1) $G = \mathbb{Z}\times\mathbb{Z} / \langle (2,2)\rangle$

Calculați $o(\widehat{(1,1)})$ și $o(\widehat{(1,0)})$ în G. Este G - grup infinit? Dar ciclic?

Sol: $H \overset{\text{not}}{=} \langle (2,2)\rangle = \{k(2,2) \mid k\in\mathbb{Z}\} = \{(2k,2k) \mid k\in\mathbb{Z}\}$.

- Caut $t\in\mathbb{N}^*$ a.î. $t\widehat{(1,1)} = \widehat{(0,0)} \iff \widehat{(t,t)} = \widehat{(0,0)}$ $\iff (t,t)-(0,0)\in H \iff (t,t)\in H \iff \exists k\in\mathbb{Z}$ a.î. $(t,t) = (2k,2k)$. Rezultă că $o(\widehat{(1,1)}) = 2$.

- Caut $t\in\mathbb{N}^*$ a.î. $t\widehat{(1,0)} = \widehat{(0,0)} \iff \widehat{(t,0)} = \widehat{(0,0)} \iff$ $\iff (t,0)\in H \iff \exists k\in\mathbb{Z}$ a.î. $(t,0) = (2k,2k)$ $\iff \exists k\in\mathbb{Z}$ a.î. $\begin{cases} t = 2k \\ 2k = 0\end{cases} \Rightarrow t = k = 0$.

 Deci $\nexists\, t\in\mathbb{N}^*$ a.î. $t\widehat{(1,0)} = \widehat{(0,0)} \overset{\text{def}}{\iff} o(\widehat{(1,0)}) = \infty$.

- $G = \mathbb{Z}\times\mathbb{Z}/\langle(2,2)\rangle$ este infinit pt că $o(\widehat{(1,0)}) = \infty$.

- G - infinit; dacă G este ciclic $\Rightarrow G \simeq \underset{\substack{\downarrow \\ \text{are toate elem} \neq 0 \\ \text{de ordin infinit}}}{\mathbb{Z}} \Rightarrow$

$\Rightarrow G$ are toate elem $\neq \widehat{(0,0)}$ de ordin infinit, dar $o(\widehat{(1,1)}) = 2$. Deci G nu este ciclic.

Obs: $\underset{\substack{\uparrow \\ \text{sistem minimal de generatori}}}{\mathbb{Z}\times\mathbb{Z} = \langle (1,0),(0,1)\rangle}$ $\quad ((a,b) = a(1,0)+b(0,1)$ $\forall a,b\in\mathbb{Z})$

$\Rightarrow \underbrace{\mathbb{Z}\times\mathbb{Z}/\langle(2,2)\rangle}_{\text{nu este ciclic}} = \langle \widehat{(1,0)}, \widehat{(0,1)}\rangle \rightarrow$ sistem minimal de generatori

2) Dacă $G = H \times K$ este produs direct de grupuri, $x \in H$ și $y \in K$ atunci: (i) $o((x,y)) = \infty$ dacă $o(x) = \infty$ sau $o(y) = \infty$.

(ii) $o((x,y)) = [o(x), o(y)]$ dacă $o(x), o(y) < \infty$.

Sol: $e_G = (e_H, e_K)$

Dacă $t \in \mathbb{N}^*$ a.î. $(x,y)^t = (e_H, e_K) \iff (x^t, y^t) = (e_H, e_K)$

$\iff \begin{cases} x^t = e_H \\ y^t = e_K \end{cases} \iff o(x), o(y) < \infty. \rightarrow$ rezolvă (i)

(ii) Fie $o(x) = m < \infty$, $o(y) = n < \infty$

• $(x,y)^{[m,n]} = \left(x^{\frac{[m,n]}{m} m}, y^{\frac{[m,n]}{n} n}\right)$

$= \left((x^m)^{\underbrace{\frac{[m,n]}{m}}_{\in \mathbb{N}^*}}, (y^n)^{\underbrace{\frac{[m,n]}{n}}_{\in \mathbb{N}^*}}\right)$

$= \left(e_H^{\frac{[m,n]}{m}}, e_K^{\frac{[m,n]}{n}}\right) = (e_H, e_K) = e_G.$

• Fie $t \in \mathbb{N}^*$ a.î. $(x,y)^t = (e_H, e_K) \iff \begin{cases} x^t = e_H \\ y^t = e_K \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} o(x) = m \mid t \\ o(y) = n \mid t \end{cases} \Rightarrow [m,n] \mid t$ ✓

3) (i) Calculați $o((\overline{2}, \hat{8}))$ din $\mathbb{Z}_{12} \times \mathbb{Z}_{72}$

(ii) ~~Det~~. elem de ordin 18 din $\mathbb{Z}_{12} \times \mathbb{Z}_{72}$.

