$f : A \to B$ funcție

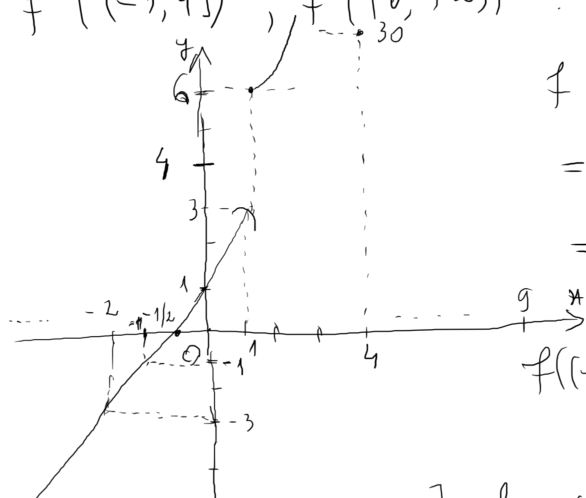
$C \subseteq A$, $D \subseteq B$

$f(C) := \{ f(c) \mid c \in C \}$, $\quad f^{-1}(D) := \{ a \in A \mid f(a) \in D \}$.

Exp: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 2x+1, & x \in (-\infty, 1) \\ x^2+3x+2, & x \in [1, +\infty) \end{cases}$

- $f([-2,4])$, $f((-\infty, 9])$, $\operatorname{Im} f = ?$
- $f^{-1}((-1,4])$, $f^{-1}((0,+\infty)) = ?$

Sol:



$f([-2,4]) = f([-2,1) \cup [1,4])$

$= f([-2,1)) \cup f([1,4])$

$= [-3,3) \cup [6,30]$

$f((-\infty,9]) = f((-\infty,1)) \cup f([1,9])$

$= (-\infty,3) \cup [6,110]$

$\operatorname{Im} f = (-\infty,3) \cup [6,+\infty)$

$f^{-1}((-1,4]) = \{ x \in \mathbb{R} \mid f(x) \in (-1,4] \}$ $\quad = \mathbb{R} \setminus [3,6)$

$= (-1, 1)$

$f^{-1}((0,+\infty)) = \{ x \in \mathbb{R} \mid f(x) \in (0,+\infty) \}$

$= \left(-\frac{1}{2}, +\infty\right)$.

1) $f : A \to B$, f = funcție

$f(X_1 \cap X_2) = f(X_1) \cap f(X_2) \ \forall X_1, X_2 \subseteq A \iff f$ = inj.

<u>Sol:</u> $f(X_1 \cap X_2) \subseteq f(X_1) \cap f(X_2), \forall X_1, X_2 \subseteq A$

$X_1 \cap X_2 \subseteq X_1, X_2 \Rightarrow$

"$\Rightarrow$" Fie $a_1, a_2 \in A$ a. î. $f(a_1) = f(a_2)$. Vreau $a_1 = a_2$.

Iau $X_1 = \{a_1\}$, $X_2 = \{a_2\}$; $X_1, X_2 \subseteq A$

$$f(X_1) \cap f(X_2) = \{f(a_1)\} \cap \{f(a_2)\} = \{f(a_1)\}$$

$$= \{f(a_2)\}.$$

$$\Rightarrow f(X_1 \cap X_2) = f(X_1) \cap f(X_2) = \{f(a_1)\} = \{f(a_2)\} \neq \emptyset$$

$$\Rightarrow X_1 \cap X_2 \neq \emptyset \Rightarrow \{a_1\} \cap \{a_2\} \neq \emptyset \Rightarrow a_1 = a_2.$$

„$\Leftarrow$" Știu că f = inj. Vreau $f(X_1 \cap X_2) \supseteq f(X_1) \cap f(X_2)$, $\forall X_1, X_2 \subseteq A$

Fie $b \in f(X_1) \cap f(X_2) \Leftrightarrow b \in f(X_1),\ b \in f(X_2)$

$\Leftrightarrow \exists x_1 \in X_1,\ x_2 \in X_2$ a.î. $b = f(x_1) = f(x_2)$

$f = \text{inj} \Rightarrow x_1 = x_2 \in X_1 \cap X_2 \Rightarrow b \in f(X_1 \cap X_2)$.

