$\boxed{C_{12} - GAL}$ 13.05.2022.

Hipercuadrice. Formă canonică

Conice. Locuri geometrice

Forma canonică pt conice cu centru unic ($\delta \neq 0$)

Def Fie $(\mathbb{R}^n, \mathbb{R}^n/\mathbb{R}, \varphi)$, resp $(\mathbb{R}^n, (\mathbb{R}^n, g_0), \varphi)$

$\mathcal{R} = \{O; e_1, .., e_n\}$ reper cartezian.

S.n. hipercuadrică în $\mathbb{R}^n$ LG al punctelor $P(x_1, .., x_n)$ care verifică:

$$\Gamma : f(x) = a_{11} x_1^2 + \dots + a_{nn} x_n^2 + 2a_{12} x_1 x_2 + \dots + 2a_{n-1\,n} x_{n-1} x_n + 2b_1 x_1 + \dots + 2b_n x_n + c = 0$$

$$f(x) = X^T A X + 2BX + c = 0, \text{ unde}$$

$A = (a_{ij})_{i,j=\overline{1,n}} = A^T$, $\tilde{A} = \begin{pmatrix} A & B^T \\ B & c \end{pmatrix}$, $B = (b_1 .. b_n)$

$\delta = \det A$, $\Delta = \det \tilde{A}$

$r = \operatorname{rg} A$, $r' = \operatorname{rg} \tilde{A}$, $r \le r'$, $r+2$

$\Delta \neq 0 \implies \Gamma$ hipercuadrică nedegenerată

$\Delta = 0 \implies$ degenerată.

$n = 2$ Γ = conică

$n = 3$ Γ = cuadrică

- $(\mathbb{R}^n, \mathbb{R}^n/\mathbb{R}, \varphi)$

$\Gamma_1 \sim \Gamma_2$ afin echivalente $\iff$ $\tau : \mathbb{R}^n \to \mathbb{R}^n$ transf. afină

$\Gamma_2 = \tau(\Gamma_1)$ $\quad \tau : X' = CX + D$, $C \in GL(n, \mathbb{R})$

- $(\mathbb{R}^n, (\mathbb{R}^n, g_0), \varphi)$

$\Gamma_1 \simeq \Gamma_2$ congruente metric $\Leftrightarrow \exists\, \tau: \mathbb{R}^n \to \mathbb{R}^n$ izometrie

$\tau: X' = CX + D,$
$C \in O(n)$

Invarianți afini: $\frac{\Delta}{\delta}, r', r$

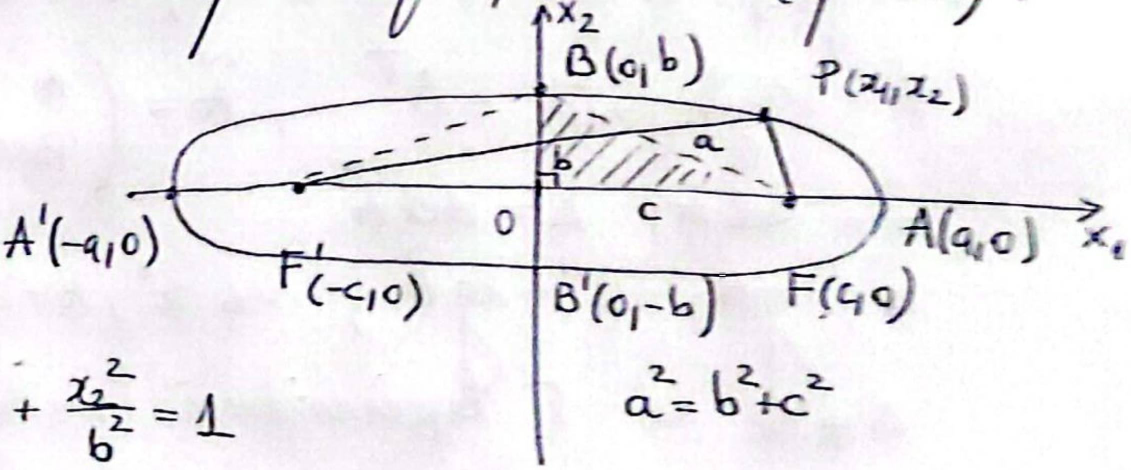
Invarianți metrici, $\frac{\Delta}{\delta}, r', r, \Delta, \delta$

$n = 2.$

$\Gamma: f(x) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 + 2b_1x_1 + 2b_2x_2 + c = 0$

Conice (nedegenerate). L.G.

