Group Theory to Solve Rubik Cube

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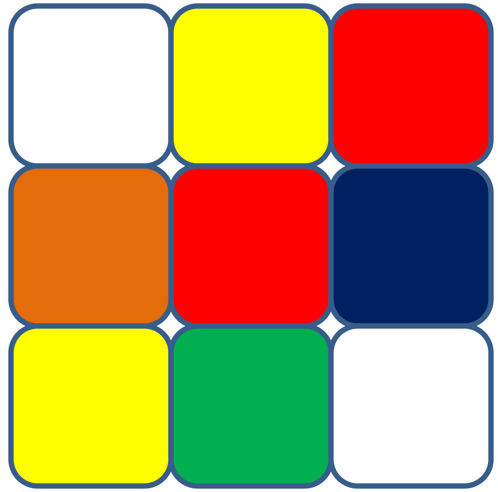


SOLVING RUBIK'S CUBE

Dr. Moloy De

Rubik's Cube

Rubik's Cube is a 3-D combination puzzle invented in 1974 by Hungarian sculptor and professor of architecture Ernő Rubik.

Originally called the Magic Cube, the puzzle was licensed by Rubik to be sold by Ideal Toy Corp. in 1980 via businessman Tibor Laczi and Seven Towns founder Tom Kremer, and won the German Game of the Year special award for Best Puzzle that year.

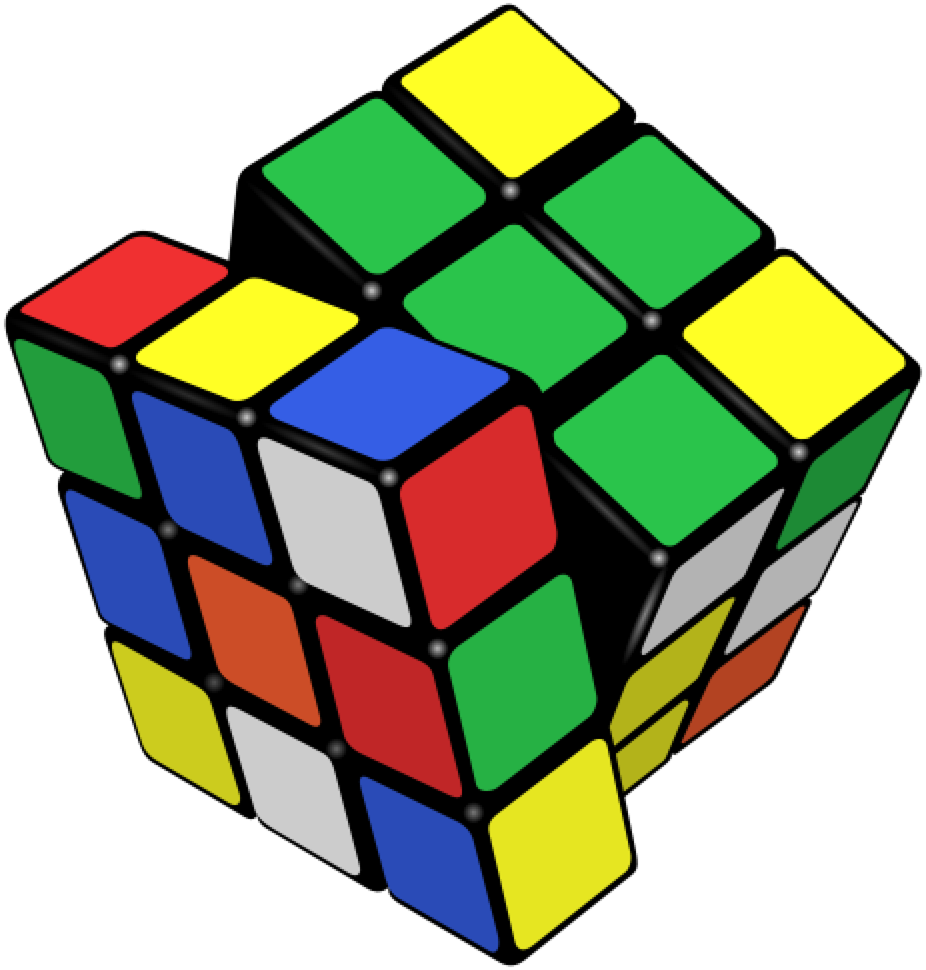
As of January 2009, 350 million cubes had been sold worldwide making it the world's top-selling puzzle game. It is widely considered to be the world's best-selling toy.

Rubik's Cube

Although the Rubik's Cube reached its height of mainstream popularity in the 1980s, it is still widely known and used.

Many speedcubers continue to practice it and other twisty puzzles and compete for the fastest times in various categories.

