Seminar 7

1) $f \in End(\mathbb{R}^3)$, $\mathcal{R}_0 = \{e_1, e_2, e_3\}$ reper canonic în $\mathbb{R}^3$

a) $f(e_1) = e_2$

$f(e_2) = e_1 + e_2 + e_3$

$f(e_3) = e_2$

Precizați $\mathcal{R}$ reper în $\mathbb{R}^3$ a.î. $[f]_{\mathcal{R},\mathcal{R}}$ este matrice diagonală.

$$[f]_{\mathcal{R},\mathcal{R}} = A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$f(x) = y \iff AX = Y \Rightarrow f(x) = (x_2, x_1 + x_2 + x_3, x_2)$

$$P(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} \underset{=}{\overset{C_1 - C_3}{=}} \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ \lambda & 1 & -\lambda \end{vmatrix} =$$

$$= \lambda \cdot \begin{vmatrix} -1 & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} \underset{=}{\overset{L_1+L_3}{=}} \lambda \cdot \begin{vmatrix} 0 & 2 & -\lambda \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = \lambda \cdot \begin{vmatrix} 2 & -\lambda \\ 1-\lambda & 1 \end{vmatrix} =$$

$$\cancel{= \lambda \cdot (2 + 1 - \lambda) = \lambda(3 - \lambda)}$$

$$= \lambda(2 + \lambda - \lambda^2) = \lambda(\lambda+1)(2-\lambda)$$

$\lambda_1 = 0$, $m_1 = 1 = \dim V_{\lambda_1}$

$\lambda_2 = -1$, $m_2 = 1 = \dim V_{\lambda_2}$

$\lambda_3 = 2$, $m_3 = 1 = \dim V_{\lambda_3}$

$V_{\lambda_1} = \{x \in \mathbb{R}^3 / f(x) = \lambda_1 x = 0\} = \ker f$.

$$AX = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\dim V_{\lambda_1} = 3 - rg A = 3 - 2 = 1$

$$\begin{cases} x_2 = 0 \\ x_1 + x_2 = -x_3 \end{cases} \Leftrightarrow \begin{cases} x_2 = 0 \\ x_1 = -x_3 \end{cases}.$$

$\Rightarrow V_{\lambda_1} = \{(-x_3, 0, x_3) / x_3 \in \mathbb{R}\} = \langle \{(-1, 0, 1)\} \rangle$

$\Rightarrow \mathcal{R}_1 = \{(-1, 0, 1)\}$ reper în V_{λ_1}.

$V_{\lambda_2} = \{ x \in \mathbb{R}^3 / f(x) = \lambda_2 x = -x \}$

$$\begin{cases} x_2 = x_1 \\ x_1 + x_2 + x_3 = -x_2 \\ x_2 = -x_3 \end{cases} \Leftrightarrow \begin{cases} x_2 + x_1 = 0 \\ x_1 + 2x_2 + x_3 = 0 \\ x_2 + x_3 = 0 \end{cases} \qquad rg \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} = 2$$

$\Rightarrow \dim V_{\lambda_2} = 1.$

$$\begin{cases} x_2 = -x_3 \\ x_1 = -x_2 = x_3 \end{cases}.$$

$\cancel{\Rightarrow V_{\lambda_2} = \{(x_3, -x_3, x_3) / x_3 \in \mathbb{R}\} = \langle \{(1, -1, 1)\}}$

$\Rightarrow V_{\lambda_2} = \{(x_3, -x_3, x_3) / x_3 \in \mathbb{R}\} = \langle \{(1, -1, 1)\} \rangle$

$\Rightarrow \mathcal{R}_2 = \{1, -1, 1\}$ reper în V_{λ_2}.

$V_{\lambda_3} = \{ x \in \mathbb{R}^3 / f(x) = \lambda_3 x = 2x \}$

$AX = 2X$

$$(A - 2I_3) \cdot X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \dim V_{\lambda_3} = 3-2=1.$

$$\begin{cases} x_1 - x_2 = -x_3 \\ x_2 = 2x_3 \end{cases}$$

$x_1 = x_3$

$\Rightarrow V_{\lambda_3} = \{(x_3, 2x_3, x_3) / x_3 \in \mathbb{R}\} = \langle \{(1,2,1)\} \rangle$

$\Rightarrow \mathcal{R}_3 = \{1,2,1\}$ reper în V_{λ_3}

$\Rightarrow \mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 = \{(-1,0,1), (1,-1,1), (1,2,1)\}$

$$A' = [f]_{\mathcal{R},\mathcal{R}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

