C_{13} - GAL

20.05.2022.

§1 Aducerea conicelor la o formă canonică ($\delta=0$)

§2 Cuadrice studiate pe ecuații reduse.

§1. $(\mathbb{R}^2, \mathbb{R}^2/_{\mathbb{R}}, \varphi)$ / $(\mathbb{R}^2, (\mathbb{R}^2, g_0), \varphi)$

$\Gamma: f(x) = X^T A X + 2BX + c = 0$

$f(x) = \underline{a_{11} x_1^2 + 2a_{12} x_1 x_2 + a_{22} x_2^2} + \underline{2b_1 x_1 + 2b_2 x_2} + c = 0$

1. $\delta \neq 0$ (Γ are centru unic)
2. $\delta = 0$ (Γ nu are centru unic)

$\mathcal{R} = \{O; e_1, e_2\} \longrightarrow \mathcal{R}' = \{O; e_1', e_2'\} \longrightarrow \mathcal{R}'' = \{P; e_1', e_2'\}$.

$Q: \mathbb{R}^2 \longrightarrow \mathbb{R}$, $Q(x) = a_{11} x_1^2 + 2a_{12} x_1 x_2 + a_{22} x_2^2$

Aducem Q la o formă canonică

(în sp. afin: met Gauss, Jacobi)

În sp. punctual euclidian utilizăm met val pr.

$P(\lambda) = \det(A - \lambda I_2) = 0$

$\lambda^2 - Tr(A)\lambda + \det A = 0$ $\quad \lambda_1 \neq 0$, $\lambda_2 = 0$.

$\delta = \det A = 0$, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$

e_1', e_2' versori proprii coresp. val. pr λ_1, λ_2.

$e_k' = (l_k, m_k)$, $k = \overline{1,2}$

$R = \begin{pmatrix} l_1 & l_2 \\ m_1 & m_2 \end{pmatrix} \in SO(2)$ (o alegem)

Fie rotația: $\theta: X = RX'$

$$Q(x) = \lambda_1 x_1'^2 + 0 \cdot x_2'^2 \qquad \begin{pmatrix} \lambda_1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\theta: \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} l_1 & l_2 \\ m_1 & m_2 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$$

$$\begin{cases} x_1 = l_1 x_1' + l_2 x_2' \\ x_2 = m_1 x_1' + m_2 x_2' \end{cases}$$

$$\theta(\Gamma): \lambda_1 x_1'^2 + 2b_1(l_1 \underline{x_1'} + l_2 \underline{\underline{x_2'}}) + 2b_2(m_1 \underline{x_1'} + m_2 \underline{\underline{x_2'}}) + c = 0$$

$$\underline{\lambda_1 x_1'^2} + 2b_1' x_1' + 2b_2' x_2' + c = 0$$

$$\Delta = \begin{vmatrix} \lambda_1 & 0 & b_1' \\ 0 & 0 & b_2' \\ b_1' & b_2' & c \end{vmatrix} = -b_2' \begin{vmatrix} \lambda_1 & 0 \\ b_1' & b_2' \end{vmatrix} = -\lambda_1 b_2'^2$$

① $\Delta \neq 0$ (Γ nedegenerată) $\Rightarrow b_2' \neq 0$.

$$\theta(\Gamma): \lambda_1 \left(x_1'^2 + 2\frac{b_1'}{\lambda_1} x_1' + \frac{b_1'^2}{\lambda_1^2}\right) + \underline{2b_2'} x_2' + c' = 0$$

$$c' = c - \frac{b_1'^2}{\lambda_1}$$

$$\lambda_1 \left(x_1' + \frac{b_1'}{\lambda_1}\right)^2 + 2b_2' \left(x_2' + \frac{c'}{2b_2'}\right) = 0$$

Fie $\begin{cases} x_1'' = x_1' + \dfrac{b_1'}{\lambda_1} \\ x_2'' = x_2' + \dfrac{c'}{2b_2'} \end{cases}$

Considerăm translația $\tau: X' = X'' + X_0$, $X_0 = \begin{pmatrix} -\dfrac{b_1'}{\lambda_1} \\ -\dfrac{c'}{2b_2'} \end{pmatrix}$

$$\tau(\theta(\Gamma)): \lambda_1 x_1''^2 + 2b_2' x_2'' = 0 \text{ (parabolă)}$$

② $\Delta = 0$ (Γ degenerată) $\Rightarrow b_2' = 0$.

