CURS 4 - GAL

Subspații vectoriale

Prop Fie $A \in \mathcal{M}_{m,n}(\mathbb{R})$

$S(A) = \{ x \in \mathbb{R}^n \mid \underset{(m,n)}{A}\underset{(n,1)}{X} = \underset{(m,1)}{0} \} \subset \mathbb{R}^n$

$A = (a_{ij})_{\substack{i=\overline{1,m} \\ j=\overline{1,n}}}$, $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\quad 0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathcal{M}_{m,1}(\mathbb{R})$

a) $S(A)$ subspațiu vectorial ?

b) $\dim S(A) \overset{?}{=} n - rg(A)$

Dem

a) Fie $x, y \in S(A)$, $a, b \in \mathbb{R}$ $\overset{?}{\Longrightarrow} ax + by \in S(A)$

$\left.\begin{matrix} AX = 0 \\ AY = 0 \end{matrix}\right\} \Rightarrow A(aX + bY) = 0 \Rightarrow ax + by \in S(A)$

$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \; Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$\Rightarrow S(A) \subset \mathbb{R}^n$ subsp. vect.

b) $r = rg(A)$

Alegem (fără a restrânge generalitatea) $x_1, \dots, x_r =$ variabile principale

$x_{r+1} = \lambda_1, \dots, x_n = \lambda_p$ var. secundare $p = n - r$

$$\begin{cases} x_1 = \alpha_{11}\lambda_1 + \dots + \alpha_{1p}\lambda_p \\ \vdots \\ x_r = \alpha_{r1}\lambda_1 + \dots + \alpha_{rp}\lambda_p \end{cases}$$

$(x_1, \dots, x_r, x_{r+1}, \dots, x_n) =$

$= (\alpha_{11}\underline{\lambda_1} + \dots + \alpha_{1p}\underline{\underline{\lambda_p}}, \dots, \alpha_{r1}\underline{\lambda_1} + \dots + \alpha_{rp}\underline{\underline{\lambda_p}}, \underline{\lambda_1}, \dots, \underline{\underline{\lambda_p}}) =$

$= \lambda_1 \underbrace{(\alpha_{11}, \dots, \alpha_{r1}, 1, 0, \dots, 0)}_{y_1} + \dots + \lambda_p \underbrace{(\alpha_{1p}, \dots, \alpha_{rp}, 0, \dots, 0, 1)}_{y_p}$

$(x_1, \dots, x_n) \in \langle \{ \overline{y_1, \dots, y_p} \} \rangle \Longrightarrow$

$\mathcal{R} = \{ y_1, \dots, y_p \}$ SG pt $S(A)$

Dem că $\mathcal{R}$ este SLI

Fie $\lambda_1, \dots, \lambda_p \in \mathbb{R}$ aî $\lambda_1 y_1 + \dots + \lambda_p y_p = 0_{\mathbb{R}^n}$.

$(x_1, \dots, x_r, \underline{\lambda_1, \dots, \lambda_p}) = (0, \dots, 0) \Rightarrow \lambda_1 = \dots = \lambda_p = 0$

Deci $\mathcal{R}$ este bază în $S(A) \Longrightarrow$

$\dim_{\mathbb{R}} S(A) = n - r = n - rg\, A$.

<u>Aplicație</u>

$(\mathbb{R}^3, +, \cdot)/\mathbb{R}$, $V' = \left\{ x \in \mathbb{R}^3 \;\middle|\; \begin{matrix} x_1 - x_2 = 0 \\ x_1 + x_3 = 0 \end{matrix} \right\} = S(A)$

a) $\dim V' = 3 - rg A = 3 - 2 = 1$. $\quad A = \left(\boxed{\begin{matrix} 1 & -1 \\ 1 & 0 \end{matrix}} \; \begin{matrix} 0 \\ 1 \end{matrix} \middle| \begin{matrix} 0 \\ 0 \end{matrix} \right)$

(V' = dreaptă care trece prin origine).

b) Det un reper în V'

$x_1 = -x_3$

$x_2 = -x_3$

$V' = \{ (-x_3, -x_3, x_3) \mid x_3 \in \mathbb{R} \} = \langle \{ (-1, -1, 1) \} \rangle$

$\mathcal{R}' = \{ (-1, -1, 1) \}$ SG + SLI $\Rightarrow$ reper.