2) $f : A \to B$, $X \subseteq A$, $Y \subseteq B$; (i) $f(f^{-1}(Y)) \subseteq Y$, cu egalitate doar dacă f = surj

(ii) $f^{-1}(f(X)) \supseteq X$, cu egalitate doar dacă f = inj.

Sol: (i) $b \in f(\underbrace{f^{-1}(Y)})$ $\Rightarrow \exists a \in f^{-1}(Y)$ a.î. $b = f(a)$

$a \in f^{-1}(Y) \overset{def}{\Leftrightarrow} f(a) \in Y \Leftrightarrow b \in Y$. Deci $f(f^{-1}(Y)) \subseteq Y$.

Dacă f = surj $\Rightarrow f(f^{-1}(Y)) \supseteq Y$, deci am egalitate.

Fie $b \in Y \subseteq B \overset{f = surj}{\Longrightarrow} \exists a \in A$ a.î. $f(a) = b \in Y$.

Rezultă că $a \in f^{-1}(Y) \Rightarrow b = f(a) \in f(\underset{a}{\underset{\cup}{f^{-1}(Y)}})$

Temă: Dacă $f(f^{-1}(Y)) = Y$ $\forall Y \subseteq B$, atunci f = surj.

(ii) $f^{-1}(f(X)) \supseteq X,\ \forall X \subseteq A.$

Fie $x \in X \Rightarrow f(x) \in f(X) \overset{def}{\Longleftrightarrow} x \in f^{-1}(f(X)).$

Dacă, în plus, $f =$ inj atunci $f^{-1}(f(X)) \subseteq X,\ \forall X \subseteq A$

Fie $a \in f^{-1}(\underbrace{f(X)}_{:=Y_0}) \overset{def}{\Longleftrightarrow} f(a) \in f(X) \overset{def}{\Longleftrightarrow} \exists x \in X$ a.î.

$f(a) = f(x)$. Cum $f =$ inj $\Rightarrow a = x \in X$ ✓

Temă: $f^{-1}(f(X)) = X\ \forall X \subseteq A \Rightarrow f =$ inj.

3) Fie A, B mulțimi finite (cu $|A| = |B| = n < \infty$) și $f: A \to B$ funcție. Atunci UASE.

(i) $f =$ inj ; (ii) $f =$ surj ; (iii) $f =$ bij

Sol: (iii) $\Rightarrow$ (ii) $\qquad A = \{a_1, \dots, a_n\},\ a_i \neq a_j\ \forall i \neq j$

$\Rightarrow$ (i) $\qquad B = \{b_1, \dots, b_n\},\ b_i \neq b_j\ \forall i \neq j$

(i) $\Rightarrow$ (iii): $\operatorname{Im} f = \{f(a_i) \mid 1 \leq i \leq n\}$

$f =$ inj $\Rightarrow f(a_i) \neq f(a_j)\ \forall i \neq j \Rightarrow |\operatorname{Im} f| = n = |B|$

Cum $\operatorname{Im} f \subseteq B \Rightarrow \operatorname{Im} f = B \Rightarrow f =$ surj $\Rightarrow f =$ bij.

