Data Structures and Algorithms

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Operations on/with Trees

Queries on Rooted Trees:

- Least-common ancestors
- Distances

Tree Decompositions Methods

- Heavy-Path (HP)
- Centroid

Tree-Based Data Structures:

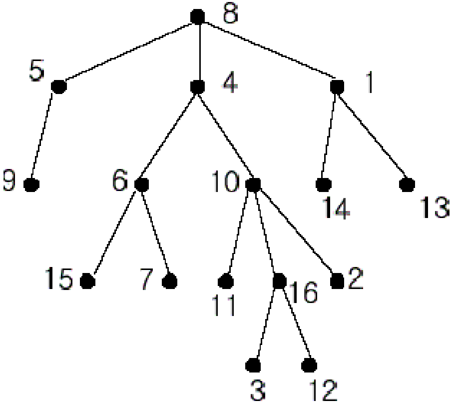
- Cartesian Trees

Least common ancestor (LCA) queries

Every two nodes x, y have a common ancestor (*e.g.*, the root)

Definition

$LCA(x, y) =$ **deepest** common ancestor (furthest from the root)



Examples:

`LCA(15,7) = 6`

`LCA(15,12) = 4`

`LCA(15,9) = 8`

Pre-processing time? Query time?

Naive resolution

While nodes x and y are distinct, replace the deepest node by its parent.

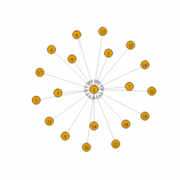
```
node *lca(node *x, node *y) {
   while(x != y){
      if(x->level >= y->level)
         x = x->father;
      else y = y->father
   }
   return x; //==y
}
```

Pre-processing: $\mathcal{O}(n)$ – Computation of the levels

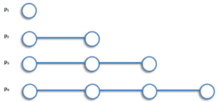
Query Complexity: Linear in the height of the tree

Extremal cases

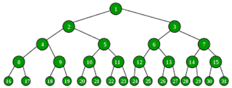
- Stars have height $= 1$



- But Paths ($\sim$ Lists) have height $= n - 1$



- Bounded-degree trees (*e.g.*, Binary) have height in $\Omega(\log n)$



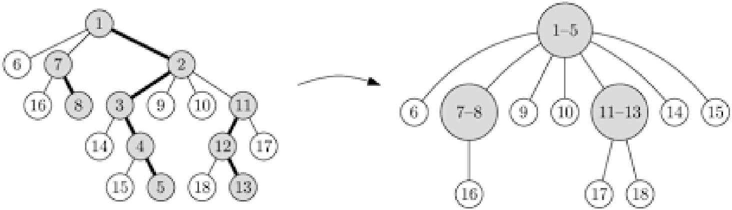
Heavy-path decomposition

- For each non-leaf node, select a child with maximum number of descendants in its rooted subtree

  – variant: select the unique child y of x s.t. $y.order > x.order/2$, if any.

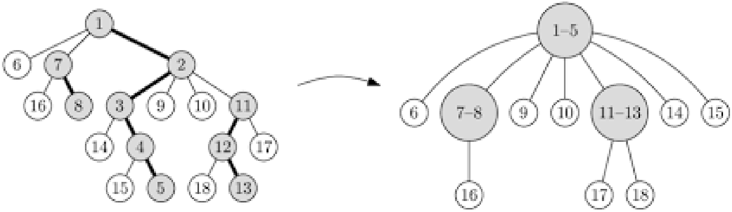
$\Longrightarrow$ Partition of the nodes in so-called "heavy" paths.

HP-tree: rooted subtree obtained by contracting each HP in one node.



Encoding

- We can associate to each node a number (ID of its HP).
- We can store in an array the highest/deepest node of each HP.



- We may also associate, to each node, a pointer to its child in the same HP (possibly, null).

Implementation

//k denotes the ID of the current HP
//first and last store ends of each HP
//hp denotes HP, next_hp points to next child in HP

```
void compute_hp(node *n, int& k, vector<node*>& first, vector<node*>& last){

    n->hp = k; last[k] = n;

    if(n->child != nullptr){

        node *c = n->child;
        for(node *p = c->next; p != nullptr; p = p->next)
            if(p->order > c->order) c = p;

        n->next_hp = c; compute_hp(c,k,first,last);

        for(node *p = n->child; p != nullptr; p = p->next)
            if(p != c) { //new HP
                first.push_back(p); last.push_back(p);
                compute_hp(p,++k,first,last);
            }
    }
}
```