2) a) $f \in End(\mathbb{R}^3)$, $\mathcal{R}_0 = \{e_1, e_2, e_3\}$ reper canonic din $\mathbb{R}^3$

$$A = [f]_{\mathcal{R}_0, \mathcal{R}_0} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Determinați un reper $\mathcal{R}$ în $\mathbb{R}^3$ a.î. $[f]_{\mathcal{R},\mathcal{R}}$ este matrice diagonală

$$P(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} \underline{L_1 + L_2 + L_3}$$

$$= \begin{vmatrix} 2-\lambda & 2-\lambda & 2-\lambda \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = (2-\lambda) \cdot \begin{vmatrix} 1 & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} =$$

$$\frac{c_2 - c_1}{c_3 - c_1} \begin{vmatrix} 1 & 0 & 0 \\ 1 & -\lambda-1 & 0 \\ 1 & 0 & -\lambda-1 \end{vmatrix} \cdot (2-\lambda) = (2-\lambda) \cdot \begin{vmatrix} -\lambda-1 & 0 \\ 0 & -\lambda-1 \end{vmatrix} =$$

$= (2-\lambda) \cdot (\lambda+1)^2$

$\lambda_1 = 2, \ m_1 = 1 = \dim V_{\lambda_1}$

$\lambda_2 = -1, \ m_2 = 2 = \dim V_{\lambda_2}$

$V_{\lambda_1} = \{x \in \mathbb{R}^3 / f(x) = \lambda_1 x = 2x\}$

$AX = 2X$

$\Rightarrow (A - 2I_3) \cdot X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} \boxed{\begin{matrix} -2 & 1 \\ 1 & -2 \end{matrix}} & \begin{matrix} 1 \\ 1 \end{matrix} \\ \begin{matrix} 1 & 1 \end{matrix} & -2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\dim V_{\lambda_1} = 3 - rg(A - 2I_3) = 3 - 2 = 1$

$$\begin{cases} -2x_1 + x_2 = -x_3 & | \cdot 2 \\ x_1 - 2x_2 = -x_3 \end{cases}$$

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$-3x_1 = -3x_3 \Rightarrow x_1 = x_3.$

$x_2 = x_3.$

$\Rightarrow V_{\lambda_1} = \langle \{(1,1,1)\} \rangle$

$R_1 = \{(1,1,1)\}$ reper in V_{λ_1}

$V_{\lambda_2} = \{x \in \mathbb{R}^3 / f(x) = -x\}.$

$AX = -X$

$\Leftrightarrow (A + I_3) \cdot X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} \boxed{1} & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow x_1 = -x_2 - x_3.$$

$\Rightarrow \dim V_{\lambda_2} = 3 - rg(A + I_3) = 3 - 1 = 2.$

$\Rightarrow V_{\lambda_2} = \{(-x_2-x_3, x_2, x_3) / x_2, x_3 \in \mathbb{R}\} = <\{(-1,1,0), (-1,0,1)\}>$

$\qquad (-x_2, x_2, 0) + (-x_3, 0, x_3)$

$\mathcal{R}_2 = \{(-1,1,0), (-1,0,1)\}$ reper în V_{λ_2} (SG + $\dim V_{\lambda_2} = 2$)

$$[f]_{\mathcal{R},\mathcal{R}} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Obs: $\mathcal{R}_0 \xrightarrow{C} \mathcal{R}$

$[f]_{\mathcal{R}_0,\mathcal{R}_0} = A \qquad [f]_{\mathcal{R}_0,\mathcal{R}_0} = A'$

$A' = C^{-1} A C$

$$C = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$A' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 2^n & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & (-1)^n \end{pmatrix}$$

$\Rightarrow A = C A' C^{-1}$

$\Rightarrow A^n = C A'^n \cdot C^{-1}$

5) $f \in End(\mathbb{R}^3)$, $\lambda_1 = 3$, $\lambda_2 = -2$, $\lambda_3 = 1$ val. proprii

$v_1 = (-3, 2, 1)$, $v_2 = (-2, 1, 0)$, $v_3 = (-6, 3, 1)$ vect. proprii

$A = [f]_{\mathcal{R}_0,\mathcal{R}_0} = ?$

$f(v_1) = \lambda_1 v_1$

$f(v_2) = \lambda_2 v_2$

$f(v_3) = \lambda_3 v_3$.

$\mathcal{R} = \{v_1, v_2, v_3\}$ SLI (vectori proprii corespunzători la valori proprii distincte) $|\mathcal{R}| = 3 = \dim_{\mathbb{R}} \mathbb{R}^3$ $\Rightarrow \mathcal{R}$ reper în $\mathbb{R}^3$

$$A' = [f]_{\mathcal{R}}, \mathcal{R} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{R}_0 \xrightarrow{C} \mathcal{R}$$

$$A' = C^{-1} A C \Rightarrow A = C A' C^{-1}$$

$v_1 = -3e_1 + 2e_2 + e_3$

$v_2 = -2e_1 + e_2$

$v_3 = -6e_1 + 3e_2 + e_3$

$$C = \begin{pmatrix} -3 & -2 & -6 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\det C \overset{C_3 - C_1}{=\!=} \begin{vmatrix} -3 & -2 & -3 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} -2 & -3 \\ 1 & 1 \end{vmatrix} = 1 \neq 0.$$

$$C^t = \begin{vmatrix} -3 & 2 & 1 \\ -2 & 1 & 0 \\ -6 & 3 & 1 \end{vmatrix}$$

$$C^* = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & -3 \\ -1 & -2 & 1 \end{pmatrix} = C^{-1}$$

$$A = \begin{pmatrix} -3 & -2 & -6 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & -3 \\ -1 & -2 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} -9 & 4 & -6 \\ 6 & -2 & 3 \\ 3 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & -3 \\ -1 & -2 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 6 & -18 \\ 1 & 0 & 9 \\ 2 & 4 & 1 \end{pmatrix}$$

