Seminar 14

1. Folosind eventual funcțiile Γ și B determinați
$$\int_0^{\frac{\pi}{2}} (\sin t)^{\frac{5}{2}} (\cos t)^{\frac{3}{2}}\, dt.$$

Sol.: $B(x,y) = 2\int_0^{\frac{\pi}{2}} (\sin t)^{2x-1} \cdot (\cos t)^{2y-1}\, dt \quad \forall\, x,y \in (0,\infty).$

$2x-1 = \frac{5}{2} \Leftrightarrow x = \frac{7}{4}.$

$2y-1 = \frac{3}{2} \Leftrightarrow y = \frac{5}{4}.$

$$\int_0^{\frac{\pi}{2}} (\sin t)^{\frac{5}{2}} (\cos t)^{\frac{3}{2}}\, dt = \frac{1}{2} \cdot \underbrace{2\int_0^{\frac{\pi}{2}} (\sin t)^{2\cdot\frac{7}{4}-1} (\cos t)^{2\cdot\frac{5}{4}-1}\, dt}_{= B\left(\frac{7}{4},\frac{5}{4}\right)} =$$

$$= \frac{1}{2} B\left(\frac{7}{4},\frac{5}{4}\right).$$

$$B\left(\frac{7}{4},\frac{5}{4}\right) = \frac{\Gamma\left(\frac{7}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma(3)}.$$

$\Gamma(3) = 2! = 2.$

$\Gamma\left(\frac{7}{4}\right) = \Gamma\left(1+\frac{3}{4}\right) = \frac{3}{4}\Gamma\left(\frac{3}{4}\right).$

$\Gamma\left(\frac{5}{4}\right) = \Gamma\left(1+\frac{1}{4}\right) = \frac{1}{4}\Gamma\left(\frac{1}{4}\right).$

$$\Gamma\left(\frac{7}{4}\right)\cdot\Gamma\left(\frac{5}{4}\right) = \frac{3}{4}\Gamma\left(\frac{3}{4}\right)\frac{1}{4}\Gamma\left(\frac{1}{4}\right) = \frac{3}{16}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{3}{16}\cdot\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) =$$

$$= \frac{3}{16}\cdot\frac{\pi}{\sin\frac{\pi}{4}} = \frac{3}{16}\cdot\frac{\pi}{\frac{\sqrt{2}}{2}} = \frac{\sqrt{2}\,3}{\cancel{16}\,8}\cdot\frac{\cancel{2\pi}}{\sqrt{2}} = \frac{3\pi\sqrt{2}}{16}.$$

Deci $B\left(\frac{7}{4},\frac{5}{4}\right) = \frac{3\pi\sqrt{2}}{16}\cdot\frac{1}{2} = \frac{3\pi\sqrt{2}}{32}.$

Așadar $\int_0^{\frac{\pi}{2}} (\sin t)^{\frac{5}{2}}(\cos t)^{\frac{3}{2}}\,dt = \frac{1}{2}\cdot\frac{3\pi\sqrt{2}}{32} = \frac{3\pi\sqrt{2}}{64}.$ $\square$

2. Studiați convergența (natura) următoarelor integrale improprii:

a) $\int_1^{\infty} \frac{1}{1+x^4}\,dx.$

Sol.: Fie $f,g:[1,\infty)\to[0,\infty)$, $f(x)=\frac{1}{1+x^4}$, $g(x)=\frac{1}{x^4}$.

Avem $0\le f(x)\le g(x)\ \forall\, x\in[1,\infty)$.

$$\int_1^{\infty} g(x)\,dx = \int_1^{\infty}\frac{1}{x^4}\,dx = \int_1^{\infty} x^{-4}\,dx = \lim_{b\to\infty}\int_1^b x^{-4}\,dx =$$

$$= \lim_{b\to\infty}\left(\frac{x^{-3}}{-3}\Big|_1^b\right) = \lim_{b\to\infty}\left(-\frac{1}{3b^3}+\frac{1}{3}\right) = \frac{1}{3}.$$

Deci $\int_1^{\infty} g(x)\,dx$ este convergentă.

Conform Crit. de comp. cu ineg. rezultă că $\int_1^{\infty} f(x)\,dx$ este conv. $\square$

b) $\displaystyle\int_2^\infty \frac{1}{\sqrt{x}-1}\,dx.$

Sol.: Rezolvați-l voi! $\square$

c) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}+1}\,dx.$

Sol.: Fie $f, g : [1,\infty) \to (0,\infty)$, $f(x) = \dfrac{1}{\sqrt{x}+1}$, $g(x) = \dfrac{1}{\sqrt{x}}$.

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x}+1} = 1 \in (0,\infty).$$

Conform crit. de comp. cu limită rezultă că

$$\int_1^\infty f(x)\,dx \sim \int_1^\infty g(x)\,dx.$$

$$\int_1^\infty g(x)\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty} \int_1^b x^{-\frac{1}{2}}\,dx =$$

$$= \lim_{b\to\infty} \left(\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \Bigg|_1^b \right) = \lim_{b\to\infty} \left(2\sqrt{b} - 2 \right) = \infty.$$

Deci $\displaystyle\int_1^\infty g(x)\,dx$ este divergentă.

Așadar $\displaystyle\int_1^\infty f(x)\,dx$ este divergentă. $\square$

d) $\displaystyle\int_1^\infty \sin\frac{1}{x^{12}}\,dx.$

Sol.: Fie $f : [1,\infty) \to [0,\infty)$, $f(x) = \sin\dfrac{1}{x^{12}}$.

f este funcție descrescătoare.

Conform Crit. integral al lui Cauchy rezultă că

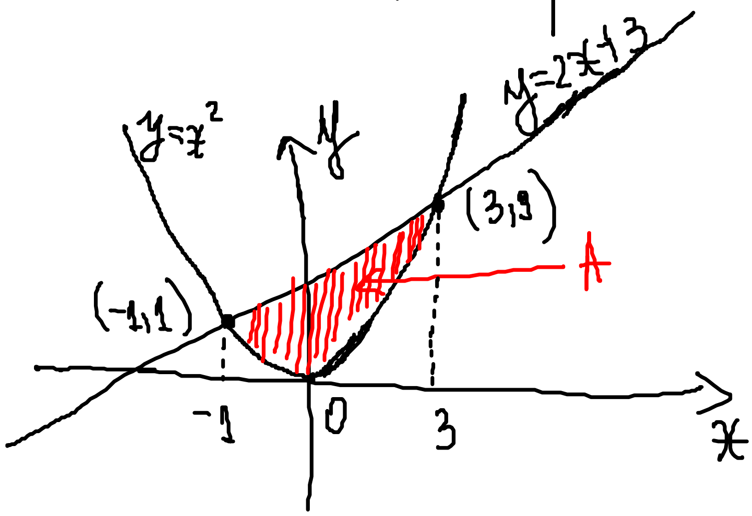
$$\int_1^\infty f(x)\,dx \sim \sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} \sin\frac{1}{n^{12}} \sim \sum_{n=1}^{\infty} \frac{1}{n^{12}} \text{ conv.}$$

(serie armonică generalizată cu $\alpha = 12$). $\square$

3. Determinați:

a) $\iint_A x\,dx\,dy$, unde A este mulțimea plană limitată de $y = x^2$ și $y = 2x+3$.

Sol.:



Determinăm punctele de intersecție dintre dreapta $y=2x+3$ și parabola $y=x^2$.

$$\begin{cases} y = 2x+3 \\ y = x^2 \end{cases} \Leftrightarrow \begin{cases} x^2 - 2x - 3 = 0 \\ y = x^2 \end{cases}.$$

$x^2 - 2x - 3 = 0$.

$\Delta = 4+12 = 16$,

$\sqrt{\Delta} = 4$.

$x_1 = \frac{2+4}{2} = 3 \left(\Leftrightarrow y_1 = 9\right)$.

$x_2 = \frac{2-4}{2} = -1 \left(\Leftrightarrow y_2 = 1\right)$.

$A = \{(x,y) \in \mathbb{R}^2 \mid x \in [-1,3],\ x^2 \le y \le 2x+3\}$.

Fie $\alpha, \beta : [-1,3] \to \mathbb{R}$, $\alpha(x) = x^2$, $\beta(x) = 2x+3$.

α, β continue.

A este mulțime măsurabilă Jordan și compactă.

