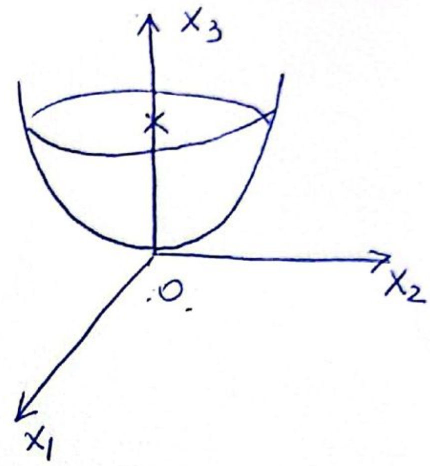
C14 - GAL

27.05.2022.

§1. Cuadrice studiate pe ec. reduse (II)

- Paraboloidul eliptic

$\mathcal{P}_e: \dfrac{x_1^2}{a^2} + \dfrac{x_2^2}{b^2} = 2x_3, \; a>0, \; b>0$



Ec. parametrice

$x_1 = a\cos\theta \cdot t$

$x_2 = b_2 \sin\theta \cdot t$

$x_3 = \frac{1}{2}(t^2), \; t\in\mathbb{R}, \theta\in[0,2\pi)$

$\cap$ cu plane // cu planele de coord.

1) $x_3 = \gamma \in [0,\infty)$ $\quad \dfrac{x_1^2}{a^2} + \dfrac{x_2^2}{b^2} = 2\gamma$

a) $\gamma > 0 \implies$ Elipsă

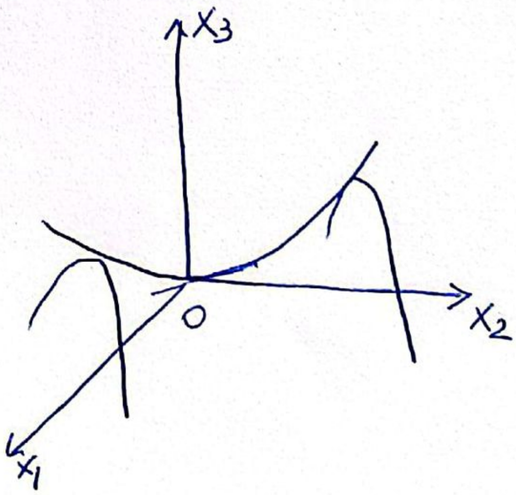
b) $\gamma = 0 \implies O(0,0,0)$

2) $x_2 = \beta \in \mathbb{R}$ $\quad \dfrac{x_1^2}{a^2} = 2x_3 - \dfrac{\beta^2}{b^2} = 2\left(x_3 - \dfrac{\beta^2}{2b^2}\right)$ parabolă

3) $x_1 = \alpha \in \mathbb{R}$ Analog cu 2)

- Paraboloidul hiperbolic

$\mathcal{P}_h: \dfrac{x_1^2}{a^2} - \dfrac{x_2^2}{b^2} = 2x_3, \; a>0, \; b>0$



Ec param.

$$\begin{cases} x_1 = at \\ x_2 = bs \\ x_3 = \frac{1}{2}(t^2 - s^2), \; t, s\in\mathbb{R}. \end{cases}$$

$\cap$ cu plane $\parallel$ cu planele de coord.

1) $x_3 = \gamma$ $\qquad \dfrac{x_1^2}{a^2} - \dfrac{x_2^2}{b^2} = 2\gamma$

- $\gamma > 0 \Rightarrow$ Hiperbolă
- $\gamma = 0 \Rightarrow x_2 = \pm \dfrac{b}{a} x_1$ drepte concurente (în origine)

2) $x_2 = \beta \in \mathbb{R}$ $\qquad \dfrac{x_1^2}{a^2} = 2x_3 + \dfrac{\beta^2}{b^2}$ parabolă

1) $x_1 = \alpha \in \mathbb{R}$ $\qquad \dfrac{x_2^2}{b^2} = -2x_3 + \dfrac{\alpha^2}{a^2}$ parabolă.

<u>Teoremă</u> $\mathcal{P}_h$ = cuadrică dublu riglată și prin fiecare pct $\in \mathcal{P}_h$ trece câte o dreaptă din fiecare familie de generatoare.

