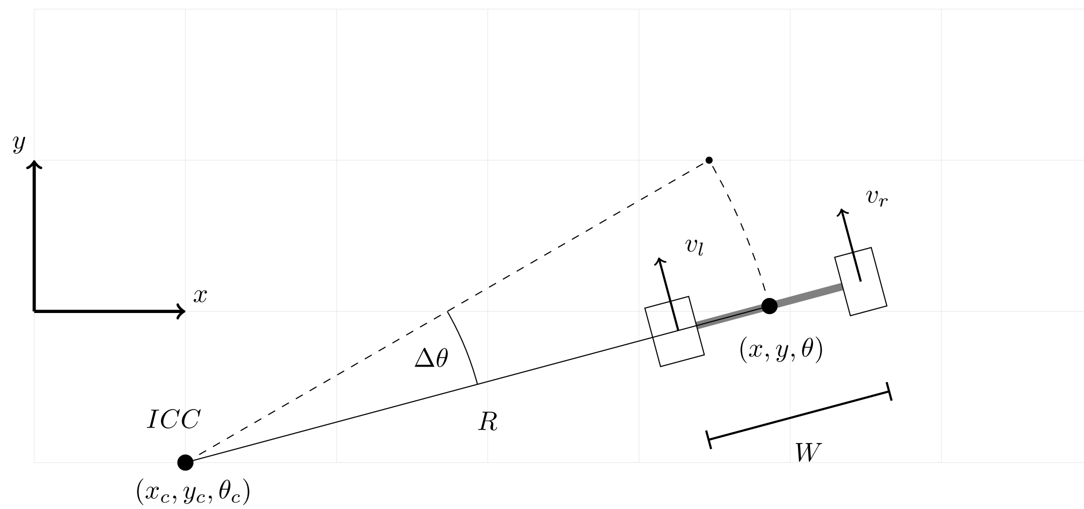
Dyanmics Equations for Smartmouse 2018

Peter Mitrano

September 19, 2017

Deriving the kinematic equations of a differential drive robot.

-



At an instant in time we say that the robot is turning around some point ICC, the instanteous center. The radius R about that point is what we want to solve for. We start with the knowedge that since the robot doesn't tear itself apart while driving, the rate ω at which both wheels (and the robot) move around ICC is the same.

$$\omega_l = \omega_r = \omega \tag{1}$$

Let R_l and R_l be the radius to the right and left wheels.

$$\omega_l = \frac{v_l}{R_l}, \omega_r = \frac{v_r}{R_r} \tag{2}$$

We can then combine 1 and 2

$$\frac{v_l}{R_l} = \frac{v_r}{R_r} \tag{3}$$

We can then substitute $R_l = R - \frac{W}{2}$ and $R_r = R + \frac{W}{2}$

$$\frac{v_l}{R - \frac{W}{2}} = \frac{v_r}{R + \frac{W}{2}} \tag{4}$$

Now do some algebra...

$$\frac{v_l}{R - \frac{W}{2}} = \frac{v_r}{R + \frac{W}{2}}$$

$$\frac{R - \frac{W}{2}}{v_l} = \frac{R + \frac{W}{2}}{v_r}$$

$$\frac{R}{v_l} - \frac{W}{2v_l} = \frac{R}{v_r} + \frac{W}{2v_r}$$

$$\frac{R}{v_l} - \frac{R}{v_r} = \frac{W}{2v_r} + \frac{W}{2v_l}$$

$$\frac{2R(v_r - v_l)}{v_r v_l} = \frac{W(v_l + v_r)}{2v_r v_l}$$

$$R(v_r - v_l) = W(v_l + v_r)$$

$$R = \frac{W(v_l + v_r)}{2(v_r - v_l)}$$

Next we need to figure out how to update our x, y, and θ given R.