Fie $f : A \to \mathbb{R}$, $f(x,y) = x$.

f continuă.

$$\iint_A f(x,y)\,dx\,dy = \int_{-1}^{3} \left(\int_{x^2}^{2x+3} x\,dy \right) dx =$$

$$= \int_{-1}^{3} \left(xy \Big|_{y=x^2}^{y=2x+3} \right) dx = \int_{-1}^{3} x\left(2x+3-x^2\right) dx =$$

$$= \int_{-1}^{3} \left(2x^2+3x-x^3\right) dx = 2\,\frac{x^3}{3}\Big|_{x=-1}^{x=3} + 3\,\frac{x^2}{2}\Big|_{x=-1}^{x=3} - \frac{x^4}{4}\Big|_{x=-1}^{x=3} =$$

$= \frac{2}{3}\cdot 28 + \frac{3}{\cancel{2}}\cdot \cancel{8}^{4} - \frac{80}{4} = \frac{56}{3} + 12 - 20 = \frac{56}{3} - 8 = \frac{56-24}{3} =$

$= \frac{32}{3}$. □

b) $\iint_A x\,dx\,dy$, unde A este mulțimea plană mărginită de

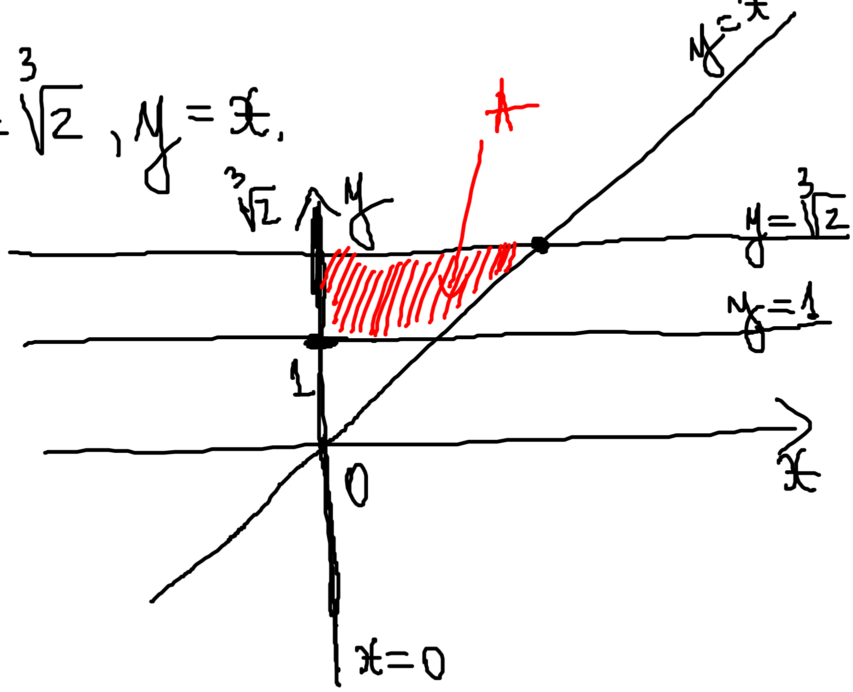
$y = -x^2 - x + 2$ și $y = x - 1$.

Sol.: Rezolvați-l voi! □

c) $\iint_A y\,dx\,dy$, unde A este mulțimea plană mărginită

de $x=0$, $y=1$, $y=\sqrt[3]{2}$, $y=x$.

Sol.:



$A = \{(x,y)\in\mathbb{R}^2 \mid y\in[1,\sqrt[3]{2}],\ 0\le x\le y\}$.

Fie $\varphi, \psi : [1,\sqrt[3]{2}] \to \mathbb{R}$, $\varphi(y) = 0$, $\psi(y) = y$.

φ, ψ continue.

A este mulțime măsurabilă Jordan și compactă.

Fie $f: A \to \mathbb{R}$, $f(x,y) = y$.

f continuă.

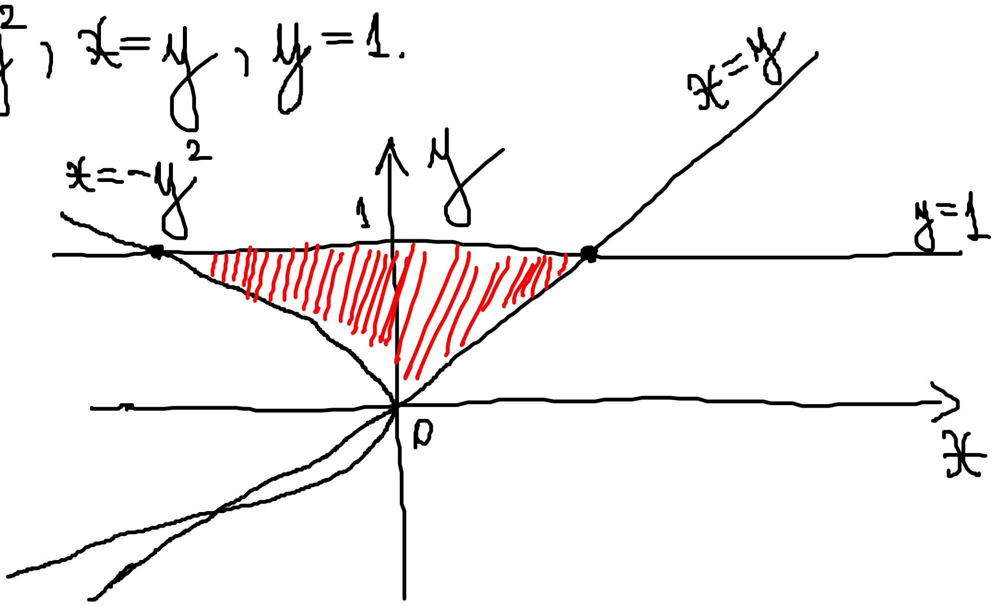
$$\iint_A f(x,y)\,dx\,dy = \int_1^{\sqrt[3]{2}} \left(\int_0^y y\,dx\right) dy =$$

$$= \int_1^{\sqrt[3]{2}} \left(y x \Big|_{x=0}^{x=y}\right) dy = \int_1^{\sqrt[3]{2}} y(y-0)\,dy = \frac{y^3}{3}\Big|_{y=1}^{y=\sqrt[3]{2}} =$$

$$= \frac{1}{3}. \ \square$$

d) $\iint_A y\,dx\,dy$, unde A este mulțimea plană limitată de $x = -y^2$, $x = y$, $y = 1$.

Sol.:



$$A = \{(x,y) \in \mathbb{R}^2 \mid y \in [0,1],\ -y^2 \le x \le y\}.$$

Fie $\varphi, \psi: [0,1] \to \mathbb{R}$, $\varphi(y) = -y^2$, $\psi(y) = y$.

φ, ψ continue.

A este mulțime măsurabilă Jordan și compactă.

Fie $f: A \to \mathbb{R}$, $f(x,y) = y$.

f continuă.

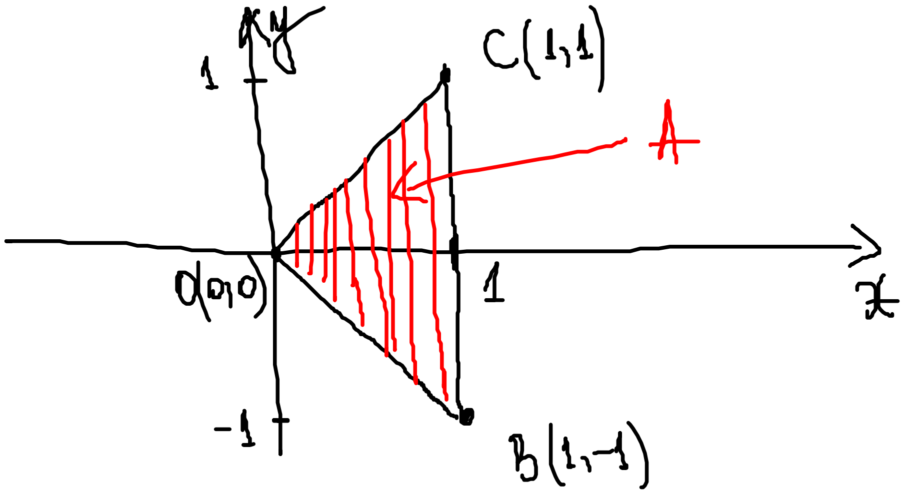
$$\iint_A f(x,y)\,dx\,dy = \int_0^1 \left(\int_{-y^2}^{y} y\,dx \right) dy =$$

$$= \int_0^1 \left(yx \Big|_{x=-y^2}^{x=y} \right) dy = \int_0^1 y(y+y^2)\,dy = \frac{y^3}{3}\Big|_{y=0}^{y=1} +$$

$$+ \frac{y^4}{4}\Big|_{y=0}^{y=1} = \frac{{}^{4)}1}{3} + \frac{{}^{3)}1}{4} = \frac{7}{12}. \quad \square$$

e) $\iint_A x\,dx\,dy$, unde A este mulțimea plană mărginită de triunghiul OBC, $O(0,0)$, $B(1,-1)$, $C(1,1)$.

