1) ~~Det~~ morfismele de grupuri $f : \mathbb{Z} \to G$ când $G \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}\}$; $(G,+)$ = grup

Sol: $\mathbb{Z} = \langle 1 \rangle$, $f \equiv f(1) \overset{not}{=} x \in G$
$\uparrow$ complet determinat

$f(n) = f(\underbrace{1+\ldots+1}_{n\text{-ori}}) = \underbrace{f(1)+\ldots+f(1)}_{n\text{-ori}} = nx, \forall n \in \mathbb{N}$

$f(-n) = -f(n) = -nx, \forall n \in \mathbb{N}$

Deci $f(k) = kx, \forall k \in \mathbb{Z}$ (o notez cu f_x, $x \in G$)
$f_x(k) = kx, \forall k \in \mathbb{Z}$

- $Hom_{gr}(\mathbb{Z}, \mathbb{Z}) = \{f_x \mid x \in \mathbb{Z}\}$
$\uparrow$ singurele izom aici sunt $f_1 = 1_{\mathbb{Z}}$ și $f_{-1} = -1_{\mathbb{Z}}$
$f_1(k) = k$ $\quad f_{-1}(k) = -k$

- $Hom_{gr}(\mathbb{Z}, \mathbb{Q}) = \{f_x \mid x \in \mathbb{Q}\}$.
$\uparrow$ izom sunt $\subseteq \{f_x \mid x \in \mathbb{Q}^*\}$; $f_x^{-1} : \mathbb{Q} \to \mathbb{Z}$?

Sunt x-urile din $\mathbb{Q}^*$: $\forall r \in \mathbb{Q}$, $\exists k \in \mathbb{Z}$ a.î. $kx = r$.
$\Updownarrow$
$\frac{r}{x} \in \mathbb{Z}$ $\forall r \in \mathbb{Q}$
$\Updownarrow$
$x\mathbb{Z} = \mathbb{Q}$

$\mathbb{Q} = \langle x \rangle \iff$
$\downarrow$
i.e. $(\mathbb{Q},+)$ e ciclic, fals!

Dacă $x = \frac{m}{n}$ cu $m, n \in \mathbb{Z}$, $n \neq 0$. $\Rightarrow \mathbb{Q} = \langle x \rangle = \langle \frac{m}{n} \rangle$
$(m,n) = 1$

Dacă p e prim și $p \nmid n$, $\frac{1}{p} \in \mathbb{Q}$ dar $\frac{1}{p} \notin \langle \frac{m}{n} \rangle$

(altfel, $\exists \alpha \in \mathbb{Z}$ a.î. $\frac{1}{p} = \alpha \cdot \frac{m}{n} \iff p\alpha m = n$
$\Downarrow$
$p \mid n$ ~~abs~~).

Deci f_x nu e morf, $\forall x \in \mathbb{Q}^*$. Deci $(\mathbb{Z},+) \ncong (\mathbb{Q},+)$

- $Hom_{gr}(\mathbb{Z}, \mathbb{R}) = \{f_x \mid x \in \mathbb{R}\}$

  $\uparrow$ nu e bij $\forall x \in \mathbb{R}$ pt că $\begin{cases} \to \mathbb{Z} \sim \mathbb{N} \text{ (numărabilă)} \\ \to \mathbb{R} \not\sim \mathbb{N} \text{ (nu este numărabilă)} \end{cases}$

- $Hom_{gr}(\mathbb{Z}, \mathbb{C}) = \{f_z \mid z \in \mathbb{C}\}$

  $\uparrow$ nu e bij (ca mai sus)

2) $Hom_{gr}(\mathbb{Z}_n, G)$, G = grup arbitrar.

Sl: Fie $f : \mathbb{Z}_n \to G$ morf de gr. ; $(G, \cdot)$

$\mathbb{Z}_n = \langle \hat{1} \rangle$, $f \equiv f(\hat{1}) \overset{not}{=} x \in G$.

$f(\hat{k}) = f(\underbrace{\hat{1} + \dots + \hat{1}}_{k-ori}) = \underbrace{f(\hat{1}) \dots f(\hat{1})}_{k-ori} = x^k, \forall \hat{k} \in \mathbb{Z}_n$

f = bine definit $\Leftrightarrow x^n = e$ #

"$\Rightarrow$" f = bine def: $\hat{k} = \hat{t} \Rightarrow x^k = x^t$

$\Updownarrow \qquad\qquad \Updownarrow$

$n \mid k - t \Rightarrow x^{k-t} = e$

Iau $k = n, t = 0 \Rightarrow x^n = e$.

"$\Leftarrow$" $n \mid k - t \Rightarrow k - t = n\alpha, \alpha \in \mathbb{Z} \Rightarrow x^{k-t} = x^{n\alpha}$

$= (x^n)^\alpha = e^\alpha$

$= e$ ✓.

Deci $Hom_{gr}(\mathbb{Z}_n, G) = \{f_x \mid x \in G \text{ cu } x^n = e\}$,

unde $f_x : \mathbb{Z}_n \to G$, $f_x(\hat{k}) = x^k \ \forall \hat{k} \in \mathbb{Z}_n$.