<u>Dem</u>

$$\mathcal{P}_h : \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 2x_3 \Rightarrow \left(\frac{x_1}{a} - \frac{x_2}{b}\right)\left(\frac{x_1}{a} + \frac{x_2}{b}\right) = x_3 \cdot 2$$

$$G_1\ d_\lambda : \begin{cases} \dfrac{x_1}{a} + \dfrac{x_2}{b} = \lambda \cdot x_3 \\ \lambda\left(\dfrac{x_1}{a} - \dfrac{x_2}{b}\right) = 2 \end{cases} \qquad d_\infty : \begin{cases} x_3 = 0 \\ \dfrac{x_1}{a} - \dfrac{x_2}{b} = 0 \end{cases}$$

$$G_2 : \bar{d}_\mu : \begin{cases} \dfrac{x_1}{a} - \dfrac{x_2}{b} = \mu x_3 \\ \mu\left(\dfrac{x_1}{a} + \dfrac{x_2}{b}\right) = 2 \end{cases} \qquad \bar{d}_\infty : \begin{cases} x_3 = 0 \\ \dfrac{x_1}{a} + \dfrac{x_2}{b} = 0. \end{cases}$$

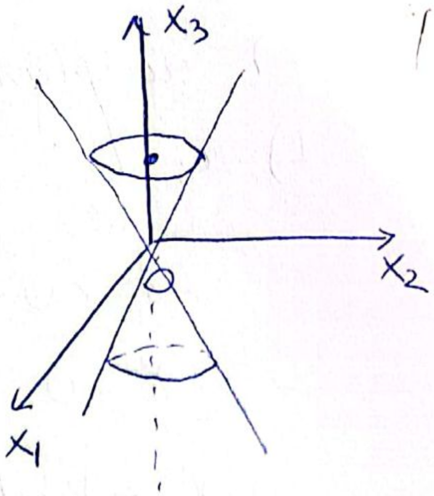
$\lambda, \mu \in \mathbb{R}^*$

$$d_\lambda \cap \bar{d}_\mu : \quad \lambda x_3 = \frac{2}{\mu} \Rightarrow x_3 = \frac{2}{\lambda\mu}.$$

$$\begin{cases} \dfrac{x_1}{a} + \dfrac{x_2}{b} = \dfrac{2}{\mu} \\ \dfrac{x_1}{a} - \dfrac{x_2}{b} = \dfrac{2}{\lambda} \end{cases} \Rightarrow \begin{aligned} \frac{x_1}{a} &= \frac{1}{\mu} + \frac{1}{\lambda} = \frac{\lambda+\mu}{\lambda\mu} \\ \frac{x_2}{b} &= \frac{1}{\mu} - \frac{1}{\lambda} = \frac{\lambda-\mu}{\lambda\mu}. \end{aligned}$$

$$P\left(a \cdot \frac{\lambda+\mu}{\lambda\mu},\ b \cdot \frac{\lambda-\mu}{\lambda\mu},\ \frac{2}{\lambda\mu}\right)$$

- Conul pătratic



Con: $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - \frac{x_3^2}{c^2} = 0$

$a>0, b>0, c>0$

$\cap$ plane // cu planele de coord.

1) $x_3 = \gamma \in \mathbb{R} \Rightarrow \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = \frac{\gamma^2}{c^2}$

- $\gamma \neq 0 \Rightarrow$ Elipsă
- $\gamma = 0 \Rightarrow O(0,0,0)$

2) $x_2 = \beta$ $\quad \frac{x_3^2}{c^2} - \frac{x_1^2}{a^2} = \frac{\beta^2}{b^2}$

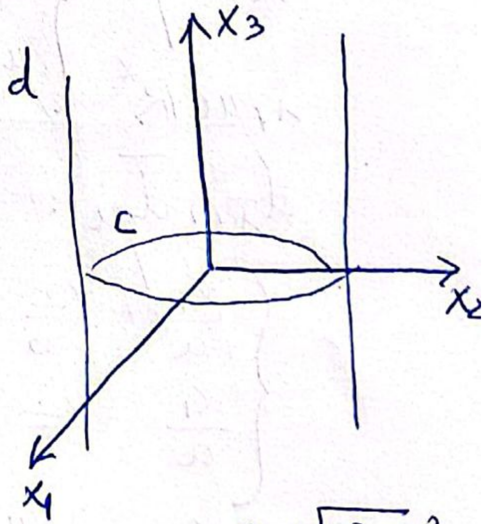
- $\beta \neq 0 \Rightarrow$ Hiperbolă
- $\beta = 0 \Rightarrow x_1 = \pm \frac{a}{c} x_3$ (drepte concurente)

3) $x_1 = \alpha$. Analog cu 2)

- <u>Cilindrul</u> = cuadrică riglată generată de o dreaptă d (de direcție dată), numită generatoare, care mai este supusă unei cond: de ex: intersectează o curbă dată c, numită curbă directoare.

