Seminar 13

1. Studiați posibilitatea aplicării Teoremei de permutare a limitei cu integrala pentru limitele de mai jos și apoi calculați-le:

a) $\lim\limits_{n\to\infty} \int_0^1 \frac{x\sin(nx)}{n^2+nx^2+x}\,dx$.

Sol.: Fie $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = \frac{x\sin(nx)}{n^2+nx^2+x}$ $\forall n \in \mathbb{N}^*$.

f_n continuă $\forall n \in \mathbb{N}^* \Rightarrow f_n$ integrabilă Riemann $\forall n \in \mathbb{N}^*$.

Convergența simplă

Fie $x \in [0,1]$.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x\sin(nx)}{n^2+nx^2+x} = ?$$

$$0 \le |f_n(x)| = \frac{|x\sin(nx)|}{n^2+nx^2+x} \le \frac{1}{n^2} \quad \forall n \in \mathbb{N}^* \Rightarrow$$

$$0$$

$$\Rightarrow \lim_{n\to\infty} |f_n(x)| = 0 \Rightarrow \lim_{n\to\infty} f_n(x) = 0 \Rightarrow f_n \xrightarrow[n\to\infty]{s} f,$$

unde $f:[0,1]\to\mathbb{R}$, $f(x)=0$.

<u>Conv. unif.</u>

$$\sup_{x\in[0,1]} |f_n(x)-f(x)| = \sup_{x\in[0,1]} \left| \frac{x\sin(nx)}{n^2+nx^2+x} - 0 \right| =$$

$$= \sup_{x\in[0,1]} \frac{|x\sin(nx)|}{n^2+nx^2+x} \le \frac{1}{n^2} \xrightarrow[n\to\infty]{} 0 \Rightarrow f_n \xrightarrow[n\to\infty]{u} f.$$

Deci putem aplica Teorema de permutare a limitei cu integrala.

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0. \quad \square$$

b) $\lim_{n\to\infty} \int_0^1 \ln(1+x^n)\,dx$.

<u>Sol</u>: Fie $f_n:[0,1]\to\mathbb{R}$, $f_n(x)=\ln(1+x^n)$ $\forall n\in\mathbb{N}^*$.

f_n cont. $\forall n\in\mathbb{N}^* \Rightarrow f_n$ integr. R $\forall n\in\mathbb{N}^*$.

<u>Conv. simplă</u>

Fie $x\in[0,1]$.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \ln(1+x^n) = \begin{cases} 0\,; & x\in[0,1) \\ \ln 2\,; & x=1 \end{cases} \Rightarrow$$

$$\Rightarrow f_n \xrightarrow[n\to\infty]{s} f, \text{ unde } f:[0,1]\to\mathbb{R},\ f(x)=\begin{cases} 0\,; & x\in[0,1) \\ \ln 2\,; & x=1. \end{cases}$$

<u>Conv. unif.</u>

f_n continuă $\forall n\in\mathbb{N}^*$

f nu e continuă (în 1) $\Bigg\} \Rightarrow f_n \xrightarrow[n\to\infty]{\cancel{u}} f.$

Deci nu putem aplica Teorema de permutare a limitei cu integrala.

$$\lim_{n\to\infty} \int_0^1 \ln(1+x^n)\,dx = ?$$

$$0 \le \ln(1+x^n) \le x^n \quad \forall x\in[0,1],\ \forall n\in\mathbb{N}^* \Rightarrow$$

$$\Rightarrow 0 \le \int_0^1 \ln(1+x^n)\,dx \le \int_0^1 x^n\,dx = \frac{x^{n+1}}{n+1}\Bigg|_0^1 = \frac{1}{n+1} \quad \forall n\in\mathbb{N}^*.$$

$$\Rightarrow 0$$

$$\text{Deci } \lim_{n\to\infty} \int_0^1 \ln(1+x^n)\,dx = 0 \quad \left(= \int_0^1 f(x)\,dx\right). \ \square$$

c) $\displaystyle \lim_{n\to\infty} \int_0^1 n x (1-x^2)^n\,dx.$

Ex.: Fie $f_n: [0,1] \to \mathbb{R}$, $f_n(x) = nx(1-x^2)^n$ $\forall n \in \mathbb{N}^*$.

f_n continuă $\forall n \in \mathbb{N}^* \Rightarrow f_n$ integr. R $\forall n \in \mathbb{N}^*$.