c) Precizați un subspațiu V'' complementar lui V',

i.e. $\mathbb{R}^3 = V' \oplus V''$

Extindem $\mathcal{R}'$ la un reper în $\mathbb{R}^3$

$\det \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \neq 0 \qquad \mathcal{R}'' = \{ e_1, e_3 \}$

$\mathcal{R} = \mathcal{R}' \cup \mathcal{R}''$ reper în $\mathbb{R}^3$

$V'' = \langle \mathcal{R}'' \rangle$ subsp 2-dim.

c) Să se descompună $x = (1,1,1)$ în raport cu $\mathbb{R}^3 = V' \oplus V''$

$$(1,1,1) = \overbrace{a(-1,-1,1)}^{v' \in V'} + \overbrace{b(1,0,0) + c(0,0,1)}^{v'' \in V''} = (-a+b, -a, a+c)$$

$\begin{cases} -a+b = 1 \\ -a = 1 \\ a+c = 1 \end{cases} \Rightarrow \begin{cases} a = -1 \\ b = 0 \\ c = 2 \end{cases} \qquad \begin{matrix} v' = (1, 1, -1) \\ v'' = (0, 0, 2) \end{matrix} \qquad x = v' + v''$ (scriere unică)

<u>OBS</u> $V' \subseteq V$ subsp vect

$\Rightarrow$ coordonatele vectorilor din V', în rapor cu $\forall$ reper, sunt soluțiile unui SLO i.e A a.î.

$V' = S(A)$.

<u>Aplicație</u>

$(\mathbb{R}^4, +, \cdot)_{\mathbb{R}}$, $V' = \langle \{(1,1,0,0), (1,0,1,-1)\} \rangle$

a) Să se descrie V' printr-un sist. de ec. liniare.

b) $\mathbb{R}^4 = V' \oplus V''$, $V'' = ?$

<u>SOL</u>

a) $x = (x_1, x_2, x_3, x_4) \in V' \Leftrightarrow \exists\, a, b \in \mathbb{R}$ a.î. $\quad rg\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} = 2.$

$x = a(1,1,0,0) + b(1,0,1,-1)$

$(x_1, x_2, x_3, x_4) = (a+b, a, b, -b)$

$$\begin{cases} a + b = x_1 \\ a = x_2 \\ b = x_3 \\ -b = x_4 \end{cases} \qquad A = \left(\begin{array}{cc} \boxed{\begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}} \\ 0 & 1 \\ 0 & -1 \end{array}\right) \left|\begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix}\right. \qquad \Delta_p = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \neq 0$$

$rg\, A = 2$

$$rg A = rg \bar{A} \Longrightarrow \begin{cases} \Delta_1 = \begin{vmatrix} 1 & 1 & x_1 \\ 1 & 0 & x_2 \\ 0 & 1 & x_3 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 & 1 & x_1 \\ 0 & -1 & x_2 - x_1 \\ 0 & 1 & x_3 \end{vmatrix} = 0 \\ \qquad -x_3 - x_2 + x_1 = 0 \\ \Delta_2 = \begin{vmatrix} 1 & 1 & x_1 \\ 1 & 0 & x_2 \\ 0 & -1 & x_4 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 & 1 & x_1 \\ 0 & -1 & x_2 - x_1 \\ 0 & -1 & x_4 \end{vmatrix} = 0 \\ \qquad -x_4 + x_2 - x_1 = 0 \end{cases}$$

$$V' = \left\{ x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \,\middle|\, \begin{cases} x_1 - x_2 - x_3 = 0 \\ -x_1 + x_2 - x_4 = 0 \end{cases} \right\} = S(B)$$

$$B = \begin{pmatrix} 1 & -1 & -1 & 0 \\ -1 & 1 & 0 & -1 \end{pmatrix}$$

$R' = \{(1,1,0,0), (1,0,1,-1)\}$ reper în V'.
Îl extindem la un reper în $\mathbb{R}^4$.

$$\det\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \neq 0 \quad R'' = \{e_1, e_4\} \text{ reper în } V''$$

$$V'' = \langle R'' \rangle$$

$R = R' \cup R''$ reper în $\mathbb{R}^4$.

