Forme biliniare

$g: V\times V \to \mathbb{K}$ formă bilin. sim

$Q(x) = g(x,x)$, $g(x,y) = X^T G Y$, $G = G^T$, $g(x,y) = \sum_{i,j=1}^{n} g_{ij} x_i y_j$

$Q(x) = X^T G X = \sum_{i=1}^{n} g_{ii} x_i^2 + 2 \sum_{i<j} g_{ij} x_i y_j$

Metoda Gauss $\quad$ g nedeg. $\Leftrightarrow \det G \neq 0$

$G = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}$ $\quad$ $Q(x) = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_3x_1$

$Q(x) = \left(x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3\right)^2 - \frac{1}{4}x_2^2 - \frac{1}{4}x_3^2 - \frac{1}{2}x_2x_3 + x_2^2 + x_3^2 + x_2x_3$

$Q(x) = \dots\dots$ – se formează pătrate

$Q(x) = \cancel{\dots} \; a y_1^2 + b y_2^2 + c y_3^2$ $\quad$ schimbare de reper (de 2 ori)

Signatura $(nr +, nr -)$ $\quad$. Poz. def. $\Leftrightarrow \nexists -, \nexists 0$

$\Downarrow$ nedeg.

Metoda Jacobi

$\Delta_1, \Delta_2, \Delta_3 \neq 0$

$Q(x) = \frac{1}{\Delta_1} x_1'^2 + \frac{\Delta_1}{\Delta_2} x_2'^2 + \frac{\Delta_2}{\Delta_3} x_3'^2$ $\quad$ Poz. def $\Leftrightarrow \Delta_1, \Delta_2, \Delta_3 > 0$

SVER $\quad$ $\|x\| = \sqrt{g(x,x)}$

$g: V\times V \to \mathbb{R}$ produs scalar $\Leftrightarrow g \in L^s(V,V;\mathbb{R})$, g poz. def

$R = \{e_1, \dots, e_n\}$ rep. – ortogonal $\Leftrightarrow g(e_i, e_j) = 0, \forall i \neq j$

– ortonormat $\Leftrightarrow g(e_i, e_j) = \delta_{ij}, \forall i, j$

$R \xrightarrow{A} R' \Rightarrow A \in O(n)$: $A \cdot A^T = I_n$

rep. ortonorm.

$U \subseteq V$ ssp. vect $\Rightarrow U^\perp = \{y \in V \mid g(x,y) = 0, \forall x \in U\}$

$\{x\}^\perp = \{y \mid g(x,y) = 0\}$

* $x \times y = \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$

* $x \wedge y \wedge z = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = g_0(x, y \times z)$

Gram-Schmidt – ortonormarea rep. $\{f_1, f_2, f_3\}$

$e_1 = f_1$

$e_2 = f_2 - \frac{\langle f_2, e_1\rangle}{\langle e_1, e_1\rangle} e_1$

$e_3 = f_3 - \frac{\langle f_3, e_1\rangle}{\langle e_1, e_1\rangle} e_1 - \frac{\langle f_3, e_2\rangle}{\langle e_2, e_2\rangle} e_2$

Rep. ortonorm.: $\frac{1}{\|e_1\|} e_1, \frac{1}{\|e_2\|} e_2, \frac{1}{\|e_3\|} e_3$

Scanned with CamScanner

Transformări ortogonale

f transf. ortog. $\Rightarrow \langle f(x), f(y)\rangle = \langle x, y\rangle$

f end., $f \in O(E) \Leftrightarrow \|f(x)\| = \|x\|$

$f \in O(E) \Leftrightarrow A = [f]_{R,R} \in O(n) \Leftrightarrow$ schimbare de repere ortonormate

$f \in O(E) \Rightarrow \lambda_R = \pm 1.$

① $\dim E = 1 \Rightarrow O(E) = \{id_E, -id_E\}$

② $\dim E = 2$ a) $\det A = 1$ $A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$

b) $\det A = -1$ $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

③ $\dim E = 3$ a) $\det A = 1$ $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$

$\operatorname{Tr} A = 1 + 2\cos\varphi$, Axa: $f(x) = x$

b) $\det A = -1$ $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$

$\operatorname{Tr} A = -1 + 2\cos\varphi$, Axa: $f(x) = -x$

$f \in Sim(E) \Leftrightarrow \langle x, f(y)\rangle = \langle f(x), y\rangle \Leftrightarrow A$ sim $(A = A^T)$

$f \in Sim(E) \Rightarrow \exists R$ rep. ortonorm. a.î. $[f]_{R,R}$ diag.

