Seminar 8

1. Arătați că seria de funcții $\sum_{n=1}^{\infty} \operatorname{arctg} \frac{2x}{x^2+n^4}$ converge uniform.

<u>Sol.</u>: $\frac{x^2+n^4}{2} \geq \sqrt{x^2 n^4} = |x| n^2 \Leftrightarrow x^2+n^4 \geq 2|x| n^2 \Leftrightarrow$

$\Leftrightarrow \frac{2\cdot|x|\cdot n^2}{x^2+n^4} \leq 1 \Leftrightarrow \frac{2|x|}{x^2+n^4} \leq \frac{1}{n^2} \Leftrightarrow -\frac{1}{n^2} \leq \frac{2x}{x^2+n^4} \leq \frac{1}{n^2}$

$\forall n \in \mathbb{N}^*, \forall x \in \mathbb{R}.$

Deoarece funcția arctg este (strict) crescătoare avem că

$-\operatorname{arctg} \frac{1}{n^2} \leq \operatorname{arctg} \frac{2x}{x^2+n^4} \leq \operatorname{arctg} \frac{1}{n^2} \quad \forall n \in \mathbb{N}^*, \forall x \in \mathbb{R},$

deci $\left|\operatorname{arctg} \frac{2x}{x^2+n^4}\right| \leq \operatorname{arctg} \frac{1}{n^2} \quad \forall n \in \mathbb{N}^*, \forall x \in \mathbb{R}.$

Fie $\alpha_n = \operatorname{arctg} \frac{1}{n^2} \quad \forall n \in \mathbb{N}^*.$

$\alpha_n > 0 \quad \forall n \in \mathbb{N}^*.$

Arătăm că $\sum_{n=1}^{\infty} \alpha_n$ este convergentă.

Fie $\beta_n = \frac{1}{n^2} \quad \forall n \in \mathbb{N}^*.$

$\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = \lim_{n\to\infty} \frac{\operatorname{arctg} \frac{1}{n^2}}{\frac{1}{n^2}} = 1 \in (0,\infty).$

Conform Crit. de comp. cu limită rezultă că

$\sum_{n=1}^{\infty} \alpha_n \sim \sum_{n=1}^{\infty} \beta_n$.

$\sum_{n=1}^{\infty} \beta_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ conv. (serie armonică generalizată cu $\alpha=2$).

Deci $\sum_{n=1}^{\infty} \alpha_n$ este conv.

Conform Teoremei lui Weierstrass rezultă că

$\sum_{n=1}^{\infty} \operatorname{arctg} \frac{2x}{x^2+n^4}$ converge uniform. □

2. Determinați mulțimea de convergență pentru următoarele serii de puteri:

a) $\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n} x^n$.

Sol.: $a_n = \frac{1}{n \cdot 2^n} \quad \forall n \in \mathbb{N}^*$.

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n \cdot 2^n}} = \lim_{n\to\infty} \frac{1}{(\sqrt[n]{n}) \cdot 2} = \frac{1}{2}.$$

Deci $R = \frac{1}{\frac{1}{2}} = 2$.

Fie A mulțimea de convergență a seriei de puteri din enunț.

Avem $(-R,R)\subset A\subset[-R,R]$, i.e. $(-2,2)\subset A\subset[-2,2]$.

Dacă $x=2$ seria devine $\sum_{n=1}^{\infty}\frac{1}{n\cdot \cancel{2^n}}\cdot \cancel{2^n}=\sum_{n=1}^{\infty}\frac{1}{n}$ div.

(serie armonică generalizată cu $\alpha=1$).

Așadar $2\notin A$.

Dacă $x=-2$ seria devine $\sum_{n=1}^{\infty}\frac{1}{n\cdot 2^n}\cdot(-2)^n=$

$=\sum_{n=1}^{\infty}\frac{1}{n\cdot \cancel{2^n}}\cdot(-1)^n\cdot \cancel{2^n}=\sum_{n=1}^{\infty}(-1)^n\cdot\frac{1}{n}$ conv.

(Crit. lui Leibniz).

Așadar $-2\in A$.

Prin urmare $A=[-2,2)$. $\square$

b) $\sum_{n=1}^{\infty}\frac{n!\,x^n}{(a+1)\cdot\ldots\cdot(a+n)}$, $a>1$.

<u>Sol.</u>: $a_n=\frac{n!}{(a+1)\cdot\ldots\cdot(a+n)}$ $\forall\, n\in\mathbb{N}^*$.

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\cancel{(n+1)!}^{\,n+1}}{\cancel{(a+1)\cdot\ldots\cdot(a+n)}(a+n+1)} \cdot \frac{\cancel{(a+1)\cdot\ldots\cdot(a+n)}}{\cancel{n!}} =$$

$$= \lim_{n\to\infty} \frac{n+1}{a+n+1} = 1.$$

Deci $R = \frac{1}{1} = 1.$

Fie A mulțimea de convergență a seriei de puteri din enunț.

Avem $(-R,R) \subset A \subset [-R,R]$, i.e. $(-1,1) \subset A \subset [-1,1]$.

