Seminar 4

1) $(\mathbb{R}^3, +, \cdot)/_{\mathbb{R}}$ și $R_0 = \{e_1, e_2, e_3\}$ reper canonic

$R' = \{ e_1' = e_1 + 2e_2 + e_3 ;\ e_2' = e_1 + 7e_2 + e_3,\ e_3' = -e_1 + e_2 + e_3 \}$

a) R' reper în $\mathbb{R}^3$; $R_0 \xrightarrow{A} R'$ $A = ?$ (mat. de trecere)

b) Coordonatele vectorului $x = (3, 2, 1)$ în rap. cu R'.

a) $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 7 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$$\det A \overset{L_1 - L_3}{=\!=} \begin{vmatrix} 0 & 0 & -2 \\ 2 & 7 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -2 \cdot (-1)^{3+1} \cdot \begin{vmatrix} 2 & 7 \\ 1 & 1 \end{vmatrix} =$$

$$= -2 \cdot (2 - 7) = 10 \neq 0 \Rightarrow \operatorname{rg} A = 3 \text{ (maxim)}$$

$\xRightarrow{\text{Crit-LI}}$ R' este SLI.

$$\left.\begin{array}{l} \dim_{\mathbb{R}}(\mathbb{R}^3) = \operatorname{card}(R') \\ R' - SLI \end{array}\right\} \Rightarrow R' \text{ reper}$$

b) $(3, 2, 1) = x_1' \cdot e_1' + x_2' \cdot e_2' + x_3' \cdot e_3'$

$= x_1' \cdot (1, 2, 1) + x_2' \cdot (1, 7, 1) + x_3' \cdot (-1, 1, 1)$

$= (x_1' + x_2' - x_3',\ 2x_1' + 7x_2' + x_3',\ x_1' + x_2' + x_3')$

$$\begin{cases} x_1' + x_2' - x_3' = 3 \\ 2x_1' + 7x_2' + x_3' = 2 \\ x_1' + x_2' + x_3' = 1 \end{cases}$$

ec. 3 - ec$_1$ $\Rightarrow 2x_3' = -2 \Rightarrow x_3' = -1$.

$$\begin{cases} x_1' + x_2' = 2 \;/\cdot(-2) \\ 2x_1' + 7x_2' = 3 \end{cases} \Leftrightarrow \begin{cases} -2x_1' - 2x_2' = -4 \\ 2x_1' + 7x_2' = 3 \end{cases} +$$

$\Rightarrow 5x_2' = -1 \Rightarrow x_2' = -\frac{1}{5}$

$$x_1' + x_2' = 2 \Rightarrow x_1' = 2 + \frac{1}{5} \Rightarrow x_1' = \frac{11}{5}$$

$$(x_1', x_2', x_3') = \left(\frac{11}{5}, -\frac{1}{5}, -1\right)$$

Obs: $x = Ax'$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 7 & 1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}$$

2) $(\mathbb{R}_2[x], +, \cdot)/\mathbb{R}$

$R_0 = \{e_1 = 1, e_2 = x, e_3 = x^2\}$ reper canonic

$R' = \{-1 + 2x + 3x^2 = e_1', x - x^2 = e_2', x - 2x^2 = e_3'\}$

a) R' reper în $\mathbb{R}_2[x]$

$R_0 \xrightarrow{A} R'$, $A = ?$

b) $p = 3 - x + x^2$ coordonate în rap. cu R'

a) $A = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 1 \\ 3 & -1 & -2 \end{pmatrix}$

$$\det A = \begin{vmatrix} -1 & 0 & 0 \\ 2 & 1 & 1 \\ 3 & -1 & -2 \end{vmatrix} = (-1)\cdot(-1)^{1+1} \cdot \begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix} =$$

$= -1 \cdot (-2+1) = 1 \neq 0 \Rightarrow rg\, A = 3$ (maxim)

$\xRightarrow{\text{crit. LI}} R'$ SLi

$\left.\begin{array}{l} R' \text{ SLi} \\ \dim_{\mathbb{R}}(\mathbb{R}_2[x]) = 3 \end{array}\right\} \Rightarrow R'$ ~~baza~~ reper

b) $P = 3 - x + x^2 = x_1' \cdot e_1' + x_2' \cdot e_2' + x_3' \cdot e_3' =$

$= x_1' \cdot (-1 + 2x + 3x^2) + x_2'(x - x^2) + x_3'(x - 2x^2)$

$= -x_1' + x(2x_1' + x_2' + x_3') + x^2(3x_1' - x_2' - 2x_3')$

$$\begin{cases} -x_1' = 3 \Rightarrow x_1' = -3. \\ 2x_1' + x_2' + x_3' = -1 \\ 3x_1' - x_2' - 2x_3' = 1 \end{cases}$$

$$\begin{cases} x_2' + x_3' = 5 \\ x_2' + 2x_3' = -10 \end{cases} \quad -$$

$x_3' = -15 \Rightarrow x_2' = 20.$

$\Rightarrow (x_1', x_2', x_3') = (-3, 20, -15)$ coordonatele lui P în raport cu R'.