Concret: $\hat{x} \quad \bar{x}$

- $Hom_{gr}(\underset{n}{\mathbb{Z}_6}, \underset{G}{\mathbb{Z}_{14}}) \equiv \{\bar{x} \in \mathbb{Z}_{14} \mid 6\bar{x} = \bar{0}\} = \{\bar{x} \in \mathbb{Z}_{14} \mid 14 \mid 6x\}$

  $= \{\bar{x} \in \mathbb{Z}_{14} \mid 7 \mid x\} = \{\bar{0}, \bar{7}\}$

$f_{\bar{0}} : \mathbb{Z}_6 \to \mathbb{Z}_{14}$, $f_{\bar{0}}(\hat{k}) = \bar{0}, \forall \hat{k} \in \mathbb{Z}_6$

$f_{\bar{7}} : \mathbb{Z}_6 \to \mathbb{Z}_{14}$, $f_{\bar{7}}(\hat{k}) = \widehat{7k}$, $\forall \hat{k} \in \mathbb{Z}_6$

• $Hom_{gr}(\mathbb{Z}_n, \mathbb{Z}_n) \equiv \{\hat{a} \in \mathbb{Z}_n \mid n\hat{a} = \hat{0}\} = \mathbb{Z}_n$

$f_{\hat{a}} : \mathbb{Z}_n \to \mathbb{Z}_n \quad , \quad f_{\hat{a}}(\hat{k}) = \widehat{ka} \quad \forall \hat{k} \in \mathbb{Z}_n.$

$f_{\hat{a}} = bij \iff \hat{a} \in U(\mathbb{Z}_n) \iff (a, n) = 1. \; ($

$\Rightarrow \exists\, \varphi(n)$ automorfisme ale lui $\mathbb{Z}_n)$.

"$\Rightarrow$" $\hat{1} \in \mathbb{Z}_n \xRightarrow{f_{\hat{a}} = surj} \exists\, \hat{k} \in \mathbb{Z}_n$ a.î. $f_{\hat{a}}(\hat{k}) = \hat{1} \iff$

$\iff \widehat{ka} = \hat{1} \iff \hat{k} \cdot \hat{a} = \hat{1} \Rightarrow \hat{a} \in U(\mathbb{Z}_n).$

"$\Leftarrow$" $\hat{a} \in U(\mathbb{Z}_n)$; $\exists\, f_{\hat{a}}^{-1} = f_{\hat{a}^{-1}} \iff f_{\hat{a}} = bij$

3) $Aut_{gr}(\mathbb{Z}_2\times\mathbb{Z}_2) = ?$

Sol: $f:\mathbb{Z}_2\times\mathbb{Z}_2 \to \mathbb{Z}_2\times\mathbb{Z}_2$, f = automorfism de grupuri

$$\mathbb{Z}_2\times\mathbb{Z}_2 = \{\underset{\text{ordin } 2}{u=(\hat{1},\hat{0})},\ \underset{\text{ordin } 2}{v=(\hat{0},\hat{1})},\ \underset{\text{ordin } 1}{e=(\hat{0},\hat{0})},\ \underset{\text{ordin } 2}{u+v=(\hat{1},\hat{1})}\}$$

f = bij $\Leftrightarrow$ f = inj

$f(\hat{0},\hat{0}) = (\hat{0},\hat{0})$

$\{f(u), f(v), f(u+v) = f(u)+f(v)\} = \{u, v, u+v\}$

distincte

Caz 1) $f(u)=u$, $f(v)=v$, $f(u+v)=u+v \rightsquigarrow f_1 = 1_{\mathbb{Z}_2\times\mathbb{Z}_2}$

2) $f(u)=v$, $f(v)=u$, $f(u+v)=u+v \rightsquigarrow f_2$

3) $f(u)=v$, $f(v)=u+v$, $f(u+v)=u \rightsquigarrow f_3$

$\vdots$ $\rightsquigarrow f_4$

$\rightsquigarrow f_5$

6) $\rightsquigarrow f_6$

Deci $\underset{\text{grup în raport cu „}\circ\text{"}}{Aut(\mathbb{Z}_2\times\mathbb{Z}_2)} = \{f_1, \dots, f_6\} \simeq S_3$

4) $S_3 = \langle \tau, \sigma\rangle$, $\tau = (1\,2)$ și $\sigma = \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$

$$\langle\tau,\sigma\rangle = \{x_1^{\alpha_1}\dots x_n^{\alpha_n} \mid \begin{matrix} n\in\mathbb{N}^* \\ x_1,\dots,x_n \in\{\tau,\sigma\},\ \alpha_i\in\mathbb{Z} \\ 1\le i\le n\end{matrix}\}$$

$\tau\sigma = \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix} = (2\,3)$

$\sigma^2 = \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix} \Rightarrow \sigma^2\tau = \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix} = (2\,3)$ $\Bigg\} \Rightarrow \tau\sigma = \sigma^2\tau$

$\tau\sigma^2 = \tau\sigma\sigma = \sigma^2\tau\sigma = \sigma^2\sigma^2\tau = \sigma^4\tau = \sigma\tau$

$\tau^2 = e = \sigma^3 \Rightarrow o(\tau) = 2,\ o(\sigma) = 3$

$\Rightarrow \langle\tau,\sigma\rangle = \{\sigma^i\tau^j \mid i,j\in\mathbb{Z}\}$

$= \{\sigma^i\tau^j \mid 0\le i\le 2,\ 0\le j\le 1\}$

$= \{e, \sigma, \sigma^2, \tau, \sigma\tau, \sigma^2\tau = \tau\sigma\} = S_3$.

