29.04.2022

$\boxed{C_{10} - \text{GAL}}$

Spații vectoriale euclidiene. Endomorfisme simetrice

Teorema (Cauchy - Buniakowski - Schwarz)

$(E, \langle \cdot, \cdot \rangle)$ s.v.e.r., $\forall x, y \in E \Rightarrow$

$$|\langle x, y\rangle| \leq \|x\| \cdot \|y\| \quad (*)$$

Dem

a) Dacă $x = 0_E$ sau $y = 0_E \Rightarrow 0 \leq 0$ (A)

b) Dacă $x \neq 0$ și $y \neq 0$

Fie $\lambda \in \mathbb{R} \Rightarrow \langle x + \lambda y, x + \lambda y\rangle \geq 0$.

$\lambda^2 \|y\|^2 + 2\lambda \langle x, y\rangle + \|x\| \geq 0, \ \forall \lambda \in \mathbb{R}$.

$\Rightarrow \underset{4\langle x,y\rangle^2 - 4\|x\|^2 \cdot \|y\|^2}{\overset{\|}{\Delta_\lambda}} \leq 0 \Rightarrow |\langle x, y\rangle| \leq \|x\| \cdot \|y\|$

$\boxed{\text{OBS}}$ „=" în $\Leftrightarrow$ sist $\{x, y\}$ SLD.

„$\Rightarrow$" „=" $\Delta_\lambda = 0$

$\exists \lambda_0 \in \mathbb{R}$ a.î. $\langle x + \lambda_0 y, x + \lambda_0 y\rangle = 0 \xRightarrow{\langle \cdot,\cdot \rangle \text{ poz def}} x + \lambda_0 y = 0$

$\Rightarrow \{x, y\}$ SLD

„$\Leftarrow$" $\{x, y\}$ SLD $\Rightarrow \exists a \in \mathbb{R}^*$ a.î. $y = ax$.

$|\langle x, y\rangle| = |\langle x, ax\rangle| = |a| \cdot \|x\|^2$

$\|x\| \cdot \|y\| = \|x\| \cdot \|ax\| = |a| \cdot \|x\|^2$ $\Rightarrow$ „="

<u>Def</u> $(E, \langle \cdot, \cdot \rangle)$ s.v.e.r, $f \in End(E)$

f s.n. <u>simetric</u> $\Leftrightarrow$ $\langle x, f(y) \rangle = \langle f(x), y \rangle, \forall x, y \in E$

<u>Prop</u>

$f \in Sim(E) \Leftrightarrow$ matricea asociată în raport cu $\forall$ reper ortonormat este simetrică.

<u>Dem</u>

$\mathcal{R} = \{e_1, .., e_n\}$ reper $\forall$ ortonormat

$f \in Sim(E) \Leftrightarrow \langle e_i, f(e_j) \rangle = \langle f(e_i), e_j \rangle, \forall i, j = \overline{1,n}$

$A = [f]_{R,R}$ $\qquad f(e_j) = \sum_{r=1}^{n} a_{rj} e_r \qquad f(e_i) = \sum_{k=1}^{n} a_{ki} e_k$

$\sum_{r=1}^{n} a_{rj} \underbrace{\langle e_i, e_r \rangle}_{\delta_{ir}} = \sum_{k=1}^{n} a_{ki} \underbrace{\langle e_k, e_j \rangle}_{\delta_{kj}} \Leftrightarrow a_{ij} = a_{ji}$

$A = A^T \qquad \forall i, j = \overline{1,n}$

$\mathcal{R} = \{e_1, .., e_n\} \xrightarrow{C} \mathcal{R}' = \{e_1', .., e_n'\}$ repere ortonormate

$A' = [f]_{R', R'} = C^{-1} A C = C^T A C$

$A'^T = (C^T A C)^T = C^T A^T (C^T)^T = C^T A C = A' \Rightarrow A'$ simetrică

<u>OBS</u> În general, $f_1, f_2 \in Sim(E) \Rightarrow f_1 \circ f_2 \notin Sim(E)$

$\langle f_1 \circ f_2(x), y \rangle = \langle f_2(x), f_1(y) \rangle = \langle x, f_2 \circ f_1(y) \rangle$

$f_1 \circ f_2 \in Sim(E) \Leftrightarrow f_2 \circ f_1 = f_1 \circ f_2$

$A_2 A_1 = A_1 A_2.$

<u>Prop</u> $f \in Sim(E) \Rightarrow$ vectorii proprii coresp. la valori proprii dist. sunt ortogonali

<u>Dem</u> Fie $x, y \in E \setminus \{0\}$ a.î. $f(x) = \lambda x$, $f(y) = \mu y$. $\lambda \neq \mu$.

