Seminar 10

1. Fie $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=\begin{cases}\dfrac{x^7y^8}{\sqrt{x^{28}+y^{28}}}; & (x,y)\neq(0,0)\\ 0; & (x,y)=(0,0).\end{cases}$

a) Studiați continuitatea funcției f;

b) Det. $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$;

c) Studiați diferențiabilitatea funcției f.

Sol.: a) f continuă pe $\mathbb{R}^2\setminus\{(0,0)\}$ (operații cu funcții elementare).

Studiem continuitatea lui f în $(0,0)$.

Fie $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$.

$$|f(x,y)-f(0,0)|=\left|\frac{x^7y^8}{\sqrt{x^{28}+y^{28}}}-0\right|=\frac{|x^7y^8|}{\sqrt{x^{28}+y^{28}}}=$$

$$=|y|\underbrace{\frac{|x^7y^7|}{\sqrt{x^{28}+y^{28}}}}_{\leq\frac{1}{\sqrt{2}}}\leq\frac{1}{\sqrt{2}}|y|\xrightarrow[(x,y)\to(0,0)]{}0\Rightarrow f \text{ cont. în } (0,0).$$

(Explicație: $\dfrac{x^{28}+y^{28}}{2}\geq\sqrt{x^{28}y^{28}}=$
$=|x^{14}y^{14}|=x^{14}y^{14}\Leftrightarrow x^{28}+y^{28}\geq 2x^{14}y^{14}\Leftrightarrow\sqrt{x^{28}+y^{28}}\geq$
$\geq\sqrt{2}\sqrt{x^{14}y^{14}}=\sqrt{2}|x^7y^7|\Leftrightarrow\dfrac{|x^7y^7|}{\sqrt{x^{28}+y^{28}}}\leq\dfrac{1}{\sqrt{2}}$)

b) Fie $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$.

$$\frac{\partial f}{\partial x}(x,y)=\frac{(x^7y^8)'_x\sqrt{x^{28}+y^{28}}-x^7y^8\left(\sqrt{x^{28}+y^{28}}\right)'_x}{\left(\sqrt{x^{28}+y^{28}}\right)^2}=$$

$$=\frac{7x^6y^8\sqrt{x^{28}+y^{28}}-x^7y^8\cdot\frac{1}{2\sqrt{x^{28}+y^{28}}}\cdot 28x^{27}}{x^{28}+y^{28}}.$$

$$\frac{\partial f}{\partial y}(x,y)=\frac{(x^7y^8)'_y\sqrt{x^{28}+y^{28}}-x^7y^8\left(\sqrt{x^{28}+y^{28}}\right)'_y}{\left(\sqrt{x^{28}+y^{28}}\right)^2}=$$

$$=\frac{8x^7y^7\sqrt{x^{28}+y^{28}}-x^7y^8\frac{1}{2\sqrt{x^{28}+y^{28}}}\cdot 28y^{27}}{x^{28}+y^{28}}.$$

$$\frac{\partial f}{\partial x}(0,0)=\lim_{t\to 0}\frac{f((0,0)+te_1)-f(0,0)}{t}=$$

$$=\lim_{t\to 0}\frac{f((0,0)+t(1,0))-f(0,0)}{t}=\lim_{t\to 0}\frac{f(t,0)-f(0,0)}{t}=$$

$$=\lim_{t\to 0}\frac{\frac{t^7\cdot 0^8}{\sqrt{t^{28}+0^{28}}}-0}{t}=0.$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f((0,0)+t e_2) - f(0,0)}{t} =$$

$$= \lim_{t\to 0} \frac{f((0,0)+t(0,1)) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(0,t) - f(0,0)}{t} =$$

$$= \lim_{t\to 0} \frac{\frac{0^7 \cdot t^8}{\sqrt{0^{28}+t^{28}}} - 0}{t} = 0.$$

c) $\left.\begin{array}{l}\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \text{ cont. pe } \mathbb{R}^2 \setminus \{(0,0)\} \\ \mathbb{R}^2 \setminus \{(0,0)\} \text{ deschisă}\end{array}\right| \Longrightarrow f \text{ dif. pe } \mathbb{R}^2 \setminus \{(0,0)\}.$

Studiem diferențiabilitatea lui f în $(0,0)$.