① Elipsa - LG al punctelor $P(x_1, x_2)$ din plan care verifică: $PF + PF' = 2a$, $a > 0$, unde F, F' = puncte fixe, distincte (focare).



$\mathcal{E}: \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$ $\qquad a^2 = b^2 + c^2$

A, A', B, B' = vârfurile elipsei.

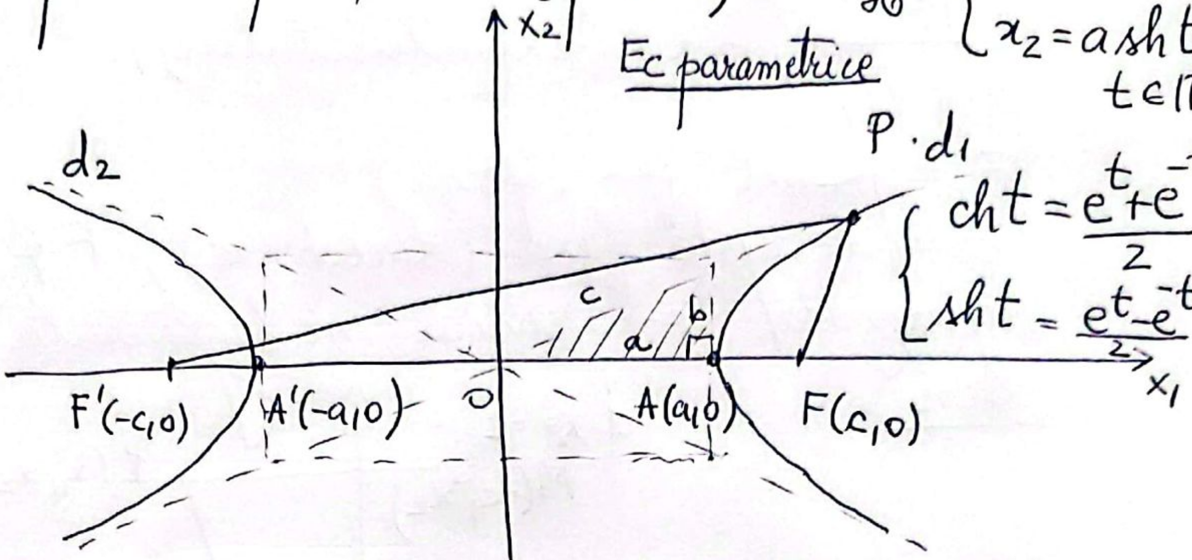
$\mathcal{E}: \begin{cases} x = a\cos t \\ y = b\sin t \end{cases}, t \in \mathbb{R}.$

② <u>Hiperbola</u> = LG al punctelor $P(x_1, x_2)$ din plan care verifică: $|PF - PF'| = 2a,\ a > 0$

F, F' = puncte fixe, dist (focare).

$\mathcal{H}: \begin{cases} x_1 = a\,\mathrm{ch}\,t \\ x_2 = a\,\mathrm{sh}\,t \end{cases}\quad t \in \mathbb{R}$

<u>Ec parametrice</u>



$\begin{cases} \mathrm{ch}\,t = \dfrac{e^t + e^{-t}}{2} \\ \mathrm{sh}\,t = \dfrac{e^t - e^{-t}}{2} \end{cases}$

$\mathcal{H}: \dfrac{x_1^2}{a^2} - \dfrac{x_2^2}{b^2} = 1,\ a > 0,\ b > 0$ $\quad \boxed{c^2 = a^2 + b^2}$