Morfisme de spații vectoriale

Fie $(V_i, +, \cdot)/_{\mathbb{K}_i}$, $i = \overline{1,2}$ spații vectoriale.
Aplicația $f: V_1 \to V_2$ s.n. <u>aplicație semi-liniară</u>
$\iff$ 1) $f(x+y) = f(x) + f(y)$, $\forall x, y \in V_1$
2) $\exists\, \theta: \mathbb{K}_1 \to \mathbb{K}_2$ izomorfism de corpuri
a.î. $f(\alpha x) = \theta(\alpha) f(x)$, $\forall \alpha \in \mathbb{K}_1$, $\forall x \in V_1$.

Dacă $\mathbb{K}_1 = \mathbb{K}_2$ și $\theta = id_{\mathbb{K}}$, atunci f s.n.
<u>aplicație liniară</u> sau morfism de spații vectoriale.

<u>Exemple</u>

1) $(V_i, +, \cdot)/\mathbb{R}$, $i = \overline{1,2}$ sp. vect
$\theta: \mathbb{R} \to \mathbb{R}$ automorfism de corpuri $\Rightarrow \theta = id_{\mathbb{R}}$.
$f: V_1 \to V_2$ aplicație semi-liniară $\Rightarrow f$ liniară.

2) $(\mathbb{C}^n, +, \cdot)/\mathbb{C}$
$f: \mathbb{C}^n \to \mathbb{C}^n$, $f(z) = \overline{z}$, $\forall z \in \mathbb{C}^n$
$\theta: \mathbb{C} \to \mathbb{C}$, $\theta(\alpha) = \overline{\alpha}$, $\forall \alpha \in \mathbb{C}$. automorfism de corpuri

$f(z+u) = \overline{z+u} = \overline{z} + \overline{u} = f(z) + f(u)$.

$f(\alpha z) = \overline{\alpha z} = \overline{\alpha}\,\overline{z} = \theta(\alpha) f(z)$

f aplicație semi-liniară ; nu este liniară

Aplicații liniare

• $f: V_1 \to V_2$ aplicație liniară

f s.n. izomorfism de sp. vect dacă e bijectivă

• $(V, +, \cdot)/\mathbb{K}$ sp. vect

$f \in End(V) \Leftrightarrow f: V \to V$ liniară

$f \in Aut(V) \Leftrightarrow f \in End(V) + f$ bijecție.

OBS a) $f: V_1 \to V_2$ aplicație liniară $\Rightarrow$

$f: (V_1, +) \to (V_2, +)$ morfism de grupuri $\Rightarrow f(0_{V_1}) = 0_{V_2}$

b) $\begin{tikzcd} V_1 \arrow[r, "f"] \arrow[rr, bend right, "h"'] & V_2 \arrow[r, "g"] & V_3 \end{tikzcd}$ f, g liniare

$\Rightarrow h = g \circ f$ liniară.

Exemple

1) $f: V \to V$, $f(x) = 0_V$ sau $f(x) = x$ apl. lin.

2) $f: \mathbb{R}^n \to \mathbb{R}^m$, $f(x) = y$ apl. lin.

$Y = AX$ $\quad \begin{pmatrix} y_1 \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} \dots a_{1n} \\ a_{m1} \dots a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$\qquad\quad (m, 1) \qquad\quad (m, n) \qquad\quad (n, 1)$

3) $f: \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}$, $f(A) = Tr(A)$ apl. lin.

$f(A) = \det(A)$ <u>nu</u> este apl. liniară

<u>Prop de caracterizare</u> $(V_i, +, \cdot)/_{\mathbb{K}}$, $i=\overline{1,2}$ sp. vect.

$f: V_1 \to V_2$ aplicatie.

f liniară $\Leftrightarrow f(ax+by) = af(x)+bf(y), \forall x,y \in V_1$
$\forall a,b \in \mathbb{K}$

$\Leftrightarrow f\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i f(x_i)$

$\forall a_1, .., a_n \in \mathbb{K}, \forall x_1, .., x_n \in V_1$

<u>Dem</u>

„$\Rightarrow$" Ip. f liniară 1) $f(x+y) = f(x)+f(y), \forall x,y \in V_1$

2) $f(\alpha x) = \alpha f(x), \forall \alpha \in \mathbb{K}, \forall x \in V_1$

$\forall a \in \mathbb{K}, x \in V_1 \Rightarrow ax \in V_1$

$\forall b \in \mathbb{K}, y \in V_1 \Rightarrow by \in V_1$ $\Rightarrow f(ax+by) \overset{1)}{=} f(ax)+f(by)$

$\overset{2)}{=} af(x)+bf(y)$.