$\theta(\Gamma): \lambda_1 x_1'^2 + 2b_1' x_1' + c = 0.$

$$\lambda_1 \left(x_1' + \frac{b_1'}{\lambda_1}\right)^2 + c' = 0.$$

Fie $\begin{cases} x_1'' = x_1' + \dfrac{b_1'}{\lambda_1} \\ x_2'' = x_2' \end{cases}$

Fie translația $\tau: X' = X'' + X_0$, $X_0 = \begin{pmatrix} -\frac{b_1'}{\lambda_1} \\ 0 \end{pmatrix}$

$\tau(\theta(\Gamma))$?: $\lambda_1 x_1''^2 + c' = 0.$

$\theta: X = RX'$

$\tau: X' = X'' + X_0$

$X = RX'' + RX_0.$ $\quad RX_0 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

$P(\alpha, \beta)$ (în raport cu reperul canonic)

Δ (natura)	δ (genul)	Tipul conicei
$\Delta \neq 0$	$\delta > 0$	Elipsă, ϕ
	$\delta < 0$	Hiperbolă
	$\delta = 0$.	Parabolă
$\Delta = 0$	$\delta > 0$	Punct dublu.
	$\delta < 0$	Drepte concurente
	$\delta = 0$	Drepte confundate, ϕ, Drepte // .

Aplicație 2 Fie conica (în sp. punctual euclidian)

$\Gamma: f(x) = x_1^2 - 4x_1x_2 + 4x_2^2 - 6x_1 + 2x_2 + 1 = 0$

Să se aducă la o formă canonică, utilizând izometrii. Reprez. grafică.

SOL

$A = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$ $\quad \delta = 4 - 4 = 0$ (centrul nu e unic)

$\tilde{A} = \begin{pmatrix} 1 & -2 & -3 \\ -2 & 4 & 1 \\ -3 & 1 & 1 \end{pmatrix}$

$\Delta = \begin{vmatrix} 1 & -2 & -3 \\ -2 & 4 & 1 \\ -3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -2 & 0 & -5 \\ -3 & -5 & -5 \end{vmatrix} = -25 \neq 0$

Γ nedegenerată (parabolă)

$Q: \mathbb{R}^2 \to \mathbb{R}, \quad Q(x) = x_1^2 - 4x_1x_2 + 4x_2^2$

Aducem Q la o f. canonică (met. val. pr.)

$\lambda^2 - 5\lambda + 0 = 0 \Rightarrow \lambda_1 = 5, \lambda_2 = 0.$

- $V_{\lambda_1} = \{x \in \mathbb{R}^2 \mid AX = 5X\}$

$(A - 5I_2)X = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-2x_1 - x_2 = 0 \Rightarrow x_2 = -2x_1.$