Pentru a afla f:

$$A \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Forme biliniare

$g : V \times V \to \mathbb{K}$

$g_{ij} = g(e_i, e_j)$ $\qquad \mathcal{R} \xrightarrow{C} \mathcal{R}'$

$G = (g_{ij})_{i,j=\overline{1,n}}$ în $\mathcal{R}$.

$G' = (g'_{ij})_{i,j=\overline{1,n}}$ în $\mathcal{R}'$

$\Rightarrow G' = C^T G C$

7) $g : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ formă biliniară antisimetrică

$\mathcal{R}_0 = \{e_1, e_2\}$ reper canonic în $\mathbb{R}^2$

$g(e_1, e_2) = 5$

a) $G = ?$ (matricea asociată lui g în rap. cu $\mathcal{R}_0$).

b) $g = ?$

a) $\left.\begin{array}{l} g_{ij} = g(e_i, e_j) \\ G^T = -G \end{array}\right\} \Rightarrow g_{11} = g_{22} = 0$

$\left.\begin{array}{l} g_{11} = g_{22} = 0 \\ g_{12} = 5 \Rightarrow g_{21} = -5 \end{array}\right\} \Rightarrow G = \begin{pmatrix} 0 & 5 \\ -5 & 0 \end{pmatrix}$.

$g(x, y) = g(x_1 e_1 + x_2 e_2, y_1 e_1 + y_2 e_2)$

$g_{11} x_1 y_1 + g_{12} x_1 y_2 + g_{21} x_2 y_1 + g_{22} x_2 y_2 = X^T G Y$

$g(x, y) = 5 x_1 y_2 - 5 x_2 y_1$

$$\boxed{g(x,y)=\sum_{i,j=1}^{n} g_{ij}x_iy_j \quad \text{ÎN GENERAL.} \Rightarrow g\text{ - biliniară}}$$

8) $g: \mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}$, $g(x,y)=x_1y_1-x_2y_2-x_1y_3-x_3y_1+$
$+2x_2y_3+2x_3y_2 = X^TGY \Rightarrow g$ biliniară (1)

a) $g\in L^s(\mathbb{R}^3,\mathbb{R}^3;\mathbb{R})$

b) $G=?$ (matricea asociată lui g în rap. cu $\mathcal{R}_0=\{e_1,e_2,e_3\}$ reper canonic

c) $\ker g=?$ Este g nedegenerată?

d) Să se afle G' (matricea asociată lui g în rap. cu
$\mathcal{R}'=\{e_1'=(1,1,1), e_2'=(1,2,1), e_3'=(0,0,1)\}$.
$G'=CᵀGC$

a,b) $G=\begin{pmatrix}1 & 0 & -1\\ 0 & -1 & 2\\ -1 & 2 & 0\end{pmatrix}$; $G=G^T$ (2)

Din (1),(2) $\Rightarrow g\in L^s(\mathbb{R}^3;\mathbb{R}^3;\mathbb{R})$.

c) $\ker g=\{x\in\mathbb{R}^3 / g(x,y)=0, \forall y\in\mathbb{R}^3\}$.

$$\begin{cases} g(x,e_1)=0\\ g(x,e_2)=0\\ g(x,e_3)=0.\end{cases} \Leftrightarrow \begin{cases} x_1-x_3=0\\ -x_2+2x_3=0\\ -x_1+2x_2=0\end{cases} \quad \begin{pmatrix}1 & 0 & -1\\ 0 & -1 & 2\\ -1 & 2 & 0\end{pmatrix}=G$$

$$\det G=\begin{vmatrix}1 & 0 & -1\\ 0 & -1 & 2\\ -1 & 2 & 0\end{vmatrix} \underset{L_3+L_1}{=} \begin{vmatrix}1 & 0 & -1\\ 0 & -1 & 2\\ 0 & 2 & -1\end{vmatrix}\neq 0.$$

g nedegenerată $\Leftrightarrow \ker g = 0_{\mathbb{R}^3} \Leftrightarrow \det G\neq 0.$

d) $\mathcal{R}_0 \xrightarrow{C} \mathcal{R}'$ $\quad C = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

$$G' = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 2 \\ -1 & 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 3 \\ -1 & 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 3 & 3 \\ 1 & 3 & 0 \end{pmatrix}$$