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$\boxed{\text{Seminar 1}}$ - GAL.

Geometrie și algebră liniară

1). Determinanți. Rang. Sisteme liniare
2) Spații vectoriale.
3) Spații vectoriale euclidiene
4) Geometrie analitică euclidiană
5) Conice și cuadrice.

1). Determinanți.

$det : \mathcal{M}_n(\mathbb{K}) \to \mathbb{K}$, $(\mathbb{K}, +, \cdot)$ corp comutativ
$\mathbb{K} = \mathbb{R}, \mathbb{C}$

$$det(A) = \sum_{\sigma \in S_n} \varepsilon_\sigma a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

$(S_n, \cdot)$ grupul permutărilor, $\sigma = \begin{pmatrix} 1 \dots n \\ \sigma(1) \dots \sigma(n) \end{pmatrix}$

$\sigma : \{1, .., n\} \to \{1, .., n\}$ bijecție.

Ex. $f : \mathcal{M}_n(\mathbb{K}) \to \mathbb{K}$, $f(A) = det A$. Studiați inj/surj.

• Fie $A \in \mathcal{M}_n(\mathbb{K})$

a) minor de ordin p

$$\Delta_p = \begin{vmatrix} a_{i_1 j_1} \dots & a_{i_1 j_p} \\ a_{i_p j_1} \dots & a_{i_p j_p} \end{vmatrix} \quad 1 \le i_1 < .. < i_p \le n$$
$$1 \le j_1 < .. < j_p \le n$$

b) minor complementar lui Δ_p

Δ_c = obținut din A, suprimând $l_{i_1}, .., l_{i_p}$ (linii)
$c_{j_1} \dots c_{j_p}$ (coloane)

c) complement algebric pt Δ_p

$$c = (-1)^{i_1 + \dots + i_p + j_1 + \dots + j_p} \Delta_c$$

Caz particular $p=1$ $A = i \cdot \left(\cdots a_{ij} \cdots \right)$

un element

$\Delta_1 = \det(a_{ij})$

$\Delta_c = \Delta_{ij}$

$c_{ij} = (-1)^{i+j} \Delta_{ij}$ (complement algebric pt a_{ij})

Teorema Laplace

$\Delta_A = \det(A) =$ suma produselor minorilor de ordinul p (pentru p linii fixate, resp. p coloane fixate) cu complemenţii algebrici corespunzători

Caz particular $p=1$, $i \in \{1, .., n\}$ fixat

$\det A = a_{i1} c_{i1} + a_{i2} c_{i2} + \ldots + a_{in} c_{in}$

(dezvoltarea de pe linia l_i; analog pt dezv. de pe coloana j)

Aplicaţii

① $A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 3 & 4 \\ 2 & 5 & 1 & -1 \\ -1 & -2 & 2 & 4 \end{pmatrix}$

a) $p=2$, l_1, l_2 fixate ; b) $p=2$ l_2, l_3 fixate.

Calculaţi Δ_A

② Fie $A = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$

Calculaţi $\det A$

③ Fie $A = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}$

Calculaţi $\det A$

- $A \in \mathcal{M}_n(\mathbb{K})$ nesingulară $\Leftrightarrow \det A \neq 0$.

  A inversabilă $\Leftrightarrow \exists A^{-1} \in \mathcal{M}_n(\mathbb{K}) : AA^{-1} = A^{-1}A = I_n$.

Prop A inversabilă $\Leftrightarrow$ A nesingulară.

