Seminar 9

1) $g : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, $g(x,y) = ax_1y_1 + bx_1y_2 + bx_2y_1 + cx_2y_2$

a) $g \in L^s(\mathbb{R}^2, \mathbb{R}^2, \mathbb{R})$

b) g produs scalar $\iff \begin{cases} a > 0 \\ ac - b^2 > 0 \end{cases} \iff (\mathbb{R}^2, g)$ s. v.e.r.

$$g(x,y) = \sum_{i,j=1}^{2} g_{ij} x_i y_j \qquad G = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \Rightarrow g \text{ formă biliniară}$$

$$G = G^T$$

$\Rightarrow g \in L^s(\mathbb{R}^2, \mathbb{R}^2, \mathbb{R})$

$Q : \mathbb{R}^2 \to \mathbb{R} : Q(x) = ax_1^2 + 2b\, x_1x_2 + cx_2^2$

Q poz. def. $\iff \Delta_1 = a > 0$

$\Delta_2 = \det G = ac - b^2 > 0$. (Metoda Iacobi)

$\Rightarrow \operatorname{sgn} = (2,0)$

2) $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ formă biliniară

$G = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{pmatrix}$ matricea asociată în rap. cu R_0.

Este $(\mathbb{R}^3, g)$ sp. v.e.r. ?

$G = G^T \Rightarrow g \in L^s(\mathbb{R}^3, \mathbb{R}^3, \mathbb{R})$

$Q: \mathbb{R}^3 \to \mathbb{R}$ forma pătratică asociată

$Q(x) = 3x_1^2 + 4x_1x_2 + 2x_2^2 + 4x_2x_3 + x_3^2$

Q este poz. def. ?

Metoda Iacobi

$\Delta_1 = 3$

$\Delta_2 = 2.$

$\Delta_3 = 6 - 12 - 4 = -10$.

$\Rightarrow sgn = (2, 1)$

$\exists \mathcal{R}'$ a.î. $Q(x) = \frac{1}{3}x_1^2 + \frac{3}{2}x_2^2 - \frac{1}{5}x_3^2$

$\Rightarrow$ Q nu e poz. def.

$\Rightarrow$ g nu e produs scalar.

$\Rightarrow (\mathbb{R}^3, g)$ nu e s.v.e.r.

Metoda Gauss.

$Q(x) = \frac{1}{3}(9x_1^2 + 12x_1x_2) + 2x_2^2 + 4x_2x_3 + x_3^2 =$

$= \frac{1}{3}(3x_1 + 2x_2)^2 - \frac{4}{3}x_2^2 + 2x_2^2 + 4x_2x_3 + x_3^2$

$= \frac{1}{3}(3x_1 + 2x_2)^2 + \frac{2}{3}x_2^2 + 4x_2x_3 + x_3^2 =$

$= \frac{1}{3}(3x_1 + 2x_2)^2 + \frac{2}{3}(x_2^2 + 6x_2x_3) + x_3^2 =$

$= \frac{1}{3}(3x_1 + 2x_2)^2 + \frac{2}{3}\cdot(x_2 + 3x_3)^2 - 5x_3^2$

3) $(\mathbb{R}^3, g_0)$, $g_0 : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$

$g_0(x,y) = x_1 y_1 + x_2 y_2 + x_3 y_3$

$U = \{x \in \mathbb{R}^3 / x_1 + x_2 - x_3 = 0\} = \{x \in \mathbb{R}^3 / g_0(x,y) = 0\}$ unde $y = (1,1,-$

a) $U^\perp$

b) $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ reper ortonormat în $\mathbb{R}^3$

$\mathcal{R}_1$ - reper ortonormat în U

$\mathcal{R}_2$ = reper ortonormat în $U^\perp$

a) $U^\perp = \{y \in \mathbb{R}^3 / g(x,y) = 0, \forall x \in U\}$

$U^\perp = \langle \{1, 1, -1\} \rangle$

b) $U = \langle \{(1,0,1), (0,1,1)\} \rangle$

$U = \{(x_1, x_2, x_1 + x_2) / x_1, x_2 \in \mathbb{R}\}$

$x_1(1,0,1) + x_2(0,1,1)$

$\dim U = 2$.

$\{f_1, f_2\}$ reper în U.

Obs pentru a)

$g_0(y, f_1) = 0$, $g_0(y, f_2) = 0$.

$$\begin{cases} y_1 + y_3 = 0 & \Rightarrow y_1 = -y_3 \\ y_2 + y_3 = 0. & y_2 = -y_3 \end{cases}$$

Aplicăm Gram-Schmidt.