„$\Leftarrow$" Ip. $f(ax+by) = af(x)+bf(y), \forall a,b \in \mathbb{K}$
$x,y \in V_1$

Fie $a = b = 1_{\mathbb{K}}$

$f(1_{\mathbb{K}}x + 1_{\mathbb{K}}y) = 1_{\mathbb{K}}f(x) + 1_{\mathbb{K}}f(y) \Rightarrow f(x+y) = f(x)+f(y)$.

$b = 0_{\mathbb{K}} \Rightarrow f(ax+0_{\mathbb{K}}y) = af(x)+0_{\mathbb{K}}f(y)$

$f(ax) = af(x)$

<u>OBS</u> $f: V_1 \to V_2$ apl. liniară

$V' \subset V_1$ subsp vect $\Rightarrow f(V') \subset V_2$ subsp. vect.

$\forall y_1, y_2 \in f(V'), \forall a,b \in \mathbb{K} \overset{?}{\Rightarrow} ay_1+by_2 \in f(V')$

$\exists x_1, x_2 \in V'$ aî $y_1 = f(x_1)$
$y_2 = f(x_2)$

$ay_1+by_2 = af(x_1)+bf(x_2) = f(ax_1+bx_2) = f(\underset{V'}{x})$

Def $f: V_1 \to V_2$ apl. liniară

a) $\operatorname{Ker} f = \{x \in V_1 \mid f(x) = 0_{V_2}\}$ nullspace, Kernel (nucleul lui f)

b) $\operatorname{Im} f = \{y \in V_2 \mid \exists x \in V_1$ aî $f(x) = y\}$ imaginea lui f.

Prop $f: V_1 \to V_2$ liniară

a) $\operatorname{Ker} f \subset V_1$, $\operatorname{Im} f \subset V_2$ subspații vectoriale

b) f injectivă $\Leftrightarrow \operatorname{Ker} f = \{0_{V_1}\}$

c) f surjectivă $\Leftrightarrow \dim \operatorname{Im} f = \dim V_2$.

Dem

a) Fie $x_1, x_2 \in \operatorname{Ker} f$, $a, b \in \mathbb{K}$ $\overset{?}{\Longrightarrow} a x_1 + b x_2 \in \operatorname{Ker} f$

$$f(ax_1 + bx_2) = a \underset{0_{V_2}}{\underset{\parallel}{f(x_1)}} + b \underset{0_{V_2}}{\underset{\parallel}{f(x_2)}} = 0_{V_2} \Rightarrow \operatorname{Ker} f \subset V_1 \text{ ssp vect}$$

$\operatorname{Im} f = f(V_1) \subset V_2$ subsp vect (cf obs)

b) "$\Rightarrow$" Ipoteza: f injectivă.

Fie $x \in \operatorname{Ker} f \Rightarrow f(x) = 0_{V_2}$, dar $f(0_{V_1}) = 0_{V_2}$ $\overset{inj}{\Longrightarrow} x = 0_{V_1} \Rightarrow \operatorname{Ker} f = \{0_{V_1}\}$.

"$\Leftarrow$" Ipoteza: $\operatorname{Ker} f = \{0_{V_1}\}$

Fie $x_1, x_2 \in V_1$ aî $f(x_1) = f(x_2) \Rightarrow \underset{f(x_1 - x_2)}{\underset{\parallel}{f(x_1) - f(x_2)}} = 0_{V_2}$

$\Rightarrow x_1 - x_2 \in \operatorname{Ker} f = \{0_{V_1}\} \Rightarrow x_1 = x_2 \Rightarrow f$ inj

c) "$\Rightarrow$" f surjectivă $\Rightarrow \operatorname{Im} f = V_2 \Rightarrow \dim \operatorname{Im} f = \dim V_2$

"$\Leftarrow$" $\dim \operatorname{Im} f = \dim V_2$.