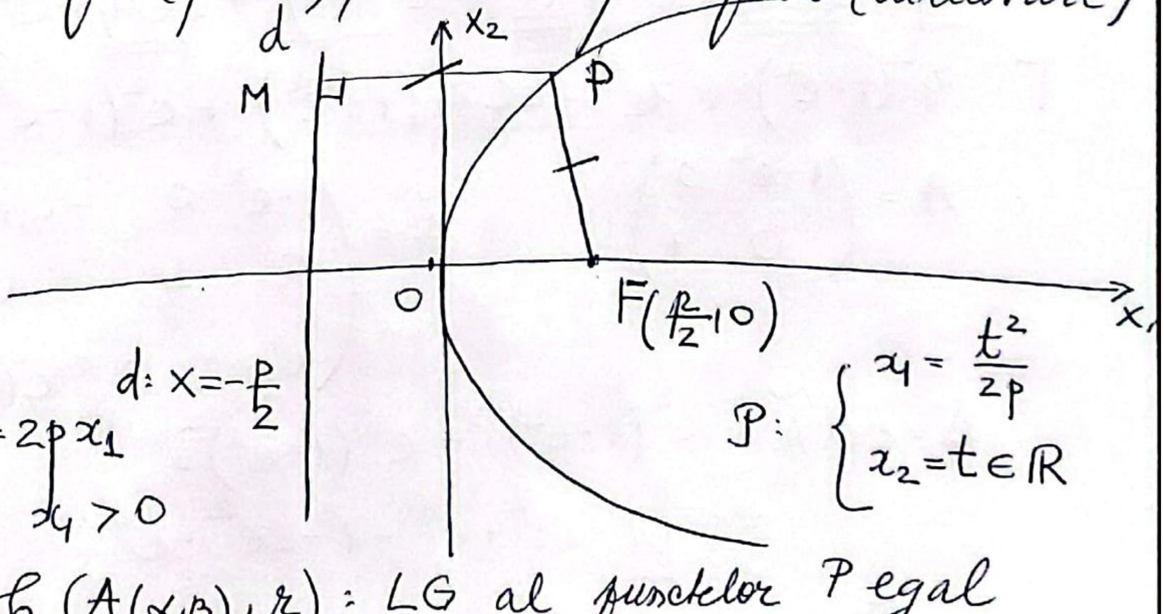
$d_1 \cup d_2: x_2 = \pm \dfrac{b}{a} x_1$ (asimptotele)

A, A' vârfuri

③ <u>Parabola</u> = LG al punctelor $P(x_1, x_2)$ din plan care verifică $\dfrac{PF}{dist(P, d)} = 1$, unde

F punct fix (focar), d = dreaptă fixă (directoare)

$F \notin d$.



$\mathcal{P}: x_2^2 = 2p x_1$

$p > 0,\ x_1 > 0$

$\mathcal{P}: \begin{cases} x_1 = \dfrac{t^2}{2p} \\ x_2 = t \in \mathbb{R} \end{cases}$

<u>OBS</u> $\mathcal{C}(A(\alpha, \beta), r)$: LG al punctelor P egal depărtate de punctul fix A

$AP = r: (x_1 - \alpha)^2 + (x_2 - \beta)^2 = r^2$

<u>Teoremă</u> (caracterizarea unitară a conicelor nedeg.

L.G al punctelor $P(x_1, x_2)$ din plan care verif

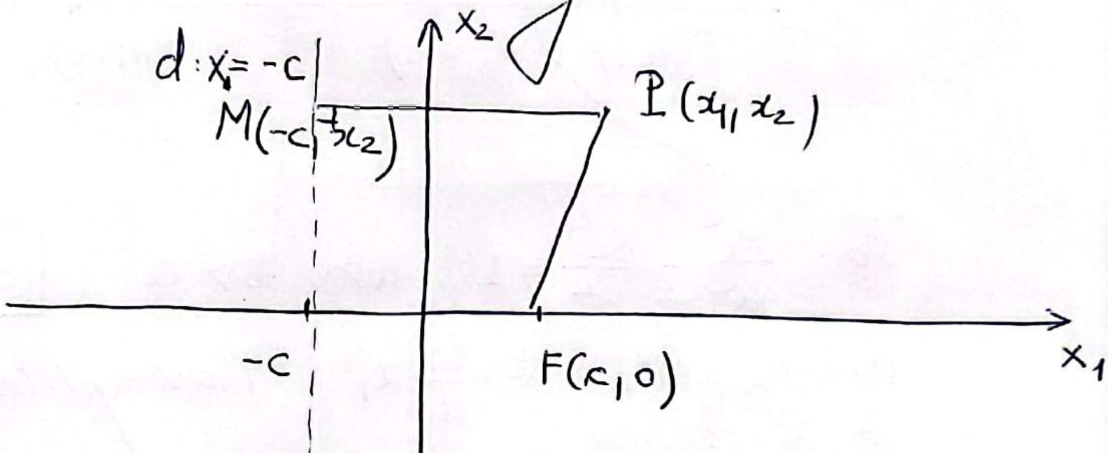
$$\frac{PF}{dist(P,d)} = e \text{ (excentricitatea)},$$

F = punct fix (focar)

d = dreaptă fixă (directoare), $F \notin d$

reprezintă o conică nedegenerată.