Complexity: $\mathcal{O}(n)$

LCA queries with HP

Logarithmic-time version

- Compute levels in the HP-tree.

```
void levels_hp(node *n, vector<int>& hp_lvl) {
    if(first[n->hp] == n) {
        if(n->father == nullptr) hp_lvl[n->hp] = 0;
        else hp_lvl[n->hp] = hp_lvl[n->father->hp] + 1;
    }
    for(node *c = n->child; c != nullptr; c = c->next) { levels_hp(c, hp_lvl); }
}
```

- Simulation of the naive algorithm on the HP-tree.

```
//for ease of writing, we assume first and last to be global arrays

node *lca_hp(node *x, node *y) {
    while(x->hp != y->hp){
        if(x->level >= y->level)
            x = ( (x == first[x->hp]) ?  x->father :  first[x->hp] );
        else y = ( (y == first[y->hp]) ?  y->father :  first[y->hp] )
    }
    return ( (x->level >= y->level) ?  x :  y );
}
```

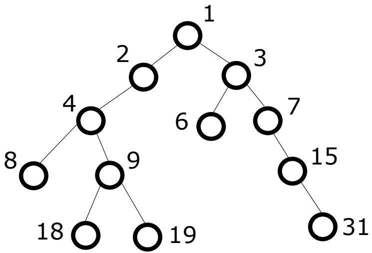
Property: The HP-tree has height in $\mathcal{O}(\log n)$.

Improvements: Binary Lifting

Binary tree of height in $\mathcal{O}(\log n)$.

1) Label the root with 1. If a node is labelled with i, then label its left (resp. right) child by $2i$ (resp., $2i + 1$).
$\rightarrow$ Keep the pairs (label,node) in a hash table

2) A node at level i has a $(i + 1)$-bit label. The j^{th} closest ancestor of a node x has label `x.label >> j` (division by 2^j)



Remark: if the LCA of two nodes x, y is at level i, then the $i + 1$ most significant bits of their respective labels are identical.

Binary Lifting: Application to LCA

Input: nodes x, y
s.t. `x.level <= y.level`.

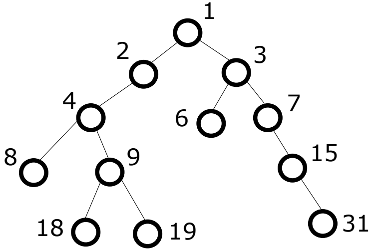
1) `a = x.label` and `b = y.label`

2) If `i = y.level - x.level` is > 0, then `b = b >> i` (division by 2^i).

3) Let `c = a ^ b` (bitwise XOR)

4) Let $j = \lfloor \log c \rfloor + 1$ (most significant bit)

5) LCA(x,y) is the node with label `a >> j`.

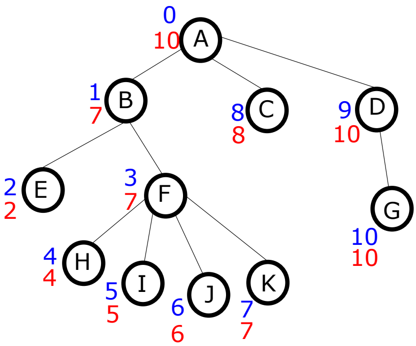


Complexity: $\mathcal{O}(1)$ assuming bitwise operators (but $\mathcal{O}(\log n)$ otherwise).

Intermezzo: A problem of **intervals**

Consider a pre-order of the nodes.

Observation: a rooted subtree = an interval



LCA(x,y), $x < y \implies$ **shortest** interval that contains $[x, y]$

DFS computation

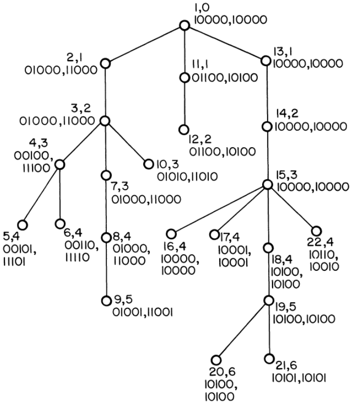
```
void compute_intervals(node *n, int& num){
   n->start = num++;
   for(node *c = n->child; c != nullptr; c = c->next)
      compute_intervals(c,num);
   n->end = num;
}
```

Complexity: $\mathcal{O}(n)$

Remark: Post-ordering = nodes ordered by increasing end value (break ties by decreasing height. . .)

Binary Lifting + HP-tree

Pre-processing



1) Compute a preorder + preorder-intervals [start,end] for each node

2) For each node v, let its inlabel denote the node in its subtree whose preorder number has maximum number of rightmost 0s in its binary representation.

3) **HP decomposition using inlabels** (same HP = same inlabel)

Pre-processing time: $\mathcal{O}(n)$ + computation of inlabels

Computation of inlabels

- We compute, for each node v, the number of rightmost 0s in `v.preorder`.

$\rightarrow$ For that, we simulate the increment of a binary counter with a stack.

