# Seminar 6

1) $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x_1, x_2, x_3) = (2x_1 + 2x_2, x_1 + x_3, x_1 + 3x_2 - 2x_3)$

a) $f$ nu este izomorfism de sp. vect.

Fie $\mathcal{R}_0 = \{e_1, e_2, e_3\}$ reperul canonic din $\mathbb{R}^3$

$$[f]_{\mathcal{R}_0, \mathcal{R}_0} = A \qquad f(x) = y \iff Y = AX$$

$f(e_1) = (2, 1, 1) = 2e_1 + e_2 + e_3$

$f(e_2) = (2, 0, 3) = 2e_1 + 3e_3$

$f(e_3) = (0, 1, -2) = e_2 - 2e_3$

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$\det A = \begin{vmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{vmatrix} = 0 \Rightarrow \nexists A^{-1} \Rightarrow f$ nu e bijectivă.

b) $f/_{V'} : V' \to V''$ este izomorfism

$V' = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 - x_3 = 0\}$

$V'' = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 3x_1 - 4x_2 - 2x_3 = 0\}$

~~$V' = \{(x_1, x_2, x_3) \in \mathbb{R}^3 /$~~

$V' = \{(x_1, x_2, x_1 + x_2) \in \mathbb{R}^3\}$

$(x_1, x_2, x_1 + x_2) = x_1 \cdot (1, 0, 1) + x_2 \cdot (0, 1, 1)$.

$\Rightarrow V' = \langle \{(1,0,1), (0,1,1)\} = \mathcal{R}'$ SG în $V'$

$\Rightarrow \dim V' = 3-1 = 2$

$f(1,0,1) = (2,2,-1) \in V''$ $(3\cdot 2 - 4\cdot 2 + 2 = 0)$

$f(0,1,1) = (2,1,1) \in V''$ $(3\cdot 2 - 4\cdot 1 - 2 = 0)$

$\mathcal{R}'' = \{(2,2,-1), (2,1,1)\}$ reper în $V''$

$\dim V'' = 3 - 1 = 2$

$rg\begin{pmatrix} 2 & 2 \\ 2 & 1 \\ -1 & 1 \end{pmatrix} = 2$ (maxim)

$\Rightarrow \mathcal{R}''$ reper în $V''$

c) Să se afle $f(V' \cap V'')$

$V' \cap V'' = \{x \in \mathbb{R}^3 / \begin{cases} x_1 + x_2 - x_3 = 0 \\ 3x_1 - 4x_2 - 2x_3 = 0 \end{cases}\} = S(B)$

$f: V' \cap V'' \to \mathbb{R}^3$

$1 = \dim(\mathrm{Ker} f/_{V'\cap V''}) + \dim(\mathrm{Im} f/_{V'\cap V''})$

$B = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 3 & -4 & -2 & 0 \end{array}\right)$

$\dim V' \cap V'' = 3 - rg B = 3 - 2 = 1.$

$x_3$ - variabilă secundară

$\begin{cases} x_1 + x_2 = x_3 \\ 3x_1 - 4x_2 = 2x_3 \end{cases}$

$4(1) + (2)$

$7x_1 = 6x_3 \Rightarrow x_1 = \frac{6}{7}x_3.$

$x_2 = \frac{1}{7}x_3.$

$\Rightarrow V' \cap V'' = \{ (\frac{6}{7} x_3, \frac{1}{7} x_3, x_3) / x_3 \in \mathbb{R} \} =$

$= \{ \frac{x_3}{7} (6, 1, 7) / x_3 \in \mathbb{R} \} = \langle \{ (6, 1, 7) \} \rangle \to SG.$

un singur vector $\Rightarrow$ SLI

$f(6,1,7) = (14, 13, -5) \neq (0,0,0)$

d) $\mathbb{R}^3 = V' \oplus W$

$p: \mathbb{R}^3 \to \mathbb{R}^3$ proiecția pe $V'$

$p(v) = p(v_1 + v_2) = v_1 \in V', v_2 \in W.$

$s: \mathbb{R}^3 \to \mathbb{R}^3$ simetria față de $V'$.