Cz 1.

Fie $x \in [0,1]$.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nx(1-x^2)^n.$$

Dacă $x=0$, atunci $f_n(x) = 0 \xrightarrow[n\to\infty]{} 0$.

Dacă $x=1$, atunci $f_n(x) = 0 \xrightarrow[n\to\infty]{} 0$.

Fie $x \in (0,1)$.

$$\lim_{n\to\infty} \frac{f_{n+1}(x)}{f_n(x)} = \lim_{n\to\infty} \frac{(n+1)\cancel{x}(1-x^2)^{\cancel{n+1}}}{n\cancel{x}\cancel{(1-x^2)^n}} =$$

$$= 1-x^2 < 1.$$

Conform Crit. rap. pt. șiruri cu termeni strict pozitivi rezultă că $\lim_{n\to\infty} f_n(x) = 0$.

Deci $f_n \xrightarrow[n\to\infty]{s} f$, unde $f: [0,1] \to \mathbb{R}$, $f(x) = 0$.

C. u.

$$\sup_{x\in[0,1]} |f_n(x)-f(x)| = \sup_{x\in[0,1]} |nx(1-x^2)^n - 0| =$$

$$= \sup_{x\in[0,1]} nx(1-x^2)^n \underset{\uparrow}{\geq} \cancel{n}\cdot\frac{1}{\cancel{n}}\left(1-\frac{1}{n^2}\right)^n = \left(1-\frac{1}{n^2}\right)^n \quad \forall n\in\mathbb{N}^*.$$

$$x=\frac{1}{n}$$

Deoarece $\left(1-\frac{1}{n^2}\right)^n > 0 \;\forall n\in\mathbb{N}^*\setminus\{1\}$ și $\lim_{n\to\infty}\left(1-\frac{1}{n^2}\right)^n =$

$= e^0 = 1 \neq 0$ rezultă că $f_n \overset{u}{\underset{n\to\infty}{\not\rightrightarrows}} f$.

Deci nu putem aplica teorema de permutare a limitei cu integrala.

$$\lim_{n\to\infty}\int_0^1 nx(1-x^2)^n\,dx = \lim_{n\to\infty}\left[-\frac{1}{2}n\int_0^1(-2x)(1-x^2)^n\,dx\right] =$$

$$= \lim_{n\to\infty}\left[-\frac{n}{2}\cdot\frac{(1-x^2)^{n+1}}{n+1}\Bigg|_0^1\right] = \lim_{n\to\infty}\left[-\frac{n}{2}\left(0-\frac{1}{n+1}\right)\right] =$$

$$= \lim_{n\to\infty}\frac{n}{2(n+1)} = \frac{1}{2} \quad \left(\neq \int_0^1 f(x)\,dx\right). \quad \square$$

Propoziție. Fie $f:[a,b]\to[0,\infty)$ o funcție continuă ($a<b$) a.î. $\int_a^b f(x)\,dx=0$. Atunci $f(x)=0$ $\forall x\in[a,b]$.

2. Fie $f:[a,b]\to\mathbb{R}$ ($a<b$) o funcție continuă a.î. $\int_a^b f(x)\,dx=0$ și $\int_a^b x^n f(x)\,dx=0$ $\forall n\in\mathbb{N}^*$. Arătați că $f\equiv 0$.

