Seminar 4

1. Studiați natura seriilor:

a) $\sum_{n=1}^{\infty} \frac{\cos nx}{n^{\lambda}}$, $x \in \mathbb{R}$, $\lambda > 0$.

Sol.: Vom folosi Criteriul Abel-Dirichlet (I).

Fie $x_n = \frac{1}{n^{\lambda}}$ $\forall n \in \mathbb{N}^*$ și $y_n = \cos nx$ $\forall n \in \mathbb{N}^*$.

Șirul $(x_n)_n$ este descrescător și $\lim_{n\to\infty} x_n = 0$. (1)

$\exists M > 0$ a.î. $\forall n \in \mathbb{N}^*$, avem $|y_1 + \dots + y_n| \leq M$?

M nu depinde de n, dar poate să depindă de x.

$|y_1 + y_2 + \dots + y_n| = |\cos x + \cos 2x + \dots + \cos nx|$.

Fie $z = \cos x + i \sin x$.

Avem: $z^2 = \cos 2x + i \sin 2x$.

$z^3 = \cos 3x + i \sin 3x$.

....

$z^n = \cos nx + i \sin nx$.

$\underbrace{\cos x}_{y_1} + \dots + \underbrace{\cos nx}_{y_n} = \mathrm{Re}(z + z^2 + \dots + z^n)$.

$$z+z^2+\dots+z^n=?$$

Presupunem că $z\neq 1$, i.e. $\cos x+i\sin x\neq 1$, i.e.

$x\in\mathbb{R}\setminus\{2k\pi \mid k\in\mathbb{Z}\}$.

$$z+z^2+\dots+z^n=z\cdot\frac{z^n-1}{z-1}=\frac{z^{n+1}-z}{z-1}=$$

$$=\frac{\cos(n+1)x+i\sin(n+1)x-\cos x-i\sin x}{\cos x+i\sin x-1}=$$

$$=\frac{(\cos(n+1)x-\cos x)+i(\sin(n+1)x-\sin x)}{(\cos x-1)+i\sin x}=$$

$$=\frac{-2\sin\frac{n+2}{2}x\sin\frac{n}{2}x+i\cdot 2\cdot\cos\frac{n+2}{2}x\sin\frac{n}{2}x}{(\cos x-\cos 0)+i(\sin x-\sin 0)}=$$

$$=\frac{-\cancel{2}\sin\frac{n+2}{2}x\sin\frac{n}{2}x+\cancel{2}i\cos\frac{n+2}{2}x\sin\frac{n}{2}x}{-\cancel{2}\sin\frac{x}{2}\sin\frac{x}{2}+\cancel{2}i\cos\frac{x}{2}\sin\frac{x}{2}}=$$

$$=\frac{\sin\frac{n}{2}x}{\sin\frac{x}{2}}\cdot\frac{-\sin\frac{n+2}{2}x+i\cos\frac{n+2}{2}x}{-\sin\frac{x}{2}+i\cos\frac{x}{2}}=$$

i

$$= \frac{\sin \frac{n}{2} x}{\sin \frac{x}{2}} \cdot \frac{\cos \frac{n+2}{2} x + i \sin \frac{n+2}{2} x}{\cos \frac{x}{2} + i \sin \frac{x}{2}} =$$

$$= \frac{\sin \frac{n}{2} x}{\sin \frac{x}{2}} \cdot \frac{\left(\cos \frac{x}{2} + i \sin \frac{x}{2}\right)^{\cancel{n+2}\ n+1}}{\cancel{\cos \frac{x}{2} + i \sin \frac{x}{2}}} =$$

$$= \frac{\sin \frac{n}{2} x}{\sin \frac{x}{2}} \cdot \left(\cos \frac{n+1}{2} x + i \sin \frac{n+1}{2} x\right).$$

Deci $\operatorname{Re}(z + z^2 + \dots + z^n) = \dfrac{\sin \frac{n}{2} x}{\sin \frac{x}{2}} \cos \dfrac{n+1}{2} x.$

$$|y_1 + \dots + y_n| = |\operatorname{Re}(z + z^2 + \dots + z^n)| = \frac{\left|\sin \frac{n}{2} x\right|}{\left|\sin \frac{x}{2}\right|} \left|\cos \frac{n+1}{2} x\right| \le$$

$$\le \frac{1}{\left|\sin \frac{x}{2}\right|}.$$

Alegem $M = \dfrac{1}{\left|\sin \frac{x}{2}\right|}$ și avem $\forall n \in \mathbb{N}^*$, $|y_1 + \dots + y_n| \le M$. (2)

Din (1) și (2) rezultă, conform Criteriului Abel-Dirichlet [1], că seria $\sum_{n=1}^{\infty} x_n \cdot y_n = \sum_{n=1}^{\infty} \dfrac{\cos nx}{n^\lambda}$ este conv.