Sol.:



Scriem ecuațiile dreptelor OB, OC și BC.

$$OB: \frac{y-y_0}{y_B-y_0} = \frac{x-x_0}{x_B-x_0} \Leftrightarrow \frac{y-0}{-1-0} = \frac{x-0}{1-0} \Leftrightarrow -y = x \Leftrightarrow$$

$\Leftrightarrow y=-x$.

$OC: \dfrac{y-y_O}{y_C-y_O}=\dfrac{x-x_O}{x_C-x_O} \Leftrightarrow \dfrac{y-0}{1-0}=\dfrac{x-0}{1-0} \Leftrightarrow y=x$.

$BC: \dfrac{y-y_B}{y_C-y_B}=\dfrac{x-x_B}{x_C-x_B} \Leftrightarrow \dfrac{y+1}{1+1}=\dfrac{x-1}{1-1} \Leftrightarrow$

$\Leftrightarrow x-1=0 \Leftrightarrow x=1$.

$A=\{(x,y)\in\mathbb{R}^2 \mid x\in[0,1],\ -x\le y\le x\}$.

Fie $\alpha,\beta:[0,1]\to\mathbb{R}$, $\alpha(x)=-x$, $\beta(x)=x$.

α,β continue.

A este mulțime măsurabilă Jordan și compactă.

Fie $f:A\to\mathbb{R}$, $f(x,y)=x$.

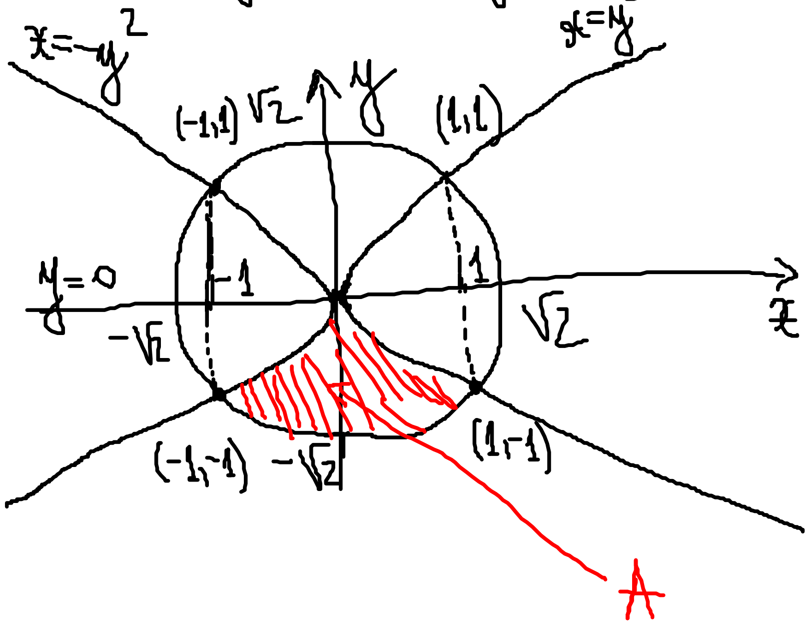
$$\iint_A f(x,y)\,dx\,dy=\int_0^1\left(\int_{-x}^{x} x\,dy\right)dx=\int_0^1\left(xy\Big|_{y=-x}^{y=x}\right)dx=$$

$$=\int_0^1 x(x+x)\,dx=2\,\frac{x^3}{3}\Big|_{x=0}^{x=1}=\frac{2}{3}.\ \square$$

4. Determinați:

a) $\iint_A y\,dx\,dy$, unde $A=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 2,\ x\le y^2,\ x\ge -y^2,\ y\le 0\}$.

Sol.:



Determinăm punctele de intersecție dintre parabola $x=-y^2$ și cercul $x^2+y^2=2$, respectiv dintre parabola $x=y^2$ și cercul $x^2+y^2=2$.

$$\begin{cases} x=-y^2 \\ x^2+y^2=2 \end{cases} \Leftrightarrow \begin{cases} y^2=-x \\ x^2-x-2=0. \end{cases}$$

$x^2-x-2=0.$

$\Delta=1+8=9.$

$\sqrt{\Delta}=3.$

$x_1=\dfrac{1+3}{2}=2.$

$x_2=\dfrac{1-3}{2}=-1.$

$y^2=-x \Rightarrow x\le 0.$

Deci $x = -1$.

$y^2 = 1 \Rightarrow y = \pm 1$.

$$\begin{cases} x = y^2 \\ x^2 + y^2 = 2 \end{cases} \Leftrightarrow \begin{cases} x = y^2 \\ x^2 + x - 2 = 0 \end{cases}.$$

$x^2 + x - 2 = 0$.

$\Delta = 1 + 8 = 9$.

$\sqrt{\Delta} = 3$.

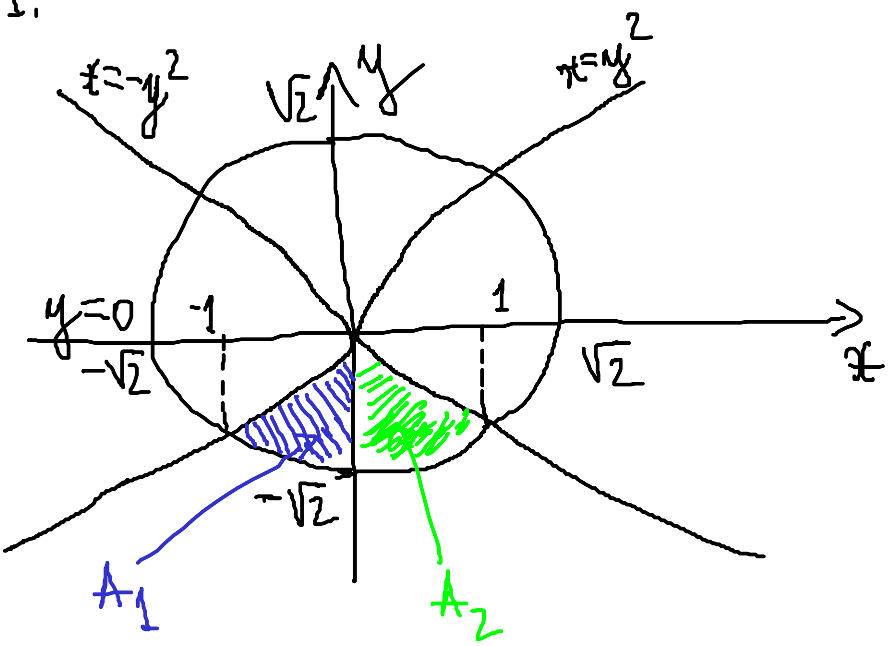
$x_1 = \dfrac{-1+3}{2} = 1$.

$x_2 = \dfrac{-1-3}{2} = -2$.

$y^2 = x \Rightarrow x \geq 0$.

Deci $x = 1$.

$y^2 = 1 \Rightarrow y = \pm 1$.



$x^2+y^2 \le 2 \Leftrightarrow y^2 \le 2-x^2 \Leftrightarrow -\sqrt{2-x^2} \le y \le \sqrt{2-x^2}.$

$x \ge -y^2 \Leftrightarrow y^2 \ge -x \Leftrightarrow y \in (-\infty, -\sqrt{-x}] \cup [\sqrt{-x}, +\infty).$

$A_1 = \{(x,y) \in \mathbb{R}^2 \mid x \in [-1,0], -\sqrt{2-x^2} \le y \le -\sqrt{-x}\}.$

$A_2 = \{(x,y) \in \mathbb{R}^2 \mid x \in [0,1], -\sqrt{2-x^2} \le y \le -\sqrt{x}\}.$

Fie $\alpha, \beta : [-1,0] \to \mathbb{R}$, $\alpha(x) = -\sqrt{2-x^2}$, $\beta(x) = -\sqrt{-x}$.

α, β continue.

A_1 este mulțime măsurabilă Jordan și compactă.

Fie $\gamma, \delta : [0,1] \to \mathbb{R}$, $\gamma(x) = -\sqrt{2-x^2}$, $\delta(x) = -\sqrt{x}$.

γ, δ continue.

A_2 este mulțime măsurabilă Jordan și compactă.

$A = A_1 \cup A_2.$

$\mu(A_1 \cap A_2) = 0.$

Fie $f : A \to \mathbb{R}$, $f(x,y) = y$.

f continuă.