Dacă f ar fi dif. în $(0,0)$, atunci $df(0,0): \mathbb{R}^2 \to \mathbb{R}$,

$$df(0,0)(u,v) = \left[\begin{pmatrix} \underbrace{\frac{\partial f}{\partial x}(0,0)}_{=0} & \underbrace{\frac{\partial f}{\partial y}(0,0)}_{=0} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \right] = 0.$$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - df(0,0)((x,y)-(0,0))}{\|(x,y)-(0,0)\|} =$$

$$= \lim_{(x,y)\to(0,0)} \frac{\frac{x^7 y^8}{\sqrt{x^{28}+y^{28}}} - 0 - 0}{\sqrt{x^2+y^2}} =$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^7 y^8}{\sqrt{x^{28}+y^{28}}\sqrt{x^2+y^2}}.$$

Alegem $(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n}\right)$ $\forall n \in \mathbb{N}^*$. Avem $\lim_{n\to\infty}(x_n, y_n) = (0,0)$ și

$$\lim_{n\to\infty} \frac{x_n^7 y_n^8}{\sqrt{x_n^{28}+y_n^{28}}\sqrt{x_n^2+y_n^2}} = \lim_{n\to\infty} \frac{\frac{1}{\cancel{n^{15}}}}{\frac{\sqrt{2}}{\cancel{n^{14}}}\cdot\frac{\sqrt{2}}{\cancel{n}}} = \frac{1}{2} \neq 0.$$

Deci $\lim_{(x,y)\to(0,0)} \frac{x^7 y^8}{\sqrt{x^{28}+y^{28}}\sqrt{x^2+y^2}} \neq 0$, i.e. f nu e dif. în $(0,0)$. $\square$

2. Fie $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} xy \sin\frac{1}{x^2+y^2}; & (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\} \\ 0 & ; (x,y) = (0,0). \end{cases}$

a) Stud. cont. funcției f;

b) Det. $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ și stud. cont. lor;

c) Stud. diferențiabilitatea funcției f.

<u>Sol</u>: a) f cont. pe $\mathbb{R}^2 \setminus \{(0,0)\}$ (operații cu funcții elementare).

Studiem continuitatea lui f în $(0,0)$.

Fie $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

$$\left|f(x,y) - f(0,0)\right| = \left|xy \sin\frac{1}{x^2+y^2} - 0\right| = |xy| \underbrace{\left|\sin\frac{1}{x^2+y^2}\right|}_{\leq 1} \leq$$

$$\leq |xy| \xrightarrow[(x,y)\to(0,0)]{} 0 \Rightarrow f \text{ cont. în } (0,0).$$

b) Fie $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

$$\frac{\partial f}{\partial x}(x,y) = y \sin\frac{1}{x^2+y^2} + xy\left(\cos\frac{1}{x^2+y^2}\right) \cdot \left(-\frac{1}{(x^2+y^2)^2} \cdot 2x\right) =$$

$$= y \sin\frac{1}{x^2+y^2} - \frac{2x^2y}{(x^2+y^2)^2} \cos\frac{1}{x^2+y^2}.$$

$$\frac{\partial f}{\partial y}(x,y) = x \sin\frac{1}{x^2+y^2} + xy\left(\cos\frac{1}{x^2+y^2}\right) \cdot \left(-\frac{2y}{(x^2+y^2)^2}\right) =$$

$$= x \sin\frac{1}{x^2+y^2} - \frac{2xy^2}{(x^2+y^2)^2} \cos\frac{1}{x^2+y^2}.$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f((0,0)+te_1) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(t,0) - f(0,0)}{t} =$$

$$= \lim_{t\to 0} \frac{0-0}{t} = 0.$$

$$\frac{\partial f}{\partial y}(0,0)=\lim_{t\to 0}\frac{f((0,0)+te_2)-f(0,0)}{t}=\lim_{t\to 0}\frac{f(0,t)-f(0,0)}{t}=$$

$$=\lim_{t\to 0}\frac{0-0}{t}=0.$$

Studiem continuitatea derivatelor parțiale.

