Ecuații diofantice de grad 1

Ec în $\mathbb{Z}\times\mathbb{Z}$ de forma $ax+by=c$ (D)

Prop: Ec. (D) are sol. $\iff (a,b) \mid c$

Dem: „$\Rightarrow$" $d=(a,b)$; $d \mid a \wedge d \mid b$

$\exists (x_0,y_0)\in\mathbb{Z}\times\mathbb{Z}$ a.î. $ax_0+by_0=c$ $\Big\} \Rightarrow d \mid ax_0+by_0=c$.

„$\Leftarrow$" $d=(a,b)$; $d \mid c \Rightarrow c=dc'$ cu $c'\in\mathbb{Z}$

$a=da'$, $b=db'$; $(a',b')=1$

Ec. (D) devine $a'x+b'y=c'$ cu $(a',b')=1$.

$(a',b')=1 \iff \exists\, \alpha,\beta\in\mathbb{Z}$ a.î. $\alpha a'+\beta b'=1$.

$\Rightarrow (x_0=\alpha c',\ y_0=\beta c')$ e sol a Ec. (D)).

[In general, dacă $d=(a,b)$ atunci $\exists\, \alpha,\beta\in\mathbb{Z}$
a.î. $\alpha a+\beta b=d$]

Algoritmul lui Euclid: $a, b \in \mathbb{Z}, |a| > |b| \neq 0$

$a = b q_1 + r_1, \quad 0 \leq r_1 < |b| \quad \Rightarrow r_1 = a - b q_1$

$b = r_1 q_2 + r_2, \quad 0 \leq r_2 < r_1 \quad \Rightarrow r_2 = b - (a - b q_1) q_2$

$= -a q_2 + (1 + q_1 q_2) b$

$r_1 = r_2 q_3 + r_3, \quad 0 \leq r_3 < r_2 \quad \Rightarrow r_3 = a - b q_1 - q_3 (-a q_2 +$

$\vdots$

$+ (1 + q_1 q_2) b)$

$r_{n-2} = r_{n-1} q_n + r_n, \quad 0 \leq r_n < r_{n-1}$

$= \alpha_3 a + \beta_3 b$

$r_{n-1} = r_n q_{n+1} + 0$

$\alpha_3 = 1 + q_2 q_3$

$\beta_3 = (1 + q_1 q_2)(-q_3) - q_1$

$\vdots$

Im: $0 \leq r_n < r_{n-1} < \ldots < r_2 < r_1 < |b|$

$d = r_n = (a, b)$

$\Rightarrow d = \underbrace{\alpha_n}_{\alpha} a + \underbrace{\beta_n}_{\beta} b$

Rez. ecuației (D) $\quad d = (a, b) \mid c$.

Fie (x_0, y_0) sol. particulară și (x, y) altă sol. pt. (D)

$a x_0 + b y_0 = a x + b y = c \mid : d$

$a' x_0 + b' y_0 = a' x + b' y \; (= c')$

$\underbrace{a'(x - x_0)}_{\vdots a'} = b'(y_0 - y) \; \Big\} \overset{(a', b') = 1}{\Longrightarrow} a' \mid y_0 - y \Rightarrow$

$\Rightarrow \exists t \in \mathbb{Z}$ a.î. $y_0 - y = a' t \Leftrightarrow y = y_0 - a' t$

$a'(x - x_0) = b' a' t \mid : a' \neq 0 \Rightarrow x = x_0 + b' t$

Sol gen. a ec (D) este $\begin{cases} x = x_0 + b' t \\ y = y_0 - a' t \end{cases}, \; t \in \mathbb{Z}$

Exemple: i) $281x - 133y = 3$

$281 = 133 \cdot 2 + 15 \Rightarrow 15 = 281 - 133 \cdot 2$

$133 = 15 \cdot 8 + 13 \Rightarrow 13 = 133 - 15 \cdot 8$

$= 133 - (281 - 133 \cdot 2) \cdot 8$

$= -8 \cdot 281 + 17 \cdot 133$

$15 = 13 \cdot 1 + 2 \Rightarrow 2 = 15 - 13 = 9 \cdot 281 - 19 \cdot 133$

$13 = 2 \cdot 6 + \boxed{1} \Rightarrow 1 = 13 - 6 \cdot 2 = 131 \cdot 133 - 62 \cdot 281$