$\iint_A f(x,y)\,dx\,dy = \iint_{A_1} f(x,y)\,dx\,dy + \iint_{A_2} f(x,y)\,dx\,dy.$

$$\iint_{A_1} f(x,y)\,dx\,dy = \int_{-1}^{0}\left(\int_{-\sqrt{2-x^2}}^{-\sqrt{-x}} y\,dy\right)dx =$$

$$= \int_{-1}^{0}\left(\frac{y^2}{2}\Bigg|_{y=-\sqrt{2-x^2}}^{y=-\sqrt{-x}}\right)dx = \int_{-1}^{0}\frac{1}{2}\cdot\left(-x-2+x^2\right)dx =$$

$$= -\frac{1}{2}\cdot\frac{x^2}{2}\Bigg|_{x=-1}^{x=0} - x\Bigg|_{x=-1}^{x=0} + \frac{x^3}{6}\Bigg|_{x=-1}^{x=0} =$$

$$= \overset{3)}{\frac{1}{4}} - \overset{12)}{1} + \overset{2)}{\frac{1}{6}} = \frac{3-12+2}{12} = -\frac{7}{12}.$$

$$\iint_{A_2} f(x,y)\,dx\,dy = \int_{0}^{1}\left(\int_{-\sqrt{2-x^2}}^{-\sqrt{x}} y\,dy\right)dx =$$

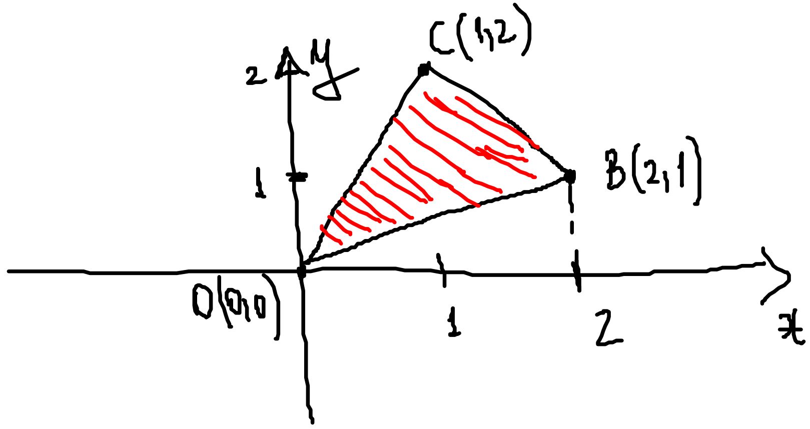
$$= \int_{0}^{1}\left(\frac{y^2}{2}\Bigg|_{y=-\sqrt{2-x^2}}^{y=-\sqrt{x}}\right)dx = \int_{0}^{1}\frac{1}{2}\left(x-2+x^2\right)dx =$$

$$= \frac{1}{2}\cdot\frac{x^2}{2}\Bigg|_{x=0}^{x=1} - x\Bigg|_{x=0}^{x=1} + \frac{x^3}{6}\Bigg|_{x=0}^{x=1} = \overset{3)}{\frac{1}{4}} - \overset{12)}{1} + \overset{2)}{\frac{1}{6}} = \frac{3-12+2}{12} = -\frac{7}{12}.$$

$$\iint_{A} f(x,y)\,dx\,dy = -\frac{7}{12} - \frac{7}{12} = -\frac{14}{12}^{(2} = -\frac{7}{6}. \ \square$$

b) $\iint_A x\,dx\,dy$, unde A este mulțimea plană mărginită de triunghiul OBC, $O(0,0)$, $B(2,1)$, $C(1,2)$.

Sol.:



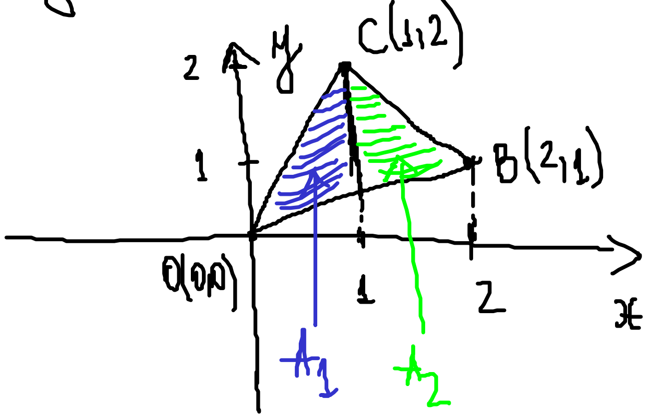
$$OB: \frac{y-y_0}{y_B-y_0} = \frac{x-x_0}{x_B-x_0} \Leftrightarrow \frac{y-0}{1-0} = \frac{x-0}{2-0} \Leftrightarrow y = \frac{x}{2}.$$

$$OC: \frac{y-y_0}{y_C-y_0} = \frac{x-x_0}{x_C-x_0} \Leftrightarrow \frac{y-0}{2-0} = \frac{x-0}{1-0} \Leftrightarrow \frac{y}{2} = x \Leftrightarrow$$

$$\Leftrightarrow y = 2x.$$

$$BC: \frac{y-y_B}{y_C-y_B} = \frac{x-x_B}{x_C-x_B} \Leftrightarrow \frac{y-1}{2-1} = \frac{x-2}{1-2} \Leftrightarrow$$

$$\Leftrightarrow y-1 = 2-x \Leftrightarrow y = 3-x.$$



$$A_1 = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; x \in [0,1],\ \frac{x}{2} \le y \le 2x \right\}.$$

$$A_2 = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; x \in [1,2],\ \frac{x}{2} \le y \le 3-x \right\}.$$

Fie $\alpha,\beta:[0,1]\to\mathbb{R}$, $\alpha(x)=\frac{x}{2}$, $\beta(x)=2x$.

α,β continue.

A_1 mulțime măsurabilă Jordan și compactă.

Fie $\gamma,\delta:[1,2]\to\mathbb{R}$, $\gamma(x)=\frac{x}{2}$, $\delta(x)=3-x$.

A_2 mulțime măsurabilă Jordan și compactă.

$A=A_1\cup A_2$.

$\mu(A_1\cap A_2)=0$.

Fie $f:A\to\mathbb{R}$, $f(x,y)=x$.

f continuă.

$$\iint_A f(x,y)\,dx\,dy=\iint_{A_1} f(x,y)\,dx\,dy+\iint_{A_2} f(x,y)\,dx\,dy.$$

$$\iint_{A_1} f(x,y)\,dx\,dy=\int_0^1\left(\int_{\frac{x}{2}}^{2x} x\,dy\right)dx=\int_0^1\left(xy\Big|_{y=\frac{x}{2}}^{y=2x}\right)dx=$$

$$=\int_0^1 x\left(2x-\frac{x}{2}\right)dx=\int_0^1\frac{3}{2}x^2\,dx=\frac{\cancel{3}}{2}\cdot\frac{x^3}{\cancel{3}}\Big|_{x=0}^{x=1}=\frac{1}{2}.$$

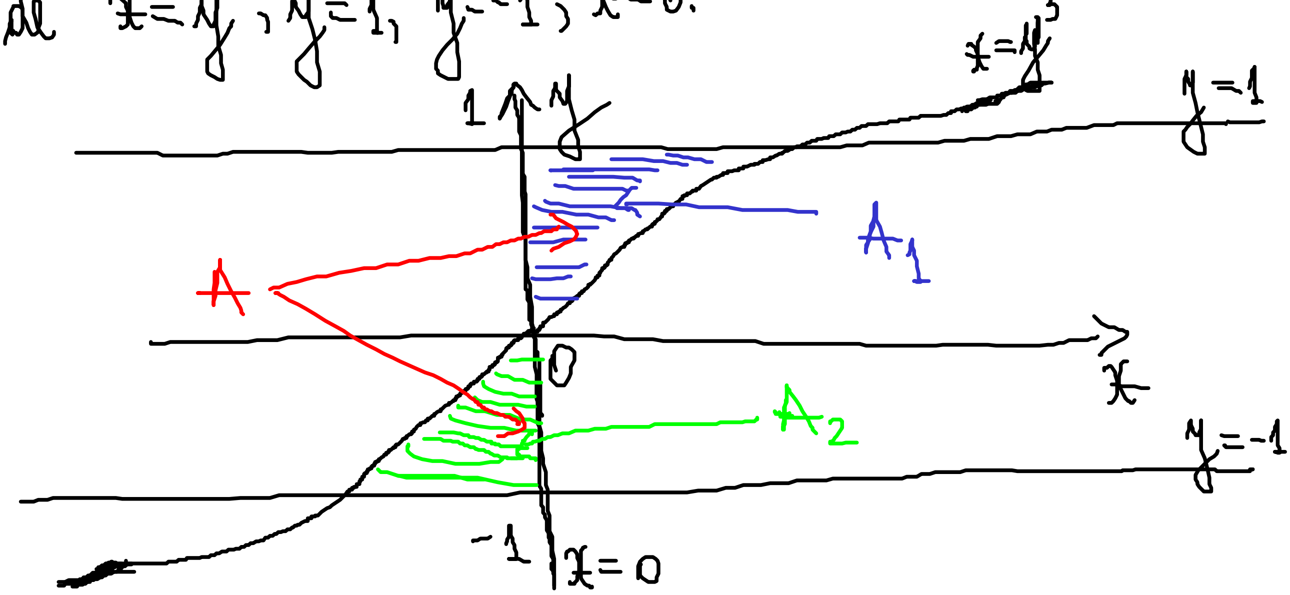
$$\iint_{A_2} f(x,y)\,dx\,dy=\int_1^2\left(\int_{\frac{x}{2}}^{3-x} x\,dy\right)dx=\int_1^2\left(xy\Big|_{y=\frac{x}{2}}^{y=3-x}\right)dx=$$