Obs: $X = AX' \Rightarrow X' = A^{-1} \cdot X$

$$X = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

4) $(\mathbb{R}_3[x], +, \cdot)/\mathbb{R}$

$V_1 = \{ P \in \mathbb{R}_3[x] \mid P(0) = 0 \}$
$V_2 = \{ P \in \mathbb{R}_3[x] \mid P(1) = 0 \}$
$V_3 = \{ P \in \mathbb{R}_3[x] \mid P(0) = P(1) = 0 \}$

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a) $V_i \subset \mathbb{R}_3[x]$, $\forall\, i = \overline{1,3}$ subspații vectoriale.

Fie $\left.\begin{matrix} P, Q \in V_1 \\ \alpha, \beta \in \mathbb{R} \end{matrix}\right\} \Rightarrow \alpha \cdot P + \beta \cdot Q \in V_1$

$(\alpha P + \beta Q)(0) = \alpha \cdot P(0) + \beta \cdot Q(0) = 0$

$P = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$P(0) = 0 \rightarrow a_0 = 0.$

Analog pt. V_2 și V_3

b) Precizați R_i reper în V_i, $i = \overline{1,3}$

$V_1 \ni P = a_1 x + a_2 x^2 + a_3 x^3 \in \langle \{ x, x^2, x^3 \} \rangle$

$\left.\begin{matrix} \mathcal{R}_1 = \{ x, x^2, x^3 \} \text{ SG pt. } V_1 \\ \left.\begin{matrix} \mathcal{R}_1 \subset \mathcal{R}_0 = \{ 1, x, x^2, x^3 \} \\ \mathcal{R}_0 \text{ SLI} \end{matrix}\right\} \Rightarrow \mathcal{R}_1 \text{ SLI} \end{matrix}\right\} \Rightarrow \mathcal{R}_1 \text{ reper în } V_1.$

$V_2: \quad P = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$P(1) = 0 \Rightarrow a_0 + a_1 + a_2 + a_3 = 0 \Rightarrow a_0 = -(a_1 + a_2 + a_3)$

$P = a_1(x-1) + a_2(x^2-1) + a_3(x^3-1) \in \langle \{x-1, x^2-1, x^3-1\} \rangle$

$\Rightarrow \mathcal{R}_2$ este SG

$\mathcal{R}_0 = \{1, x, x^2, x^3\}$

$$A = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

matricea componentelor vectorilor din $\mathcal{R}_2$ în raport cu $\mathcal{R}_0$.

$rg\, A = 3$ (maxim)

$\xrightarrow{\text{crit. Li}} \mathcal{R}_2$ SLi

$\mathcal{R}_2$ SG $\Big\} \Rightarrow \mathcal{R}_2$ reper în V_2

$V_3: \quad P = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$P(0) = 0 \Rightarrow a_0 = 0$

$P(1) = 0 \Rightarrow a_1 + a_2 + a_3 = 0 \Rightarrow a_1 = -(a_2 + a_3)$

$P = a_2(x^2 - x) + a_3(x^3 - x) \in \langle \{x^2 - x, x^3 - x\} \rangle$

$\mathcal{R}_3 = \{x^2 - x, x^3 - x\}$ este SG

$$B = \begin{pmatrix} 0 & 0 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow rg\, B = 2 \text{ (maxim)}$$

$\xrightarrow{\text{crit Li}} \mathcal{R}_3$ SLi

$\mathcal{R}_3$ SG $\Big\} \Rightarrow \mathcal{R}_3$ reper

$\dim V_1 = \dim V_2 = 3$

$\dim V_2 = 2$

c) $P_1 = x + 2x^2 + 3x^3$ în raport cu $\mathcal{R}_1 = \{x, x^2, x^3\} \Rightarrow (1,$

$P_2 = 1 + 2x^2 - 3x^3$ în raport cu $\mathcal{R}_2 = \{x-1, x^2-1, x^3-1\}$.

$P_2 = 0 \cdot (x-1) + 2(x^2-1) - 3(x^3-1) \Rightarrow (0, 2, -3)$

$P_3 = x + 3x^2 - 4x^3$ în raport cu $\mathcal{R}_3 = \{x^2-x, x^3-x\}$.

