Seminar 5

1. Fie $n \in \mathbb{N}^*$ și $d_2 \overset{not.}{=} d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$,

$$d(\underset{(x_1,\dots,x_n)}{x}, \underset{(y_1,\dots,y_n)}{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$ Arătați că d este metrică pe $\mathbb{R}^n$.

Sol.: 1) $d(x,y) \geq 0 \quad \forall x,y \in \mathbb{R}^n$ (evident).

2) $d(x,y) = 0 \Leftrightarrow \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2} = 0 \Leftrightarrow$

$\Leftrightarrow \sum_{i=1}^{n}(x_i-y_i)^2 = 0 \Leftrightarrow (x_i-y_i)^2 = 0 \ \forall i=\overline{1,n} \Leftrightarrow x_i = y_i \ \forall$

$\forall i=\overline{1,n} \Leftrightarrow x = y \ \forall x,y \in \mathbb{R}^n$.

3) $d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2} = \sqrt{\sum_{i=1}^{n}(-1)^2(y_i-x_i)^2} =$

$= \sqrt{\sum_{i=1}^{n}(y_i-x_i)^2} = d(y,x) \ \forall x,y \in \mathbb{R}^n$.

4) Fie $x,y,z \in \mathbb{R}^n$.

$d(x,z) = \sqrt{\sum_{i=1}^{n}(x_i-z_i)^2} = \sqrt{\sum_{i=1}^{n}(x_i-y_i+y_i-z_i)^2}$.

Folosim inegalitatea Cauchy-Buniakowski-Schwarz (C.B.S.): $\forall n \in \mathbb{N}^*$, $\forall a_1,\dots,a_n \in \mathbb{R}$, $\forall b_1,\dots,b_n \in \mathbb{R}$, avem $\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$.

$d(x,z) = \sqrt{\sum_{i=1}^{n}(\underbrace{x_i-y_i}+\underbrace{y_i-z_i})^2} =$

$$= \sqrt{\sum_{i=1}^{n}\left[(x_i-y_i)^2+(y_i-z_i)^2+2(x_i-y_i)(y_i-z_i)\right]} =$$

$$= \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2+\sum_{i=1}^{n}(y_i-z_i)^2+2\sum_{i=1}^{n}(x_i-y_i)(y_i-z_i)} \leq$$

$$\underset{\text{C.B.S.}}{\leq} \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2+\sum_{i=1}^{n}(y_i-z_i)^2+2\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}\sqrt{\sum_{i=1}^{n}(y_i-z_i)^2}} =$$

$$= \sqrt{\left(\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}+\sqrt{\sum_{i=1}^{n}(y_i-z_i)^2}\right)^2} =$$

$$= \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}+\sqrt{\sum_{i=1}^{n}(y_i-z_i)^2} = d(x,y)+d(y,z).$$

Deci d este metrică pe $\mathbb{R}^n$. $\square$

2. Fie $n \in \mathbb{N}^*$, $d_1, d_2 : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$,

$$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i| \text{ și } d_2(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

Arătați că există $a, b > 0$ a.î. $a\,d_1(x,y) \leq d_2(x,y) \leq b\,d_1(x,y)$ $\forall\, x, y \in \mathbb{R}^n$.

Sol.: Fie $x, y \in \mathbb{R}^n$.

$$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} |x_i - y_i| \cdot 1 \underset{\text{C.B.S.}}{\leq} \left(\sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}\right) \cdot$$

$$\cdot \sqrt{\sum_{i=1}^{n} 1^2} = \left(\sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}\right) \cdot \sqrt{n} = \sqrt{n}\, d_2(x,y) \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{\sqrt{n}}\, d_1(x,y) \leq d_2(x,y).$$

Alegem $a = \dfrac{1}{\sqrt{n}}$.

$$d_2(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} =$$

$$= \sqrt{|x_1 - y_1|^2 + \ldots + |x_n - y_n|^2} \leq \sqrt{\left(|x_1 - y_1| + \ldots + |x_n - y_n|\right)^2} =$$

$= |x_1-y_1|+\ldots+|x_n-y_n| = d_1(x,y).$

Alegem $b=1$.

Am obținut $\frac{1}{\sqrt{n}} d_1(x,y) \le d_2(x,y) \le d_1(x,y)$. $\square$

3. Fie $n\in\mathbb{N}^*$, d_1 ca mai sus și $d_\infty : \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$, $d_\infty(x,y) = \max\{|x_i-y_i| \mid i=\overline{1,n}\}$. Arătați că există $a,b>0$ a.î. $a\,d_1(x,y) \le d_\infty(x,y) \le b\,d_1(x,y)$ $\forall\, x,y\in\mathbb{R}^n$.

Sol.: Fie $x,y\in\mathbb{R}^n$.

