Seminar 5

Partea 1. Subspații vectoriale

2) $(\mathbb{R}^3, +, \cdot)/\mathbb{R}$, $S = \{(1,2,3), (-1,1,5)\}$
$S' = \{(1,5,11), (2,1,-2), (3,6,9)\}$

a) $\langle S \rangle = \langle S' \rangle = V'$

b) Să se descrie V' printr-un sistem de ec. liniare

c) Să se det. V'' a.î. $\mathbb{R}^3 = V' \oplus V''$

$$\text{rg}\begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 3 & 5 \end{pmatrix} = 2 \text{ (maxim)} \xRightarrow{\text{crit LI}} S \text{ este SLI}$$

$$\Rightarrow \dim \langle S \rangle = 2$$

$$\det\begin{pmatrix} \boxed{\begin{matrix} 1 & 2 \\ 5 & 1 \end{matrix}} & \begin{matrix} 3 \\ 6 \end{matrix} \\ \begin{matrix} 11 & -2 \end{matrix} & 9 \end{pmatrix} = 0$$

$S'' = \{(1,5,11), (2,1,-2)\}$ este SLI maximal în S'

$\Rightarrow \langle S' \rangle = \langle S'' \rangle \Rightarrow \dim \langle S' \rangle = 2.$

Fie $u = (1,2,3)$, $v = (-1,1,5)$
$u' = (1,5,11)$, $v' = (2,1,-2)$
$u \in S' \Leftrightarrow \exists\, a, b \in \mathbb{R}$ a.î. $u = au' + bv'$.

$$\begin{cases} a + 2b = 1 \\ 5a + b = 2 \\ 11a - 2b = 3 \end{cases} \qquad A = \left(\begin{matrix} 1 & 2 \\ 5 & 1 \\ 11 & -2 \end{matrix}\right)\left|\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}\right. \Rightarrow \text{rg} A = 2$$

$$\Delta c = \begin{vmatrix} 1 & 2 & 1 \\ 5 & 1 & 2 \\ 11 & -2 & 3 \end{vmatrix} = 3+44-10-11+4-30=0$$

$\Rightarrow rg\,\overline{A} = 2 \Rightarrow$ SCD

$v \in S' \Leftrightarrow \exists\, a', b' \in \mathbb{R}$ a.î. $v = a'u' + b'v'$

$$\begin{cases} a'+2b' = -1 \\ 5a'+b' = 1 \\ 11a'-2b' = 5 \end{cases} \qquad A' = \left(\begin{matrix} 1 & 2 \\ 5 & 1 \\ 11 & -2 \end{matrix}\right) \left|\begin{matrix} -1 \\ 1 \\ 5 \end{matrix}\right. \Rightarrow rg A' = 2$$

$$\Delta c = \begin{vmatrix} 1 & 2 & -1 \\ 5 & 1 & 1 \\ 11 & -2 & 5 \end{vmatrix} = 5+22+10+11+2-50 = 0$$

$\Rightarrow rg\,\overline{A}' = 2 \Rightarrow$ SCD

$\langle S \rangle \subset \langle S' \rangle$

$\dim \langle S \rangle = \dim \langle S' \rangle = 2$

$\Rightarrow$ „$=$".

b) $V' = \langle \{(1,5,11), (2,1,-2)\} \rangle =$

$= \{ (x,y,z) \in \mathbb{R}^3 \,/\, \exists\, a, b \in \mathbb{R}$ a.î. $(x,y,z) = a\cdot(1,5,11) + b\cdot(2,1,-2) \}$

$$\begin{cases} a+2b = x \\ 5a+b = y \\ 11a-2b = z. \end{cases} \qquad A = \left(\begin{matrix} 1 & 2 \\ 5 & 1 \\ 11 & -2 \end{matrix}\right) \left|\begin{matrix} x \\ y \\ z \end{matrix}\right. \Rightarrow rg A = 2$$

$$\Delta c = \begin{vmatrix} 1 & 2 & x \\ 5 & 1 & y \\ 11 & -2 & z \end{vmatrix} = 0,$$

$$x \cdot (-21) - y \cdot (-24) + z \cdot (1-10) =$$

$$= -21x + 24y - 9z = 0 /:3 \Leftrightarrow -7x + 8y - 3z = 0$$

$$V' = \{(x,y,z) \in \mathbb{R}^3 / -7x + 8y - 3z = 0\}.$$

c) $\mathcal{R}' = \{(1,5,11), (2,1,-2)\}$ reper în V'

Extindem la un reper în $\mathbb{R}^3$.

$$rg \begin{pmatrix} 1 & 2 & 1 \\ 5 & 1 & 0 \\ 11 & -2 & 0 \end{pmatrix} = 3$$

$\rightarrow \mathcal{R}' \cup \{e_1\}$ reper în $\mathbb{R}^3$

$\rightarrow V'' = \langle \{e_1\} \rangle$

3) $(\mathbb{R}^3, +, \cdot)/\mathbb{R}$, $V' = \{(x,y,z) \in \mathbb{R}^3 / \begin{cases} x - y + 2z = 0 \\ 2x + y + z = 0 \end{cases}\} = S(A)$.