$$e_1 = f_1 = (1,0,1)$$

$$e_2 = f_2 - \frac{<f_2, e_1>}{<e_1, e_1>} \cdot e_1 = (0,1,1) - \frac{1}{2}(1,0,1) = \left(-\frac{1}{2}, 1, \frac{1}{2}\right) = $$
$$= \frac{1}{2}(-1,2,1)$$

$$<f_2, e_1> = <f_2, f_1> = 1$$
$$<e_1, e_1> = <f_1, f_1> = 2.$$

$$\{f_1, f_2\} \longrightarrow \{e_1, e_2\} \longrightarrow \{e_1', e_2'\}.$$

$$e_1' = \frac{e_1}{\|e_1\|} = \frac{1}{\sqrt{2}} \cdot (1,0,1).$$

$$e_2' = \frac{e_2}{\|e_2\|} = \frac{1}{\sqrt{6}}(-1,2,1)$$

Obs. $v = \alpha \cdot u$, $\alpha > 0$.
$$\frac{v}{\|v\|} = \frac{u}{\|u\|}$$

$\mathcal{R}_2 = \{\frac{1}{\sqrt{3}}(1,1,-1)\}$ reper în $U^\perp$

4) $(\mathbb{C}, +, \cdot)_{\mathbb{R}}$, $g : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ formă biliniară

$G = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ mat. asociată lui g în rap. cu $\mathcal{R}_0 = \{1, i\}$.

a) $(\mathbb{C}, g)$ s.v.e.r.

$z = x_1 + i x_2$

$G = G^T \Rightarrow g \in L^s(\mathbb{C}, \mathbb{C}; \mathbb{R})$

$Q : \mathbb{C} \to \mathbb{R}$, $Q(z) = g(z,z) = x_1^2 + 4x_1x_2 + 5x_2^2$

Metoda Iacobi

$\Delta_1 = 1 > 0$
$\Delta_2 = 1 > 0.$

$\Rightarrow \exists\, \mathcal{R}'$ a.î. $Q(x) = x_1'^2 + x_2'^2$

$\Rightarrow sgn = (2,0)$

$\Rightarrow Q$ poz. def $\Rightarrow g$ prod. scalar.

Met. Gauss

$Q(x) = x_1^2 + 4x_1x_2 + 5x_2^2 = (x_1 + 2x_2)^2 + x_2^2$

$\begin{cases} x_1' = x_1 + 2x_2 \\ x_2' = x_2 \end{cases} \Rightarrow Q(x) = x_1'^2 + x_2'^2$

$z = x_1 + x_2$

$z' = y_1 + y_2.$

b) ~~#~~

$u = 2 - i$ versor în rap. cu g. ?

$\|u\|_g = \sqrt{g(u,u)} = \sqrt{Q(u)} = \sqrt{4-8+5} = 1 \Rightarrow u$ versor.

c) $\langle \{u\} \rangle^{\perp} = \{ \underset{x_1+ix_2}{\overset{\|}{z}} \in \mathbb{C} \,/\, g(z,u) = 0 \} = \cancel{R}$

$g(z, z') = x_1y_1 + 2x_1y_2 + 2x_2y_1 + 5x_2y_2.$

$\Rightarrow y_1 = 2$
$\quad y_2 = -1$

$g(z,u) = 2x_1 - 2x_1 + 4x_2 - 5x_2 = -x_2 = 0.$

d) Să se ortonormeze R_0 în rap. cu g (. Să se aplice Gram - Schmidt)

$R_0 = \{1, i\} = \{f_1, f_2\} \longrightarrow R^* = \{e_1, e_2\} \longrightarrow R^2 \{e_1', e_2'\}$

$e_1 = f_1 = 1.$

$e_2 = f_2 - \dfrac{g(f_2, e_1)}{g(e_1, e_1)} \cdot e_1 = i - \dfrac{2}{1} \cdot 1 = -2 + i$

$g(\underset{0+1\cdot i}{\overset{\|}{f_2}}, \underset{1+0\cdot i}{\overset{\|}{e_1}}) = 2 \cdot 1 = 2.$

$g_0(\underset{x_1+ix_2}{\overset{\|}{z}}, \underset{y_1+iy_2}{\overset{\|}{z'}}) = x_1y_1 + x_2y_2$

$g(e_1, e_1) = 1.$

$$e_1' = \frac{e_1}{\|e_1\|_g} = e_1 \qquad e_2' = \frac{e_2}{\|e_2\|_g} = e_2$$

$$\|e_2\|_g = \sqrt{g(e_2, e_2)} = \cancel{\sqrt{g(-2+i)}} =$$

$$= \sqrt{g(-2+i, -2+i)} = \sqrt{g(-u, -u)} = \sqrt{(-1)^2 \cdot g(u,u)} = 1.$$

e) Să se afle intersecția dintre cercul unitate în $(\mathbb{C}, g_0)$ și $(\mathbb{C}, g)$

$$S^1_{g_0} = \{z \in \mathbb{C} \mid \|z\|_{g_0} = 1\} = \{z = x_1 + i x_2 \mid x_1^2 + x_2^2 = 1\}$$

$$S^1_g = \{z \in \mathbb{C} \mid \|z\|_g = 1\} = \{z = x_1 + i x_2 \mid x_1^2 + 4x_1x_2 + 5x_2^2 = 1\}$$