$d_1(x,y) = |x_1-y_1|+\ldots+|x_n-y_n| \le n\cdot\max\{|x_i-y_i| \mid i=\overline{1,n}\} =$

$= n\,d_\infty(x,y) \Leftrightarrow \frac{1}{n} d_1(x,y) \le d_\infty(x,y).$

Alegem $a=\frac{1}{n}$.

$d_\infty(x,y) = \max\{|x_i-y_i| \mid i=\overline{1,n}\} \le |x_1-y_1|+\ldots+|x_n-y_n| =$

$= d_1(x,y).$

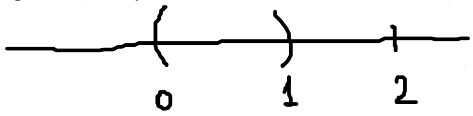
Alegem $b=1$.

Am obținut $\frac{1}{n} d_1(x,y) \le d_\infty(x,y) \le d_1(x,y)$. $\square$

4. Faceți analiza topologică a mulțimii $A\subset\mathbb{R}$, unde:
(Determinați $\mathring{A}, \bar{A}, Fr(A), A', Izo(A)$)

a) $A=(0,1)\cup\{2\}$.

Sol.:



$x\in\mathring{A} \Leftrightarrow \exists\, r>0$ a.î. $\underbrace{B(x,r)}_{=(x-r,\,x+r)}\subset A$.

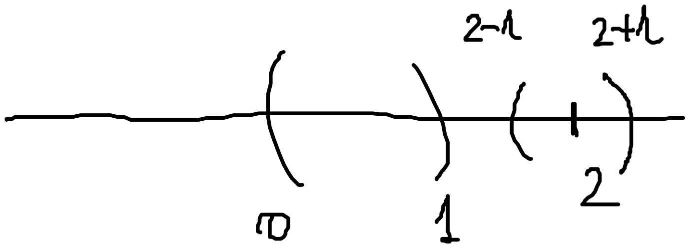
$\mathring{A}\subset A$

$\left.\begin{array}{l}(0,1) \text{ deschisă}\\ (0,1)\subset A\end{array}\right\} \Rightarrow (0,1)\subset\mathring{A}$.

Avem $(0,1)\subset\mathring{A}\subset(0,1)\cup\{2\}$.

Studiem dacă $2\in\mathring{A}$.

$2\in\mathring{A} \Leftrightarrow \exists\, r>0$ a.î. $(2-r,\,2+r)\subset A$.



Deci $2\notin\mathring{A}$.

Așadar $\mathring{A}=(0,1)$.

2) $\bar{A}=?$

$x\in\bar{A} \Leftrightarrow \forall\, r>0$, avem $\underbrace{B(x,r)}_{=(x-r,\,x+r)}\cap A\neq\emptyset$.

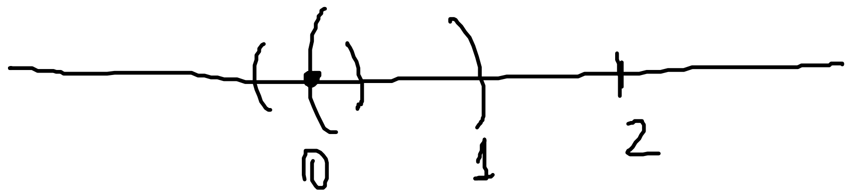
$A\subset\bar{A}$

$\left.\begin{array}{l}[0,1]\cup\{2\} \text{ închisă}\\ A\subset[0,1]\cup\{2\}\end{array}\right\} \Rightarrow \bar{A}\subset[0,1]\cup\{2\}$.

Am obținut $(0,1)\cup\{2\}\subset\overline{A}\subset[0,1]\cup\{2\}$.

Studiem dacă $0\in\overline{A}$ și $1\in\overline{A}$.

$0\in\overline{A}\Leftrightarrow\forall r>0$, avem $\underset{\underset{(-r,\,r)}{\parallel}}{(0-r,\,0+r)}\cap A\neq\emptyset$.



Deci $0\in\overline{A}$.

Analog $1\in\overline{A}$.

Așadar $\overline{A}=[0,1]\cup\{2\}$.

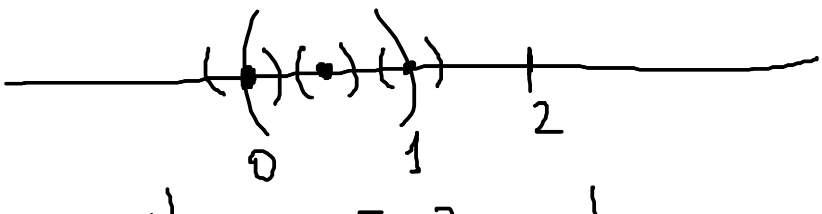
3) $A'=?$

$x\in A'\Leftrightarrow\forall r>0$, avem $(x-r,\,x+r)\cap(A\setminus\{x\})\neq\emptyset$.

$A'\subset\overline{A}=[0,1]\cup\{2\}$.

