1) A = inel com, $a \in A$; $\dfrac{A[X]}{(X-a)A[X]} \simeq A$, izom de inele

Sol: $A[X] \xrightarrow{\varphi} A$, $\varphi(f) = \tilde{f}(a)$

$\left(f = \sum_{i=0}^{n} a_i X^i \in A[X],\ \tilde{f}(a) = \sum_{i=0}^{n} a_i a^i \in A\right)$

φ = morf unitar de inele: $\varphi(f+g) = \varphi(f) + \varphi(g)$

$\varphi(fg) = \varphi(f)\varphi(g)$

$\varphi(1) = 1$

φ = surj : $\forall\ x \in A$, $\exists\ f \in A[X]$ a.î. $\tilde{f}(a) = x$.

(orice $f = (X-a)q + x$, $q \in A[X]$, este bun)

$\tilde{f}(a) = x$

$\operatorname{Ker}\varphi = (X-a)A[X]$

„$\supseteq$" $f \in (X-a)A[X] \Rightarrow \exists\ g \in A[X]$ a.î. $f = (X-a)g$

$\Rightarrow \varphi(f) = \tilde{f}(a) = 0 \cdot \tilde{g}(a) = 0 \Rightarrow f \in \operatorname{Ker}\varphi$.

„$\subseteq$" fie $f \in \operatorname{Ker}\varphi \Leftrightarrow \varphi(f) = 0 \Leftrightarrow \tilde{f}(a) = 0$

$X-a$ monic $\xRightarrow[\text{rest}]{\text{t. împ cu}}$ $\exists\ q, r \in A[X]$ a.î. $f = (X-a)q + r$

cu $\operatorname{grad}(r) < \operatorname{grad}(X-a) = 1$

Avem $r \in A$ și $\tilde{f}(a) = r \in A$. Cum $\tilde{f}(a) = 0 \Rightarrow r = 0 \Rightarrow$

$\Rightarrow f = (X-a)q$ cu $q \in A[X] \Rightarrow f \in (X-a)A[X]$.

T.F.I pt inele $\Rightarrow \dfrac{A[X]}{\operatorname{Ker}\varphi} \simeq \operatorname{Im}\varphi$, izom de inele unitare

$\dfrac{A[X]}{\operatorname{Ker}\varphi} = \dfrac{A[X]}{(X-a)}$, $\operatorname{Im}\varphi = A$

2) $\dfrac{\mathbb{R}[X]}{(X^2-1)\mathbb{R}[X]} \simeq \mathbb{R}\times\mathbb{R}$, izom de inele unitare

Sol: $\mathbb{R}[X] \xrightarrow{\varphi} \mathbb{R}\times\mathbb{R}$

$\varphi(f) = (\tilde{f}(1), \tilde{f}(-1))$

φ - morf de inele unitare $\varphi(f+g) = (\widetilde{f+g}(1), \widetilde{f+g}(-1))$

$= (\tilde{f}(1)+\tilde{g}(1), \tilde{f}(-1)+\tilde{g}(-1))$

$= (\tilde{f}(1), \tilde{f}(-1)) + (\tilde{g}(1), \tilde{g}(-1))$

$= \varphi(f) + \varphi(g)$ etc.

φ = surj: $\forall (a,b)\in\mathbb{R}\times\mathbb{R}$, $\exists f\in\mathbb{R}[X]$ a.î. $\varphi(f) = (a,b)$

$\Leftrightarrow \tilde{f}(1) = a$, $\tilde{f}(-1) = b$.

Caut $\tilde{f}$ de forma $mX+n$ cu $m,n\in\mathbb{R}$, a.î. $\begin{cases} m+n=a \\ -m+n=b \end{cases}$

$\Rightarrow \tilde{f} = \dfrac{a-b}{2}X + \dfrac{a+b}{2} \in \mathbb{R}[X]$. ✓

(orice pol de forma $(X^2-1)\underset{\in\,\mathbb{R}[X]}{q} + \dfrac{a-b}{2}X + \dfrac{a+b}{2}$ e bun).

$\operatorname{Ker}\varphi = (X^2-1)\mathbb{R}[X]$ #

„$\supseteq$" $f\in(X^2-1)\mathbb{R}[X] \Rightarrow f = (X^2-1)g$, $g\in\mathbb{R}[X] \Rightarrow$

$\Rightarrow \tilde{f}(1) = 0 = \tilde{f}(-1) \Rightarrow \varphi(f) = (0,0)\in\mathbb{R}\times\mathbb{R} \Rightarrow f\in\operatorname{Ker}\varphi$.

