# Seminar 2

Ex

Ex. 1) $A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 0 & a & 1 \\ 0 & 1 & 3 & b \end{pmatrix} \in \mathcal{M}_{3,4}(\mathbb{R})$

$a, b = ?$ a.î. $rg\, A = 2$.

$$\Delta_1 = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & a \\ 0 & 1 & 3 \end{vmatrix} = 0 + 6 + 0 - 0 - a - 12 = -6 - a$$

$\Delta_1 = 0 \Rightarrow -6 - a = 0 \Rightarrow a = -6$

$$\Delta_2 = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & b \end{vmatrix} = 0 + 2 + 0 - 0 - 1 - 4b = -4b + 1.$$

$\Delta_2 = 0 \Rightarrow -4b + 1 = 0 \Rightarrow b = \frac{1}{4}$

Ex 2) $A = \begin{pmatrix} a & 1 & 2 \\ 1 & 1 & 1 \\ -1 & 1 & 1-a \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$

$$\det A = \begin{vmatrix} a & 1 & 2 \\ 1 & 1 & 1 \\ -1 & 1 & 1-a \end{vmatrix} = a(1-a) - 1 + 2 + 2 - a - 1 + a =$$
$$= a - a^2 + 2$$

$\Delta = a^2 - a + 2a + 2 = -a(a+1) + 2(a+1) = (a+1)(2-a)$

Caz I: $\Delta \neq 0 \Rightarrow a \in \mathbb{R} \setminus \{-1; 2\} \Rightarrow rg\, A = 3$

Caz II: $\Delta = 0$

$$\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1 - 2 = -1 \neq 0 \Rightarrow rg A = 2 \quad \forall a \in \{-1; 2\}.$$

Ex. 3) $A = \begin{pmatrix} 1 & 0 & 1 & m \\ m & 1 & 2 & -1 \\ m & -2 & -1 & 1 \end{pmatrix} \in \mathcal{M}_{3,4}$, $m \in \mathbb{R}$, $rg\,A = ?$

$$\Delta_1 = \begin{vmatrix} 0 & 1 & m \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{vmatrix} = 0 + 2 - m + 4m - 0 - 1 = $$
$$= 3m + 1$$

$$\Delta_2 = \begin{vmatrix} 1 & 1 & m \\ m & 2 & -1 \\ m & -1 & 1 \end{vmatrix} = 2 - m - m^2 - 2m^2 - 1 - m =$$
$$= -3m^2 - 2m - 1$$

Caz I) $m \neq -\frac{1}{3} \iff \Delta_1 \neq 0 \Rightarrow rg\,A = 3$

Caz II) $m = -\frac{1}{3} \Rightarrow \Delta_2 \neq 0 \Rightarrow rg\,A = 3.$

$\Rightarrow rg\,A = 3 \ \forall\, m \in \mathbb{R}$

Ex. 4) Fie $A \in \mathcal{M}_3(\mathbb{R})$, $A^{2022} - 2022A - I_3 = O_3$.

a) $rg\,A = ?$

b) $rg(2022A + I_3) = ?$

a) $A^{2022} - 2022A = I_3$

$A(A^{2021} - 2022 I_3) = I_3. \ /\det$

$\det A \cdot \det(A^{2021} - 2022 I_3) = 1.$

$\Rightarrow \det A \neq 0 \Rightarrow rg\,A = 3.$

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b) $A^{2022} = 2022A + I_3$ / det.

$\det(A^{2022}) = \det(2022A + I_3)$

$\Rightarrow [\det(A)]^{2022} = \det(2022A + I_3)$

$\det A \neq 0$

$\Rightarrow rg(2022A + I_3) = 3$.

Ex. 6) $A, B \in \mathcal{M}_n(\mathbb{C})$ inversabile

$\Rightarrow rg(A^{-1} + B^{-1}) = rg(A+B)$

$rg(A^{-1} + B^{-1}) = rg(A \cdot (A^{-1} + B^{-1}) \cdot B) =$
$= rg(B + A)$

Ex. 7) $A, B \in \mathcal{M}_n(\mathbb{C})$

$AB = BA$

$A^7 = I_n$

$B^5 = I_n$

$\Rightarrow rg(A+B) = n$.