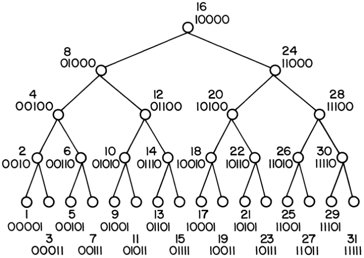
- We consider the nodes in preorder
- At the time we consider the i^{th} node, a stack S memorizes the position of all 1's in decreasing order (from right to left) in the binary representation of $i-1$.

  Incrementation: pop until we find two nonconsecutive 1's or S becomes empty. **Amortized complexity** $\mathcal{O}(1)$
- Number of rightmost zeros = position of the rightmost zero (access to top element)

- Inlabels by dynamic programming on the tree. $\Longrightarrow \mathcal{O}(n)$

Binary Lifting + HP-tree

Properties



1) Identify the inlabels with the inorder numbers of some nodes in the smallest **complete binary tree** B with $> n$ nodes.

2) **Descendance-preservation property**: if x is a descendant of y, then `x.inlabel` is a descendant of `y.label` (the converse is false in general)

3) In a complete binary rooted tree of order $2^{h+1} - 1$, for a node x at level i, we can compute its j^{th} closest ancestor by suppressing the $i, i-1, \ldots, i-j+1$ most significant bits and adding 0s to the right.

Consequences: if $z = LCA(x, y)$, then `z.inlabel` is an ancestor of `w = LCA(x.inlabel,y.inlabel)` in the complete binary rooted tree B. Furthermore, we can compute w using binary lifting.

Binary Lifting + HP-tree

Query (1/2)

Input: nodes x, y with `x.inlabel != y.inlabel`.

1) Compute `LCA(x.inlabel,y.inlabel)` in B

- Compute the difference of level between `x.inlabel` and `LCA(x.inlabel,y.inlabel)`
  (from the most significant bit of `x.inlabel XOR y.inlabel`)
- Binary lifting: Compute `a = (x.inlabel << j) >> j` (most significant bits), `b = (x.inlabel - a)` then `c = b - ((b << k) >> k)` (suppression of the bit subsequence). Output `a + c`.

  Remark: requires level of `x.inlabel` in B, that can be deduced from the number of rightmost 0s (already computed)

In what follows, let `w = LCA(x.inlabel,y.inlabel)`

The **ascendant** numbers

Reminder: for every node `r`, we know the level of `r.inlabel` in B

- We define r.ascendant so that its j^{th} bit is set to 1 if and only if it has an ancestor s in T whose inlabel is at level j in B.

We can compute the ascendant numbers by dynamic programming on the tree T.

Sketch: Let `p` denote the parent of `r`. We obtain `r.acendant` from `t.ascendant` by setting one new bit to 1 (namely, the level of `r.inlabel` in B). This can be done using bitwise operators:

```
r.ascendant = t.ascendant + (1 >> i)
```

Additional pre-processing time: $\mathcal{O}(n)$

Binary Lifting + HP-tree

Query (2/2)

Input: nodes x, y with `x.inlabel != y.inlabel`.

2) Let $z = LCA(x, y)$ (we do not know z at this point). In order to compute `z.inlabel` from `w`, we compute the level of `z.inlabel` in B.

$\rightarrow$ XOR on `x.ascendant`, `y.ascendant`

3) To compute `z`, we need to find the closest ancestors of `x,y` in the heavy-path corresponding to `z.inlabel`. Let us detail for `x`. We assume x.inlabel != z.inlabel. Let P_x, P_z denote the two HPs of x, z. We just need to find the neighbour of P_z on the unique $P_z P_x$-path in the HP-tree.

$\rightarrow$ The inlabel of this HP can be retrieved from `x.ascendant`! (again using bitwise operators).

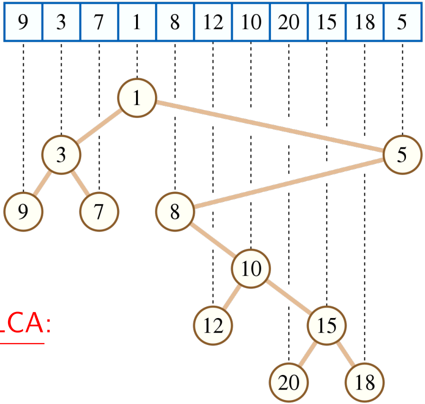
Query time: $\mathcal{O}(1)$ using bitwise operators, and $\mathcal{O}(\log n)$ otherwise.

An application to Range queries: Cartesian trees

Input: an n-size vector `v[]`

Output: an n-node binary rooted tree T *whose nodes are the elements of* `v[]`, such that:

- The root is a min/max element
- the left/right subtrees are Cartesian trees for the left/right subvectors to the root.



