Seminar 7

1. Studiați convergența simplă și uniformă pentru următoarele șiruri de funcții:

a) $f_n : [0,\infty) \to \mathbb{R}$, $f(x) = \frac{x}{x+n}$ $\forall n \in \mathbb{N}^*$.

<u>Sol.</u>: <u>Convergența simplă</u>

Fie $x \in [0,\infty)$.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{x+n} = 0 \Rightarrow f_n \xrightarrow[n\to\infty]{s} f, \text{ unde}$$

$f : [0,\infty) \to \mathbb{R}$, $f(x) = 0$.

<u>Convergența uniformă</u>

$$\sup_{x\in[0,\infty)} |f_n(x) - f(x)| = \sup_{x\in[0,\infty)} \left| \frac{x}{x+n} - 0 \right| = \sup_{x\in[0,\infty)} \frac{x}{x+n} \underset{x=n}{\geq}$$

$$\underset{x=n}{\geq} \frac{n}{n+n} = \underset{\substack{\vee \\ 0}}{\frac{1}{2}} \not\xrightarrow[n\to\infty]{} 0 \Rightarrow f_n \not\xrightarrow[n\to\infty]{u} f. \quad \square$$

b) $f_n : [2,3] \to \mathbb{R}$, $f_n(x) = \frac{x}{x+n}$ $\forall n \in \mathbb{N}$.

<u>Sol.</u>: <u>Convergența simplă</u>

Fie $x \in [2,3]$.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{x+n} = 0 \Rightarrow f_n \xrightarrow[n\to\infty]{s} f, \text{ unde } f:[2,3]\to\mathbb{R},\ f(x)=0.$$

Convergența uniformă

$$\sup_{x\in[2,3]} |f_n(x)-f(x)| = \sup_{x\in[2,3]} \left|\frac{x}{x+n}-0\right| = \sup_{x\in[2,3]} \frac{x}{x+n}.$$

Fie $f_n:[2,3]\to\mathbb{R}$, $f_n(x)=\dfrac{x}{x+n}$.

$$f_n'(x)=\frac{\cancel{x}+n-\cancel{x}}{(x+n)^2} = \frac{n}{(x+n)^2} \geq 0 \ \forall\, x\in[2,3],\ \forall\, n\in\mathbb{N}.$$

Deci f_n este crescătoare $\forall\, n\in\mathbb{N}$.

x	2		3
$f_n'(x)$	$+$	$+\ +\ +\ +\ +\ +\ +$	$+$
$f_n(x)$	$\frac{2}{2+n}$	$\nearrow$	$\frac{3}{3+n}$

Deci $\displaystyle\sup_{x\in[2,3]} |f_n(x)-f(x)| = \frac{3}{3+n} \xrightarrow[n\to\infty]{} 0.$

Așadar $f_n \xrightarrow[n\to\infty]{u} f$. $\square$

c) $f_n:[0,\infty)\to\mathbb{R}$, $f_n(x)=\sqrt{x^2+\frac{1}{n}}$ $\forall\, n\in\mathbb{N}^*$.

<u>Sol.</u>: <u>Convergența simplă</u>

Fie $x\in[0,\infty)$.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\sqrt{x^2+\frac{1}{n}} = |x| = x \Rightarrow f_n \xrightarrow[n\to\infty]{s} f, \text{ unde}$$

$f:[0,\infty)\to\mathbb{R},\ f(x)=x.$

<u>Convergența uniformă</u>

$$\sup_{x\in[0,\infty)}|f_n(x)-f(x)|=\sup_{x\in[0,\infty)}\left|\sqrt{x^2+\frac{1}{n}}-x\right|=$$

$$=\sup_{x\in[0,\infty)}\left(\sqrt{x^2+\frac{1}{n}}-x\right)=\sup_{x\in[0,\infty)}\frac{\cancel{x^2}+\frac{1}{n}-\cancel{x^2}}{\sqrt{x^2+\frac{1}{n}}+x}=$$

$$=\sup_{x\in[0,\infty)}\frac{\frac{1}{n}}{\sqrt{x^2+\frac{1}{n}}+x}=\frac{\frac{1}{n}}{\sqrt{0^2+\frac{1}{n}}+0}=\frac{\frac{1}{n}}{\sqrt{\frac{1}{n}}}=\sqrt{\frac{1}{n}}\xrightarrow[n\to\infty]{}0.$$