$$= \int_1^2 x\left(3-x-\frac{x}{2}\right)dx = \int_1^2 \left(3x - \frac{3}{2}x^2\right)dx = 3\frac{x^2}{2}\Big|_{x=1}^{x=2} - \frac{3}{2}\cdot\frac{x^3}{3}\Big|_{x=1}^{x=2} =$$

$$= \frac{9}{2} - \frac{7}{2} = 1.$$

$$\iint_A f(x,y)\,dx\,dy = \frac{1}{2} + 1 = \frac{3}{2}. \quad \square$$

c) $\iint_A e^{y^4}\,dx\,dy$, unde A este mulțimea plană mărginită de $x=y^3$, $y=1$, $y=-1$, $x=0$.



$A = A_1 \cup A_2$, unde $A_1 = \{(x,y)\in\mathbb{R}^2 \mid y\in[0,1],\ 0\le x\le y^3\}$ și

$A_2 = \{(x,y)\in\mathbb{R}^2 \mid y\in[-1,0],\ y^3\le x\le 0\}$.

Fie $\varphi,\psi:[0,1]\to\mathbb{R}$, $\varphi(y)=0$, $\psi(y)=y^3$.

φ,ψ continue.

A_1 este mulțime măsurabilă Jordan și compactă.

Fie $\omega, \theta : [-1,0] \to \mathbb{R}$, $\omega(y) = y^3$, $\theta(y) = 0$.

ω, θ continue.

A_2 este mulțime măsurabilă Jordan și compactă.

$\mu(A_1 \cap A_2) = 0$.

Fie $f : A \to \mathbb{R}$, $f(x,y) = e^{y^4}$.

f continuă.

$$\iint_A f(x,y)\,dx\,dy = \iint_{A_1} f(x,y)\,dx\,dy + \iint_{A_2} f(x,y)\,dx\,dy.$$

$$\iint_{A_1} f(x,y)\,dx\,dy = \int_0^1 \left(\int_0^{y^3} e^{y^4}\,dx \right) dy = \int_0^1 e^{y^4} y^3\,dy =$$

$$= \frac{1}{4} e^{y^4} \Big|_{y=0}^{y=1} = \frac{1}{4}(e-1).$$

$$\iint_{A_2} f(x,y)\,dx\,dy = \int_{-1}^{0} \left(\int_{y^3}^{0} e^{y^4}\,dx \right) dy = \int_{-1}^{0} e^{y^4}(-y^3)\,dy =$$

$$= -\frac{1}{4} e^{y^4} \Big|_{y=-1}^{y=0} = -\frac{1}{4}(1-e) = \frac{1}{4}(e-1).$$

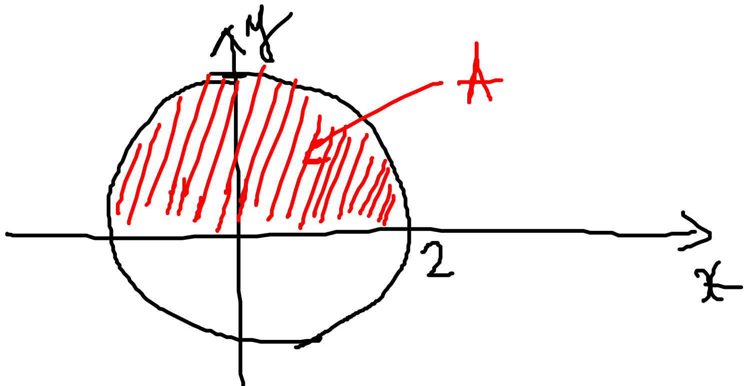
$$\iint_A f(x,y)\,dx\,dy = \frac{1}{4}(e-1) + \frac{1}{4}(e-1) = \frac{1}{2}(e-1). \ \square$$

Obs.: În exercițiile în care calculăm integrale prin schimbare de variabilă nu vom mai arăta că A este mulțime măsurabilă Jordan (și nici compactă) și că f este integrabilă Riemann.

5. Determinați:

a) $\iint_A e^{-x^2-y^2}\,dx\,dy$, unde $A=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 4,\ y\ge 0\}$.

Sol.:



Fie $f:A\to\mathbb{R}$, $f(x,y)=e^{-x^2-y^2}$.

S.V. $\begin{cases} x = r\cos\theta \\ y=r\sin\theta\end{cases}$, $r\in[0,\infty)$, $\theta\in[0,2\pi]$.

$(x,y)\in A \iff \begin{cases}x^2+y^2\le 4\\ y\ge 0\end{cases} \iff \begin{cases} r^2\le 4\\ r\sin\theta\ge 0\end{cases} \iff \begin{cases} r\in[0,2]\\ \theta\in[0,\pi]\end{cases}$.

$B=[0,2]\times[0,\pi]$.

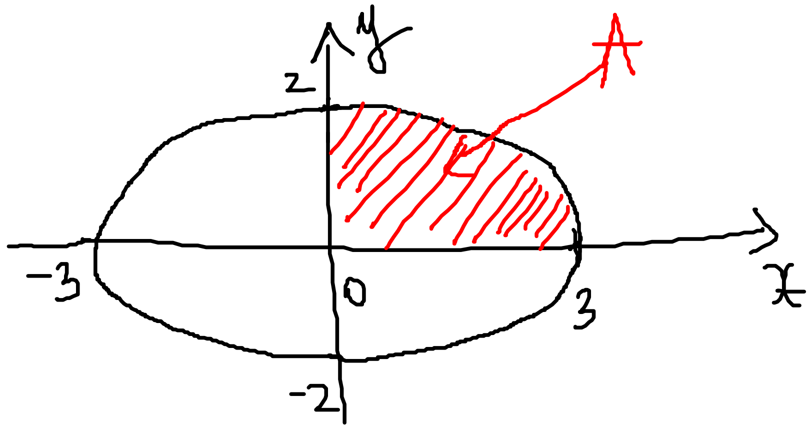
$\iint_A f(x,y)\,dx\,dy = \iint_B r\, f(r\cos\theta, r\sin\theta)\,dr\,d\theta =$

$$= \int_0^2 \left(\int_0^{\pi} r e^{-r^2} \, d\theta \right) dr = \int_0^2 \left(\pi r e^{-r^2} \right) dr = -\frac{\pi}{2} \int_0^2 (-2r) e^{-r^2} \, dr =$$

$$= -\frac{\pi}{2} e^{-r^2} \Big|_{r=0}^{r=2} = -\frac{\pi}{2}\left(e^{-4}-1\right) = \frac{\pi}{2}\left(1-e^{-4}\right). \quad \square$$

b) $\iint_A \sqrt{1-\frac{x^2}{9}-\frac{y^2}{4}} \, dx\, dy$, unde $A = \{(x,y)\in\mathbb{R}^2 \mid \frac{x^2}{9}+\frac{y^2}{4}\le 1,$
$x \ge 0, y \ge 0\}$.

<u>Sol</u>.:



Fie $f : A \to \mathbb{R}$, $f(x,y) = \sqrt{1-\frac{x^2}{9}-\frac{y^2}{4}}$.