$s(v) = s(v_1 + v_2) = v_1 - v_2$

$s = 2p - id_{\mathbb{R}^3}$

$p(1,3,6) = ?, \ s(1,3,6) = ?$

$V' = \langle \{ (1,0,1), (0,1,1) \} \rangle.$

$\det \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \neq 0.$

$\mathcal{R}' \cup \{e_1\}$ reper în $\mathbb{R}^3$, $W = \langle \{e_1\} \rangle$

$(1,3,6) = \underbrace{a \cdot (1,0,1) + b \cdot (0,1,1)}_{v_1} + c \cdot \underbrace{(1,0,0)}_{v_2}$

$(1,3,6) = (a+c, b, a+b)$

$\begin{cases} a+c=1 \\ b=3 \\ a+b=6 \end{cases} \Leftrightarrow \begin{cases} a=3 \\ b=3 \\ c=-2. \end{cases}$

$v_1 = (3,3,6)$

$v_2 = (-2,0,0)$

$p(1,3,6) = (3,3,6)$

$s(1,3,6) = v_1 - v_2 = (5,3,6)$

3) $f : \mathbb{R}_2[x] \to \mathbb{R}_1[x]$, $f(Q) = Q'$

a) $\mathcal{R} = \{x^2, 1+x, 2-x\}$ reper în $\mathbb{R}_2[x]$

$\mathcal{R}' = \{x, 1+3x\}$ reper în $\mathbb{R}_1[x]$.

$[f]_{\mathcal{R},\mathcal{R}'} = ?$

$f(x^2) = 2x = a \cdot x + b(1+3x)$

$a = 2,\ b = 0$

$$A = \begin{pmatrix} 2 & -3 & 3 \\ 0 & 1 & -1 \end{pmatrix}$$

$f(1+x) = 1 = -3 \cdot x + 1(1+3x)$

$f(2-x) = -1 = 3 \cdot x - 1 \cdot (1+3x)$

b) $\mathbb{R}_2[x] = \operatorname{Ker} f \oplus W$

$p_1, p_2 : \mathbb{R}_2[x] \to \mathbb{R}_2[x]$ proiecții pe Ker, respectiv W.

$p_1(\underset{\substack{\uparrow \\ \operatorname{Ker} f}}{1} \underbrace{- x + 3x^2}_{\Downarrow}),\ p_2 = (2x + x^2) = ?$

$\operatorname{Ker} f = \{Q \in \mathbb{R}_2[x] \,/\, f(Q) = 0\} = \langle \{1\} \rangle$

Completam $\{1\}$ la un reper în $\mathbb{R}_2[x]$

$\mathcal{R}_0 = \{1, x, x^2\}$

$W = \langle \{x, x^2\} \rangle$

$p1(1-x+3x^2)=1$

$p2(2x+x^2)=2x+x^2$

4) $f: \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2^s(\mathbb{R})$, $f(A)=A+A^t$

$\mathcal{R}_0 = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ reper în $\mathcal{M}_2(\mathbb{R})$

$\mathcal{R}_0' = \{E_{11}, E_{12}+E_{21}, E_{22}\}$ reper în $\mathcal{M}_2^s(\mathbb{R})$

a) $[f]_{\mathcal{R}_0, \mathcal{R}_0'} = ?$

b) Ker $f$, Im $f$

c) $f(V)=?$, $V = \left\{ \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}, c, d \in \mathbb{R} \right\}$

$f(E_{11}) = f\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2E_{11}$

$f(E_{12}) = f\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = E_{12}+E_{21}$

$[f]_{\mathcal{R}_0, \mathcal{R}_0'} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

$f(E_{21}) = f\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = E_{12}+E_{21}$

$f(E_{22}) = f\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 2E_{22}$

b) $\text{Ker} f = \{A \in \mathcal{M}_2(\mathbb{R}) / \underbrace{f(A) = O_2}_{A=-A^t}\} = \left\langle \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} \right.$