$e_1' = \frac{1}{\sqrt{5}}(1,-2) \qquad V_{\lambda_1} = \langle\{e_1'\}\rangle$

- $V_{\lambda_2} = \{x \in \mathbb{R}^2 \mid AX = 0 \cdot X\}$

$\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad x_1 - 2x_2 = 0.$

$e_2' = \frac{1}{\sqrt{5}}(2,1), \qquad V_{\lambda_2} = \langle\{e_2'\}\rangle$

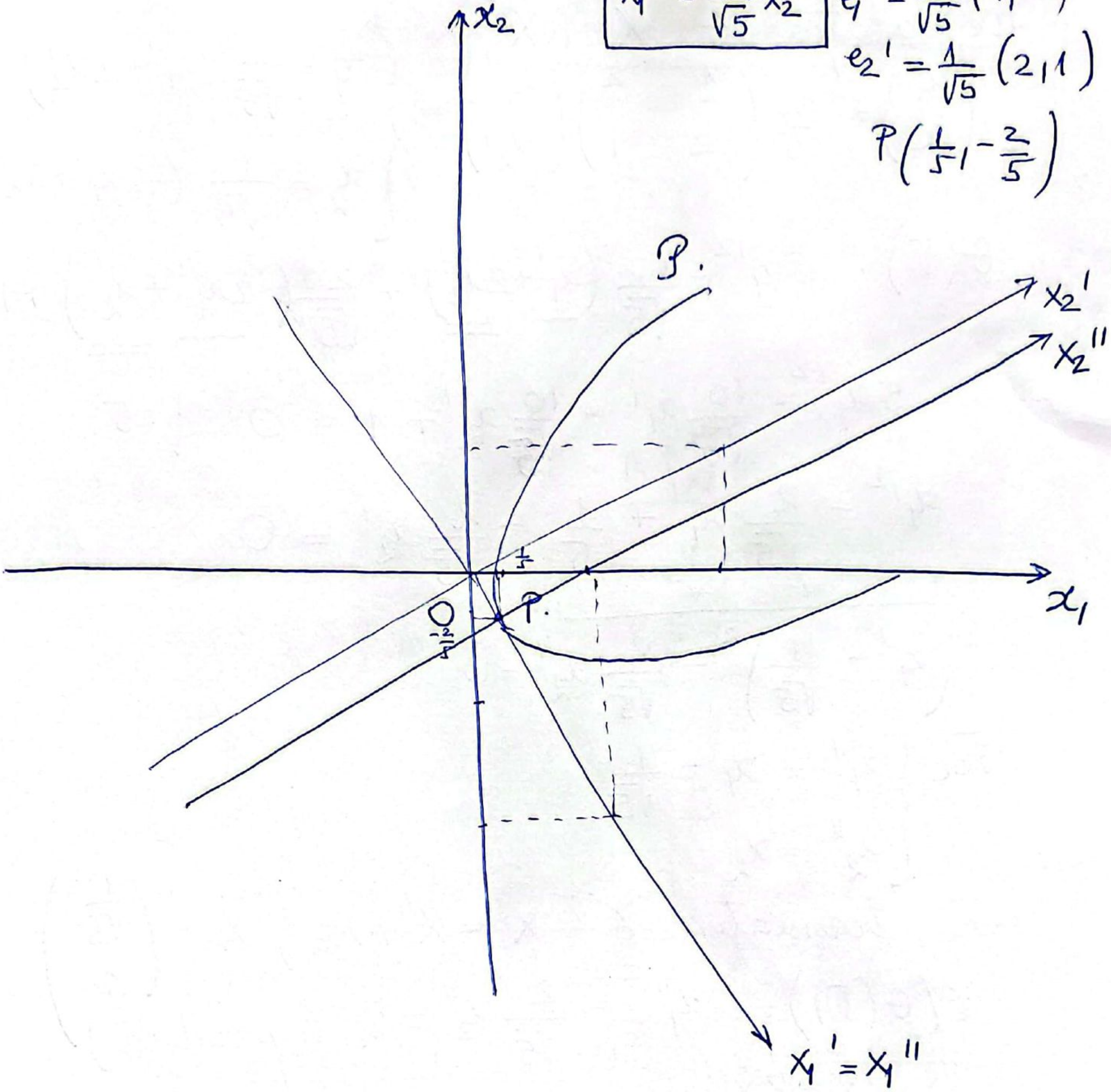
$R = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \in SO(2)$