Since 2003, The World Cube Association, the Rubik's Cube's international governing body, has organised competitions worldwide and kept the official world records.



Top Five Speedcubing Records

1. Collin Burns 5.25s at Doylestown Spring 2015
2. Feliks Zemdegs 5.39s at World Championship 2015
3. Mats Valk 5.55s at Zonhoven Open 2013
4. Pavan Ravindra 5.58s at US Nationals 2015
5. Bill Wang 5.72s at Canadian Open 2015

Interesting Competitions

1. Blindfolded solving
2. Solving the Cube with one person blindfolded and the other person saying what moves to make, known as "Team Blindfold"
3. Multiple blindfolded solving, or "multi-blind", in which the contestant solves any number of cubes blindfolded in a row
4. Solving the Cube underwater in a single breath
5. Solving the Cube using a single hand
6. Solving the Cube with one's feet
7. Solving the Cube in the fewest possible moves

Rubik's Invention

In the mid-1970s, Ernő Rubik worked at the Department of Interior Design at the Academy of Applied Arts and Crafts in Budapest.

Although it is widely reported that the Cube was built as a teaching tool to help his students understand 3D objects, his actual purpose was solving the structural problem of moving the parts independently without the entire mechanism falling apart.

Rubik did not realise that he had created a puzzle until the first time he scrambled his new Cube and then tried to restore it.

Rubik obtained Hungarian patent HU170062 for his "Magic Cube" in 1975. Rubik's Cube was first called the Magic Cube (Bűvös kocka) in Hungary.

Rubik's Invention

The puzzle had not been patented internationally within a year of the original patent. Patent law then prevented the possibility of an international patent.

Ideal Toy Corp. wanted at least a recognizable name to trademark; of course, that arrangement put Rubik in the spotlight because the Magic Cube was renamed after its inventor in 1980.

According to Rubik's Brand Ltd., Professor Ernő Rubik holds "the copyright" for the Rubik's Cube and gave them the licensing and enforcement rights.

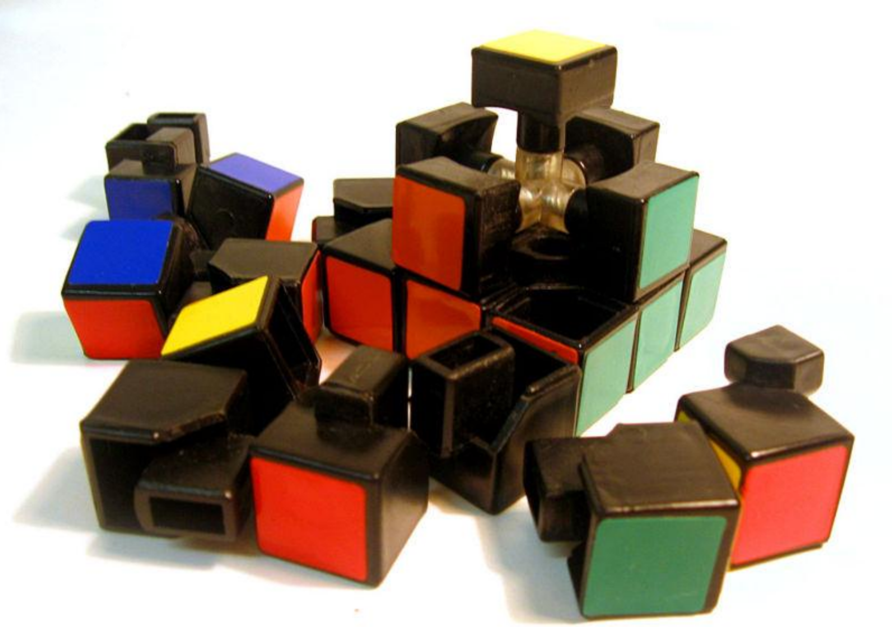
However copyright does not extend to ideas, inventions or physical objects, so presumably this refers to the manufacturing instructions and diagrams which Ernő Rubik wrote when producing his samples.

Cube Mechanics

The puzzle consists of twenty-six unique miniature cubes, also called "cubies" or "cubelets".

Each of these includes a concealed inward extension that interlocks with the other cubes, while permitting them to move to different locations.

However, the centre cube of each of the six faces is merely a single square façade; all six are affixed to the core mechanism. These provide structure for the other pieces to fit into and rotate around.



Cube Mechanics

So there are twenty-one pieces: a single core piece consisting of three intersecting axes holding the six centre squares in place but letting them rotate, and twenty smaller plastic pieces which fit into it to form the assembled puzzle.

Each of the six centre pieces pivots on a screw (fastener) held by the centre piece, a "3-D cross".