1) C_e : $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$

<u>Cilindrul eliptic</u>



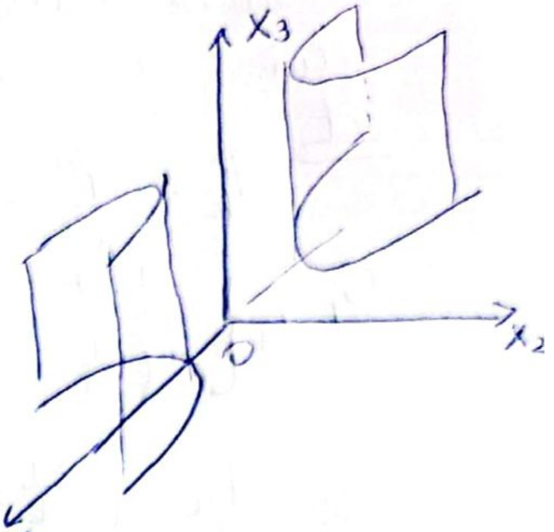
$d \parallel Ox_3$ $\quad C: \begin{cases} \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1 \\ x_3 = 0 \end{cases}$

- $x_3 = \gamma$ $\quad \begin{cases} \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1 \\ x_3 = \gamma \end{cases}$ Elipsă

- $\underset{[-b,b]}{x_2 = \beta}$ $\quad \frac{x_1^2}{a^2} = 1 - \frac{\beta^2}{b^2} \geq 0 \Rightarrow \underset{\text{drepte} \parallel}{x_1 = \pm \frac{a}{b}\sqrt{b^2 - \beta^2}}$

- $x_1 = \alpha$ analog cu cazul precedent

2) Cilindrul hiperbolic



$C_h : \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 1.$

$d \parallel Ox_3$ $\quad C : \begin{cases} \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 1 \\ x_3 = 0 \end{cases}$

- $x_3 = \gamma \in \mathbb{R}$ $\quad \begin{cases} \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 1 \\ x_3 = \gamma \end{cases}$ Hiperbolă

- $x_2 = \beta \in \mathbb{R}$ $\quad \frac{x_1^2}{a^2} = 1 + \frac{\beta^2}{b^2} \Rightarrow x_1 = \pm \frac{a}{b}\sqrt{b^2+\beta^2}$ drepte //

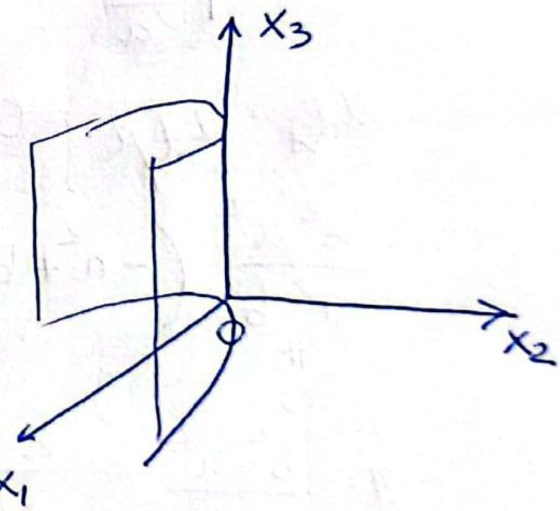
- $x_1 = \alpha$ $\quad \frac{x_2^2}{b^2} = \frac{\alpha^2}{a^2} - 1 \geq 0 \Rightarrow x_2 = \pm \frac{b}{a}\sqrt{\alpha^2 - a^2}$

$\alpha \in (-\infty, -a] \cup [a, \infty)$

$\alpha \neq \pm a$ 2 drepte //

$\alpha = \pm a$ câte 1 dr.

