Seminar 7

1. Fie $f:\left(0, \frac{2}{\pi}\right] \to \mathbb{R}$, $f(x)=\sin\frac{1}{x}$. Arătați că f nu este uniform continuă.

Sol.: Conform unei propoziții de la curs, următoarele afirmații sunt echivalente:

1) f este uniform continuă.

2) $\exists \tilde{f}: \left[0, \frac{2}{\pi}\right] \to \mathbb{R}$, $\tilde{f}$ continuă a.î. $\tilde{f}\Big|_{\left(0, \frac{2}{\pi}\right]} = f$.

Presupunem prin absurd că f este uniform continuă.

Deci $\exists \tilde{f}: \left[0, \frac{2}{\pi}\right] \to \mathbb{R}$, $\tilde{f}$ continuă a.î. $\tilde{f}\Big|_{\left(0, \frac{2}{\pi}\right]} = f$.

$\tilde{f}$ continuă $\Rightarrow \lim\limits_{x\to 0} \tilde{f}(x) = \tilde{f}(0) \Rightarrow \lim\limits_{x\to 0} f(x) = \tilde{f}(0) \Rightarrow$

$\Rightarrow \lim\limits_{x\to 0} \sin\frac{1}{x} = \tilde{f}(0)$.

Deci $\exists \lim\limits_{x\to 0} \sin\frac{1}{x}$.

Alegem $x_n = \frac{1}{n\pi}$ $\forall n \in \mathbb{N}^*$ și $y_n = \frac{1}{2n\pi + \frac{\pi}{2}}$ $\forall n \in \mathbb{N}^*$.

Avem $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = 0$ și $\lim\limits_{n\to\infty} \sin\frac{1}{x_n} = \lim\limits_{n\to\infty} \sin n\pi =$

$= 0$, $\lim\limits_{n\to\infty} \sin\frac{1}{y_n} = \lim\limits_{n\to\infty} \sin\left(2n\pi + \frac{\pi}{2}\right) = 1$, deci $\nexists \lim\limits_{x\to 0} \sin\frac{1}{x}$, contradicție.

Prin urmare f nu este uniform continuă. $\square$

2. Fie $x_0 \in \mathbb{R}$, $f, g : \mathbb{R} \to \mathbb{R}$ două funcții derivabile în x_0 și $h : \mathbb{R} \to \mathbb{R}$, $h(x) = \begin{cases} f(x) ; & x \in \mathbb{Q} \\ g(x) ; & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$. Arătați că h este derivabilă în x_0 dacă și numai dacă $f(x_0) = g(x_0)$ și $f'(x_0) = g'(x_0)$.

Sol.: Rezolvați-l voi! $\square$

3. Studiați convergența simplă și uniformă pentru următoarele șiruri de funcții:

a) $f_n : [0, \infty) \to \mathbb{R}$, $f_n(x) = \dfrac{x}{x+n}$ $\forall\, n \in \mathbb{N}^*$.

Sol.: Convergența simplă

Fie $x \in [0, \infty)$.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{x+n} = 0 \Rightarrow f_n \xrightarrow[n\to\infty]{\Delta} f, \text{ unde}$$

$f : [0, \infty) \to \mathbb{R}$, $f(x) = 0$.

Convergența uniformă

$$\sup_{x\in[0,\infty)} |f_n(x) - f(x)| = \sup_{x\in[0,\infty)} \left| \frac{x}{x+n} - 0 \right| = \sup_{x\in[0,\infty)} \frac{x}{x+n} \underset{x=n}{\geq}$$

$$\underset{\substack{\uparrow\\ x=n}}{\geq} \frac{n}{n+n} = \underset{\substack{\vee\\ 0}}{\frac{1}{2}} \not\xrightarrow[n\to\infty]{} 0 \Rightarrow f_n \not\xrightarrow[n\to\infty]{u} f. \ \square$$

b) $f_n : [2,3] \to \mathbb{R}$, $f_n(x) = \dfrac{x}{x+n}$ $\forall n \in \mathbb{N}$.

Sol.: C.s.

Fie $x \in [2,3]$.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{x+n} = 0 \Rightarrow f_n \xrightarrow[n\to\infty]{s} f \text{, unde}$$

$f : [2,3] \to \mathbb{R}$, $f(x) = 0$.

