac{2}{3\ln 3}\sqrt{4^3} = \frac{16}{3\ln 3}$$

Ответ: $\frac{16}{3\ln 3}$

№ 14.7

$\rho = 4\sin^2\varphi$



φ	0	$\frac{\pi}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
ρ	0	4	1	2	3

$4\sin^2\varphi = 4$

$4\sin^2\frac{\pi}{6} = 4\cdot\frac{1}{4} = 1 \qquad 4\sin^2\frac{\pi}{3} = 4\cdot\frac{3}{4} = 3$

$4\sin^2\frac{\pi}{4} = 4\cdot\frac{1}{2} = 2$

Фигура симметрична поэтому $S = 2S_1 \quad 0 \le \varphi \le \frac{\pi}{2} \qquad \frac{1}{2}\int_\alpha^\beta \rho^2(\varphi)\,d\varphi = S_1$

$$2\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} 16\sin^4\varphi\,d\varphi = 16\int_0^{\frac{\pi}{2}}\sin^4\varphi\,d\varphi = 16\int_0^{\frac{\pi}{2}}\frac{(1-\cos 2\varphi)^2}{4}\,d\varphi = 4\int_0^{\frac{\pi}{2}}d\varphi - 4\int_0^{\frac{\pi}{2}}2\cos 2\varphi\,d\varphi +$$

$$+4\int_0^{\frac{\pi}{2}}(\cos 2\varphi)^2\,d\varphi = 2\pi - 2\left(\frac{\sin\pi}{2} - \frac{\sin 0}{2}\right) + 4\int_0^{\frac{\pi}{2}}\frac{1+\cos 4\varphi}{2}\,d\varphi = 2\pi + 2\int_0^{\frac{\pi}{2}}(1+\cos 4\varphi)\,d\varphi =$$

$$= 2\pi + \pi + 2\left(\frac{\sin 2\pi}{4} - \frac{\sin 0}{4}\right) = 3\pi$$

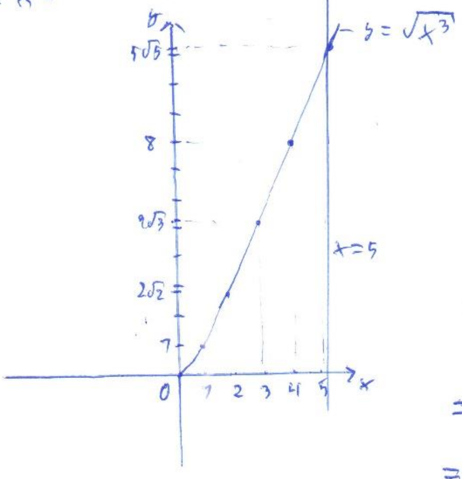
Ответ: 3π

ТР №19.7

$y^2 = x^3$, $x = 5$

$y = \sqrt{x^3} = x^{\frac{3}{2}}$

y	0	1	$2\sqrt{2}$	$3\sqrt{3}$	8	$5\sqrt{5}$
x	0	1	2	3	4	5



$$l = \int_a^b \sqrt{1 + (f'(x))^2}$$

$$y' = (x^{\frac{3}{2}})' = \frac{3}{2} x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$$

$$l = \int_0^5 \sqrt{1 + \frac{9}{4}x} = \left| \begin{array}{l} d(\underbrace{1 + \frac{9}{4}x}_{=u}) = \frac{9}{4}dx \\ 0 \le x \le 5 \\ 0 \le \frac{9}{4}x \le \frac{45}{4} \\ 1 \le \frac{9}{4}x + 1 \le \frac{49}{4} \end{array} \right| =$$

$$= \frac{4}{9}\int_1^{\frac{49}{4}} \sqrt{u}\, du = \frac{4}{9} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_1^{\frac{49}{4}} =$$

$$= \frac{8}{27}\left(\sqrt{\left(\frac{49}{4}\right)^3} - \sqrt{1^3}\right) = \frac{8}{27}\left(\sqrt{\frac{7^6}{2^6}} - 1\right) =$$

$$= \frac{8}{27}\left(\frac{7^3}{2^3} - 1\right) = \frac{343}{27} - \frac{8}{27} = \frac{335}{27}$$

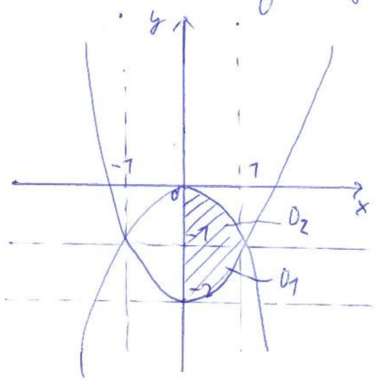
Ответ: $\frac{335}{27}$

Кратные интегралы ТР № 1

$$\int_{-2}^{-1} dy \int_{0}^{\sqrt{2+y}} f\,dx + \int_{-1}^{0} dy \int_{0}^{\sqrt{-y}} f\,dx =$$

