CURS 7
FUNCTII DIFERENTIABILE

A) APLICATII LINIARE SI CONTINUE PE SPATII LINIARE NORMATE

Definitia 1. O functie $T : (X, \| \|_X) \to (Y, \| \|_Y)$ se numeste aplicatie liniara daca $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ $\forall \alpha, \beta \in \mathbb{R}, \forall x, y \in X$.

Teorema 1. O aplicatie liniara $T : (X, \| \|_X) \to (Y, \| \|_Y)$ este functie continua pe X daca si numai daca $\exists \lambda > 0$ astfel incat $\|T(x)\|_Y \leq \lambda \|x\|_X$ $\forall x \in X$.

Notatie.$\mathcal{L}(X, Y) = \{T : X \to Y | \, T$ *aplicatie liniara si continua*$\}$

Pe spatiul liniar real $\mathbb{R}^n$ se considera baza canonica $B = \{e_1, e_2, ..., e_n\}$, unde

$$e_1 = (1, 0, 0, ..., 0)$$

$$e_2 = (0, 1, 0, ..., 0)$$

$$.$$

$$.$$

$$.$$

$$e_n = (0, 0, 0, ..., 1)$$

Oricare ar fi $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ are loc egalitatea $x = x_1 e_1 + x_2 e_2 + ... + x_n e_n$.

Teorema 2. Fie $n, m \in \mathbb{N}^*$. Orice aplicatie liniara $T : (\mathbb{R}^n, \| \|_2) \to (\mathbb{R}^m, \| \|_2)$ este functie continua pe $\mathbb{R}^n$.

Teorema 3. Functia $T : \mathbb{R} \to \mathbb{R}^m$ este aplicatie liniara daca si numai daca $\exists! u \in \mathbb{R}^m$ astfel incat $T(x) = xu$ $\forall x \in \mathbb{R}$.

$$T = id_{\mathbb{R}} \cdot u$$

$$id_{\mathbb{R}} \overset{not}{=} dx \Longrightarrow T = dx \cdot u$$

Teorema 4. Fie $n \geq 2$. Functia $T : \mathbb{R}^n \to \mathbb{R}^m$ este aplicatie liniara daca si numai daca $\exists! \; \lambda_1, \lambda_2, ...\lambda_n \in \mathbb{R}^m$ astfel incat

$$T(x_1, x_2, ...x_n) = x_1 \lambda_1 + x_2 \lambda_2 + ... + x_n \lambda_n \; \forall (x_1, x_2, ...x_n) \in \mathbb{R}^n.$$

Definim aplicatiile liniare
$pr_1 = dx_1 : \mathbb{R}^n \to \mathbb{R}, pr_1(x_1, x_2, ...x_n) = x_1 \; \forall (x_1, x_2, ...x_n) \in \mathbb{R}^n$
$pr_2 = dx_2 : \mathbb{R}^n \to \mathbb{R}, pr_2(x_1, x_2, ...x_n) = x_2 \; \forall (x_1, x_2, ...x_n) \in \mathbb{R}^n$
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$pr_n = dx_n : \mathbb{R}^n \to \mathbb{R}, pr_n(x_1, x_2, ...x_n) = x_n \; \forall (x_1, x_2, ...x_n) \in \mathbb{R}^n$
Aplicatia liniara $T : \mathbb{R}^n \to \mathbb{R}^m$ se descrie in felul urmator

$$T = dx_1 \cdot \lambda_1 + dx_2 \cdot \lambda_2 + ... + dx_n \cdot \lambda_n$$

B) DERIVATELE PARTIALE ALE FUNCTIILOR DE MAI MULTE VARIABILE REALE

Se considera $n \geq 2$ si functia $f = (f_1, f_2, ...f_m) : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$.

Definitia 2. Spunem ca functia $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ admite derivata partiala in raport cu variabila $x_i, 1 \leq i \leq n$, in punctul $x_0 \in D \cap D'$ daca $\exists \lim\limits_{t \to 0} \frac{f(x_0+te_i)-f(x_0)}{t} \in \mathbb{R}^m$.

