Seminar 11

1. Determinați punctele de extrem local ale funcțiilor de mai jos și precizați natura lor.

a) $f:\mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^3 + 8y^3 - 2xy$.

<u>Sol.</u>: $\mathbb{R}^2$ deschisă.

Determinăm punctele critice ale lui f.

f continuă pe $\mathbb{R}^2$.

$$\frac{\partial f}{\partial x} = 3x^2 - 2y.$$

$$\frac{\partial f}{\partial y} = 24y^2 - 2x.$$

$$\left.\begin{array}{l} \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \text{ cont. pe } \mathbb{R}^2 \\ \mathbb{R}^2 \text{ deschisă} \end{array}\right\} \Longrightarrow f \text{ dif. pe } \mathbb{R}^2.$$

$$\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases} \Leftrightarrow \begin{cases} 3x^2 - 2y = 0 \\ 24y^2 - 2x = 0 \end{cases} \Leftrightarrow \begin{cases} 3 \cdot 144 y^4 - 2y = 0 \,\Big|\, :2 \\ x = 12y^2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 216y^4 - y = 0 \\ x = 12y^2 \end{cases} \Leftrightarrow \begin{cases} y(216y^3 - 1) = 0 \\ x = 12y^2 \end{cases} \Leftrightarrow \begin{cases} y = 0 \\ x = 0 \end{cases} \text{ sau}$$

$$\begin{cases} y = \frac{1}{6} \\ x = \frac{1}{3}. \end{cases}$$

Punctele critice ale lui f sunt $(0,0)$ și $\left(\frac{1}{3}, \frac{1}{6}\right)$.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 6x.$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 48y.$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -2 = \frac{\partial^2 f}{\partial y \partial x}.$$

Lema lui Schwarz

Toate derivatele parțiale de ordinul doi ale lui f sunt continue pe $\mathbb{R}^2$.

$$H_f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 6x & -2 \\ -2 & 48y \end{pmatrix} \quad \forall (x,y) \in \mathbb{R}^2.$$

$$H_f(0,0) = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}.$$

$\Delta_1 = 0$

$\Delta_2 = \begin{vmatrix} 0 & -2 \\ -2 & 0 \end{vmatrix} = -4 < 0$

$\not\Rightarrow$ $(0,0)$ nu este punct de extrem local al lui f.

$$H_f\left(\frac{1}{3}, \frac{1}{6}\right) = \begin{pmatrix} 2 & -2 \\ -2 & 8 \end{pmatrix}.$$

$$\left.\begin{array}{l} \Delta_1 = 2 > 0 \\ \Delta_2 = \begin{vmatrix} 2 & -2 \\ -2 & 8 \end{vmatrix} = 16 - 4 = 12 > 0 \end{array}\right| \Rightarrow \left(\frac{1}{3}, \frac{1}{6}\right) \text{ punct de minim local al lui } f. \ \square$$

b) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = 4xy - x^4 - y^4$.

Sol.: $\mathbb{R}^2$ deschisă.

Determinăm punctele critice ale lui f.

f cont. pe $\mathbb{R}^2$.

$$\frac{\partial f}{\partial x} = 4y - 4x^3.$$

$$\frac{\partial f}{\partial y} = 4x - 4y^3.$$

$$\left.\begin{array}{l} \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \text{ cont. pe } \mathbb{R}^2 \\ \mathbb{R}^2 \text{ deschisă} \end{array}\right| \Rightarrow f \text{ dif. pe } \mathbb{R}^2.$$

$$\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases} \Leftrightarrow \begin{cases} 4y - 4x^3 = 0 \mid :4 \\ 4x - 4y^3 = 0 \mid :4 \end{cases} \Leftrightarrow \begin{cases} y - x^3 = 0 \\ x - y^3 = 0 \end{cases} \Leftrightarrow \begin{cases} y - y^9 = 0 \\ x = y^3 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y(1-y^8)=0 \\ x=y^3 \end{cases} \Leftrightarrow \begin{cases} y=0 \\ x=0 \end{cases} \text{ sau } \begin{cases} y=-1 \\ x=-1 \end{cases} \text{ sau } \begin{cases} y=1 \\ x=1 \end{cases}.$$

Punctele critice ale lui f sunt $(0,0)$, $(-1,-1)$ şi $(1,1)$.