$a^{2n+1} + b^{2n+1} = (a+b)(a^{2n} - a^{2n-1} \cdot b + \dots + b^{2n})$.

$A^{35} + B^{35} = (A+B)(\underbrace{A^{34} - A^{33} \cdot B + \dots + B^{34}}_{C}) = 2I_n$. / det.

$\Rightarrow \det(A+B) \cdot \det C = 2^n$

$\Rightarrow rg(A+B) = n$.

Ex. 5) $A \in \mathcal{M}_n(\mathbb{R})$

$$A^3 - 6A^2 + 12A = O_n$$

$rg(2I_n - A)$

$$(2I_n - A)^3 = 8I_n - 12A + 6A^2 - A^3 =$$

$$= \underbrace{-A^3 + 6A^2 - 12A}_{O_n} + 8I_n . = 8I_n$$

$$\det(2I_n - A)^3 = \det(8I_n) = 8^n \neq 0$$

$$\Rightarrow rg(2I_n - A) = n.$$

Ex. 10) $\begin{cases} x + \alpha y + z = 1 \\ \alpha x - y + z = 1 \\ x + y - z = 2. \end{cases}$ $\alpha \in \mathbb{R}$

$$A = \begin{pmatrix} 1 & \alpha & 1 \\ \alpha & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \left| \begin{matrix} 1 \\ 1 \\ 2 \end{matrix} \right.$$

$$\det A = 1 + \alpha + \alpha + 1 - 1 + \alpha^2 = \alpha^2 + 2\alpha + 1 = (\alpha+1)^2$$

Caz I) $\Delta \neq 0 \Leftrightarrow \alpha \in \mathbb{R} \setminus \{-1\}$

$\Leftrightarrow rg A = rg \bar{A} = 3 \Rightarrow$ SCD.

$$x = \frac{\Delta x}{\Delta}$$

$$\Delta x = \begin{vmatrix} 1 & \alpha & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = 1 + 2\alpha + 1 + 2 - 1 + \alpha = 3\alpha + 3 =$$

$$= 3(\alpha + 1)$$

$$\Rightarrow x = \frac{\Delta x}{\Delta} = \frac{3(\alpha+1)}{(\alpha+1)^2} = \frac{3}{\alpha+1}$$

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$y = \frac{\Delta y}{\Delta}$

$$\Delta y = \begin{vmatrix} 1 & 1 & 1 \\ \alpha & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = -1 + 1 + 2\alpha - 1 - 2 + \alpha = 3\alpha - 3$$

$$y = \frac{3\alpha - 3}{(\alpha+1)^2}$$

$z = \frac{\Delta z}{\Delta}$

$$\Delta z = \begin{vmatrix} 1 & \alpha & 1 \\ \alpha & -1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = -2 + \alpha + \alpha + 1 - 1 - 2\alpha^2 =$$

$$= -2\alpha^2 + 2\alpha - 2 =$$

$$= -2(\alpha^2 - \alpha + 1)$$

$$\Rightarrow z = \frac{-2(\alpha^2 - \alpha + 1)}{(\alpha+1)^2}$$

$$\Rightarrow (x, y, z) = \left( \frac{3}{\alpha+1}, \frac{3(\alpha-1)}{(\alpha+1)^2}, \frac{-2(\alpha^2-\alpha+1)}{(\alpha+1)^2} \right)$$

Caz II) $\Delta = 0 \Rightarrow \alpha = -1$

$$A = \left( \begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 2 \end{array} \right)$$

$$\Delta p = \begin{vmatrix} 1 & -1 \\ -1 & -1 \end{vmatrix} = -1 - 1 = -2 \neq 0$$

$$\Rightarrow rg\, A = 2$$

$$\Delta c = \begin{vmatrix} 1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & 2 \end{vmatrix} \xrightarrow{L_1 - L_2} \begin{vmatrix} 2 & 0 & 0 \\ -1 & -1 & 1 \\ 1 & 1 & 2 \end{vmatrix} =$$