Notatie. $\frac{\partial f}{\partial x_i}(x_0) \overset{not}{=} \lim\limits_{t \to 0} \frac{f(x_0+te_i)-f(x_0)}{t}$

Teorema 5. Functia $f = (f_1, f_2, ...f_m) : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ admite derivata partiala in raport cu variabila $x_i, 1 \leq i \leq n$, in punctul $x_0 \in D \cap D'$daca si numai daca functiile $f_1, f_2, ..., f_m : D \subseteq \mathbb{R}^n \to \mathbb{R}$ admit derivata partiala in raport cu variabila $x_i, 1 \leq i \leq n$, in punctul $x_0 \in D \cap D'$. In plus, $\frac{\partial f}{\partial x_i}(x_0) = \left(\frac{\partial f_1}{\partial x_i}(x_0), \frac{\partial f_2}{\partial x_i}(x_0), ..., \frac{\partial f_m}{\partial x_i}(x_0)\right)$.

Exemplu. $f : \mathbb{R}^2 \to \mathbb{R}, f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, (x,y) \neq (0,0) \\ 0, (x,y) = (0,0) \end{cases}$

Derivatele partiale se calculeaza pe $\mathbb{R}^2 \backslash \{(0,0)\}$ in felul urmator.

$\frac{\partial f}{\partial x}(x,y) = \left(\frac{xy}{x^2+y^2}\right)'_x = \frac{(xy)\prime_x (x^2+y^2) - xy(x^2+y^2)\prime_x}{(x^2+y^2)^2} = \frac{y(x^2+y^2)-xy \cdot 2x}{(x^2+y^2)^2} = \frac{y^3-x^2y}{(x^2+y^2)^2} \forall (x,y) \in \mathbb{R}^2 \backslash \{(0,0)\}$

$\frac{\partial f}{\partial y}(x,y) = \left(\frac{xy}{x^2+y^2}\right)'_y = \frac{(xy)\prime_y (x^2+y^2) - xy(x^2+y^2)\prime_y}{(x^2+y^2)^2} = \frac{x(x^2+y^2)-xy \cdot 2y}{(x^2+y^2)^2} = \frac{x^3-y^2x}{(x^2+y^2)^2} \forall (x,y) \in \mathbb{R}^2 \backslash \{(0,0)\}$

In $(0,0)$ derivatele partiale se calculeaza folosind definitia.

$\lim\limits_{t \to 0} \frac{f((0,0)+te_1)-f(0,0)}{t} = \lim\limits_{t \to 0} \frac{f(t,0)-f(0,0)}{t} = \lim\limits_{t \to 0} \frac{0-0}{t} = 0 \in \mathbb{R} \Rightarrow$f admite derivata partiala in raport cu variabila x in punctul $(0,0)$ si $\frac{\partial f}{\partial x}(0,0) = 0$

$\lim\limits_{t \to 0} \frac{f((0,0)+te_2)-f(0,0)}{t} = \lim\limits_{t \to 0} \frac{f(0,t)-f(0,0)}{t} = \lim\limits_{t \to 0} \frac{0-0}{t} = 0 \in \mathbb{R} \Rightarrow$f admite derivata partiala in raport cu variabila

$y inpunctul (0,0)$ si $\frac{\partial f}{\partial y}(0,0) = 0$.

C) FUNCTII DIFERENTIABILE

Definitia 3. Spunem ca functia $f : D \subseteq (X, \| \|_X) \to (Y, \| \|_Y)$ este diferentiabila in punctul$x_0 \in D \cap D'$ daca $\exists T \in \mathcal{L}(X,Y)$ astfel incat

$$\lim_{x \to x_0} \frac{\|f(x)-f(x_0)-T(x-x_0)\|_Y}{\|x-x_0\|_X} = 0.$$

Observatie. Aplicatia liniara si continua $T \in \mathcal{L}(X,Y)$ din definitia 3 este unica.

Notatie. $T \overset{not}{=} df(x_0)$ -diferentiala functiei f in punctul x_0.

Definitia 4. Spunem ca functia $f : D \subseteq (X, \| \|_X) \to (Y, \| \|_Y)$ este diferentiabila pe multimea $A \subseteq D \cap D'$ daca f este diferentiabila in orice punct al multimii A.

Notatie. $df : A \to \mathcal{L}(X, Y)$ -diferentiala functiei f pe multimea $A \subseteq D \cap D\prime$.