$P_3 = 3(x^2 - x) - 4(x^3 - x) \Rightarrow (3, -4)$

d) $\mathbb{R}_3[x] = V_i \oplus V_i'$, $i = \overline{1,3}$

V_i' - subspațiu complementar

$V_1' = \langle \{1\} \rangle$

$\mathbb{R}_3[x] = V_2 \oplus V_2'$

$$A = \begin{pmatrix} -1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\det A \neq 0 \Rightarrow V_2' = \langle \{1\} \rangle$

$\mathbb{R}_3[x] = V_3 \oplus V_3'$

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$\det A \neq 0$

$V_3' = \langle \{-x, 1\} \rangle$

e) $\mathbb{R}_3[x] = U_1 \oplus U_2 \oplus U_3$

$U_1 = \langle \{x^2 - x, x^3 - x\} \rangle$

$U_2 = \langle \{-x\} \rangle$

$U_3 = \langle \{-1\} \rangle$

$U_1' = \langle \{x^2 - x\} \rangle$

$U_1'' = \langle \{x^3 - x\} \rangle$

$\mathbb{R}_3[x] = U_1' \oplus U_1'' \oplus U_2 \oplus U_3$

6) $(\mathbb{R}^3, +, \cdot)/\mathbb{R}$, $V' = \{x \in \mathbb{R}^3 / \begin{cases} 2x_1 + x_2 = 0 \\ x_1 + 4x_3 = 0 \end{cases}\} = S(A)$

$A = \left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 1 & 0 & 4 & 0 \end{array}\right)$

a) o bază în V'

b) un subspațiu complementar V'' al lui V' i.e. $\mathbb{R}^3 = V' \oplus V''$

c) să se descompună $x = (1,1,2)$ în raport cu $\mathbb{R}^3 = V' \oplus V''$.
(adică $\exists!\ u \in V'$, $v \in V''$ a.î. $x = u + v$.)

a) $\dim_{\mathbb{R}} V' = 3 - \operatorname{rg} A = 3 - 2 = 1$.

$x_1 = -4x_3$

$x_2 = 8x_3$

$V' = \{(-4x_3, 8x_3, x_3) / x_3 \in \mathbb{R}\} = \langle \{(-4, 8, 1)\} \rangle$ SG

$R' = \{-4, 8, 1\}$ SLI

$\Rightarrow$ reper

b) $\mathbb{R}^3 = V' \oplus V''$

$$rg\begin{pmatrix} -4 & 1 & 0 \\ 8 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = 3$$

$\Rightarrow V'' = \langle \{ e_2, e_3 \} \rangle$

$R = \{ (-4,8,1), (1,0,0), (0,1,0) \}$

$$x = (1,1,2) = \underbrace{a\cdot(-4,8,1)}_{u} + \underbrace{b\cdot(1,0,0) + c\cdot(0,1,0)}_{v}$$

$$(1,1,2) = (-4a+b,\ 8a+c,\ a)$$

$\Rightarrow a = 2$

$b = 9$

$c = -15$

$u = 2(-4,8,1) = (-8,16,2)$

$v = (9,0,0) + (0,-15,0) = (9,-15,0)$

$x = (1,1,2) = (-8,16,2) + (9,-15,0)$

7) $(\mathbb{R}^4, +, \cdot)/\mathbb{R}$, $V' = \{ (x,y,z,t) \in \mathbb{R}^4 / x+y+z-2t=0 \}$

$V'' = \{ (x,y,z,t) \in \mathbb{R}^4 / x+y-2z+t=0 \}$

Să se arate că $\mathbb{R}^4 = V' + V''$

Justificați că suma nu e directă.

$A' = (1 \quad 1 \quad 1 \quad -2)$

$A'' = (1 \quad 1 \quad -2 \quad 1)$

$\dim_{\mathbb{R}} V' = \dim_{\mathbb{R}} V'' = 4-1=3$ (V' și V'' sunt hiperplane)

<u>Teorema Grassmann</u>: $\dim_{\mathbb{R}}(V'+V'') = \dim_{\mathbb{R}} V' + \dim_{\mathbb{R}} V'' - \dim_{\mathbb{R}}(V' \cap V'')$

$V' \cap V'' = \{ (x,y,z,t) \in \mathbb{R}^4 / \begin{cases} x+y+z-2t=0 \\ x+y-2z+t=0 \end{cases} \}$

$rg\, A = 2$

$$A = \left(\begin{array}{cccc} 1 & 1 & 1 & -2 \\ 1 & 1 & -2 & 1 \end{array}\right) \Bigg| \begin{array}{c} 0 \\ 0 \end{array}$$

$\dim_{\mathbb{R}}(V' \cap V'') = 4-2=2$

$\Rightarrow \dim_{\mathbb{R}}(V'+V'') = 3+3-2=4$

$V'+V'' = \langle V' \cup V'' \rangle \subset \mathbb{R}^4$ subspațiu vectorial

$\dim_{\mathbb{R}}(V'+V'') = \dim_{\mathbb{R}} \mathbb{R}^4 = 4$ $\Rightarrow V'+V'' = \mathbb{R}^4$

~~Dimensiu~~ $\dim_{\mathbb{R}}(V' \cap V'') = 2$

$\oplus \Leftrightarrow V' \cap V'' = \{ 0_{\mathbb{R}^4} \}$