3) Cilindrul parabolic



$C_p : x_2^2 = 2px_1, \; p > 0.$

$d \parallel Ox_3$ $\quad C : \begin{cases} x_2^2 = 2px_1 \\ x_3 = 0. \end{cases}$

- $x_3 = \gamma \in \mathbb{R}$ $\quad \begin{cases} x_2^2 = 2px_1 \\ x_3 = \gamma \end{cases}$ Parabolă

- $x_2 = \beta$ $\quad x_1 = \frac{\beta^2}{2p}.$ 1 dreaptă

- $x_1 = \alpha \geq 0$ $\quad x_2^2 = 2p\alpha$

$\alpha > 0$ $\quad x_2 = \pm\sqrt{2p\alpha}$. drepte //

$\alpha = 0$ $\quad x_2 = 0$

<u>Teoremă</u> $\forall$ suprafață dublu riglată = planul, $\mathcal{H}_1$, P_h

Aplicatie

Să se determine LG al punctelor $\in \mathcal{P}_h$ din care se pot duce generatoare $\perp$.

SOL. $\mathcal{P}_h: \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 2x_3.$

$G_1: d_\lambda \begin{cases} \frac{x_1}{a} + \frac{x_2}{b} = \lambda x_3 \\ \lambda\left(\frac{x_1}{a} - \frac{x_2}{b}\right) = 2 \end{cases} ; d_\infty: \begin{cases} x_3 = 0 \\ \frac{x_1}{a} - \frac{x_2}{b} = 0 \end{cases}$

$G_2: \bar{d}_\mu: \begin{cases} \frac{x_1}{a} - \frac{x_2}{b} = \mu x_3 \\ \mu\left(\frac{x_1}{a} + \frac{x_3}{b}\right) = 2 \end{cases} ; \bar{d}_\infty: \begin{cases} x_3 = 0 \\ \frac{x_1}{a} + \frac{x_2}{b} = 0. \end{cases}$

$u_{d_\lambda} = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{1}{a} & \frac{1}{b} & -\lambda \\ \frac{\lambda}{a} & -\frac{\lambda}{b} & 0 \end{vmatrix} = \left(-\frac{\lambda^2}{b}, -\frac{\lambda^2}{a}, \frac{-2\lambda}{ab}\right)$

$u_{\bar{d}_\mu} = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{1}{a} & -\frac{1}{b} & -\mu \\ \frac{\mu}{a} & \frac{\mu}{b} & 0 \end{vmatrix} = \left(\frac{\mu^2}{b}, -\frac{\mu^2}{a}, \frac{2\mu}{ab}\right)$

$u_{d_\lambda} \cdot u_{\bar{d}_\mu} = 0 \Rightarrow \frac{-\lambda^2\mu^2}{b^2} + \frac{\lambda^2\mu^2}{a^2} - \frac{4\lambda\mu}{a^2b^2} = 0.$

$\underset{\neq 0}{\frac{\lambda^2\mu^2}{a^2b^2}}\left(-a^2+b^2-\frac{4}{\lambda\mu}\right) = 0 \Rightarrow -a^2+b^2 = \frac{4}{\lambda\mu}.$

$\pi: \frac{b^2-a^2}{2} = \frac{2}{\lambda\mu} = x_3 ; \quad P\left(a\frac{\lambda+\mu}{\lambda\mu}, b\frac{\lambda-\mu}{\lambda\mu}, \frac{2}{\lambda\mu}\right)$

(π = planul Monge) $\quad d_\lambda \cap \bar{d}_\mu = \{P\}.$

$\pi \cap \mathcal{P}_h \quad \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = \frac{b^2-a^2}{\cancel{2}} \cdot \cancel{2} = b^2 - a^2$

- $b \neq a$ Hiperbolă.
- $b = a$. $\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 0 \Rightarrow \begin{matrix} x_2 = \pm\frac{b}{a}x_1 \\ x_3 = 0 \end{matrix}$ drepte concurente

Recapitulare

1) T. Hamilton-Cayley. Sist. lin, rang
2) Sp. vect., SLI / SLD, Sist. gen, Bază, dim.
$\forall$ SLI se poate prelungi la o bază
din $\forall$ SG se poate extrage o bază.
3) Repere, Coordonate, crit LI
Sch. reperelor. Subsp. vect. Th. Grassman.
$\oplus$
4) $\dim S(A) = n - rg A$
Apl. lin. Ker, Im f Inj; surj; bij. Th. dim.
5) Matricea asoc. unei apl. lin. Modif matricei la sch. reperelor, inj / surj / bij., $V^* \simeq V$, proiecții, simetrii
6) Vect pr, valori pr, Diagonalizare.
Forme bil., forme pătratice; Matricea asoc.
7) Forma canonică. Th. Gauss, Met Jacobi.
forme bil sim, nedeg, poz def, signatura
8) Sp v euclidiene. P. Gram-Schmidt. Produs scalar.
Produs vect, mixt. $E = U \oplus U^\perp$
Transf. ortog, $A \in O(n)$