„$\subseteq$" $f\in\operatorname{Ker}\varphi \Leftrightarrow \varphi(f) = (0,0) \Leftrightarrow (\tilde{f}(1), \tilde{f}(-1)) = (0,0)$

T. împ cu rest în $\mathbb{R}[X]$: $f = (X^2-1)q + r$ cu $q, r\in\mathbb{R}[X]$, grad $r < 2$

$\Rightarrow f = (X^2-1)q + uX + v$ cu $u,v\in\mathbb{R}$

$\Rightarrow \begin{cases} \tilde{f}(1) = u+v \\ \tilde{f}(-1) = -u+v \end{cases} \Rightarrow \begin{cases} u+v = 0 \\ -u+v = 0 \end{cases} \Rightarrow u = v = 0$

$\Rightarrow f = (X^2-1)q$ cu $q\in\mathbb{R}[X] \Rightarrow f\in(X^2-1)\mathbb{R}[X]$.

TFI pt inele: $\underset{\| \; \mathbb{R}[X]/(X^2-1)\mathbb{R}[X]}{\dfrac{\mathbb{R}[X]}{\operatorname{Ker}\varphi}} \simeq \underset{\| \; \mathbb{R}\times\mathbb{R}}{\operatorname{Im}\varphi}$, izom de inele unitare ✓.

Lema chineză a resturilor (pentru inele)

A = inel comutativ unitar; $I, J \subseteq A$ ideale a.î. $\boxed{I+J=A}$ $\Leftrightarrow$

ideal

$\Leftrightarrow 1 \in I+J \Leftrightarrow \exists\, i \in I, j \in J$ a.î. $1 = i+j$.

$\left(I+J = \{i+j \mid i \in I, j \in J\} \subseteq A \text{ ideal}\right)$.

Atunci $A/_{I \cap J} \simeq A/_I \times A/_J$, izom de inele unitare.

Dem: $\varphi : A \longrightarrow A/_I \times A/_J$, $\varphi(a) = (\hat{a}, \bar{a}) \in A/_I \times A/_J$

φ = morf de inele unitare.

$\operatorname{Ker}\varphi = \{a \in A \mid \varphi(a) = (\hat{0}, \bar{0})\} = \left\{a \in A \;\middle|\; \begin{array}{l}\hat{a} = \hat{0} \\ \bar{a} = \bar{0}\end{array}\right\} =$

$= \{a \in A \mid a \in I, a \in J\} = I \cap J$

φ = surj: $\forall\, (\hat{b}, \bar{c}) \in A/_I \times A/_J$, $\exists\, a \in A$ a.î. $\varphi(a) =$

$= (\hat{b}, \bar{c}) \Leftrightarrow \hat{a} = \hat{b}$ și $\bar{a} = \bar{c}$.

Iau $a = jb + ic$ (cu $i \in I, j \in J$ a.î. $i+j=1$).

$i+j=1 \Rightarrow \widehat{i+j} = \hat{1}$ în $A/_I \Rightarrow \hat{j} = \hat{1}$

$\overline{i+j} = \bar{1}$ în $A/_J \Rightarrow \bar{i} = \bar{1}$ $\Rightarrow$

$\Rightarrow \hat{a} = \widehat{jb+ic} = \underbrace{\hat{j}}_{=\hat{1}}\, \hat{b} + \underbrace{\hat{i}}_{=\hat{0}}\, \hat{c} = \hat{1} \cdot \hat{b} + \hat{0} \cdot \hat{c} = \hat{b}$

$\bar{a} = \overline{jb+ic} = \underbrace{\bar{j}}_{=\bar{0}}\, \bar{b} + \underbrace{\bar{i}}_{=\bar{1}} \cdot \bar{c} = \bar{0} \cdot \bar{b} + \bar{1} \cdot \bar{c} = \bar{c}$ ✓

Aplic TFI pentru inele $\Rightarrow \underbrace{A/_{\operatorname{Ker}\varphi}}_{= A/_{I \cap J}} \simeq \underbrace{\operatorname{Im}\varphi}_{= A/_I \times A/_J}$, izom unitar de inele

Dacă $(m,n)=1$ în $\mathbb{Z}$ atunci $\begin{cases} x \equiv a \pmod{m} \\ x \equiv b \pmod{n} \end{cases}$ are sol în $\mathbb{Z}$

$\uparrow$

Lema chineză a resturilor (varianta clasică)

$\Leftrightarrow$ $\left.\begin{array}{l} \mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}_m \times \mathbb{Z}_n \\ \quad x \longmapsto (\hat{x}, \bar{x}) \end{array}\right)$ este surjectivă, dacă $(m,n)=1$

φ = morf de inele cu $\operatorname{Ker}\varphi = m\mathbb{Z} \cap n\mathbb{Z} = [m,n]\mathbb{Z}$

$= mn\mathbb{Z}$

TFI pt inele $\Rightarrow$ $\mathbb{Z}/mn\mathbb{Z} \simeq \mathbb{Z}_m \times \mathbb{Z}_n$