$$= 2 \cdot (-1)^{1+1} \cdot \begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} = 2 \cdot (-2 - 1) = 2 \cdot (-3) = -6 \neq 0$$

$$\Rightarrow rg\, \bar{A} = 3 \Rightarrow S.I.$$

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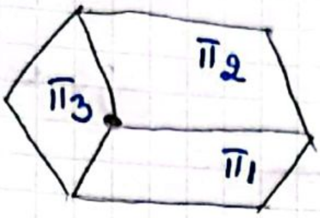
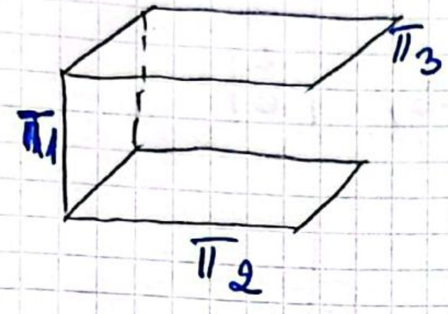
Cazul I) $\Delta \neq 0$ S.C.D.




Cazul II) $\Delta = 0 \Rightarrow$ S.I.




Ex 12) $\cancel{\begin{cases} x+y+z=0 \\ ax+by+cz=0 \\ (b+c)\cdot x+(a+c)\cdot y+(a+b)\cdot z=0. \end{cases}}$

Ex 11) $\begin{cases} x+2y+3z=0 \\ 4x+5y+6z=0 \\ x+\lambda^2 z=0. \end{cases}$

$$A=\left(\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 0 & \lambda^2 \end{array}\right) \left|\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right.$$

$$\Delta = \det A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 0 & \lambda^2 \end{vmatrix} = 5\lambda^2+12+0-15-0-8\lambda^2 =$$
$$= -3\lambda^2-3 = -3(\lambda^2+1) \neq 0$$

$\Rightarrow$ SCD. și există soluție unică nulă

Ex. 12).

$$\begin{cases} x+y+z=0 \\ ax+by+cz=0 \\ (b+c)\cdot x+(a+c)\cdot y+(a+b)\cdot z=0 \end{cases} \qquad a,b,c \in \mathbb{R}$$

Dacă $a \neq b$, să se rezolve.

$$A=\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ a & b & c & 0 \\ b+c & a+c & a+b & 0 \end{array}\right)$$

$$\Delta = \det A = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & a+c & a+b \end{vmatrix}$$

$$\Delta = \det A = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & a+c & a+b \end{vmatrix} \xrightarrow{L_3+L_2} \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a+b+c & a+b+c & a+b+c \end{vmatrix} =$$

$$=(a+b+c)\cdot \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = (a+b+c)\cdot 0 = 0.$$

$\Rightarrow$ S C N. sau S.i.

$$a \neq b \Rightarrow \Delta p = \begin{vmatrix} 1 & 1 \\ a & b \end{vmatrix} = b-a \neq 0$$

$\Rightarrow rg\, A = 2$

$$\Delta c = \begin{vmatrix} 1 & 1 & 0 \\ a & b & 0 \\ b+c & a+c & 0 \end{vmatrix} = 0.$$

$\Rightarrow rg\, \bar{A} = 2$

$\Rightarrow$ S C N.

$z = \alpha$ variabilă secundară

$x, y$ variabile principale.

$$\begin{cases} x+y = -\alpha \quad / \cdot(-a) \\ ax+by = -c\alpha \end{cases}$$

$$0 + y(b-a) = \alpha(a-c)$$

$$\Rightarrow y = \frac{\alpha(a-c)}{b-a}$$

$$\Rightarrow x = -\alpha - \frac{\alpha(a-c)}{b-a} = -\alpha\left(1 + \frac{a-c}{b-a}\right) =$$

$$= -\alpha\left(\frac{b-a+a-c}{b-a}\right) = -\alpha \cdot \frac{b-c}{b-a} = \alpha \cdot \frac{c-b}{a-b}$$

$$\Rightarrow (x, y, z) \in \left\{ \left(\alpha \cdot \frac{c-b}{a-b}, \frac{\alpha(a-c)}{b-a}, \alpha\right), \alpha \in \mathbb{R} \right\}$$

