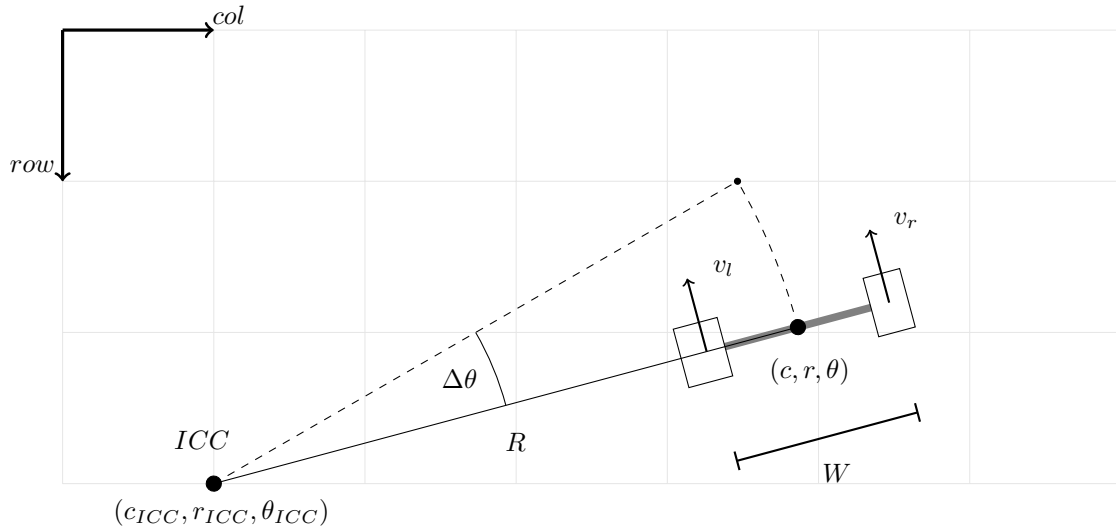# Dyanmics Equations for Smartmouse 2018

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1   Forward Kinematics

Deriving the kinematic equations of a differential drive robot.

It's worth noting up front that we do not consider an X/Y plane. It is not a useful coordinate system for smartmouse. We think instead of row/column, where rows increase going down and columns increase going left to right. The top left cell is 0,0. Normally, we think of these as integer values, but we can also think of them as floating point values if we want to talk about the exact position in a cell. With that out of the way, let's do the derivation.

At an instant in time we say that the robot is turning around some point ICC, the instantaneous center. The radius R about that point is what we want to solve for. We start with the knowedge that since the robot doesn't tear itself apart while driving, the rate ω at which both wheels (and the robot) move around ICC is the same.

$$\omega_l = \omega_r = \omega \tag{1}$$

Let R_l and R_l be the radius to the right and left wheels.

$$\omega = \frac{v_l}{R_l} = \frac{v_r}{R_r} \tag{2}$$

We can then combine 1 and 2

$$\frac{v_l}{R_l} = \frac{v_r}{R_r} \tag{3}$$

We can then substitute $R_l = R - \frac{W}{2}$ and $R_r = R + \frac{W}{2}$

$$\frac{v_l}{R - \frac{W}{2}} = \frac{v_r}{R + \frac{W}{2}} \tag{4}$$

Now do some algebra...

$$\frac{v_l}{R - \frac{W}{2}} = \frac{v_r}{R + \frac{W}{2}}$$

$$\frac{R - \frac{W}{2}}{v_l} = \frac{R + \frac{W}{2}}{v_r}$$

$$\frac{R}{v_l} - \frac{W}{2v_l} = \frac{R}{v_r} + \frac{W}{2v_r}$$

$$\frac{R}{v_l} - \frac{R}{v_r} = \frac{W}{2v_r} + \frac{W}{2v_l}$$

$$\frac{2R(v_r - v_l)}{v_r v_l} = \frac{W(v_l + v_r)}{v_r v_l}$$

$$2R(v_r - v_l) = W(v_l + v_r)$$

$$R = \frac{W(v_l + v_r)}{2(v_r - v_l)}$$

We can use this result to get a cleaner equation for ω, which right now requires diving by R. This is problematic because R can be 0.