- $A^{-1} = \frac{1}{\det A} \cdot A^*$, $\det A \neq 0$

  A^*_{ij} = complemenții algebrici pt a_{ji}

Prop a) $\det(A^{-1}) = \frac{1}{\det A}$

b) $\det(A^*) = \det(A)^{n-1}$, $\forall A \in \mathcal{M}_n(\mathbb{K})$

Aplicații

① Fie $A = \begin{pmatrix} 2 & -1 & 3m+4 \\ 1 & m & 1 \\ -1 & -1 & 0 \end{pmatrix} \in \mathcal{M}_3(\mathbb{Z})$

$m = ?$ a.î. $A^{-1} \in \mathcal{M}_3(\mathbb{Z})$

Aceeași cerință pt $A = \begin{pmatrix} m & 1 & 2 \\ 3 & -1 & 2 \\ 1 & 0 & -1 \end{pmatrix}$

② Fie $A = \begin{pmatrix} 1+a^2 & ba & ca \\ ba & 1+b^2 & cb \\ ca & bc & 1+c^2 \end{pmatrix}$

Să se afle $\det(A^*)$

- $(GL(n, \mathbb{K}) = \{A \in \mathcal{M}_n(\mathbb{K}) \mid \det A \neq 0\}, \cdot)$ grupul general liniar

  $(O(n) = \{A \in \mathcal{M}_n(\mathbb{K}) \mid A \cdot A^T = I_n\}, \cdot)$ grupul ortogonal

  $(SO(n) = \{A \in O(n) \mid \det A = 1\}, \cdot)$ grupul special ortogonal.

- $A \in \mathcal{M}_n(\mathbb{K})$

  $P_A(X) = \det(A - XI_n) = (-1)^n \left[X^n - \sigma_1 X^{n-1} + \ldots + (-1)^n \sigma_n\right]$

  polinomul caracteristic asociat lui A

  σ_k = suma minorilor diagonali de ordin k, $k = \overline{1,n}$

$$\sigma_1 = \sum_{i=1}^{n} a_{ii} = Tr(A)$$

$$\sigma_2 = \sum_{i<j} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}$$

$$\sigma_3 = \sum_{i<j<k} \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix}$$

$\vdots$

$$\sigma_n = \det(A)$$

<u>Teorema HAMILTON-CAYLEY</u>

$\forall A \in \mathcal{M}_n(\mathbb{K})$

$P_A(A) = O_n \iff A^n - \sigma_1 A^{n-1} + \ldots + (-1)^n I_n \sigma_n = O_n$.

<u>Aplicații</u>

1) Fie $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ / $A = \begin{pmatrix} 3 & 2 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix}$

Să se calculeze A^{-1}, utilizând th H-C.

2)* Fie $A \in SO(3)$ ai $A^{100} = aA^2 + bA + cI_3$, $a, b, c \in \mathbb{R}$

Dacă $\varepsilon = \frac{-1 + i\sqrt{3}}{2}$ este răd. a polinomului caracteristic asociat lui A, atunci să se afle a, b, c.

3) Fie $A = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$ și $B = A^4 - 3A^3 + 3A^2 - 2A + 8I_2$

Să se afle $a, b \in \mathbb{R}$, unde $B = aA + bI_2$.

4) $A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ -1 & 0 & -2 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix}$

a) Să se scrie P_A

b) Să se calculeze A^{100}, utilizând th H-C.

Ex*. $A=\begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$

Dacă $a^2+b^2+c^2=1$, atunci $|\det(A)| \leq 1$.

Ex*. Fie $A, B \in \mathcal{M}_2(\mathbb{R})$ a.î. $AB=BA$

a) Să se arate că $\det(A^2+B^2) \geq 0$

b) Dacă $\det(A^2+B^2)=0$, at $\det A=\det B$

c) $\det(I_2+A^2) \geq 0$; d) $\det(A^2+A+I_2)$; e)* Dacă $A, B, C \in \mathcal{M}_2(\mathbb{R})$ comută 2 câte 2, atunci $\det(A^2+B^2+C^2-AB-AC-BC) \geq 0$

Ex $A=\begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix} \in \mathcal{M}_4(\mathbb{R})$

a) $A \cdot A^T = \alpha I_4$

b) Dacă $A \neq O_4$, at A inversabilă

Ex. $A \in \mathcal{M}_3(\mathbb{C})$

a) Fie $B=A-A^T \Rightarrow \det B=0$

b) Dacă $B \neq O_3$ și $\exists x, y \in \mathbb{C}$ a.î. $xA+yA^T$ inversabilă, at $x+y \neq$