<u>Dem</u>



$$\frac{PF}{PM} = e \Rightarrow PF = e\,PM.$$

$$(x_1-c)^2 + x_2^2 = e^2\left[(x_1+c)^2\right]$$

$$x_1^2 - 2cx_1 + c^2 + x_2^2 = e^2\,(x_1^2 + 2cx_1 + c^2)$$

$$\Gamma: x_1^2(1-e^2) + x_2^2 - 2cx_1(1+e^2) + c^2(1-e^2) = 0$$

$$A = \begin{pmatrix} 1-e^2 & 0 \\ 0 & 1 \end{pmatrix} \qquad \tilde{A} = \begin{pmatrix} 1-e^2 & 0 & -c(1+e^2) \\ 0 & 1 & 0 \\ -c(1+e^2) & 0 & c^2(1-e^2) \end{pmatrix}$$

$$\Delta = \det\tilde{A} = c^2(1-e^2)^2 - e^2(1+e^2)^2 =$$

$$= c^2\left[\cancel{1} - 2e^2 + \cancel{e^4} - \cancel{1} - 2e^2 - \cancel{e^4}\right] = -4c^2e^2 \neq 0$$

$\Rightarrow \Gamma$ nedegenerată

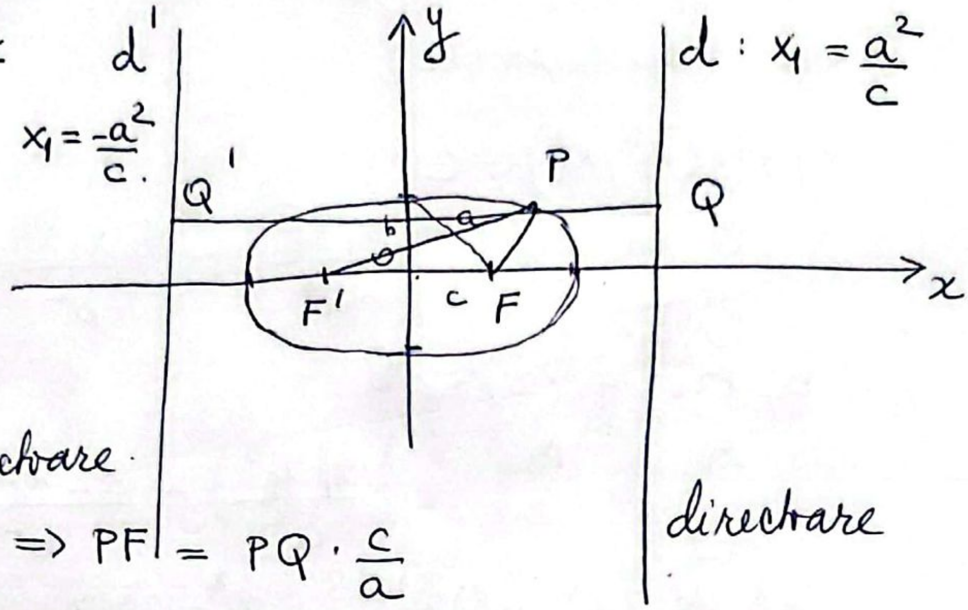
OBS

$a^2=b^2+c^2 \Rightarrow c^2=a^2-b^2$

a) Elipsa

$e=\frac{c}{a}<1$

$d: x_1=\frac{a^2}{c}$



$\frac{PF}{PQ}=e=\frac{c}{a} \Rightarrow PF=PQ\cdot\frac{c}{a}$