Teorema 6. Orice functie $f : D \subseteq (X, \| \|_X) \to (Y, \| \|_Y)$ diferentiabila in punctul $x_0 \in D \cap D\prime$ este continua in x_0.

Demonstratie. $f : D \subseteq (X, \| \|_X) \to (Y, \| \|_Y)$ diferentiabila in punctul $x_0 \in D \cap D' \Rightarrow \exists T \in \mathcal{L}(X, Y)$ astfel incat

$$\lim_{x \to x_0} \frac{\|f(x)-f(x_0)-T(x-x_0)\|_Y}{\|x-x_0\|_X} = 0.$$

$T \in \mathcal{L}(X, Y) \Rightarrow \exists \lambda > 0$ astfel incat $\|T(x)\|_Y \leq \lambda \|x\|_X \ \forall x \in X$

Evaluam

$$\|f(x) - f(x_0)\|_Y = \|f(x) - f(x_0) - T(x - x_0) + T(x - x_0)\|_Y \leq$$

$$\leq \|f(x) - f(x_0) - T(x - x_0)\|_Y + \|T(x - x_0)\|_Y =$$

$$= \frac{\|f(x)-f(x_0)-T(x-x_0)\|_Y}{\|x-x_0\|_X} \cdot \|x - x_0\|_X + \|T(x - x_0)\|_Y \leq$$

$$\leq \frac{\|f(x)-f(x_0)-T(x-x_0)\|_Y}{\|x-x_0\|_X} \cdot \|x - x_0\|_X + \lambda \|x - x_0\|_X \ \forall x \in D, x \neq x_0.$$

Avem ca $0 \leq \|f(x) - f(x_0)\|_Y \leq \frac{\|f(x)-f(x_0)-T(x-x_0)\|_Y}{\|x-x_0\|_X} \cdot \|x - x_0\|_X + \lambda \|x - x_0\|_X$ $\forall x \in D, x \neq x_0$.

Folosind criteriul clestelui pentru limite de functii, obtinem ca

$$\lim_{x \to x_0} \|f(x) - f(x_0)\|_Y = 0 \Rightarrow \lim_{x \to x_0} f(x) = f(x_0) \Rightarrow f \text{ este continua in } x_0.$$

Teorema 7. (Operatii cu functii diferentiabile)

a) Fie $f, g : D \subseteq (X, \| \|_X) \to (Y, \| \|_Y)$ doua functii diferentiabile in punctul $x_0 \in D \cap D'$. Atunci functiile $f + g, f - g, \alpha f : D \subseteq (X, \| \|_X) \to (Y, \| \|_Y)$ sunt diferentiabile in x_0 si sunt adevarate egalitatile

$$d(f + g)(x_0) = df(x_0) + dg(x_0)$$

$$d(f - g)(x_0) = df(x_0) - dg(x_0)$$

$$d(\alpha f)(x_0) = \alpha df(x_0), \alpha \in \mathbb{R}.$$

b) Fie $f : D \subseteq (X, \| \|_X) \to B \subseteq (Y, \| \|_Y)$ o functie diferentiabila in punctul $x_0 \in D \cap D'$ si $g : B \subseteq (Y, \| \|_Y) \to (Z, \| \|_Z)$ o functie diferentiabila in punctul $y_0 = f(x_0) \in B \cap B'$. Atunci functia $g \circ f : D \subseteq (X, \| \|_X) \to (Z, \| \|_Z)$ este diferentiabila in punctul $x_0 \in D \cap D'$ si

$$d(g \circ f)(x_0) = dg(y_0) \circ df(x_0).$$

Teorema 8. a) Fie $f : (X, \| \|_X) \to (Y, \| \|_Y)$ o aplicatie liniara si continua pe X. Atunci f este diferentiabila pe X si $df(x) = f\ \forall x \in X$.

b) Fie $f : (X, \| \|_X) \to (Y, \| \|_Y)$ o functie constanta pe X. Atunci f este diferentiabila pe X si $df(x) = 0\ \forall x \in X$.