Ex. Fie $A, B \in \mathcal{M}_2(\mathbb{R})$, $\operatorname{Tr} A = \operatorname{Tr} B$ și $\operatorname{Tr}(A^2)=\operatorname{Tr}(B^2) \Rightarrow \det A=\det B$

Ex. $A \in \mathcal{M}_n(\mathbb{R})$, $A^2=O_n \Rightarrow I_n-A, I_n+A$ inversabile

Ex $A \in \mathcal{M}_n(\mathbb{R})$, $A^3=O_n \Rightarrow I_n-A, I_n+A$ inversabile.

Ex. Fie x_1, x_2, x_3 sol ec $x^3+ax+b=0$, $a, b \in \mathbb{R}$

Dacă $A=\begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{pmatrix}$, iar $\Delta=\det A$, să se arate că

$\Delta^2=-4a^3-27b^2$

Ex. Să se rez în $\mathcal{M}_2(\mathbb{R})$ ec $X^n=A$, $A=\begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$

EX
$$\begin{vmatrix} a^3 & 3a^2 & 3a & 1 \\ a^2 & a^2+2a & 2a+1 & 1 \\ a & 2a+1 & a+2 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix} = (a-1)^6$$

EX
$$\begin{vmatrix} 1-n & 1 & 1 \\ 1 & 1-n & 1 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 1-n \end{vmatrix}$$

EX $\Delta_n = \det(a_{ij}) = 1$, unde $a_{ij} = \min\{i, j\}$ $1 \le i, j \le n$.

Rezolvări $(S_1 - \text{GAL})$

① $f : \mathcal{M}_n(\mathbb{K}) \to \mathbb{K}$, $f(A) = \det A$

Studiați inj/surj.

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SOL.

• $\left.\begin{array}{l} f(O_n) = f(A) = 0 \\ A = \begin{pmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 1 \\ 1 & \cdots & 1 \end{pmatrix} \end{array}\right\} \Rightarrow f$ nu e inj

• $\forall y \in \mathbb{K}$, $\exists A = \underbrace{\begin{pmatrix} y_1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}}_{\text{diag}(y, 1 \dots 1)} \in \mathcal{M}_n(\mathbb{K})$ aî $f(A) = \det A = y$

② Fie $A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 3 & 4 \\ 2 & 5 & 1 & -1 \\ -1 & -2 & 2 & 4 \end{pmatrix}$

Să se calculeze $\det A$, utilizând Th. Laplace

a) $p = 2$, ℓ_1, ℓ_2 fixate

b) $p = 2$, ℓ_2, ℓ_3 fixate.

SOL

a) $$\det A = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} (-1)^{1+2+1+2} \begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} (-1)^{1+2+1+3} \begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix}$$

$$+ \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} (-1)^{1+2+1+4} \begin{vmatrix} 5 & 1 \\ -2 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} (-1)^{1+2+2+3} \begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix} +$$

$$+ \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} (-1)^{1+2+2+4} \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} (-1)^{1+2+3+4} \begin{vmatrix} 2 & 5 \\ -1 & -2 \end{vmatrix} = -5$$

b) $$\det A = \begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix} (-1)^{2+3+1+2} \begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} (-1)^{2+3+1+3} \begin{vmatrix} 1 & 3 \\ -2 & 4 \end{vmatrix} +$$

$$+ \begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix} (-1)^{2+3+1+4} \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 5 & 1 \end{vmatrix} (-1)^{2+3+2+3} \begin{vmatrix} 1 & 3 \\ -1 & 4 \end{vmatrix} +$$

$$+ \begin{vmatrix} 1 & 4 \\ 5 & -1 \end{vmatrix} (-1)^{2+3+2+4} \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} (-1)^{2+3+3+4} \begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix} = -5$$