$$\frac{\partial f}{\partial x}(x,y)=\begin{cases} y\cdot\sin\frac{1}{x^2+y^2}-\frac{2x^2y}{(x^2+y^2)^2}\cos\frac{1}{x^2+y^2} & ;(x,y)\neq(0,0)\\ 0 & ;(x,y)=(0,0).\end{cases}$$

$\frac{\partial f}{\partial x}$ cont. pe $\mathbb{R}^2\setminus\{(0,0)\}$ (operații cu funcții elementare).

Studiem continuitatea lui $\frac{\partial f}{\partial x}$ în $(0,0)$.

Alegem $(x_n,y_n)=\left(\frac{1}{2\sqrt{n\pi}},\frac{1}{2\sqrt{n\pi}}\right)\ \forall n\in\mathbb{N}^*$. Avem

$\lim_{n\to\infty}(x_n,y_n)=(0,0)$ și $\lim_{n\to\infty}\frac{\partial f}{\partial x}(x_n,y_n)=$

$$=\lim_{n\to\infty}\left(\frac{1}{2\sqrt{n\pi}}\underbrace{\sin(2n\pi)}_{\substack{\parallel\\0}}-\frac{\cancel{2}\cdot\frac{1}{4n\pi}\cdot\frac{1}{\cancel{2}\sqrt{n\pi}}}{\left(\frac{1}{4n\pi}+\frac{1}{4n\pi}\right)^2}\underbrace{\cos(2n\pi)}_{\substack{\parallel\\1}}\right)=$$

$$=\lim_{n\to\infty}\left(-\frac{1}{4n\pi\sqrt{n\pi}}\cdot\frac{4n^2\pi^2}{1}\right)=-\infty\neq 0=\frac{\partial f}{\partial x}(0,0).$$

Deci $\frac{\partial f}{\partial x}$ nu este cont. în $(0,0)$.

Analog se arată că $\frac{\partial f}{\partial y}$ este cont. pe $\mathbb{R}^2\setminus\{(0,0)\}$ și nu este cont. în $(0,0)$ (folosim $((x_n,y_n))_n$ de mai sus).

c) $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ cont. pe $\mathbb{R}^2\setminus\{(0,0)\}$
$\mathbb{R}^2\setminus\{(0,0)\}$ deschisă $\Bigg\} \Rightarrow f$ dif. pe $\mathbb{R}^2\setminus\{(0,0)\}$.

Studiem dif. lui f în $(0,0)$.

Dacă f ar fi dif. în $(0,0)$, atunci $df(0,0):\mathbb{R}^2\to\mathbb{R}$,

$$df(0,0)(u,v)=\left[\left(\underbrace{\frac{\partial f}{\partial x}(0,0)}_{=0}\quad \underbrace{\frac{\partial f}{\partial y}(0,0)}_{=0}\right)\begin{pmatrix}u\\ v\end{pmatrix}\right]^t=0.$$

$$\lim_{(x,y)\to(0,0)}\frac{f(x,y)-f(0,0)-df(0,0)((x,y)-(0,0))}{\|(x,y)-(0,0)\|}=$$

$$=\lim_{(x,y)\to(0,0)}\frac{xy\sin\frac{1}{x^2+y^2}-0-0}{\sqrt{x^2+y^2}}=\lim_{(x,y)\to(0,0)}\frac{xy\sin\frac{1}{x^2+y^2}}{\sqrt{x^2+y^2}}.$$

Fie $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$.

