Seminar 8

1) $Q:\mathbb{R}^3\to\mathbb{R}$

$Q(x)=x_1^2+x_2^2+x_3^2+x_1x_2+x_1x_3+x_2x_3$

a) $G=$ mat. asociată în rap cu R_0.

b) $g:\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}$ forma polară asociată

c) formă canonică a lui Q (met. Gauss, Jacobi)

poz. definită? Generalizare

Teorie

$g: V\times V\to K$ formă bil. sim. $\quad R\to R'\to R''$

$Q: V\to K$ f. pătratică

$Q(x)=g(x,x),\ \forall x\in V$

$g(x,y)=2^{-1}[Q(x+y)-Q(x)-Q(y)]$

$\mathrm{ch} K\neq 2$

$g(x,y)=X^T G Y,\ G=G^T,\ g(x,y)=\sum_{i,j=1}^{n} g_{ij}x_iy_j$

$Q(x)=X^TGX=\sum_{i,j=1}^{n} g_{ij}x_ix_j=\sum_{i=1}^{n} g_{ii}x_i^2+2\sum_{i<j} g_{ij}x_ix_j$

$$G=\begin{pmatrix}1&\frac12&\frac12\\ \frac12&1&\frac12\\ \frac12&\frac12&1\end{pmatrix}$$

$g(x,y)=x_1y_1+\frac12x_1y_2+\frac12x_1y_3+\frac12x_2y_1+x_2y_2+$

$+\frac12x_2y_3+\frac12x_3y_1+\frac12x_3y_2+x_3y_3$.

c) Metoda Gauss:

$Q(x)=\left(x_1+\frac12x_2+\frac12x_3\right)^2-\frac14x_2^2-\frac14x_3^2+x_2^2+x_3^2+x_2x_3-$

$-\frac12x_2x_3=\left(x_1+\frac12x_2+\frac12x_3\right)^2+\underbrace{\frac34x_2^2+\frac34x_3^2+\frac12x_2x_3}=$

$\frac34\left(x_2^2+\frac23x_2x_3+\frac19x_3^2\right)-\frac1{12}x_3^2$

$+\frac34x_3^2$

$$Q(x) = \left(x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3\right)^2 + \frac{3}{4}\left(x_2 + \frac{1}{3}x_3\right)^2 + \frac{2}{3}x_3^2.$$

Schimbam reperul $\begin{cases} x_1' = x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 \\ x_2' = x_2 + \frac{1}{3}x_3 \\ x_3' = x_3 \end{cases}$

$$\Rightarrow Q(x) = x_1'^2 + \frac{3}{4}x_2'^2 + \frac{2}{3}x_3'^2$$

$$\Rightarrow \begin{cases} x_1'' = x_1' \\ x_2'' = \frac{\sqrt{3}}{2}x_2' \\ x_3'' = \sqrt{\frac{2}{3}}\,x_3' \end{cases}$$

$$\Rightarrow Q(x) = x_1''^2 + x_2''^2 + x_3''^2$$

$(3,0)$ signatura

Metoda Jacobi

$$G = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}$$

$\Delta_1 = 1 \neq 0$

$\Delta_2 = \begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} = \frac{3}{4} \neq 0$

$\Delta_3 = \begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{vmatrix}$

$$= \begin{vmatrix} \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{vmatrix} = \frac{1}{2} \neq 0.$$

$\exists\, \mathcal{R}'$ a.î. $Q(x) = \frac{1}{\Delta_1}x_1'^2 + \frac{\Delta_1}{\Delta_2}x_2'^2 + \frac{\Delta_2}{\Delta_3}x_3'^2 =$

$= x_1'^2 + \frac{4}{3}x_2'^2 + \frac{3}{2}x_3'^2$

$(3,0)$ signatura (invariant)

Generalizare

$$Q(x) = \sum_{i=1}^{n} x_i^2 + \sum_{i<j} x_i x_j \text{ poz. definită}$$

2) $Q: \mathbb{R}^3 \to \mathbb{R}$, $Q(x) = 2x_1x_2 - 6x_1x_3 - 6x_2x_3$

Să se aducă la o formă canonică

$$G = \begin{pmatrix} 0 & \boxed{1} & -3 \\ 1 & 0 & -3 \\ -3 & -3 & 0 \end{pmatrix}$$

Fie schimbarea de reper

$$\begin{cases} x_1' = x_1 + x_2 \\ x_2' = x_1 - x_2 \\ x_3' = x_3 \end{cases} \Leftrightarrow \begin{cases} x_1 = \frac{1}{2}(x_1' + x_2') \\ x_2 = \frac{1}{2}(x_1' - x_2') \end{cases}$$

$$\Rightarrow Q(x) = \frac{1}{2}\left(x_1'^2 - x_2'^2\right) - 6x_3' \cdot x_1'$$

$$= \frac{1}{2}\underbrace{\left(x_1'^2 - 12x_1'x_3' + 36x_3'^2\right)}_{(x_1' - 6x_3')^2} - 18x_3'^2 - \frac{1}{2} \cdot x_2'^2$$

$x_1'' = x_1' - 6x_3'$

$x_2'' = x_2'$

$x_3'' = x_3'$

$$\Rightarrow Q(x) = \frac{1}{2}x_1''^2 - \frac{1}{2} \cdot x_2''^2 - 18x_3''^2$$

$(1, 2)$ signatura

Pt. pb. 8. $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$, $Y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$

Fac înmulțirea și iese g_{ij} apoi scriem Q (unde vedem y punem x)

4. fie $Q: \mathbb{R}^3 \to \mathbb{R}$ formă pătratică

$G = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{pmatrix}$ matricea asociată în rap. cu R_0

diag Q?

$$Q(X) = x_1^2 + 3x_2^2 + x_3^2 + 4x_1x_2 + 2x_1x_3 + 4x_2x_3$$

$$Q(X) = (x_1 + 2x_2 + x_3)^2 - x_2^2$$

$$\begin{cases} x_1' = x_1 + 2x_2 + x_3 \\ x_2' = x_2 \\ x_3' = x_3 \end{cases}$$

$$Q(X) = x_1'^2 - x_2'^2$$

signatura $(\underset{+}{1}, \underset{-}{1})$

$\det G = 0 \Rightarrow \Delta_3 = 0 \Rightarrow$ Jacobi nu merge

8. fie $g: M_2(\mathbb{R}) \times M_2\mathbb{R} \to \mathbb{R}$

$g(X, Y) = 2\,Tr(X \cdot Y) - Tr(X)\,Tr(Y)$, $\forall X, Y \in M_2(\mathbb{R})$

a. $g \in L^s(M_2(\mathbb{R}), M_2(\mathbb{R}); \mathbb{R})$

$g(X, Y) = g(Y, X)$

$g(aX + bY + Y) = a \cdot g(X, Y) + bg(Z, Y)$

b. G în rap. cu $R_0 = \{E_{ij}\}_{i,j=\overline{1,2}}$

$g(E_{11}, E_{11}) = 2\,Tr(E_{11}E_{11}) - 2\,Tr(E_{11})$

$g(E_{11}, E_{12}) = 2\,Tr(E_{11}, E_{12}) - Tr(E_{11}) - Tr(E_{12}) = 1$

SAU $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$, $Y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$