Seminar 8

1. Arătați că seria de funcții $\sum_{n=1}^{\infty} \operatorname{arctg} \frac{2x}{x^2+n^4}$ converge uniform.

Sol.: $\frac{x^2+n^4}{2} \geq \sqrt{x^2 n^4} = |x| \cdot n^2 \Leftrightarrow x^2+n^4 \geq 2|x|n^2 \Leftrightarrow$

$\Leftrightarrow 1 \geq \frac{2|x|n^2}{x^2+n^4} \Leftrightarrow \frac{1}{n^2} \geq \frac{2|x|}{x^2+n^4} \Leftrightarrow \frac{2|x|}{x^2+n^4} \leq \frac{1}{n^2} \Leftrightarrow$

$\Leftrightarrow -\frac{1}{n^2} \leq \frac{2x}{x^2+n^4} \leq \frac{1}{n^2}$ $\forall x \in \mathbb{R}, \forall n \in \mathbb{N}^*$.

Deoarece arctg este o funcție (strict) crescătoare avem

$-\operatorname{arctg}\frac{1}{n^2} \leq \operatorname{arctg}\frac{2x}{x^2+n^4} \leq \operatorname{arctg}\frac{1}{n^2}$ $\forall x \in \mathbb{R}, \forall n \in \mathbb{N}^*$.

Deci $\left|\operatorname{arctg}\frac{2x}{x^2+n^4}\right| \leq \operatorname{arctg}\frac{1}{n^2}$ $\forall x \in \mathbb{R}, \forall n \in \mathbb{N}^*$.

Fie $\alpha_n = \operatorname{arctg}\frac{1}{n^2}$ $\forall n \in \mathbb{N}^*$.

Arătăm că $\sum_{n=1}^{\infty} \alpha_n$ este convergentă.

$\alpha_n > 0$ $\forall n \in \mathbb{N}^*$.

Fie $\beta_n = \frac{1}{n^2}$ $\forall n \in \mathbb{N}^*$.

$$\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = \lim_{n\to\infty} \frac{\operatorname{arctg}\frac{1}{n^2}}{\frac{1}{n^2}} = 1 \in (0,\infty).$$

Conform crit. de comparație cu limită avem că

$$\sum_{n=1}^{\infty} \alpha_n \sim \sum_{n=1}^{\infty} \beta_n .$$

$\sum\limits_{n=1}^{\infty} \beta_n = \sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ conv. (serie armonică generalizată cu $\alpha=2$).

Deci $\sum\limits_{n=1}^{\infty} \alpha_n$ este convergentă.

Conform Teoremei lui Weierstrass rezultă că $\sum\limits_{n=1}^{\infty} \operatorname{arctg} \frac{2x}{x^2+n^4}$ converge uniform. □

2. Det. mulțimea de conv. pentru următoarele serii de puteri:

a) $\sum\limits_{n=1}^{\infty} \frac{1}{n \cdot 2^n} \cdot x^n$.

Sol.: $a_n = \frac{1}{n \cdot 2^n}$ $\forall$ $n \in \mathbb{N}^*$.

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n \cdot 2^n}} = \lim_{n\to\infty} \frac{1}{(\sqrt[n]{n}) \cdot 2} = \frac{1}{2}.$$

Deci $R = \frac{1}{\frac{1}{2}} = 2$.

Fie A mulțimea de conv. a seriei de puteri din enunț.

Avem $(-R, R) \subset A \subset [-R, R]$, i.e. $(-2, 2) \subset A \subset [-2, 2]$.

Dacă $x=2$ seria devine $\sum_{n=1}^{\infty} \frac{1}{n\cdot \cancel{2^n}}\cdot \cancel{2^n} = \sum_{n=1}^{\infty}\frac{1}{n}$ div.

(serie armonică generalizată cu $\alpha=1$).

Așadar $2 \notin A$.

Dacă $x=-2$ seria devine $\sum_{n=1}^{\infty} \frac{1}{n\cdot 2^n}\cdot (-2)^n = \sum_{n=1}^{\infty}\frac{1}{n\cdot \cancel{2^n}}\cdot (-1)^n \cancel{2^n} =$

$= \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$ conv. (Crit. lui Leibniz).

Așadar $-2 \in A$.

Prin urmare $A=[-2,2)$. $\square$

b) $\sum_{n=1}^{\infty} \frac{n!\, x^n}{(a+1)(a+2)\cdot\ldots\cdot(a+n)}$, $a>1$.

Sol.: $a_n = \frac{n!}{(a+1)\cdot\ldots\cdot(a+n)}$ $\forall n\in\mathbb{N}^*$.

$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\cancel{(n+1)}!^{\,n+1}}{\cancel{(a+1)\cdot\ldots\cdot(a+n)}(a+n+1)}\cdot \frac{\cancel{(a+1)\cdot\ldots\cdot(a+n)}}{\cancel{n!}} =$

$= \lim_{n\to\infty} \frac{n+1}{a+n+1} = 1.$

Deci $R=\frac{1}{1}=1$.