C.u.

$$\sup_{x\in[2,3]} |f_n(x) - f(x)| = \sup_{x\in[2,3]} \left|\frac{x}{x+n} - 0\right| = \sup_{x\in[2,3]} \left|\frac{x}{x+n}\right| =$$

$$= \sup_{x\in[2,3]} \frac{x}{x+n}.$$

Fie $f_n : [2,3] \to \mathbb{R}$, $f_n(x) = \dfrac{x}{x+n}$ $\forall n \in \mathbb{N}$.

$$f_n'(x) = \frac{\cancel{x} + n - \cancel{x}}{(x+n)^2} = \frac{n}{(x+n)^2} \geq 0 \quad \forall x \in [2,3], \forall n \in \mathbb{N}.$$

Deci f_n este crescătoare $\forall n \in \mathbb{N}$.

x	2			3
$f_n'(x)$	$+ + + + + + + + +$			
$f_n(x)$	$\frac{2}{2+n}$	$\nearrow$	$\nearrow$	$\frac{3}{3+n}$

Deci $\sup\limits_{x\in[2,3]} |f_n(x)-f(x)| = \frac{3}{3+n} \xrightarrow[n\to\infty]{} 0.$

Așadar $f_n \xrightarrow[n\to\infty]{u} f$. $\square$

c) $f_n : [0,\infty) \to \mathbb{R}$, $f_n(x) = \sqrt{x^2+\frac{1}{n}}$ $\forall\, n\in\mathbb{N}^*$.

Sol.: C.S.

Fie $x\in[0,\infty)$.

$\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \sqrt{x^2+\frac{1}{n}} = \sqrt{x^2} = |x| = x \Rightarrow$

$\Rightarrow f_n \xrightarrow[n\to\infty]{s} f$, unde $f : [0,\infty) \to \mathbb{R}$, $f(x) = x$.

C.U.

$$\sup_{x\in[0,\infty)} |f_n(x)-f(x)| = \sup_{x\in[0,\infty)} \left| \sqrt{x^2+\frac{1}{n}} - x \right| =$$

$$= \sup_{x\in[0,\infty)} \left(\sqrt{x^2+\frac{1}{n}} - x\right) = \sup_{x\in[0,\infty)} \left(\frac{\cancel{x^2}+\frac{1}{n}-\cancel{x^2}}{\sqrt{x^2+\frac{1}{n}}+x} \right) =$$

$$= \sup_{x\in[0,\infty)} \frac{\frac{1}{n}}{\sqrt{x^2+\frac{1}{n}}+x} = \frac{\frac{1}{n}}{\sqrt{0^2+\frac{1}{n}}+0} = \frac{\frac{1}{n}}{\sqrt{\frac{1}{n}}} = \sqrt{\frac{1}{n}} \xrightarrow[n\to\infty]{} 0.$$

Deci $f_n \xrightarrow[n\to\infty]{u} f$. $\square$

d) $f_n : (0,1] \to \mathbb{R}$, $f_n(x) = x^n$ $\forall n \in \mathbb{N}$.

Sol.: C.S.

Fie $x \in (0,1]$.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} 0 & ; x\in(0,1) \\ 1 & ; x=1 \end{cases} \Rightarrow f_n \xrightarrow[n\to\infty]{s} f,$$

unde $f : (0,1] \to \mathbb{R}$, $f(x) = \begin{cases} 0 ; & x\in(0,1) \\ 1 ; & x=1 \end{cases}$.

C.U.

f_n continuă $\forall n \in \mathbb{N}$

f nu e continuă (în 1) $\Bigg| \Rightarrow f_n \overset{u}{\underset{n\to\infty}{\not\longrightarrow}} f$. $\square$

e) $f_n : [0,\infty) \to \mathbb{R}$, $f_n(x) = \dfrac{n}{n+x}$ $\forall n \in \mathbb{N}^*$.

Sol.: C.S.