Deci $f_n\xrightarrow[n\to\infty]{u}f$. □

d) $f_n:[0,\infty)\to\mathbb{R},\ f_n(x)=\dfrac{n}{n+x}\quad\forall n\in\mathbb{N}^*.$

<u>Sol.</u>: <u>Conv. simplă</u>

Fie $x\in[0,\infty)$.

$$\lim_{n\to\infty}f_n(x)=\lim_{n\to\infty}\frac{n}{n+x}=1\Rightarrow f_n\xrightarrow[n\to\infty]{s}f,\text{ unde}$$

$f:[0,\infty)\to\mathbb{R},\ f(x)=1.$

$$\sup_{x\in[0,\infty)} |f_n(x)-f(x)| = \sup_{x\in[0,\infty)} \left|\frac{n}{n+x} - 1\right| = \sup_{x\in[0,\infty)} \left|\frac{\cancel{n}-\cancel{n}-x}{n+x}\right| =$$

$$= \sup_{x\in[0,\infty)} \left|-\frac{x}{n+x}\right| = \sup_{x\in[0,\infty)} \frac{x}{n+x}.$$

Fie $g_n:[0,\infty)\to\mathbb{R}$, $g_n(x)=\frac{x}{n+x}$ $\forall n\in\mathbb{N}^*$.

$$g_n'(x) = \frac{\cancel{x}+n-\cancel{x}}{(x+n)^2} = \frac{n}{(x+n)^2} > 0 \quad \forall x\in[0,\infty), \forall n\in\mathbb{N}^*.$$

x	0		∞
$g_n'(x)$	$+$	$+ + + + + + +$	$+$
$g_n(x)$	0	$\nearrow$	1

Deci $\sup_{x\in[0,\infty)} \frac{x}{x+n} = \lim_{x\to\infty} \frac{x}{x+n} = 1 \not\xrightarrow[n\to\infty]{} 0.$

Așadar $f_n \not\xrightarrow[n\to\infty]{u} f$. $\square$

e) $f_n:(0,1]\to\mathbb{R}$, $f_n(x)=x^n$.

<u>Sol.</u>: <u>C. S.</u>

Fie $x\in(0,1]$.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} 0 \ ; \ x\in(0,1) \\ 1 \ ; \ x=1 \end{cases} \Rightarrow f_n \xrightarrow[n\to\infty]{s} f,$$

unde $f:(0,1]\to\mathbb{R}$, $f(x)=\begin{cases} 0 \ ; \ x\in(0,1) \\ 1 \ ; \ x=1. \end{cases}$

<u>C.U.</u>

$$\left.\begin{array}{l} f_n \text{ continuă } \forall n\in\mathbb{N} \\ f \text{ nu e continuă (în 1)} \end{array}\right\} \Rightarrow f_n \overset{u}{\underset{n\to\infty}{\not\longrightarrow}} f. \quad \square$$

f) $f_n:\left[\frac{1}{2},1\right]\to\mathbb{R}$, $f_n(x)=\dfrac{(1+x)^n}{e^{2nx}}$ $\forall n\in\mathbb{N}^*$.

<u>Sol</u>: <u>C.S.</u>

Fie $x\in\left[\frac{1}{2},1\right]$.

$$f_n(x)=\left(\frac{1+x}{e^{2x}}\right)^n \ \forall n\in\mathbb{N}^*.$$

Fie $g:\left[\frac{1}{2},1\right]\to\mathbb{R}$, $g(x)=1+x-e^{2x}$.