$$\boxed{x_1''^2 = \frac{2}{\sqrt{5}} x_2''}$$

$e_1' = \frac{1}{\sqrt{5}}(1,-2)$

$e_2' = \frac{1}{\sqrt{5}}(2,1)$

$P\left(\frac{1}{5}, -\frac{2}{5}\right)$



Fie rotatia $\theta : X = RX'$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \Rightarrow \begin{cases} x_1 = \frac{1}{\sqrt{5}} (x_1' + 2x_2') \\ x_2 = \frac{1}{\sqrt{5}} (-2x_1' + x_2') \end{cases}$$

$$\theta(\Gamma) : \ 5x_1'^2 - \frac{6}{\sqrt{5}} (\underline{x_1'} + \underline{\underline{2x_2'}}) + \frac{2}{\sqrt{5}} (\underline{-2x_1'} + \underline{\underline{x_2'}}) + 1 = 0.$$

$$5x_1'^2 - \frac{10}{\sqrt{5}} x_1' - \frac{10}{\sqrt{5}} x_2' + 1 = 0. \ \Big| :5$$

$$\underbrace{x_1'^2 - \frac{2}{\sqrt{5}} x_1' + \frac{1}{5}} - \frac{2}{\sqrt{5}} x_2' = 0$$

$$\left(x_1' - \frac{1}{\sqrt{5}}\right)^2 - \frac{2}{\sqrt{5}} x_2' = 0$$

Fie $\begin{cases} x_1'' = x_1 - \frac{1}{\sqrt{5}} \\ x_2'' = x_2 \end{cases}$

Fie translația $\tau : X' = X'' + X_0, \ X_0 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ 0 \end{pmatrix}$

$$\tau(\theta(\Gamma)) : \ x_1''^2 - \frac{2}{\sqrt{5}} x_2'' = 0 \ (\text{parabolă}).$$

$\theta : X = RX'$

$\tau : X' = X'' + X_0 \Rightarrow X = RX'' + \underbrace{RX_0}$

$$RX_0 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} \\ 0 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$P\left(\frac{1}{5}, -\frac{2}{5}\right)$ (în rap cu $\mathcal{R}$)

$$\mathcal{R} = \{O; e_1, e_2\} \longrightarrow \mathcal{R}' = \{O; e_1', e_2'\} \longrightarrow \mathcal{R}'' = \{P; e_1', e_2'\}$$

$e_1' = \frac{1}{\sqrt{5}} (1, -2)$

$e_2' = \frac{1}{\sqrt{5}} (2, 1)$

§2. Cuadrice studiate pe ecuații reduse.

1) Sfera

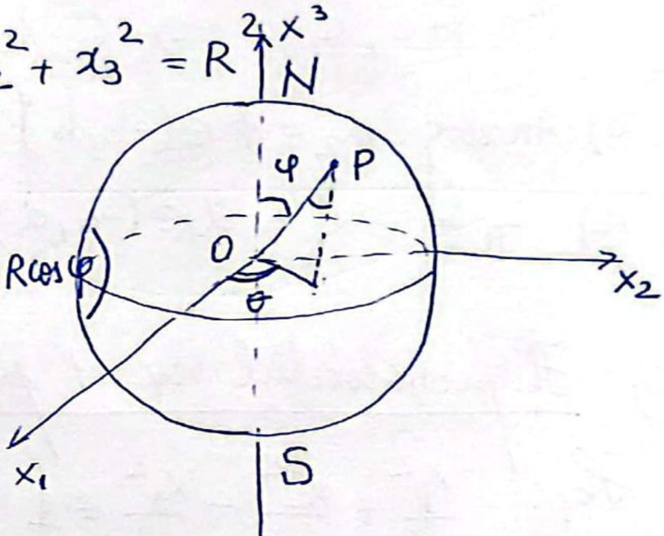
$\mathcal{S}(A(a,b,c), R): \ (x_1-a)^2+(x_2-b)^2+(x_3-c)^2-R^2=0.$

$$f(x_1,x_2,x_3)=x_1^2+x_2^2+x_3^2-2ax_1-2bx_2-2cx_3+\alpha=0$$

$$\alpha=a^2+b^2+c^2-R^2$$

În particular, $A=O$

$\mathcal{S}(O, R): \ x_1^2+x_2^2+x_3^2=R^2$



$P(R\sin\varphi\cos\theta, R\sin\varphi\sin\theta, R\cos\varphi)$

$\theta\in[0,2\pi)$

$\varphi\in[0,\pi]$.