Sol.: Fie $P(x)=a_0+a_1x+\ldots+a_nx^n$.

$$\int_a^b P(x)f(x)\,dx=a_0\int_a^b f(x)\,dx+a_1\int_a^b xf(x)\,dx+\ldots+a_n\int_a^b x^nf(x)\,dx=$$

$$=0.$$

f continuă $\xrightarrow[\text{Bernstein}]{\text{Teorema lui}}$ $\exists(P_n)_n$ un șir de funcții polinomiale ($P_n:[a,b]\to\mathbb{R}$ $\forall n\in\mathbb{N}$) a.î. $P_n\xrightarrow[n\to\infty]{u}f$ (i.e. $\lim_{n\to\infty}\left(\sup_{x\in[a,b]}|P_n(x)-f(x)|\right)=0$).

Avem $\int_a^b P_n(x)f(x)\,dx=0$ $\forall n\in\mathbb{N}$.

Arătăm că $P_nf\xrightarrow[n\to\infty]{u}f^2$.

$$\sup_{x\in[a,b]}|P_n(x)f(x)-f^2(x)|=\sup_{x\in[a,b]}\left(|f(x)|\cdot|P_n(x)-f(x)|\right).$$

$$\left.\begin{array}{l} f \text{ continuă} \\ [a,b] \text{ mulțime compactă} \end{array}\right\} \Rightarrow f \text{ mărginită și își atinge marginile} \Rightarrow \exists\, M>0 \text{ a.î. } |f(x)| \le M \ \forall\, x \in [a,b].$$

$$\sup_{x\in[a,b]} \left|P_n(x) f(x) - f^2(x)\right| = \sup_{x\in[a,b]} \Big(|f(x)| \cdot |P_n(x) - f(x)|\Big) \le$$

$$\le M \sup_{x\in[a,b]} |P_n(x) - f(x)| \xrightarrow[n\to\infty]{} 0.$$

Deci $P_n f \xrightarrow[n\to\infty]{u} f^2$.

P_n, f continue $\forall\, n\in\mathbb{N} \Rightarrow P_n f$ continuă $\forall\, n\in\mathbb{N} \Rightarrow$

$\Rightarrow P_n \cdot f$ integrabilă R $\forall\, n\in\mathbb{N}$.

Conform Teoremei de permutare a limitei cu integrala avem că

$$\underbrace{\lim_{n\to\infty} \int_a^b P_n(x) f(x)\, dx}_{=0} = \int_a^b f^2(x)\, dx.$$

Deci $\int_a^b f^2(x)\, dx = 0$.

Cum f^2 este continuă și $f^2 : [a,b] \to [0,\infty)$ rezultă, conform propoziției anterioare, că $f^2 \equiv 0$, deci $f \equiv 0$. $\square$

3. Determinați:

a) $\int_0^\infty \frac{1}{1+x^2} dx$.

<u>Sol</u>: $\int_0^\infty \frac{1}{1+x^2} dx = \lim\limits_{b\to\infty} \int_0^b \frac{1}{1+x^2} dx = \lim\limits_{b\to\infty} \left(\operatorname{arctg} x \Big|_0^b \right) =$

$= \lim\limits_{b\to\infty} (\operatorname{arctg} b - \underbrace{\operatorname{arctg} 0}_{0}) = \frac{\pi}{2}$. $\square$

b) $\int_{-\infty}^0 e^x dx$.

<u>Sol</u>: Rezolvați-l voi! $\square$

c) $\int_0^1 \frac{1}{\sqrt{1-x^2}} dx$.

<u>Sol</u>: $\int_0^1 \frac{1}{\sqrt{1-x^2}} dx = \lim\limits_{\substack{b\to 1\\ b<1}} \int_0^b \frac{1}{\sqrt{1-x^2}} dx =$

$= \lim\limits_{\substack{b\to 1\\ b<1}} \left(\arcsin x \Big|_0^b \right) = \lim\limits_{\substack{b\to 1\\ b<1}} (\arcsin b - \underbrace{\arcsin 0}_{0}) =$

$= \arcsin 1 = \frac{\pi}{2}$. $\square$

d) $\int_0^{\frac{1}{2}} \frac{1}{x\ln^2 x}\,dx$.