$$\left|\frac{xy\sin\frac{1}{x^2+y^2}}{\sqrt{x^2+y^2}}-0\right| = \frac{|xy|\overbrace{\left|\sin\frac{1}{x^2+y^2}\right|}^{\le 1}}{\sqrt{x^2+y^2}} \le \frac{|xy|}{\sqrt{x^2+y^2}} =$$

$$= |x|\underbrace{\frac{|y|}{\sqrt{x^2+y^2}}}_{\le 1} \le |x| \xrightarrow[(x,y)\to(0,0)]{} 0,$$

≤ 1 (Explicație: $\sqrt{x^2+y^2} \ge \sqrt{y^2} = |y|$)

Deci $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy\sin\frac{1}{x^2+y^2}}{\sqrt{x^2+y^2}} = 0$, i.e. f dif. în $(0,0)$. $\square$

3. Fie $\varphi: \mathbb{R}^2 \to \mathbb{R}$ o funcție diferențiabilă și $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = \varphi(xy, x^2+y^2-z^2)$. Arătați că $xz\frac{\partial f}{\partial x}(x,y,z) - yz\frac{\partial f}{\partial y}(x,y,z) + (x^2-y^2)\frac{\partial f}{\partial z}(x,y,z) = 0 \ \forall (x,y,z) \in \mathbb{R}^3$.

Sol.:
$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "g"] \arrow[rr, bend right, "f=\varphi\circ g"'] & \mathbb{R}^2 \arrow[r, "\varphi"] & \mathbb{R} \end{tikzcd}$$

Fie $g: \mathbb{R}^3 \to \mathbb{R}^2$, $g(x,y,z) = (xy, x^2+y^2-z^2)$.

Fie $u, v: \mathbb{R}^3 \to \mathbb{R}$, $u(x,y,z) = xy$, $v(x,y,z) = x^2+y^2-z^2$.

Avem $f = \varphi\circ g$.

$$\frac{\partial g}{\partial x}(x,y,z)=\left(\frac{\partial u}{\partial x}(x,y,z),\ \frac{\partial v}{\partial x}(x,y,z)\right)=(y,\ 2x)\quad \forall (x,y,z)\in\mathbb{R}^3.$$

$$\frac{\partial g}{\partial y}(x,y,z)=\left(\frac{\partial u}{\partial y}(x,y,z),\ \frac{\partial v}{\partial y}(x,y,z)\right)=(x,\ 2y)\quad \forall (x,y,z)\in\mathbb{R}^3.$$

$$\frac{\partial g}{\partial z}(x,y,z)=\left(\frac{\partial u}{\partial z}(x,y,z),\ \frac{\partial v}{\partial z}(x,y,z)\right)=(0,\ -2z)\quad \forall (x,y,z)\in\mathbb{R}^3.$$

$$\left.\begin{array}{l}\dfrac{\partial g}{\partial x},\ \dfrac{\partial g}{\partial y},\ \dfrac{\partial g}{\partial z}\ \text{cont. pe } \mathbb{R}^3\\ \mathbb{R}^3 \text{ deschisă}\end{array}\right| \Longrightarrow g \text{ dif. pe } \mathbb{R}^3.$$

$$\left.\begin{array}{l}\varphi \text{ dif. pe } \mathbb{R}^2\\ g \text{ dif. pe } \mathbb{R}^3\end{array}\right| \Longrightarrow f=\varphi\circ g \text{ dif. pe } \mathbb{R}^3.$$

$$\begin{tikzcd}
\mathbb{R}^3 \arrow[r, "g"] \arrow[rr, bend right, "f=\varphi\circ g"'] & \mathbb{R}^2 \arrow[r, "\varphi"] & \mathbb{R}
\end{tikzcd}$$