$\cap$ cu plane // cu planele de coord.

$x_3=\gamma\in(-R,R) \Rightarrow x_1^2+x_2^2=R^2-\gamma^2$ cerc.

Dacă $\gamma\in\{-R,R\} \Rightarrow N(0,0,R),\ S(0,0,-R)$

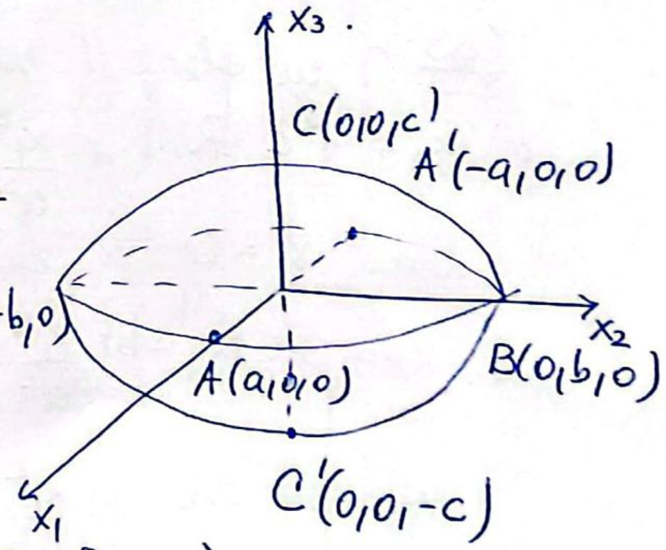
Analog pt $x_2=\beta$

$x_1=\alpha$

2) Elipsoidul

$$\mathcal{E}: \ \frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}+\frac{x_3^2}{c^2}=1$$

$a,b,c>0$ (semiaxe) $B'(0,-b,0)$



$$\begin{cases} x_1=a\sin\varphi\cos\theta \\ x_2=b\sin\varphi\sin\theta \\ x_3=c\cos\varphi, \quad \varphi\in[0,\pi],\ \theta\in[0,2\pi) \end{cases}$$

OBS a) planele de simetrie = planele de coordonate.

b) axe de simetrie = axele de coordonate.

c) centrul de simetrie = orig. axelor.

OBS $\cap$ cu plane // cu planele de coord.

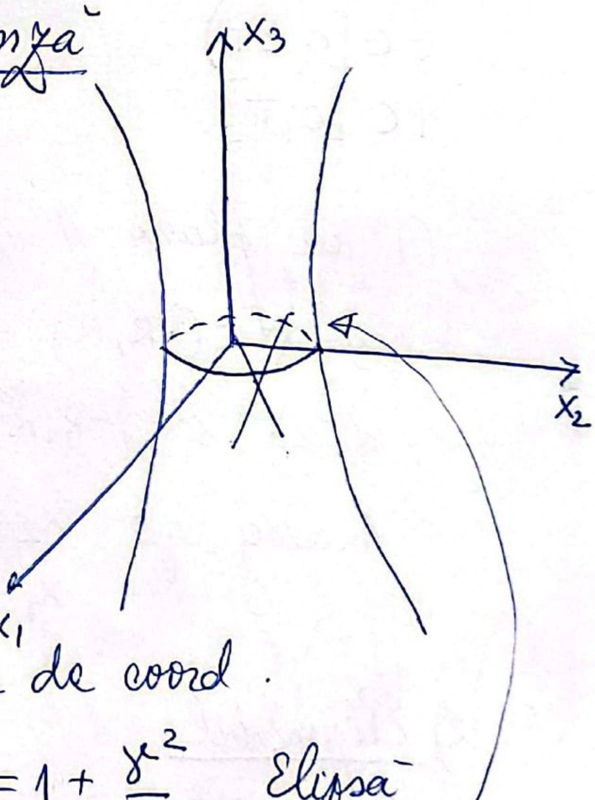
a) $x_3 = \gamma \in (-c, c)$: $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1 - \frac{\gamma^2}{c^2}$ Elipsa