$$\omega = \frac{v_l}{R - \frac{W}{2}}$$

$$= \frac{v_l}{\frac{W(v_l + v_r)}{2(v_r - v_l)} - \frac{W}{2}}$$

$$= \frac{v_l}{\frac{W(v_l + v_r)}{2(v_r - v_l)} - \frac{W(v_r - v_l)}{2(v_r - v_l)}}$$

$$= \frac{v_l}{\frac{W(v_l + v_r) - W(v_r - v_l)}{2(v_r - v_l)}}$$

$$= \frac{2v_l(v_r - v_l)}{W(v_l + v_r) - W(v_r - v_l)}$$

$$= \frac{2v_l(v_r - v_l)}{Wv_l + Wv_r - Wv_r + Wv_l}$$

$$= \frac{2v_l(v_r - v_l)}{2Wv_l}$$

$$= \frac{v_r - v_l}{W}$$

Next we need to figure out how to update our c, r, and θ given R.

Before we can solve any of these, we need to figure out what $\Delta\theta$ is equal to. We will assume we are following this arc of constant radius for a given time step Δt. This is a good assumption if our time step is equal to our controller time step, during which presumably the speeds of the motors aren't changing, and so neither will our turning radius. If so, then $\Delta\theta = \omega\Delta t$, and we know

$$\omega = \frac{v_r - v_l}{W}$$

Lets just pick v_l, and put them together to get $\Delta\theta$.

$$\Delta\theta = \frac{v_r - v_l}{W}\Delta t$$

So what are our update equations?

Theta is obvious: $\theta \leftarrow \theta + \Delta\theta$.

To discover the rest, we will consider how to c and r change when we rotate around the ICC.

The coordinates of the ICC are:

$$(c_{ICC}, r_{ICC}) = (c - R\sin\theta, r - R\cos\theta)$$

You can see this if you think that the vector to ICC is 90 degrees offset from θ and do the trig. Let's assume in the situation diagramed above, $\theta = 105°$, $R = 4$, and $c = 4.8, r = 0$, so $ICC = (4.8 - 4\sin(105), 0 - 4\cos(105)) \approx (1, 1)$. For another example, pretend the robot is right below the origin facing east at $(0, 1, 0)$ and $R = 1$. ICC should be $(0, 0)$ so let's check. $ICC = (0 - 1\sin(0), 1 - 1\cos(0)) = (0, 0)$. Great, the math seems to check out. Now we just rotate the robot coordinates around the point ICC by $\Delta\theta$, which we can do easily with a rotation matrix. Of course, we're not rotating around the origin, we're roating around ICC, so we first subtract out the coordinates of ICC to get $(c_o, r_o) = (c - c_{ICC}, r - r_{ICC})$. Recall that to rotate around the origin by $\Delta\theta$, we multiply our point (c_o, r_o) as follows:

$$\begin{bmatrix} \cos\Delta\theta & -\sin\Delta\theta \\ \sin\Delta\theta & \cos\Delta\theta \end{bmatrix} \begin{bmatrix} c_o \\ r_o \end{bmatrix} = \begin{bmatrix} c_{new} \\ r_{new} \end{bmatrix} \tag{5}$$