<u>Sol</u>.: $\int_0^{\frac{1}{2}} \frac{1}{x\ln^2 x}\,dx = \lim\limits_{\substack{a\to 0\\ a>0}} \int_a^{\frac{1}{2}} \frac{1}{x\ln^2 x}\,dx =$

$$= \lim_{\substack{a\to 0\\ a>0}} \int_a^{\frac{1}{2}} (\ln x)'(\ln x)^{-2}\,dx = \lim_{\substack{a\to 0\\ a>0}} \left[\frac{(\ln x)^{-1}}{-1} \Bigg|_a^{\frac{1}{2}} \right] =$$

$$= \lim_{\substack{a\to 0\\ a>0}} \left(-\frac{1}{\ln\frac{1}{2}} + \frac{1}{\ln a} \right) = -\frac{1}{\ln\frac{1}{2}} = \frac{1}{\ln 2}. \square$$

e) $\int_{-\infty}^{\infty} \frac{x}{1+x^4}\,dx$.

<u>Sol</u>.: $\int_{-\infty}^{0} \frac{x}{1+x^4}\,dx \underset{\uparrow}{=} \int_{\infty}^{0} \frac{1}{1+y^2}\cdot\frac{1}{2}\,dy =$

S.V. $y = x^2$

$dy = 2x\,dx \Leftrightarrow x\,dx = \frac{1}{2}\,dy$

$x \to -\infty \Leftrightarrow y \to +\infty$

$x = 0 \Leftrightarrow y = 0$

$$= \frac{1}{2} \lim_{a \to \infty} \int_a^0 \frac{1}{1+y^2} dy = \frac{1}{2} \lim_{a \to \infty} \left(\operatorname{arctg} y \Big|_a^0 \right) =$$

$$= \frac{1}{2} \lim_{a \to \infty} \left(\underbrace{\operatorname{arctg} 0}_{0} - \operatorname{arctg} a \right) = -\frac{\pi}{4}.$$

$$\int_0^\infty \frac{x}{1+x^4} dx = \frac{\pi}{4} \text{ (Rezolvați voi!)}.$$

$$\int_{-\infty}^\infty \frac{x}{1+x^4} dx = \int_{-\infty}^0 \frac{x}{1+x^4} dx + \int_0^\infty \frac{x}{1+x^4} dx = -\frac{\pi}{4} + \frac{\pi}{4} = 0. \square$$

f) $\int_0^1 \frac{1}{x^5} dx$.

<u>Sol</u>: $\int_0^1 \frac{1}{x^5} dx = \lim_{\substack{a \to 0 \\ a > 0}} \int_a^1 x^{-5} dx = \lim_{\substack{a \to 0 \\ a > 0}} \left(\frac{x^{-4}}{-4} \Big|_a^1 \right) =$

$$= \lim_{\substack{a \to 0 \\ a > 0}} \left(-\frac{1}{4} + \frac{1}{4a^4} \right) = -\frac{1}{4} + \infty = \infty. \square$$

4. Folosind eventual funcțiile Γ și B determinați:

a) $\int_0^\infty e^{-x^2} dx$.

Sol.: $\int_0^\infty e^{-x^2}dx = \int_0^\infty e^{-t}\cdot\frac{1}{2\sqrt{t}}dt = \frac{1}{2}\int_0^\infty t^{-\frac{1}{2}}e^{-t}dt =$

S.V. $x^2=t \Leftrightarrow x=\sqrt{t}$

$dx = \frac{1}{2\sqrt{t}}dt$

$x=0 \Leftrightarrow t=0$

$x\to\infty \Leftrightarrow t\to\infty$

$= \frac{1}{2}\int_0^\infty t^{\frac{1}{2}-1}e^{-t}dt = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi} = \frac{\sqrt{\pi}}{2}$. □

b) $\int_{-\infty}^\infty e^{-x^2}dx$.

Sol.: Rezolvați-l voi! □

c) $\int_0^\infty x^6 e^{-2x}dx$.