Fie $x \in [0,\infty)$.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n}{n+x} = 1 \Rightarrow f_n \xrightarrow[n\to\infty]{s} f, \text{ unde}$$

$f : [0,\infty) \to \mathbb{R},\ f(x) = 1.$

<u>C.U.</u>

$$\sup_{x\in[0,\infty)} |f_n(x) - f(x)| = \sup_{x\in[0,\infty)} \left|\frac{n}{n+x} - 1\right| =$$

$$= \sup_{x\in[0,\infty)} \left|\frac{n-n-x}{n+x}\right| = \sup_{x\in[0,\infty)} \left|\frac{-x}{n+x}\right| =$$

$$= \sup_{x\in[0,\infty)} \frac{x}{n+x}.$$

Fie $g_n : [0,\infty) \to \mathbb{R},\ g_n(x) = \dfrac{x}{n+x}\ \forall n\in\mathbb{N}^*.$

$$g_n'(x) = \frac{n+\cancel{x}-\cancel{x}}{(n+x)^2} = \frac{n}{(n+x)^2} > 0\ \forall x\in[0,\infty), \forall n\in\mathbb{N}^* \Rightarrow$$

$\Rightarrow g_n$ este strict crescătoare $\forall n\in\mathbb{N}^*$.

x	0	$+\infty$
$g_n'(x)$	$+ + + + + + + + +$	
$g_n(x)$	$0 \nearrow \nearrow \nearrow$	1

Deci $\sup\limits_{x\in[0,\infty)} \frac{x}{x+n} = \lim\limits_{x\to\infty}\frac{x}{x+n} = 1 \not\xrightarrow[n\to\infty]{} 0.$

Așadar $f_n \not\xrightarrow[n\to\infty]{u} f.$ □

f) $f_n:\left[\frac{1}{2},1\right]\to\mathbb{R}$, $f_n(x)=\frac{(1+x)^n}{e^{2nx}}$ $\forall n\in\mathbb{N}^*$.

<u>Sol:</u> <u>C.S.</u>

Fie $x\in\left[\frac{1}{2},1\right]$.

Avem $f_n(x)=\left(\frac{1+x}{e^{2x}}\right)^n=\left(f_1(x)\right)^n$ $\forall n\in\mathbb{N}^*$.

Fie $g:\left[\frac{1}{2},1\right]\to\mathbb{R}$, $g(x)=1+x-e^{2x}$.

$g'(x)=1-2e^{2x}<0$ $\forall x\in\left[\frac{1}{2},1\right] \Rightarrow g$ este strict descrescătoare.

x	$\frac{1}{2}$				1
$g'(x)$	$-$ $-$	$-$ $-$	$-$ $-$	$-$ $-$	
$g(x)$	$\frac{3}{2}-e$	$\searrow$	$\searrow$	$\searrow$	$2-e^2$

Deci $g(x)<0$ $\forall x\in\left[\frac{1}{2},1\right]$, i.e. $1+x<e^{2x}$ $\forall x\in\left[\frac{1}{2},1\right]$, i.e.

$$0<\underbrace{\frac{1+x}{e^{2x}}}_{\parallel \atop f_1(x)}<1 \quad \forall x\in\left[\frac{1}{2},1\right].$$

Așadar $\lim\limits_{n\to\infty} f_n(x)=\lim\limits_{n\to\infty}\left(f_1(x)\right)^n=0$. Prin urmare

$f_n \xrightarrow[n\to\infty]{s} f$, unde $f: [\frac{1}{2}, 1] \to \mathbb{R}$, $f(x) = 0$.

C.U.

$0 < f_1(x) < 1 \ \forall x \in [\frac{1}{2}, 1] \Rightarrow \underbrace{(f_1(x))^n}_{f_n(x)} > \underbrace{(f_1(x))^{n+1}}_{f_{n+1}(x)} \ \forall n \in \mathbb{N}^*, \forall x \in [\frac{1}{2}, 1]$.

Deci $(f_n)_n$ este (strict) descrescător.

Avem: 1) $[\frac{1}{2}, 1]$ mulțime compactă.

2) f_n continuă $\forall n \in \mathbb{N}^*$, f continuă.

3) $(f_n)_n$ monoton.

4) $f_n \xrightarrow[n\to\infty]{s} f$.

Conform Teoremei lui Dini rezultă că $f_n \xrightarrow[n\to\infty]{u} f$. □

g) $f_n: [\frac{1}{2}, \frac{\pi}{2}] \to \mathbb{R}$, $f_n(x) = \cos^n x \ \forall n \in \mathbb{N}^*$.