We then just add back the coordinates of ICC. Great! Now we know how to update all our coordinates. Let's summarize:

$$\theta \leftarrow \theta + \Delta\theta \tag{6}$$
$$c \leftarrow \cos(\Delta\theta)(c - c_{ICC}) - \sin(\Delta\theta)(r - r_{ICC}) + c_{ICC} \tag{7}$$
$$r \leftarrow \sin(\Delta\theta)(c - c_{ICC}) + \cos(\Delta\theta)(r - r_{ICC}) + r_{ICC} \tag{8}$$

where

$$c_{ICC} = c - R\sin\theta \tag{9}$$
$$r_{ICC} = r - R\cos\theta \tag{10}$$
$$\Delta\theta = \frac{v_r - v_l}{W}\Delta t \tag{11}$$
$$R = \frac{W(v_l + v_r)}{2(v_r - v_l)} \tag{12}$$

We can expand and simplify the update equations for c.

$$c \leftarrow \cos\left(\frac{v_r - v_l}{W}\Delta t\right)(c - c + R\sin\theta) - \sin\left(\frac{v_r - v_l}{W}\Delta t\right)(r - r + R\cos\theta) + c - R\sin\theta \tag{13}$$

$$c \leftarrow \cos\left(\frac{v_r - v_l}{W}\Delta t\right)(R\sin\theta) - \sin\left(\frac{v_r - v_l}{W}\Delta t\right)(R\cos\theta) + c - R\sin\theta \tag{14}$$

$$c \leftarrow R\cos\left(\frac{v_r - v_l}{W}\Delta t\right)\sin\theta - R\sin\left(\frac{v_r - v_l}{W}\Delta t\right)\cos\theta + c - R\sin\theta \tag{15}$$

$$c \leftarrow c + R\cos\left(\frac{v_r - v_l}{W}\Delta t\right)\sin\theta - R\sin\left(\frac{v_r - v_l}{W}\Delta t\right)\cos\theta - R\sin\theta \tag{16}$$

$$c \leftarrow c + R\left(\cos\left(\frac{v_r - v_l}{W}\Delta t\right)\sin\theta - \sin\left(\frac{v_r - v_l}{W}\Delta t\right)\cos\theta\right) - R\sin\theta \tag{17}$$

let $\frac{v_r - v_l}{W} = a$, and $\theta = b$, we can use $\cos a \sin b - \sin a \cos b = -\sin(a - b)$

$$c \leftarrow c - R\left(\sin\left(\frac{v_r - v_l}{W}\Delta t - \theta\right)\right) - R\sin\theta \tag{18}$$

$$c \leftarrow c + -R\left(\sin\left(\frac{v_r - v_l}{W}\Delta t - \theta\right) + \sin\theta\right) \tag{19}$$

We can apply the same process for r.

$$r \leftarrow \sin\left(\frac{v_r - v_l}{W}\Delta t\right)(c - c + R\sin\theta) + \cos\left(\frac{v_r - v_l}{W}\Delta t\right)(r - r + R\cos\theta) + r - R\cos\theta \tag{20}$$

$$r \leftarrow \sin\left(\frac{v_r - v_l}{W}\Delta t\right)(R\sin\theta) + \cos\left(\frac{v_r - v_l}{W}\Delta t\right)(R\cos\theta) + r - R\cos\theta \tag{21}$$

$$r \leftarrow R\sin\left(\frac{v_r - v_l}{W}\Delta t\right)(\sin\theta) + R\cos\left(\frac{v_r - v_l}{W}\Delta t\right)(\cos\theta) + r - R\cos\theta \tag{22}$$

$$r \leftarrow r + R\sin\left(\frac{v_r - v_l}{W}\Delta t\right)(\sin\theta) + R\cos\left(\frac{v_r - v_l}{W}\Delta t\right)(\cos\theta) - R\cos\theta \tag{23}$$

$$r \leftarrow r + R\left(\sin\left(\frac{v_r - v_l}{W}\Delta t\right)(\sin\theta) + \cos\left(\frac{v_r - v_l}{W}\Delta t\right)(\cos\theta) - \cos\theta\right) \tag{24}$$

$$r \leftarrow r + R\left(\cos\left(\frac{v_r - v_l}{W}\Delta t - \theta\right) - \cos\theta\right) \tag{25}$$

