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Math 21a: Multivariable calculus Fall 2019

Homework 2: Vectors and Dot product

This homework is due on Friday, 9/13 at the beginning of class.

1 A **kite surfer** gets pulled with a force $\vec{F} = [7, 1, 4]$. She moves with velocity $\vec{v} = [4, -2, 1]$. The dot product of $\vec{F}$ with $\vec{v}$ is **power**.

(SEE BACK) a) What is the angle between the $\vec{F}$ and $\vec{v}$?

b) Find the **vector projection** of the $\vec{F}$ onto $\vec{v}$. (SEE BACK)

Wording of question was confusing

2 Light shines long the vector $\vec{a} = [a_1, a_2, a_3]$ and reflects at the three coordinate planes where the angle of incidence equals the angle of reflection. Verify that the reflected ray is $-\vec{a}$. **Hint.** Reflect first at the xy-plane. What happens with the vector $\vec{a}$?

3 a) In order to see whether two data points $\vec{v} = [1, 1, -2]$ and (SEE BACK) $\vec{w} = [1, -2, 1]$ are correlated, we compute the cosine of the angle between the two vectors. Do this for the vectors $\vec{v}$ and $\vec{w}$.

b) Find two vectors $\vec{a}$ and $\vec{b}$ for which all coordinates are positive such that the angle between them is $\pi/4 = 45°$. In statistics the dot product between $\vec{v}$ and $\vec{w}$ is also called the **covariance** and the lengths $|\vec{v}|$ and $|\vec{w}|$ are called the **standard deviations** of $\vec{v}$ and $\vec{w}$. A data scientist calls the cosine of the angle the **correlation**.

Answer key is incorrect!

(SEE BACK)

4 a) Find the angle between a space diagonal of a cube and the diagonal in one of its faces. (SEE BACK)

b) The **hypercube** is also called the **tesseract**. It has vertices $(\pm1, \pm1, \pm1, \pm1)$. Find the angle between the hyper diagonal connecting $(1, 1, 1, 1)$ with $(-1, -1, -1, -1)$ and the space diagonal connecting $(1, 1, 1, 1)$ with $(-1, -1, -1, 1)$.

5 a) Verify that if $\vec{a}, \vec{b}$ are nonzero vectors, then $\vec{c} = |\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ bisects the angle between $\vec{a}, \vec{b}$ if $\vec{c}$ is not zero. (SEE BACK OF NEXT PAGE)

① A. $\cos^{-1}\left(\frac{\vec{F}\cdot\vec{v}}{|\vec{F}||\vec{v}|}\right) = \theta = \cos^{-1}\left(\frac{30}{\sqrt{66}\sqrt{21}}\right) \approx \boxed{0.634 \text{ radians}}$.

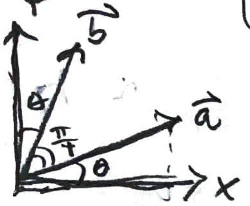
B. $\underbrace{(|\vec{F}||\vec{v}|\cos\theta)}_{\vec{F}\cdot\vec{v}} \frac{\vec{v}}{|\vec{v}|^2} = \boxed{\frac{30}{21}\langle 4,-2,1\rangle}$

②

After the first reflection, the light is along $\langle a_1, a_2, -a_3\rangle$. Repeating the process reflecting off of xz-plane next gives $\langle a_1, -a_2, -a_3\rangle$. The final reflection gives $-\vec{a}$.

③ A. $\cos\theta = \frac{\vec{w}\cdot\vec{v}}{|\vec{w}||\vec{v}|} = \boxed{-\frac{1}{2} \Rightarrow \theta = \frac{4}{6}\pi = \frac{2\pi}{3}}$

B. Find $\vec{a}$ & $\vec{b}$ where all components are positive and θ between them is $\frac{\pi}{4}$ rad. $\left(\frac{\sqrt{2}}{2}\right)\sqrt{(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)} = a_1b_1 + a_2b_2 + a_3b_3$



In the XY-plane $\langle \cos\pi/8, \sin\frac{\pi}{8}, 0\rangle$ and $\langle \sin\frac{\pi}{8}, \cos\frac{\pi}{8}, 0\rangle$ work but $a_3 = b_3 \neq 0$. To preserve the ∡ between $\vec{a}$ & $\vec{b}$ as we ↑ a_3, b_3 (equally) the ∡ between the projections of $\vec{a}$ & $\vec{b}$ on XY-plane will need to increase as well! Lets set $a_1 = \cos\frac{\pi}{16}$, $a_2 = \sin\frac{\pi}{16}$, $b_1 = \sin\frac{\pi}{16}$, $b_2 = \cos\frac{\pi}{16}$ and $a_3 = b_3 = x \Rightarrow$