OBS

$l_2' = l_2 - l_1;\ l_3' = l_3 - 2l_1;\ l_4' = l_4 + l_1.$

$p = 1.$

$$\begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 3 & 4 \\ 2 & 5 & 1 & -1 \\ -1 & -2 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & \cancel{1} & \cancel{2} & \cancel{3} \\ 0 & 0 & 1 & 1 \\ 0 & 3 & -3 & -7 \\ 0 & -1 & 4 & 7 \end{vmatrix}$$

$$= 1(-1)^{1+1} \begin{vmatrix} 0 & 1 & 1 \\ 3 & -3 & -7 \\ -1 & 4 & 7 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 3 & 4 & -7 \\ -1 & -3 & 7 \end{vmatrix} = 1(-1)^{1+3} \begin{vmatrix} 3 & 4 \\ -1 & -3 \end{vmatrix}$$

$c_2' = c_2 - c_3$

$= -9 + 4 = -5$

③ Fie $A = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$ (matrice circulară)

Calculați $\det A$

SOL

$$\det A = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$l_1' = l_1 + l_2 + l_3$

$$= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix} = (a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ b & c-b & a-b \\ c & a-c & b-c \end{vmatrix}$$

$c_2' = c_2 - c_1$
$c_3' = c_3 - c_1$

$$= (a+b+c) \cdot 1(-1)^{1+1} \begin{vmatrix} c-b & a-b \\ a-c & b-c \end{vmatrix} =$$

$$= (a+b+c)\left[-(b-c)^2 - (a-b)(a-c)\right] =$$

$$= (a+b+c)(-b^2 - c^2 - a^2 + ab + ac + bc) = -(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$$

$$= -\frac{1}{2}(a+b+c)\left[(a^2-2ab+b^2) + (a^2-2ac+c^2) + (b^2-2bc+c^2)\right]$$

$$\Rightarrow \Delta = -\frac{1}{2}(a+b+c)\left[(a-b)^2 + (a-c)^2 + (b-c)^2\right].$$

OBS a) $a^2+b^2+c^2-ab-ac-bc \geq 0 \Rightarrow ab+ac+bc \leq a^2+b^2+c^2$

b) Dacă $a+b+c \neq 0$ și $\Delta = 0$, atunci $a=b=c$.

(4)
$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \quad (\text{det Vandermonde})$$

SOL

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & (b-a)(b+a) & (c-a)(c+a) \end{vmatrix} = 1(-1)^{1+1} \begin{vmatrix} b-a & c-a \\ (b-a)(b+a) & (c-a)(c+a) \end{vmatrix}$$

$c_2' = c_2 - c_1$
$c_3' = c_3 - c_1$

$$= (b-a)(c-a) \underbrace{\begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix}}_{c-b}$$

$$\Delta = (c-b)(c-a)(b-a)$$

OBS

a) $A \cdot A^{-1} = I_n \mid \det$

$\det A \cdot \det(A^{-1}) = 1 \Rightarrow \boxed{\det(A^{-1}) = \frac{1}{\det A}}$

b) $A^{-1} = \frac{1}{\det A} \cdot A^* \quad (\det A \neq 0)$

$\det(A^{-1}) = \det\left(\frac{1}{\det A} \cdot A^*\right)$; $\det(\alpha A) = \alpha^n \det A$, $\forall A \in \mathcal{M}_n(\mathbb{C})$

$\frac{1}{\det A} = \frac{1}{(\det A)^n} \cdot \det(A^*) \Rightarrow \boxed{\det A^* = (\det A)^{n-1}}$

(5) Fie $A = \begin{pmatrix} 1+a^2 & ba & ca \\ ab & 1+b^2 & cb \\ ac & bc & 1+c^2 \end{pmatrix}$

Să se calc. $\det(A^*)$.