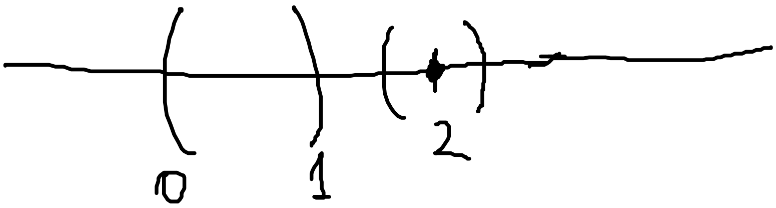
Fie $x\in[0,1]$.

$x\in A'\Leftrightarrow\forall r>0$, avem $(x-r,\,x+r)\cap(A\setminus\{x\})\neq\emptyset$.



Deci $x\in A'$, i.e. $[0,1]\subset A'$.

$2\in A'\Leftrightarrow\forall r>0$, avem $(2-r,\,2+r)\cap(A\setminus\{2\})\neq\emptyset$.



Deci $2 \notin A'$.

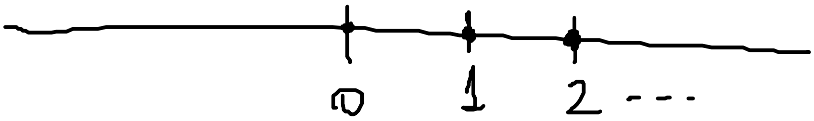
Așadar $A' = [0,1]$.

4) $Fr(A) = \partial A = \bar{A} \setminus \mathring{A} = \{0,1,2\}$

5) $Izo(A) = {}'A = \bar{A} \setminus A' = \{2\}$. $\square$

b) $A = \mathbb{N}$.

Sol.:



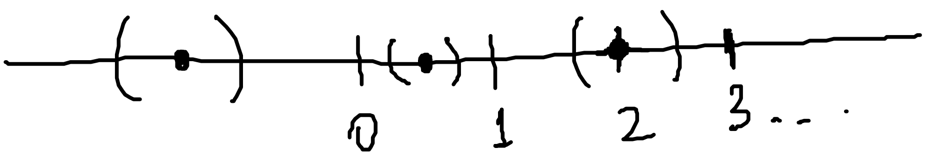
1) $\mathring{A} = ?$

$x \in \mathring{A} \Leftrightarrow \exists r > 0$ a.î. $(x-r, x+r) \subset A$.

Deci $\mathring{A} = \emptyset$ (deoarece A nu conține intervale nedegenerate).

2) $A' = ?$

$x \in A' \Leftrightarrow \forall r > 0$, avem $(x-r, x+r) \cap (A \setminus \{x\}) \neq \emptyset$.



Deci $A' = \emptyset$.

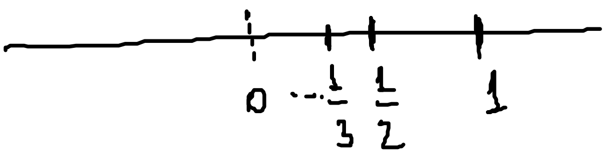
3) $\overline{A} = A \cup A' = \mathbb{N} \cup \emptyset = \mathbb{N}$.

4) $Fr(A) = \partial A = \overline{A} \setminus \mathring{A} = \mathbb{N} \setminus \emptyset = \mathbb{N}$.

5) $Izo(A) = {}'A = \overline{A} \setminus A' = \mathbb{N} \setminus \emptyset = \mathbb{N}$. □

c) $A = \left\{ \frac{1}{n} \mid n \in \mathbb{N}^* \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \dots \right\}$.

Sol.:



1) $\mathring{A} = ?$

$x \in \mathring{A} \Leftrightarrow \exists r > 0$ a.î. $(x-r, x+r) \subset A$.

$\mathring{A} = \emptyset$ (deoarece A nu conține intervale nedegenerate).

2) $A' = ?$

$x \in A' \Leftrightarrow \exists (x_n)_n \subset A \setminus \{x\}$ a.î. $\lim\limits_{n \to \infty} x_n = x$.

Fie $x_n = \frac{1}{n}$ $\forall n \in \mathbb{N}^*$. Avem $(x_n)_n \subset A \setminus \{0\}$ și $\lim\limits_{n \to \infty} x_n = 0$. Deci $0 \in A'$.

Fie $(y_n)_n \subset A$ un șir convergent. Atunci $(y_n)_n$

are toți termenii egali cu un element din A de la un rang încolo sau $\lim_{n\to\infty} y_n = 0$ (deoarece $\lim_{n\to\infty} \frac{1}{n} = 0$).

Prin urmare $A' = \{0\}$.

3) $\overline{A} = A \cup A' = \{0, 1, \frac{1}{2}, \dots\}$.

4) $Fr(A) = \partial A = \overline{A} \setminus \mathring{A} = \{0, 1, \frac{1}{2}, \dots\}$.

5) $Izo(A) = {}'A = \overline{A} \setminus A' = \{1, \frac{1}{2}, \dots\}$. $\square$