Seminar 4

1. Studiați natura seriilor:

a) $\sum_{n=1}^{\infty} \frac{\cos nx}{n^\lambda}$, $x \in \mathbb{R}$, $\lambda > 0$.

Sol.: Vom folosi Criteriul Abel-Dirichlet (I).

Fie $x_n = \frac{1}{n^\lambda}$ $\forall n \in \mathbb{N}^*$ și $y_n = \cos nx$ $\forall n \in \mathbb{N}^*$.

Șirul $(x_n)_n$ este descrescător și $\lim_{n\to\infty} x_n = 0$. (1)

$\exists M > 0$ a.î. $\forall n \in \mathbb{N}^*$, avem $|y_1 + \ldots + y_n| \leq M$?
M nu depinde de n, dar poate depinde de x.
$|y_1 + y_2 + \ldots + y_n| = |\cos x + \cos 2x + \ldots + \cos nx|$.

Fie $z = \cos x + i \sin x$.

Avem: $z^2 = \cos 2x + i \sin 2x$

$z^3 = \cos 3x + i \sin 3x$

$\ldots\ldots\ldots\ldots$

$z^n = \cos nx + i \sin nx$.

$\cos x + \ldots + \cos nx = \mathrm{Re}(z + z^2 + \ldots + z^n)$.

Presupunem că $z \neq 1$, i.e. $\cos x + i \sin x \neq 1$, i.e.

$x \in \mathbb{R} \setminus \{2k\pi \mid k \in \mathbb{Z}\}$.

$z + z^2 + \ldots + z^n = z \cdot \frac{z^n - 1}{z - 1} = \frac{z^{n+1} - z}{z - 1} =$

$$= \frac{\cos(n+1)x + i\sin(n+1)x - \cos x - i\sin x}{\cos x + i\sin x - 1} =$$

$$= \frac{\left(\cos(n+1)x - \cos x\right) + i\left(\sin(n+1)x - \sin x\right)}{\left(\cos x - 1\right) + i\sin x} =$$

$$= \frac{-2\sin\frac{n+2}{2}x\,\sin\frac{n}{2}x + i\,2\cdot\cos\frac{n+2}{2}x\,\sin\frac{n}{2}x}{\left(\cos x - \cos 0\right) + i\left(\sin x - \sin 0\right)} =$$

$$= \frac{-\cancel{2}\sin\frac{n+2}{2}x\,\sin\frac{n}{2}x + \cancel{2}\,i\cos\frac{n+2}{2}x\,\sin\frac{n}{2}x}{-\cancel{2}\sin\frac{x}{2}\sin\frac{x}{2} + i\,\cancel{2}\cos\frac{x}{2}\sin\frac{x}{2}} =$$

$$= \frac{\sin\frac{n}{2}x}{\sin\frac{x}{2}} \cdot \frac{\overset{-i}{\cancel{\ }}-\sin\frac{n+2}{2}x + i\cos\frac{n+2}{2}x}{-\sin\frac{x}{2} + i\cos\frac{x}{2}} =$$

$$= \frac{\sin\frac{n}{2}x}{\sin\frac{x}{2}} \cdot \frac{\cos\frac{n+2}{2}x + i\sin\frac{n+2}{2}x}{\cos\frac{x}{2} + i\sin\frac{x}{2}} =$$

$$= \frac{\sin\frac{n}{2}x}{\sin\frac{x}{2}} \cdot \frac{\left(\cos\frac{x}{2} + i\sin\frac{x}{2}\right)^{\overset{n+1}{\cancel{n+2}}}}{\cancel{\cos\frac{x}{2} + i\sin\frac{x}{2}}} =$$

$$= \frac{\sin\frac{n}{2}x}{\sin\frac{x}{2}}\left(\cos\frac{n+1}{2}x + i\sin\frac{n+1}{2}x\right).$$

$$|y_1+\dots+y_n| = |\operatorname{Re}(z+z^2+\dots+z^n)| = \frac{\left|\sin\frac{n}{2}x\right|}{\left|\sin\frac{x}{2}\right|}\cdot\left|\cos\frac{n+1}{2}x\right| \le$$

$$\le \frac{1}{\left|\sin\frac{x}{2}\right|}\cdot 1 = \frac{1}{\left|\sin\frac{x}{2}\right|}.$$