OBS! Fie $V' \subset V$ subspațiu vect.
Dacă $\dim V' = \dim V = n$ $\Big\} \Rightarrow V' = V$

$\left.\begin{array}{l} \operatorname{Im} f \subseteq V_2 \text{ subsp vect} \\ \dim \operatorname{Im} f = \dim V_2 \end{array}\right\} \Rightarrow \operatorname{Im} f = V_2 \Rightarrow f$ surjectivă

Consecință $f: V_1 \to V_2$ liniară

f izomorfism de sp vect $\Leftrightarrow \begin{cases} \operatorname{Ker} f = \{0_{V_1}\} \\ \dim \operatorname{Im} f = \dim V_2 \end{cases}$

Teorema dimensiunii $(V_i, +, \cdot)/\mathbb{K}$, $i = \overline{1,2}$ sp. vect

$f: V_1 \to V_2$ apl. liniară

$\Longrightarrow \dim_{\mathbb{K}} V_1 = \dim_{\mathbb{K}}(\operatorname{Ker} f) + \dim_{\mathbb{K}}(\operatorname{Im} f)$

Dem

$\dim_{\mathbb{K}} \operatorname{Ker} f = k$, $\dim_{\mathbb{K}} V_1 = n$, $k \leq n$.

Fie $R_0 = \{e_1, .., e_k\}$ reper în $\operatorname{Ker} f$.

Extindem la $R_1 = \{e_1, .., e_k, \underline{e_{k+1}, .., e_n}\}$ reper în V_1.

Considerăm $R = \{f(e_{k+1}), .., f(e_n)\}$

Dem că R este reper în $\operatorname{Im} f$

1) R este SLI

Fie $a_{k+1}, .., a_n \in \mathbb{K}$ aî $\sum_{i=k+1}^{n} a_i f(e_i) = 0_{V_2}$. $\Rightarrow$

$f\left(\sum_{i=k+1}^{n} a_i e_i\right) = 0_{V_2} \Rightarrow \sum_{i=k+1}^{n} a_i e_i \in \operatorname{Ker} f = \langle R_0 \rangle$.

$\exists a_1, .., a_k \in \mathbb{K}$ aî $\sum_{i=k+1}^{n} a_i e_i = \sum_{j=1}^{k} a_j e_j \Rightarrow$

$\sum_{j=1}^{k} a_j e_j - \sum_{i=k+1}^{n} a_i e_i = 0_{V_1} \xrightarrow[\text{reper SLI}]{R_1} a_j = 0, j = \overline{1,k}$, $\boxed{a_i = 0, i = \overline{k+1,n}}$

$\Rightarrow R$ este SLI

2) R este SG. i.e $\operatorname{Im} f = \langle R \rangle$

$\forall y \in \operatorname{Im} f$, $\exists x \in V_1 = \langle R_1 \rangle$ a.î. $y = f(x)$

$$= f\Big(\sum_{k=1}^{n} a_k e_k\Big) = f\Big(\sum_{j=1}^{k} a_j e_j + \sum_{i=k+1}^{n} a_i e_i\Big)$$

$$= f\Big(\underbrace{\sum_{j=1}^{k} a_j e_j}_{\in \operatorname{Ker} f}\Big) + f\Big(\sum_{i=k+1}^{n} a_i e_i\Big) = 0_{V_2} + \sum_{i=k+1}^{n} a_i f(e_i)$$

$y \in \langle \{ f(e_{k+1}), \ldots, f(e_n) \} \rangle = \langle R \rangle$

$\dim_{\mathbb{K}} V_1 = n = k + n - k = \dim_{\mathbb{K}}(\operatorname{Ker} f) + \dim_{\mathbb{K}}(\operatorname{Im} f)$

<u>Prop</u> $f: V_1 \to V_2$ liniară

a) f injectivă $\Leftrightarrow \dim V_1 = \dim \operatorname{Im} f$

b) f surjectivă $\Leftrightarrow \dim V_1 = \dim \operatorname{Ker} f + \dim V_2$

c) f bijectivă $\Leftrightarrow \dim V_1 = \dim V_2$.

<u>Teoremă</u> $(V_i, +, \cdot)/\mathbb{K}$ sp. vect, $i = \overline{1,2}$

$V_1 \simeq V_2$ (sp. vect izomorfe) $\Leftrightarrow \dim V_1 = \dim V_2$

<u>Dem</u>

"$\Rightarrow$" $\exists f: V_1 \to V_2$ izomorfism de sp. vect $\xRightarrow{\text{Prop c)}}$ $\dim V_1 = \dim V_2$

"$\Leftarrow$" Ip: $\dim V_1 = \dim V_2 = n$.

