Part III

Localization, Mapping, and Navigation


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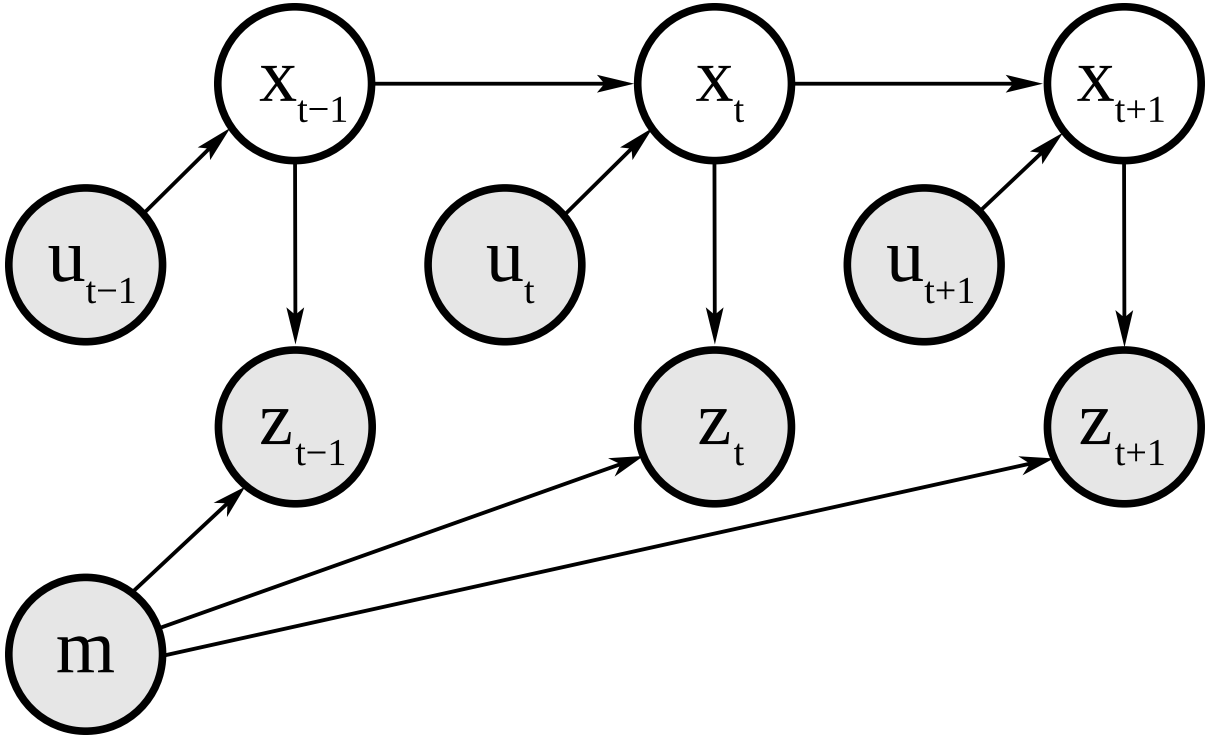
Localization Problem

Find robot's location given a map of the environment, robot actions, and sensor measurements

$$p(x_{0:t}|z_{1:t}, u_{1:t}, m)$$

Robot Localization

Under the Markov assumption, we have that



u, z, m are observed. x is unknown

Bayes Filter

Family of approaches for state estimation

It considers two steps

1. Prediction step: The belief of the robot's pose is updated based on its previous state and action.

$$\bar{bel}(x_t) = \int p(x_t|u_t, x_{t-1}) bel(x_{t-1}) dx_{t-1}$$

2. Correction step: Where the belief is refined based on sensor observations $bel(x_t)$

$$bel(x_t) = \eta p(z|x_t) \bar{bel}(x_t)$$

η is a normalization constant, and $bel(\cdot)$ is a short hand notation for:

$\bar{bel}(x_t) = p(x_t|x_{0:t-1}, u_{0:t}, z_{0:t-1})$ $bel(x_t) = p(x_t|x_{0:t-1}, u_{0:t}, z_{0:t})$

Local vs Global Localization

Bayes Filter is a Recursive estimator as to compute $\mathrm{bel}(x_t)$ we need the estimation at the previous time $t-1$. We can do this back to $t=0$.

Therefore, we need to define the distribution of $p(x_0) = bel(x_0)$.