Sol.: C.S.

Fie $x \in [\frac{1}{2}, \frac{\pi}{2}]$.

$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \cos^n x \underset{\substack{\uparrow \\ \cos x \in [0,1) \ \forall x \in [\frac{1}{2}, \frac{\pi}{2}]}}{=} 0 \Rightarrow f_n \xrightarrow[n\to\infty]{s} f$, unde

$f:\left[\frac{1}{2},\frac{\pi}{2}\right]\to\mathbb{R},\ f(x)=0.$

C.U.

$x \longmapsto \cos x$ (strict) descrescătoare $\Rightarrow$

$\cap$

$\left[\frac{1}{2},\frac{\pi}{2}\right]$

$\Rightarrow \forall n\in\mathbb{N}^*$, f_n este funcție (strict) descrescătoare.

Avem: 1) $f_n:\left[\frac{1}{2},\frac{\pi}{2}\right]\to\mathbb{R},\ f_n(x)=\cos^n x\ \forall n\in\mathbb{N}^*.$

2) f_n monotonă $\forall n\in\mathbb{N}^*.$

3) $f_n \xrightarrow[n\to\infty]{s} f.$

4) f continuă.

Conform Teoremei lui Polya rezultă că $f_n \xrightarrow[n\to\infty]{u} f.$ П

4. Studiați convergența simplă și uniformă pentru $(f_n)_n$ și $(f_n')_n$, unde:

a) $f_n:[0,\pi]\to\mathbb{R},\ f_n(x)=\frac{\cos nx}{n}\ \forall n\in\mathbb{N}^*.$

Sol: Pentru $(f_n)_n$

C.S.

Fie $x \in [0,\pi]$.

$$-\frac{1}{n} \leq \frac{\cos nx}{n} \leq \frac{1}{n} \quad \forall\, n \in \mathbb{N}^*.$$

$$\searrow \quad \downarrow \quad \swarrow$$

$$0$$

Deci $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \frac{\cos nx}{n} = 0$. Așadar $f_n \xrightarrow[n\to\infty]{s} f$,

unde $f : [0,\pi] \to \mathbb{R}$, $f(x) = 0$.

C.U.

$$\sup_{x\in[0,\pi]} |f_n(x) - f(x)| = \sup_{x\in[0,\pi]} \left| \frac{\cos nx}{n} - 0 \right| =$$

$$= \sup_{x\in[0,\pi]} \frac{|\cos nx|}{n} \leq \frac{1}{n} \xrightarrow[n\to\infty]{} 0 \Rightarrow f_n \xrightarrow[n\to\infty]{u} f.$$

Pentru $(f_n')_n$

$$f_n'(x) = \frac{1}{\cancel{n}}(-\sin(nx)) \cdot \cancel{n} = -\sin(nx) \quad \forall\, x \in [0,\pi], \forall\, n \in \mathbb{N}^*.$$

C.S.

Alegem $x = \frac{\pi}{2}$.

Arătăm că $\left(f_n'\left(\frac{\pi}{2}\right)\right)_n$ nu are limită.

$$f'_{4n}\left(\frac{\pi}{2}\right) = -\sin\left(\cancel{4}^{2}n\cdot\frac{\pi}{\cancel{2}}\right) = -\sin(2n\pi) = 0 \xrightarrow[n\to\infty]{} 0.$$

$$f'_{4n+1}\left(\frac{\pi}{2}\right) = -\sin\left(\cancel{4}^{2}n\cdot\frac{\pi}{\cancel{2}} + \frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1 \xrightarrow[n\to\infty]{} -1.$$

Deci $\nexists \lim\limits_{n\to\infty} f'_n\left(\frac{\pi}{2}\right)$.

Așadar $(f'_n)_n$ nu converge simplu.

<u>C.U.</u>

$(f'_n)_n$ nu converge simplu $\Rightarrow$ $(f'_n)_n$ nu converge uniform. □

b) $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \dfrac{\operatorname{arctg} nx}{n}$ $\forall n \in \mathbb{N}^*$.

<u>Sol</u>: Rezolvați-l voi! □