$$\frac{\partial^2 f}{\partial x^2} = -12x^2.$$

$$\frac{\partial^2 f}{\partial y^2} = -12y^2.$$

$$\frac{\partial^2 f}{\partial x \partial y} = 4 \underset{\uparrow}{=} \frac{\partial^2 f}{\partial y \partial x}.$$

Lema lui Schwarz

Toate derivatele parţiale de ordinul doi ale lui f sunt cont. pe $\mathbb{R}^2$.

$$H_f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} -12x^2 & 4 \\ 4 & -12y^2 \end{pmatrix} \quad \forall (x,y) \in \mathbb{R}^2.$$

$$H_f(0,0) = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$$

$$\left.\begin{array}{l}\Delta_1 = 0 \\ \Delta_2 = \begin{vmatrix} 0 & 4 \\ 4 & 0 \end{vmatrix} = -16 < 0\end{array}\right| \Longrightarrow (0,0)$$ nu este punct de extrem local al lui f.

$$H_f(-1,-1) = \begin{pmatrix} -12 & 4 \\ 4 & -12 \end{pmatrix}.$$

$$\left.\begin{array}{l}\Delta_1 = -12 < 0 \\ \Delta_2 = \begin{vmatrix} -12 & 4 \\ 4 & -12 \end{vmatrix} = 144 - 16 = 128 > 0\end{array}\right| \Longrightarrow (-1,-1)$$ punct de maxim local al lui f.

$$H_f(1,1) = \begin{pmatrix} -12 & 4 \\ 4 & -12 \end{pmatrix}.$$

$$\left.\begin{array}{l}\Delta_1 = -12 < 0 \\ \Delta_2 = \begin{vmatrix} -12 & 4 \\ 4 & -12 \end{vmatrix} = 144 - 16 = 128 > 0\end{array}\right| \Longrightarrow (1,1)$$ punct de maxim local al lui f. $\square$

c) $f : (0,\infty)^3 = (0,\infty) \times (0,\infty) \times (0,\infty) \longrightarrow \mathbb{R}$,

$$f(x,y,z) = \frac{1}{x} + \frac{x}{y} + \frac{y}{z} + z.$$

__Sol.__: $(0,\infty)^3$ deschisă.

Determinăm punctele critice ale lui f.

f cont. pe $(0,\infty)^3$.

$$\frac{\partial f}{\partial x} = -\frac{1}{x^2} + \frac{1}{y}.$$

$$\frac{\partial f}{\partial y} = -\frac{x}{y^2} + \frac{1}{z}.$$

$$\frac{\partial f}{\partial z} = -\frac{y}{z^2} + 1.$$

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ cont. pe $(0,\infty)^3$

$(0,\infty)^3$ deschisă

$\Longrightarrow f$ dif. pe $(0,\infty)^3$.

$$\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \\ \frac{\partial f}{\partial z} = 0 \end{cases} \Longleftrightarrow \begin{cases} -\frac{1}{x^2} + \frac{1}{y} = 0 \\ -\frac{x}{y^2} + \frac{1}{z} = 0 \\ -\frac{y}{z^2} + 1 = 0 \end{cases} \Longleftrightarrow \begin{cases} x^2 = y \\ -\frac{x}{y^2} + \frac{1}{z} = 0 \\ y = z^2 \end{cases} \Longleftrightarrow$$

$$\Longleftrightarrow \begin{cases} x^2 = y \\ -\frac{x}{y^2} + \frac{1}{z} = 0 \\ y = z^2 = x^2 \end{cases} \overset{x,y,z\in(0,\infty)}{\Longleftrightarrow} \begin{cases} x^2 = y \\ -\frac{x}{x^4} + \frac{1}{x} = 0 \\ x = z \end{cases} \Longleftrightarrow \begin{cases} x^2 = y \\ -\frac{1}{x^2} = -1 \\ x = z \end{cases} \overset{x,y,z\in(0,\infty)}{\Longleftrightarrow} \begin{cases} y = 1 \\ x = 1 \\ z = 1. \end{cases}$$

Singurul punct critic al lui f este $(1,1,1)$.