$\langle f(x), y\rangle = \langle x, f(y)\rangle \Leftrightarrow \langle \lambda x, y\rangle = \langle x, \mu y\rangle$

$\Leftrightarrow \lambda\langle x,y\rangle - \mu\langle x,y\rangle = 0 \Leftrightarrow \langle x,y\rangle \underbrace{(\lambda-\mu)}_{\neq 0} = 0$

$\Leftrightarrow \langle x, y\rangle = 0$.

<u>Teorema</u> $f \in Sim(E) \Rightarrow$ toate răd polinomului caracteristic sunt reale.

<u>Prop</u> Dacă $f \in Sim(E)$
$U \subseteq E$ subsp. vect. invariant al lui f $\Rightarrow$
$U^\perp \subseteq E$ — // —

<u>Dem</u>

$U \subseteq E$ ssp invar: $\forall x \in U \Rightarrow f(x) \in U$

Dem că $\forall y \in U^\perp \Rightarrow f(y) \in U^\perp$

$\langle \underset{\cap\atop U}{x}, f(y)\rangle = \langle \underset{\cap\atop U}{f(x)}, \underset{\cap\atop U^\perp}{y}\rangle = 0 \Rightarrow f(y) \in U^\perp$

Mai mult $f/_{U^\perp}$ este endom. simetric.

<u>Teorema</u>

$f \in Sim(E) \Rightarrow \exists R$ reper ortonormat format versori proprii ai $[f]_{R,R}$ = diagonală

<u>Dem</u>

Fie R_0 = reper ortonormat arbitrar

$A = [f]_{R_0,R_0}$

$P(\lambda) = \det(A - \lambda I_n) = 0 \xrightarrow{f \in Sim(E)}$ toate răd sunt reale.

Fie λ_1 o răd și e_1 = versorul propriu coresp.

$f(e_1) = \lambda_1 e_1 \Rightarrow \langle\{e_1\}\rangle \subset E$ subsp. invar

$\Rightarrow \langle\{e_1\}\rangle^\perp$ subsp. invar și $f/_{\langle\{e_1\}\rangle^\perp}$ endom sim.

Fie λ_2 = valoare proprie, coresp. versorului propriu $\hat{e}_2$ pt $f/\langle\{e_1\}\rangle^\perp$

$f(e_2) = \lambda_2 e_2$. $\qquad e_2 \in \langle\{e_1\}\rangle^\perp \Downarrow$

$\langle\{e_1, e_2\}\rangle \subset E$ subsp. invar. $\quad (e_1 \perp e_2)$

$\langle\{e_1, e_2\}\rangle^\perp$ — // — , $f/\langle\{e_1,e_2\}\rangle^\perp$ end. sim.

Repetăm raționamentul și după un nr. finit de pași construim $R = \{e_1, .., e_n\} \Rightarrow$ SLI (sistem de versori, mutual ortogonali)

$\dim E = |R| = n \Rightarrow R = \{e_1, .., e_n\}$ reper ortonormat

$$[f]_{R,R} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$\boxed{\text{OBS}}$ a) $f \in Sim(E) \Rightarrow \dim V_{\lambda_i} = m_i$, $i = \overline{1,r}$, unde $\lambda_1, .., \lambda_r$ distincte

$m_1 + \dots + m_r = n$.

b) $R_0 = \{e_1^0, .., e_n^0\} \xrightarrow{C} R = \{e_1, .., e_n\}$

$C \in O(n)$

$h \in O(E)$, $h(e_i^0) = e_i$, $\forall i = \overline{1,n}$

$[h]_{R_0, R_0} = C$.

c) $A = A^T \to f \in Sim(E)$

$\searrow Q : E \to \mathbb{R}$ formă pătratică.

