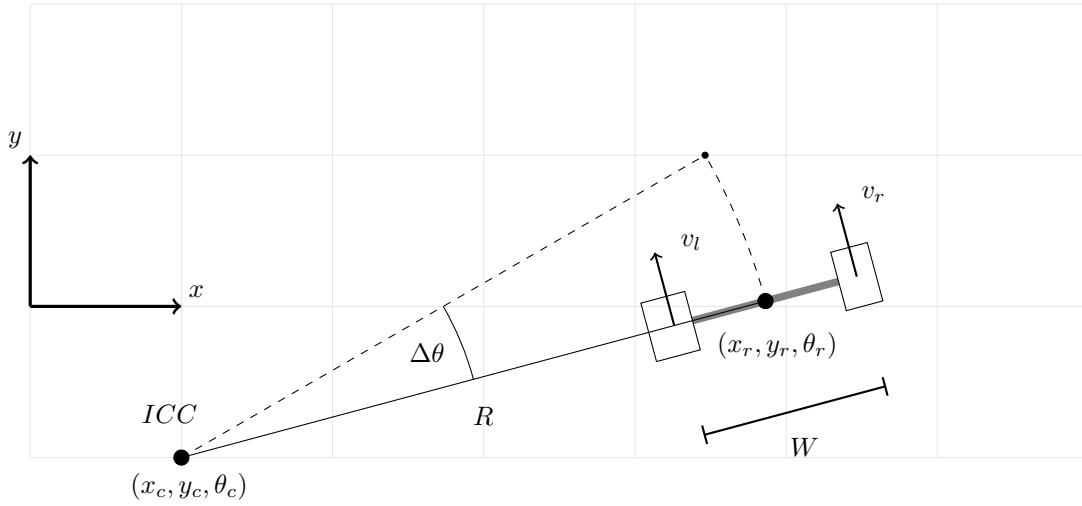


Dyanmics Equations for Smartmouse 2018

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September 14, 2017

Deriving the kinematic equations of a differential drive robot.



At an instant in time we say that the robot is turning around some point ICC , the instantaneous center. The radius R about that point is what we want to solve for. We start with the knowledge that since the robot doesn't tear itself apart while driving, the rate w at which both wheels (and the robot) move around ICC is the same.

$$\omega_l = \omega_r = w \quad (1)$$

Let R_l and R_r be the radius to the right and left wheels.

$$\omega_l = \frac{v_l}{R_l}, \omega_r = \frac{v_r}{R_r} \quad (2)$$

We can then combine 1 and 2

$$\frac{v_l}{R_l} = \frac{v_r}{R_r} \quad (3)$$

We can then substitute $R_l = R - \frac{w}{2}$ and $R_r = R + \frac{w}{2}$

$$\frac{v_l}{R - \frac{w}{2}} = \frac{v_r}{R + \frac{w}{2}} \quad (4)$$

No do some algebra...

$$\frac{v_l}{R - \frac{w}{2}} = \frac{v_r}{R + \frac{w}{2}}$$

$$\frac{R - \frac{w}{2}}{v_l} = \frac{R + \frac{w}{2}}{v_r}$$

$$\frac{R}{v_l} - \frac{w}{2v_l} = \frac{R}{v_r} + \frac{w}{2v_r}$$

$$\frac{R}{v_l} - \frac{R}{v_r} = \frac{w}{2v_r} + \frac{w}{2v_l}$$

$$\frac{R(v_r - v_l)}{v_r v_l} = \frac{w(v_l + v_r)}{2v_r v_l}$$

$$R(v_r - v_l) = w(v_l + v_r)$$

$$R = \frac{w(v_l + v_r)}{2(v_r - v_l)}$$

Next we need to figure out how to update our x_r , y_r , and θ_r given R .

Before we can solve any of these, we need to figure out what $\Delta\theta$ is equal to. We will assume we are following this arc of constant radius for a given time step Δt . This is a good assumption if our time step is equal to our controller time step, during which presumably the speeds of the motors aren't changing, and so neither will our turning radius. If so, then $\Delta\theta = \omega\Delta t$, and we know

$$\omega = \frac{v_l}{R - \frac{w}{2}} = \frac{v_r}{R + \frac{w}{2}}$$

Lets just pick v_l , and put them together to get $\Delta\theta$.

$$\Delta\theta = \frac{v_l}{R - \frac{w}{2}} \Delta t$$

Theta is obvious: $\theta_r \leftarrow \theta_r + \Delta\theta$. The coordinates of $ICC = (x_r - R \sin \theta_r, y_r + R \cos \theta_r)$. You can see this if you think that the vector to ICC is 90 degrees offset from θ_r and do the trig. Let's assume in the situation diagramed above, $\theta_r = 105^\circ$, $R = 4$, and $x = 4.8, y = 0$, so $ICC = (4.8 - 4 \sin(105), 0 + 4 \cos(105)) \approx (1, -1)$. For another example, pretend the robot is right below the origin facing east at $(0, -1, 0)$ and $R = 1$. ICC should be $(0, 0)$ so let's check. $ICC = (0 - 1 \sin(0), -1 + 1 \cos(0)) = (0, 0)$. Great, the math seems to check out. Now we just rotate the robot coordinates around the point ICC by $\Delta\theta$, which we can do easily with a rotation matrix. Of course, we're not rotating around the origin, we're rotating around ICC , so we first subtract out the coordinates of ICC to get $(x_o, y_o) = (x_r - ICC_x, y_r - ICC_y)$. Recall that to rotate around the origin by $\Delta\theta$, we multiply our point (x_o, y_o) as follows:

$$\begin{bmatrix} \cos \Delta\theta & \sin \Delta\theta \\ -\sin \Delta\theta & \cos \Delta\theta \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix} = \begin{bmatrix} x_{new} \\ y_{new} \end{bmatrix} \quad (5)$$

We then just add back the coordinates of ICC . Great! Now we know how to update all our coordinates. Let's summarize:

$$\theta_r \leftarrow \theta_r + \Delta\theta \quad (6)$$

$$x_r \leftarrow \cos(\Delta\theta)(x_r - ICC_x) + \sin(\Delta\theta)(y_r - ICC_y) + ICC_x \quad (7)$$

$$y_r \leftarrow -\sin(\Delta\theta)(x_r - ICC_x) + \cos(\Delta\theta)(y_r - ICC_y) + ICC_y \quad (8)$$

where

$$ICC_x = x_r - R \sin \theta_r \quad (9)$$

$$ICC_y = y_r + R \cos \theta_r \quad (10)$$

$$\Delta\theta = \frac{v_l}{R - \frac{W}{2}} \Delta t \quad (11)$$

In full expanded form, with some algebraic simplifications:

$$\theta_r \leftarrow \theta_r + \frac{v_l}{R - \frac{W}{2}} \Delta t \quad (12)$$

$$x_r \leftarrow \cos\left(\frac{v_l}{R - \frac{W}{2}} \Delta t\right)(R \sin \theta_r) + \sin\left(\frac{v_l}{R - \frac{W}{2}} \Delta t\right)(-R \cos \theta_r) + x_r - R \sin \theta_r \quad (13)$$

$$y_r \leftarrow -\sin\left(\frac{v_l}{R - \frac{W}{2}} \Delta t\right)(R \sin \theta_r) + \cos\left(\frac{v_l}{R - \frac{W}{2}} \Delta t\right)(-R \cos \theta_r) + y_r + R \cos \theta_r \quad (14)$$