$\gamma \in \{-c, c\} \Rightarrow C$ şi C'

b) Analog $x_2 = \gamma \in (-b, b)$ Elipsa

c) — $x_1 = \alpha \in (-a, a)$ Elipsa.

3). Hiperboloidul cu 1 pânză



$\mathcal{H}_1$: $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - \frac{x_3^2}{c^2} = 1$

$$\begin{cases} x_1 = a \cos\theta \, ch\varphi \\ x_2 = b \sin\theta \, ch\varphi \\ x_3 = c \, sh\varphi \end{cases}$$

$\varphi \in \mathbb{R}$, $\theta \in [0, 2\pi]$

OBS $\cap$ cu plane // cu planele de coord.

a) $x_3 = \gamma \in \mathbb{R}$. $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1 + \frac{\gamma^2}{c^2}$ Elipsa

Dacă $\gamma = 0 \Rightarrow$ Elipsa colier : $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$

b) $x_2 = \beta \in \mathbb{R} \setminus \{\pm b\}$ $\frac{x_1^2}{a^2} - \frac{x_3^2}{c^2} = 1 - \frac{\beta^2}{b^2}$ Hiperbola

$x_2 = \beta = \pm b$ $\frac{x_1^2}{a^2} - \frac{x_3^2}{c^2} = 0 \Rightarrow x_3 = \pm \frac{c}{a} x_1$.

(drepte)

e) $x_1 = \alpha \in \mathbb{R} \setminus \{\pm a\}$ $\dfrac{x_2^2}{b^2} - \dfrac{x_3^2}{c^2} = 1 - \dfrac{\alpha^2}{a^2}$ Hiperbolă.

$x_1 = \alpha = \pm a$ $\dfrac{x_2^2}{b^2} - \dfrac{x_3^2}{c^2} = 0 \Rightarrow x_3 = \pm \dfrac{c}{b} x_2$

(Drepte)

<u>Teoremă</u>

$\mathcal{H}_1$ = cuadrică dublu riglată i.e $\exists$ 2 familii de generatoare G_1 și G_2 și prin fiecare punct al cuadricei trece câte o dreaptă din fiecare familie.

<u>Dem</u>

$$\mathcal{H}_1 : \frac{x_1^2}{a^2} - \frac{x_3^2}{c^2} = 1 - \frac{x_2^2}{b^2} \Rightarrow$$

$$\underbrace{\left(\frac{x_1}{a} - \frac{x_3}{c}\right)}_{f_1} \underbrace{\left(\frac{x_1}{a} + \frac{x_3}{c}\right)}_{f_2} = \underbrace{\left(1 - \frac{x_2}{b}\right)}_{g_1} \underbrace{\left(1 + \frac{x_2}{b}\right)}_{g_2}.$$

$$\mathcal{D} : \begin{cases} \alpha f_1 = \beta g_1 \\ \beta f_2 = \alpha g_2 \end{cases} \qquad \alpha \beta f_1 f_2 = \alpha \beta g_1 g_2$$

$\mathcal{D} \subset \mathcal{H}_1$.

