1) $f: A \to B$

Pe $A$ : $a_1 \rho_f a_2 \iff f(a_1) = f(a_2)$ ; $\rho_f$ = rel de echiv.

Concret: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - x$. Descrieți $\mathbb{R}/\rho_f$

Sol: $\hat{x} = \{ y \in \mathbb{R} \mid y \rho_f x \} = \{ y \in \mathbb{R} \mid f(y) = f(x) \}$

$= \{ y \in \mathbb{R} \mid y^3 - y = x^3 - x \}$

$= \{ y \in \mathbb{R} \mid (y-x)(y^2 + yx + x^2 - 1) = 0 \}$

$= \{x\} \cup \{ y \in \mathbb{R} \mid y^2 + x \cdot y + x^2 - 1 = 0 \}$

$y^2 + x \cdot y + x^2 - 1 = 0$ ; $\Delta = x^2 - 4x^2 + 4 = 4 - 3x^2$

- $\Delta < 0 \iff x \in \left(-\infty, -\frac{2}{\sqrt{3}}\right) \cup \left(\frac{2}{\sqrt{3}}, +\infty\right)$, atunci $\hat{x} = \{x\}$.
- $\Delta = 0 \iff x \in \left\{ \pm \frac{2}{\sqrt{3}} \right\}$, atunci $y = -\frac{x}{2}$

Deci $\hat{x} = \left\{ x, -\frac{x}{2} \right\}$, dacă $x \in \left\{ \pm \frac{2}{\sqrt{3}} \right\}$.

- $\Delta > 0 \iff x \in \left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$, atunci $y_{1,2} = \frac{-x \pm \sqrt{4-3x^2}}{2}$

distincte

$\hat{x} = \left\{ x, \frac{-x \pm \sqrt{4-3x^2}}{2} \right\}$

$3x = \pm\sqrt{4-3x^2} \implies 12x^2 = 4 \implies x \in \left\{ \pm \frac{1}{\sqrt{3}} \right\}$.

Deci $\hat{x} = \begin{cases} \{x\}, & x \in \left(-\infty, -\frac{2}{\sqrt{3}}\right) \cup \left(\frac{2}{\sqrt{3}}, +\infty\right) \\ \left\{\frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right\}, & x \in \left\{\frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right\} \\ \left\{-\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}, & x \in \left\{-\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\} \\ \left\{ x, \frac{-x \pm \sqrt{4-3x^2}}{2} \right\}, & x \in \left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) \setminus \left\{ \pm\frac{1}{\sqrt{3}} \right\} \end{cases}$

$\mathbb{R}/\rho = \left\{ \{x\} \mid x \in \left(-\infty, -\frac{2}{\sqrt{3}}\right) \cup \left(\frac{2}{\sqrt{3}}, +\infty\right) \right\} \cup \left\{ \left\{-\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\} \right\}$

$\cup \left\{ \left\{\frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right\} \right\} \cup \left\{ \left\{ x, \frac{-x \pm \sqrt{4-3x^2}}{2} \right\} \;\middle|\; x \in \left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) \setminus \left\{ \pm\frac{1}{\sqrt{3}} \right\} \right\}$

$f: \mathbb{R}/\rho \xrightarrow{\sim} \mathbb{R}$, $f(\hat{x}) = x^3 - x$

$f$ = bine definită: $\hat{x} = \hat{y} \implies x^3 - x = y^3 - y$

$x \rho_f y \iff f(x) = f(y) \iff x^3 - x = y^3 - y$

„⇒" dă buna definire a lui f
„⇐" dă injectivitatea lui f
$f: \mathbb{R}/\rho \to \mathbb{R},\ f(\hat{x}) = x^3 - x$.
$f$ = surj, pt că $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}$ a.î. $x^3 - x - y = 0$ (o ec
de grad 3 cu coef reali are cel puțin o răd reală)
Deci $f$ = bij $\Rightarrow \mathbb{R}/\rho \simeq \mathbb{R}$.