Luăm $M = \dfrac{1}{\left|\sin\frac{x}{2}\right|}$ și avem $|y_1+\dots+y_n| \le M$ $\forall n\in\mathbb{N}^*$. (2)

Din (1) și (2) rezultă, conform Criteriului Abel-Dirichlet (I), că seria $\sum\limits_{n=1}^{\infty} x_n\cdot y_n = \sum\limits_{n=1}^{\infty}\frac{\cos nx}{n^\lambda}$ este conv.

Am tratat mai sus doar cazul $x\in\mathbb{R}\setminus\{2k\pi \mid k\in\mathbb{Z}\}$.

Fie $x\in\{2k\pi \mid k\in\mathbb{Z}\}$.

Seria din enunț este $\sum\limits_{n=1}^{\infty}\frac{\cos n\cdot(2k\pi)}{n^\lambda} =$

$$= \sum_{n=1}^{\infty}\frac{1}{n^\lambda} \begin{cases} \text{conv.}; & \text{dacă } \lambda\in(1,\infty) \\ \text{div.}; & \text{dacă } \lambda\in(0,1]. \end{cases}$$

$\lambda>0$

□

b) $\sum_{n=1}^{\infty} \frac{\cos n \cos\frac{1}{n}}{n}$.

Sol.: Fie $x_n = \cos\frac{1}{n}$ $\forall n \in \mathbb{N}^*$ și $y_n = \frac{\cos n}{n}$ $\forall n \in \mathbb{N}^*$.

$-1 \leq x_n \leq 1$ $\forall n \in \mathbb{N}^* \Rightarrow (x_n)_n$ mărginit.

$x \longmapsto \cos x$ este descrescătoare
$\cap$
$(0, \frac{\pi}{2})$

$\frac{1}{n} \in (0, \frac{\pi}{2})$ $\forall n \in \mathbb{N}^*$.

$\left(\frac{1}{n}\right)_n$ descrescător

$\Rightarrow \left(\cos\frac{1}{n}\right)_n$ crescătoare.

Deci $(x_n)_n$ este monoton și mărginit. (1)

$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{\cos n}{n}$ conv. (Vezi a): $x = 1 \neq 2k\pi$, $\lambda = 1$). (2)

Din (1) și (2) rezultă, conform Crit. Abel-Dirichlet (II), că seria $\sum_{n=1}^{\infty} x_n y_n$ este conv. $\square$

$$\sum_{n=1}^{\infty} x_n y_n = \sum_{n=1}^{\infty} \frac{\cos n \cos\frac{1}{n}}{n}$$

c) $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}+1}{n}$.

<u>Sol</u>.: $x_n = \frac{(-1)^n \sqrt{n}+1}{n}$ $\forall n \in \mathbb{N}^*$.

$x_n = \frac{(-1)^n \cancel{\sqrt{n}}}{\underset{\sqrt{n}}{\cancel{n}}} + \frac{1}{n} = (-1)^n \frac{1}{\sqrt{n}} + \frac{1}{n}$ $\forall n \in \mathbb{N}^*$.

$\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ conv. (Criteriul lui Leibniz)

$\sum_{n=1}^{\infty} \frac{1}{n}$ div. (serie armonică generalizată cu $\alpha=1$).

Deci $\sum_{n=1}^{\infty} x_n$ este divergentă. $\square$

d) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$, $x \in (-1, 1)$.

<u>Sol</u>.: $x_n = \frac{x^n}{n^2}$ $\forall n \in \mathbb{N}^*$.

$|x_n| = \left|\frac{x^n}{n^2}\right| = \frac{|x|^n}{n^2}$ $\forall n \in \mathbb{N}^*$.

<u>Cazul 1: $x=0$</u>

$|x_n| = \frac{|0|^n}{n^2} = 0$ $\forall n \in \mathbb{N}^*$.