Ex. 13)
$$\begin{cases} x+y+z=0 \\ (b+c)\cdot x + (a+c)\cdot y + (a+b)\cdot z = 0 \\ b\cdot c\cdot x + a\cdot c\cdot y + a\cdot b\cdot z = 0 \end{cases}$$

$$A = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ b+c & a+c & a+b & 0 \\ b\cdot c & a\cdot c & a\cdot b & 0 \end{array}\right)$$

$$\Delta = \det A = \begin{vmatrix} 1 & 1 & 1 \\ b+c & a+c & a+b \\ b\cdot c & a\cdot c & a\cdot b \end{vmatrix} \xrightarrow[C_3-C_1]{C_2-C_1} \begin{vmatrix} 1 & 0 & 0 \\ b+c & a-b & a-c \\ b\cdot c & c(a-b) & b(a-c) \end{vmatrix}$$

$$= 1\cdot(-1)^{1+1} \cdot \begin{vmatrix} a-b & a-c \\ c(a-b) & b(a-c) \end{vmatrix} =$$

$$= (a-b)\cdot(a-c)\cdot \begin{vmatrix} 1 & 1 \\ c & b \end{vmatrix} = (a-b)(a-c)(b-c)$$

$\Rightarrow$ SCD $\Rightarrow \Delta \neq 0 \Rightarrow a \neq b \neq c \neq a$

Ex 16) $\begin{cases} x+2y = m+1 \\ 2x-3y = m-1 \\ mx+y = 3 \end{cases}$ , $m \in \mathbb{R}$

$m = ?$ a.î. Si. (Interpretare geometrică)

$$A = \begin{pmatrix} 1 & 2 \\ 2 & +3 \\ m & 1 \end{pmatrix} \left| \begin{matrix} m+1 \\ m-1 \\ 3 \end{matrix} \right.$$

$$\Delta p = \begin{vmatrix} 1 & 2 \\ 2 & +3 \end{vmatrix} = +3-4 = -1 \neq 0$$

$\Rightarrow rg\, A = 2$

$$\Delta c = \begin{vmatrix} 1 & 2 & m+1 \\ 2 & +3 & m-1 \\ m & 1 & 3 \end{vmatrix} = +9 + 2m(m-1) + 2(m+1) + 3m(m+1) - m+1 - 12 =$$

$$= +9 + 2m^2 - 2m + 2m + 2 - 3m^2 - 3m - m - 11 =$$

$$= -m^2 - 4m = m(-m-4)$$

Si $\Rightarrow \Delta c \neq 0 \Rightarrow m \in \mathbb{R} \setminus \{0, -4\}$

Ex. 17) $\sum_{i=1}^{k} (1+i)\cdot x_i + \sum_{i=1}^{4-k} i\cdot x_{i+k} = 0, \ \forall k=\overline{1,3}$

$k=1: \ 2x_1 + 1\cdot x_2 + 2\cdot x_3 + 3\cdot x_4 = 0.$

$k=2: \ 2x_1 + 3x_2 + \cdot x_3 + 2x_4 = 0.$

$k=3: \ 2x_1 + 3x_2 + 4x_3 + x_4 = 0.$

$$A = \left(\begin{array}{cccc} 2 & 1 & 2 & 3 \\ 2 & 3 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{array}\right) \left| \begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right.$$

$$\Delta p = \begin{vmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 2 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 2 & 2 \end{vmatrix} \neq 0.$$

$\Rightarrow rg\ A = 3 = rg\ \bar{A} \Rightarrow SCN.$

$x_4 = \alpha$ variabilă secundară

$x_1, x_2, x_3$ variabile principale.

$$\begin{cases} 2x_1 + x_2 + 2x_3 = -3\alpha. \\ 2x_1 + 3x_2 + x_3 = -2\alpha. \\ 2x_1 + 3x_2 + 4x_3 = -\alpha. \end{cases}$$

Se aplică ~~Crammer~~ Cramer. ( $SC_{\text{simplu}}N$ )