Seminar 2

1. Det. $\underline{\lim}\, x_n$, $\overline{\lim}\, x_n$ și precizați dacă există $\lim\limits_{n\to\infty} x_n$, unde:

a) $x_n = 1 + 2(-1)^{n+1} + 3(-1)^{\frac{n(n+1)}{2}}$, $\forall n \in \mathbb{N}$.

Sol.: $x_{4n} = 1+2(-1)^{4n+1} + 3\cdot(-1)^{\frac{\overset{2}{\cancel{4n}}(4n+1)}{\cancel{2}}} = 1-2+3 = 2 \xrightarrow[n\to\infty]{} 2$.

$x_{4n+1} = 1+2(-1)^{4n+2} + 3(-1)^{\frac{(4n+1)(\overset{2}{\cancel{4n+2}})^{1}}{\cancel{2}}} = 1+2-3 = 0 \xrightarrow[n\to\infty]{} 0$.

$$x_{4n+2} = 1+2(-1)^{4n+3} + 3(-1)^{\frac{(\overset{2}{\cancel{4}}n+\overset{1}{\cancel{2}})(4n+3)}{\cancel{2}}} = 1-2-3=-4 \xrightarrow[n\to\infty]{} -4.$$

$$x_{4n+3} = 1+2(-1)^{4n+4} + 3(-1)^{\frac{(4n+3)(\overset{2}{\cancel{4}}n+\overset{2}{\cancel{4}})}{\cancel{2}}} = 1+2+3=6 \xrightarrow[n\to\infty]{} 6.$$

Cum $\mathbb{N} = 4\mathbb{N} \cup (4\mathbb{N}+1) \cup (4\mathbb{N}+2) \cup (4\mathbb{N}+3)$ rezultă că $\mathcal{L}((x_n)_n) = \{-4,\ 0,\ 2,\ 6\}$.

Deci $\underline{\lim}\, x_n = -4$ și $\overline{\lim}\, x_n = 6$.

Deoarece $\underline{\lim}\, x_n \neq \overline{\lim}\, x_n$ rezultă că nu există $\lim\limits_{n\to\infty} x_n$. $\square$

b) $x_n = \sin \frac{n\pi}{3}$ $\forall n \in \mathbb{N}$.

<u>Sol.</u>: $\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$

$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$

$\sin(n\pi) = 0$

$\cos(n\pi) = (-1)^n$.

$$x_{6n} = \sin\frac{\cancel{6}^{2}n\pi}{\cancel{3}} = \sin 2n\pi = 0 \xrightarrow[n\to\infty]{} 0.$$

$$x_{6n+1} = \sin\frac{(6n+1)\pi}{3} = \sin\left(2n\pi + \frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \xrightarrow[n\to\infty]{} \frac{\sqrt{3}}{2}.$$

$$x_{6n+2} = \sin\frac{(6n+2)\pi}{3} = \sin\left(2n\pi + \frac{2\pi}{3}\right) = \sin\frac{2\pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right) =$$

$$= \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \xrightarrow[n\to\infty]{} \frac{\sqrt{3}}{2}.$$

$$x_{6n+3} = \sin\frac{(6n+3)\pi}{3} = \sin(2n+1)\pi = 0 \xrightarrow[n\to\infty]{} 0.$$

$$x_{6n+4} = \sin\frac{(6n+4)\pi}{3} = \sin\left(2n\pi + \frac{4\pi}{3}\right) = \sin\frac{4\pi}{3} = \sin\left(\pi + \frac{\pi}{3}\right) =$$

$$= \underbrace{\sin\pi\cos\frac{\pi}{3}}_{=0} + \cos\pi\,\sin\frac{\pi}{3} = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2} \xrightarrow[n\to\infty]{} -\frac{\sqrt{3}}{2}.$$

$$x_{6n+5} = \sin\frac{(6n+5)\pi}{3} = \sin\left(2n\pi + \frac{5\pi}{3}\right) = \sin\frac{5\pi}{3} = \sin\left(2\pi - \frac{\pi}{3}\right) =$$

$$= \underbrace{\sin 2\pi}_{=0} \cos\frac{\pi}{3} - \cos 2\pi \sin\frac{\pi}{3} = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2} \xrightarrow[n\to\infty]{} -\frac{\sqrt{3}}{2}.$$

Cum $\mathbb{N} = 6\mathbb{N} \cup (6\mathbb{N}+1) \cup \dots \cup (6\mathbb{N}+5)$ rezultă că

$$\mathcal{L}((x_n)_n) = \left\{-\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}\right\}.$$

Deci $\underline{\lim}\, x_n = -\frac{\sqrt{3}}{2}$ şi $\overline{\lim}\, x_n = \frac{\sqrt{3}}{2}$.

