1) Fie $\alpha = \left(\mathbb{Q}, \mathbb{Q}, \underbrace{\left\{\left(\frac{a}{b}, \frac{a+1}{b}\right) \middle| a, b \in \mathbb{Z} \text{ cu } b \neq 0\right\}}_{\overset{\text{not}}{=} \sigma}\right)$.

Este α funcție?

Def: α = funcție $\Leftrightarrow \forall r \in \mathbb{Q}, \exists! r' \in \mathbb{Q}$ a.i. $(r', r) \in \sigma$

$$\frac{1}{2} = \frac{2}{4}$$

$$\frac{1+1}{2} = 1 \qquad \frac{2+1}{4} = \frac{3}{4}$$

$\Rightarrow \alpha$ nu este funcție.

2) $\beta = \left(\mathbb{Q}, \mathbb{Q}, \left\{\left(\frac{a}{b}, \frac{a+1}{b}\right) \middle| \begin{array}{l} a, b \in \mathbb{Z} \text{ cu } b \neq 0 \\ (a, b) = 1 \end{array}\right\}\right)$

β = funcție?

Def:

$$\frac{-7}{-4} = \frac{7}{4}$$

$\Rightarrow \beta$ nu este funcție.

$$\frac{-7+1}{-4} = \frac{3}{2} \qquad \frac{7+1}{4} = 2$$

3) $\gamma = \left(\mathbb{Q}, \mathbb{R}, \left\{ \left(\frac{a}{b}, \frac{a+1}{b} \right) \middle| \begin{array}{l} a \in \mathbb{Z}, b \in \mathbb{N}^* \\ (a,b)=1 \end{array} \right\} \right)$

γ = funcție ?

Sol: $\frac{a}{b} = \frac{c}{d}$ cu $a, c \in \mathbb{Z}$; $b, d \in \mathbb{N}^*$, $(a,b) = 1 = (c,d)$ $\Rightarrow \begin{cases} a = c \\ b = d \end{cases} \Rightarrow$ *

$\Rightarrow \frac{a+1}{b} = \frac{c+1}{d}$, deci γ = funcție

Avem $ad = bc \;\vdots\, a$ $\Rightarrow \left.\begin{array}{l} a \mid bc \\ (a,b)=1 \end{array}\right\} \Rightarrow a \mid c \Rightarrow c = ac', c' \in \mathbb{Z}$

$ad = bc$

$\left.\begin{array}{l} \left.\begin{array}{l} c \mid bc = ad \\ (c,d)=1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} c \mid a \\ a \mid c \end{array}\right\} \Rightarrow a = \pm c \end{array}\right\} \Rightarrow \left.\begin{array}{l} d = \pm b \\ d, b \in \mathbb{N}^* \end{array}\right\} \Rightarrow$

$\Rightarrow d = b \Rightarrow a = c$.

Este γ = inj, surj sau bij ?

$\gamma : \left\{ \frac{a}{b} \middle| \begin{array}{l} a \in \mathbb{Z}, b \in \mathbb{N}^* \\ (a,b)=1 \end{array} \right\} \to \mathbb{Q}$

$\gamma\left(\frac{a}{b}\right) = \frac{a+1}{b}$

$\gamma \neq$ inj : $\gamma\left(\frac{3}{2}\right) = \gamma\left(\frac{7}{4}\right) = 2 \Rightarrow \gamma \neq$ bij

γ = surj: $\forall \frac{c}{d} \in \mathbb{Q}$, $\exists a \in \mathbb{Z}, b \in \mathbb{N}^*$ cu $(a,b)=1$

a.î. $f\left(\frac{a}{b}\right) = \frac{c}{d} \Leftrightarrow \frac{a+1}{b} = \frac{c}{d}$?

$(a+1)d = bc$; pp $(c,d)=1$, altfel simplific fracția $\frac{c}{d}$.

$\left.\begin{array}{l} c \mid bc = (a+1)d \\ (c,d)=1 \end{array}\right\} \Rightarrow c \mid a+1 \Rightarrow a+1 = ck, k \in \mathbb{Z} \Rightarrow a = ck-1$

$ckd = bc \Rightarrow b = kd$

$\forall c, d$ cu $(c,d)=1$, $\exists k, a, b$ cu $(a,b)=1$ a.î. $\begin{cases} a = ck-1 \\ b = kd \end{cases}$

Altfel spus, $\exists k \in \mathbb{Z}$ a.î. $(ck-1, kd) = 1$

$\exists k \in \mathbb{Z}$ și $\alpha, \beta \in \mathbb{Z}$ a.î. $\alpha(ck-1) + \beta kd = 1$

$\alpha k \cdot \underline{\underline{c}} + \beta k \cdot \underline{\underline{d}} = \alpha + 1$

$(c,d) = 1$ $\quad\quad k(\alpha c + \beta d) = \alpha + 1$

$\Updownarrow$ $\exists u, v \in \mathbb{Z}$ a.î. $uc + vd = 1 \mid \cdot (\alpha+1) \Rightarrow u(\alpha+1)c + v(\alpha+1)d = \alpha+1$

Iau $k = \alpha+1$, $\alpha = u$, $\beta = v$; deci $\alpha = u$, $\beta = v$, $k = u+1$.