$Q(x) = X^T A X$, $\quad f(x) = y$, $Y = AX$

$$\langle f(x), x \rangle = Q(x) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$$

Scanned with CamScanner

$E = V_{\lambda_1} \oplus \ldots \oplus V_{\lambda_r}$

$R = R_1 \cup \ldots \cup R_r$ reper orton în E,
unde R_i reper orton. în V_{λ_i}

Aplicație

$(\mathbb{R}^3, g_0)$, $f \in End(\mathbb{R}^3)$, $[f]_{R_0, R_0} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} = A$

a) $f \in Sim(\mathbb{R}^3)$

b) $Q : \mathbb{R}^3 \longrightarrow \mathbb{R}$ f. pătratică asociată
Să se aducă Q la o formă canonică, printr-o transf. ortogonală

SOL

a) $A = A^T \Rightarrow f \in Sim(\mathbb{R}^3)$

b) $Q(x) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 - 2x_1x_3 - 2x_2x_3$.

Aplicăm metoda valorilor proprii

$P(\lambda) = \det(A - \lambda I_3) = 0$

$\lambda^3 - \sigma_1 \lambda^2 + \sigma_2 \lambda - \sigma_3 = 0$, $\sigma_3 = 0$

$\sigma_1 = 3$

$\sigma_2 = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} =$

$= 0 + 0 + 0 = 0$

$\lambda^3 - 3\lambda^2 = 0 \Rightarrow \lambda^2(\lambda - 3) = 0$.

$\lambda_1 = 0,\ m_1 = 2$

$\lambda_2 = 3,\ m_2 = 1$

$V_{\lambda_1} = \{x \in \mathbb{R}^3 \mid f(x) = 0\} = \ker f$.

$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $f(x) = (x_1 + x_2 - x_3, x_1 + x_2 - x_3, -x_1 - x_2 + x_3)$

$AX = 0$ $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$ $rg A = 1$.

$x_3 = x_1 + x_2$

$V_{\lambda_1} = \{(x_1, x_2, x_1 + x_2) = x_1 \underbrace{(1,0,1)}_{f_1} + x_2 \underbrace{(0,1,1)}_{f_2},\ x_1, x_2 \in \mathbb{R}\}$

Aplicăm Gram - Schmidt

$f_1 = (1,0,1),\ f_2 = (0,1,1)$

$e_1' = f_1 = (1,0,1),\quad e_1 = \frac{1}{\sqrt{2}}(1,0,1)$

$e_2' = f_2 - \frac{\langle f_2, e_1'\rangle}{\langle e_1', e_1'\rangle} e_1' = (0,1,1) - \frac{1}{2}(1,0,1)$

$\cancel{= \left(0, \frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}(0,1,1),\ e_2 = \frac{1}{\sqrt{2}}(0,1,1)}$

$= \left(-\frac{1}{2}, 1, \frac{1}{2}\right) = \frac{1}{2}(-1,2,1),\quad e_2 = \frac{1}{\sqrt{6}}(-1,2,1)$

$R_1 = \left\{ e_1 = \frac{1}{\sqrt{2}}(1,0,1),\ e_2 = \frac{1}{\sqrt{6}}(-1,2,1) \right\}$ reper ortonormat în V_{λ_1}

$V_{\lambda_2} = \{ x \in \mathbb{R} \ / \ f(x) = 3x \}$

$AX = 3X \Rightarrow (A - 3I_3)X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\underbrace{\begin{pmatrix} -2 & 1 & -1 \\ 1 & -2 & -1 \\ -1 & -1 & -2 \end{pmatrix}}_{B} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -2x_1 + x_2 - x_3 = 0 \\ x_1 - 2x_2 - x_3 = 0 \\ -x_1 - x_2 - 2x_3 = 0 \end{cases}$

$\det B = 0,\quad \dim V_{\lambda_2} = 3 - 2 = 1.$

$\begin{cases} -2x_1 + x_2 = x_3 \\ x_1 - 2x_2 = x_3 \end{cases} \Big| \ 2$

$-3x_1 \quad / \quad = 3x_3$

$x_1 = -x_3$

$x_2 = x_3 - 2x_3 = -x_3.$

$V_{\lambda_2} = \langle \{(-1,-1,1)\} \rangle,\quad R_2 = \left\{ e_3 = \frac{1}{\sqrt{3}}(-1,-1,1) \right\}$ reper ortonormat în V_{λ_2}