$2 = 1 \cdot 2 + 0$

Deci $-62 \cdot 281 + 131 \cdot 133 = 1 \,\Big|\, \cdot 3 \Rightarrow$

$\Rightarrow x_0 = -3 \cdot 62 = -186,\ y_0 = -3 \cdot 131 = -393$

este sol. partic. a ec. $\boxed{281x - 133y = 3}$

Sol gen este $\begin{cases} x = -186 + 133t \\ y = -393 + 281t \end{cases}, t \in \mathbb{Z}$

iii) $A=\{3n-2 \mid n\in\mathbb{N}\}$, $B=\{1003-2m \mid m\in\mathbb{N}\}$

$C=\{6p+1 \mid p\in\mathbb{N}\cap[0,166]\}$.

$A\cap B\cap C=?$

Sol: $x\in A\cap B \iff \exists n,m\in\mathbb{N}$ a.î. $x=3n-2=1003-2m$

$3n+2m=1005$

Sol partic: $3\cdot 3+2\cdot(-4)=1 \mid \cdot 1005 \Rightarrow \begin{cases} n_0=3015 \\ m_0=-4020 \end{cases} \Rightarrow$

sol generale $\begin{cases} n=3015+2t\in\mathbb{N} \\ m=-4020-3t\in\mathbb{N} \end{cases} \Rightarrow t\in\mathbb{Z}\cap[-1507,-1340]$

$x=3n-2=9043+6t$, $t\in\mathbb{Z}\cap[-1507,-1340]$

$A\cap B=\{6t+9043 \mid t\in\mathbb{Z}\cap[-1507,-1340]\}$

$=\{6q+1 \mid q\in\mathbb{Z}\cap[0,167]\}$.

$(q=1507+t)$

$A\cap B\cap C=\{6q+1 \mid q\in\mathbb{N}\cap[0,167]\}\cap\{6p+1 \mid p\in\mathbb{N}\cap[0,166]\}$

$=\{6p+1 \mid p\in\mathbb{N}\cap[0,166]\}=C$.

Temă: $M = \{x \in \mathbb{Q} \mid x = \frac{n^2+3}{n^2+n}, n \in \mathbb{N} \cap [1,50]\}$

$|M| = ?$

Relații:

1) $A = \{2,4,6,8\}$, $B = \{1,3,5,7\}$

$\rho = \{(x,y) \mid x \geq 6 \vee y \leq 1\} \subseteq A \times B$

Explicitați ρ.

Sol: $\rho = \{(x,y) \mid x \in \{6,8\} \vee y = 1\}$

$= \{(6,b) \mid b \in B\} \cup \{(8,b) \mid b \in B\} \cup$

$\cup \{(2,1); (4,1)\} \subseteq A \times B$

2) $A = B = \mathbb{N}$; $\rho = \{(3,5), (5,3), (3,3), (5,5)\}$

$\sigma = \{(x,y) \mid x \leq y\} \subseteq \mathbb{N} \times \mathbb{N}$

$\gamma = \{(x,y) \mid y - x = 12\} \subseteq \mathbb{N} \times \mathbb{N}$

a) $\rho^{-1}, \sigma^{-1}, \gamma^{-1}$

$\rho^{-1} = \{(5,3), (3,5), (3,3), (5,5)\} = \rho$

$\sigma^{-1} = \{(x,y) \mid (y,x) \in \sigma\} = \{(x,y) \mid y \leq x\}$

$\gamma^{-1} = \{(x,y) \mid (y,x) \in \gamma\} = \{(x,y) \mid x - y = 12\}$

$= \{(y+12, y) \mid y \in \mathbb{N}\}$.

b) $\rho^2, \sigma \circ \rho, \rho \circ \sigma = ?$

$\rho^2 = \rho \circ \rho = \{(a,c) \mid \exists b$ a.î. $(a,b) \in \rho, (b,c) \in \rho\}$

