Seminar 5

1. Fie $n\in\mathbb{N}^*$, $d_1, d_2 : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$,

$$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i| \; , \quad d_2(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \; .$$

Arătați că există $a, b > 0$ a.î. $a\,d_1(x,y) \le d_2(x,y) \le b\,d_1(x,y)$ $\forall\, x, y \in \mathbb{R}^n$.

Sol: Fie $x, y \in \mathbb{R}^n$.

$$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} |x_i - y_i|\cdot 1 \underset{\text{C.B.S.}}{\le} \left(\sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}\right) \cdot$$

$$\cdot \sqrt{\sum_{i=1}^{n} 1^2} = \left(\sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}\right) \cdot \sqrt{n} = \sqrt{n}\, d_2(x,y) \iff$$

$$\iff \frac{1}{\sqrt{n}}\, d_1(x,y) \le d_2(x,y).$$

Alegem $a = \frac{1}{\sqrt{n}}$.

$$d_2(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} =$$

$$= \sqrt{|x_1 - y_1|^2 + \ldots + |x_n - y_n|^2} \le \sqrt{\left(|x_1 - y_1| + \ldots + |x_n - y_n|\right)^2} =$$

$= |x_1 - y_1| + \ldots + |x_n - y_n| = d_1(x,y).$

Alegem $b = 1$.

Am obținut $\frac{1}{\sqrt{n}} d_1(x,y) \leq d_2(x,y) \leq d_1(x,y)$. $\square$

2. Fie $n \in \mathbb{N}^*$, d_1 ca mai sus și $d_\infty : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$,
$d_\infty(x,y) = \max\{|x_i - y_i| \mid i = \overline{1,n}\}$. Arătați că există $a, b > 0$
a.î. $a d_1(x,y) \leq d_\infty(x,y) \leq b d_1(x,y)$ $\forall x, y \in \mathbb{R}^n$.

Sol.: Fie $x, y \in \mathbb{R}^n$.

$d_1(x,y) = |x_1 - y_1| + \ldots + |x_n - y_n| \leq n \cdot \max\{|x_i - y_i| \mid i = \overline{1,n}\} =$

$= n d_\infty(x,y) \Leftrightarrow \frac{1}{n} d_1(x,y) \leq d_\infty(x,y).$

Alegem $a = \frac{1}{n}$.

$d_\infty(x,y) = \max\{|x_i - y_i| \mid i = \overline{1,n}\} \leq |x_1 - y_1| + \ldots + |x_n - y_n| =$

$= d_1(x,y).$

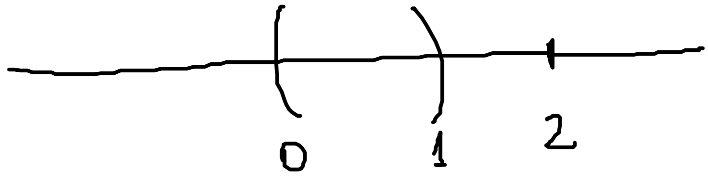
Alegem $b = 1$.

Am obținut $\frac{1}{n} d_1(x,y) \leq d_\infty(x,y) \leq d_1(x,y)$. $\square$

3. Faceți analiza topologică a mulțimii $A \subset \mathbb{R}$, unde:
(Determinați $\mathring{A}$, $\overline{A}$, A', $Fr(A)$ și $Izo(A)$)

a) $A = (0,1) \cup \{2\}$.

Sol.:



1) $\mathring{A} = ?$

$x \in \mathring{A} \iff \exists\, r > 0$ avem $B(x,r) \subset A$.

$$B(x,r) = (x-r, x+r)$$

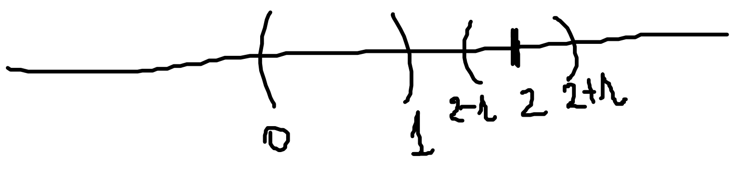
$\mathring{A} \subset A$

$$\left.\begin{array}{l}(0,1) \text{ deschisă} \\ (0,1) \subset A\end{array}\right\} \Rightarrow (0,1) \subset \mathring{A}.$$

Deci $(0,1) \subset \mathring{A} \subset (0,1) \cup \{2\}$.

Studiem dacă $2 \in \mathring{A}$.

$2 \in \mathring{A} \iff \exists\, r > 0$ a.î. $(2-r, 2+r) \subset A$.



Deci $2 \notin \mathring{A}$.

Așadar $\mathring{A} = (0,1)$.