Am tratat mai sus doar cazul $x \in \mathbb{R} \setminus \{2k\pi \mid k \in \mathbb{Z}\}$.

Fie $x \in \{2k\pi \mid k \in \mathbb{Z}\}$.

Seria devine $\sum_{n=1}^{\infty} \frac{\cos n(2k\pi)}{n^\lambda} = \sum_{n=1}^{\infty} \frac{1}{n^\lambda}$ → conv., dacă $\lambda \in (1, \infty)$; → div., dacă $\lambda \in (0, 1]$. □

$\lambda > 0$

b) $\sum_{n=1}^{\infty} \frac{\cos n \cos \frac{1}{n}}{n}$.

<u>Sol.</u>: Fie $x_n = \cos \frac{1}{n}\ \forall n \in \mathbb{N}^*$ și $y_n = \frac{\cos n}{n}\ \forall n \in \mathbb{N}^*$.

$-1 \leq x_n \leq 1\ \forall n \in \mathbb{N}^* \Rightarrow (x_n)_n$ mărginit.

$x \longmapsto \cos x$ descrescătoare

$\cap$

$(0, \frac{\pi}{2})$

$\frac{1}{n} \in (0, \frac{\pi}{2})\ \forall n \in \mathbb{N}^*$

$(\frac{1}{n})_n$ descrescator

$\Rightarrow (\cos \frac{1}{n})_n$ este crescător.

Deci $(x_n)_n$ este monoton și mărginit. (1)

$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{\cos n}{n}$ conv. (Vezi a): $x = 1$ și $\lambda = 1$). (2)

$\neq 2k\pi$

Din (1) și (2) rezultă, conform criteriului Abel-Dirichlet (II), că $\sum_{n=1}^{\infty} x_n \cdot y_n = \sum_{n=1}^{\infty} \frac{\cos n \cos\frac{1}{n}}{n}$ este conv. $\square$

c) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$, $x \in (-1, 1)$.

<u>Sol.</u>: $x_n = \frac{x^n}{n^2}$ $\forall n \in \mathbb{N}^*$.

$|x_n| = \left| \frac{x^n}{n^2} \right| = \frac{|x|^n}{n^2}$ $\forall n \in \mathbb{N}^*$.

Fie $y_n = \frac{1}{n^2}$ $\forall n \in \mathbb{N}^*$.

$|x_n| = \frac{|x|^n}{n^2} < \frac{1}{n^2} = y_n$ $\forall n \in \mathbb{N}^*$.

$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ conv. (serie armonică generalizată cu $\alpha = 2$).

Conform crit. de comparație cu ineg. rezultă că $\sum_{n=1}^{\infty} |x_n|$ este conv. Deci $\sum_{n=1}^{\infty} x_n$ este absolut conv.

Prin urmare $\sum_{n=1}^{\infty} x_n$ este conv. $\square$

2. Fie $n \in \mathbb{N}^*$ şi $d_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$,

$$\underset{\substack{\| \\ (x_1,\dots,x_n)}}{d_1(x}, \underset{\substack{\| \\ (y_1,\dots,y_n)}}{y)} = |x_1 - y_1| + \dots + |x_n - y_n| = \sum_{i=1}^{n} |x_i - y_i|.$$

a) Arătaţi că d_1 este metrică pe $\mathbb{R}^n$.

Sol.: $\mathbb{R}^n = \{(x_1,\dots,x_n) \mid x_i \in \mathbb{R} \ \forall i=\overline{1,n}\} =$

$$= \underbrace{\mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R}}_{\text{de } n \text{ ori}}.$$

1) $d_1(x,y) \geq 0 \quad \forall x,y \in \mathbb{R}^n$ (evident)

2) $d_1(x,y) = 0 \Leftrightarrow \sum_{i=1}^{n} |x_i - y_i| = 0 \Leftrightarrow |x_i - y_i| = 0 \ \forall i = \overline{1,n} \Leftrightarrow$

$\Leftrightarrow x_i = y_i \ \forall i = \overline{1,n} \Leftrightarrow x = y \ \forall x,y \in \mathbb{R}^n$.