$g'(x)=1-2e^{2x}<0 \ \forall x\in\left[\frac{1}{2},1\right]$.

x	$\frac{1}{2}$			1
$g'(x)$	$-$	$-$	$-$	$-$
$g(x)$	$\frac{3}{2}-e$	$\searrow$	$\searrow$	$2-e^2$

Deci $g(x) < 0 \ \forall x \in \left[\frac{1}{2}, 1\right]$, i.e. $1+x < e^{2x} \ \forall x \in \left[\frac{1}{2}, 1\right]$,

i.e. $0 < \frac{1+x}{e^{2x}} < 1 \ \forall x \in \left[\frac{1}{2}, 1\right]$.

Avem $f_n(x) = \underbrace{\left(f_1(x)\right)^n}_{= \left(\frac{1+x}{e^{2x}}\right)^n} \xrightarrow[n\to\infty]{} 0$. Deci

$f_n \xrightarrow[n\to\infty]{s} f$, unde $f : \left[\frac{1}{2}, 1\right] \to \mathbb{R}$, $f(x) = 0$.

<u>C.U.</u>

Avem: 1) $\left[\frac{1}{2}, 1\right]$ mulțime compactă.

2) f_n continuă $\forall n \in \mathbb{N}^*$, f continuă.

3) $0 < \frac{x+1}{e^{2x}} < 1 \ \forall x \in \left[\frac{1}{2}, 1\right] \Rightarrow \underbrace{\left(\frac{x+1}{e^{2x}}\right)^n}_{= f_n(x)} > \underbrace{\left(\frac{x+1}{e^{2x}}\right)^{n+1}}_{= f_{n+1}(x)} \ \forall n \in \mathbb{N}^* \ \forall x \in \left[\frac{1}{2}, 1\right] \Rightarrow$

$\Rightarrow (f_n)_n$ (strict) descrescător.

4) $f_n \xrightarrow[n\to\infty]{s} f$.

Conform Teoremei lui Dini rezultă că $f_n \xrightarrow[n\to\infty]{u} f$. $\square$

g) $f_n : \left[\frac{1}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$, $f_n(x) = \cos^n x$ $\forall n \in \mathbb{N}^*$.

Sol.: C.S.

Fie $x \in \left[\frac{1}{2}, \frac{\pi}{2}\right]$.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \cos^n x \underset{\uparrow}{=} 0 \Rightarrow f_n \xrightarrow[n\to\infty]{s} f, \text{ unde}$$

$\cos x \in [0, 1)$ $\forall x \in \left[\frac{1}{2}, \frac{\pi}{2}\right]$

$f : \left[\frac{1}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$, $f(x) = 0$.

C.U.

$\underset{\left[\frac{1}{2}, \frac{\pi}{2}\right]}{x} \longmapsto \cos x$ este descrescătoare $\Rightarrow f_n$ descrescătoare $\forall n \in \mathbb{N}^*$.

Avem: 1) $f_n : \left[\frac{1}{2}, 1\right] \to \mathbb{R}$, $f_n(x) = \cos^n x$ $\forall n \in \mathbb{N}^*$.

2) f_n descrescătoare $\forall n \in \mathbb{N}^*$.

3) $f_n \xrightarrow[n\to\infty]{s} f$.

4) f continuă.

Conform Teoremei lui Polya rezultă că $f_n \xrightarrow[n\to\infty]{u} f$. □

2. Studiați convergența simplă și uniformă pentru $(f_n)_n$ și $(f_n')_n$, unde:

a) $f_n : [0,\pi] \to \mathbb{R}$, $f_n(x) = \frac{\cos nx}{n}$ $\forall n \in \mathbb{N}^*$.

Sol.: Pentru $(f_n)_n$

C.S.

Fie $x \in [0,\pi]$.

$$-\frac{1}{n} \le \frac{\cos nx}{n} \le \frac{1}{n} \quad \forall x \in [0,\pi], \forall n \in \mathbb{N}^*.$$

$$\to 0$$

Deci $\lim\limits_{n\to\infty} f_n(x) = 0$. Așadar $f_n \xrightarrow[n\to\infty]{s} f$, unde

$f : [0,\pi] \to \mathbb{R}$, $f(x) = 0$.