Deoarece $\underline{\lim} x_n \neq \overline{\lim} x_n$ rezultă că nu există $\lim\limits_{n\to\infty} x_n$. $\square$

c) $x_n = \dfrac{n \cos \frac{n\pi}{2}}{n^2+1} \quad \forall n \in \mathbb{N}$.

$\underline{\text{Sol}}$.: $-1 \leq \cos \frac{n\pi}{2} \leq 1 \quad \forall n \in \mathbb{N} \Leftrightarrow -\dfrac{n}{n^2+1} \leq \dfrac{n \cdot \cos \frac{n\pi}{2}}{n^2+1} \leq \dfrac{n}{n^2+1} \quad \forall n \in \mathbb{N}$.

$$\underbrace{-\frac{n}{n^2+1}}_{\to 0} \leq \underbrace{\frac{n\cos\frac{n\pi}{2}}{n^2+1}}_{\to 0} \leq \underbrace{\frac{n}{n^2+1}}_{\to 0}$$

Conform criteriului cleștelui $\lim\limits_{n\to\infty} x_n = 0$ (există $\lim\limits_{n\to\infty} x_n$).

Deci $\underline{\lim} x_n = \overline{\lim} x_n = 0$. $\square$

2. Det. suma seriei $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$ și precizați dacă este convergentă.

Sol.: $x_n = \frac{n}{(n+1)!} \quad \forall n \in \mathbb{N}^*$.

$$s_n = x_1 + x_2 + \ldots + x_n = \sum_{k=1}^{n} x_k = \sum_{k=1}^{n} \frac{k}{(k+1)!} = \sum_{k=1}^{n} \frac{k+1-1}{(k+1)!} =$$

$$= \sum_{k=1}^{n} \left(\frac{\cancel{k+1}}{\underbrace{\cancel{(k+1)!}}_{k!}} - \frac{1}{(k+1)!} \right) = \sum_{k=1}^{n} \left(\frac{1}{k!} - \frac{1}{(k+1)!} \right) =$$

$$= \left(\frac{1}{1!} - \cancel{\frac{1}{2!}}\right) + \left(\cancel{\frac{1}{2!}} - \cancel{\frac{1}{3!}}\right) \cancel{+} \ldots \cancel{+} \left(\cancel{\frac{1}{n!}} - \frac{1}{(n+1)!}\right) =$$

$$= \underline{1 - \frac{1}{(n+1)!}} \quad \forall\, n \in \mathbb{N}^*.$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{(n+1)!}\right) = 1.$$

Deci $\sum_{n=1}^{\infty} x_n = 1$, i.e. $\sum_{n=1}^{\infty} x_n$ este convergentă. $\square$

3. Studiați convergența (natura) seriilor:

a) $$\sum_{n=1}^{\infty} \frac{1\cdot 4\cdot 7\cdot \ldots \cdot (3n-2)}{3\cdot 6\cdot 9\cdot \ldots \cdot (3n)} \cdot \frac{1}{2^n}.$$

<u>Sol.</u>: $$x_n = \frac{1\cdot 4\cdot 7\cdot \ldots \cdot (3n-2)}{3\cdot 6\cdot 9\cdot \ldots \cdot (3n)} \cdot \frac{1}{2^n} \quad \forall n \in \mathbb{N}^*.$$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\cancel{1\cdot 4\cdot 7\cdot \ldots \cdot (3n-2)}\,(3(n+1)-2)}{\cancel{3\cdot 6\cdot 9\cdot \ldots \cdot (3n)}\,(3(n+1))} \cdot \frac{1}{\underset{2}{\cancel{2^{n+1}}}} \cdot \frac{\cancel{3\cdot 6\cdot 9\cdot \ldots \cdot (3n)}}{\cancel{1\cdot 4\cdot 7\cdot \ldots \cdot (3n-2)}} \cdot \frac{\cancel{2^n}}{1} =$$

$$= \lim_{n\to\infty} \frac{3n+1}{3n+3} \cdot \frac{1}{2} = \frac{1}{2} < 1.$$

Conform Crit. rap. rezultă că $\sum_{n=1}^{\infty} x_n$ este conv. $\square$

b) $\sum_{n=1}^{\infty} \frac{\sqrt{n-1}}{n^2}$.

<u>Sol</u>: $x_n = \frac{\sqrt{n-1}}{n^2}$ $\forall n \in \mathbb{N}^*$.

Fie $y_n = \frac{\sqrt{n}}{n^2} = \frac{\cancel{\sqrt{n}}}{n \cdot \cancel{n}\sqrt{n}} = \frac{1}{n\sqrt{n}} = \frac{1}{n^{\frac{3}{2}}}$ $\forall n \in \mathbb{N}^*$.

Avem $x_n < y_n \ \forall n \in \mathbb{N}^*$.

$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ convergentă (serie armonică generalizată cu $\alpha = \frac{3}{2}$).