Before we can solve any of these, we need to figure out what $\Delta\theta$ is equal to. We will assume we are following this arc of constant radius for a given time step Δt. This is a good assumption if our time step is equal to our controller time step, during which presumably the speeds of the motors aren't changing, and so neither will our turning radius. If so, then $\Delta\theta = \omega\Delta t$, and we know

$$\omega = \frac{v_l}{R - \frac{W}{2}} = \frac{v_r}{R + \frac{W}{2}}$$

Lets just pick v_l, and put them together to get $\Delta\theta$.

$$\Delta\theta = \frac{v_l}{R - \frac{W}{2}}\Delta t$$

Theta is obvious: $\theta \leftarrow \theta + \Delta\theta$. The coordinates of $ICC = (x - R\sin\theta, y + R\cos\theta)$. You can see this if you think that the vector to ICC is 90 degrees offset from θ and do the trig. Let's assume in the situation diagramed above, $\theta = 105°$, $R = 4$, and $x = 4.8, y = 0$, so $ICC = (4.8 - 4\sin(105), 0 + 4\cos(105)) \approx (1, -1)$. For another example, pretend the robot is right below the origin facing east at $(0, -1, 0)$ and $R = 1$. ICC should be $(0, 0)$ so let's check. $ICC = (0 - 1\sin(0), -1 + 1\cos(0)) = (0, 0)$. Great, the math seems to check out. Now we just rotate the robot coordinates around the point ICC by $\Delta\theta$, which we can do easily with a rotation matrix. Of course, we're not rotating around the origin, we're roating around ICC, so we first subtract out the coordinates of ICC to get $(x_o, y_o) = (x - ICC_x, y - ICC_y)$. Recall that to rotate around the origin by $\Delta\theta$, we multiply our point (x_o, y_o) as follows:

$$\begin{bmatrix} \cos\Delta\theta & \sin\Delta\theta \\ -\sin\Delta\theta & \cos\Delta\theta \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix} = \begin{bmatrix} x_{new} \\ y_{new} \end{bmatrix} \tag{5}$$

We then just add back the coordinates of ICC. Great! Now we know how to update all our coordinates. Let's summarize:

$$\theta \leftarrow \theta + \Delta\theta \tag{6}$$
$$x \leftarrow \cos(\Delta\theta)(x - ICC_x) + \sin(\Delta\theta)(y - ICC_y) + ICC_x \tag{7}$$
$$y \leftarrow -\sin(\Delta\theta)(x - ICC_x) + \cos(\Delta\theta)(y - ICC_y) + ICC_y \tag{8}$$

where

$$ICC_x = x - R\sin\theta \tag{9}$$
$$ICC_y = y + R\cos\theta \tag{10}$$
$$\Delta\theta = \frac{v_l}{R - \frac{W}{2}}\Delta t \tag{11}$$
$$R = \frac{W(v_l + v_r)}{2(v_r - v_l)} \tag{12}$$

We can expand and simplify the udpate equations for x.

$$x \leftarrow \cos(\frac{v_l}{R - \frac{W}{2}}\Delta t)(x - x + R\sin\theta) + \sin(\frac{v_l}{R - \frac{W}{2}}\Delta t)(y - y - R\cos\theta) + x - R\sin\theta \tag{13}$$
$$x \leftarrow \cos(\frac{v_l}{R - \frac{W}{2}}\Delta t)(R\sin\theta) + \sin(\frac{v_l}{R - \frac{W}{2}}\Delta t)(-R\cos\theta) + x - R\sin\theta \tag{14}$$
$$x \leftarrow R\cos(\frac{v_l}{R - \frac{W}{2}}\Delta t)\sin\theta - R\sin(\frac{v_l}{R - \frac{W}{2}}\Delta t)\cos\theta + x - R\sin\theta \tag{15}$$
$$x \leftarrow x + R\cos(\frac{v_l}{R - \frac{W}{2}}\Delta t)\sin\theta - R\sin(\frac{v_l}{R - \frac{W}{2}}\Delta t)\cos\theta - R\sin\theta \tag{16}$$
$$x \leftarrow x + R\Big(\cos(\frac{v_l}{R - \frac{W}{2}}\Delta t)\sin\theta - \sin(\frac{v_l}{R - \frac{W}{2}}\Delta t)\cos\theta\Big) - R\sin\theta \tag{17}$$