C.U.

$$\sup_{x\in[0,\pi]} |f_n(x) - f(x)| = \sup_{x\in[0,\pi]} \left| \frac{\cos nx}{n} - 0 \right| =$$

$$= \sup_{x\in[0,\pi]} \frac{|\cos nx|}{n} \le \frac{1}{n} \xrightarrow[n\to\infty]{} 0 \Rightarrow f_n \xrightarrow[n\to\infty]{u} f.$$

Pentru $(f_n')_n$

$$f_n'(x) = \frac{1}{\cancel{n}} \cdot (-\sin nx) \cdot \cancel{n} = -\sin(nx) \quad \forall x \in [0,\pi], \forall n \in \mathbb{N}^*.$$

C.S.

Alegem $x = \frac{\pi}{2} \in [0, \pi]$.

Arătăm că $(f_n'(\frac{\pi}{2}))_n$ nu este conv.

$$f_{4n}'\left(\frac{\pi}{2}\right) = -\sin\left(4n \cdot \frac{\pi}{2}\right) = 0 \xrightarrow[n\to\infty]{} 0.$$

$$f_{4n+1}'\left(\frac{\pi}{2}\right) = -\sin\left(4n\frac{\pi}{2} + \frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1 \xrightarrow[n\to\infty]{} -1.$$

Deci $\nexists \lim\limits_{n\to\infty} f_n'(\frac{\pi}{2})$.

Așadar $(f_n')_n$ nu este simplu convergent.

C.U.

Deoarece $(f_n')_n$ nu este simplu convergent rezultă că $(f_n')_n$ nu este uniform convergent. □

b) $f_n : \mathbb{R} \longrightarrow \mathbb{R}$, $f_n(x) = \frac{\operatorname{arctg} nx}{n}$ $\forall n \in \mathbb{N}^*$.

Sol.: Pentru $(f_n)_n$

C.S.

Fie $x \in \mathbb{R}$.

$$-\frac{\frac{\pi}{2}}{n} \le \frac{\operatorname{arctg} nx}{n} \le \frac{\frac{\pi}{2}}{n} \quad \forall n \in \mathbb{N}^*.$$

$$\searrow \quad \downarrow \quad \swarrow$$

$$0$$

Deci $\lim\limits_{n\to\infty} f_n(x) = 0$. Așadar $f_n \xrightarrow[n\to\infty]{s} f$, unde

$f:\mathbb{R}\to\mathbb{R}$, $f(x)=0$.

C.U.

$$\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \sup_{x\in\mathbb{R}} \left| \frac{\operatorname{arctg} nx}{n} - 0 \right| =$$

$$= \sup_{x\in\mathbb{R}} \frac{|\operatorname{arctg} nx|}{n} \le \frac{\frac{\pi}{2}}{n} = \frac{\pi}{2n} \xrightarrow[n\to\infty]{} 0 \Rightarrow$$

$$\Rightarrow f_n \xrightarrow[n\to\infty]{u} f. \ \square$$

Pentru $(f_n')_n$.

$$f_n'(x) = \frac{1}{\cancel{n}} \cdot \frac{1}{1+n^2x^2} \cdot \cancel{n} = \frac{1}{1+n^2x^2} \quad \forall x\in\mathbb{R}, \forall n\in\mathbb{N}^*.$$

C.S.

Fie $x\in\mathbb{R}$.

$$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \frac{1}{1+n^2x^2} = \begin{cases} 1; & x=0 \\ 0; & x\neq 0 \end{cases} \Rightarrow$$

$$\Rightarrow f_n' \xrightarrow[n\to\infty]{s} g, \text{ unde } g:\mathbb{R}\to\mathbb{R},\ g(x)=\begin{cases}1; & x=0\\ 0; & x\neq 0.\end{cases}$$

<u>C.M.</u>

$$\left.\begin{array}{l} f_n' \text{ continuă } \forall n\in\mathbb{N}^* \\ g \text{ nu e continuă (în } 0) \end{array}\right| \Rightarrow f_n' \not\xrightarrow[n\to\infty]{u} g. \ \square$$