Final Solution to the forward Kinematics:

$$\theta \leftarrow \theta + \frac{v_r - v_l}{W}\Delta t \tag{26}$$

$$c \leftarrow c + -R\left(\sin\left(\frac{v_r - v_l}{W}\Delta t - \theta\right) + \sin\theta\right) \tag{27}$$

$$r \leftarrow r + R\left(\cos\left(\frac{v_r - v_l}{W}\Delta t - \theta\right) - \cos\theta\right) \tag{28}$$

However, there is also the case where the robot is going perfectly straight. This has to be handled seperately, because otherwise the equations above involve R, but $R = \infty$ if we're going straight. Luckily, the equations for moving straight are trivial:
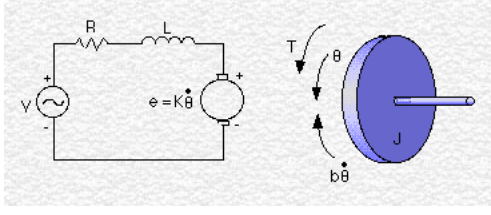
$$\theta \leftarrow \theta \tag{29}$$
$$c \leftarrow c + v\Delta t \cos(theta) \tag{30}$$
$$r \leftarrow r - v\Delta t \sin(theta) \tag{31}$$

Here, v can be v_l, or v_r since they should be the same. In code, a simple average is used to correct for any slight numerical inconsistencies.

2 DC Motor Modeling

We must also consider how to model the DC Motors.

The motor torque, T, is related to the armature current, i, by a constant factor K. The back emf, e, is related to the rotational velocity by the following equations:

$$T = Ki$$

$$e = K\dot{\theta}$$

From the figure above we can write the following equations based on Newton's law combined with Kirchhoff's law:

$$J\frac{d}{dt}\dot{\theta} + b\dot{\theta} = Ki$$

$$L\frac{d}{dt}i + Ri = V - K\dot{\theta}$$

In state-space form, the equations above can be expressed by choosing the rotating speed and electric current as the state variables and the voltage as an input. The output is chosen to be the rotating speed. The general linear state-space form is $\dot{X} = AX + BW$. We can get the two equations above in this form with some algebra.

$$J\frac{d}{dt}\dot{\theta} + b\dot{\theta} = Ki \tag{32}$$

$$\frac{d}{dt}\dot{\theta} + \frac{b\dot{\theta}}{J} = \frac{Ki}{J} \tag{33}$$

$$\frac{d}{dt}\dot{\theta} = \frac{Ki}{J} - \frac{b\dot{\theta}}{J} \tag{34}$$

$$L\frac{d}{dt}i + Ri = V - K\dot{\theta} \tag{35}$$

$$\frac{d}{dt}i + \frac{Ri}{L} = \frac{V}{L} - \frac{K\dot{\theta}}{L} \tag{36}$$

$$\frac{d}{dt}i = \frac{V}{L} - \frac{K\dot{\theta}}{L} - \frac{Ri}{L} \tag{37}$$

Next we combine these in a matrix.

$$\frac{d}{dt}\begin{bmatrix}\dot{\theta} \\ i\end{bmatrix} = \begin{bmatrix}\frac{-b}{J} & \frac{K}{J} \\ \frac{-K}{L} & \frac{-R}{L}\end{bmatrix}\begin{bmatrix}\dot{\theta} \\ i\end{bmatrix} + \begin{bmatrix}0 \\ \frac{1}{L}\end{bmatrix}V \tag{38}$$

What this gives us is a nice matrix-y formula for the change in the current and angle of our motor over time. We can use this equation to simulate the behavior of the motor. If we assume the motor start at some initial $\dot{\theta}$, i_0 and $\dot{i}$, and we know all our motor constants (K, R, L, b, J) we can figure out the right hand side of the equation, and the result will be how much i and $\dot{\theta}$ changes over some discrete time step. That is what we do in the smartmouse simulator server code.