$\parallel$

$\mathbb{Z}_{mn}$

$I = m\mathbb{Z}$, $J = n\mathbb{Z}$

$(m,n)=1 \Rightarrow I+J = m\mathbb{Z}+n\mathbb{Z} = (m,n)\mathbb{Z} = \mathbb{Z}$

$I \cap J = m\mathbb{Z} \cap n\mathbb{Z} = [m,n]\mathbb{Z} = mn\mathbb{Z}$

$A/I\cap J \simeq A/I \times A/J = \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$

$\mathbb{Z}/mn\mathbb{Z} = \mathbb{Z}_{mn}$ $\qquad\qquad \mathbb{Z}_m \times \mathbb{Z}_n$

Obs: 1) $\mathbb{R}[X]/(X^2-1)\mathbb{R}[X]$

$I=(X-1)\mathbb{R}[X]$, $J=(X+1)\mathbb{R}[X]$

$I+J=(X-1,X+1)\underset{\substack{\Uparrow \\ \text{corp}}}{\mathbb{R}}[X]=\underline{\underline{\mathbb{R}[X]}}$

$I\cap J=(X-1)\mathbb{R}[X]\cap(X+1)\mathbb{R}[X]=[X-1,X+1]\mathbb{R}[X]$

$=(X-1)(X+1)\mathbb{R}[X]=(X^2-1)\mathbb{R}[X]$

Lema chinezească $\Rightarrow \mathbb{R}[X]/(X^2-1)\mathbb{R}[X]\simeq \mathbb{R}[X]/(X-1)\mathbb{R}[X]\times \mathbb{R}[X]/(X+1)\mathbb{R}[X]$

$\overset{\text{problema 1)}}{\simeq}\mathbb{R}\times\mathbb{R}$

2) La fel ca la 1), se arată că $\mathbb{Q}[X]/(X^2-1)\mathbb{Q}[X]\simeq\mathbb{Q}\times\mathbb{Q}$

(direct: se aplică TFI lui $\varphi:\mathbb{Q}[X]\to\mathbb{Q}\times\mathbb{Q}$)

$\varphi(f)=(\tilde{f}(1),\tilde{f}(-1))$

Lema chinezească: $(X-1)\mathbb{Q}[X]+(X+1)\mathbb{Q}[X]=\mathbb{Q}[X]$

$$1=(X-1)\cdot\underset{\substack{\cap \\ \mathbb{Q}[X]}}{\left(-\frac{1}{2}\right)}+(X+1)\cdot\underset{\substack{\cap \\ \mathbb{Q}[X]}}{\frac{1}{2}}$$

$\mathbb{C}[X]/(X^2-1)\mathbb{C}[X]\underset{\Uparrow}{\simeq}\mathbb{C}\times\mathbb{C}.$

$(X-1)\mathbb{C}[X]+(X+1)\mathbb{C}[X]=\overset{\text{corp}}{\mathbb{C}}[X]$

3) $\mathbb{Z}[X]/(X^2-1)\mathbb{Z}[X]\not\simeq\mathbb{Z}\times\mathbb{Z}$

$(X-1)\mathbb{Z}[X]+(X+1)\underset{\substack{\uparrow \\ \text{nu e corp}}}{\mathbb{Z}}[X]\neq\mathbb{Z}[X]$

$\mathrm{Idemp}(\mathbb{Z}\times\mathbb{Z})=\{(a,b)^2=(a,b)\mid a,b\in\mathbb{Z}\}$

$=\{(a^2,b^2)=(a,b)\mid a,b\in\{0,1\}\}$

$=\{(0,0),(1,1),(0,1),(1,0)\}.$

$\hat{f} \in \mathbb{Z}[X]/(X^2-1)\mathbb{Z}[X]$, $f \in \mathbb{Z}[X]$.

X^2-1 monic $\xRightarrow[\text{rest}]{\text{t. împ. cu}} \exists q, r \in \mathbb{Z}[X]$ a.î. $f = (X^2-1)q + r$, $\operatorname{grad}(r) < 2$ $\Rightarrow$

$\Rightarrow \exists q \in \mathbb{Z}[X]$ și $a, b \in \mathbb{Z}$ a.î. $f = (X^2-1)q + aX + b$

$\Rightarrow \hat{f} = \underbrace{\widehat{(X^2-1)q}}_{=\hat{0}} + \widehat{aX+b} = \widehat{aX+b} \Rightarrow$

$\Rightarrow \mathbb{Z}[X]/(X^2-1)\mathbb{Z}[X] = \{\widehat{aX+b},\ a, b \in \mathbb{Z}\}$.