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9) pr. ortog, sim. ortog Clasif. transf. ortog $n = 1, 2, 3$.

sp 1 $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$	sp 2 $\begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$
$Tr A = 1 + 2\cos\varphi$; Axa: $f(x) = x$	$Tr A = -1 + 2\cos\varphi$, Axa: $f(x) = -x$

10) Endom. sim. $A = A^T$, toate răd pol car $\in \mathbb{R}$.
$\exists$ un reper format din versori proprii ai A se poate diag

$$A = A^T \begin{cases} f \in Sim(E) \\ Q : E \to \mathbb{R} \end{cases}$$

11) Geom. anal. euclidiana
Ec dr, ec plane, ec hiperplane. , $d \cap \pi$
$\perp$ comună a 2 dr. necoplanare, Arii, volume, dist
12) Conice ca LG (elipsă, hiperbolă, parabolă). Caract unitară conice nedeg.
Forma canonică $\delta \neq 0$.
13) $\delta = 0$. Cuadrice pe ec reduse.
14) Sferă, elipsoid, H_1, H_2, P_e, P_h, Con, Cil_e, Cil_h, Cil_p. Cuadrice riglate (dublu riglate)

Cuadrice. Formă canonică

$$\Gamma: f(x) = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3 + 2b_1x_1 + 2b_2x_2 + 2b_3x_3 + c = 0$$

$A = (a_{ij})_{i,j=\overline{1,3}} = A^T$ $\qquad \delta = \det A$

$$\tilde{A} = \left(\begin{array}{ccc|c} & A & & \begin{matrix} b_1 \\ b_2 \\ b_3 \end{matrix} \\ \hline b_1 & b_2 & b_3 & c \end{array}\right) \qquad \Delta = \det \tilde{A}$$

① $\delta \neq 0$ ($\exists!$ centrul : $\begin{cases} \frac{\partial f}{\partial x_1} = 0 \\ \frac{\partial f}{\partial x_2} = 0 \\ \frac{\partial f}{\partial x_3} = 0 \end{cases}$ $P_0(x_1^0, x_2^0, x_3^0)$)

$\theta: X = X' + X_0$.

$\theta(\Gamma): X'^T A X' + \frac{\Delta}{\delta} = 0$

$Q: \mathbb{R}^3 \to \mathbb{R}$, $Q(x) = X'^T A X'$

$P(\lambda) = 0 \Rightarrow \lambda^3 - I\lambda^2 + J\lambda - \delta = 0$. $\lambda_1, \lambda_2, \lambda_3 (\neq 0)$

$e_i' = (l_i, m_i, n_i)$, $i = \overline{1,3}$ versori proprii

<u>OBS</u> λ_1 $m_1 = 2$ V_{λ_1} se aplică Gram-Schmidt

$\tau: X' = RX''$ $R = \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{pmatrix}$

$\tau(\theta(\Gamma)): \lambda_1 x_1''^2 + \lambda_2 x_2''^2 + \lambda_3 x_3''^2 + \frac{\Delta}{\delta} = 0.$

- $\Delta \neq 0$ $\mathcal{E}, \mathcal{H}_1, \mathcal{H}_2, \phi$
- $\Delta = 0.$ Con, pct dublu.