$f \in Sim(E) \Rightarrow \lambda_R \in \mathbb{R}$, $\dim V_{\lambda_R} = m_R$.

C.B.S.

$|\langle x, y\rangle| \le \|x\| \cdot \|y\|$

Geometrie analitică euclidiană

① Ecuația unei drepte afine

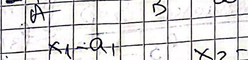
a) 

$V_D = \langle\{v\}\rangle$

$\vec{OA} = \sum a_i e_i$, $\vec{OX} = \sum x_i e_i$

$D: \dfrac{x_1 - a_1}{v_1} = \dfrac{x_2 - a_2}{v_2} = \dfrac{x_3 - a_3}{v_3} = t \Rightarrow$ ec. param.

$V_D = \langle\{\vec{AB}\}\rangle$

b) 

$D: \dfrac{x_1 - a_1}{b_1 - a_1} = \dfrac{x_2 - a_2}{b_2 - a_2} = \dfrac{x_3 - a_3}{b_3 - a_3}$

* $\mathcal{D}_1 \parallel \mathcal{D}_2 \Leftrightarrow V_{\mathcal{D}_1} = V_{\mathcal{D}_2} \Leftrightarrow \exists \alpha$ a.î. $v' = \alpha v$

* $\mathcal{D}_1 \nparallel \mathcal{D}_2$ $\quad \mathcal{D}_1 : x_i - a_i = t v_i$

$\mathcal{D}_2 : x_i - b_i = t' v_i'$

$$\boxed{\left(\begin{array}{cc|c} v_1 & -v_1' & b_1 - a_1 \\ v_2 & -v_2' & b_2 - a_2 \\ v_3 & -v_3' & b_3 - a_3 \end{array}\right) \quad \Delta_c}$$

$\Delta_c = 0 \Rightarrow$ drepte concurente coplanare

$\Delta_c \neq 0 \Rightarrow$ drepte necoplanare

② Ecuația unui plan afin

a) π $(A \in \pi, V_\pi = \langle \{u, v\} \rangle)$
$\quad\quad \parallel \{\vec{AM}, M \in \pi\}$

$\exists t, s$ a.î. $\vec{AM} = tu + sv$ $\quad \vec{OA} = \sum a_i e_i \quad \vec{OM} = \sum x_i e_i$

$x_i - a_i = t u_i + s v_i$

$$\pi : \begin{vmatrix} x_1 - a_1 & u_1 & v_1 \\ x_2 - a_2 & u_2 & v_2 \\ x_3 - a_3 & u_3 & v_3 \end{vmatrix} = 0$$

$N = u \times v = (A_1, A_2, A_3)$

$\pi : A_1 (x_1 - a_1) + A_2 (x_2 - a_2) + A_3 (x_3 - a_3) = 0$

$A_1 x_1 + A_2 x_2 + A_3 x_3 + A_0 = 0$

b) π $(A, B, C \in \pi)$. $V_\pi = \langle \{\vec{AB}, \vec{AC}\} \rangle$

$\pi : x_i - a_i = t(b_i - a_i) + s(c_i - a_i)$

$$\pi : \begin{vmatrix} x_1 & x_2 & x_3 & 1 \\ a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \end{vmatrix} = 0$$

c) $A(a_1, \dots, a_n) \in \pi$, $\mathcal{D} \perp \pi$

$\mathcal{D} : \dfrac{x_1 - x_1^0}{u_1} = \dfrac{x_2 - x_2^0}{u_2} = \dfrac{x_3 - x_3^0}{u_3}$