Exemplu:

$$\frac{21}{14} = \frac{3}{2} = \frac{a+1}{b} \text{ cu } (a,b)=1.$$

$(3,2)=1$: $\exists u, v$ a.î. $3u+2v=1$ $(u=1, v=-1)$

Avem $k = u+1 = 2$, $a = ck-1 = 6-1 = 5$

$b = kd = 4$

Deci r = surj.

4) $f: \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$, $f((x,y)) = x^2 - y^2$

f = inj, surj, bij?, $\operatorname{Im} f = ?$

Sol: $f((-1,0)) = f((1,0)) = 1$, dar $(-1,0) \neq (1,0)$

Deci $f \neq$ inj $\Rightarrow f \neq$ bij

f = surj? $2 \notin \operatorname{Im} f$, altfel $\exists (x,y) \in \mathbb{Z}$ a.î.

$f((x,y)) = 2 \iff x^2 - y^2 = 2 \iff (x+y)(x-y) = 2$ în $\mathbb{Z}$

$\cdot \begin{cases} x+y = \pm 1 \\ x-y = \pm 2 \end{cases} \Rightarrow y \notin \mathbb{Z}$; $\cdot \begin{cases} x+y = \pm 2 \\ x-y = \pm 1 \end{cases} \Rightarrow y \notin \mathbb{Z}$ $\quad$ Ab.

Deci $f \neq$ surj

$\operatorname{Im} f = ?$

$f((x,y)) = x^2 - y^2$

$\operatorname{Im} f = \mathbb{Z} \setminus \{4k+2 \mid k \in \mathbb{Z}\}$.

$x, y \in \mathbb{Z} \to x = 2k,\ y = 2l \Rightarrow x^2 - y^2 \in 4\mathbb{Z} = \{4k \mid k \in \mathbb{Z}\}$

$\to x = 2k+1,\ y = 2l+1 \Rightarrow x^2 - y^2 \in 4\mathbb{Z}$

$\to x = 2k+1,\ y = 2l \Rightarrow x^2 - y^2 \in 4\mathbb{Z}+1 =$

$= \{4k+1 \mid k \in \mathbb{Z}\}$.

$\to x = 2k,\ y = 2l+1 \Rightarrow x^2 - y^2 \in 4\mathbb{Z}+3$

Deci $\operatorname{Im} f \subseteq \mathbb{Z} \setminus (4\mathbb{Z}+2)$

Reciproc, $4k = (k+1)^2 - (k-1)^2$

$4k+1 = (2k+1)^2 - (2k)^2$, $\forall k \in \mathbb{Z} \Rightarrow \mathbb{Z} \setminus (4\mathbb{Z}+2) \subseteq \operatorname{Im} f$.

$4k+3 = (2k+2)^2 - (2k+1)^2$

5) $f: \mathbb{Z}\times\mathbb{Z} \to (0,+\infty)$, $f((x,y)) = (x-\sqrt{3})^2 + \left(y-\frac{1}{3}\right)^2$

Este f inj, surj, bij ?

Sol: f = funcție: $f((x,y)) > 0$ $\forall (x,y) \in \mathbb{Z}\times\mathbb{Z}$ (este ≥ 0

și nu este $= 0$, altfel $\begin{cases} x-\sqrt{3}=0 \\ y-\frac{1}{3}=0 \end{cases}$ $\cancel{\text{ñ}}$ cu $x,y \in \mathbb{Z}$)

f = inj: $f((x,y)) = f((a,b)) \Rightarrow (x,y) = (a,b)$?

$$(x-\sqrt{3})^2 + \left(y-\frac{1}{3}\right)^2 = (a-\sqrt{3})^2 + \left(b-\frac{1}{3}\right)^2 \text{ cu } x,y,a,b \in \mathbb{Z}$$

$$x^2 - 2\sqrt{3}\cdot x + y^2 - \frac{2}{3}y = a^2 - 2\sqrt{3}\cdot a + b^2 - \frac{2}{3}b$$

$$\underbrace{2\sqrt{3}}_{\in \mathbb{R}\setminus\mathbb{Q}}\underbrace{(a-x)}_{\in\mathbb{Z}} = \underbrace{a^2 - x^2 - \frac{2}{3}(b-y) + b^2 - y^2}_{\in\mathbb{Q}}$$

$\Rightarrow a - x = 0 \Rightarrow x = a \Rightarrow b^2 - y^2 - \frac{2}{3}(b-y) = 0$

$$(b-y)\underbrace{\left(\underbrace{b+y}_{\in\mathbb{Z}} - \overset{\mathbb{Q}\setminus\mathbb{Z}}{\frac{2}{3}}\right)}_{\neq 0} = 0$$

$\Rightarrow b - y = 0 \Rightarrow y = b$.