Să se descompună $x = (-1, 3, 4)$ în raport cu $\mathbb{R}^3 = V' \oplus V''$.

$$A = \left(\begin{array}{cc|c} \boxed{\begin{matrix} 1 & -1 \\ 2 & 1 \end{matrix}} & \begin{matrix} 2 \\ 1 \end{matrix} \end{array}\right) \left| \begin{matrix} 0 \\ 0 \end{matrix} \right.$$

z - variabilă secundară

x, y - variabile principale.

$$\begin{cases} x - y = -2z \\ 2x + y = -z \end{cases}$$

$3x = -3z \Rightarrow x = -z$

$\Rightarrow y = z$

$V' = \{(-z, z, z) = z(-1,1,1) \,/\, z \in \mathbb{R}\} = \langle\{(-1,1,1)\}\rangle$

SG ⇒ SLi (vector nenul)

$\Rightarrow \mathcal{R}' = \{(-1,1,1)\}$ reper în V'.

Extindem $\mathcal{R}'$ la un reper în $\mathbb{R}^3$

$$rg\begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = 3$$

$\Rightarrow \mathcal{R}' \cup \{e_1, e_2\}$ reper în $\mathbb{R}^3$.

$V'' = \langle\{e_1, e_2\}\rangle$

$\exists\, a, b, c \in \mathbb{R}$ a.î. $(-1,3,4) = a\cdot(-1,1,1) + b\cdot(1,0,0) + c\cdot(0,1,0)$

$$\begin{cases} -a + b = -1 \\ a + c = 3 \\ a = 4 \end{cases} \Rightarrow \begin{cases} a = 4 \\ b = 3 \\ c = -1 \end{cases}$$

$$(-1,3,4) = 4\cdot\underbrace{(-1,1,1)}_{\substack{x' \\ \cap \\ V'}} + \underbrace{3\cdot(1,0,0) - 1\cdot(0,1,0)}_{\substack{x'' \\ \cap \\ V''}} =$$

$= x' + x''$, $\quad x' = (-4,4,4)$

$\quad x'' = (3,-1,0)$

Partea a 2-a. Aplicații liniare

Ex 1) $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (x_1 + x_2, -x_2)$

$f \in Aut(\mathbb{R}^2)$ (liniară și bijectivă)

I) f liniară $\Leftrightarrow$ a) $f(x+y) = f(x) + f(y)$
b) $f(a \cdot x) = a \cdot f(x)$, $\forall x, y \in \mathbb{R}^2, \forall a \in \mathbb{R}$

a) $f(x+y) = f(x_1+y_1, x_2+y_2) = (x_1+y_1+x_2+y_2, -x_2-y_2)$

$= (x_1 + x_2, -x_2) + (y_1 + y_2, -y_2) = f(x) + f(y)$

b) $f(a \cdot x) = f(ax_1, ax_2) = (ax_1 + ax_2, -ax_2) =$

$= a \cdot (x_1 + x_2, -x_2) = a \cdot f(x)$

II) $\ker f = \{x \in \mathbb{R}^2 / f(x) = 0_{\mathbb{R}^2}\}$.

$$\begin{cases} x_1 + x_2 = 0 \\ -x_2 = 0 \end{cases} \Rightarrow x_1 = x_2 = 0.$$

$\ker f = \{0_{\mathbb{R}^2}\} \Rightarrow f$ inj

Teorema dimensiunii: $\dim \mathbb{R}^2 = \underbrace{\dim \ker f}_{0} + \dim Im f$.

$\Rightarrow \dim Im f = 2$.

$$\left.\begin{array}{l} Im f \subset \mathbb{R}^2 \text{ subspațiu vectorial} \\ \dim Im f = \dim \mathbb{R}^2 = 2 \end{array}\right\} \Rightarrow Im f = \mathbb{R}^2 \Rightarrow \text{surjectivă}$$

f bijecție + liniară $\Rightarrow f \in Aut(\mathbb{R}^2)$

Ex 2) $f : \mathbb{R}^3 \to \mathbb{R}^3$

$f(x_1, x_2, x_3) = (\underbrace{x_1+2x_2+x_3}_{y_1}, \underbrace{2x_1+5x_2+3x_3}_{y_2}, \underbrace{-3x_1-7x_2-4x_3}_{y_3})$

a) f liniară

b) $\operatorname{Ker} f = ?$ reper în Ker

c) $\operatorname{Im} f = ?$ reper în $\operatorname{Im} f$

d) $[f]_{\mathcal{R}_0, \mathcal{R}_0} = A$, $\mathcal{R}_0$ reper canonic în $\mathbb{R}^3$.