Position Tracking

We assume we known the robot's starting location (often by considering $p(x_0)$ to be a gausian distribution around [0, 0, 0] with known variance),

Global Localization

We make no assumptions regarding the robot's starting location other than it is somewhere within the mapped area

Kalman Filters

Kalman Filter: Gaussian World

Kalman Filters model makes two (rather strict) assumptions

1. Everything is modeled as multivariate Gaussians,

$$p(x) = \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu) \right)$$

2. All models are linear
   i. Linear state transition probability
   ii. Linear measurement probability

Under these restrictions, KF is the optimal estimator (statistical optimal)

Linear state transition probability

Model x_t as a linear function (※)

$$x_t = Ax_{t-1} + Bu_t + \epsilon_x$$

where

- ϵ_x is white noise with variance R
- x is a nx1 vector (n=3 for planar motion),
- u is a mx1 vector (m=3 for the odometry model, 2 for velocity model).
- A and B are a nxn and nxm matrices, respectively

Probabilistic model

$$p(x_t | u_t, x_{t-1}) = \mathcal{N}(Ax_{t-1} + Bu_t, R_t)$$

`Note`

※This has not been our assumption with either the velocity or the odometry models, but an approximation that speeds computations considerably

Prediction Step

$$\bar{bel}(x_t) = \int p(x_t | u_t, x_{t-1}) bel(x_{t-1}) dx_{t-1}$$

To compute $bel(x_t)$ we need to remember that $bel(x_{t-1})$ has a multivariate Gaussian distribution

$$bel(x_t) = \mathcal{N}(\mu_{t-1}, \Sigma_{t-1})$$

Applying properties for the linear combination of Gaussians, and convolution of two multivariate Gaussians distributions we get

$$\bar{bel}(x_t) = \mathcal{N}(\bar{\mu}_t, \bar{\Sigma}_t)$$

where,

$$\bar{\mu}_t = A\mu_{t-1} + Bu_t \qquad \bar{\Sigma}_t = A\Sigma_{t-1}A^T + R$$

`Note` A full proof can be found at Probabilistic Robotics, section 3.2

Linear measurement probability

Model z_t as a linear function linear function

$$z_t = Cx_t + \epsilon_z$$

where

- z is a kx1 vector,
- C is a kxn matrix,
- ϵ is white noise with covariance Q

Probabilistic model

$$p(z_t|x_t) = \mathcal{N}(Cx_t, Q)$$

Correction Step

$$bel(x_t) = \eta p(z_t|x_t)\bar{bel}(x_t)$$

Rewriting,

$$bel(x_t) = \eta \det(2\pi Q)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(z_t - Cx_t)^T Q^{-1}(z_t - Cx_t)\right) \det(2\pi\bar{\Sigma})^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x_t - \bar{\mu}_t)^T \bar{\Sigma}^{-1}(x_t - \bar{\mu}_t)\right)$$

Checking only the exponential component

$$bel(x_t) = \eta_2 \exp(-J)$$

$$J_t = \frac{1}{2}(z_t - Cx_t)^T Q^{-1}(z_t - Cx_t) + \frac{1}{2}(x_t - \bar{\mu}_t)^T \bar{\Sigma}^{-1}(x_t - \bar{\mu}_t)$$

To calculate the quadratic equation in x_t that defines the Gaussian Distribution, we can calculate its first two derivatives.

$$\frac{\partial J}{\partial x_t} = -C^T Q^{-1}(z_t - Cx_t) + \bar{\Sigma}_t^{-1}(x_t - \bar{\mu}_t)$$

$$\frac{\partial^2 J}{\partial x_t^2} = -C^T Q^{-1} C + \bar{\Sigma}_t^{-1}$$

The covariance of bel, Σ_t, is equal to the inverse of the second derivative.

$$\Sigma = (-C^T Q^{-1} C + \bar{\Sigma}_t^{-1})^{-1}$$

Using the matrix inversion lemma (*)

$$\Sigma = \bar{\Sigma}_t - \bar{\Sigma}_t C^T (Q + C\bar{\Sigma}_t C^T)^{-1} C^T \bar{\Sigma}_t$$

which we can rewrite as

$$\Sigma_t = (I - K_t C)\bar{\Sigma}_t$$

with $K_t = \bar{\Sigma}_t C^T (Q + C\bar{\Sigma}_t C^T)^{-1}$

(*) Table 3.2 Probabilistic Robotics

Optional

Inversion Lemma. For any invertible quadratic matrices R and Q and any matrix P with appropriate dimension, the following holds true

$$(R + P\,Q\,P^T)^{-1} \quad = \quad R^{-1} - R^{-1}\,P\,(Q^{-1} + P^T\,R^{-1}\,P)^{-1}\,P^T\,R^{-1}$$

assuming that all above matrices can be inverted as stated.