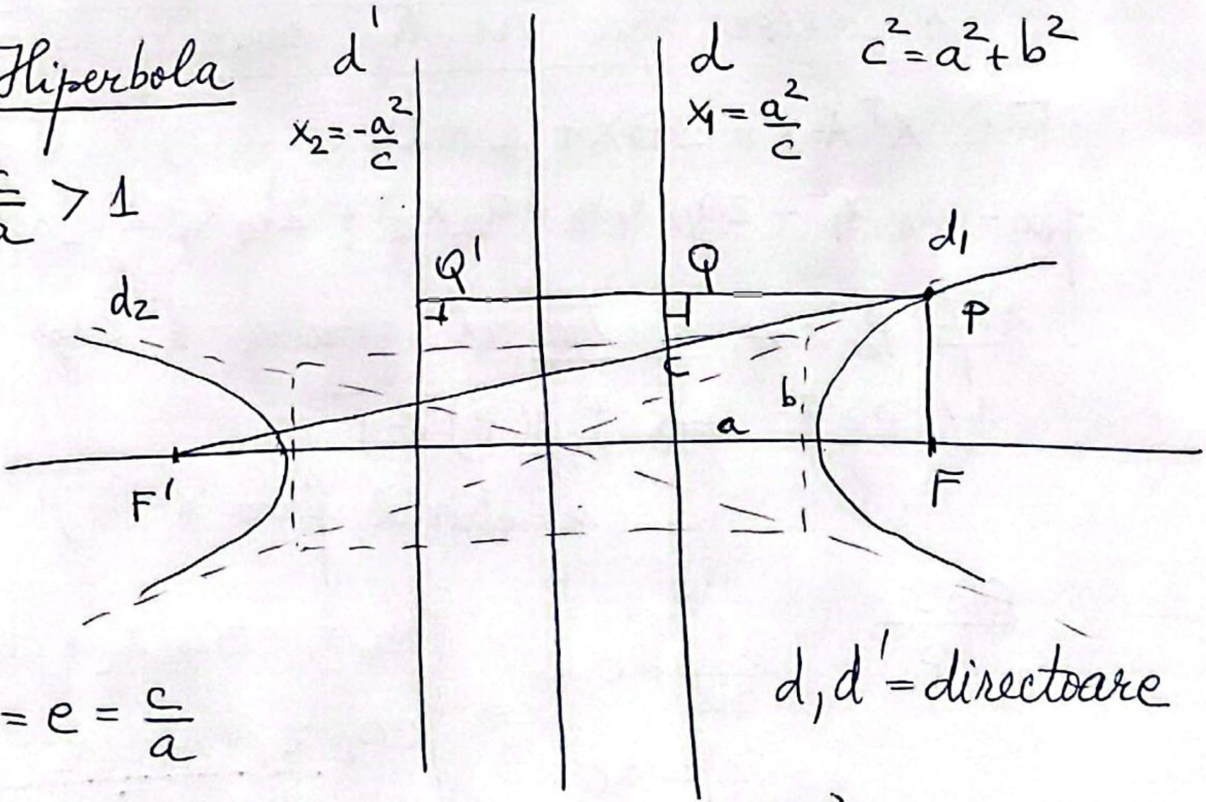
$\frac{PF'}{PQ'}=e=\frac{c}{a} \Rightarrow PF'=PQ'\,\frac{c}{a}$ (+)

$PF+PF'=\frac{c}{a}\cdot\frac{2a^2}{c}=2a$

b) Hiperbola

$e=\frac{c}{a}>1$

$c^2=a^2+b^2$



d, d' = directoare

$\frac{PF}{PQ}=e=\frac{c}{a}$

$\frac{PF'}{PQ'}=e=\frac{c}{a}$ $\quad |PF-PF'|=\frac{c}{a}\,\frac{2a^2}{c}=2a.$

OBS Tg într-un punct al unei conice (proced. de dedublare)

a) $M_0(x_1^0, x_2^0) \in \mathcal{E}$

tg în M_0 $\frac{x_1 x_1^0}{a^2} + \frac{x_2 x_2^0}{b^2} = 1$

b) $M_0(x_1^0, x_2^0) \in \mathcal{H}$

tg în M_0 : $\frac{x_1 x_1^0}{a^2} - \frac{x_2 x_2^0}{b^2} = 1$

c) $M_0(x_1^0; x_2^0) \in \mathcal{P}$: $x_2^2 = 2p x_1$

tg în M_0 : $x_2 x_2^0 = p(x_1 + x_1^0)$

Aducerea la o formă canonică a conicelor cu centru unic ($\delta \neq 0$)

Γ : $X^T A X + 2BX + c = 0$.