Sol: (i) $o((\overline{2}, \hat{8})) = [o(\overline{2}), o(\hat{8})] = [6, 9] = 18$.

Obs: $o(x^k) = \dfrac{o(x)}{(k, o(x))} \Rightarrow \underset{\mathbb{Z}_n}{o(\hat{a})} = o(a\hat{1}) = \dfrac{o(\hat{1})}{(a, o(\hat{1}))} = \dfrac{n}{(a,n)}$

(ii) $\underset{\mathbb{Z}_{12} \times \mathbb{Z}_{72}}{o((\overline{a}, \hat{b}))} = 18 \Leftrightarrow [o(\overline{a}), o(\hat{b})] = 18$

$o(\overline{a}) \mid 12$

$o(\hat{b}) \mid 72$

Caz 1 $o(\overline{a}) = 1$, $o(\hat{b}) = 18 \Rightarrow \overline{a} = \overline{0}$, iar 18 este cel mai mic nr nat pt care $18\hat{b} = \hat{0} \Leftrightarrow 4 \mid b \Leftrightarrow b = 4k$

$\hat{b} \in \{\hat{4}, \widehat{20}, \widehat{28}, \widehat{44}, \widehat{52}, \widehat{68}\}$.

$18 = o(\hat{b}) = \dfrac{72}{(4k, 72)} \Leftrightarrow 4(k, 18) = 4 \Leftrightarrow (k, 18) = 1$

Caz 2 $o(\overline{a}) = 2$, $o(\hat{b}) = 9$ sau 18 ~~rezolvat mai sus~~

$\dfrac{12}{(a,12)} = 2$ $\Leftrightarrow$ $(a, 12) = 6 \Leftrightarrow a = 6$

$\hat{b} \in \{\widehat{8k} \mid 3 \nmid k\} = \{\hat{8}, \widehat{16}, \widehat{32}, \dots\}$

Caz 3 $o(\overline{a}) = 3$, $o(\hat{b}) = 18$ ~~rezolvat~~

$\Leftrightarrow (a, 12) = 4 \Leftrightarrow a \in \{4, 8\}$

Caz 4 $o(\overline{a}) = 4$; $\nexists\ \hat{b} \in \mathbb{Z}_{72}$ a.î. $[4, o(\hat{b})] = 18$

⋮

ș.a.m.d.

4) $G = \mathbb{Z}_9 \times \mathbb{Z}_{18} \underset{\supseteq}{\overset{?}{=}} \langle (\bar{1}, \hat{3}), (\bar{3}, \hat{5}) \rangle$

Sol: ? „$\subseteq$"

$$\langle (\bar{1}, \hat{3}), (\bar{3}, \hat{5}) \rangle \underset{\uparrow}{=} \{ \alpha (\bar{1}, \hat{3}) + \beta (\bar{3}, \hat{5}) \mid \alpha, \beta \in \mathbb{Z} \}$$

generatorii comută

$$= \{ (\overline{\alpha + 3\beta}, \widehat{3\alpha + 5\beta}) \mid \alpha, \beta \in \mathbb{Z} \}$$

„$\subseteq$" $\iff \forall a, b \in \mathbb{Z}$, $\exists \alpha, \beta \in \mathbb{Z}$ a.î.

$$(\bar{a}, \hat{b}) = (\overline{\alpha + 3\beta}, \widehat{3\alpha + 5\beta}) \iff \begin{cases} \alpha + 3\beta \equiv a \pmod 9 \\ 3\alpha + 5\beta \equiv b \pmod{18} \end{cases}$$

$$\overset{(2,9)=1}{\iff} \begin{cases} 2\alpha + 6\beta \equiv 2a \pmod{18} \\ 3\alpha + 5\beta \equiv b \pmod{18} \end{cases} \overset{(!)}{\Longrightarrow} \alpha - \beta \equiv b - 2a \pmod{18}$$

$$\Rightarrow \alpha \equiv b - 2a + \beta \pmod{18} \quad (*)$$

$$3\alpha + 5\beta \equiv b \pmod{18} \Rightarrow 3b - 6a + 8\beta \equiv b \pmod{18}$$

$$\Rightarrow 8\beta \equiv 6a - 2b \pmod{18} \Rightarrow 4\beta \equiv 3a - b \pmod 9$$

$$\left.\begin{aligned} &\Rightarrow \bar{4} \cdot \bar{\beta} = \overline{3a - b} \ (\text{în } \mathbb{Z}_9) \\ &\exists\, \bar{4}^{-1} = \bar{7} \end{aligned}\right\} \Rightarrow \bar{\beta} = \overline{21a - 7b} \text{ în } \mathbb{Z}_9$$