$\widehat{aX+b} \in \operatorname{Idemp}\left(\mathbb{Z}[X]/(X^2-1)\mathbb{Z}[X]\right) \Leftrightarrow \widehat{aX+b}^2 = \widehat{aX+b}$

$\Leftrightarrow \widehat{a^2X^2 + 2abX + b^2} = \widehat{aX+b}$

$\widehat{a^2X^2 + 2abX + b^2} = \widehat{a^2(X^2-1) + 2abX + a^2 + b^2} = \widehat{2ab\,X + a^2 + b^2}$

$\Leftrightarrow \underbrace{(2ab - a)X + a^2 + b^2 - b}_{\text{de grad } 1} \in (X^2-1)\mathbb{Z}[X]$

$\Rightarrow \begin{cases} 2ab = a \\ a^2 + b^2 = b \end{cases} \xRightarrow{a, b \in \mathbb{Z}} \begin{cases} a = 0 \\ b \in \{0, 1\} \end{cases} \Rightarrow \operatorname{Idemp}\left(\mathbb{Z}[X]/(X^2-1)\mathbb{Z}[X]\right) = \{\hat{0}, \hat{1}\}$.

Dar $|\operatorname{Idemp}(\mathbb{Z}\times\mathbb{Z})| = 4 \Rightarrow \mathbb{Z}[X]/(X^2-1)\mathbb{Z}[X] \not\simeq \mathbb{Z}\times\mathbb{Z}$

3) $I \subseteq J$ ideale în A, atunci $J/I \overset{\text{not}}{=} \{\hat{x} \mid x \in J\}$ este ideal în A/I și $(A/I)/(J/I) \simeq A/J$ izom. de inele unitare

(A inel com. unitar)

Sol: $I \subseteq J \Rightarrow A/I \xrightarrow{\varphi} A/J$, $\varphi(\hat{a}) = \bar{a}$

este bine def. și morf. surj. de inele unitare

(bine def.: $\hat{a} = \hat{b} \Rightarrow \bar{a} = \bar{b}$)

$a - b \in I \subseteq J \Rightarrow a - b \in J$

$\operatorname{Ker}\varphi = \{\hat{a} \mid \varphi(\hat{a}) = \bar{0}\} = \{\hat{a} \mid \bar{a} = \bar{0}\} = \{\hat{a} \mid a \in J\} = J/I$.

TFI pt. inele $\Rightarrow (A/I)/(J/I) \simeq \operatorname{Im}\varphi = A/J$ ✓.

Exp: $\dfrac{\mathbb{Z}[X]}{(3, X^3 - X^2 + 2X + 1)} \simeq \dfrac{\mathbb{Z}_3[X]}{(X^3 - X^2 + \hat{2}X + \hat{1})}$, pt că:

$$\frac{\mathbb{Z}[X]}{(3, X^3 - X^2 + 2X+1)} \simeq \frac{\underbrace{\mathbb{Z}[X]}_{A} / \underbrace{3\mathbb{Z}[X]}_{I}}{\underbrace{(3, X^3 - X^2 + 2X+1)}_{J} / \underbrace{3\mathbb{Z}[X]}_{I}} \quad (*) \quad (I \subseteq J \text{ ideale în } \mathbb{Z}[X])$$

În general $I \subseteq R$ ideal într-un inel com. unitar $\Rightarrow$

$$\Rightarrow R[X] / IR[X] \simeq (R/I)[X]$$

$\Uparrow$ TFI pt inele

$$R[X] \xrightarrow{\varphi} (R/I)[X]$$

$$\varphi\left(f = \sum_{i=0}^{n} a_i X^i\right) = \hat{f} \left(\overset{def}{=} \sum_{i=0}^{n} \hat{a}_i X^i \right)$$

$$IR[X] = \left\{ g = \sum_{j=0}^{m} b_j X^j \;\middle|\; b_j \in I \;\; \forall j = \overline{0,m} \right\}$$

Atunci $(*) \simeq \dfrac{\mathbb{Z}/3\mathbb{Z}\,[X]}{(\underset{\underset{\hat{0}}{\shortparallel}}{\hat{3}}, X^3 - X^2 + \hat{2}X + \hat{1})} = \dfrac{\mathbb{Z}_3[X]}{(X^3 - X^2 + \hat{2}X + \hat{1})}$

La fel:

$$\frac{\mathbb{Z}[X]}{(2, X-1)} = \frac{\mathbb{Z}[X] / 2\mathbb{Z}[X]}{(2, X-1) / 2\mathbb{Z}[X]} \simeq \frac{\mathbb{Z}/2\mathbb{Z}\,[X]}{(\hat{2}, X - \hat{1})} = \frac{\overbrace{\mathbb{Z}_2[X]}^{A}}{(X - \hat{1})\mathbb{Z}_2[X]}$$

probl 1) $\to$ IS^a $\mathbb{Z}_2$