$f(x) = a_{11} x_1^2 + 2a_{12} x_1 x_2 + a_{22} x_2^2 + 2b_1 x_1 + 2b_2 x_2 + c = 0$

Def P_0 s.n. centru al conicei $\Gamma \Leftrightarrow$

$\forall P \in \Gamma \Rightarrow S_{P_0}(P) \in \Gamma$

(simetricul față de P_0)

OBS

P_0 : $\begin{cases} \frac{\partial f}{\partial x_1} = 0 \\ \frac{\partial f}{\partial x_2} = 0 \end{cases} \Rightarrow \begin{cases} 2a_{11} x_1 + 2a_{12} x_2 + 2b_1 = 0 \\ 2a_{12} x_1 + 2a_{22} x_2 + 2b_2 = 0 \end{cases}$

$\boxed{AX_0 + B^T = O_{2,1}}$

(*) $\begin{cases} a_{11} x_1 + a_{12} x_2 = -b_1 \\ a_{12} x_1 + a_{22} x_2 = -b_2 \end{cases}$ $A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -b_1 \\ -b_2 \end{pmatrix}$

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = A^T$

(*) are sol unică $\Leftrightarrow \delta = \det A \neq 0$.

$P_0(x_1^0, x_2^0)$ centrul unic

Prop $f(x_1^0, x_2^0) = \frac{\Delta}{\delta}$, $P_0(x_1^0, x_2^0)$ centrul unic al conicei ($\delta \neq 0$)

- $(\mathbb{R}^2, \mathbb{R}^2/_{\mathbb{R}}, \varphi)$

$$\mathcal{R} = \{O; e_1, e_2\} \xrightarrow[\text{translatie}]{\theta} \mathcal{R}' = \{P_0; e_1, e_2\} \xrightarrow[\text{transf. afina}]{\tau} \mathcal{R}'' = \{P_0; e_1', e_2'\}.$$

$$\theta: X = X' + X_0 \Leftrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} + \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}$$

$$\theta(\Gamma): (X' + X_0)^T A (X' + X_0) + 2B(X' + X_0) + C = 0$$

$$\Rightarrow \underbrace{X'^T A X'} + \frac{\Delta}{\delta} = 0$$

Fie $Q : \mathbb{R}^2 \to \mathbb{R}$, $Q(x) = X'^T A X' =$
$= a_{11} x_1'^2 + 2a_{12} x_1' x_2' + a_{22} x_2'^2$

formă pătratică

Aducem Q la o formă canonică (met Gauss/Jacobi)

$$Q(x) = \lambda_1 x_1''^2 + \lambda_2 x_2''^2$$

$$\tau: X' = C X''$$

$$\tau \circ \theta(\Gamma): \lambda_1 x_1''^2 + \lambda_2 x_2''^2 + \frac{\Delta}{\delta} = 0.$$

$$\tau \circ \theta: \quad X = CX'' + X_0$$

- $(\mathbb{R}^2, (\mathbb{R}^2, g), \varphi)$

Prima etapă este identică (translația)

Q se aduce la o formă canonică, prin met. valorilor proprii.

$$P(\lambda) = \det(A - \lambda I_2) = 0 \Rightarrow \lambda^2 - Tr(A)\lambda + \det(A) = 0$$

- $\lambda_1 \neq \lambda_2$ $\quad e_1', e_2'$ versori proprii $(e_1' \perp e_2')$,

$e_k' = (\ell_k, m_k)$

$\tau: X' = RX''$ $\quad R = \begin{pmatrix} \ell_1 & \ell_2 \\ m_1 & m_2 \end{pmatrix} \in O(2)$

(putem alege $R \in SO(2)$)

Schimbări de repere ortonormate.

$\tau \circ \theta: X = RX'' + X_0$.

- $\lambda_1 x_1''^2 + \lambda_2 x_2''^2 + \frac{\Delta}{\delta} = 0$

$\left(\lambda_1 = \lambda_2 \quad V_{\lambda_1} = \mathbb{R}^2 \right.$ (Gram-Schmidt) $\Big|$

Exemplu ($\delta \neq 0$)

În sp. punctual euclidian $\mathcal{E}_2$ se consideră conica:

$\Gamma: f(x_1, x_2) = \underline{7x_1^2 - 8x_1x_2 + x_2^2} - 6x_1 - 12x_2 - 9 = 0$.

Să se aducă la o formă canonică, utilizând izometrii. Reprez. grafică.

SOL

$A = \begin{pmatrix} 7 & -4 \\ -4 & 1 \end{pmatrix}$ $\quad \delta = \det A = 7 - 16 = -9$.