Sol.: $\int_0^\infty x^6e^{-2x}dx = \int_0^\infty \left(\frac{t}{2}\right)^6 e^{-t}\cdot\frac{1}{2}dt = \frac{1}{2^7}\int_0^\infty t^6e^{-t}dt =$

$2x=t \Leftrightarrow x=\frac{t}{2}$

$dx=\frac{1}{2}dt$

$x=0 \Leftrightarrow t=0$

$x\to\infty \Leftrightarrow t\to\infty$

$= \frac{1}{2^7}\int_0^\infty t^{7-1}e^{-t}dt = \frac{1}{2^7}\Gamma(7) = \frac{6!}{2^7} = \frac{720}{2^7}$. □

d) $\int_0^\infty \sqrt{x}\, e^{-x^3} dx$.

<u>Sol.</u>: $\int_0^\infty \sqrt{x}\, e^{-x^3} dx = \int_0^\infty t^{\frac{1}{6}} e^{-t} \cdot \frac{1}{3} t^{-\frac{2}{3}} dt =$

$x^3 = t \Leftrightarrow x = t^{\frac{1}{3}}$

$dx = \frac{1}{3} t^{-\frac{2}{3}} dt$

$x = 0 \Leftrightarrow t = 0$

$x \to \infty \Leftrightarrow t \to \infty$

$$= \frac{1}{3} \int_0^\infty t^{-\frac{3}{6}} e^{-t} dt = \frac{1}{3} \int_0^\infty t^{-\frac{1}{2}} e^{-t} dt =$$

$$= \frac{1}{3} \int_0^\infty t^{\frac{1}{2}-1} e^{-t} dt = \frac{1}{3} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{3}. \ \square$$

e) $\int_0^2 \frac{x^2}{\sqrt{2-x}} dx$.

<u>Sol.</u>: $\int_0^2 \frac{x^2}{\sqrt{2-x}} dx = \int_0^2 x^2 (2-x)^{-\frac{1}{2}} dx =$

$$= \int_0^2 x^2 \cdot 2^{-\frac{1}{2}} \left(1 - \frac{x}{2}\right)^{-\frac{1}{2}} dx = \frac{1}{\sqrt{2}} \int_0^1 2^2 t^2 (1-t)^{-\frac{1}{2}} 2\, dt =$$

S.V. $\frac{x}{2} = t \Leftrightarrow x = 2t$

$dx = 2dt$

$x = 0 \Leftrightarrow t = 0$

$x \to 2 \Leftrightarrow t \to 1$

$$= \frac{8}{\sqrt{2}}\int_0^1 t^2(1-t)^{-\frac{1}{2}}\,dt = \frac{8}{\sqrt{2}}\int_0^1 t^{3-1}(1-t)^{\frac{1}{2}-1}\,dt =$$

$$= \frac{8}{\sqrt{2}}\, B\left(3, \frac{1}{2}\right).$$

$$B\left(3, \frac{1}{2}\right) = \frac{\Gamma(3)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(3+\frac{1}{2}\right)}.$$

$$\Gamma(3) = 2! = 2.$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

$$\Gamma\left(3+\frac{1}{2}\right) = \Gamma\Big(1+\underbrace{2+\frac{1}{2}}\Big) = \left(2+\frac{1}{2}\right)\Gamma\left(2+\frac{1}{2}\right) = \frac{5}{2}\,\Gamma\Big(1+\underbrace{1+\frac{1}{2}}\Big) =$$

$$= \frac{5}{2}\left(1+\frac{1}{2}\right)\Gamma\Big(1+\underbrace{\frac{1}{2}}\Big) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{15}{8}\sqrt{\pi}.$$

Deci $B\left(3, \frac{1}{2}\right) = \dfrac{2\cdot\cancel{\sqrt{\pi}}}{\frac{15}{8}\cdot\cancel{\sqrt{\pi}}} = \dfrac{16}{15}.$

Așadar $\displaystyle\int_0^2 \frac{x^2}{\sqrt{2-x}}\,dx = \overset{\sqrt{2}}{\frac{8}{\sqrt{2}}}\cdot\frac{16}{15} = \frac{8\cdot\overset{8}{\cancel{16}}\cdot\sqrt{2}}{\cancel{2}\cdot 15} = \frac{64\sqrt{2}}{15}.$