$$\frac{\partial^2 f}{\partial x^2} = \frac{2}{x^3}.$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{2x}{y^3}.$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{2y}{z^3}.$$

$$\frac{\partial^2 f}{\partial x \partial y} = -\frac{1}{y^2} \underset{\substack{\uparrow \\ \text{Lema lui Schwarz}}}{=} \frac{\partial^2 f}{\partial y \partial x}.$$

$$\frac{\partial^2 f}{\partial x \partial z} = 0 \underset{\substack{\uparrow \\ \text{Lema lui Schwarz}}}{=} \frac{\partial^2 f}{\partial z \partial x}.$$

$$\frac{\partial^2 f}{\partial y \partial z} = -\frac{1}{z^2} \underset{\substack{\uparrow \\ \text{Lema lui Schwarz}}}{=} \frac{\partial^2 f}{\partial z \partial y}.$$

Toate derivatele parțiale de ordinul doi ale lui f sunt continue pe $(0,\infty)^3$.

$$H_f(x,y,z) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{2}{x^3} & -\frac{1}{y^2} & 0 \\ -\frac{1}{y^2} & \frac{2x}{y^3} & -\frac{1}{z^2} \\ 0 & -\frac{1}{z^2} & \frac{2y}{z^3} \end{pmatrix} \quad \forall\ (x,y,z) \in (0,\infty)^3.$$

$$H_f(1,1,1) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

$$\Delta_1 = 2 > 0$$

$$\Delta_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 > 0$$

$$\Delta_3 = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 8 + 0 + 0 - 0 - 2 - 2 = 4 > 0$$

$\Rightarrow$

$\Rightarrow (1,1,1)$ punct de minim local al lui f. $\square$

d) $f:\mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x^2+y^2+z^2-xy+x-2z$.

Sol.: Rezolvați-l voi! $\square$

e) $f:\mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^4+y^4$.

Sol.: $\mathbb{R}^2$ deschisă.

Det. punctele critice ale lui f.

f cont. pe $\mathbb{R}^2$.

$$\frac{\partial f}{\partial x} = 4x^3.$$

$$\frac{\partial f}{\partial y} = 4y^3.$$

$\left.\begin{array}{l}\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y} \text{ cont. pe } \mathbb{R}^2 \\ \mathbb{R}^2 \text{ deschisă}\end{array}\right| \Rightarrow f$ dif. pe $\mathbb{R}^2$.

$$\begin{cases}\dfrac{\partial f}{\partial x} = 0 \\ \dfrac{\partial f}{\partial y} = 0\end{cases} \Leftrightarrow \begin{cases}4x^3 = 0 \\ 4y^3 = 0\end{cases} \Leftrightarrow \begin{cases}x = 0 \\ y = 0.\end{cases}$$

Singurul punct critic al lui f este $(0,0)$.

$$\frac{\partial^2 f}{\partial x^2} = 12x^2.$$

$$\frac{\partial^2 f}{\partial y^2} = 12y^2.$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0 \underset{\uparrow}{=} \frac{\partial^2 f}{\partial y \partial x}.$$

Lema lui Schwarz

Toate derivatele parțiale de ordinul doi ale lui f sunt cont. pe $\mathbb{R}^2$.

$$H_f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{pmatrix} \quad \forall (x,y) \in \mathbb{R}^2.$$

$$H_f(0,0) = \begin{pmatrix} \boxed{0} & 0 \\ 0 & 0 \end{pmatrix}.$$

$$\left.\begin{array}{l} \Delta_1 = 0 \\ \Delta_2 = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0 \end{array}\right\} \Rightarrow \text{Criteriul nu decide.}$$

$$\underbrace{f(x,y)}_{x^4+y^4} \geq \underbrace{f(0,0)}_{0^4+0^4=0} \quad \forall (x,y)\in\mathbb{R}^2 \Rightarrow (0,0) \text{ punct de}$$

minim global al lui $f \Rightarrow$

$\Rightarrow (0,0)$ punct de minim lo-

cal al lui f. □

f) $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=-x^4-y^4$.

<u>Sol</u>.: Rezolvați-l voi! □

g) $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=x^4-y^4$.

<u>Sol</u>.: $\mathbb{R}^2$ deschisă.