Reduction of min/max range queries to LCA:

The min/max elements between indices i and j is the element in position LCA(i,j).

Construction

- It suffices to compute the vector `father` encoding the parent of each node.

- We scan the vector from left to right and maintain potential candidates for parent nodes in a stack. We repeatedly pop out smaller/bigger elements. $\Longrightarrow$ left neighbour

Interpretation: if the father of a i is at an earlier position $j < i$, then it must be its left neighbour.

- Compute the right neighbours in the same way by scanning from right to left.

- Father of a node: the largest/least value amongst its left and right neighbours.

Implementation

Min. version

```
vector<int> compute_cartesian_tree(const vector<int>& v) {
   vector<int> left(v.size());
   stack<int> candidates;
   for(int i = 0; i < v.size(); i++) {
      while(!candidates.empty() && v[candidates.top()] > v[i])
         candidates.pop();
      left[i] = (candidates.empty()) ? -1 : candidates.top();
      candidates.push(i);
   }
   /* right neighbours */
   vector<int> father(v.size());
   for(int i = 0; i < v.size(); i++) {
      father[i] = (left[i] <= right[i]) ? right[i] : left[i];
   }
   return father;
}
```

Complexity: $\mathcal{O}(n)$ (Potential: size of the stack)

Beyond LCA: Distance queries

- Distance between two nodes x and y: number of edges on the (unique) path in the tree between x and y

- Application: Routing in Tree Networks (or in general networks using a spanning trees)

- All distances can be computed in $\mathcal{O}(n^2)$ time by varying the root and applying BFS.

Can we do better?

Reduction to LCA

Input: nodes x, y.

1) Let $z = LCA(x, y)$.

2) Node z must be on the xy-path

$\Longrightarrow d(x, y) = d(x, z) + d(z, x)$.

3) Since z is an ancestor of x (resp., y), we have
$d(x, z) = x.height - z.height$ (resp., $d(y, z) = y.height - z.height$).

Pre-processing: $\mathcal{O}(n)$ (compute the heights + LCA pre-processing)

Query time: $\mathcal{O}(1)$

Centroids in trees

Definition

A centroid in an n-node tree T is a node c such that every subtree of $T \setminus \{c\}$ has order $\leq n/2$.

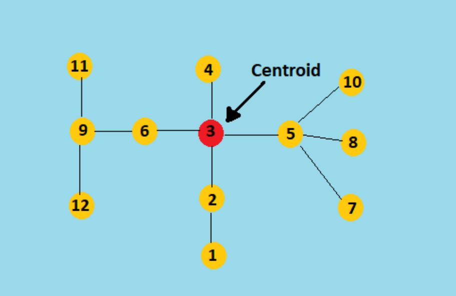
A centroid always exists.

Proof:

1) There are $n - 1$ edges (all nodes but the root have a parent)

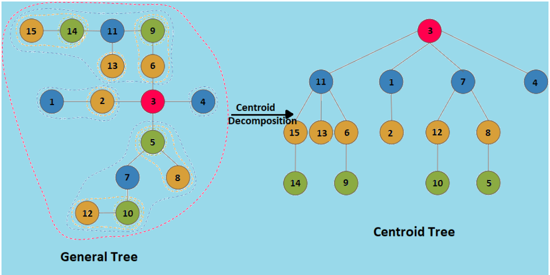
2) For each node n, choose a heaviest subtree of $T \setminus \{n\}$ and orient the edge from n to this subtree.

3) One edge is oriented in both directions!



Centroid of a Tree

Centroid decomposition



1) Compute a centroid c

2) Compute a centroid decomposition for each subtree of $T \setminus \{c\}$

(output = rooted tree)

3) Merge all decompositions in one rooted tree with c as root.

The result is a rooted tree T' with same node-set as the original tree T.

Property: the centroid decomposition outputs a tree of height in $\mathcal{O}(\log n)$

Application to LCA/Distance queries: store for each node its path-to-root in the centroid decomposition + distances in T.

Computation

- A centroid can be computed in $\mathcal{O}(n)$ time.
  - Pre-compute the order of each rooted subtree.
  - **Local search**. Start from any node. If each subtree of $T \setminus \{c\}$ has order $\leq n/2$ then output c. Otherwise, go to a neighbour in a heaviest subtree.

    Remark: for each child c' of c we know the order of its rooted subtree. If c' is the parent of c, then the subtree of $T \setminus \{c\}$ that contains c' has order $n - c -> order$.

- There are $\mathcal{O}(\log n)$ recursive steps

$\Longrightarrow$ Centroid decomposition in $\mathcal{O}(n \log n)$ time.

Questions