1) $\alpha = 0$ $\begin{cases} g_1 = 0 \\ f_2 = 0 \end{cases}$

2) $\alpha \neq 0$ $\begin{cases} f_1 = \dfrac{\beta}{\alpha} g_1 \\ \dfrac{\beta}{\alpha} f_2 = g_2 \end{cases}$ $\quad \dfrac{\beta}{\alpha} = \lambda$

$$G_1 : \; d_\lambda : \begin{cases} \dfrac{x_1}{a} - \dfrac{x_3}{c} = \lambda \left(1 - \dfrac{x_2}{b}\right) \\ \lambda \left(\dfrac{x_1}{a} + \dfrac{x_3}{c}\right) = 1 + \dfrac{x_2}{b} \end{cases} \qquad d_\infty : \begin{cases} 1 - \dfrac{x_2}{b} = 0 \\ \dfrac{x_1}{a} + \dfrac{x_3}{c} = 0 \end{cases}$$

Analog obținem familia.

$$G_2 : \; \overline{d}_\mu : \begin{cases} \dfrac{x_1}{a} - \dfrac{x_3}{c} = \mu \left(1 + \dfrac{x_2}{b}\right) \\ \mu \left(\dfrac{x_1}{a} + \dfrac{x_3}{c}\right) = 1 - \dfrac{x_2}{b} \end{cases} \qquad \overline{d}_\infty : \begin{cases} 1 + \dfrac{x_2}{b} = 0 \\ \dfrac{x_1}{a} + \dfrac{x_3}{c} = 0 \end{cases}$$

$$\lambda\left(1-\frac{x_2}{b}\right)=\mu\left(1+\frac{x_2}{b}\right) \Rightarrow \frac{x_2}{b}(\mu+\lambda)=\lambda-\mu.$$

- $\lambda+\mu=0 \Rightarrow \lambda-\mu=0 \Rightarrow \lambda=\mu=0.$

$$d_0:\begin{cases}\frac{x_1}{a}-\frac{x_3}{c}=0\\ 1+\frac{x_2}{b}=0\end{cases} \quad ; \quad \overline{d_0}:\begin{cases}\frac{x_1}{a}-\frac{x_3}{c}=0\\ 1-\frac{x_2}{b}=0\end{cases}$$

$$d_0\cap\overline{d_0}=\emptyset$$

- $\lambda+\mu\neq 0 \Rightarrow \frac{x_2}{b}=\frac{\lambda-\mu}{\lambda+\mu}$

$$1+\frac{x_2}{b}=1+\frac{\lambda-\mu}{\lambda+\mu}=\frac{2\lambda}{\lambda+\mu}$$

$$1-\frac{x_2}{b}=1-\frac{\lambda-\mu}{\lambda+\mu}=\frac{2\mu}{\lambda+\mu}$$

$$\begin{cases}\frac{x_1}{a}-\frac{x_3}{c}=\frac{2\lambda\mu}{\lambda+\mu}.\\ \frac{x_1}{a}+\frac{x_3}{c}=\frac{2}{\lambda+\mu}\end{cases} \quad \oplus$$

$$\frac{x_1}{a}=\frac{\lambda\mu+1}{\lambda+\mu}$$

$$\frac{x_3}{c}=\frac{1-\lambda\mu}{\lambda+\mu}.$$

$$d_\lambda\cap\overline{d_\mu}:\ P\left(a\cdot\frac{\lambda\mu+1}{\lambda+\mu},\ b\,\frac{\lambda-\mu}{\lambda+\mu},\ c\cdot\frac{1-\lambda\mu}{\lambda+\mu}\right)$$

$$d_\lambda\cap\overline{d_\infty}\quad\left[\left(a\,\frac{\lambda+\frac{1}{\mu}}{\frac{\lambda}{\mu}+1},\ b\,\frac{\frac{\lambda}{\mu}-1}{\frac{\lambda}{\mu}+1},\ c\,\frac{\frac{1}{\mu}-\lambda}{\frac{\lambda}{\mu}+1}\right)\right]$$

$$P(a\lambda,-b,-c\lambda)$$

$$d_\infty\cap\overline{d_\mu}\quad\left[P\left(a\,\frac{\mu+\frac{1}{\lambda}}{1+\frac{\mu}{\lambda}},\ b\,\frac{1-\frac{\mu}{\lambda}}{1+\frac{\mu}{\lambda}},\ c\,\frac{\frac{1}{\lambda}-\mu}{1+\frac{\mu}{\lambda}}\right)\right]$$

$$P(a\mu,\ b,\ -c\mu)$$

Aplicație

$\mathcal{H}_1: x_1^2 + 3x_2^2 - x_3^2 = 1.$

Să se scrie ec. generatoarelor care trec prin $M(1,0,0)$.