SOL

$$\det A = \begin{array}{c} \begin{matrix} 1 & 1' & & 2 & 2' & & 3 & 3' \end{matrix} \\ \begin{vmatrix} 1+a^2 & 0+ba & 0+ca \\ 0+ab & 1+b^2 & 0+cb \\ 0+ac & 1+bc & 1+c^2 \end{vmatrix} \end{array} =$$

$= |1\ 2\ 3| + \underbrace{|1\ 2\ 3'|}_{=0} + \underbrace{|1\ 2'\ 3|}_{=0} + \underbrace{|1\ 2'\ 3'|}_{=0} +$

$+ |1'\ 2\ 3| + \underbrace{|1'\ 2\ 3'|}_{=0} + \underbrace{|1'\ 2'\ 3|}_{=0} + \underbrace{|1'\ 2'\ 3'|}_{=0}$

OBS: $1', 2', 3'$ sunt proporționale.

$= |1\,2\,3| + |1'\,2\,3| + |1\,2'\,3| + |1\,2\,3'|$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} a^2 & 0 & 0 \\ ab & 1 & 0 \\ ac & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & ba & 0 \\ 0 & b^2 & 0 \\ 0 & bc & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & ca \\ 0 & 1 & cb \\ 0 & 0 & c^2 \end{vmatrix}$$

$= 1 + a^2 + b^2 + c^2 > 0$

$\det(A^*) = (\det A)^2 = (1 + a^2 + b^2 + c^2)^2$

$(n=3)$

(6) Fie $A = \begin{pmatrix} 2 & -1 & 3m+4 \\ 1 & m & 1 \\ -1 & -1 & 0 \end{pmatrix} \in \mathcal{M}_3(\mathbb{Z})$

$m = ?$ aî $A^{-1} \in \mathcal{M}_3(\mathbb{Z})$

<u>SOL</u>

$$\det A = \begin{vmatrix} 2 & -1 & 3m+4 \\ 1 & m & 1 \\ -1 & -1 & 0 \end{vmatrix} \underset{c_2' = c_2 - c_1}{=} \begin{vmatrix} 2 & -3 & 3m+4 \\ 1 & m-1 & 1 \\ -1 & 0 & 0 \end{vmatrix}$$

$$= (-1)(-1)^{3+1} \begin{vmatrix} -3 & 3m+4 \\ m-1 & 1 \end{vmatrix} = -\left[-3 - (m-1)(3m+4)\right]$$

$= 3 + 3m^2 - 3m + 4m - 4 = 3m^2 + m - 1.$

$$\left.\begin{aligned} &A \in \mathcal{M}_3(\mathbb{Z}) \Rightarrow \det A \in \mathbb{Z} \\ &A^{-1} \in \mathcal{M}_3(\mathbb{Z}) \Rightarrow \det(A^{-1}) = \frac{1}{\det A} \in \mathbb{Z} \end{aligned}\right\} \Rightarrow \det A = \pm 1.$$

1) $\det A = 1 \Rightarrow 3m^2 + m - 2 = 0$ $\nearrow m_1 = -1$, $\searrow m_2 = \frac{2}{3} \notin \mathbb{Z}.$

2) $\det A = -1 \Rightarrow 3m^2 + m = 0$ $\rightarrow m_1 = 0$, $\searrow m_2 = -\frac{1}{3} \notin \mathbb{Z}$

Deci $m \in \{-1, 0\}$.

<u>OBS</u>

$P_A(X) = \det(A - XI_n) = (-1)^n \left(X^n - \sigma_1 X^{n-1} + \ldots + \sigma_n (-1)^n\right)$

σ_k = suma minorilor diag. de ord k, $1 \le k \le n$, $\forall A \in \mathcal{M}_n(\mathbb{K})$

T. Hamilton - Cayley

$\forall A \in \mathcal{M}_n(\mathbb{K}) \Rightarrow P_A(A) = O_n$

$A^n - \sigma_1 A^{n-1} + \dots + (-1)^n \sigma_n I_n = O_n$

Ex 7 $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ Să se afle A^{-1}, utilizând Th H-C.