Fie A mulțimea de convergență a seriei de puteri din enunț.

Avem $(-R,R)\subset A\subset[-R,R]$, i.e. $(-1,1)\subset A\subset[-1,1]$.

Dacă $x=1$ seria devine $\sum_{n=1}^{\infty}\frac{n!}{(a+1)\cdot\ldots\cdot(a+n)}\cdot 1^n=$

$$=\sum_{n=1}^{\infty}\frac{n!}{(a+1)\cdot\ldots\cdot(a+n)}.$$

Fie $x_n=\frac{n!}{(a+1)\cdot\ldots\cdot(a+n)}\quad\forall n\in\mathbb{N}^*$.

$$\lim_{n\to\infty}n\left(\frac{x_n}{x_{n+1}}-1\right)=\lim_{n\to\infty}n\left(\frac{a+n+1}{n+1}-1\right)=$$

$$=\lim_{n\to\infty}n\cdot\frac{a+\cancel{n}+\cancel{1}-\cancel{n}-\cancel{1}}{n+1}=a>1.$$

Conform crit. Raabe-Duhamel avem că $\sum_{n=1}^{\infty}x_n$ este convergentă.

Așadar $1\in A$.

Dacă $x=-1$ seria devine $\sum_{n=1}^{\infty}\frac{n!}{(a+1)\cdot\ldots\cdot(a+n)}\cdot(-1)^n$.

$$\sum_{n=1}^{\infty} \left| \frac{n!}{(a+1)\cdot\ldots\cdot(a+n)} \cdot (-1)^n \right| = \sum_{n=1}^{\infty} \frac{n!}{(a+1)\cdot\ldots\cdot(a+n)}$$ conv. (vezi mai sus).

Deci $\sum_{n=1}^{\infty} \frac{n!}{(a+1)\cdot\ldots\cdot(a+n)} (-1)^n$ absolut conv. Prin urmare $\sum_{n=1}^{\infty} \frac{n!}{(a+1)\cdot\ldots\cdot(a+n)} (-1)^n$ conv.

Așadar $-1 \in A$.

Prin urmare $A = [-1, 1]$. $\square$

c) $\sum_{n=1}^{\infty} \frac{3^n}{\sqrt[3]{n}} (x+3)^n$.

<u>Sol.</u>: Notăm $y = x+3$. Seria devine $\sum_{n=1}^{\infty} \frac{3^n}{\sqrt[3]{n}} y^n$.

Determinăm mulțimea de convergență a seriei de puteri $\sum_{n=1}^{\infty} \frac{3^n}{\sqrt[3]{n}} y^n$.

$a_n = \frac{3^n}{\sqrt[3]{n}} \ \forall n \in \mathbb{N}^*$.

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{3^n}{\sqrt[3]{n}}} = \lim_{n\to\infty} \frac{3}{\sqrt[3]{\sqrt[n]{n}}} = 3.$$

Deci $R = \frac{1}{3}$.

Fie B mulțimea de conv. a seriei de puteri $\sum\limits_{n=1}^{\infty} \frac{3^n}{\sqrt[3]{n}} y^n$.

Avem $(-R,R) \subset B \subset [-R,R]$, i.e. $\left(-\frac{1}{3}, \frac{1}{3}\right) \subset B \subset \left[\frac{1}{3}, \frac{1}{3}\right]$.

Dacă $y = \frac{1}{3}$ seria devine $\sum\limits_{n=1}^{\infty} \frac{\cancel{3^n}}{\sqrt[3]{n}} \cdot \frac{1}{\cancel{3^n}} = \sum\limits_{n=1}^{\infty} \frac{1}{n^{\frac{1}{3}}}$ div.

(serie armonică generalizată cu $\alpha = \frac{1}{3}$).

Așadar $\frac{1}{3} \notin B$.

Dacă $y = -\frac{1}{3}$ seria devine $\sum\limits_{n=1}^{\infty} \frac{\cancel{3^n}}{\sqrt[3]{n}} \cdot \frac{(-1)^n}{\cancel{3^n}} =$

$= \sum\limits_{n=1}^{\infty} (-1)^n \cdot \frac{1}{\sqrt[3]{n}}$ conv. (Crit. lui Leibniz).

Așadar $-\frac{1}{3} \in B$.

Prin urmare $B = \left[-\frac{1}{3}, \frac{1}{3}\right)$.

Fie A mulțimea de conv. a seriei de puteri $\sum\limits_{n=1}^{\infty} \frac{3^n}{\sqrt[3]{n}} (x+3)^n$.

$$\underbrace{y \in B}_{\left[-\frac{1}{3}, \frac{1}{3}\right)} \Leftrightarrow -\frac{1}{3} \le y < \frac{1}{3} \underset{y = x+3}{\Leftrightarrow} -\frac{1}{3} \le x+3 < \frac{1}{3} \Big| -3 \Leftrightarrow$$

$\Leftrightarrow -\frac{1}{3}-3 \le x < \frac{1}{3}-3 \Leftrightarrow -\frac{10}{3} \le x < -\frac{8}{3}$.