Det. punctele critice ale lui f.

f continuă pe $\mathbb{R}^2$.

$$\frac{\partial f}{\partial x}=4x^3.$$

$$\frac{\partial f}{\partial y}=-4y^3.$$

$$\left.\begin{array}{l}\frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y} \text{ cont. pe } \mathbb{R}^2 \\ \mathbb{R}^2 \text{ deschisă}\end{array}\right\} \Rightarrow f \text{ dif. pe } \mathbb{R}^2.$$

$$\begin{cases}\frac{\partial f}{\partial x}=0\\ \frac{\partial f}{\partial y}=0\end{cases} \Leftrightarrow \begin{cases}4x^3=0\\ -4y^3=0\end{cases} \Leftrightarrow \begin{cases}x=0\\ y=0.\end{cases}$$

Singurul punct critic al lui f este $(0,0)$.

$$\frac{\partial^2 f}{\partial x^2} = 12x^2.$$

$$\frac{\partial^2 f}{\partial y^2} = -12y^2.$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0 \underset{\uparrow}{=} \frac{\partial^2 f}{\partial y \partial x}.$$

Lema lui Schwarz

Toate derivatele parțiale de ordinul 2 ale lui f sunt cont. pe $\mathbb{R}^2$.

$$H_f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 12x^2 & 0 \\ 0 & -12y^2 \end{pmatrix} \quad \forall (x,y) \in \mathbb{R}^2.$$

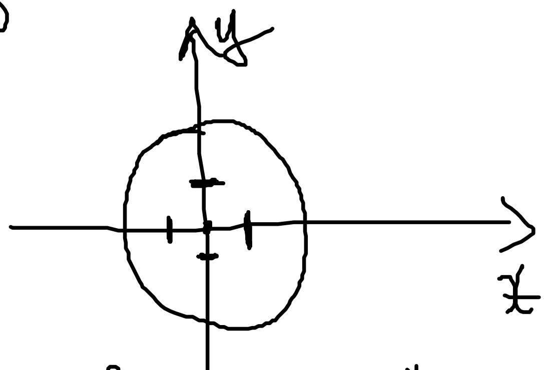
$$H_f(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

$$\left.\begin{array}{l} \Delta_1 = 0 \\ \Delta_2 = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0 \end{array}\right| \Longrightarrow \text{Criteriul nu decide.}$$

$f(x,y)$ $\quad$ $f(0,0)$

$\| \quad\quad\quad \|$

$x^4 - y^4 \quad\quad 0^4 - 0^4 = 0$



$f(x,0) = x^4 - 0^4 = x^4 > 0 = f(0,0) \ \forall\, x \in \mathbb{R}^*$

$f(0,y) = 0^4 - y^4 = -y^4 < 0 = f(0,0) \ \forall\, y \in \mathbb{R}^*$

$\Rightarrow (0,0)$ nu este punct de extrem local al lui f. $\square$

2. Să se arate că ecuația $x\cos y + y\cos z + z\cos x = 1$ definește într-o vecinătate a punctului $(1,0,0)$ unica funcție implicită $z = z(x,y)$ și det. $\frac{\partial z}{\partial x}(1,0)$, $\frac{\partial z}{\partial y}(1,0)$ și $dz(1,0)$.

<u>Sol.</u>: Fie $D = \mathbb{R}^3$ și $F: D \to \mathbb{R}$, $F(x,y,z) =$

$= x\cos y + y\cos z + z\cos x - 1$.

D deschisă.

1) $F(1,0,0) = 1\cos 0 + 0\cos 0 + 0\cos 1 - 1 = 0$.

2) $\frac{\partial F}{\partial x}(x,y,z) = \cos y - z\sin x \ \forall\, (x,y,z) \in D$.

$$\frac{\partial F}{\partial y}(x,y,z) = -x\sin y + \cos z \quad \forall (x,y,z) \in D.$$

$$\frac{\partial F}{\partial z}(x,y,z) = -y\sin z + \cos x \quad \forall (x,y,z) \in D.$$

$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$ cont. pe $D \Rightarrow F$ de clasă C^1 pe D.

3) $\frac{\partial F}{\partial z}(1,0,0) = \cos 1 \neq 0$.

Conform T.F.I. există U vecinătate deschisă a lui $(1,0)$, există V vecinătate deschisă a lui 0 și există unica funcție implicită $z: U \to V$ a.î.

a) $z(1,0) = 0$.

b) $F(x,y,z(x,y)) = 0 \quad \forall (x,y) \in U$.

c) z este de clasă C^1 pe U și

$$\frac{\partial z}{\partial x}(x,y) = -\frac{\frac{\partial F}{\partial x}(x,y,z(x,y))}{\frac{\partial F}{\partial z}(x,y,z(x,y))} \quad \forall (x,y) \in U,$$

$$\frac{\partial z}{\partial y}(x,y) = -\frac{\frac{\partial F}{\partial y}(x,y,z(x,y))}{\frac{\partial F}{\partial z}(x,y,z(x,y))} \quad \forall (x,y) \in U.$$

Pentru a determina $\frac{\partial z}{\partial x}(1,0)$ și $\frac{\partial z}{\partial y}(1,0)$ avem două variante.