D) FUNCTII DIFERENTIABILE, CAZUL $\mathbf{f} : \mathbf{D} \subseteq \mathbb{R} \to \mathbb{R}^{\mathbf{m}}$, $\mathbf{m} \in \mathbb{N}^*$

Functia $f : D \subseteq \mathbb{R} \to \mathbb{R}^m$ este definita prin $f(x) = (f_1(x), f_2(x), ..., f_m(x))\ \forall x \in D$.

Functiile $f_1, f_2, ..., f_n : D \subseteq \mathbb{R} \to \mathbb{R}$ se numesc componentele functiei f.

Notam $f = (f_1, f_2, ..., f_m)$.

Teorema 9. Functia $f = (f_1, f_2, ..., f_m) : D \subseteq \mathbb{R} \to \mathbb{R}^m$ este diferentiabila in punctul $x_0 \in D \cap D\prime$daca si numai daca f este derivabila in punctul x_0. In plus, $df(x_0) : \mathbb{R} \to \mathbb{R}^m$ este data de formula $df(x_0)(x) = x \cdot f'(x_0)\, \forall x \in \mathbb{R}$.

Notatie.$df(x_0) = id_{\mathbb{R}} \cdot f'(x_0) = dx \cdot f'(x_0)$

E) FUNCTII DIFERENTIABILE, CAZUL $\mathbf{f} : \mathbf{D}^{\mathbf{n}} \subseteq \mathbb{R} \to \mathbb{R}^{\mathbf{m}}$, $\mathbf{m} \in \mathbb{N}^*, \mathbf{n} \geq 2$

Functia $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ este definita prin $f(x) = (f_1(x), f_2(x), ..., f_m(x))\ \forall x \in D$.

Functiile $f_1, f_2, ..., f_n : D \subseteq \mathbb{R}^n \to \mathbb{R}$ se numesc componentele functiei f.

Notam $f = (f_1, f_2, ..., f_m)$.

Teorema 10. Daca functia $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ este diferentiabila in punctul $x_0 \in D \cap D\prime$, atunci f admite toate derivatele partiale in punctul x_0. In plus, $df(x_0) : \mathbb{R}^n \to \mathbb{R}^m$ este data de formula $df(x_0)(x) = df(x_0)(x_1, x_2, .., x_n) = x_1 \frac{\partial f}{\partial x_1}(x_0) + ... + x_n \frac{\partial f}{\partial x_n}(x_0)\, \forall (x_1, x_2, .., x_n) \in \mathbb{R}^n$.

Notatie. $df(x_0) = pr_1 \cdot \frac{\partial f}{\partial x_1}(x_0) + ... + pr_n \cdot \frac{\partial f}{\partial x_n}(x_0) = dx_1 \cdot \frac{\partial f}{\partial x_1}(x_0) + ... + dx_n \cdot \frac{\partial f}{\partial x_n}(x_0)$

Corolar. Daca functia $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ nu admite cel putin o derivata partiala in punctul $x_0 \in D \cap D'$, atunci f nu este diferentiabila in x_0.

Observatie. Reciproca Teoremei 10 nu este adevarata.

Functia $f : \mathbb{R}^2 \to \mathbb{R}, f(x, y) = \{ \begin{array}{c} \frac{xy}{x^2+y^2}, (x, y) \neq (0, 0) \\ 0, (x, y) = (0, 0) \end{array}$ admite toate derivatele partiale in $(0, 0)$.

$\frac{\partial f}{\partial x}(0, 0) = \frac{\partial f}{\partial y}(0, 0) = 0$

Fie $T : \mathbb{R}^2 \to \mathbb{R}, T(x, y) = x \frac{\partial f}{\partial x}(0, 0) + y \frac{\partial f}{\partial y}(0, 0) = 0\ \forall (x, y) \in \mathbb{R}^2$.

$$\lim_{(x,y)\to(0,0)} \frac{|f(x,y) - f(0,0) - T((x,y)-(0,0))|}{\|(x,y)-(0,0)\|} = \lim_{(x,y)\to(0,0)} \frac{xy}{(x^2+y^2)\sqrt{x^2+y^2}}$$

Pentru a testa existenta limitei construim cel putin doua siruri de vectori care converg catre $(0, 0)$.