SOL

$a=1,\ b=\frac{1}{\sqrt{3}},\ c=1$

$$\left(\frac{\lambda\mu+1}{\lambda+\mu},\ \frac{1}{\sqrt{3}}\frac{\lambda-\mu}{\lambda+\mu},\ \frac{1-\lambda\mu}{\lambda+\mu}\right)=(1,0,0).$$

$$\begin{cases}\lambda-\mu=0 & \Rightarrow \lambda=\mu\\ 1-\lambda\mu=0 & \Rightarrow \lambda^2=1 \Rightarrow \lambda=\pm 1.\end{cases}$$

1) $\lambda=\mu=1 \Rightarrow \frac{2}{2}=1$ ✓

2) $\lambda=\mu=-1 \Rightarrow \frac{2}{-2}=-1$ nu conv.

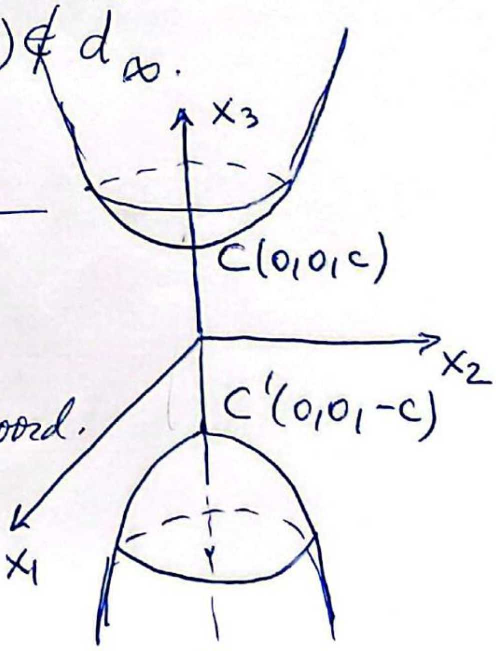
Generatoarele sunt:

$$d_1: \begin{cases} x_1 - x_3 = 1 - \dfrac{x_2}{\frac{1}{\sqrt{3}}} \\ x_1 + x_3 = 1 + \dfrac{x_2}{\frac{1}{\sqrt{3}}}\end{cases}$$

$$\bar{d}_1: \begin{cases} x_1 - x_3 = 1 + \dfrac{x_2}{\frac{1}{\sqrt{3}}} \\ x_1 + x_3 = 1 - \dfrac{x_2}{\frac{1}{\sqrt{3}}}\end{cases}.$$

OBS $M(1,0,0)\notin d_\infty$, $M(1,0,0)\notin \bar{d}_\infty$.

4) Hiperboloid cu 2 pânze



$$\mathcal{H}_2: -\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1.$$

$\cap$ cu plane // cu planele de coord.

a) $x_3 = \gamma \in (-\infty, -c) \cup (c, \infty)$

$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = \frac{\gamma^2}{c^2} - 1$ Elipsa

$\gamma \in \{-c, c\} \Rightarrow C, C'$

b) $x_2 = \beta$ $\qquad$ $-\dfrac{x_1^2}{a^2} + \dfrac{x_3^2}{c^2} = 1 + \dfrac{\beta^2}{b^2}$ Hiperbolă.

c) $x_1 = \alpha$ $\qquad$ $-\dfrac{x_2^2}{b^2} + \dfrac{x_3^2}{c^2} = 1 + \dfrac{\alpha^2}{a^2}$ Hiperbolă.