<u>Varianta 1 (Folosim c) și a)).</u>

$$\frac{\partial z}{\partial x}(x,y) = -\frac{\frac{\partial F}{\partial x}(x,y,z(x,y))}{\frac{\partial F}{\partial z}(x,y,z(x,y))} = -\frac{\cos y - z(x,y)\sin x}{-y\sin z(x,y)+\cos x} \ \forall$$

$$\forall (x,y)\in U \Rightarrow \frac{\partial z}{\partial x}(1,0) = -\frac{1-z(1,0)\sin 1}{-0\cdot \sin z(1,0)+\cos 1} \underset{\substack{\uparrow \\ z(1,0)=0}}{=} -\frac{1}{\cos 1}.$$

$$\frac{\partial z}{\partial y}(x,y) = -\frac{\frac{\partial F}{\partial y}(x,y,z(x,y))}{\frac{\partial F}{\partial z}(x,y,z(x,y))} = -\frac{-x\sin y + \cos z(x,y)}{-y\sin z(x,y)+\cos x} \ \forall$$

$$\forall (x,y)\in U \Rightarrow \frac{\partial z}{\partial y}(1,0) = -\frac{-0\cdot\sin 0+\cos z(1,0)}{-0\cdot\sin z(1,0)+\cos 1} \underset{\substack{\uparrow \\ z(1,0)=0}}{=} -\frac{1}{\cos 1}.$$

<u>Varianta 2 (Folosim b) și a)).</u>

$$F(x,y,z(x,y)) = 0 \ \forall (x,y)\in U \Leftrightarrow$$

$$\Leftrightarrow x\cos y + y\cos z(x,y) + z(x,y)\cdot\cos x - 1 = 0 \ \forall (x,y)\in U. \quad (*)$$

Derivăm parțial în raport cu x relația (*) și obținem:

$$-\cos y - y\left(\sin z(x,y)\right)\frac{\partial z}{\partial x}(x,y) + \frac{\partial z}{\partial x}(x,y)\cos x +$$

$$+z(x,y)(-\sin x) = 0 \Rightarrow \frac{\partial z}{\partial x}(x,y)\left(-y\sin z(x,y) +\right.$$

$$\left.+\cos x\right) = -\cos y + z(x,y)\sin x \Rightarrow$$

$$\Rightarrow \frac{\partial z}{\partial x}(x,y) = \frac{-\cos y + z(x,y)\sin x}{-y\sin z(x,y) + \cos x} \quad \forall (x,y)\in U \Rightarrow$$

$$\Rightarrow \frac{\partial z}{\partial x}(1,0) = \frac{-\cos 0 + z(1,0)\sin 1}{-0\sin z(1,0) + \cos 1} \underset{z(1,0)=0}{\overset{\uparrow}{=}} \frac{-1}{\cos 1}.$$

Derivăm parțial relația (*) în raport cu y și obținem:

$$-x\sin y + \cos z(x,y) + y\left(-\sin z(x,y)\right)\frac{\partial z}{\partial y}(x,y) +$$

$$+\frac{\partial z}{\partial y}(x,y)\cos x = 0 \Rightarrow \frac{\partial z}{\partial y}(x,y)\cdot\left(-y\sin z(x,y) +\right.$$

$$\left.+\cos x\right) = x\sin y - \cos z(x,y) \Rightarrow \ldots \Rightarrow \frac{\partial z}{\partial y}(1,0) = -\frac{1}{\cos 1}.$$

z de clasă C^1 pe $U \Rightarrow z$ dif. pe U.

$dz(1,0) : \mathbb{R}^2 \to \mathbb{R}$, $dz(1,0)(u,v) = \left[\begin{pmatrix} \frac{\partial z}{\partial x}(1,0) & \frac{\partial z}{\partial y}(1,0) \end{pmatrix}^t \begin{pmatrix} u \\ v \end{pmatrix}\right] =$

$= \frac{\partial z}{\partial x}(1,0)\, u + \frac{\partial z}{\partial y}(1,0)\, v = -\frac{1}{\cos 1} u - \frac{1}{\cos 1} v$, i.e.

$dz(1,0) = -\frac{1}{\cos 1} dx - \frac{1}{\cos 1} dy$. $\square$