3) $d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} |y_i - x_i| = d_1(y,x) \ \forall x,y \in \mathbb{R}^n$.

4) Fie $x,y,z \in \mathbb{R}^n$.

$$d_1(x,z) = \sum_{i=1}^{n} |x_i - z_i| = \sum_{i=1}^{n} |x_i - y_i + y_i - z_i| \leq$$

$$\leq \sum_{i=1}^{n} \left(|x_i - y_i| + |y_i - z_i|\right) = \sum_{i=1}^{n} |x_i - y_i| + \sum_{i=1}^{n} |y_i - z_i| =$$

$$= d_1(x,y) + d_1(y,z).$$

Deci d_1 este metrică pe $\mathbb{R}^n$. $\square$

b) Fie $(x^k)_k \subset \mathbb{R}^n$, $x^k = (x_1^k, x_2^k, \dots, x_n^k)$ $\forall k \in \mathbb{N}$ și $x \in \mathbb{R}^n$, $x = (x_1, \dots, x_n)$. Arătați că $\lim\limits_{k\to\infty} x^k \overset{d_1}{=} x$ dacă și numai dacă $\forall i = \overline{1,n}$, avem $\lim\limits_{k\to\infty} x_i^k = x_i$.

Sol.:

• $\forall i=\overline{1,n}$, $\lim\limits_{k\to\infty} x_i^k = x_i \Leftrightarrow \forall i=\overline{1,n}, \forall \varepsilon>0, \exists k_\varepsilon^i \in \mathbb{N}$ a.î. $\forall k \geq k_\varepsilon^i$, avem $|x_i^k - x_i| < \varepsilon$. (1)

• $\lim\limits_{k\to\infty} x^k \overset{d_1}{=} x \Leftrightarrow \underset{\substack{\cap \\ [0,\infty)}}{d_1(x^k, x)} \xrightarrow[k\to\infty]{} 0 \Leftrightarrow \forall \varepsilon > 0,$

$\exists k_\varepsilon \in \mathbb{N}$ a.î. $\forall k \geq k_\varepsilon$, avem $|d_1(x^k, x) - 0| < \varepsilon$. (2)

$$|d_1(x^k, x) - 0| = d_1(x^k, x) = \sum_{i=1}^{n} |x_i^k - x_i|$$

"$\Leftarrow$" Fie $\varepsilon > 0$. Conform (1) $\forall i = \overline{1,n}$, $\exists k_\varepsilon^i \in \mathbb{N}$ a.î. $\forall k \geq k_\varepsilon^i$, avem $|x_i^k - x_i| < \dfrac{\varepsilon}{n}$.

Alegem $k_\varepsilon = \max\{k_\varepsilon^1, \dots, k_\varepsilon^n\} \in \mathbb{N}$.

$\forall k \geq k_\varepsilon$, avem $d_1(x^k, x) = \sum_{i=1}^{n} |x_i^k - x_i| < \sum_{i=1}^{n} \frac{\varepsilon}{n} =$

$= \cancel{n} \cdot \frac{\varepsilon}{\cancel{n}} = \varepsilon.$

Deci $\lim_{k\to\infty} x^k \overset{d_1}{=} x.$

"$\Longrightarrow$"

Fie $\varepsilon > 0$. Conform (2) $\exists\, k_\varepsilon \in \mathbb{N}$ a.î. $\forall k \geq k_\varepsilon$,

avem $\sum_{i=1}^{n} |x_i^k - x_i| < \varepsilon$,

Deci $\forall i = \overline{1,n}$, $\exists\, k_\varepsilon^i = k_\varepsilon \in \mathbb{N}$ a.î. $\forall k \geq k_\varepsilon^i = k_\varepsilon$,

avem $|x_i^k - x_i| \leq \sum_{i=1}^{n} |x_i^k - x_i| < \varepsilon.$

Așadar $\forall i = \overline{1,n}$ $\lim_{k\to\infty} x_i^k = x_i$. $\square$