Iau $\beta = 21a - 7b \overset{(*)}{\Longrightarrow} \alpha \equiv 19a - 6b \pmod{18}$; iau $\alpha = 19a - 6b$

Verificare: $\alpha + 3\beta = 82a - 27b \equiv a \pmod 9$ ✓

$3\alpha + 5\beta = 162a - 53b \equiv b \pmod{18}$ ✓

Obs: (!) se reduce la sistemul $\begin{cases} \hat{2}\hat{\alpha} + \hat{6}\hat{\beta} = \widehat{2a} \\ \hat{3}\hat{\alpha} + \hat{5}\hat{\beta} = \hat{b} \end{cases}$ în $\mathbb{Z}_{18}$,

cu soluția $\hat{\alpha} = \widehat{19}\,\hat{a} - \hat{6}\,\hat{b}$, $\hat{\alpha} - \hat{\beta} = \widehat{b - 2a}$ în $\mathbb{Z}_{18}$

5) $G = \mathbb{Z}\times\mathbb{Z} / \langle (a,b)\rangle$ - ciclic $\iff (a,b)=1$.

$(a,b\in\mathbb{Z}^*$ fixate$)$

Sol.:

$\mathbb{Z}\times\mathbb{Z} = \langle (1,0),(0,1)\rangle \Rightarrow G = \langle \widehat{(1,0)}, \widehat{(0,1)}\rangle$

G-ciclic $\iff \exists (u,v)\in \mathbb{Z}\times\mathbb{Z}$ a.î. $G=\langle \widehat{(u,v)}\rangle$

$\iff \widehat{(1,0)}, \widehat{(0,1)} \in \langle\widehat{(u,v)}\rangle = \{\widehat{(ku,kv)} \mid k\in\mathbb{Z}\}$

$\iff \exists k,l\in\mathbb{Z}$ a.î. $\begin{cases}\widehat{(1,0)} = \widehat{(ku,kv)}\\ \widehat{(0,1)} = \widehat{(lu,lv)}\end{cases} \iff \begin{cases}(1-ku,-kv)\in\langle(a,b)\rangle\\ (-lu,1-lv)\in\langle(a,b)\rangle\end{cases}$

$(\langle(a,b)\rangle = \{(ta,tb)\mid t\in\mathbb{Z}\})$ $\iff \exists k,l,s,t\in\mathbb{Z}$ a.î.

$\begin{cases}(1-ku,-kv) = (ta,tb)\\ (-lu,1-lv) = (sa,sb)\end{cases} \iff \begin{cases} ku+ta=1 & (1)\\ tb=-kv & (2)\\ sa=-lu & (3)\\ sb+lv=1 & (4)\end{cases}$

Din $(1) \Rightarrow (a,u)=1$
$(3) \Rightarrow a \mid lu$ $\Big\} \Rightarrow a\mid l \Rightarrow l = al'$ cu $l'\in\mathbb{Z} \overset{(4)}{\Rightarrow}$

$\Rightarrow sb + l'va = 1 \Rightarrow (a,b)=1$.

Reciproc, dacă $(a,b)=1 \Rightarrow \exists \alpha,\beta\in\mathbb{Z}$ a.î. $\alpha a+\beta b=1$

Atunci $G = \langle \widehat{(-\beta,\alpha)}\rangle$ pt că: $\cdot\ \widehat{(1,0)} = -b\,\widehat{(-\beta,\alpha)}$

$\left(= \widehat{(b\beta,-b\alpha)} = \widehat{(1-\alpha a,-\alpha b)} = \widehat{(1,0)} - \alpha \underbrace{\widehat{(a,b)}}_{=\widehat{(0,0)}}\right)$

$\cdot\ \widehat{(0,1)} = a\,\widehat{(-\beta,\alpha)}$

$\left(= \widehat{(-a\beta,a\alpha)} = \widehat{(-a\beta,1-b\beta)} = -\beta\underbrace{\widehat{(a,b)}}_{=\widehat{(0,0)}} + \widehat{(0,1)}\right)$