Deci $A = \left[-\frac{10}{3}, -\frac{8}{3}\right)$. $\square$

d) $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n+1}(x-2)^n$,

Sol.: Rezolvați-l voi! $\square$

e) $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n} x^{2n}$.

Sol.: $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n} x^{2n} = \sum_{k=0}^{\infty} a_k x^k$.

$$a_0 = 0, \quad a_k = \begin{cases} 0 & ; k = 2n-1 \\ \underbrace{\frac{(-1)^{\frac{k}{2}}}{k}}_{= \frac{(-1)^n}{2n}} & ; k = 2n \end{cases}.$$

Deci $a_0 = 0$, $a_{2n} = \frac{(-1)^n}{2n}$ $\forall n \in \mathbb{N}^*$ și $a_{2n-1} = 0$ $\forall n \in \mathbb{N}^*$.

$\overline{\lim} \sqrt[k]{|a_k|} = ?$

$$\lim_{n\to\infty} \sqrt[2n]{|a_{2n}|} = \lim_{n\to\infty} \sqrt[2n]{\frac{1}{2n}} = 1.$$

$$\lim_{n\to\infty} \sqrt[2n-1]{|a_{2n-1}|} = \lim_{n\to\infty} \sqrt[2n-1]{0} = 0.$$

Așadar $\overline{\lim} \sqrt[k]{|a_k|} = 1$.

Deci $R = \frac{1}{1} = 1$.

Fie A mulțimea de convergență a seriei de puteri din enunț.

Avem $(-R, R) \subset A \subset [-R, R]$, i.e. $(-1,1) \subset A \subset [-1,1]$.

Dacă $x = 1$ seria devine $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n} \cdot 1^{2n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n}$ conv.

(Crit. lui Leibniz).

Așadar $1 \in A$.

Dacă $x = -1$ seria devine $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n} (-1)^{2n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n}$ conv.

(Crit. lui Leibniz).

Așadar $-1 \in A$.

Prin urmare $A = [-1, 1]$. $\square$

3. Să se dezvolte în serie de puteri ale lui x funcțiile de mai jos:

a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin x$.

Sol.: $I=\mathbb{R}=(-\infty,\infty)$.

$0\in I$.

$f\in C^\infty(I)$.

$f(x)=\sin x \Rightarrow f(0)=0.$

$f'(x)=\cos x \Rightarrow f'(0)=1.$

$f''(x)=-\sin x \quad \forall x\in\mathbb{R} \Rightarrow f''(0)=0.$

$f'''(x)=-\cos x \Rightarrow f'''(0)=-1.$

$f^{(4)}(x)=\sin x \Rightarrow f^{(4)}(0)=0.$

$\vdots \qquad \vdots$

Conform Teoremei lui Taylor cu restul sub forma lui Lagrange $\forall x\in\mathbb{R}^*$ (i.e. $x\neq 0$), $\exists c$ între 0 și x (i.e. $c\in(0,x)$ sau $c\in(x,0)$) a.î.

$$R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}.$$

Arătăm că $\lim\limits_{n\to\infty}R_n(x)=0 \ \forall x\in\mathbb{R}^*$.

Fie $x\in\mathbb{R}^*$.

Arătăm că $|R_n(x)| \xrightarrow[n\to\infty]{} 0$.

$$0 \le |R_n(x)| = \frac{\overbrace{|f^{(n+1)}(c)|}^{\le 1}}{(n+1)!}|x|^{n+1} \le \frac{|x|^{n+1}}{(n+1)!} \quad \forall n \in \mathbb{N}.$$

$\xrightarrow[n\to\infty]{} 0$

(Crit. rap. pentru șiruri cu termeni strict pozitivi)

Deci $\lim\limits_{n\to\infty} |R_n(x)| = 0$, i.e. $\lim\limits_{n\to\infty} R_n(x) = 0$.

Prin urmare $f(x) = \sum\limits_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n \ \forall x \in \mathbb{R}^*$.

Deci $f(x) = \sum\limits_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0 + \frac{1}{1!}x^1 + 0 - \frac{1}{3!}x^3 + 0 +$

$+ \frac{1}{5!}x^5 + \ldots = \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot x^{2n+1} \quad \forall x \in \mathbb{R}^*$.

Prin urmare $\sin x = \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad \forall x \in \mathbb{R}^*$.

$f(0) = \sin 0 = 0$

$\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot 0^{2n+1} = 0$

$\Rightarrow f(0) = \sin 0 = \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} 0^{2n+1}$.

Deci $\underset{\underset{\sin x}{\parallel}}{f(x)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^n \quad \forall x \in \mathbb{R}.$ $\square$

b) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \cos x$.

<u>Sol</u>: Rezolvați-l voi! $\square$