Dacă $x=1$ seria devine $\sum_{n=1}^{\infty} \frac{n!}{(a+1)\cdot\ldots\cdot(a+n)} \cdot 1^n = \sum_{n=1}^{\infty} \frac{n!}{(a+1)\cdot\ldots\cdot(a+n)}$.

Fie $x_n = \frac{n!}{(a+1)\cdot\ldots\cdot(a+n)}$ $\forall n \in \mathbb{N}^*$.

$$\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{a+n+1}{n+1} - 1\right) =$$

$$= \lim_{n\to\infty} n\, \frac{a+\cancel{n}+\cancel{1}-\cancel{n}-\cancel{1}}{n+1} = a > 1.$$

Conform Crit. Raabe-Duhamel seria $\sum_{n=1}^{\infty} x_n$ este conv.

Așadar $1 \in A$.

Dacă $x=-1$ seria devine $\sum_{n=1}^{\infty} \frac{n!}{(a+1)\cdot\ldots\cdot(a+n)}\cdot(-1)^n$.

$\sum_{n=1}^{\infty}\left|\frac{n!}{(a+1)\cdot\ldots\cdot(a+n)}(-1)^n\right| = \sum_{n=1}^{\infty}\frac{n!}{(a+1)\cdot\ldots\cdot(a+n)}$ conv. (vezi mai sus) $\Rightarrow \sum_{n=1}^{\infty}\frac{n!}{(a+1)\cdot\ldots\cdot(a+n)}(-1)^n$ absolut conv. $\Rightarrow$

$\Rightarrow \sum_{n=1}^{\infty}\frac{n!}{(a+1)\cdot\ldots\cdot(a+n)}(-1)^n$ conv.

Așadar $-1\in A$.

Prin urmare $A=[-1,1]$. $\square$

c) $\sum_{n=1}^{\infty}\frac{3^n}{\sqrt[3]{n}}(x+3)^n$.

<u>Sol</u>: Notăm $y=x+3$. Seria devine $\sum_{n=1}^{\infty}\frac{3^n}{\sqrt[3]{n}}y^n$.

Fie B mulțimea de convergență a seriei de puteri $\sum_{n=1}^{\infty}\frac{3^n}{\sqrt[3]{n}}y^n$.

$a_n=\frac{3^n}{\sqrt[3]{n}}\ \forall n\in\mathbb{N}^*$.

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\frac{\overset{3}{\cancel{3^{n+1}}}}{\sqrt[3]{n+1}}\cdot\frac{\sqrt[3]{n}}{\cancel{3^n}}=3.$$

Deci $R=\frac{1}{3}$.

Avem $(-R,R)\subset B\subset[-R,R]$, i.e. $\left(-\frac{1}{3},\frac{1}{3}\right)\subset B\subset\left[-\frac{1}{3},\frac{1}{3}\right]$.

Dacă $y=\frac{1}{3}$ seria devine $\sum_{n=1}^{\infty}\frac{3^n}{\sqrt[3]{n}}\left(\frac{1}{3}\right)^n=\sum_{n=1}^{\infty}\frac{\cancel{3^n}}{\sqrt[3]{n}}\cdot\frac{1}{\cancel{3^n}}=$

$=\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n}}=\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{3}}}$ div. (serie armonică generalizată cu $\alpha=\frac{1}{3}$).

Așadar $\frac{1}{3}\notin B$.

Dacă $y=-\frac{1}{3}$ seria devine $\sum_{n=1}^{\infty}\frac{3^n}{\sqrt[3]{n}}\cdot\left(-\frac{1}{3}\right)^n=$

$=\sum_{n=1}^{\infty}\frac{\cancel{3^n}}{\sqrt[3]{n}}\cdot\frac{(-1)^n}{\cancel{3^n}}=\sum_{n=1}^{\infty}(-1)^n\cdot\frac{1}{\sqrt[3]{n}}$ conv. (Crit. lui Leibniz)

Așadar $-\frac{1}{3}\in B$.

Prin urmare $B=\left[-\frac{1}{3},\frac{1}{3}\right)$.

Fie A mulțimea de conv. a seriei de puteri $\sum_{n=1}^{\infty}\frac{3^n}{\sqrt[3]{n}}(x+3)^n$.

$$\underset{\left[-\frac{1}{3},\frac{1}{3}\right)}{y\in B}\iff-\frac{1}{3}\le y<\frac{1}{3}\overset{y=x+3}{\iff}-\frac{1}{3}\le x+3<\frac{1}{3}\ \Big|-3\iff$$

$\Leftrightarrow -\frac{1}{3}-3 \le x < \frac{1}{3}-3 \Leftrightarrow -\frac{10}{3} \le x < -\frac{8}{3} \Leftrightarrow x \in \left[-\frac{10}{3}, -\frac{8}{3}\right).$

Deci $A = \left[-\frac{10}{3}, -\frac{8}{3}\right)$. $\square$

d) $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n+1}(x-2)^n$.