SOL

$A^3 - \sigma_1 A^2 + \sigma_2 A - \sigma_3 I_3 = O_3$.

$\sigma_1 = Tr A = 2$

$\sigma_2 = \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1 - 1 = 0$

$\sigma_3 = \det A = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1 \cdot (-1)^{1+1} \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1$

$A^3 - 2A^2 + I_3 = O_3 \quad | \cdot A^{-1}$

$A^2 - 2A + A^{-1} = O_3 \Rightarrow A^{-1} = -A^2 + 2A$.

8 $A = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$

$B = A^4 - 3A^3 + 3A^2 - 2A + 8I_2$

Să se determine $a, b \in \mathbb{R}$ a.î. $B = aA + bI_2$.

SOL

$A^2 - \sigma_1 A + \sigma_2 I_2 = O_2$ $\quad \sigma_1 = Tr A = 1$

$\sigma_2 = \det A = 2$

$A^2 - A + 2I_2 = O_2$.

$P_A = X^2 - X + 2 \in \mathbb{R}[X]$

Fie $Q = X^4 - 3X^3 + 3X^2 - 2X + 8 \in \mathbb{R}[X]$

$Q = P_A \cdot C + R$ (teorema de împ. cu rest), grad $R <$ grad P_A

$Q(A) = R(A)$ $\quad (P_A(A) = O_2)$

$$
\begin{array}{l|l}
X^4 - 3X^3 + 3X^2 - 2X + 8 & X^2 - X + 2 \\
\hline
-X^4 + X^3 - 2X^2 & X^2 - 2X - 1 \\
\hline
\quad / \quad -2X^3 + X^2 - 2X + 8 & \\
\quad\quad 2X^3 - 2X^2 + 4X & \\
\hline
\quad\quad / \quad -X^2 + 2X + 8 & \\
\quad\quad\quad X^2 - X + 2 & \\
\hline
\quad\quad\quad / \quad X + 10 &
\end{array}
$$

$R = X + 10.$

$$\left.\begin{array}{l} B = R(A) = A + 10 I_2 \\ B = aA + b I_2 \end{array}\right\} \Rightarrow \begin{array}{l} a = 1 \\ b = 10 \end{array}$$

(9) Fie $A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ -1 & 0 & -2 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix}$

a) $P_A = ?$; b) $A^{100} - ?$, utilizând th H-C

Sol.

a) $P_A(A) = A^4 - \sigma_1 A^3 + \sigma_2 A^2 - \sigma_3 A + \sigma_4 I_4 = O_4$.

$\sigma_1 = \operatorname{Tr} A = 0$

$$\sigma_2 = \begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ -1 & -2 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 1 \\ -1 & 0 \end{vmatrix}$$

$= 1 + 1 + 0 - 4 + 1 + 1 = 0$

$$\sigma_3 = \begin{vmatrix} 2 & 0 & 1 \\ 0 & -2 & 1 \\ -1 & -1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 0 \\ -1 & -2 & 1 \\ 0 & -1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 0 \\ -1 & 2 & 1 \\ 0 & -1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 1 \\ -1 & 2 & 0 \\ -1 & 0 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -2 & 1 \\ 0 & -2 & 1 \\ -1 & 0 & 0 \end{vmatrix} + (-1)(-1)^{3+2} \begin{vmatrix} 0 & 0 \\ -1 & -1 \end{vmatrix} + (-1)(-1)^{3+2} \begin{vmatrix} 0 & 0 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 1 \\ -1 & 2 & 0 \\ -1 & 2 & -2 \end{vmatrix}$$

$= 0 + 0 + 0 + 0 = 0$

$\sigma_3 = \det A = 0$

$P_A = X^4$

b) $A^4 = O_4$ (nilpotentă) $\longrightarrow A^{100} = O_4$