Fie $s_n = |x_1| + \dots + |x_n| = \underbrace{0 + \dots + 0}_{\text{de } n \text{ ori}} = 0 \cdot n = 0$ $\forall n \in \mathbb{N}^*$.

Deci $\lim_{n\to\infty} s_n = 0 \in \mathbb{R}$, i.e. $\sum_{n=1}^{\infty} |x_n|$ este conv. (1)

Cazul 2 : $x \in (-1, 1) \setminus \{0\}$

$$\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = \lim_{n\to\infty} \frac{|x|^{\cancel{n+1}}}{(n+1)^2} \cdot \frac{n^2}{\cancel{|x|^n}} = |x| \underset{x \in (-1,1)\setminus\{0\}}{<} 1.$$

Conform Crit. rap. seria $\sum_{n=1}^{\infty} |x_n|$ este conv. (2)

Din (1) și (2) rezultă că $\sum_{n=1}^{\infty} |x_n|$ este conv. $\forall x \in (-1, 1)$.
Prin urmare $\sum_{n=1}^{\infty} x_n$ este absolut conv. $\forall x \in (-1,1)$. Deci

$\sum_{n=1}^{\infty} x_n$ este conv. $\forall x \in (-1, 1)$. $\square$

2. Fie $n \in \mathbb{N}^*$ și $d_1 : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$,

$$d_1(x, y) = |x_1 - y_1| + \dots + |x_n - y_n| = \sum_{i=1}^{n} |x_i - y_i|.$$

$x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$

a) Arătați că d_1 este metrică pe $\mathbb{R}^n$.

Sol.: 1) $d_1(x, y) \geq 0 \ \forall x, y \in \mathbb{R}^n$ (evident).

2) $d_1(x,y)=0 \Leftrightarrow \sum_{i=1}^{n}|x_i-y_i|=0 \Leftrightarrow |x_i-y_i|=0 \ \forall i=\overline{1,n} \Leftrightarrow$

$\Leftrightarrow x_i=y_i \ \forall i=\overline{1,n} \Leftrightarrow x=y \ \forall x,y\in\mathbb{R}^n.$

3) $d_1(x,y)=\sum_{i=1}^{n}|x_i-y_i|=\sum_{i=1}^{n}|y_i-x_i|=d_1(y,x) \ \forall x,y\in\mathbb{R}^n.$

4) Fie $x,y,z\in\mathbb{R}^n$.

$d_1(x,z)=\sum_{i=1}^{n}|x_i-z_i|=\sum_{i=1}^{n}|x_i-y_i+y_i-z_i| \leq$

$\leq \sum_{i=1}^{n}\left(|x_i-y_i|+|y_i-z_i|\right)=\sum_{i=1}^{n}|x_i-y_i|+\sum_{i=1}^{n}|y_i-z_i|=$

$=d_1(x,y)+d_1(y,z).$

Deci d_1 este metrică pe $\mathbb{R}^n$. $\square$

b) Fie $(x^k)_k\subset\mathbb{R}^n$, $x^k=(x^k_1, x^k_2,\dots, x^k_n) \ \forall k\in\mathbb{N}$, $x\in\mathbb{R}^n$, $x=(x_1,\dots,x_n)$. Arătați că $\lim_{k\to\infty} x^k \overset{d_1}{=} x$ dacă și numai dacă $\lim_{k\to\infty} x^k_i=x_i \ \forall i=\overline{1,n}$.

Sol.: „$\Rightarrow$"

$\lim_{k\to\infty} x^k \overset{d_1}{=} x \Rightarrow \lim_{k\to\infty} d_1(x^k,x)=0 \Rightarrow \forall \varepsilon>0, \exists k_\varepsilon\in\mathbb{N}$

a.î. $\forall k \geq k_\varepsilon$, avem $\left| d_1(x^k, x) - 0 \right| < \varepsilon$

$$\left| d_1(x^k, x) - 0 \right| = d_1(x^k, x) = \sum_{i=1}^{n} |x_i^k - x_i|$$

Fie $\varepsilon > 0$ și $k_\varepsilon \in \mathbb{N}$ dat de relația de mai sus.