2) Pe $\mathbb{R}$: $x \sim y \Leftrightarrow x - y \in \mathbb{Z}$.
(i) $\sim$ = rel de echiv ; (ii) $\mathbb{R}/_\sim \simeq [0,1)$.
Sol: (i) $\cdot\ x \sim x \Leftrightarrow x - x \in \mathbb{Z} \Leftrightarrow 0 \in \mathbb{Z}$ ✓

$\cdot\ x \sim y \Leftrightarrow x - y \in \mathbb{Z} \Leftrightarrow y - x \in \mathbb{Z} \Leftrightarrow y \sim x$.

$\cdot \begin{cases} x \sim y \\ y \sim z \end{cases} \Leftrightarrow \begin{cases} x - y \in \mathbb{Z} \\ y - z \in \mathbb{Z} \end{cases} \xRightarrow{\text{„+"}} x - z \in \mathbb{Z} \Leftrightarrow x \sim z$.

(ii) $x \in \mathbb{R},\ \hat{x} = \{y \in \mathbb{R} \mid y \sim x\} = \{y \in \mathbb{R} \mid y - x \in \mathbb{Z}\}$
$= \{y \in \mathbb{R} \mid y \in x + \mathbb{Z}\} = x + \mathbb{Z}$
$= \{x + k \mid k \in \mathbb{Z}\} = \{\underbrace{[x]} + \{x\} + \underbrace{k} \mid k \in \mathbb{Z}\}$.
$= \{\{x\} + l \mid l \in \mathbb{Z}\} = \widehat{\{x\}}$.

Fie $f: \mathbb{R}/_\sim \to [0,1),\ f(\hat{x}) = \{x\}$.
$f$ = bine def: $\hat{x} = \hat{y} \Rightarrow \{x\} = \{y\}$.
$\Updownarrow$
$\widehat{\{x\}} = \widehat{\{y\}} \Leftrightarrow \{x\} \sim \{y\}$
$\Leftrightarrow \underbrace{\{x\} - \{y\}}_{\in(-1,1)} \in \mathbb{Z} \Leftrightarrow \{x\} - \{y\} = 0$

„⇒" dă buna def a lui f
„⇐" dă inj lui f
$f$ = surj: $\forall t \in [0,1), \exists x \in \mathbb{R}$ a.î. $f(\hat{x}) = t$ ?
$\Updownarrow$
$\{x\} = t$

Iau $x \in \{t + k \mid k \in \mathbb{Z}\}$.
Deci $f$ = bij ✓.

3) Pe $\mathbb{R}$: $x \sim y \iff x - y \in \mathbb{Q}$.

(i) $\sim$ este rel de echiv; (ii) $\mathbb{R}/_{\sim} \simeq \mathbb{R}$.

Sol: (i) $\sim$ rel de echiv. ✓

(ii) $\hat{x} = \{y \in \mathbb{R} \mid y \sim x\} = \{y \in \mathbb{R} \mid y - x \in \mathbb{Q}\}$

$= x + \mathbb{Q} \simeq \mathbb{Q} \Rightarrow \hat{x}$ - este mult. numărabilă

$x + r \to r$

Dacă $(x_i)_{i \in I}$ un sist complet de repr. ptr $\sim$

$\Rightarrow \{\hat{x}_i \mid i \in I\}$ - partiție a lui $\mathbb{R}$, adică

$\underbrace{\mathbb{R}}_{\text{NU este mulțime numărabilă}} = \biguplus_{i \in I} \underbrace{\hat{x}_i}_{\text{mult. numărabile}} \Rightarrow I$ nu este numărabilă și nici finită.

$I \simeq \mathbb{R}/_{\sim} \xleftarrow{\pi} \mathbb{R}$ $\Rightarrow$

surj canonică

$\Rightarrow$ compunerea lor dă o funcție surj de la $\mathbb{R}$ la $I$

$\Rightarrow \exists h : I \to \mathbb{R}$ funcție injectivă și, deci,

$\underbrace{I}_{\text{nu este nici finită, nici numărabilă}} \simeq h(I) \subseteq \mathbb{R}$.

I.C (ipoteza continuului: dacă $\mathbb{N} \subseteq X \subseteq \mathbb{R} \Rightarrow$

$\Rightarrow X \simeq \mathbb{N} \vee X \simeq \mathbb{R}$).

Sau mai: $I$ - infinită $\overset{\text{am}}{\underset{\text{arătat}}{\Longrightarrow}}$ $\mathbb{N} \hookrightarrow I \hookrightarrow \mathbb{R} \overset{I.C}{\Longrightarrow} I \simeq \mathbb{R}$.

