1) Pe $\mathbb{R}$: $x \rho y \iff (x = y \vee x + y = 7)$.

(i) ρ = rel. de echiv.

(ii) $\mathbb{R}/_{\rho} \cong ?$

Sol: (i) ρ = reflexivă, simetrică și tranzitivă.

- $\forall x \in \mathbb{R}$: $x \rho x \overset{def}{\iff} (x = x \vee x + x = 7)$
- $\forall x, y \in \mathbb{R}$: $x \rho y \Rightarrow y \rho x$

$$\begin{matrix} x \rho y & \Rightarrow & y \rho x \\ {\scriptstyle def}\Updownarrow & & \Updownarrow{\scriptstyle def} \\ (x = y \vee x + y = 7) & & (y = x \vee y + x = 7) \end{matrix}$$

- $\forall x, y, z \in \mathbb{R}$: $x \rho y \wedge y \rho z \Rightarrow x \rho z$

$$x \rho y \wedge y \rho z = (x = y \vee x + y = 7) \wedge (y = z \vee y + z = 7)$$

$$= (x = y \wedge y = z) \vee (x = y \wedge y + z = 7) \vee$$

$$\vee (x + y = 7 \wedge y = z) \vee (x + y = 7 \wedge y + z = 7)$$

$$= (x = y = z) \vee (x = y \wedge x + z = 7) \vee$$

$$\vee (x + z = 7 \wedge y = z) \vee (x = z \wedge x + y = 7)$$

$$\Rightarrow (x = z \vee x + z = 7) = x \rho z.$$

Deci ρ = rel. de echivalență.

Altfel: $x \rho y = (x = y \vee x + y = 7)$

$$= ((x-y)(x+y-7) = 0)$$

$x \rho y \iff x^2 - y^2 = 7(x-y)$.

(ii) $\mathbb{R}/\rho = ?$; $\mathbb{R}/\rho = \{\hat{x} \mid x \in \mathbb{R}\}$.

$\hat{x} = \{y \in \mathbb{R} \mid y \rho x\} = \{y \in \mathbb{R} \mid y = x \vee y + x = 7\}$

$\Rightarrow \hat{x} = \{x, 7 - x\}$, dacă $x \neq \frac{7}{2}$

$\widehat{\frac{7}{2}} = \{\frac{7}{2}\}$.

$\mathbb{R}/\rho = \{\frac{7}{2}\} \cup \{\{x, 7-x\} \mid x \in (-\infty, \frac{7}{2})\}$

$\mathbb{R}/\rho \xrightarrow{f} (-\infty, \frac{49}{4}]$

$f(\hat{x}) = x(7-x)$

f = bine def: $\hat{x} = \hat{y} \Rightarrow x(7-x) = y(7-y)$

$\Updownarrow$

$x \rho y \iff x^2 - y^2 = 7(x-y) \iff 7x - x^2 = 7y - y^2$

Am arătat că $\hat{x} = \hat{y} \iff f(\hat{x}) = f(\hat{y}) \iff x(7-x) = y(7-y)$

„$\Rightarrow$" f = bine definită

„$\Leftarrow$" f = injectivă.

$\operatorname{Im} f = \{x(7-x) \mid x \in \mathbb{R}\} = \{7x - x^2 \mid x \in \mathbb{R}\}$

$= \{-x^2 + 2 \cdot \frac{7}{2} \cdot x - \frac{49}{4} + \frac{49}{4} \mid x \in \mathbb{R}\}$ V $(\frac{7}{2}, +\frac{49}{4})$

$= \{\frac{49}{4} - (x - \frac{7}{2})^2 \mid x \in \mathbb{R}\}$.

