

Exercitii

1. Determinați $\lim x_n$, $\overline{\lim} x_n$ și precizați dacă există $\lim_{n \rightarrow \infty} x_n$, unde :

$$a) x_n = \frac{n}{2n+1} \left(\cos \frac{n\pi}{3} \right)^n \quad \forall n \in \mathbb{N}^*.$$

$$b) x_n = \frac{2 + (-1)^n}{1 + n^{(-1)^n}} + \sin \frac{n\pi}{2} \quad \forall n \in \mathbb{N}.$$

$$c) x_n = \sqrt[n]{4^{(-1)^n} + 2} \quad \forall n \in \mathbb{N}^*.$$

$$d) x_n = \frac{1}{2} \left(n - 2 - 3 \left[\frac{n-1}{3} \right] \right) \left(n - 3 - 3 \left[\frac{n-1}{3} \right] \right)$$

$\forall n \in \mathbb{N}^*$, unde $[a]$ reprezintă partea întreagă a lui a .

$$e) x_n = \frac{(1 - (-1)^n) \cdot 2^n + 1}{2^n + 3} \quad \forall n \in \mathbb{N}.$$

$$f) x_n = \frac{(1 + \cos n\pi) \ln 3n + \ln n}{\ln 2n} \quad \forall n \in \mathbb{N}^*.$$

2. Studiați convergența seriilor:

a) $\sum_{n=1}^{\infty} e^{-n^2}.$

b) $\sum_{n=1}^{\infty} \frac{(an)^n}{n!}, \quad a > 0, a \neq \frac{1}{e}.$

c) $\sum_{n=1}^{\infty} \frac{\sin(n\pi)}{2^n}, \quad \pi \in \mathbb{R}.$

d) $\sum_{n=1}^{\infty} a^n \left(1 + \frac{1}{n}\right)^n, \quad a > 0.$

e) $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}.$

f) $\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{2n}\right).$

g) $\sum_{n=1}^{\infty} \left(a \frac{n^2 + n + 1}{n^2}\right)^n, \quad a > 0.$

h) $\sum_{n=1}^{\infty} \frac{a(a+1)\dots(a+n-1)}{b(b+1)\dots(b+n-1)} (c-2)^n, \quad a > 0, b > 0, c > 2.$

i) $\sum_{n=1}^{\infty} \frac{1}{n} \sqrt[n]{(n+1)(n+2)\dots(n+n)}.$

$$j) \sum_{n=2}^{\infty} \frac{1}{\sqrt[n]{\ln n}}.$$

$$k) \sum_{n=1}^{\infty} \left(e^{\sin \frac{1}{n}} - 1 \right) x^n, \quad x > 0.$$

$$l) \sum_{n=2}^{\infty} \frac{1}{(\ln n)^3} \cdot x^n, \quad x > 0.$$

$$m) \sum_{n=1}^{\infty} \frac{1}{\sqrt{n} \sqrt[4]{n+2}} x^n, \quad x > 0.$$

$$n) \sum_{n=1}^{\infty} \frac{a^n + n}{a^{2n} + n^3}, \quad a > 0.$$

$$o) \sum_{n=1}^{\infty} \left(\sqrt{(n+1)(n+x)} - n \right)^n.$$

$$p) \sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!} x^n, \quad x > 0.$$

$$q) \sum_{n=1}^{\infty} \frac{a^n}{2^n + 5^n}, \quad a > 0.$$

$$r) \sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}.$$

$$1) \sum_{n=1}^{\infty} \sin(\pi \sqrt{n^2+1}).$$

$$t) \sum_{n=1}^{\infty} \frac{10 \cdot 18 \cdot \dots \cdot (8n+2)}{10 \cdot 19 \cdot \dots \cdot (9n+1)} x^n, \quad x > 0.$$

$$u) \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n)} \cdot \frac{1}{n}.$$

$$v) \sum_{n=1}^{\infty} \left(\operatorname{arctg} \frac{1}{n(n+1)} \right) x^n, \quad x > 0.$$

$$w) \sum_{n=1}^{\infty} \frac{\sqrt{n-1}}{n(n+1)} x^n, \quad x > 0.$$

$$x) \sum_{n=1}^{\infty} \frac{\sin(nx)}{n}, \quad x \in \mathbb{R}.$$

$$y) \sum_{n=1}^{\infty} \frac{\sin n \sin \frac{1}{n}}{n}.$$

$$z) \sum_{n=1}^{\infty} \frac{x^n}{\sqrt[3]{n+1} \sqrt[5]{n+2}}, \quad x > 0.$$

3. Faceți analiza topologică a mulțimii $A \subset \mathbb{R}$, unde :

$$a) A = \mathbb{R} \setminus \mathbb{Q}.$$

b) $A = \mathbb{Q} \cap (0, 1)$

c) $A = [0, 4] \setminus \{2\}$.

d) $A = \left\{ \frac{n}{3n+1} \mid n \in \mathbb{N} \right\}$.

4. Fie $X = C([0, 1]) = \{f: [0, 1] \rightarrow \mathbb{R} \mid f \text{ continuă}\}$,

$d_1: X \times X \rightarrow \mathbb{R}$, $d_1(f, g) = \int_0^1 |f(x) - g(x)| dx$ și

$d_\infty: X \times X \rightarrow \mathbb{R}$, $d_\infty(f, g) = \max\{|f(x) - g(x)| \mid x \in [0, 1]\}$

a) Arătați că d_1 și d_∞ sunt distanțe pe X .

b) Fie $(f_n)_n \subset X$ și $f \in X$ a.î. $\lim_{n \rightarrow \infty} f_n \stackrel{d_\infty}{=} f$.

Arătați că $\lim_{n \rightarrow \infty} f_n \stackrel{d_1}{=} f$.

c) Fie $(f_n)_n \subset X$, $f_n: [0, 1] \rightarrow \mathbb{R}$, $f_n(x) =$

$= \frac{1}{1+n^2x} \quad \forall n \in \mathbb{N}$. Arătați că $\lim_{n \rightarrow \infty} f_n \stackrel{d_1}{=} f$ și

$\lim_{n \rightarrow \infty} f_n \not\stackrel{d_\infty}{=} f$, unde $f: [0, 1] \rightarrow \mathbb{R}$, $f(x) = 0$.