$R = R_1 \cup R_2$ reper ortonormat în $\mathbb{R}^3$

$A' = [f]_{R,R} = \begin{pmatrix} 0 & & 0 \\ & 0 & \\ 0 & & 3 \end{pmatrix},\quad Q(x) = 3(x_3')^2$ $\quad (1,0)$ signatura

$R_0 = \{e_1^0, e_2^0, e_3^0\} \xrightarrow{C} R = \left\{ e_1 = \frac{1}{\sqrt{2}}(1,0,1), e_2 = \frac{1}{\sqrt{6}}(-1,2,1), e_3 = \frac{1}{\sqrt{3}}(-1,-1,1) \right\}$

$C \in O(3)$

$h \in O(\mathbb{R}^3)$, $\quad h(e_i^0) = e_i,\ \forall i = \overline{1,3}$

$[h]_{R_0, R_0} = C$

Def $f \in Sim(E)$ s.n. pozitiv def $\Leftrightarrow Q: E \to \mathbb{R}$ (forma pătratică asociată) este poz. def.

Prop $f \in Sim(E)$ poz. def $\Rightarrow \exists h \in Sim(E)$ p. def a.î. $f = h \circ h$.

Teorema (descompunere polară)

$\forall f \in Aut(E) \Rightarrow \exists h \in Sim(E)$, $t \in O(E)$ a.î. $f = h \circ t$

Geometrie afină euclidiană

Def $(\mathcal{A}, V/_{\mathbb{R}}, \varphi)$ s.n. spațiu afin $\Leftrightarrow$

1) $\mathcal{A} \neq \emptyset$ (mult. puncte)
2) $V/_{\mathbb{R}}$ spațiu vectorial (spațiu director)
3) $\varphi: \mathcal{A} \times \mathcal{A} \to V$ structură afină i.e. aplicație care verifică:
   a) $\varphi(A, B) + \varphi(B, C) = \varphi(A, C)$, $\forall A, B, C \in \mathcal{A}$
   b) $\exists O \in \mathcal{A}$ a.î. $\varphi_O: \mathcal{A} \to \mathcal{A}$, bijectivă, unde $\varphi_O(A) = \varphi(O, A)$, $\forall A \in \mathcal{A}$ (De fapt, $\exists \Rightarrow \forall$)

Not $\varphi(A, B) = \overrightarrow{AB}$ $\quad \dim \mathcal{A} = \dim V$

Caz particular $(\mathbb{R}^n, \mathbb{R}^n/_{\mathbb{R}}, \varphi)$

$\varphi: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\varphi(u, v) = v - u$, $\forall u, v \in \mathbb{R}^n$.

Def $M \subset \mathbb{R}^n$ m. de puncte.

$$\mathcal{A}f(M) = \left\{ \sum_{i=1}^{n} a_i P_i, \ \sum_{i=1}^{n} a_i = 1, \ a_i \in \mathbb{R}, \ P_i \in M, \ i = \overline{1,n} \right\}$$

combinații afine de puncte din M.

Def $\mathcal{A}' \subseteq \mathbb{R}^n = \mathcal{A}$ varietate liniară sau subspațiu afin $\Leftrightarrow \left[\forall P_1, P_2 \in \mathcal{A}' \Rightarrow \mathcal{A}f\{P_1, P_2\} \subseteq \mathcal{A}'\right]$

i.e. $\forall a_1, a_2 \in \mathbb{R}$, $a_1 + a_2 = 1$, $a_1 P_1 + a_2 P_2 \in \mathcal{A}'$

Prop a) $\mathcal{A}' \subseteq \mathbb{R}^n$ subsp. afin $\Rightarrow \exists! \; V' \subseteq V$ subsp vect. (director) a.î. $\forall P' \in \mathcal{A}'$, $V' = \{\overrightarrow{P'P}, \forall P \in \mathcal{A}'\}$.

b) Fie $P \in \mathbb{R}^n$, $V' \subseteq \mathbb{R}^n$ subsp. vect. director.

$\Rightarrow \exists! \; \mathcal{A}' \subseteq \mathbb{R}^n$ subsp afin, $P \in \mathcal{A}'$ și V' = sp. vect director

Exemplu $(\mathbb{R}^n, \mathbb{R}^n/\mathbb{R}, \varphi)$ s.af. cu str. canonică.

