1) $f = \sum_{i=0}^{n} a_i X^i \in U(R[X]) \iff a_0 \in U(R)$ și $a_1, \dots, a_n \in N(R)$.

Sol: „⇐" $u \in U(A)$ și $x \in N(A) \Rightarrow u + x \in U(A)$ (vezi cursul!)

$$\begin{cases} (1-x)(1+x+\dots+x^{n-1}) = 1 - x^n = 1, \text{ dacă } x^n = 0 \ (n \in \mathbb{N}^*) \\ u + x = \underset{\in U(A)}{u}\underbrace{(1 + u^{-1}x) = u(1 - (\underset{\in N(A)}{-u^{-1}x}))}_{\in U(A)} \in U(A) \end{cases}$$

$A = R[X]$

$$\left.\begin{array}{l} a_0 \in U(R) \subseteq U(R[X]) \\ a_1 \in N(R) \Rightarrow a_1 X \in N(R[X]) \end{array}\right\} \Rightarrow \left.\begin{array}{l} a_0 + a_1 X \in U(R[X]) \\ a_2 X^2 \in N(R[X]) \end{array}\right\} \Rightarrow$$

$\Rightarrow a_0 + a_1 X + a_2 X^2 \in U(R[X])$ ș.a.m.d. $\Rightarrow f = a_0 + a_1 X + \dots + a_n X^n \in U(R[X])$

„⇒" $f \in U(R[X]) \iff \exists g = \sum_{j=0}^{m} b_j X^j$ a.î. $fg = 1$

$\underbrace{fg}_{\sum_{k=0}^{n+m} c_k X^k} = 1 \iff c_k = 0 \ \forall k = \overline{1, n+m}$ și $c_0 = 1$, $c_k = \sum_{i+j=k} a_i b_j \ \forall k = \overline{0, n+m}$

$$\begin{cases} a_n b_m = 0 \\ a_n b_{m-1} + a_{n-1} b_m = 0 \mid \cdot a_n \Rightarrow a_n^2 b_{m-1} = 0 \\ a_n b_{m-2} + a_{n-1} b_{m-1} + a_{n-2} b_m = 0 \mid \cdot a_n^2 \Rightarrow a_n^3 b_{m-2} = 0 \\ \vdots \\ a_1 b_0 + a_0 b_1 = 0 \\ a_0 b_0 = 1 \Rightarrow a_0 \in U(R) \quad (a_0^{-1} = b_0) \end{cases}$$

Arăt prin ind. după $n = \text{grad}(f)$ că $a_1, \dots, a_n \in N(R)$.

$n = 0$, $f = a_0 \in U(R)$

Inductiv, avem $a_n^i b_{m-i+1} = 0 \ \forall i = \overline{1, m+1}$

Pt. $i = m+1 \Rightarrow a_n^{m+1} b_0 = 0 \mid \cdot a_0 \xRightarrow{a_0 b_0 = 1} a_n^{m+1} = 0 \Rightarrow$

$$\left.\begin{array}{l} \Rightarrow a_n \in N(R) \Rightarrow a_n X^n \in N(R[X]) \\ f \in U(R[X]) \end{array}\right\} \Rightarrow \underset{a_{n-1}X^{n-1} + \dots + a_1 X + a_0}{f - a_n X^n} \in U(R[X])$$

ip. ind. $\Rightarrow a_1, \dots, a_{n-1} \in N(R)$, QED

2) $f \in \mathcal{D}(R[X]) \iff \exists\, 0 \neq a \in R$ a.î. $af = 0$.

Sol: „$\Leftarrow$" evidentă $(0 \neq a \in R \subset R[X])$

„$\Rightarrow$" $f \in \mathcal{D}(R[X]) \Rightarrow \exists\, 0 \neq g \in R[X]$ a.î. $fg = 0$.

$X = \{ \text{grad}(g) \mid 0 \neq g \in R[X] \text{ a.î. } fg = 0 \} \subseteq \mathbb{N}$

$\neq \emptyset$; X are un prim element; $\exists\, 0 \neq g \in R[X]$ de grad minim a.î. $fg = 0$. Scriu $g = \sum_{j=0}^{m} b_j X^j \in R[X]$.

$fg = 0 \Rightarrow a_n b_m = 0$

$a_n f g = a_n \cdot 0 = 0 \Rightarrow f(a_n g) = 0$

$a_n g = \sum_{j=0}^{m-1} a_n b_j X^j \rightarrow$ de grad $< \text{grad}(g)$ (deoarece $a_n b_m = 0$)

$\xrightarrow[\text{lui } g]{\text{alegerea}} a_n g = 0 \Rightarrow a_n b_j = 0 \quad \forall\, j = \overline{0,m}$

$fg = 0 \Rightarrow \underbrace{a_n b_{m-1}}_{=0} + a_{n-1} b_m = 0 \Rightarrow a_{n-1} b_m = 0$

$\left.\begin{array}{l} \Rightarrow a_{n-1} f g = a_{n-1} \cdot 0 = 0 \Rightarrow f(\underbrace{a_{n-1} g}) = 0 \\ \text{de grad} \leq m-1 < \text{grad}(g) \end{array}\right\} \Rightarrow$

$\xrightarrow[\text{lui } g]{\text{alegerea}} a_{n-1} g = 0$.