Fie $R_1 = \{e_1, \ldots, e_n\}$ reper în V_1

$R_2 = \{e_1', \ldots, e_n'\}$ — " — în V_2.

Fie $f: V_1 \longrightarrow V_2$, $f(e_i) = e'_i$, $i = \overline{1,n}$

Extindem prin liniaritate.

$$f(x) = f\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} x_i \underbrace{f(e_i)}_{= e'_i} = x'$$

f bijectie:

$\forall x' = \sum_{i=1}^{n} x_i e'_i$, $\exists! x = \sum_{i=1}^{n} x_i e_i$ ai $f(x) = x'$

f izomorfism de sp. vectoriale.

<u>Exemplu</u>

$f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x_1, x_2, x_3) = (x_1 + x_2 - x_3, \ x_1 + x_2, \ x_1 + x_2 + x_3)$

a) f liniara

b) dim Ker f, dim Im f.

Precizati câte un reper în fiecare.

<u>SOL</u>

a) $f(ax + by) = a f(x) + b f(y)$

b) $\text{Ker} f = \{ x \in \mathbb{R}^3 \mid f(x) = 0_{\mathbb{R}^3} \} = S(A)$

$$\begin{cases} x_1 + x_2 - x_3 = 0 \\ x_1 + x_2 = 0 \\ x_1 + x_2 + x_3 = 0 \end{cases} \qquad A = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right)$$

$\dim \text{Ker} f = 3 - \text{rg} A = 3 - 2 = 1$

x_1 = var secundară $\quad x_2 = -x_1$

$\quad x_3 = x_1 + x_2 = 0$

$\text{Ker} f = \{ (x_1, -x_1, 0) \mid x_1 \in \mathbb{R} \} = \langle \{ (1, -1, 0) \} \rangle$

T. dim: $\dim \mathbb{R}^3 = \underbrace{\dim \text{Ker} f}_{= 1} + \dim \text{Im} f \Rightarrow \dim \text{Im} f = 2$.

$\mathcal{R}_0 = \{ (1, -1, 0) \}$ $\qquad \det \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \neq 0$

Extindem la $\mathcal{R}_1 = \{ (1, -1, 0), e_3, e_1 \}$ reper în $\mathbb{R}^3$

$\mathcal{R} = \{ \underset{\substack{\shortparallel \\ (-1,0,1)}}{f(e_3)}, \underset{\substack{\shortparallel \\ (1,1,1)}}{f(e_4)} \}$ reper în $\operatorname{Im} f$ $\qquad e_1 = (1,0,0)$

SAU $\quad y \in \operatorname{Im} f \Rightarrow \underline{\underline{\exists}}\, x \in \mathbb{R}^3$ a.î. $f(x) = y$.

$$(*) \begin{cases} x_1 + x_2 - x_3 = y_1 \\ x_1 + x_2 = y_2 \\ x_1 + x_2 + x_3 = y_3 \end{cases} \qquad A = \left(\begin{array}{ccc} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{array} \right) \left| \begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array} \right.$$

$(*)$ este compatibil $\Leftrightarrow \Delta_c = \begin{vmatrix} 1 & -1 & y_1 \\ 1 & 0 & y_2 \\ 1 & 1 & y_3 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} 1 & -1 & y_1 \\ 1 & 0 & y_2 \\ 2 & 0 & y_1 + y_3 \end{vmatrix} = 0 \qquad y_1 + y_3 - 2y_2 = 0.$$

$\operatorname{Im} f = \{ y = (y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_1 - 2y_2 + y_3 = 0 \}$

$\dim \operatorname{Im} f = 3 - 1 = 2.$

$\operatorname{Im} f = \{ (\underset{\shortparallel}{2y_2 - y_3}, y_2, y_3) \mid y_2, y_3 \in \mathbb{R} \}$

$$y_2(2,1,0) + y_3(-1,0,1)$$

$\operatorname{Im} f = \langle \{ (2,1,0), (-1,0,1) \} \rangle$

$\downarrow$

reper în $\operatorname{Im} f$.

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