Conform crit. de comparație cu inegalități rezultă că $\sum_{n=1}^{\infty} x_n$ este convergentă. $\square$

c) $\sum_{n=1}^{\infty} \frac{a^n}{\sqrt[n]{n}}$, $a>0$.

Sol.: $x_n = \frac{a^n}{\sqrt[n]{n}}$ $\forall n \in \mathbb{N}^*$.

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\cancel{a^{n+1}}^{\,a}}{\sqrt[n+1]{n+1}} \cdot \frac{\sqrt[n]{n}}{\cancel{a^n}} = \lim_{n\to\infty} a \cdot \frac{\sqrt[n]{n}}{\sqrt[n+1]{n+1}} = a.$$

$\sqrt[n]{n} \xrightarrow{n\to\infty} 1$

$\sqrt[n+1]{n+1} \xrightarrow{n\to\infty} 1$

Conform Crit. rap. avem:

1) Dacă $a<1$ (i.e. $a\in(0,1)$), atunci seria este conv.

2) Dacă $a>1$ (i.e. $a\in(1,\infty)$), atunci seria este div.

3) Dacă $a=1$, atunci Crit. rap. nu decide, dar, în acest caz, $x_n = \frac{1^n}{\sqrt[n]{n}} = \frac{1}{\sqrt[n]{n}}$ $\forall\, n\in\mathbb{N}^*$.

Seria devine $\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}}$.

$$\lim_{n\to\infty} x_n = \frac{1}{1} = 1 \neq 0.$$

Conform criteriului suficient de divergență rezultă că seria $\sum_{n=1}^{\infty} x_n$ este div.

Am obținut:
$$\sum_{n=1}^{\infty} \frac{a^n}{\sqrt[n]{n}} \begin{cases} \text{conv., dacă } a \in (0,1). \\ \text{div., dacă } a \in [1,\infty). \end{cases} \quad \square$$

d) $\sum_{n=1}^{\infty} \frac{\sqrt{n^2+1}}{\sqrt{n^3+1}}$.

Sol: $x_n = \frac{\sqrt{n^2+1}}{\sqrt{n^3+1}} \quad \forall\, n \in \mathbb{N}^*$.

Fie $y_n = \frac{\sqrt{n^2}}{\sqrt{n^3}} = \frac{\cancel{n}}{\cancel{n}\sqrt{n}} = \frac{1}{\sqrt{n}} = \frac{1}{n^{\frac{1}{2}}} \quad \forall\, n \in \mathbb{N}^*$.

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{\sqrt{n^2+1}}{\sqrt{n^3+1}} \cdot \frac{\sqrt{n^3}}{\sqrt{n^2}} = 1 \in (0, \infty).$$

Conform Crit. de comparație cu limită avem că

$$\sum_{n=1}^{\infty} x_n \sim \sum_{n=1}^{\infty} y_n.$$

$$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$$ div. (serie armonică generalizată cu $\alpha = \frac{1}{2}$).

Prin urmare $\sum_{n=1}^{\infty} x_n$ este div. $\square$

e) $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$.

<u>Sol</u>.: $x_n = \frac{1}{n\ln n}$ $\forall n \in \mathbb{N}^* \setminus \{1\}$.

$\ln n < \ln(n+1) \Longrightarrow x_n > x_{n+1}$ $\forall n \geq 2 \Rightarrow (x_n)_n$ strict descrescător.

Aplicăm crit. condensării.

Avem $\sum_{n=2}^{\infty} x_n \sim \sum_{n=2}^{\infty} 2^n x_{2^n}$.

$$\sum_{n=2}^{\infty} 2^n \cdot x_{2^n} = \sum_{n=2}^{\infty} \cancel{2^n} \cdot \frac{1}{\cancel{2^n} \ln 2^n} = \sum_{n=2}^{\infty} \frac{1}{\ln 2^n} = \sum_{n=2}^{\infty} \frac{1}{n \ln 2}.$$

Fie $y_n = \frac{1}{n \ln 2}$ $\forall n \geq 2$ și $z_n = \frac{1}{n}$ $\forall n \geq 2$.

$$\lim_{n\to\infty} \frac{y_n}{z_n} = \lim_{n\to\infty} \frac{\frac{1}{n\ln 2}}{\frac{1}{n}} = \frac{1}{\ln 2} \in (0,\infty).$$

Conform Crit. de comp. cu limită avem că

$$\sum_{n=2}^{\infty} y_n \sim \sum_{n=2}^{\infty} z_n .$$

$$\sum_{n=2}^{\infty} z_n = \sum_{n=2}^{\infty} \frac{1}{n}$$ div. (serie armonică generalizată cu $\alpha = 1$)

Așadar $\sum_{n=2}^{\infty} y_n$ este div., deci $\sum_{n=2}^{\infty} x_n$ este div. $\square$