Şi aşa mai departe, inductiv obţinem că $a_i g = 0 \;\forall i = \overline{0,n}$

$\iff a_i b_j = 0$, $\forall\, i = \overline{0,n}$ şi $j = \overline{0,m}$.

$g \neq 0 \Rightarrow \exists\, 0 \leq j_0 \leq m$ a.î. $b_{j_0} \neq 0$. Iau $a = b_{j_0} \Rightarrow$

$\Rightarrow a_i a = 0 \quad \forall i = \overline{0,n} \Rightarrow af = 0$, $a \in R \setminus \{0\}$.

3) $\mathrm{Idemp}(R[X]) = \mathrm{Idemp}(R)$

Sol: „$\supseteq$" evidentă

„$\subseteq$" $f = \sum_{i=0}^{n} a_i X^i \in \mathrm{Idemp}(R[X])$

$$f^2 = f \Rightarrow \begin{cases} a_0^2 = a_0 \Rightarrow a_0 \in \mathrm{Idemp}(R) \\ a_0 a_1 + a_1 a_0 = a_1 \Rightarrow 2a_0 a_1 = a_1 \mid \cdot a_0 \Rightarrow 2a_0 a_1 = a_0 a_1 \Rightarrow a_0 a_1 = 0 \Rightarrow a_1 = 0 \\ a_0 a_2 + a_1^2 + a_2 a_0 = a_2 \Rightarrow 2a_0 a_2 = a_2 \mid \cdot a_0 \Rightarrow \dots \Rightarrow a_2 = 0 \\ a_0 a_3 + \underbrace{a_1 a_2 + a_2 a_1}_{=0} + a_3 a_0 = a_3 \Rightarrow 2a_0 a_3 = a_3 \Rightarrow \dots \Rightarrow a_3 = 0 \\ \vdots \\ a_0 a_i + a_1 a_{i-1} + \dots + a_{i-1} a_1 + a_i a_0 = a_i \\ \vdots \end{cases}$$

Inductiv obținem că $a_i = 0 \ \forall i = \overline{1,n}$. Deci $f = a_0 \in \mathrm{Idemp}(R)$

Aplicatie: Descrieți pol. nilpotente, inversabile, divizori ai lui zero și respectiv idempotente din $\mathbb{Z}_{12}[X]$.

- $N(\mathbb{Z}_{12}) = \hat{6}\,\mathbb{Z}_{12} = \{\hat{0}, \hat{6}\}$. $12 = 2^2 \cdot 3$

$$N(\mathbb{Z}_{12}[X]) = \{\hat{0}, \hat{6}X^n + a_{n-1}X^{n-1} + \dots + a_0 \mid n \in \mathbb{N},\ a_0, \dots, a_{n-1} \in \{\hat{0}, \hat{6}\}\}$$

- $U(\mathbb{Z}_{12}) = \{\hat{a} \mid (a,12)=1\} = \{\hat{1}, \hat{5}, \hat{7}, \hat{11}\}$.

$$U(\mathbb{Z}_{12}[X]) = \left\{a_nX^n + \dots + a_1X + a_0 \;\middle|\; \begin{array}{l} n \in \mathbb{N} \\ a_1, \dots, a_n \in \{\hat{0}, \hat{6}\} \\ a_0 \in \{\hat{1}, \hat{5}, \hat{7}, \hat{11}\} \end{array}\right\}.$$

$$(\hat{6}X^3 + \hat{6}X + \hat{1})^{-1} = ?$$

$$\hat{6}^2 = \hat{0} \Rightarrow (\hat{6}X)^2 = \hat{0} \Rightarrow (\hat{1} + \hat{6}X)(\hat{1} - \hat{6}X) = \hat{1}$$

$$\Rightarrow \exists\, (\hat{1} + \hat{6}X)^{-1} = \hat{1} - \hat{6}X \quad ; u = \hat{1} + \hat{6}X,\ a = \hat{6}X^3$$

$$(\hat{6}X^3)^2 = \hat{0} \qquad (u + a) = u(1 + u^{-1}a)$$

$$(u+a)\cdot u^{-1}(1 - u^{-1}a + u^{-2}a^2 - \dots + (-1)^{n-1}u^{-n+1}a^{n-1}) = 1 - (-u^{-1}a)^n = 1$$

$$\Rightarrow \exists\, (u+a)^{-1} = u^{-1}(1 - u^{-1}a + \dots + (-1)^{n-1}u^{-n+1}a^{n-1})$$

$$(\underbrace{\hat{6}X^3}_{a} + \underbrace{\hat{6}X + \hat{1}}_{u})^{-1} = (\hat{6}X + \hat{1})^{-1}(\hat{1} - u^{-1}a)$$

$$= (\hat{1} - \hat{6}X)[\hat{1} - (\hat{1} - \hat{6}X)\hat{6}X^3]$$

$$= (\hat{1} - \hat{6}X)(\hat{1} - \hat{6}X^3)$$

$$= \hat{1} - \hat{6}X^3 - \hat{6}X = -\hat{6}X^3 - \hat{6}X + \hat{1}.$$

- $D(\mathbb{Z}_{12}[X]) = \{f = \sum_{i=0}^{n} a_iX^i \mid \exists\, 0 \neq a \in \mathbb{Z}_{12}$ a.î. $a a_i = 0\ \forall i\}$.