A spring between each screw head and its corresponding piece tensions the piece inward, so that collectively, the whole assembly remains compact, but can still be easily manipulated.

The screw can be tightened or loosened to change the "feel" of the Cube. Newer official Rubik's brand cubes have rivets instead of screws and cannot be adjusted.

Cube Mathematics

The original 3 by 3 by 3 Rubik's Cube has eight corners and twelve edges.

There are 8! (= 40,320) ways to arrange the corner cubes.

Seven can be oriented independently, and the orientation of the eighth depends on the preceding seven, giving 3^7 (= 2,187) possibilities.

There are 12!/2 (= 239,500,800) ways to arrange the edges, since an even permutation of the corners implies an even permutation of the edges as well.

When arrangements of centers are also permitted, as described below, the rule is that the combined arrangement of corners, edges, and centers must be an even permutation.

Eleven edges can be flipped independently, with the flip of the twelfth depending on the preceding ones, giving 2^{11} (= 2,048) possibilities.

Cube Mathematics

So the number of permutations equals

$$8! * 3^7 * (12!/2) * 2^{11} = 43,252,003,274,489,856,000$$

which is approximately 43 quintillion.

The puzzle is often advertised as having only "billions" of positions, as the larger numbers are unfamiliar to many.

Cube Operations

To denote a sequence of moves on the 3 by 3 by 3 Rubik's Cube, one uses "Singmaster notation", which was developed by David Singmaster.

- The letters L, R, F, B, U, D indicate a quarter clockwise turn of the left, right, front, back, up and down face respectively.
- Half turns are indicated by appending a 2.
- A quarter counterclockwise turn is indicated by appending a prime symbol (′).

Basic 90°	**180°**	**-90°**
F turns the front clockwise	F^2 turns the front clockwise twice	F' turns the front counter-clockwise
B turns the back clockwise	B^2 turns the back clockwise twice	B' turns the back counter-clockwise
U turns the top clockwise	U^2 turns the top clockwise twice	U' turns the top counter-clockwise
D turns the bottom clockwise	D^2 turns the bottom clockwise twice	D' turns the bottom counter-clockwise
L turns the left face clockwise	L^2 turns the left face clockwise twice	L' turns the left face counter-clockwise
R turns the right face clockwise	R^2 turns the right face clockwise twice	R' turns the right face counter-clockwise

Group Structure

One can identify each of the six face rotations as elements in the symmetric group on the set of non-center facets.

More concretely, we can label the non-center facets by the numbers 1 through 48, and then identify the six face rotations as elements of the symmetric group S_{48} according to how each move permutes the various facets.

The Rubik's Cube group, G, is then defined to be the subgroup of S_{48} generated by the six face (quarter) rotations {F, B, U, D, L, R}.

Group Structure

Despite the large size of G, God's Number for Rubik's Cube is 20; that is, any position can be solved in 20 or fewer moves where a half-twist is counted as a single move.

If a half-twist is counted as two quarter-twists, then God's number is 26.

The largest order of an element in G is 1260. For example, one such element of order 1260 is $RU^2D^{-1}BD^{-1}$

G is non-abelian since, for example, FR is not the same as RF. That is, not all cube moves commute with each other.

Lower Bounds

It can be proven by counting arguments that there exist positions needing at least 18 moves to solve.

To show this, first count the number of cube positions that exist in total, then count the number of positions achievable using at most 17 moves. It turns out that the latter number is smaller.

This argument was not improved upon for many years. Also, it is not a constructive proof: it does not exhibit a concrete position that needs this many moves.

It was conjectured that the so-called super-flip would be a position that is very difficult. A Rubik's Cube is in the super-flip pattern when each corner piece is in the correct position, but each edge piece is incorrectly oriented.

Lower Bounds

In 1992, a solution for the super-flip with 20 face turns was found by Dik T. Winter, of which the minimality was shown in 1995 by Michael Reid, providing a new lower bound for the diameter of the cube group.

Also in 1995, a solution for super-flip in 24 quarter turns was found by Michael Reid, with its minimality proven by Jerry Bryan.

In 1998, a new position requiring more than 24 quarter turns to solve was found. The position, which was called a 'super-flip composed with four spot' needs 26 quarter turns.

Upper Bounds

The first upper bounds were based on the 'human' algorithms. By combining the worst-case scenarios for each part of these algorithms, the typical upper bound was found to be around 100.

Perhaps the first concrete value for an upper bound was the 277 moves mentioned by David Singmaster in early 1979. He simply counted the maximum number of moves required by his cube-solving algorithm.

Later, Singmaster reported that Elwyn Berlekamp, John Conway, and Richard Guy had come up with a different algorithm that took at most 160 moves.