$\mathcal{A}' = \{x \in \mathbb{R}^n \mid AX = B\} \subset \mathbb{R}^n$ subsp. afin.

$V' = \{x \in \mathbb{R}^n \mid AX = 0\} \subset \mathbb{R}^n$ sp. vect director.

$\forall x, y \in \mathcal{A}' \Rightarrow ax + by \in \mathcal{A}'$, $a + b = 1$

$AX = B$

$AY = B$

$$A(aX + bY) = a\,AX + b\,AY = (a+b)B = B \Rightarrow ax + by \in \mathcal{A}'.$$

$$\underset{1}{\underbrace{(a+b)}}$$

Caz particular

$$\mathcal{A}' = \left\{x \in \mathbb{R}^3 \;\middle|\; \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 - x_3 = 1 \end{cases}\right\} \subset \mathbb{R}^3 \text{ subsp. af.}$$

$$V' = \left\{x \in \mathbb{R}^3 \;\middle|\; \begin{cases} x_1 + x_2 - x_3 = 0 \\ x_1 + 2x_2 - x_3 = 0 \end{cases}\right\} \subset \mathbb{R}^3 \text{ subsp vect. director}$$

Def $\mathcal{A}', \mathcal{A}'' \subset \mathcal{A}$ subsp. afine.

$\mathcal{A}' \parallel \mathcal{A}'' \Leftrightarrow V' \subseteq V''$ sau $V'' \subseteq V'$

(sp. vect. directoare)

<u>Exemplu</u> $(\mathbb{R}^3, \mathbb{R}^3/\mathbb{R}, \varphi)$

$\mathcal{A}' = \{ x \in \mathbb{R}^3 \mid x_1 - 2x_2 - 2x_3 = 2 \}$

$\mathcal{A}'' = \{ x \in \mathbb{R}^3 \mid x_1 - 2x_2 - 2x_3 = 1 \}$

$V' = V'' = \{ x \in \mathbb{R}^3 \mid x_1 - 2x_2 - 2x_3 = 0 \} \Rightarrow \mathcal{A}' \parallel \mathcal{A}''$

<u>Def</u> $(\mathcal{E}, (E, \langle \cdot, \cdot \rangle), \varphi)$ spațiu afin euclidian sau spațiu punctual euclidian $\Leftrightarrow$ este un spațiu afin și sp. vect director = sp. vect. euclidian.

<u>Def</u> $(\mathcal{E}, E, \varphi)$, $\mathcal{E}', \mathcal{E}'' \subset \mathcal{E}$ subsp. afine.

a) $\mathcal{E}_1, \mathcal{E}_2$ sunt perpendiculare $\Leftrightarrow E_1 \perp E_2$
(E_1, E_2 sp. vect directoare)

b) $\mathcal{E}_1, \mathcal{E}_2$ s.n. <u>normale</u> $\Leftrightarrow E_2 = E_1^{\perp}$

$$E = E_1 \oplus \underset{E_2}{\underset{\parallel}{E_1^{\perp}}}.$$

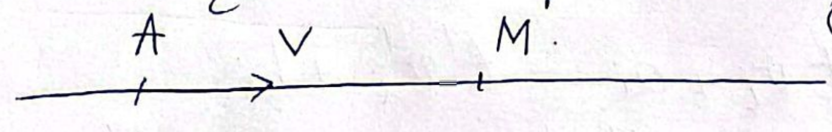
<u>Ecuații ale varietăților liniare în spațiul afin euclidian $(\mathcal{E}, E/\mathbb{R}, \varphi)$</u>

$\mathcal{R} = \{ O; e_1, \dots, e_n \}$ reper cartezian ortonormat

$O \in \mathcal{E}$, $\{ e_1, \dots, e_n \}$ reper ortonormat în E

① <u>Ecuația unei drepte</u>

a) $\mathcal{D}$



$\overrightarrow{OA} = \sum_{i=1}^{n} a_i e_i$

$V = \sum_{i=1}^{n} v_i e_i$

$V_{\mathcal{D}} = \langle \{ v \} \rangle$

$A, M \in \mathcal{D} \Rightarrow \overrightarrow{AM} \in V_{\mathcal{D}} \Rightarrow \exists t \in \mathbb{R}$ a.î.