Deci f = inj.

$\text{Im} f = \{ f((x,y)) \mid x,y \in \mathbb{Z} \}$

$= \{ x^2 - 2\sqrt{3}\cdot x + 3 + y^2 - \frac{2}{3}y + \frac{1}{9} \mid x,y\in\mathbb{Z} \}$

$\Rightarrow \text{Im} f \subseteq \{ m + n\sqrt{3} \mid m,n \in \mathbb{Q} \}$.

$\not\ni \sqrt{2}$

$\Rightarrow f$ nu e surj

$\Rightarrow f$ nu este bij.

6) $g \circ f = inj$ (resp. $surj$) $\not\Longrightarrow$ $g = inj$ (resp. $f = surj$)

<u>Ex:</u> $\mathbb{R} \xrightarrow{g} [0, +\infty) \xrightarrow{f} \mathbb{R}$

$\mathbb{R} \ni x \xrightarrow{g} x^2$; $x \xrightarrow{f} \sqrt{x}$

$(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{x}^2 = x \Rightarrow$

$\Rightarrow g \circ f = Id_{[0,+\infty)} \to bij$ (inj + surj)

$g \neq inj$ $(g(-1) = g(1))$; $f \neq surj$ $(-1 \notin Im f)$

7) $A, B \subseteq M \neq \emptyset$, $f: \mathcal{P}(M) \to \mathcal{P}(A) \times \mathcal{P}(B)$

$f(X) = (X \cap A, X \cap B)$

i) f = inj $\Leftrightarrow A \cup B = M$

ii) f = surj $\Leftrightarrow A \cap B = \emptyset$

iii) f = bij $\Leftrightarrow B = C_M A$

<u>Sol:</u> i) „$\Rightarrow$" $f(A \cup B) = ((A \cup B) \cap A, (A \cup B) \cap B) = (A, B)$

$f(M) = (M \cap A, M \cap B) = (A, B)$ $\Rightarrow$

$\xRightarrow{f = inj}$ $A \cup B = M$.

„$\Leftarrow$" Fie $X, Y \in \mathcal{P}(M)$ a.î. $f(X) = f(Y) \Leftrightarrow \begin{cases} X \cap A = Y \cap A \\ X \cap B = Y \cap B \end{cases}$

$X = X \cap M = X \cap (A \cup B) = (X \cap A) \cup (X \cap B)$

$= (Y \cap A) \cup (Y \cap B) = Y \cap (A \cup B) = Y \cap M = Y$.

ii) „$\Rightarrow$" PRA că $A \cap B \neq \emptyset \Rightarrow \exists x \in A \cap B$

$(\{x\}, B \setminus \{x\}) \in \mathcal{P}(A) \times \mathcal{P}(B) \xRightarrow{f = surj} \exists X \subseteq M$ a.î. $f(X) = (\{x\}, B \setminus \{x\})$

$\Leftrightarrow \begin{cases} X \cap A = \{x\} \Rightarrow x \in X \text{; dar } x \in B \Rightarrow x \in \underbrace{X \cap B}_{= B \setminus \{x\}} \text{ abs} \\ X \cap B = B \setminus \{x\} \end{cases}$

„$\Leftarrow$" fie $(X, Y) \in \mathcal{P}(A) \times \mathcal{P}(B)$; caut $Z \subseteq M$ a.î.

$f(Z) = (X, Y) \Leftrightarrow \begin{cases} Z \cap A = X \\ Z \cap B = Y. \end{cases}$

Iau $Z = X \cup Y$: $(X \cup Y) \cap A = (\underbrace{X \cap A}_{= X}) \cup (\underbrace{Y \cap A}_{\subseteq B \cap A = \emptyset}) = X \cup \emptyset = X$

$(X \cup Y) \cap B = (\underbrace{X \cap B}_{\subseteq A \cap B = \emptyset}) \cup (\underbrace{Y \cap B}_{= Y}) = \emptyset \cup Y = Y$.

iii) Rezultă din i) și ii)

<u>obs:</u> f = bij ; $f^{-1}: \mathcal{P}(A) \times \mathcal{P}(B) \to \mathcal{P}(M)$

$f^{-1}((X, Y)) = X \cup Y$