область интегрирования: $D = D_1 \cup D_2$

$$D_1 = \begin{cases} -2 \le y \le -1 \\ 0 \le x \le \sqrt{2+y} \end{cases} ; \quad D_2 = \begin{cases} -1 \le y \le 0 \\ \cancel{\sqrt{-y}} \, 0 \le x \le \cancel{0} \, \sqrt{-y} \end{cases}$$



$0 < x = \sqrt{2+y}$

$x^2 = 2+y$

$y = x^2 - 2$

$\cancel{\sqrt{-y} \le x < 0}$

$\cancel{-y = x^2}$

$\cancel{y = -x^2}$

$0 < x = \sqrt{-y}$

$x^2 = -y$

$y = -x^2$

$-x^2 = x^2 - 2$

$2x^2 = 2$

$x = \pm 1$

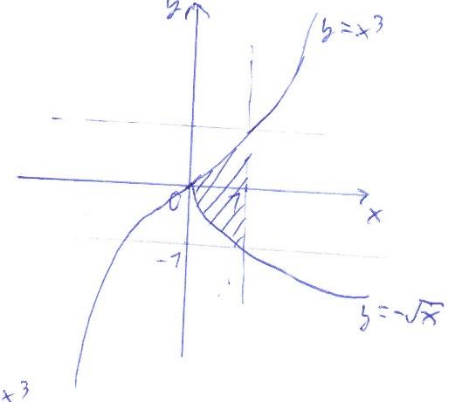
$$D: \begin{cases} 0 \le x \le 1 \\ x^2 - 2 \le y \le -x^2 \end{cases}$$

$$= \int_{0}^{1} dx \int_{x^2-2}^{-x^2} f\,dy$$

Ответ: $\int_{0}^{1} dx \int_{x^2-2}^{-x^2} f\,dy$

$$\iint\limits_D (18x^2y^2 + 32x^3y^3)\,dx\,dy$$ ТР №2

$$D: \begin{cases} x = 1 \\ y = x^3 \\ y = -\sqrt{x} \end{cases}$$



$$\int_0^1 dx \int_{-\sqrt{x}}^{x^3} (18x^2y^2 + 32x^3y^3)\,dy$$

$$\int_{-\sqrt{x}}^{x^3} (18x^2y^2 + 32x^3y^3)\,dy = 18x^2 \frac{y^3}{3}\Big|_{-\sqrt{x}}^{x^3} + 32x^3 \frac{y^4}{4}\Big|_{-\sqrt{x}}^{x^3} = 18x^2\left(\frac{x^9}{3} - \frac{-x^{\frac{3}{2}}}{2}\right) +$$

$$+ 32x^3\left(\frac{x^{12}}{4} - \frac{x^2}{4}\right) = 6x^{11} + 6x^{\frac{7}{2}} + 8x^{15} - 8x^5$$

$$\int_0^1 (6x^{11} + 6x^{\frac{7}{2}} + 8x^{15} - 8x^5)\,dx = 6\frac{x^{12}}{12}\Big|_0^1 + 6\frac{x^{\frac{9}{2}}}{\frac{9}{2}}\Big|_0^1 + 8\frac{x^{16}}{16}\Big|_0^1 - 8\frac{x^6}{6}\Big|_0^1 =$$

$$= \frac{1}{2} + \frac{4}{3} + \frac{1}{2} - \frac{4}{3} = \cancel{1 + \frac{8}{3} = \frac{11}{3}} = 1$$

Ответ: $\cancel{\frac{11}{3}}$ 1.

ТР № 3.7

$$\int \frac{6x-5}{\sqrt{-x^2+2x+3}}dx = -\int \frac{6x-5}{\sqrt{x^2-2x-3}}dx = -\int \frac{6x-5}{\sqrt{(x-1)^2-4}}dx = \begin{cases} u = x-1 \\ du = dx \\ x = u+1 \end{cases} =$$

$$= -\int \frac{6u+1}{\sqrt{u^2-4}}du = -6\int \frac{u\,du}{\sqrt{u^2-4}} - \int \frac{du}{\sqrt{u^2-4}} = -6\int \frac{u\,du}{\sqrt{u^2-4}} + \arcsin\left(\frac{u}{2}\right) \ominus$$

$$\int \frac{u\,du}{\sqrt{u^2-4}} = \begin{cases} v = u^2-4 \\ dv = 2u\,du \\ \frac{dv}{2} = u\,du \end{cases} = \frac{1}{2}\int \frac{dv}{\sqrt{v}} = \frac{1}{2}\sqrt{v}\cdot 2 + C = \sqrt{u^2-4} + C$$

$$\ominus -6\sqrt{u^2-4} + \arcsin\left(\frac{u}{2}\right) + C = \arcsin\left(\frac{x-1}{2}\right) - 6\sqrt{x^2-2x-3} + C$$

Ответ: $\arcsin\left(\frac{x-1}{2}\right) - 6\sqrt{x^2-2x-3} + C$