Signals, Spectra and Signal Processing

Laboratory 6 Digital Filter Design

6.1 Introduction

The purpose of this lab[1] is to introduce the concept of FIR and IIR filters and then concept of FIR filter design.

6.2 Digital Filters

In general, a linear and time-invariant casual digital filter with input $x(n)$ and output $y(n)$ may be specified by its difference equation

$$y(n) = \sum_{i=0}^{N-1} b_i x(n-i) - \sum_{k=1}^{M} a_k y(n-k)$$

where b_i and a_k are coefficients which parameterize the filter. This filter is said to have N zeros and M poles. Each new value of the output signal $y(n)$ is determined by past values of the output and by present and past values of the input. The impulse response, $h(n)$, is the response of the filter to an input of $\delta(n)$

$$h(n) == \sum_{i=0}^{N-1} b_i \delta x(n-i) - \sum_{k=1}^{M} a_k h(n-k)$$

There are two general classes of digital filters: infinite impulse response (IIR) and finite impulse response (FIR). The FIR case occurs when $a_k = 0$, for all k. Such a filter is said to have no poles, only zeros. In this case, the difference equation becomes

$$h(n) = \sum_{i=0}^{N-1} b_i \delta(n-i)$$

Since the equation is no longer recursive, the impulse response has finite duration N.

In the case where $a_k \neq 0$, the difference equation usually represents an IIR filter. In this case, the impulse response has non-zero values as $n \to \infty$.

[1] S. Burrus, etal., *Computer-Based Exercises for Signal Processing using Matlab*, Prentice-Hall: Englewood Cliffs, NJ, 1994, pp.29-31

6.3 FIR Filter Design

To illustrate the use of zeros in filter design, you will design a simple second order FIR filter with the two zeros on the unit circle. In order for the filters impulse response to be real, the two zeros must be complex conjugates of one another:

$$z_1 = e^{j\theta} \qquad z_2 = e^{-j\theta}$$

where θ is the angle of z_1 relative to the positive real axis. We will see later that $\theta \in [0, \pi]$ may be interpreted as the location of the zeros in the frequency response.

The transfer function for this filter is given by

$$\begin{aligned} H_f(z) &= (1 - z_1 z^1)(1 - z_2 z^{-1}) \\ &= (1 - e^{j\theta} z^{-1})(1 - e^{j\theta} z^{-1}) \\ &= 1 - 2cos(\theta) z^{-1} + z^{-2} \end{aligned}$$

Use this transfer function to determine the difference equation for this filter and compute the filters impulse response $h(n)$.

This filter is an FIR filter because it has impulse response $h(n)$ of finite duration. Any filter with only zeros and no poles other than those at 0 and $\pm\infty$ is an FIR filter. Zeros in the transfer function represent frequencies that are not passed through the filter. This can be useful for removing unwanted frequencies in a signal. The fact that $H_f(z)$ has zeros at $e^{\pm j\theta}$ implies that $H_f(e^{\pm j\theta}) = 0$. This means that the filter will not pass pure sine waves at a frequency of $\omega = \theta$.

6.3.1 EXERCISE

Use Octave to compute and plot the magnitude of the filters frequency response for the following three values of θ:

1. $\theta = \pi/6$
2. $\theta = \pi/3$
3. $\theta = \pi/3$

Put all three plots on the same figure using the subplot command.

6.4 IIR Filter Design

While zeros attenuate a filtered signal, poles amplify signals that are near their frequency. Consider the following transfer function for a second order IIR filter with complex- conjugate poles:

$$\begin{aligned} H_i(z) &= \frac{1-r}{(1-re^{j\theta}z^{-1})(1-re^{-j\theta}z^{-1})} \\ &= \frac{1-r}{(1-2rcos(\theta)z^{-1}+r^2z^{-2})} \end{aligned}$$

The poles of this filter have the form

$$p_1 = re^{j\theta} \qquad p_2 = re^{-j\theta}$$

where r is the distance from the origin, and θ is the angle of p_1 relative to the positive real axis. From the theory of Z-transforms, we know that a causal filter is stable if and only if its poles are located within the unit circle. This implies that this filter is stable if and only if $|r| < 1$. However, we will see that by locating the poles close to the unit circle, the filters bandwidth may be made extremely narrow around θ.

This two-pole system is an example of an IIR filter because its impulse response has infinite duration. Any filter with nontrivial poles (not at $z = 0$ or $\pm\infty$) is an IIR filter unless the poles are canceled by zeros.

Use Octave to get the filters frequency response for $\theta = \pi/3$ and the following three values of r.

1. r = 0.99
2. r = 0.9
3. r = 0.7

Put all three plots on the same figure using the subplot command.