(ii) $\Rightarrow$ (iii): $\forall 1 \leq i \leq n,\ A_i := \{a \in A \mid f(a) = b_i\} \subseteq A$
$= f^{-1}(\{b_i\}).$

. $f =$ surj $\Rightarrow A_i \neq \emptyset\ \ \forall 1 \leq i \leq n.$

. $\forall i \neq j,\ A_i \cap A_j = \emptyset$ (altfel, $\exists a \in A_i \cap A_j \Rightarrow$
$\Rightarrow f(a) = b_i$ și $f(a) = b_j \Rightarrow b_i = b_j$ ↯ cu $i \neq j$)

Atunci $\left|\bigcup_{i=1}^{n} A_i\right| \underset{\substack{\uparrow \\ A_i \cap A_j = \emptyset\ \forall i \neq j}}{=} \sum_{i=1}^{n} \underbrace{|A_i|}_{\geq 1} \geq n = |A|$

Cum $\bigcup_{i=1}^{n} A_i \subseteq A \Rightarrow \left|\bigcup_{i=1}^{n} A_i\right| \leq |A| = n$ $\Bigg\} \Rightarrow$

$\Rightarrow \bigcup_{i=1}^{n} A_i = A$ și, deci, $|A_i| = 1\ \ \forall 1 \leq i \leq n.$

Astfel, $\forall b \in B,\ \exists!\, a \in A$ a.î. $f(a) = b \Rightarrow f =$ bij.

4) Fie $A \neq \emptyset$. UASE: (i) A - mulț. infinită

(ii) $\exists f: A \to A$ inj dar nu surj

(iii) $\exists f: A \to A$ surj dar nu inj.

Sol: (ii) $\Rightarrow$ (i) le avem de la problema anterioară.

(iii) $\Rightarrow$

(i) $\Rightarrow$ (ii) și (i) $\Rightarrow$ (iii):

$A \neq \emptyset \Rightarrow \exists a_0 \in A$

A - infinită $\Rightarrow \exists a_1 \in A \setminus \{a_0\}$

$\exists a_2 \in A \setminus \{a_0, a_1\}$

$\vdots$

$\exists a_{n+1} \in A \setminus \{a_0, a_1, \dots, a_n\}$

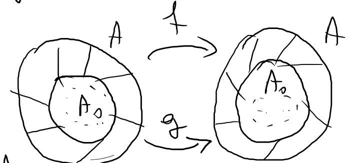
$\vdots$

Deci:

A infinită $\Rightarrow \exists A_0 = (a_i)_{i \in \mathbb{N}} \subset A$ cu $a_i \neq a_j \ \forall i \neq j$

Definesc

$f: A \to A$, $f(a) = \begin{cases} a_{i+1}, & a \in A_0, \ a = a_i \\ a, & a \in A \setminus A_0 \end{cases}$



f = inj dar nu surj ($a_0 \notin \operatorname{Im} f$).

și $g: A \to A$, $g(a) = \begin{cases} a_{i-1}, & a \in A_0, \ a = a_i \text{ cu } i \geq 1 \\ a_2, & a = a_0 \\ a, & a \in A \setminus A_0 \end{cases}$

g = surj dar nu inj $(f(a_3) = f(a_0))$

$\operatorname{Im} g = A_0 \cup (A \setminus A_0) = A$

5) Fie A, B finite cu $|A| = a$, $|B| = b$.

Nr funcțiilor injective $f : A \to B$?

Sol: $\exists f : A \to B$ inj $\iff |A| \le |B|$.

$\Updownarrow$

$|\mathrm{Im} f| = |A|$, $\mathrm{Im} f \subseteq B$

$f : A = \{x_1, \dots, x_a\} \to B$ e complet det de submulț.

ordonată $\{f(x_1), \dots, f(x_a)\}$ a lui B.

$$\text{Nr funcțiilor inj} = \begin{cases} 0, & a > b \\ A_b^a, & a \le b \end{cases}$$

Altfel: Ind. matem. după $a \ge 1$ (temă!)

6) $|A| = a$, $|B| = b$ finite

Nr. funcțiilor $f : A \to B$ bijective ?

Sol: $$\text{nr. cerut} = \begin{cases} 0, & a \ne b \\ a!, & a = b \end{cases}$$

A, B finite

$\exists f : A \to B$ bij $\iff |A| = |B|$.