$\Rightarrow \dim \text{Ker} f = 1$

$\dim \mathcal{M}_2(\mathbb{R}) = 4 = \dim \text{Ker} f + \dim \text{Im} f \Rightarrow \dim \text{Im} f = 3$ $\Big/ \Rightarrow \text{Im} f = \mathcal{M}_2^s(\mathbb{R})$

Dar $\text{Im} f \subset \mathcal{M}_2^s(\mathbb{R})$ subspațiu vectorial

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c) $f\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} + \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix} = \begin{pmatrix} 0 & c \\ c & 2d \end{pmatrix} =$

$= c \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2d \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow f(V) = \langle \underbrace{\{E_{12}+E_{21}, E_{22}\}}_{SG + SLi \text{ (submulțime în } \mathcal{R}_0')} \rangle$

$\Rightarrow \dim f(V) = 2.$

5) $f : \mathbb{R}^4 \to \mathbb{R}^4$, $f(x) = (x_2 - x_3 + x_4, x_2 - x_3 + x_4, x_4, x_4)$

a) Să se afle valorile proprii

b) Determinați subspațiile proprii

c) $\exists$ un reper $\mathcal{R}$ în $\mathbb{R}^4$ a.î. $[f]_{\mathcal{R},\mathcal{R}}$ este diagonală?

$f(e_1) = (0,0,0,0)$

$f(e_2) = (1,1,0,0) = e_1 + e_2$

$f(e_3) = (-1,-1,0,0) = -e_1 - e_2$

$f(e_4) = (1,1,1,1) = e_1 + e_2 + e_3 + e_4$

$A = \begin{pmatrix} 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$P(\lambda) = \det(A - \lambda I_4) = 0$

$\det \begin{pmatrix} -\lambda & 1 & -1 & 1 \\ 0 & 1-\lambda & -1 & 1 \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & 0 & 1-\lambda \end{pmatrix} = \lambda^2 (1-\lambda)^2 = 0$

$\Rightarrow \lambda_1 = 0, \; m_1 = 2$

$\lambda_2 = 1, \; m_2 = 2.$

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$V_{\lambda_1} = \{ x \in \mathbb{R}^4 / \ f(x) = \lambda_1 \cdot x = 0 \} = \operatorname{Ker} f$

$$\begin{cases} x_2 - x_3 + x_4 = 0 \Rightarrow x_2 = x_3. \\ x_4 \qquad = 0 \end{cases}$$

$\Rightarrow V_{\lambda_1} = \operatorname{Ker} f = \{ (x_1, x_2, x_2, 0) / x_1, x_2 \in \mathbb{R} \} =$

$= \langle \{ e_1, e_2 + e_3 \} \rangle$

$\mathcal{R}_1 = \{ e_1, e_2 + e_3 \}$ reper în $V_{\lambda_1}$ ( SG, card $(\mathcal{R}_1) = \dim V_{\lambda_1} = 2$)

$\Rightarrow \mathcal{R}_1 \Rightarrow$ reper.

$\dim V_{\lambda_1} = 4 - \operatorname{rg} A = 4 - 2 = 2 = m_1$

. $V_{\lambda_2} = \{ x \in \mathbb{R}^4 / \ f(x) = \lambda_2 \cdot x = x \}$

$$\begin{cases} x_2 - x_3 + x_4 = x_1 \\ x_2 - x_3 + x_4 = x_2 \quad \Rightarrow x_1 = x_2 \\ x_4 \qquad = x_3 \qquad x_3 = x_4 \\ x_4 \qquad = x_4 \end{cases}$$

$\Rightarrow V_{\lambda_2} = \{ (x_1, x_1, x_3, x_3) / x_1, x_3 \in \mathbb{R} \} = \langle \{ e_1 + e_2, e_3 + e_4 \} \rangle$
SG

$\operatorname{rg} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = 2$ (maxim) $\Rightarrow V_{\lambda_2}$ - SLi

$\mathcal{R}_2 = \{ e_1 + e_2, e_3 + e_4 \}$ reper în $V_{\lambda_2}$

$\exists \ \mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ reper în $\mathbb{R}^4$ a.î. $[f]_{\mathcal{R},\mathcal{R}} = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$