a) $f(ax+by) = a\cdot f(x) + b\cdot f(y)$, $\forall\, x, y \in \mathbb{R}^3$, $\forall a, b \in \mathbb{R}$

$f(ax+by) = (ax_1+by_1+2(ax_2+by_2)+ax_3+by_3,$
$2(ax_1+by_1)+5(ax_2+by_2)+3(ax_3+by_3), -3(ax_1+by_1)-$
$-7(ax_2+by_2)-4(ax_3+by_3)) = a\cdot f(x) + b\cdot f(y)$

b) $\operatorname{Ker} f = \{x \in \mathbb{R}^3 / f(x) = 0_{\mathbb{R}^3}\} = S(A$

$$\begin{cases} x_1+2x_2+x_3=0 \\ 2x_1+5x_2+3x_3=0 \\ -3x_1-7x_2-4x_3=0 \end{cases} \qquad A = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & 0 \\ -3 & -7 & -4 & 0 \end{array}\right)$$

$\det A = 0$

$\dim \operatorname{Ker} f = 3 - \operatorname{rg} A = 3 - 2 = 1$

$$\begin{cases} x_1+2x_2 = -x_3 \quad /\cdot(-2) \\ 2x_1+5x_2 = -3x_3 \end{cases}$$

$x_2 = -x_3$

$\Rightarrow x_1 = x_3$

$$\begin{cases} x_1 = x_3 \\ x_2 = -x_3 \end{cases}$$

$$S(A) = \{ \underbrace{(x_3, -x_3, x_3)}_{x_3(1,-1,1)} / x_3 \in \mathbb{R} \} = \langle \{(1,-1,1)\} \rangle$$

$\mathcal{R}' = \{(1,-1,1)\}$ reper în Ker$f$.

c) $\dim \mathbb{R}^3 = \dim \operatorname{Ker} f + \dim \operatorname{Im} f$.

$\Rightarrow \dim \operatorname{Im} f = 2$.

Metoda I) Extindem $\mathcal{R}'$ la un reper în $\mathbb{R}^3$

$$\operatorname{rg}\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} = 3$$

$\Rightarrow \mathcal{R}' \cup \{e_1, e_3\}$ reper în $\mathbb{R}^3$

$\{f(e_1), f(e_3)\}$ reper în Imf.

$f(e_1) = f(1,0,0) = (1, 2, -3)$

$f(e_3) = f(0,0,1) = (1, 3, -4)$

Metoda II)

$\operatorname{Im} f = \{ y \in \mathbb{R}^3 / \exists x \in \mathbb{R}^3 \text{ a.î. } f(x) = y \}$

$$\begin{cases} x_1 + 2x_2 + x_3 = y_1 \\ 2x_1 + 5x_2 + 3x_3 = y_2 \\ -3x_1 - 7x_2 - 4x_3 = y_3 \end{cases} \qquad A = \left(\begin{array}{ccc|c} 1 & 2 & 1 & y_1 \\ 2 & 5 & 3 & y_2 \\ -3 & -7 & -4 & y_3 \end{array}\right)$$

$\det A = 0 \qquad \Delta c \overset{SCN}{=} 0$

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$$\Rightarrow \begin{vmatrix} 1 & 2 & y_1 \\ 2 & 5 & y_2 \\ -3 & -7 & y_3 \end{vmatrix} = \cancel{5y_3 - 6y_2}$$

$$= y_1 \cdot 1 + y_2 + y_3 = y_1 + y_2 + y_3 = 0$$

$$\Rightarrow \operatorname{Im} f = \{ y \in \mathbb{R}^3 \,/\, y_1 + y_2 + y_3 = 0 \} =$$

$$= \{ \underbrace{(y_1, y_2, -y_1 - y_2)}_{\displaystyle y_1 \cdot (1,0,-1) + y_2 \cdot (0,1,-1)} \,/\, y_1, y_2 \in \mathbb{R} \}.$$

$R'' = \{(1,0,-1), (0,1,-1)\}$ reper în $\operatorname{Im} f$.

d) $R_0 = \{e_1, e_2, e_3\} \xrightarrow{A} R_0 = \{e_1, e_2, e_3\}$

$$f(e_1) = (1,2,-3) = e_1 + 2e_2 - 3e_3$$
$$f(e_2) = (2,5,-7) = 2e_1 + 5e_2 - 7e_3$$
$$f(e_3) = (1,3,-4) = e_1 + 3e_2 - 4e_3$$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ -3 & -7 & -4 \end{pmatrix}$$

$f(x) = y \iff Y = AX$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ -3 & -7 & -4 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

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