2) $\overline{A} = ?$

$x\in\overline{A} \iff \forall r>0$, avem $(x-r, x+r)\cap A\neq\emptyset$.

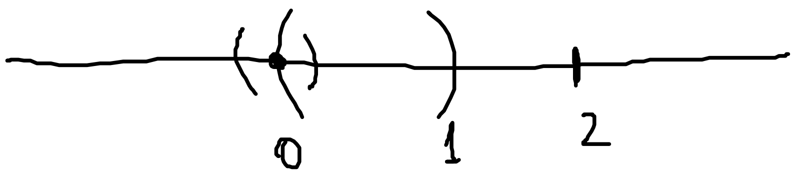
$A\subset\overline{A}$

$[0,1]\cup\{2\}$ închisă
$A\subset[0,1]\cup\{2\}$ $\Longrightarrow$ $\overline{A}\subset[0,1]\cup\{2\}$.

Deci $(0,1)\cup\{2\}\subset\overline{A}\subset[0,1]\cup\{2\}$.

Studiem dacă $0\in\overline{A}$ și $1\in\overline{A}$.

$0\in\overline{A}\iff\forall r>0$, avem $\underbrace{(0-r, 0+r)}_{=(-r,r)}\cap A\neq\emptyset$



Deci $0\in\overline{A}$.

Analog $1\in\overline{A}$.

Așadar $\overline{A}=[0,1]\cup\{2\}$.

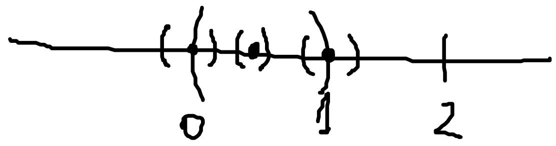
c) $A'=?$

$x\in A'\iff\forall r>0$, avem $(x-r, x+r)\cap(A\setminus\{x\})\neq\emptyset$.

$A'\subset\overline{A}=[0,1]\cup\{2\}$.

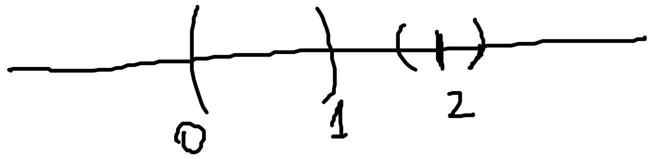
Fie $x\in[0,1]$.

$x\in A'\iff\forall r>0$ avem $(x-r, x+r)\cap(A\setminus\{x\})\neq\emptyset$.



Deci $x \in A'$, i.e. $[0,1] \subset A'$.

$2 \in A' \iff \forall r>0$, avem $(2-r, 2+r) \cap (A \setminus \{2\}) \neq \emptyset$.



Deci $2 \notin A'$.

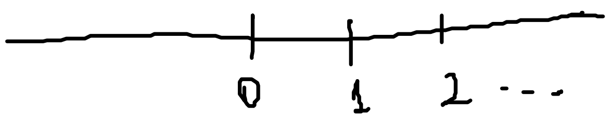
Așadar $A' = [0,1]$.

$Fr(A) = \partial A = \bar{A} \setminus \mathring{A} = \{0,1,2\}$.

$Iz(A) = {}'A = \bar{A} \setminus A' = \{2\}$. □

b) $A = \mathbb{N}$.

Sol.:



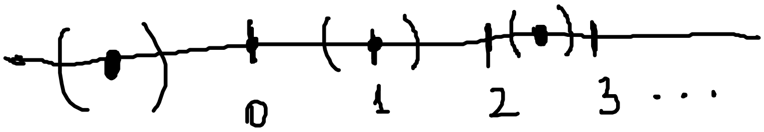
1) $\mathring{A} = ?$

$x \in \mathring{A} \iff \exists r>0$ a.î. $(x-r, x+r) \subset A$.

Deci $\mathring{A} = \emptyset$ (A nu conține intervale nedegenerate).

2) $A' = ?$

$x \in A' \iff \forall r>0$, avem $(x-r, x+r) \cap (A \setminus \{x\}) \neq \emptyset$.



Deci $A' = \emptyset$.

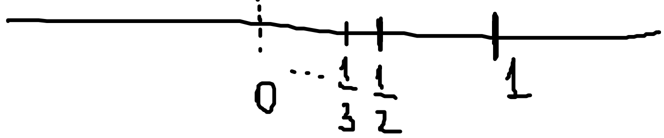
3) $\overline{A} = A \cup A' = \mathbb{N}$.

4) $Fr(A) = \partial A = \overline{A} \setminus \mathring{A} = \mathbb{N} \setminus \emptyset = \mathbb{N}$.