$\tilde{A} = \begin{pmatrix} 7 & -4 & -3 \\ -4 & 1 & -6 \\ -3 & -6 & -9 \end{pmatrix}$ $\quad \Delta = \det \tilde{A} = \begin{vmatrix} 7 & -4 & -3 \\ -4 & 1 & -6 \\ -3 & -6 & -9 \end{vmatrix}$

$= 9 \begin{vmatrix} 7 & -4 & 1 \\ -4 & 1 & 2 \\ 1 & 2 & -1 \end{vmatrix} = 9 \begin{vmatrix} 8 & -2 & 0 \\ -2 & 5 & 0 \\ 1 & 2 & -1 \end{vmatrix} = -9(40-4)$

$= -9 \cdot 36$

$\Delta = 7(-9-36) + 4(36-18) - 3(24+3) =$

$= 7\cdot(-45)+4\cdot 18-3\cdot 27$

$= 9(-35+8-9) = 9(-36) =$

$\Delta = -9\cdot 36 \qquad \frac{\Delta}{\delta} = 36.$

Det. centrul: $\begin{cases} 14x_1-8x_2-6=0 \\ -8x_1+2x_2-12=0 \end{cases} \Rightarrow \begin{cases} 7x_1-4x_2=3 \\ -4x_1+x_2=6 \end{cases}$

$x_1\underbrace{(-16+7)}_{-9} \quad / = 27$

$\begin{cases} x_1=-3. \\ x_2 = 6-12=-6 \end{cases}$

$P_0(-3,-6)$

- $R=\{0; e_1, e_2\} \longrightarrow R'=\{P_0; e_1, e_2\}.$

Considerăm translația.

$\theta: X = X'+X_0. \qquad X_0=\begin{pmatrix} -3 \\ -6 \end{pmatrix}$

$\begin{cases} x_1 = x_1'-3 \\ x_2 = x_2'-6 \end{cases}$

$\theta(\Gamma): \underbrace{7x_1'^2-8x_1'x_2'+x_2'^2}+\overset{36}{\frac{\Delta}{\delta}}=0.$

$Q: \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad Q(x) = 7x_1'^2-8x_1'x_2'+x_2'^2.$

Aplicăm met. valorilor proprii

Polinomul caracteristic:

$\det(A-\lambda I_2)=0 \iff \lambda^2 - Tr(A)\lambda + \det(A) = 0.$

$\lambda^2-8\lambda-9=0 \iff (\lambda-9)(\lambda+1)=0$

$\lambda_1 = 9, \quad \lambda_2=-1.$

$\lambda_1 x_1''^2+\lambda_2 x_2''^2+\frac{\Delta}{\delta}=0$

$9x_1''^2-x_2''^2+36=0$

$V_9 = \{x \in \mathbb{R}^2 \mid AX = 9X\}$

$(A - 9I_2)X = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -2 & -4 \\ -4 & -8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-2x_1 - 4x_2 = 0 \Rightarrow x_1 = -2x_2$

$e_1' = \frac{1}{\sqrt{5}}(+2, -1) \qquad \langle\{e_1'\}\rangle = V_9.$

$V_{-1} = \{x \in \mathbb{R}^2 \mid AX = -X\}$

$(A + I_2)X = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 8 & -4 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$8x_1 - 4x_2 = 0 \Rightarrow 2x_1 = x_2.$