$\overrightarrow{AM} = tv = t \sum_{i=1}^{n} v_i e_i$ $\qquad \overrightarrow{OM} = \sum_{i=1}^{n} x_i e_i$

$\parallel$

$\overrightarrow{OM} - \overrightarrow{OA} = \sum_{i=1}^{n} (x_i - a_i) e_i$

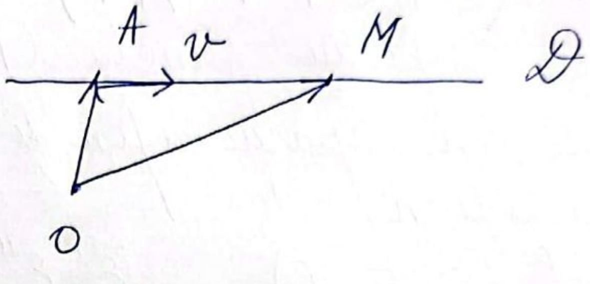
$\mathcal{D} : (x_1 - a_1, \dots, x_n - a_n) = t(v_1, \dots, v_n)$ ec. param

$$\frac{x_1 - a_1}{v_1} = \dots = \frac{x_n - a_n}{v_n} = t$$

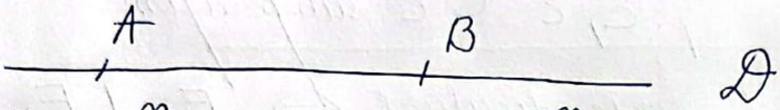
<u>Convenție</u> Dacă $\exists i \in \{1, \dots, n\}$ ai $v_i = 0$, atunci $x_i - a_i = 0$.

$\mathcal{D} : \vec{OM} = \vec{OA} + tv$

$\vec{OM} = r$, $\vec{OA} = r_0$



b)



$V = \vec{AB} = \vec{OB} - \vec{OA}$

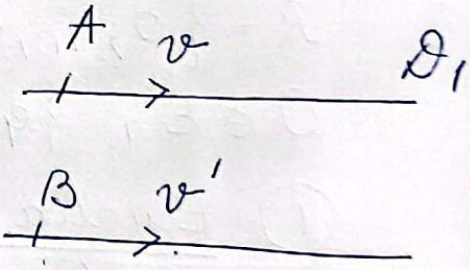
$\vec{OA} = \sum_{i=1}^{n} a_i e_i$, $\vec{OB} = \sum_{i=1}^{n} b_i e_i$ $\quad = \sum_{i=1}^{n} \underbrace{(b_i - a_i)}_{= v_i} e_i$

$\mathcal{D} : (x_1 - a_1, \dots, x_n - a_n) = t(b_1 - a_1, \dots, b_n - a_n)$

$$\frac{x_1 - a_1}{b_1 - a_1} = \dots = \frac{x_n - a_n}{b_n - a_n} = t \quad \text{ec. param.}$$

Conv. $\exists i \in \{1, \dots, n\}$ ai $b_i = a_i$, at $x_i = a_i$

<u>Poziția relativă a 2 drepte</u>



$\mathcal{D}_1 : x_i - a_i = t v_i$, $i = \overline{1, n}$

$\mathcal{D}_2 : x_i - b_i = t' v_i'$

$\mathcal{D}_1 \cap \mathcal{D}_2 : t v_i + a_i = t' v_i' + b_i$

$$t v_i - t' v_i' = b_i - a_i, \quad \forall i = \overline{1, n}.$$

$$C = \left(\begin{array}{cc|c} v_1 & -v_1' & b_1 - a_1 \\ v_2 & -v_2' & b_2 - a_2 \\ \vdots & \vdots & \vdots \\ v_n & -v_n' & b_n - a_n \end{array} \right)$$

a) $\operatorname{rg} C = \operatorname{rg} \overline{C} = 2 \Rightarrow \mathcal{D}_1, \mathcal{D}_2$ drepte concurente.

b) $\operatorname{rg} C = \operatorname{rg} \overline{C} = 1 \Rightarrow \mathcal{D}_1 = \mathcal{D}_2$.

c) $\operatorname{rg} C = 2, \operatorname{rg} \overline{C} = 3 \Rightarrow$ necoplanare.

d) $\operatorname{rg} C = 1, \operatorname{rg} \overline{C} = 2 \Rightarrow \mathcal{D}_1 \parallel \mathcal{D}_2$