S.V. $\begin{cases} x = 3r\cos\theta \\ y = 2r\sin\theta \end{cases}$, $r \in [0,\infty)$, $\theta \in [0, 2\pi]$.

$$(x,y)\in A \iff \begin{cases} \frac{x^2}{9}+\frac{y^2}{4} \le 1 \\ x \ge 0 \\ y \ge 0 \end{cases} \iff \begin{cases} \frac{\cancel{9} r^2\cos^2\theta}{\cancel{9}} + \frac{\cancel{4} r^2\sin^2\theta}{\cancel{4}} \le 1 \\ r\cos\theta \ge 0 \\ r\sin\theta \ge 0 \end{cases} \iff$$

$$\iff \begin{cases} r^2 \le 1 \\ \theta \in \left[0, \frac{\pi}{2}\right] \end{cases} \iff \begin{cases} r \in [0,1] \\ \theta \in \left[0, \frac{\pi}{2}\right]. \end{cases}$$

$$B = [0,1] \times \left[0, \frac{\pi}{2}\right].$$

$$\iint_A f(x,y)\, dx\, dy = \iint_B 3 \cdot 2 \cdot r\, f(3r\cos\theta, 2r\sin\theta)\, dr\, d\theta =$$

$$= \int_0^1 \left(\int_0^{\frac{\pi}{2}} 6r\sqrt{1-r^2}\, d\theta \right) dr = \int_0^1 \left(\left(6r\sqrt{1-r^2}\right)\theta \Big|_{\theta=0}^{\theta=\frac{\pi}{2}} \right) dr =$$

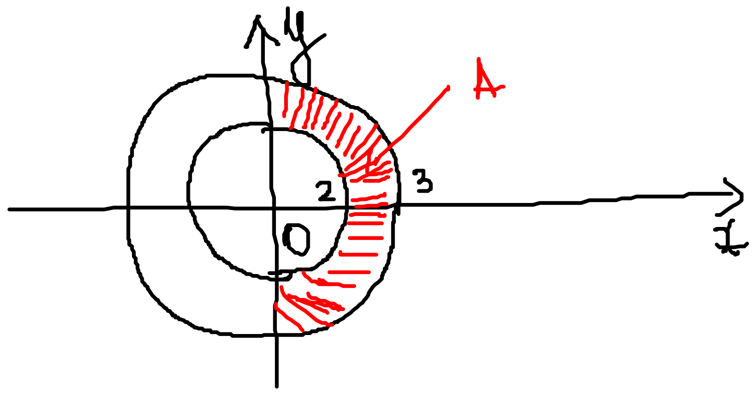
$$= \frac{\pi}{2} \int_0^1 6r\,(1-r^2)^{\frac{1}{2}}\, dr = -\frac{3\pi}{2} \int_0^1 (-2r)(1-r^2)^{\frac{1}{2}}\, dr = -\frac{3\pi}{2} \cdot \frac{(1-r^2)^{\frac{1}{2}+1}}{\frac{1}{2}+1} \Bigg|_{r=0}^{r=1} =$$

$$= -\frac{3\pi}{2}\left(0 - \frac{2}{3}\right) = \frac{\cancel{3}\pi}{\cancel{2}} \cdot \frac{\cancel{2}}{\cancel{3}} = \pi. \quad \square$$

6. determinați:

a) $\iint_A x\,dx\,dy$, unde $A=\{(x,y)\in\mathbb{R}^2 \mid 4\le x^2+y^2\le 9,\ x\ge 0\}$.

Sol.:



Fie $f: A\to\mathbb{R}$, $f(x,y)=x$.

S.V. $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$, $r\in[0,\infty)$, $\theta\in[0,2\pi]$.

$(x,y)\in A \Leftrightarrow \begin{cases} 4\le x^2+y^2\le 9 \\ x\ge 0 \end{cases} \Leftrightarrow \begin{cases} 4\le r^2\le 9 \\ r\cos\theta\ge 0 \end{cases} \Leftrightarrow \begin{cases} r\in[2,3] \\ \theta\in\left[0,\frac{\pi}{2}\right]\cup\left[\frac{3\pi}{2},2\pi\right] \end{cases}$.

$B=[2,3]\times\left(\left[0,\frac{\pi}{2}\right]\cup\left[\frac{3\pi}{2},2\pi\right]\right)$.

$$\iint_A f(x,y)\,dx\,dy=\iint_B r f(r\cos\theta, r\sin\theta)\,dr\,d\theta=$$

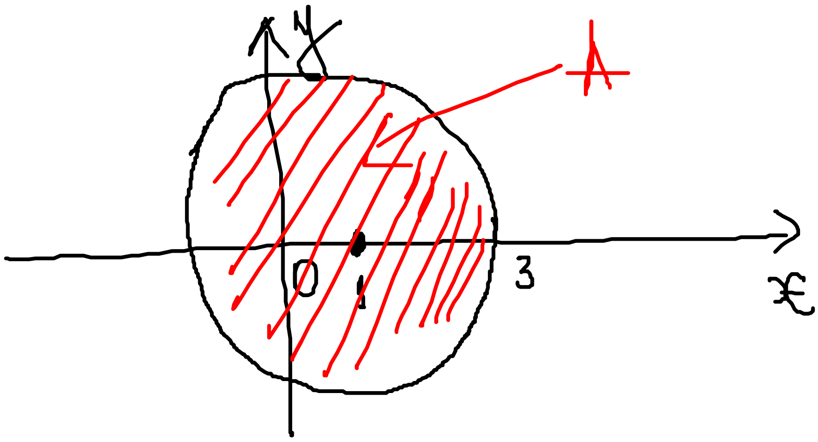
$$=\int_2^3\left(\int_0^{\frac{\pi}{2}} r\cdot r\cos\theta\,d\theta+\int_{\frac{3\pi}{2}}^{2\pi} r\cdot r\cos\theta\,d\theta\right)dr=$$

$$=\int_2^3\left(r^2\sin\theta\Big|_{\theta=0}^{\theta=\frac{\pi}{2}}+r^2\sin\theta\Big|_{\theta=\frac{3\pi}{2}}^{\theta=2\pi}\right)dr=\int_2^3\left(r^2+r^2\right)dr=$$

$$=2\,\frac{r^3}{3}\Big|_{r=2}^{r=3}=2\cdot\frac{19}{3}=\frac{38}{3}.\ \square$$

b) $\iint_A y \, dx\,dy$, unde $A = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 2x+3\}$.

Sol.: $x^2+y^2 \le 2x+3 \Leftrightarrow x^2+y^2-2x-3 \le 0 \Leftrightarrow (x^2-2x+1)-1+y^2-3 \le$
$\le 0 \Leftrightarrow (x-1)^2+y^2 \le 4$.



Fie $f : A \to \mathbb{R}$, $f(x,y) = y$.

S.V. $\begin{cases} x = 1 + r\cos\theta \\ y = r\sin\theta \end{cases}$, $r \in [0,\infty)$, $\theta \in [0,2\pi]$.

$(x,y) \in A \Leftrightarrow (x-1)^2+y^2 \le 4 \Leftrightarrow r^2 \le 4 \Leftrightarrow r \in [0,2]$.

$B = [0,2] \times [0,2\pi]$.

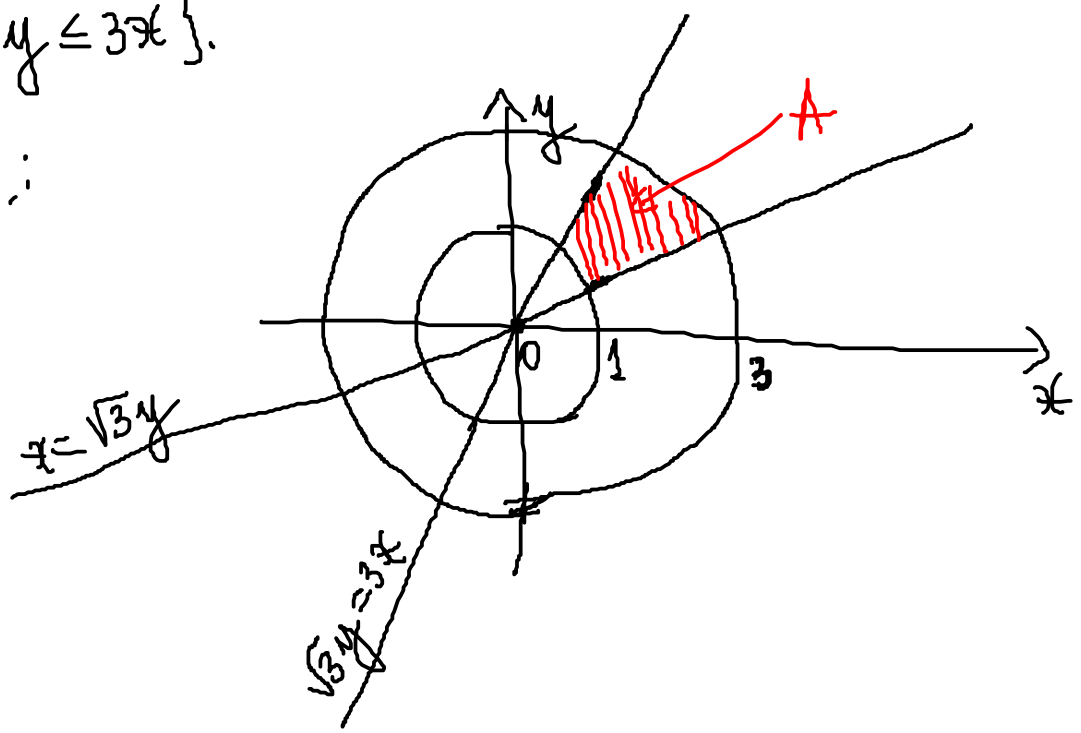
$$\iint_A f(x,y)\,dx\,dy = \iint_B r\, f(1+r\cos\theta, r\sin\theta)\,dr\,d\theta =$$

$$= \int_0^2 \left(\int_0^{2\pi} r\, r\sin\theta \, d\theta \right) dr = \int_0^2 \left(r^2(-\cos\theta) \Big|_{\theta=0}^{\theta=2\pi} \right) dr =$$

$$= \int_0^2 0 \, dr = 0.$$

c) $\iint_A \operatorname{arctg}\frac{y}{x}\,dx\,dy$, unde $A=\{(x,y)\in\mathbb{R}^2 \mid 1\le x^2+y^2\le 9,\ x\le\sqrt{3}y\le 3x\}$.