$f = \hat{6}X^2 + \hat{3}X \in D(\mathbb{Z}_{12}[X])$; $a = \hat{4}$ ($\hat{4}X + \hat{8}$ - anulează pe f)

$g = \hat{3}X^2 + \hat{4}X$ $12 \mid 3a \Rightarrow 4 \mid a$; $12 \mid 4a \Rightarrow 3 \mid a \Rightarrow 12 \mid a \Rightarrow \hat{a} = 0$

$g \notin D(\mathbb{Z}_{12}[X])$

- $Idemp(\mathbb{Z}_{12}[X]) = Idemp(\mathbb{Z}_{12}) = \{\hat{0}, \hat{1}, \hat{4}, \hat{9}\}$

4) $(X^2-1)\mathbb{Q}[X] \cap (X^3-1)\mathbb{Q}[X] = ?$

$(X^2-1)\mathbb{Q}[X] + (X^3-1)\mathbb{Q}[X] = ?$

Sol: $R = K[X]$, K-corp.

$\hookrightarrow$ orice ideal este principal.

$aR + bR = (a,b)R \qquad , aR \cap bR = [a,b]R$

$d = (X^2-1, X^3-1) = X-1 \ , \ m = [X^2-1, X^3-1] = (X+1)(X^3-1)$

$(X^2-1)\mathbb{Q}[X] + (X^3-1)\mathbb{Q}[X] = (X-1)\mathbb{Q}[X]$

$(X^2-1)\mathbb{Q}[X] \cap (X^3-1)\mathbb{Q}[X] = (X+1)(X^3-1)\mathbb{Q}[X]$

5) $m,n,p = ?$ a.î. $X^4+X^2+1 \mid X^{3m} + X^{3n+1} + X^{3p+2}$ în $\mathbb{Q}[X]$

Sol: $X^4+X^2+1 = X^4+2X^2+1-X^2 = (X^2+1)^2 - X^2$

$= \underbrace{(X^2-X+1)(X^2+X+1)}$

nu au rad în $\mathbb{Q}$, deci sunt ired în $\mathbb{Q}[X]$ (au grad 2)

Dacă $f = X^{3m} + X^{3n+1} + X^{3p+2}$, $X^4+X^2+1 \mid f \iff$

$(X^2-X+1, X^2+X+1) = 1$ $\iff$ $\begin{cases} X^2-X+1 \mid f \in \mathbb{R}[X] \\ X^2+X+1 \mid f \end{cases} \iff \begin{cases} f(w) = 0 \\ f(\eta) = 0 \end{cases}$ în $\mathbb{C}$,

unde $w^2 - w + 1 = 0$, $\eta^2+\eta+1 = 0$. (obs. că $w^3 = -1$, $\eta^3 = 1$)

$\iff \begin{cases} w^{3m} + w^{3n+1} + w^{3p+2} = 0 \\ \eta^{3m} + \eta^{3n+1} + \eta^{3p+2} = 0 \end{cases} \iff \begin{cases} (-1)^m + w \cdot (-1)^n + w^2 \cdot (-1)^p = 0 \\ 1 + \eta + \eta^2 = 0 \ \checkmark \end{cases}$

$\iff \left. \begin{array}{l} w^2 + (-1)^{n-p} w + (-1)^{m-p} = 0 \\ w^2 - w + 1 = 0 \end{array} \right\} \iff \begin{cases} n-p \equiv 1 \pmod 2 \\ m-p \equiv 0 \pmod 2 \end{cases}$

$\iff \begin{cases} m, p \text{ au aceeași paritate} \\ n, p \text{ au parități diferite.} \end{cases}$

6) Descompuneți $f = X^{56} - X^{49} - X^7 + \hat{1}$ în produs de polinoame ireductibile din $\mathbb{Z}_7[X]$.

Sol: $f = (X^8 - X^7 - X + \hat{1})^7 = [X^7(X-\hat{1}) - (X-\hat{1})]^7$

endom. lui Frobenius

$= (X-\hat{1})^7 (X^7 - \hat{1})^7$

$= (X-\hat{1})^7 (X-\hat{1})^{49}$

$= (X-\hat{1})^{56}$

R-inel com. de caract. p-prim, $R \ni x \to x^p \in R$ este morf. de inele.

$(x-y)^p = x^p - y^p$ ($\Leftarrow$ bin. lui N + $p \mid C_p^k \ \forall\, 1 \le k \le p-1$)