$\forall k \geq k_\varepsilon$, $\forall i = \overline{1,n}$ avem $\left| x_i^k - x_i \right| \leq \sum_{i=1}^{n} |x_i^k - x_i| < \varepsilon$.

Deci $\lim\limits_{k \to \infty} x_i^k = x_i \quad \forall i = \overline{1,n}$.

$\Leftarrow$ "

$\forall i = \overline{1,n} \quad \lim\limits_{k \to \infty} x_i^k = x_i \Rightarrow \forall i = \overline{1,n}, \forall \varepsilon > 0, \exists k_\varepsilon^i \in \mathbb{N}$ a.î.

$\forall k \geq k_\varepsilon^i$, avem $|x_i^k - x_i| < \dfrac{\varepsilon}{n}$.

Fie $\varepsilon > 0$. Alegem $k_\varepsilon = \max \{k_\varepsilon^1, \dots, k_\varepsilon^n\} \in \mathbb{N}$.

$\forall k \geq k_\varepsilon$, avem $\underbrace{\sum_{i=1}^{n} |x_i^k - x_i|}_{= d_1(x^k, x)} < \sum_{i=1}^{n} \dfrac{\varepsilon}{n} = \cancel{n} \cdot \dfrac{\varepsilon}{\cancel{n}} = \varepsilon$.

Deci $d_1(x^k, x) \xrightarrow[k \to \infty]{} 0$, i.e. $\lim\limits_{k \to \infty} x^k \overset{d_1}{=} x$. $\square$

3. Fie $n \in \mathbb{N}^*$ și $d_2 \stackrel{\text{not.}}{=} d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$,

$d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$. Arătați că d este metrică pe $\mathbb{R}^n$.

<u>Sol.</u>: 1) $d(x,y) \geq 0 \quad \forall x,y \in \mathbb{R}^n$ (evident)

2) $d(x,y) = 0 \Leftrightarrow \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = 0 \Leftrightarrow$

$\Leftrightarrow \sum_{i=1}^{n} (x_i - y_i)^2 = 0 \Leftrightarrow (x_i - y_i)^2 = 0 \ \forall i = \overline{1,n} \Leftrightarrow x_i = y_i \ \forall$

$\forall i = \overline{1,n} \Leftrightarrow x = y \quad \forall x,y \in \mathbb{R}^n$.

3) $d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = \sqrt{\sum_{i=1}^{n} (y_i - x_i)^2} =$

$= d(y,x) \quad \forall x,y \in \mathbb{R}^n$.

4) Fie $x,y,z \in \mathbb{R}^n$.

$d(x,z) = \sqrt{\sum_{i=1}^{n} (x_i - z_i)^2} = \sqrt{\sum_{i=1}^{n} (x_i - y_i + y_i - z_i)^2}$.

Folosim inegalitatea Cauchy-Buniakowski-Schwarz

(C.B.S.): $\forall n \in \mathbb{N}^*$, $\forall a_1, \dots, a_n \in \mathbb{R}$, $\forall b_1, \dots, b_n \in \mathbb{R}$,

avem $\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$.

$d(x,z) = \sqrt{\sum_{i=1}^{n} (x_i - y_i + y_i - z_i)^2} =$

$$= \sqrt{\sum_{i=1}^{n} \left[(x_i - y_i)^2 + (y_i - z_i)^2 + 2(x_i - y_i)(y_i - z_i) \right]} =$$

$$= \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2 + \sum_{i=1}^{n} (y_i - z_i)^2 + 2 \sum_{i=1}^{n} (x_i - y_i)(y_i - z_i)} \leq$$

$$\underset{\text{C.B.S.}}{\leq} \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2 + \sum_{i=1}^{n} (y_i - z_i)^2 + 2 \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \sqrt{\sum_{i=1}^{n} (y_i - z_i)^2}} =$$

$$= \sqrt{\left(\sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} + \sqrt{\sum_{i=1}^{n} (y_i - z_i)^2} \right)^2} =$$

$$= \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} + \sqrt{\sum_{i=1}^{n} (y_i - z_i)^2} = d(x, y) + d(y, z).$$

Deci d este metrică pe $\mathbb{R}^n$. $\square$