<u>Sol</u>:



Fie $f:A\to\mathbb{R}$, $f(x,y)=\operatorname{arctg}\frac{y}{x}$.

S.V. $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$, $r\in[0,\infty)$, $\theta\in[0,2\pi]$.

$(x,y)\in A \iff \begin{cases} 1\le x^2+y^2\le 9 \\ x\le\sqrt{3}y \\ \sqrt{3}y\le 3x \end{cases} \iff \begin{cases} 1\le r^2\le 9 \\ r\cos\theta\le\sqrt{3}\,r\sin\theta \\ r\sin\theta\le\sqrt{3}\,r\cos\theta \end{cases} \iff$

$\iff \begin{cases} r\in[1,3] \\ \operatorname{tg}\theta\ge\frac{1}{\sqrt{3}} \\ \operatorname{tg}\theta\le\sqrt{3} \end{cases} \iff \begin{cases} r\in[1,3] \\ \theta\in\left[\frac{\pi}{6},\frac{\pi}{3}\right] \end{cases}$.

$B=[1,3]\times\left[\frac{\pi}{6},\frac{\pi}{3}\right]$.

$\iint_A f(x,y)\,dx\,dy=\iint_B r\,f(r\cos\theta,r\sin\theta)\,dr\,d\theta=$

$$= \int_1^3 \left(\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} r \operatorname{arctg}\left(\frac{\cancel{r} \sin\theta}{\cancel{r} \cos\theta}\right) d\theta \right) dr = \int_1^3 \left(\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} r \operatorname{arctg}(\operatorname{tg}\theta)\, d\theta \right) dr =$$

$$= \int_1^3 \left(\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} r\theta \, d\theta \right) dr = \int_1^3 \left(r \frac{\theta^2}{2} \Big|_{\theta = \frac{\pi}{6}}^{\theta = \frac{\pi}{3}} \right) dr =$$

$$= \frac{1}{2} \int_1^3 r \left(\frac{\pi^2}{9} - \frac{\pi^2}{36} \right) dr = \frac{\cancel{3}\pi^2}{\cancel{36}_{12} \cdot 2} \int_1^3 r \, dr = \frac{\pi^2}{24} \cdot \frac{r^2}{2} \Big|_{r=1}^{r=3} =$$

$$= \frac{\pi^2}{\cancel{24}_3} \cdot \frac{\cancel{8}}{2} = \frac{\pi^2}{6}. \quad \square$$

<u>Obs.</u>: 1. Pentru acest examen, exercițiile cu integrale triple vor fi formulate astfel încât să nu fie necesară reprezentarea grafică a mulțimii A.

2. Atunci când calculăm integrale triple nu mai arătam că A este mulțime măsurabilă Jordan (și nici compactă) și că f este integrabilă Riemann.

3. Considerând toate cele discutate, singura situație în care arătam că A este măsurabilă Jordan (și compactă) și că f este integrabilă Riemann este cea în care calculăm integrale duble fără schimbare de variabilă.

7. Determinați:

a) $\iiint_A (xyz+y^2)\,dx\,dy\,dz$, unde $A=[-1,1]\times[2,3]\times[0,1]$.

Sol.: Fie $f: A\to\mathbb{R}$, $f(x,y,z)=xyz+y^2$.

$$\iiint_A f(x,y,z)\,dx\,dy\,dz=\int_{-1}^{1}\left(\int_2^3\left(\int_0^1 (xyz+y^2)\,dz\right)dy\right)dx=$$

$$=\int_{-1}^{1}\left(\int_2^3\left(xy\frac{z^2}{2}\Big|_{z=0}^{z=1}+y^2z\Big|_{z=0}^{z=1}\right)dy\right)dx=$$

$$=\int_{-1}^{1}\left(\int_2^3\left(\frac{xy}{2}+y^2\right)dy\right)dx=\int_{-1}^{1}\left(\frac{x}{2}\cdot\frac{y^2}{2}\Big|_{y=2}^{y=3}+\frac{y^3}{3}\Big|_{y=2}^{y=3}\right)dx=$$

$$=\int_{-1}^{1}\left(\frac{5}{4}x+\frac{19}{3}\right)dx=\frac{5}{4}\cdot\frac{x^2}{2}\Big|_{x=-1}^{x=1}+\frac{19}{3}x\Big|_{x=-1}^{x=1}=0+\frac{19}{3}\cdot 2=$$

$$=\frac{38}{3}.\ \square$$

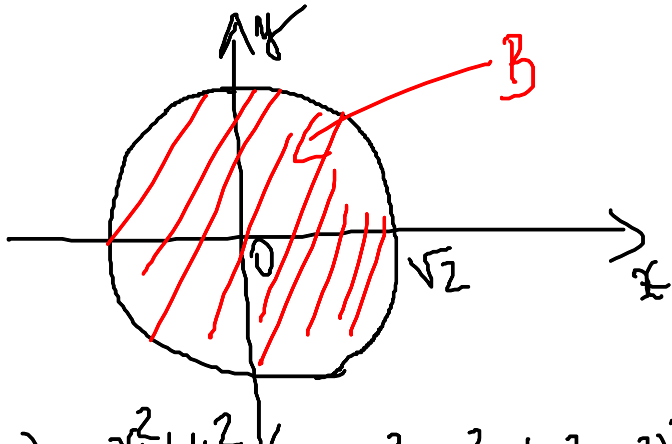
b) $\iiint_A (x^2+y^2)z\,dx\,dy\,dz$, unde $A=\{(x,y,z)\in\mathbb{R}^3 \mid (x,y)\in B,\ x^2+y^2\le z\le\sqrt{6-x^2-y^2}\}$ și $B=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 2\}$.

Sol.: Fie $f: A\to\mathbb{R}$, $f(x,y,z)=(x^2+y^2)z$.

$$\iiint_A f(x,y,z)\,dx\,dy\,dz=\iint_B\left(\int_{x^2+y^2}^{\sqrt{6-x^2-y^2}}(x^2+y^2)z\,dz\right)dx\,dy=$$

$$= \iint_B \left((x^2+y^2) \frac{z^2}{2} \Big|_{z = x^2+y^2}^{z=\sqrt{6-x^2-y^2}} \right) dx\,dy =$$

$$= \iint_B \frac{x^2+y^2}{2} \left(6 - x^2 - y^2 - (x^2+y^2)^2 \right) dx\,dy.$$



Fie $g : B \to \mathbb{R}$, $g(x,y) = \frac{x^2+y^2}{2}\left(6 - x^2 - y^2 - (x^2+y^2)^2\right)$.

S. V. $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $r \in [0, \infty)$, $\theta \in [0, 2\pi]$.

$(x,y) \in B \Leftrightarrow x^2+y^2 \leq 2 \Leftrightarrow r^2 \leq 2 \Leftrightarrow r \in [0, \sqrt{2}]$.