<u>Sol</u>.: Rezolvați-l voi! $\square$

e) $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n} x^{2n} = \frac{(-1)^1}{2}x^2 + \frac{(-1)^2}{4}x^4 + \dots$

<u>Sol</u>.: $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n} x^{2n} = \sum_{k=0}^{\infty} a_k x^k = a_0x^0 + a_1x^1 + a_2x^2 + \dots$

$a_0 = 0, \quad a_k = \begin{cases} 0 \ ; & k = 2n-1 \\ \frac{(-1)^n}{2n} \ ; & k = 2n \end{cases} \quad \forall n \in \mathbb{N}^*.$

Deci $a_0 = 0$, $a_{2n} = \frac{(-1)^n}{2n}$ $\forall n \in \mathbb{N}^*$ și $a_{2n-1} = 0$ $\forall n \in \mathbb{N}^*$.

$\lim_{n\to\infty} \sqrt[2n]{|a_{2n}|} = \lim_{n\to\infty} \sqrt[2n]{\frac{1}{2n}} = \lim_{n\to\infty} \frac{1}{\sqrt[2n]{2n}} = 1.$

$\lim_{n\to\infty} \sqrt[2n-1]{|a_{2n-1}|} = 0.$

Așadar $\overline{\lim} \sqrt[k]{|a_k|} = 1.$

Deci $R=\frac{1}{1}$.

Fie A mulțimea de convergență a seriei de puteri din enunț.

Avem $(-R,R)\subset A\subset[-R,R]$, i.e. $(-1,1)\subset A\subset[-1,1]$.

Dacă $x=1$ seria devine $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{2n}\cdot 1^{2n}=$

$=\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{2n}$ conv. (Crit. lui Leibniz).

Așadar $1\in A$.

Dacă $x=-1$ seria devine $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{2n}\cdot(-1)^{2n}=$

$=\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{2n}$ conv. (Crit. lui Leibniz).

Așadar $-1\in A$.

Prin urmare $A=[-1,1]$. $\square$

3. Să se dezvolte în serie de puteri ale lui x funcțiile de mai jos:

a) $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\sin x$.

Sol.: $I = \mathbb{R} = (-\infty, \infty)$.

$0 \in I$.

$f \in C^\infty(I)$.

$$
\left.\begin{array}{ll}
f(x) = \sin x & \Rightarrow f(0) = 0. \\
f'(x) = \cos x & \Rightarrow f'(0) = 1. \\
f''(x) = -\sin x \quad \forall x \in \mathbb{R} & \Rightarrow f''(0) = 0. \\
f'''(x) = -\cos x & \Rightarrow f'''(0) = -1. \\
f^{(4)}(x) = \sin x & \Rightarrow f^{(4)}(0) = 0. \\
\vdots & \quad \vdots
\end{array}\right.
$$

Conform Teoremei lui Taylor cu restul sub forma lui Lagrange $\forall x \in \mathbb{R}^*$ (i.e. $x \neq 0$), $\exists c$ între 0 și x (i.e. $c \in (0, x)$ sau $c \in (x, 0)$) a.î.

$$f(x) = \underbrace{f(0) + \frac{f'(0)}{1!}(x-0) + \ldots + \frac{f^{(n)}(0)}{n!}(x-0)^n}_{= T_n(x)} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-0)^{n+1}}_{= R_n(x)}.$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \quad \forall x \in \mathbb{R}^*, \forall n \in \mathbb{N}.$$

Arătăm că $\lim\limits_{n\to\infty} R_n(x) = 0 \ \forall x \in \mathbb{R}^*$.

Fie $x \in \mathbb{R}^*$.

Arătăm că $\lim\limits_{n\to\infty} |R_n(x)| = 0$.

$$0 \le |R_n(x)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}|x|^{n+1} \le \frac{1}{(n+1)!}|x|^{n+1} \ \forall n \in \mathbb{N}.$$

$$0 \xrightarrow{n\to\infty} 0, \qquad \frac{1}{(n+1)!}|x|^{n+1} \xrightarrow{n\to\infty} 0$$

(Crit. rap. pentru șiruri cu termeni strict pozitivi)

Deci $\lim\limits_{n\to\infty} |R_n(x)| = 0$, i.e. $\lim\limits_{n\to\infty} R_n(x) = 0$.

Așadar $f(x) = \sum\limits_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n \ \forall x \in \mathbb{R}^*$.

Prin urmare $\underset{\sin x}{\underset{\|}{f(x)}} = \sum\limits_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n =$

$$= 0 + \frac{1}{1!}x^1 + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 + \ldots =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot x^{2n+1} \ \forall x \in \mathbb{R}^*.$$

$f(0) = \sin 0 = 0.$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} 0^{2n+1} = 0.$$

$$\Rightarrow \underbrace{f(0)}_{\sin 0} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} 0^{2n+1}.$$

Deci $\underbrace{f(x)}_{\sin x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad \forall x \in \mathbb{R}.$ $\square$

b) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cos x$.

<u>Sol.</u>: Rezolvați-l voi! $\square$