$\sigma = \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$, $\sigma^2 = \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$, $\tau = (1\,2)$, $\sigma\tau = (1\,3)$, $\sigma^2\tau = (2\,3)$

5) $Aut_{gr}(\mathbb{Z}_2 \times \mathbb{Z}_2) \simeq S_3$, izom de grupuri

Sol: $Aut_{gr}(\mathbb{Z}_2 \times \mathbb{Z}_2) = \{f_1 = 1_{\mathbb{Z}_2 \times \mathbb{Z}_2}, f_2, \dots, f_6\}$

$f_2(u) = v$, $f_2(v) = u$, $f_2(u+v) = u+v$

$f_3(u) = v$, $f_3(v) = u+v$, $f_3(u+v) = u$

$f_2^2 = f_2 \circ f_2 = 1_{\mathbb{Z}_2 \times \mathbb{Z}_2} = f_1 \Rightarrow o(f_2) = 2$

$f_3^2(u) = f_3(v) = u+v \Rightarrow f_3^3(u) = f_3(u+v) = u$

La fel, $f_3^3(v) = v$ și $f_3^3(u+v) = u+v \Rightarrow o(f_3) = 3.$

Avem $f_2 \circ f_3 = f_3^2 \circ f_2$ #

$f_2 \circ f_3(u) = f_2(v) = u$

$f_3^2 \circ f_2(u) = f_3^2(v) = u$ ✓

(la fel celelalte!)

Rezultă că $Aut_{gr}(\mathbb{Z}_2 \times \mathbb{Z}_2) = \{f_1, f_3, f_3^2, f_2, f_3 \circ f_2, \underbrace{f_3^2 \circ f_2}_{f_2 \circ f_3}\}$

izom de gr. $\longrightarrow$ $\Downarrow$ etc.

$S_3 = \{e, \sigma, \sigma^2, \tau, \sigma\tau, \sigma^2\tau = \tau\sigma\}.$

$Aut_{gr}(\mathbb{Z}_2 \times \mathbb{Z}_2) \ni f_3^i \circ f_2^j \xrightarrow{\sim} \sigma^i \tau^j \in S_3.$

$(0 \le i \le 2,\ 0 \le j \le 1)$

6) $Aut_{gr}(S_3) = ?$

Sol: $f: \underset{\shortparallel}{S_3} \to S_3$ izom de gr.

$\langle \tau, \sigma \rangle = \{e, \sigma, \sigma^2, \tau, \sigma\tau, \sigma^2\tau = \tau\sigma\}$.

ordine: 1 3 3 2 2 2

f izom de gr ($f: G \to G'$) $\Rightarrow$ $o(x) = o(f(x)), \forall x \in G$

$f(e) = e$

$f(\sigma) \in \{\sigma, \sigma^2\}$, $f(\tau) \in \{\tau, \sigma\tau, \sigma^2\tau\}$. $\Rightarrow$

$f(\sigma^2) = f(\sigma)^2$, $f(\sigma\tau) = f(\sigma)f(\tau)$, $f(\underset{\tau\sigma}{\underset{\shortparallel}{\sigma^2\tau}}) = f(\sigma)^2 f(\tau) = f(\tau)f(\sigma)$

$\Rightarrow$ $\exists$ cel mult 6 izom de grupuri de la S_3 la S_3

- $f_1 = 1_{S_3}$ ($f(\sigma) = \sigma$, $f(\tau) = \tau$)
- $f_2(\sigma) = \sigma^2$, $f_2(\tau) = \tau$, $f_2(\sigma^2) = \sigma$, $f_2(\sigma\tau) = \sigma^2\tau$

  $f_2(\sigma^2\tau) = \sigma\tau$ ($f_2(\tau)f_2(\sigma) = \tau\sigma^2 = \overbrace{\tau\sigma}^{\sigma^2\tau}\sigma = \sigma\tau = f_2(\sigma^2\tau)\underline{v}$)

- $f_3(\sigma) = \sigma^2$, $f_3(\tau) = \sigma\tau$, etc...

Temă: $Aut_{gr}(S_3) = \langle \underset{\tau}{\underset{|||}{f_2}}, \underset{\sigma}{\underset{|||}{f_3}} \rangle \xrightarrow{\sim} S_3$, izom de grupuri

Obs: $\mathbb{Z}_2 \times \mathbb{Z}_2 \not\xrightarrow{\sim} S_3$, dar $Aut_{gr}(\mathbb{Z}_2 \times \mathbb{Z}_2) \underset{\text{izom de grupuri}}{\simeq} \overset{S_3}{\overset{\wr}{Aut_{gr}(S_3)}}$