$u = N$. $\forall M$, $\langle \vec{AM}, N \rangle = 0$

$u_1 (x_1 - a_1) + u_2 (x_2 - a_2) + u_3 (x_3 - a_3) = 0$

③ $\perp$ comună a 2 drepte necoplanare

$\mathcal{D}_1 : x_i - a_i = t v_i \quad \mathcal{D}_2 : x_i - b_i = t' v_i'$

$P_1 (a_1 + t v_1, a_2 + t v_2, a_3 + t v_3)$, $P_2 (b_1 + t' v_1', b_2 + t' v_2', b_3 + t' v_3')$

$\begin{cases} \langle \vec{P_1 P_2}, v \rangle = 0 \\ \langle \vec{P_1 P_2}, v' \rangle = 0 \end{cases} \Rightarrow t, t' \Rightarrow P_1, P_2$

$$\begin{vmatrix} v_1 & v_1' & b_1 - a_1 \\ v_2 & v_2' & b_2 - a_2 \\ v_3 & v_3' & b_3 - a_3 \end{vmatrix} \neq 0$$

$\underset{v}{\uparrow} \quad \underset{v'}{\uparrow} \quad \underset{\vec{AB}}{\uparrow} \Rightarrow \mathcal{D}_1, \mathcal{D}_2$ necop.

④ Poz. relativă a 2 hiperplane

$\mathcal{H}: \ A_1x_1 + A_2x_2 + A_3x_3 + A_0 = 0$

$N = (A_1, A_2, A_3)$

$\mathcal{H}_1 \parallel \mathcal{H}_2 \iff \frac{A_1}{A_1'} = \frac{A_2}{A_2'} = \frac{A_3}{A_3'} \neq \frac{A_0}{A_0'}$

$\mathcal{H}_1 = \mathcal{H}_2 \iff$ ——— " ——— $= \frac{A_0}{A_0'}$

⑤ Intersecția unei drepte cu un plan

$\mathcal{D}: \frac{x_1 - a_1}{u_1} = \frac{x_2 - a_2}{u_2} = \frac{x_3 - a_3}{u_3} = t$

$\pi: A_1x_1 + A_2x_2 + A_3x_3 + A_0 = 0$

$(*)\ \mathcal{D} \cap \pi: \ t(A_1u_1 + A_2u_2 + A_3u_3) + A_1a_1 + A_2a_2 + A_3a_3 + A_0 = 0$

a) $\mathcal{D} \parallel \pi: \ \langle N, u \rangle = 0 \Rightarrow \sum A_i u_i = 0$

$\mathcal{D} \not\subset \pi \quad \sum A_i a_i + A_0 \neq 0 \iff$ nu are sol

b) $\langle N, u \rangle = 0, \ \sum A_i a_i + A_0 = 0 \iff$ o inf de sol.

c) $\mathcal{D} \cap \pi = \{P\} \iff$ sol unică

$t = -\frac{\sum A_i a_i + A_0}{\sum A_i u_i}$

Arii, volume, distanțe

① $S_{\triangle ABC} = \frac{1}{2} \ \|\vec{AB} \times \vec{AC}\|$

② $d(A, \mathcal{D}) = \frac{\|\vec{AB} \times \vec{AC}\|}{\|\vec{BC}\|} \qquad B, C \in \mathcal{D}$

sau $d(A,\mathcal{D}) = d(A, M), \ \pi \perp \mathcal{D}, \ A \in \pi, \ \mathcal{D} \cap \pi = \{M\}$

③ $V_{ABCD} = \frac{1}{6} \cdot |\Delta|, \quad \Delta = \begin{vmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \\ d_1 & d_2 & d_3 & 1 \end{vmatrix} \qquad \Delta = 0 \iff A, B, C, D$ coplanare