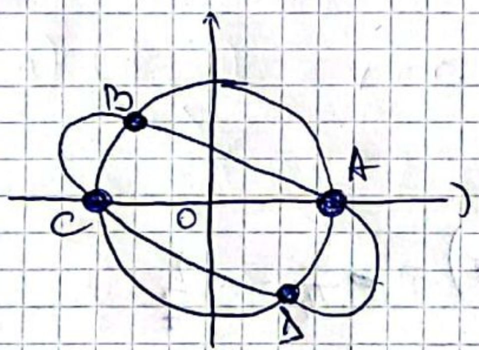
$$\begin{cases} x_1^2 + x_2^2 = 1 \\ x_1^2 + 4x_1x_2 + 5x_2^2 = 1 \end{cases} \Rightarrow z = \cos t + i \sin t.$$

$$4x_2^2 + 4x_1x_2 = 0.$$

$$\Rightarrow 4x_2(x_1 + x_2) = 0. \Rightarrow \sin t = 0 \Leftrightarrow x_2 = 0.$$

$$\Rightarrow t = k\pi$$

$$x_1 + x_2 = 0 \Rightarrow \sin t + \cos t = 0 \,/ : \cos t \Rightarrow tg\, t = -1 \Leftrightarrow t = -\frac{\pi}{4} + k\pi, k \in \mathbb{Z}$$



$$z_A = 1$$

$$z_B = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

$$z_C = -1$$

$$z_D = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i$$

5) $(\mathbb{R}^3, g_0)$, $\mathcal{R} = \{f_1 = (1,2,3), f_2 = (0,1,1), f_3 = (1,2,5)\}$

$g_0 : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$

$g_0(x,y) = x_1 y_1 + x_2 y_2 + x_3 y_3$

a) $\mathcal{R}$ reper. Să se ortonormeze

b) $f_1 \times f_2$

c) $f_1 \wedge f_2 \wedge f_3$

a)

$$\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 5 \end{vmatrix} = 2 \neq 0 \Rightarrow \mathcal{R} \text{ SLI}$$

$\dim \mathcal{R} = \dim \mathbb{R}^3$ $\Rightarrow \mathcal{R}$ reper

$$\boxed{e_1 = f_1 = (1,2,3)}$$

$$e_2 = f_2 - \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} \cdot e_1$$

$$e_3 = f_3 - \frac{\langle f_3, e_1 \rangle}{\langle e_1, e_1 \rangle} \cdot e_1 - \frac{\langle f_3, e_2 \rangle}{\langle e_2, e_2 \rangle} e_2$$

$\langle f_2, e_1 \rangle = \langle f_2, f_1 \rangle = 0 + 2 + 3 = 5$

$\langle e_1, e_1 \rangle = \langle f_1, f_1 \rangle = 1 + 4 + 9 = 14$

$$e_2 = (0,1,1) - \frac{5}{14} \cdot (1,2,3) = \left(-\frac{5}{14}, \frac{4}{14}, -\frac{1}{14}\right) = \boxed{\frac{1}{14}(-5, 4, -1)}$$

$\langle f_3, e_1 \rangle = \langle f_3, f_1 \rangle = 20$

$\langle f_3, e_2 \rangle = \frac{1}{14}(-5 + 8 - 5) = -\frac{2}{14} = -\frac{1}{7}$

$\langle e_2, e_2 \rangle = \frac{1}{14^2}(25 + 16 + 1) = \frac{42}{14^2} = \frac{3}{14}$

$$e_3 = (1,2,5) - \frac{\cancel{20}^{10}}{\cancel{14}_{7}} \cdot (1,2,3) - \frac{-\frac{1}{7}}{\frac{3}{\cancel{14}}} \cdot \frac{1}{\cancel{14}}(-5, 4, -1)$$

$\Rightarrow e_3 = \left(1 - \frac{10}{7} - \frac{5}{21},\ 2 - \frac{20}{7} + \frac{4}{21},\ 5 - \frac{30}{7} - \frac{1}{21}\right) =$

$= \frac{1}{21}\left(21 - 30 - 5,\ 42 - 60 + 4,\ 105 - 90 - 1\right) =$

$= \frac{1}{21}\left(-14, -14, 14\right) = \frac{14}{21}(-1, -1, 1) = \boxed{\frac{2}{3}(-1, -1, 1)}$

$e_1' = \frac{e_1}{\|e_1\|} = \frac{(1,2,3)}{\sqrt{14}}$

$e_2' = \frac{e_2}{\|e_2\|} = \frac{1}{\sqrt{42}} \cdot e_2 = \frac{1}{\sqrt{42}} \cdot (-5, 4, -1)$

$e_3' = \frac{e_3}{\|e_3\|} = \frac{1}{\sqrt{3}}(-1, -1, 1)$

b) $f_1 \times f_2 = \begin{vmatrix} e_1^0 & e_2^0 & e_3^0 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix} = e_1^0 \cdot \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} - e_2^0 \cdot \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} +$

$+ e_3^0 \cdot \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = (-1, -1, 1)$

c) $f_1 \wedge f_2 \wedge f_3 = f_3 \wedge f_1 \wedge f_2 = \langle f_3, f_1 \times f_2 \rangle =$

$= \cancel{(-1, -2, 1)} = -1 - 2 + 5 = 2$

sau $f_1 \wedge f_2 \wedge f_3 = \begin{vmatrix} 1 & 2 & 5 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix} = 2$

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