Proof. It suffices to show that

$$(R^{-1} - R^{-1}\,P\,(Q^{-1} + P^T\,R^{-1}\,P)^{-1}\,P^T\,R^{-1})\,(R + P\,Q\,P^T) \quad = \quad I$$

This is shown through a series of transformations:

$$\begin{aligned}
&= \underbrace{R^{-1}\,R}_{=I} + R^{-1}\,P\,Q\,P^T - R^{-1}\,P\,(Q^{-1} + P^T\,R^{-1}\,P)^{-1}\,P^T\,\underbrace{R^{-1}\,R}_{=I} \\
&\quad - R^{-1}\,P\,(Q^{-1} + P^T\,R^{-1}\,P)^{-1}\,P^T\,R^{-1}\,P\,Q\,P^T \\
&= I + R^{-1}\,P\,Q\,P^T - R^{-1}\,P\,(Q^{-1} + P^T\,R^{-1}\,P)^{-1}\,P^T \\
&\quad - R^{-1}\,P\,(Q^{-1} + P^T\,R^{-1}\,P)^{-1}\,P^T\,R^{-1}\,P\,Q\,P^T \\
&= I + R^{-1}\,P\,[Q\,P^T - (Q^{-1} + P^T\,R^{-1}\,P)^{-1}\,P^T \\
&\quad - (Q^{-1} + P^T\,R^{-1}\,P)^{-1}\,P^T\,R^{-1}\,P\,Q\,P^T] \\
&= I + R^{-1}\,P\,[Q\,P^T - (Q^{-1} + P^T\,R^{-1}\,P)^{-1}\,\underbrace{Q^{-1}\,Q}_{=I}\,P^T \\
&\quad - (Q^{-1} + P^T\,R^{-1}\,P)^{-1}\,P^T\,R^{-1}\,P\,Q\,P^T] \\
&= I + R^{-1}\,P\,[Q\,P^T - \underbrace{(Q^{-1} + P^T\,R^{-1}\,P)^{-1}\,(Q^{-1} + P^T\,R^{-1}\,P)}_{=I}\,Q\,P^T] \\
&= I + R^{-1}\,P\,\underbrace{[Q\,P^T - Q\,P^T]}_{=0} \quad = \quad I
\end{aligned}$$

Table 3.2 The (specialized) inversion lemma.

The mean of bel, μ_t, is calculated by setting the first derivative to zero

$$C^T Q^{-1}(z_t - C\mu_t) = \bar{\Sigma}_t^{-1}(\mu_t - \bar{\mu}_t)$$

Using several matrix manipulations we get

$$\mu_t = \bar{\mu}_t + K_t(z_t - C_t\bar{\mu}_t)$$

`Note` A full proof can be found at Probabilistic Robotics, section 3.2

Kalman Filter

Prediction Step

1. $\bar{\mu}_t = A\mu_{t-1} + Bu_t$
2. $\bar{\Sigma}_t = A\Sigma_{t-1}A^T + R$

Correction Step

3. $K_t = \bar{\Sigma}_t C^T (Q + C\bar{\Sigma}_t C^T)^{-1}$
4. $\mu_t = \bar{\mu}_t + K_t(z_t - C_t\bar{\mu}_t)$
5. $\Sigma_t = (I - K_t C)\bar{\Sigma}_t$

Multivariate Gaussian distribution times Scalar Matrix

For x distributed as $\mathcal{N}(\mu, \Sigma)$ and a scalar matrix M,

$$Mx = \mathcal{N}(M\mu, M\Sigma M^T)$$

Addition of two Multivariate Gaussian distribution

For two multivariate Gaussian distributions $x = \mathcal{N}(\mu_x, \Sigma_x)$, $y = \mathcal{N}(\mu_y, \Sigma_y)$

$$x + y = \mathcal{N}(\mu_x + \mu_y, \Sigma_x + \Sigma_y)$$