$e_2' = \frac{1}{\sqrt{5}}(1, 2) \qquad \langle\{e_2'\}\rangle = V_{-1}$

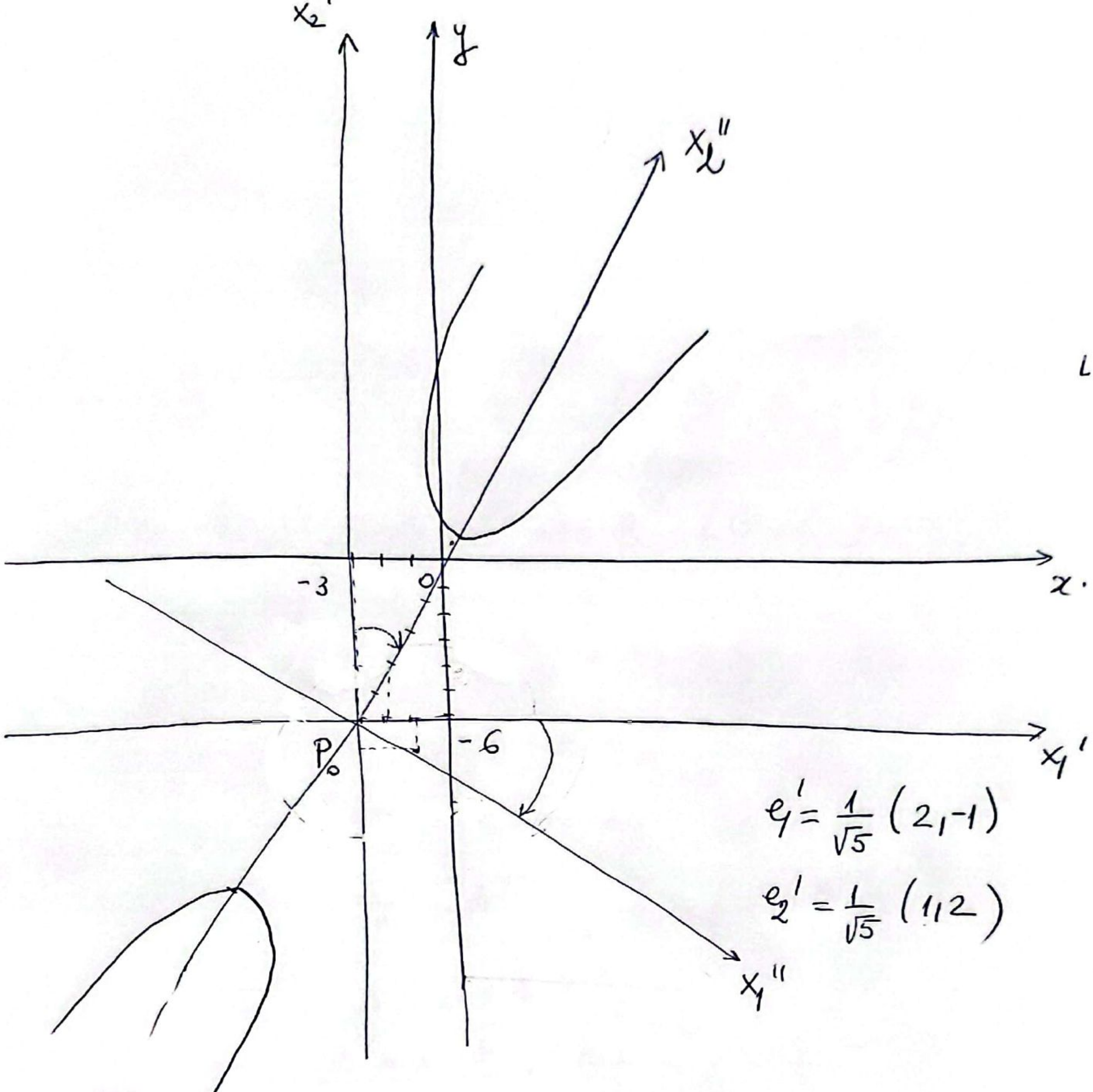
$\tau : X' = RX'' \qquad R = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \in SO(2)$

$\mathcal{R}' = \{P_0; e_1, e_2\} \longrightarrow \mathcal{R}'' = \{P_0; e_1', e_2'\}.$

$\tau \circ \theta(\Gamma) : 9x_1''^2 - x_2''^2 + 36 = 0$

$9x_1''^2 - x_2''^2 = -36 \Rightarrow$

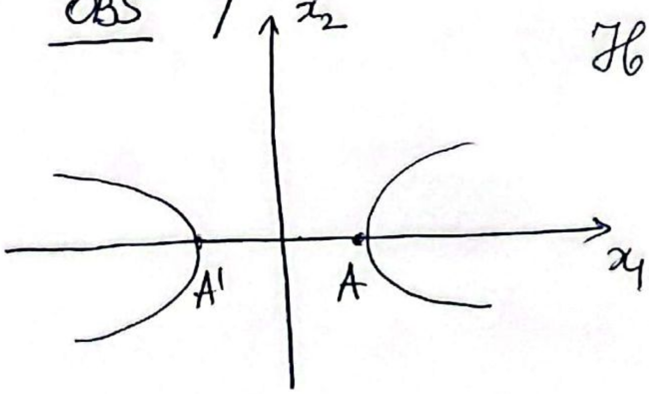
$-\frac{x_1''^2}{4} + \frac{x_2''^2}{36} = 1.$ (hiperbolă)



$$e_1' = \frac{1}{\sqrt{5}}(2,-1)$$

$$e_2' = \frac{1}{\sqrt{5}}(1,2)$$

<u>OBS</u>



$$\mathcal{H} : \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 1.$$

$$\boxed{\mathcal{H} : \frac{-x_1''^2}{4} + \frac{x_2''^2}{36} = 1.}$$