(x,y,z) $\qquad\qquad$ (u,v)

$$\frac{\partial f}{\partial x}(x,y,z)=\frac{\partial(\varphi\circ g)}{\partial x}(x,y,z)=\frac{\partial \varphi}{\partial u}(g(x,y,z))\cdot\frac{\partial u}{\partial x}(x,y,z)+$$

$$+\frac{\partial \varphi}{\partial v}(g(x,y,z))\,\frac{\partial v}{\partial x}(x,y,z)=\frac{\partial \varphi}{\partial u}(xy,\ x^2+y^2-z^2)\cdot y+$$

$$+\frac{\partial \varphi}{\partial v}(xy,\ x^2+y^2-z^2)\cdot 2x\quad \forall (x,y,z)\in\mathbb{R}^3.$$

$$\frac{\partial f}{\partial y}(x,y,z) = \frac{\partial(\varphi\circ g)}{\partial y}(x,y,z) = \frac{\partial \varphi}{\partial u}\left(g(x,y,z)\right)\cdot \frac{\partial u}{\partial y}(x,y,z) +$$

$$+ \frac{\partial \varphi}{\partial v}\left(g(x,y,z)\right)\cdot \frac{\partial v}{\partial y}(x,y,z) = \frac{\partial \varphi}{\partial u}\left(xy, x^2+y^2-z^2\right)\cdot x +$$

$$+ \frac{\partial \varphi}{\partial v}\left(xy, x^2+y^2-z^2\right)\cdot 2y \quad \forall (x,y,z)\in\mathbb{R}^3.$$

$$\frac{\partial f}{\partial z}(x,y,z) = \frac{\partial(\varphi\circ g)}{\partial z}(x,y,z) = \frac{\partial \varphi}{\partial u}\left(g(x,y,z)\right)\cdot \frac{\partial u}{\partial z}(x,y,z) +$$

$$+ \frac{\partial \varphi}{\partial v}\left(g(x,y,z)\right)\cdot \frac{\partial v}{\partial z}(x,y,z) = \frac{\partial \varphi}{\partial u}\left(xy, x^2+y^2-z^2\right)\cdot 0 +$$

$$+ \frac{\partial \varphi}{\partial v}\left(xy, x^2+y^2-z^2\right)\cdot(-2z) \quad \forall (x,y,z)\in\mathbb{R}^3.$$

$$xz\,\frac{\partial f}{\partial x}(x,y,z) - yz\,\frac{\partial f}{\partial y}(x,y,z) + (x^2-y^2)\,\frac{\partial f}{\partial z}(x,y,z) =$$

$$= xz\left(\frac{\partial \varphi}{\partial u}\left(xy, x^2+y^2-z^2\right)\cdot y + \frac{\partial \varphi}{\partial v}\left(xy, x^2+y^2-z^2\right)\cdot 2x\right) -$$

$$- yz\left(\frac{\partial \varphi}{\partial u}\left(xy, x^2+y^2-z^2\right)\cdot x + \frac{\partial \varphi}{\partial v}\left(xy, x^2+y^2-z^2\right)\cdot 2y\right) +$$

$$+ (x^2-y^2)\left(\frac{\partial \varphi}{\partial u}\left(xy, x^2+y^2-z^2\right)\cdot 0 + \frac{\partial \varphi}{\partial v}\left(xy, x^2+y^2-z^2\right)\cdot(-2z)\right) =$$

$$= \cancel{xyz\,\frac{\partial \varphi}{\partial u}(xy,\, x^2+y^2-z^2)} + \cancel{2x^2 z\,\frac{\partial \varphi}{\partial v}(xy,\, x^2+y^2-z^2)} -$$

$$- \cancel{xyz\,\frac{\partial \varphi}{\partial u}(xy,\, x^2+y^2-z^2)} - \cancel{2y^2 z\,\frac{\partial \varphi}{\partial v}(xy,\, x^2+y^2-z^2)} -$$

$$- \cancel{2x^2 z\,\frac{\partial \varphi}{\partial v}(xy,\, x^2+y^2-z^2)} + \cancel{2y^2 z\,\frac{\partial \varphi}{\partial v}(xy,\, x^2+y^2-z^2)} = 0. \ \square$$