$= (-\infty, \frac{49}{4}]$

2) $A=\{0,1,2,3\}$

$\rho=\{(1,2),(3,3),(1,0)\}\subseteq A\times A.$

$R(\rho),\ S(\rho),\ T(\rho),\ E(\rho)=?$

Sol: (i) $R(\rho)=\rho\cup\Delta_A=\{(1,2),(1,0),(0,0),(1,1),(2,2),(3,3)\}$

(ii) $S(\rho)=\rho\cup\rho^{-1}$

$=\{(1,2),(2,1),(3,3),(1,0),(0,1)\}.$

(iii) $T(\rho)=\bigcup\limits_{n\geq 1}\rho^n$

$\rho=\{(1,2),(3,3),(1,0)\}$

$\rho^2=\{(a,c)\mid \exists b\in A:\ (\underset{3}{a},\underset{3}{b}),(\underset{3}{b},\underset{3}{c})\in\rho\}=\{(3,3)\}.$

$\rho^3=\{(a,c)\mid \exists b\in A:\ (\underset{3}{a},\underset{3}{b})\in\rho^2\wedge(\underset{3}{b},\underset{3}{c})\in\rho\}=\{(3,3)\}.$

Inductiv, $\rho^n=\{(3,3)\}\ \forall n\geq 2.$

$T(\rho)=\rho$

(iv) $E(\rho)=\bigcup\limits_{n\in\mathbb{N}}(\rho\cup\rho^{-1})^n$; $\tau\overset{not}{=}\rho\cup\rho^{-1}=S(R)$

$=\{(1,2),(2,1),(3,3),$

$(1,0),(0,1)\}.$

$\tau^2=\{(a,c)\in A\times A\mid \exists b\in A:\ (a,b),(b,c)\in\tau\}$

$$\begin{array}{cc|cc} a & b & b & c \\ 0 & 1 & 1 & 2,0 \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 0,2 \\ 3 & 3 & 3 & 3 \end{array}$$

$=\{(0,0),(0,2),(1,1),(2,0),$

$(2,2),(3,3)\}=\Delta_A\cup\{(0,2),(2,0)\}.$

$\tau^3=\{(a,c)\mid \exists b\in A:\ (a,b)\in\tau^2,\ (b,c)\in\tau\}$

$\in\Delta_A,\ (0,2)\wedge(2,1)$

$(2,0)\wedge(0,1)$

$=\{(a,c)\in\tau\}\cup\{(0,1),(2,1)\}=\tau\cup\{(0,1),(2,1)\}=\tau$

Inductiv, $\tau^{2n}=\tau^2\ \forall n\geq 1$

$\tau^{2n+1}=\tau\ \forall n\geq 0$ $\Rightarrow E(\rho)=\Delta_A\cup\tau\cup\tau^2$

$=\tau\cup\Delta_A\cup\{(0,2),(2,0)\}$

$=\Delta_A\cup\{(1,2),(2,1),(0,1),(1,0),(0,2),(2,0)\}$

3) $\rho = \{(a,2a) \mid a \in \mathbb{N}\} \subseteq \mathbb{N} \times \mathbb{N}$

$E(\rho) = ?$, un sist complet de repr. pt. $E(\rho)$.

Sol: $E(\rho) = \bigcup_{n \in \mathbb{N}} (\rho \cup \rho^{-1})^n = \bigcup_{n \in \mathbb{N}} \tau^n$, $\tau = \rho \cup \rho^{-1}$

$= \{(a,2a), (2a,a) \mid a \in \mathbb{N}^*\} \cup \{(0,0)\}$

$\tau^2 = \{(x,z) \mid \exists y$ a.î. $(x,y), (y,z) \in \tau\}$

$x \; 2x \quad 2x \; x, 4x$

$2x \; x \quad x \; 2x$

$4x\, 2x \quad 2x \quad x$

$= \{(x,x), (x,4x), (2x,2x), (4x,x)\} = \Delta_{\mathbb{N}} \cup \{(x,4x), (4x,x) \mid x \in \mathbb{N}^*\}$

$\tau^3 = \{(x,z) \mid \exists y$ a.î. $(x,y) \in \tau^2, (y,z) \in \tau\}$.