Soon after, Conway's Cambridge Cubists reported that the cube could be restored in at most 94 moves.

Thistlethwaite's Algorithm

Thistlethwaite came up with a famous solution to the Rubik's Cube. The way the algorithm works is by restricting the positions of the cubes into groups of cube positions that can be solved using a certain set of moves. The groups are:

- $G_0 = <L, R, F, B, U, D>$. This group contains all possible positions of the Rubik's Cube.
- $G_1 = <L, R, F, B, U^2, D^2>$. This group contains all positions that can be reached (from the solved state) with quarter turns of the left, right, front and back sides of the Rubik's Cube, but only double turns of the up and down sides.
- $G_2 = <L, R, F^2, B^2, U^2, D^2>$. In this group, the positions are restricted to ones that can be reached with only double turns of the front, back, up and down faces and quarter turns of the left and right faces.
- $G_3 = <L^2, R^2, F^2, B^2, U^2, D^2>$. Positions in this group can be solved using only double turns on all sides.
- $G_4 = \{I\}$. The final group contains only one position, the solved state of the cube.

Thistlethwaite's Algorithm

The cube is solved by moving from group to group, using only moves in the current group.

For example, a scrambled cube likely lies in group G_0.
A look up table of possible permutations is used that uses quarter turns of all faces to get the cube into group G_1.
Once in group G_1, quarter turns of the up and down faces are disallowed in the sequences of the look-up tables, and the tables are used to get to group G_2, and so on, until the cube is solved.

Initially, Thistlethwaite showed that any configuration could be solved in at most 85 moves.
In January 1980 he improved his strategy to yield a maximum of 80 moves.
Later that same year, he reduced the number to 63, and then again to 52.
By exhaustively searching the coset spaces it was later found that the best possible number of moves for each stage was 7, 10, 13, and 15 giving a total of 45 moves at most.

Kociemba's Algorithm

Thistlethwaite's algorithm was improved by Herbert Kociemba in 1992. He reduced the number of intermediate groups to only two:

- $G_0 = \langle U, D, L, R, F, B \rangle$
- $G_1 = \langle U, D, L^2, R^2, F^2, B^2 \rangle$
- $G_2 = \{I\}$

As with Thistlethwaite's Algorithm, he would search through the right coset space to take the cube from G_0 to G_1.

Next he searched the optimal solution for group G_1.

Both the searches in were done with a method equivalent to IDA, which is known as “Iterative deepening A” or Korf's algorithm.

Kociemba's Algorithm

The first search needs at most 12 moves and the second search at most 18 moves, as Michael Reid showed in 1995.

By generating also suboptimal solutions that take the cube to group G_1 and looking for short solutions in G_1, one usually gets much shorter overall solutions.

Using this algorithm solutions are typically found of fewer than 21 moves, though there is no proof that it will always do so.

In 1995 Michael Reid proved that using these two groups every position can be solved in at most 29 face turns, or in 42 quarter turns.

This result was improved by Silviu Radu in 2005 to 40.

Further Improvements

In 2006, Silviu Radu further improved his methods to prove that every position can be solved in at most 27 face turns or 35 quarter turns.

Daniel Kunkle and Gene Cooperman in 2007 used a supercomputer to show that all unsolved cubes can be solved in no more than 26 moves in face-turn metric. Instead of attempting to solve each of the billions of variations explicitly, the computer was programmed to bring the cube to one of 15,752 states, each of which could be solved within a few extra moves.

All were proved solvable in 29 moves, with most solvable in 26. Those that could not initially be solved in 26 moves were then solved explicitly, and shown that they too could be solved in 26 moves.

Further Improvements

Tomas Rokicki reported in a 2008 computational proof that all unsolved cubes could be solved in 25 moves or fewer. This was later reduced to 23 moves.

In August 2008 Rokicki announced that he had a proof for 22 moves.

Finally, in 2010, Tomas Rokicki, Herbert Kociemba, Morley Davidson, and John Dethridge gave the final computer-assisted proof that all cube positions could be solved with a maximum of 20 face turns.

In 2009, Tomas Rokicki proved that 29 moves in the quarter-turn metric is enough to solve any scrambled cube.

And in 2014, Tomas Rokicki and Morley Davidson proved that the maximum number of quarter-turns needed to solve the cube is 26.

References

1. Rubik's Cube - Wikipedia
2. Rubik's Cube group - Wikipedia
3. How to Solve the Rubik's Cube - Wikibooks
4. Optimal solutions for Rubik's Cube – Wikipedia
5. Group Theory and the Rubik’s Cube - PDF
6. Algorithms for Solving Rubik's Cubes - PDF
7. Twenty-Five Moves Suffice for Rubik’s Cube - arXiv


THANK
YOU!