let $\frac{v_l}{R - \frac{W}{2}} = a$, and $\theta = b$, we can use $\cos a \sin b - \sin a \cos b = -\sin(a - b)$

$$x \leftarrow x + R\Bigg(-\sin\Big(\frac{v_l}{R - \frac{W}{2}}\Delta t - \theta\Big)\Bigg) - R\sin\theta \tag{18}$$
$$x \leftarrow x + R\Bigg(-\sin\Big(\frac{v_l}{R - \frac{W}{2}}\Delta t - \theta\Big) - \sin\theta\Bigg) \tag{19}$$
$$x \leftarrow x - R\Bigg(\sin\Big(\frac{v_l}{R - \frac{W}{2}}\Delta t - \theta\Big) + \sin\theta\Bigg) \tag{20}$$

sub in for R in the denominator

$$x \leftarrow x - R\Bigg(\sin\Big(\frac{v_l}{\frac{W(v_l+v_r)}{2(v_r-v_l)} - \frac{W}{2}}\Delta t - \theta\Big) + \sin\theta\Bigg) \tag{21}$$
$$x \leftarrow x - R\Bigg(\sin\Big(\frac{v_l}{\frac{W(v_l+v_r)}{2(v_r-v_l)} - \frac{W(v_r-v_l)}{2(v_r-v_l)}}\Delta t - \theta\Big) + \sin\theta\Bigg) \tag{22}$$
$$x \leftarrow x - R\Bigg(\sin\Big(\frac{v_l}{\frac{Wv_l+Wv_r-Wv_r+Wv_l)}{2(v_r-v_l)}}\Delta t - \theta\Big) + \sin\theta\Bigg) \tag{23}$$
$$x \leftarrow x - R\Bigg(\sin\Big(\frac{v_l}{\frac{Wv_l}{(v_r-v_l)}}\Delta t - \theta\Big) + \sin\theta\Bigg) \tag{24}$$
$$x \leftarrow x - R\Bigg(\sin\Big(\frac{v_r - v_l}{W}\Delta t - \theta\Big) + \sin\theta\Bigg) \tag{25}$$

We can apply the same process for y.

$$y \leftarrow y - R\left(\cos\left(\frac{v_r - v_l}{W}\Delta t - \theta\right) - \cos\theta\right) \tag{26}$$

Final Solution to the forward Kinematics:

$$\theta \leftarrow \theta + \frac{v_l}{R - \frac{W}{2}}\Delta t \tag{27}$$

$$x \leftarrow x - R\left(\sin\left(\frac{v_r - v_l}{W}\Delta t - \theta\right) + \sin\theta\right) \tag{28}$$

$$y \leftarrow y - R\left(\cos\left(\frac{v_r - v_l}{W}\Delta t - \theta\right) - \cos\theta\right) \tag{29}$$

However, there is also the case where the robot is going perfectly straight. This has to be handled seperately, because otherwise the equations above involve R, but $R = \infty$ if we're going straight. Luckily, the equations for moving straight are trivial:

$$\theta \leftarrow \theta \tag{30}$$

$$x \leftarrow x + v\Delta t\cos(theta) \tag{31}$$

$$y \leftarrow y + v\Delta t\sin(theta) \tag{32}$$

$$\tag{33}$$

Here, v can be v_l, or v_r since they should be the same. In code, a simple average is used.

We must also consider how to model the DC Motors. First, we model static friction by saying if the abstract force on the wheel (0-255) is less than a certain threshold that it will not move. Second, we subtract $\mu_k * \omega$ from wheel velocity since kinetic friction is proportional to wheel velocity. Lastly, we subtract $J * \alpha$ from wheel acceleration since inertia is proportional to acceleration. I know that last sentence didn't make much sense...