$\in \Delta_{\mathbb{N}}$

$(x,4x), (4x, 2x)$
$8x$

$= \tau \cup \{(x,8x), (8x,x) \mid x \in \mathbb{N}^*\}$

$(4x,x), (x,z) \in \tau$
$2x$

$(8x, 2x), (2x, x)$

Afirm că $E(\rho) = \{(x,y) \mid \exists k \in \mathbb{Z}$ a.î. $y = 2^k x\}$

$\Leftrightarrow \overset{not}{=} \sigma$ rel de echiv pe $\mathbb{N}$ (i)

$\rho \subseteq \sigma$ (ii)

$\forall \theta$ altă rel de echiv pe (iii)

$\mathbb{N}$ a.î. $\rho \subseteq \theta \Rightarrow \sigma \subseteq \theta$.

(i) $\cdot\, x \sigma x \;\forall x \in \mathbb{N}$ (iau $k=0$)

$\cdot\, (x,y) \in \sigma \Rightarrow \exists k \in \mathbb{Z}$ a.î. $y = 2^k x \Rightarrow x = 2^{-k} y$ cu $-k \in \mathbb{Z} \Rightarrow (y,x) \in \sigma$.

$\cdot\, (x,y), (y,z) \in \sigma \Rightarrow \exists k, l \in \mathbb{Z}$ a.î. $\begin{cases} y = 2^k x \\ z = 2^l y \end{cases}$

$\Rightarrow z = 2^{k+l} x$ cu $k+l \in \mathbb{Z}$.

(ii) $\rho = \{(x,2x) \mid x \in \mathbb{N}\} \subset \tau = \{(x,y) \mid \exists k \in \mathbb{Z}$ a.î. $y = 2^k \cdot x\}$

(iii) E suf să arăt că $\sigma \subseteq E(\rho) = \bigcup_{n \in \mathbb{N}} (\rho \cup \rho^{-1})^n$

rel de echiv pe $\mathbb{N}$ ce conține pe ρ

cea mai mică rel de echiv pe $\mathbb{N}$ ce conține pe ρ

$\tau = \{(x,y) \mid \exists k \in \mathbb{Z}$ a.î. $y = 2^k x\} \subseteq E(\rho) = \bigcup_{n \in \mathbb{N}} (\rho \cup \rho^{-1})^n$

$y = 2^k x,\ k \in \mathbb{Z} \Rightarrow (x,y) \in (\rho \cup \rho^{-1})^k = \tau^k \ (\cancel{=} \tau^{-k})$

$\Updownarrow$

$\exists s_1, \dots, s_{k-1} \in \mathbb{N}$ a.î. $(x, s_1), (s_1, s_2), \dots, (s_{k-1}, y) \in \tau$.

Dacă $\cdot\ k \in \mathbb{Z}^*_-$: $x = 2^{-k} y$, $\tilde{s}_1 = 2^{-k-1} y$, ..., $\tilde{s}_{-k+1} = 2y$ ($y = 2^k x$)

$\cdot\ k = 0$: $x = y$ și $(x,y) \in \tau^0 = \Delta_{\mathbb{N}}$.

$\cdot\ k \in \mathbb{N}$: $s_1 = 2x$, $s_2 = 2^2 x$, ..., $s_{k-1} = 2^{k-1} x$

Deci $E(\rho) = \{(x,y) \mid \exists k \in \mathbb{Z}$ a.î. $y = 2^k x\}$

Un SCR pentru ρ este:

$$\hat{x} = \{y \in \mathbb{N} \mid y\, E(\rho)\, x\} = \{2^k \cdot x \mid k \in \mathbb{Z}\} \cap \mathbb{N}$$

$x = 2^{n_x} \cdot x'$ cu $n_x \in \mathbb{N}$ și x' = impar (din $2\mathbb{N}+1$)

$$\hat{x} = \{2^{n_x + k} \cdot x' \mid k \in \mathbb{Z} \text{ a.î. } n_x + k \geq 0\}$$

$= \hat{x'}$, $\forall x \in \mathbb{N}^*$ iar $\hat{0} = \{0\}$.

Un SCR pentru $E(\rho)$ este $\{0\} \cup (2\mathbb{N}+1) =$

$= \{0, 1, 3, 5, 7, \dots\}$.