3 5   5 3,5
5 3   3 3,5
3 3   3 3,5
5 5   5 3,5

$= \{(3,3); (3,5); (5,3); (5,5)\} = \rho$

$\rho^2 = \rho \Rightarrow \rho^n = \rho, \forall n \in \mathbb{N}^*$

$\sigma \circ \rho = \{(a,c) \mid \exists b$ a.î. $(a,b) \in \rho, (b,c) \in \sigma\}$

3 5   5, $c \geq 5$
5 3   3, $c \geq 3$
3 3   3, $c \geq 3$
5 5   5, $c \geq 5$

$= \{(3,c) \mid c \geq 5\} \cup \{(5,c) \mid c \geq 3\}$

$\cup \{(3,c) \mid c \geq 3\} \cup \{(5,c) \mid c \geq 5\}$

$= \{(3,c) \mid c \geq 3\} \cup \{(5,c) \mid c \geq 3\}$.

$\rho \circ \sigma = \{(a,c) \mid \exists b$ a.î. $(a,b) \in \sigma, (b,c) \in \rho\}$

$a \leq b$ 3 5, 5 3, 3 3, 5 5

$= \{(a,5) \mid a \leq 3\} \cup \{(a,3) \mid a \leq 5\} \cup \{(a,3) \mid a \leq 3\} \cup \{(a,5) \mid a \leq 5\}$

$= \{(a,3) \mid a \leq 5\} \cup \{(a,5) \mid a \leq 5\} \neq \sigma \circ \rho$

3) $\rho = \{(3,5),(5,3),(3,3),(5,5)\} \subseteq \mathbb{N}\times\mathbb{N}$

$\gamma = \{(x,y) \mid y - x = 12\} \subseteq \mathbb{N}\times\mathbb{N}$

$\sigma = \{(x,y) \mid x \leq y\} \subseteq \mathbb{N}\times\mathbb{N}$

$\gamma \circ \rho,\ \sigma \circ \gamma = ?$

<u>Sol:</u> $\gamma \circ \rho = \{(x,z) \mid \exists y$ a.î. $(x,y) \in \rho,\ (y,z) \in \gamma\}$

3 5	5, 17
5 3	3, 15
3 3	3 15
5 5	5 17

$= \{(3,17),(5,15),(3,15),(5,17)\}.$

$\sigma \circ \gamma = \{(a,c) \mid \exists b$ a.î. $(a,b) \in \gamma,\ (\underbrace{b}_{},c) \in \sigma\}$

$b = a+12$ $\qquad a+12 \leq c$

$= \{(a,c) \mid a+12 \leq c\}$

4) $\rho = \{(2a, 3b) \mid a, b \in \mathbb{Z}\} \subseteq \mathbb{Z} \times \mathbb{Z}$

$\rho^n = ?, \ n \in \mathbb{N}^* \quad \left(\rho^0 \stackrel{def}{=} \Delta_{\mathbb{Z}} \right)$

$\rho^2 = \{(x, z) \mid \exists y$ a.î. $(\underset{2a}{x}, \underset{3b}{y}) \in \rho, (\underset{3b}{y}, z) \in \rho\}$

$= \{(2a, z) \mid \exists c$ a.î. $(\underset{2 \cdot 3c}{6c}, \underset{3d}{z}) \in \rho\} = \{(2a, 3d) \mid a, d \in \mathbb{Z}\}$
$(b = 2c)$
$= \rho$

$\Rightarrow \rho^n = \rho, \ \forall n \in \mathbb{N}^*.$

<u>Temă</u> Dacă $\rho = \{(a, a+3) \mid a \in \mathbb{N}\} \subseteq \mathbb{N} \times \mathbb{N}$,

calculați ρ^n, $n \in \mathbb{N}^*$.

<u>Indicație:</u> $\rho^2 = \{(a, c) \mid \exists b$ a.î. $(a, \underset{a+3}{b}), (\underset{a+3}{b}, \underset{a+6}{c}) \in \rho\}$

$= \{(a, a+6) \mid a \in \mathbb{N}\} \neq \rho.$