$I \overset{\text{III}}{\ } h(I)$

nu este numărabilă

$\Rightarrow \mathbb{R}/_{\sim} \simeq I \simeq \mathbb{R}$.

4) $A \neq \emptyset$ , $\emptyset \neq B \subseteq A$

Pe $\mathcal{P}(A)$ ; $X \sim Y \iff X \cap B = Y \cap B$.

(i) $\sim$ = rel de echiv ; $\mathcal{P}(A)/_{\sim} \xrightarrow{\simeq} \mathcal{P}(B)$

Sol: (i) • $X \sim X \iff X \cap B = X \cap B$ ✓

• $X \sim Y \iff X \cap B = Y \cap B \iff Y \sim X$

• $\left\{\begin{array}{l} X \sim Y \\ Y \sim Z \end{array}\right. \iff \left\{\begin{array}{l} X \cap B = Y \cap B \\ Y \cap B = Z \cap B \end{array}\right. \Rightarrow X \cap B = Z \cap B \Rightarrow X \sim Z$.

(ii) $\mathcal{P}(A)/_{\sim} \xrightarrow{\simeq} \mathcal{P}(B)$

$f(\hat{X}) = X \cap B$

buna def a lui f : $\hat{X} = \hat{Y} \Rightarrow X \cap B = Y \cap B$

$\hat{X} = \hat{Y} \iff X \sim Y \overset{def}{\iff} X \cap B = Y \cap B$

"$\Rightarrow$" buna def

"$\Leftarrow$" f = inj

f = surj : $\forall Y \subseteq B, \exists X \subseteq A$ a.î. $f(\hat{X}) = Y \iff X \cap B = Y$

Iau $X = Y \; (\subseteq B \subseteq A)$

Deci f = bij ✓.

5) Pe $\mathbb{Z}^*$ : $a \sim b \iff \frac{a}{(a,b)}$ și $\frac{b}{(a,b)}$ au aceeași paritate

(i) $\sim$ = rel de echiv ; (ii) găsiți un SCR pt $\sim$

sistem complet de reprezentanți.

Sol: (i) $\to$ problema e la tranzitivitate.

Arăt că $a \sim b \iff$ puterea maximă la care apare 2 în descompunerile lui $a$ și $b$ (în produse de prime distincte) este aceeași.

Fie $a = 2^\alpha a'$ cu $a'$ = impar , $b = 2^\beta b'$ cu $b'$ = impar.

$(a,b) = 2^{\min(\alpha,\beta)} (a', b') \Rightarrow \frac{a}{(a,b)} = 2^{\alpha - \min(\alpha,\beta)} \frac{a'}{(a',b')}$

$\frac{b}{(a,b)} = 2^{\beta - \min(\alpha,\beta)} \frac{b'}{(a',b')}$

$\frac{a'}{(a',b')}$ , $\frac{b'}{(a',b')}$ = nr. impare

Fie $\alpha - \min(\alpha,\beta)$, fie $\beta - \min(\alpha,\beta)$ este 0.

Dacă $\alpha \neq \beta$, atunci $\frac{a}{(a,b)}$ și $\frac{b}{(a,b)}$ au parități diferite. \&.

Deci $a \sim b \iff \alpha = \beta$.

E clar acum că $\sim$ este rel de echiv pe $\mathbb{Z}^*$.

(ii) SCR pt $\sim$ : $a = 2^\alpha a'$ cu $a'$ = impar

$a \sim 2^\alpha \Rightarrow \hat{a} = \widehat{2^\alpha}$

Exemple: $\hat{1} = 2\mathbb{Z}+1$

$\hat{2} = \{2a' \mid a' = \text{impar}\} = 4\mathbb{Z}+2$

$\hat{3} = \hat{1} = \hat{5}$

$\hat{4} = \{2^2 a' \mid a' = \text{impar}\} = 2^3\mathbb{Z} + 2^2$

$\hat{6} = \hat{2}$ ş.a.m.d.

Atunci un SCR este $\{1, 2^\alpha \mid \alpha \in \mathbb{N}^*\}$ $= \{2^\alpha \mid \alpha \in \mathbb{N}\}$