$C = [0, \sqrt{2}] \times [0, 2\pi]$.

$$\iint_B g(x,y)\,dx\,dy = \iint_C r\, g(r\cos\theta, r\sin\theta)\,dr\,d\theta =$$

$$= \int_0^{\sqrt{2}} \left(\int_0^{2\pi} r \cdot \frac{r^2}{2} \cdot (6 - r^2 - r^4)\, d\theta \right) dr = \int_0^{\sqrt{2}} \left(\frac{6r^3 - r^5 - r^7}{2}\, \theta \Big|_{\theta=0}^{\theta=2\pi} \right) dr =$$

$$= \pi \int_0^{\sqrt{2}} (6r^3 - r^5 - r^7)\, dr = \pi \cdot \cancel{6}^{\,3} \frac{r^4}{\cancel{4}_{\,2}} \Big|_{r=0}^{r=\sqrt{2}} - \pi \frac{r^6}{6} \Big|_{r=0}^{r=\sqrt{2}} - \pi \frac{r^8}{8} \Big|_{r=0}^{r=\sqrt{2}} =$$

$$= \frac{12\pi}{2} - \frac{\cancel{8}^{4}\pi}{\cancel{6}_{3}} - \frac{16\pi}{8} = 6\pi - \frac{4\pi}{3} - 2\pi = \overset{3}{\frown}4\pi - \frac{4\pi}{3} = \frac{8\pi}{3}. \square$$

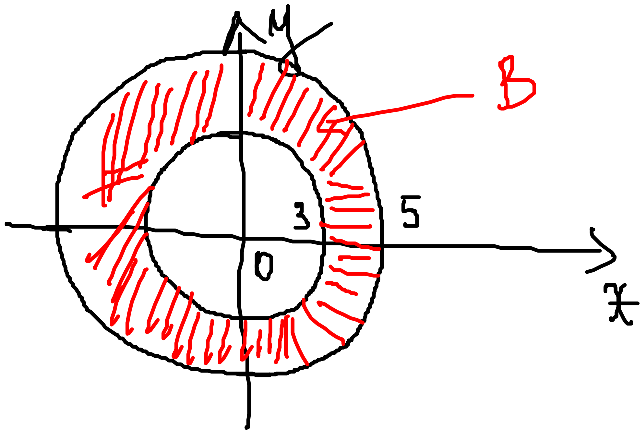
c) $\iiint_A xyz\,dx\,dy\,dz$, unde $A = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in B,$ $\sqrt{x^2+y^2} \leq z \leq 5\}$ și $B = \{(x,y) \in \mathbb{R}^2 \mid 9 \leq x^2+y^2 \leq 25\}$.

<u>Sol</u>.: Fie $f: A \to \mathbb{R}$, $f(x,y,z) = xyz$.

$$\iiint_A f(x,y,z)\,dx\,dy\,dz = \iint_B \left(\int_{\sqrt{x^2+y^2}}^{5} xyz\,dz \right) dx\,dy =$$

$$= \iint_B \left(xy \frac{z^2}{2} \Bigg|_{z=\sqrt{x^2+y^2}}^{z=5} \right) dx\,dy = \iint_B \frac{xy}{2}(25-x^2-y^2)\,dx\,dy.$$

Fie $g: B \to \mathbb{R}$, $g(x,y) = \frac{xy}{2}(25-x^2-y^2)$.



S.V. $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $r \in [0,\infty)$, $\theta \in [0,2\pi]$.

$(x,y) \in B \iff 9 \le x^2+y^2 \le 25 \iff 9 \le r^2 \le 25 \iff r \in [3,5]$.

$C = [3,5] \times [0,2\pi]$.

$$\iint_B g(x,y)\,dx\,dy = \iint_C r\, g(r\cos\theta, r\sin\theta)\,dr\,d\theta =$$

$$= \int_3^5 \left(\int_0^{2\pi} r \cdot \frac{r\cos\theta\, r\sin\theta}{2} (25-r^2)\,d\theta \right) dr =$$

$$= \int_3^5 \left(\int_0^{2\pi} \frac{r^3(25-r^2)}{2} \sin\theta (\sin\theta)'\,d\theta \right) dr =$$

$$= \int_3^5 \left(\frac{r^3(25-r^2)}{2} \cdot \frac{\sin^2\theta}{2} \Bigg|_{\theta=0}^{\theta=2\pi} \right) dr = \int_3^5 0\,dr = 0. \quad \square$$

d) $\iiint_A x\,dx\,dy\,dz$, unde $A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \le 1, y \ge 0\}$.

Sol.: Fie $f: A \to \mathbb{R}$, $f(x,y,z) = x$.

S.V. $\begin{cases} x = r\cos\theta \sin\varphi \\ y = r\sin\theta\sin\varphi, \quad r \in [0,\infty),\ \theta \in [0,2\pi],\ \varphi \in [0,\pi]. \\ z = r\cos\varphi \end{cases}$

$(x,y,z) \in A \iff \begin{cases} x^2+y^2+z^2 \le 1 \\ y \ge 0 \end{cases} \iff \begin{cases} r^2 \le 1 \\ r\sin\theta\sin\varphi \ge 0 \end{cases} \iff \begin{cases} r \in [0,1] \\ \theta \in [0,\pi]. \end{cases}$

$$C = [0,1] \times [0,\pi] \times [0,\pi].$$

$$\iiint_A f(x,y,z)\,dx\,dy\,dz =$$

$$= \iiint_C r^2 \sin\varphi \, f(r\cos\theta\sin\varphi, r\sin\theta\sin\varphi, r\cos\varphi)\,dr\,d\theta\,d\varphi =$$

$$= \int_0^1 \left(\int_0^\pi \left(\int_0^\pi r^2 \sin\varphi \, r\cos\theta \sin\varphi \, d\varphi \right) d\theta \right) dr =$$

$$= \int_0^1 \left(\int_0^\pi \underbrace{\left(\int_0^\pi r^3 \cos\theta \sin^2\varphi \, d\varphi \right)} d\theta \right) dr = (*).$$

$$\underbrace{\int_0^\pi r^3 \cos\theta \sin^2\varphi \, d\varphi} = r^3\cos\theta \int_0^\pi \frac{1-\cos 2\varphi}{2}\,d\varphi = \frac{\pi r^3 \cos\theta}{2} -$$

$$1 - 2\sin^2\varphi = \cos 2\varphi$$

$$\sin^2\varphi = \frac{1-\cos 2\varphi}{2}$$

$$-\frac{r^3\cos\theta}{2}\int_0^\pi \cos 2\varphi \, d\varphi = \frac{\pi r^3 \cos\theta}{2} - \frac{r^3\cos\theta}{2} \cdot \left. \frac{\sin 2\varphi}{2} \right|_{\varphi=0}^{\varphi=\pi} =$$

$$= \underbrace{\frac{\pi r^3 \cos\theta}{2}}.$$

$$(*) = \int_0^1 \left(\int_0^\pi \frac{\pi r^3 \cos\theta}{2}\,d\theta \right) dr = \int_0^1 \left(\frac{\pi r^3}{2} \left. \sin\theta \right|_{\theta=0}^{\theta=\pi} \right) dr =$$

$= \int_0^1 0 \, dr = 0. \square$

e) $\iiint_A \left(\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16}\right) dx\,dy\,dz$, unde $A = \{(x,y,z) \in \mathbb{R}^3 \mid \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} \le 1, z \le 0\}$.

Sol.: Fie $f: A \to \mathbb{R}$, $f(x,y,z) = \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16}$.

S.V. $\begin{cases} x = 2r\cos\theta\sin\varphi \\ y = 3r\sin\theta\sin\varphi \\ z = 4r\cos\varphi \end{cases}$, $r \in [0,\infty)$, $\theta \in [0, 2\pi]$, $\varphi \in [0,\pi]$.

$(x,y,z) \in A \Leftrightarrow \begin{cases} \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} \le 1 \\ z \le 0 \end{cases} \Leftrightarrow \begin{cases} r^2 \le 1 \\ 4r\cos\varphi \le 0 \end{cases} \Leftrightarrow \begin{cases} r \in [0,1] \\ \varphi \in [\frac{\pi}{2}, \pi] \end{cases}$.

$C = [0,1] \times [0, 2\pi] \times [\frac{\pi}{2}, \pi]$.

$\iiint_A f(x,y,z)\,dx\,dy\,dz =$

$= \iiint_C 2 \cdot 3 \cdot 4\, r^2 \sin\varphi\, f(2r\cos\theta\sin\varphi, 3r\sin\theta\sin\varphi, 4r\cos\varphi)\,dr\,d\theta\,d\varphi =$

$= \int_0^1 \left(\int_0^{2\pi} \left(\int_{\frac{\pi}{2}}^{\pi} 24 r^2 (\sin\varphi) r^2 \, d\varphi\right) d\theta\right) dr =$

$$= \int_0^1 \left(\int_0^{2\pi} \left(24r^4 (-\cos\varphi) \Big|_{\varphi=\frac{\pi}{2}}^{\varphi=\pi} \right) d\theta \right) dr =$$

$$= \int_0^1 \left(\int_0^{2\pi} 24r^4 \, d\theta \right) dr = \int_0^1 24r^4 \cdot 2\pi \, dr = 48\pi \, \frac{r^5}{5} \Big|_{r=0}^{r=1} = \frac{48\pi}{5} . \square$$