② $\delta = 0$ ($\nexists$ centru unic)

$Q: \mathbb{R}^3 \to \mathbb{R}$, $Q(x) = X^T A X$

$\lambda_1 = 0$ (~~cel puțin una dintre~~ $\lambda_1 = 0$ sau $\lambda_1 = \lambda_2 = 0$)

$\theta: X = RX''$

$\theta(\Gamma): \lambda_2 x_2'^2 + \lambda_3 x_3'^2 + 2b_1' x_1' + 2b_2' x_2' + 2b_3' x_3' + c = 0$

$$\tilde{A} = \begin{pmatrix} 0 & 0 & 0 & b_1' \\ 0 & \lambda_2 & 0 & b_2' \\ 0 & 0 & \lambda_3 & b_3' \\ b_1' & b_2' & b_3' & c \end{pmatrix}$$

$\Delta = \det \tilde{A} = + b_1' \lambda_3 (-\lambda_2 b_1') = - b_1'^2 \lambda_2 \lambda_3.$

a) $\Delta \neq 0$ (Γ nedeg)

$$\lambda_2 \underbrace{\left(x_2' + \frac{b_2'}{\lambda_2}\right)}_{x_2''}^2 + \lambda_3 \underbrace{\left(x_3' + \frac{b_3'}{\lambda_3}\right)}_{x_3''}^2 + 2b_1' \underbrace{\left(x_1' + \frac{c'}{2b_1'}\right)}_{x_1''} = 0$$

$\tau: X' = X'' + X_0$

$\tau(\theta(\Gamma)): \lambda_2 x_2''^2 + \lambda_3 x_3''^2 + 2b_1' x_1'' = 0$

$\mathcal{P}_e, \mathcal{P}_h.$

b) $\Delta = 0$ (Γ cuadrica degenerată)

$\Delta = -b_1'^2 \lambda_2 \lambda_3$.

b_1). $b_1' = 0$, λ_2, λ_3 nenule.

$$\lambda_2 \underbrace{\left(x_2' + \frac{b_2'}{\lambda_2}\right)}_{x_2''}^2 + \lambda_3 \underbrace{\left(x_3' + \frac{b_3'}{\lambda_3}\right)}_{x_3''}^2 + c' = 0. \qquad x_1'' = x_1'$$

$\tau : X' = X'' + X_0$.

$\tau(\theta(\Gamma)) : \lambda_2 x_2''^2 + \lambda_3 x_3''^2 + c' = 0$.

ϕ, C_e, C_h, dreaptă dublă, drepte concurente.

b_2) $\lambda_2 = 0$, $b_1' \neq 0$, $\lambda_3 \neq 0$.

$$\underline{\lambda_3 x_3'^2} + 2b_1' x_1' + 2b_2' x_2' + \overline{2b_3' x_3'} + c = 0.$$

$$\lambda_3 \underbrace{\left(x_3' + \frac{b_3'}{\lambda_3}\right)}_{x_3''}^2 + 2b_1' x_1' + 2b_2' x_2' + c' = 0$$

$$x_2'' = x_2', \quad x_1'' = x_1'$$

$\tau : X' = X'' + X_0$.

$\tau(\theta(\Gamma)) : \lambda_3 x_3''^2 + 2b_1' x_1'' + 2b_2' x_2'' + c' = 0$.

$\nu : X'' = \tilde{R} X'''$

$$R = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{cases} x_1'' = \cos\alpha\, x_1'' - \sin\alpha\, x_2'' \\ x_2''' = -\sin\alpha\, x_1'' + \cos\alpha\, x_2'' \\ x_3''' = x_3'' \end{cases}$$

$$\lambda_3 x_3'''^2 + 2b_1'(\cos\alpha\, \underline{x_1''} - \sin\alpha\, \underline{\underline{x_2''}}) + 2b_2'(-\sin\alpha\, \underline{x_1''} + \cos\alpha\, \underline{\underline{x_2''}}) + c = 0$$

$$2x_1''\underbrace{(b_1'\cos\alpha - b_2'\sin\alpha)}_{=0} + 2x_2''(-b_1'\sin\alpha + b_2'\cos\alpha)$$

$\lambda_3 x_3'''^2 + 2x_2'' b_1'' = 0$. Cil $\mathcal{P}$.

$b_3)$ $\lambda_2 = 0$, $b_1' = 0$.

$\underline{\lambda_3 x_3'^2} + 2b_2' x_2' + 2\underline{b_3' x_3'} + c = 0$

$$\lambda_3 \underbrace{\left(x_3' + \frac{b_3'}{\lambda_3}\right)}_{x_3''}^2 + 2b_2' \underbrace{\left(x_2' + \frac{c'}{2b_2'}\right)}_{x_2''} = 0 \quad , \quad x_1'' = x_1$$

$\tau : X' = X'' + X_0$.

$\tau(\theta(\Gamma)) : \lambda_3 x_3''^2 + 2b_2' x_2'' = 0$ Cil $\mathcal{P}$.