5) $Izo(A) = {}'A = \overline{A} \setminus A' = \mathbb{N} \setminus \emptyset = \mathbb{N}$. □

c) $A = \left\{ \frac{1}{n} \mid n \in \mathbb{N}^* \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \dots \right\}$.

Sol.:



1) $\mathring{A} = ?$

$x \in \mathring{A} \iff \exists\, r > 0$ a.î. $(x - r, x + r) \subset A$.

$\mathring{A} = \emptyset$ (A nu conține intervale nedegenerate).

2) $A' = ?$

$x \in A' \iff \exists\, (x_n)_n \subset A \setminus \{x\}$ a.î. $\lim\limits_{n \to \infty} x_n = x$.

Considerăm $x_n = \frac{1}{n}$. Avem $(x_n)_n \subset A \setminus \{0\}$ și

$\lim\limits_{n \to \infty} x_n = 0$. Deci $0 \in A'$.

Fie $(y_n)_n \subset A$ un șir convergent. Atunci limita sa poate fi un element din A (dacă șirul are termenii egali cu acel element de la un rang încolo) sau poate fi 0 (deoarece $\lim_{n\to\infty} \frac{1}{n} = 0$).

Așadar $A' = \{0\}$.

3) $\bar{A} = A \cup A' = \{0, 1, \frac{1}{2}, \ldots\}$.

4) $Fr(A) = \partial A = \bar{A} \setminus \mathring{A} = \{0, 1, \frac{1}{2}, \ldots\}$.

5) $Izo(A) = {}'A = \bar{A} \setminus A' = \{1, \frac{1}{2}, \ldots\}$. $\square$

4. Fie $x_0 \in \mathbb{R}$, $f, g : \mathbb{R} \to \mathbb{R}$ două funcții continue în x_0 și $h : \mathbb{R} \to \mathbb{R}$, $h(x) = \begin{cases} f(x); & x \in \mathbb{Q} \\ g(x); & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

Arătați că h e cont. în x_0 dacă și numai dacă $f(x_0) = g(x_0)$ $(= h(x_0))$.

<u>Sol.</u>: „$\Rightarrow$"

Ip. că h e cont. în x_0. Arătăm că $f(x_0) = g(x_0)$.

$\begin{pmatrix} \bar{\mathbb{Q}} = \mathbb{R} \\ x_0 \in \mathbb{R} = \bar{\mathbb{Q}} \end{pmatrix} \Rightarrow \exists (a_n)_n \subset \mathbb{Q}$ a.î. $\lim_{n\to\infty} a_n = x_0$.

$\left(\begin{array}{l}\overline{\mathbb{R}\setminus\mathbb{Q}} = \mathbb{R} \\ x_0 \in \mathbb{R} = \overline{\mathbb{R}\setminus\mathbb{Q}}\end{array}\right) \Rightarrow \exists (b_n)_n \subset \mathbb{R}\setminus\mathbb{Q}$ a.î. $\lim\limits_{n\to\infty} b_n = x_0$.

h cont. în $x_0 \Rightarrow \lim\limits_{n\to\infty} h(a_n) = h(x_0)$.

$$\lim_{n\to\infty} h(a_n) = \lim_{n\to\infty} f(a_n) \underset{\substack{\uparrow \\ f \text{ cont. în } x_0}}{=} f(x_0).$$

Deci $h(x_0) = f(x_0)$.

Analog $h(x_0) = g(x_0)$.

Așadar $f(x_0) = g(x_0) \; (= h(x_0))$.

"$\Leftarrow$"

Fie că $f(x_0) = g(x_0) \; (= h(x_0))$. Arătăm că h e continuă în x_0.

Fie $(z_n)_n \subset \mathbb{R}$ a.î. $\lim\limits_{n\to\infty} z_n = x_0$.

Arătăm că $\lim\limits_{n\to\infty} h(z_n) = h(x_0)$.

$$|h(z_n) - h(x_0)| = \begin{cases} |f(z_n) - f(x_0)| \; ; \; z_n \in \mathbb{Q} \\ |g(z_n) - g(x_0)| \; ; \; z_n \in \mathbb{R}\setminus\mathbb{Q} \end{cases} \le$$

$$\leq \underbrace{|f(z_n)-f(x_0)|}_{\substack{\downarrow n\to\infty \\ 0 \\ (f \text{ cont. in } x_0)}} + \underbrace{|g(z_n)-g(x_0)|}_{\substack{\downarrow n\to\infty \\ 0 \\ (g \text{ cont. in } x_0)}} \xrightarrow[n\to\infty]{} 0.$$

Deci $\lim\limits_{n\to\infty} \left(h(z_n)-h(x_0)\right) = 0$, i.e.

$\lim\limits_{n\to\infty} h(z_n) = h(x_0)$, i.e. h cont. in x_0. $\square$