④ $d(A, \pi) = \frac{|a \cdot a_1 + b \cdot a_2 + c \cdot a_3 + d|}{\sqrt{a^2 + b^2 + c^2}}, \quad \pi = ax_1 + bx_2 + cx_3 + d = 0$, $A(a_1, a_2, a_3)$

sau $d(A, \pi) = d(A, M), \ A \in \mathcal{D}, \ \mathcal{D} \perp \pi, \ \mathcal{D} \cap \pi = \{M\}$

⑤ $d(\mathcal{D}_1, \mathcal{D}_2) = \frac{|\langle \vec{AB}, N \rangle|}{\|N\|}, \quad A \in \mathcal{D}_1, B \in \mathcal{D}_2, \ N = u \times v$, $u = u_{\mathcal{D}_1}, \ v = u_{\mathcal{D}_2}$

$V_{paralelipiped} = \langle u, v \times w \rangle = |u \wedge v \wedge w| \qquad u = u_{\vec{AB}}, \ v = u_{\vec{AC}}, \ w = u_{\vec{AD}}$

Unghiuri

$\measuredangle(\mathcal{D}_1,\mathcal{D}_2) = \measuredangle(u_1,u_2) = \measuredangle\varphi \in [0;\pi]$ $\quad u_1 \perp u_2 \Leftrightarrow \langle u_1,u_2\rangle = 0$

$\cos\varphi = \dfrac{\langle u_1,u_2\rangle}{\|u_1\|\cdot\|u_2\|}$ $\quad \mathcal{D}_k$ orientată de u_k

$\measuredangle(\Pi_1,\Pi_2) = \measuredangle(N_1,N_2) = \measuredangle\varphi$

$\cos\varphi = \dfrac{\langle N_1,N_2\rangle}{\|N_1\|\cdot\|N_2\|}$ $\quad \Pi_k$ orientat de N_k

$\measuredangle(\mathcal{D},\Pi) = \measuredangle(\mathcal{D},\mathcal{D}') = \measuredangle\varphi = 90^\circ - \measuredangle(\mathcal{D},N)$

$\mathcal{D}$ orientată de u, Π orientat de N, $\mathcal{D}' = pr_\Pi \mathcal{D}$

Conice

$\Gamma: f(x) = a_{11}x_1^2 + a_{22}x_2^2 + 2a_{12}x_1x_2 + 2b_1x_1 + 2b_2x_2 + c = 0$

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = A^T$, $\tilde{A} = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix}$, $B = (b_1, b_2)$

$\delta = \det A$, $\Delta = \det\tilde{A}$, $r = rg A$, $r' = rg\tilde{A}$, $r \le r' \le r+2$

$\Delta = 0 \Rightarrow \Gamma$ conică degenerată, $\Delta \neq 0 \Rightarrow$ nedegenerată

Centru unic

$\delta \neq 0 \rightarrow$ centru unic

$\begin{cases} \dfrac{\partial f}{\partial x_1} = 0 \\ \dfrac{\partial f}{\partial x_2} = 0 \end{cases} \Rightarrow P(x_1^0, x_2^0)$ $\quad f(x_1^0, x_2^0) = \dfrac{\Delta}{\delta}$

$\mathcal{R} = \{O, e_1, e_2\} \xrightarrow[\text{translație}]{\theta} \mathcal{R}' = \{P_0, e_1, e_2\} \xrightarrow[\text{transf. afină}]{\tau} \mathcal{R}'' = \{P_0, e_1'', e_2''\}$

a) $\theta: X = X' + X_0$

$\theta(\Gamma) = X'^T A X' + \dfrac{\Delta}{\delta} = 0$

b) $Q(x) = X'^T A X'$ form. pătratică $\longrightarrow$ form. can. (Gauss/Jacobi)

$Q(x) = \lambda_1 x_1''^2 + \lambda_2 x_2''^2$

$\tau: X' = CX''$

$\underset{\Gamma''}{\tau\circ\theta(\Gamma)}$: $X = CX'' + X_0$

$\lambda_1 x_1''^2 + \lambda_2 x_2''^2 + \dfrac{\Delta}{\delta} = 0$