$$\lim_{n\to\infty} \left(\frac{1}{n}, \frac{1}{n}\right) = (0, 0) \Rightarrow \lim_{n\to\infty} \frac{\frac{1}{n} \cdot \frac{1}{n}}{\left(\frac{1}{n^2} + \frac{1}{n^2}\right)\sqrt{\frac{1}{n^2} + \frac{1}{n^2}}} = +\infty$$

$$\lim_{n\to\infty} \left(\frac{1}{n}, 0\right) = (0, 0) \Rightarrow \lim_{n\to\infty} \frac{\frac{1}{n} \cdot 0}{\left(\frac{1}{n^2} + 0\right)\sqrt{\frac{1}{n^2} + 0}} = 0$$

Limitele functiei pe sirurile alese sunt diferite, rezulta ca limita functiei nu exista cand $(x, y) \to (0, 0)$.

Folosind definitia, deducem ca f nu este diferentiabila in punctul $(0, 0)$.

Teorema 11. (Criteriu de diferentiabilitate) Fie $f : D = \overset{0}{D} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ o functie, $x_0 \in D$ si $V \in V_{\tau_{\mathbb{R}^n}}(x_0) \subseteq D$ astfel ca f admite toate derivatele partiale pe multimea V si acestea sunt continue in punctul x_0. Atunci f este diferentiabila in x_0.

Corolar. Fie $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ o functie si $A = \overset{0}{A} \subseteq D$ o multime nevida pe care f admite toate derivatele partiale si acestea sunt continue. Atunci f este diferentiabila pe multimea A.

Definitia 5. Spunem ca functia $f : D = \overset{0}{D} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ este de clasa C^1 pe multimea D daca f admite toate derivatele partiale pe D si acestea sunt functii continue pe D.

Notatie. $C^1(D) = \left\{ f : D = \overset{0}{D} \subseteq \mathbb{R}^n \to \mathbb{R}^m \middle| f \text{ functie de clasa } C^1 \text{ pe } D \right\}$

Observatie. Daca $f \in C^1(D)$, atunci f este diferentiabila pe D.

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Definitia 6. Fie $f : D \subseteq (X, \| \|_X) \to (Y, \| \|_Y)$ o functie. Elementul $x_0 \in D \cap D'$ se numeste punct critic al functiei f daca f este diferentiabila in x_0 si $df(x_0) = 0 \in \mathcal{L}(X, Y)$.

Teorema 12. a) Se considera functia $f : D \subseteq \mathbb{R} \to \mathbb{R}^m$ si $x_0 \in D \cap D'$. Elementul x_0 este punct critic al functiei f daca si numai daca f este derivabila in x_0 si $f'(x_0) = 0_{\mathbb{R}^m}$.

b) Se considera functia $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m, n \geq 2$ si $x_0 \in D \cap D'$. Elementul x_0 este punct critic al functiei f daca si numai daca f este diferentiabila in x_0 si $\frac{\partial f}{\partial x_1}(x_0) = ... = \frac{\partial f}{\partial x_n}(x_0) = 0_{\mathbb{R}^m}$.

Definitia 7. a) Fie $f = (f_1, f_2, ..., f_m) : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ o functie diferentiabila in $x_0 \in D \cap D'$. Matricea $J_f(x_0) = \left(\frac{\partial f_i}{\partial x_j}(x_0) \right)_{\substack{1 \leq i \leq m, \\ 1 \leq j \leq n}} \in M_{m,n}(\mathbb{R})$ se numeste matricea Jacobi a functiei f in punctul x_0.

b) Daca $m = n$, $\det J_f(x_0) \overset{not}{=} \frac{D(f_1, f_2, ..., f_n)}{D(x_1, x_2, ..., x_n)}(x_0) \in \mathbb{R}$ se numeste Jacobianul functiei f in punctul x_0.

Observatie. a) $d(f)(x_0)(x_1, x_2, ..., x_n) = \left[J_f(x_0) \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{pmatrix} \right]^t \forall (x_1, x_2, ..., x_n) \in \mathbb{R}^n$.

b) $J_{f \pm g}(x_0) = J_f(x_0) \pm J_g(x_0)$
$J_{\alpha f}(x_0) = \alpha J_f(x_0)$
$J_{g \circ f}(x_0) = J_g(f(x_0)) J_f(x_0)$.