$\sqrt{(1+x^2)(1+x^2)} = \frac{2}{\sqrt{2}}\left(2\sin\left(\frac{\pi}{16}\right)\cos\left(\frac{\pi}{16}\right) + x^2\right) \Leftrightarrow \boxed{a_3 = b_3 = \sqrt{\frac{1-\frac{4}{\sqrt{2}}\sin\frac{\pi}{16}\cos\frac{\pi}{16}}{\left(\frac{2}{\sqrt{2}}-1\right)}}}$

A. Using a unit cube, the desired angle θ is the ∡ between $\langle 1,1,1\rangle$ and $\langle 1,1,0\rangle \Rightarrow |\langle 1,1,1\rangle||\langle 1,1,0\rangle|\cos\theta = 2 \Leftrightarrow \boxed{\theta = \cos^{-1}\sqrt{\frac{2}{3}}}$

$\vec{a}$:= vector from $\langle 1,1,1,1\rangle$ and $\langle -1,-1,-1,-1\rangle$
$\vec{b}$:= " " " and $\langle -1,-1,-1,1\rangle$ $\Bigg\}\ \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \cos\theta$

$= \langle -2,-2,-2,-2\rangle$
$= \langle -2,-2,-2,0\rangle$ $\Bigg\} \Rightarrow \frac{\cancel{12}\,3}{\sqrt{16}\sqrt{12}} = \cos\theta \Rightarrow \boxed{\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \theta}$.

▸ § ◂

b) Verify the parallelogram law $|\vec{a}+\vec{b}|^2+|\vec{a}-\vec{b}|^2 = 2|\vec{a}|^2+2|\vec{b}|^2$.

(SEE BACK)

Main definitions

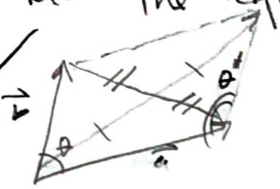
Two points $P = (a, b, c)$ and $Q = (x, y, z)$ define a **vector** $\vec{v} = [x - a, y - b, z - c]$. We also write $\vec{v} = \vec{PQ}$. The numbers v_1, v_2, v_3 in $\vec{v} = [v_1, v_2, v_3]$ are the **components** of $\vec{v}$. The **length** $|\vec{v}|$ of a vector $\vec{v} = \vec{PQ}$ is defined as the distance $d(P, Q)$ from P to Q. A vector of length 1 is called a **unit vector**. The **addition** is $\vec{u} + \vec{v} = [u_1, u_2, u_3] + [v_1, v_2, v_3] = [u_1+v_1, u_2+v_2, u_3+v_3]$. The **scalar multiple** $\lambda\vec{u} = \lambda[u_1, u_2, u_3] = [\lambda u_1, \lambda u_2, \lambda u_3]$. The difference $\vec{u} - \vec{v}$ can be seen as $\vec{u} + (-\vec{v})$.

The **dot product** of two vectors $\vec{v} = [a, b, c]$ and $\vec{w} = [p, q, r]$ is defined as $\vec{v} \cdot \vec{w} = ap + bq + cr$. The **Cauchy-Schwarz inequality** tells $|\vec{v} \cdot \vec{w}| \leq |\vec{v}||\vec{w}|$.

The **angle** between two nonzero vectors is defined as the unique $\alpha \in [0, \pi]$ satisfying $\vec{v} \cdot \vec{w} = |\vec{v}|\ |\vec{w}| \cos(\alpha)$. Two vectors are called **orthogonal** or **perpendicular** if $\vec{v} \cdot \vec{w} = 0$. The zero vector $\vec{0}$ is orthogonal to any vector. For example, $\vec{v} = [2, 3]$ is orthogonal to $\vec{w} = [-3, 2]$. The vector $\mathrm{P}(\vec{v}) = \frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\vec{w}$ is called the **projection** of $\vec{v}$ onto $\vec{w}$. The **scalar projection** $\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|}$ is plus or minus the length of the projection of $\vec{v}$ onto $\vec{w}$. The vector $\vec{b} = \vec{v} - P(\vec{v})$ is a vector orthogonal to $\vec{w}$. **Pythagoras tells:** if $\vec{v}$ and $\vec{w}$ are orthogonal, then $|v - w|^2 = |v|^2 + |w|^2$.

⑤A. $\vec{c}$ bisects the angle θ between $\vec{a}$ & $\vec{b}$ iff., by the half-angle cosine identity, we can show that $\frac{\vec{a}\cdot\vec{c}}{|\vec{a}||\vec{c}|}$ is equivalent to $\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{1-\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}}{2}}$,

↳ $$\frac{1+\frac{(\vec{a}\cdot\vec{b})}{|\vec{a}||\vec{b}|}}{2} = \frac{(\vec{a}\cdot\vec{c})^2}{|\vec{a}|^2|\vec{c}|^2}.$$ Substituting $\vec{c} = |\vec{b}|\vec{a} + |\vec{a}|\vec{b}$ makes the equality true.

B. 

Sum of lengths^2 of a parallelogram's diagonals are equal to $2 \times$ (sum of edge lengths^2)

$$\Updownarrow$$

$$\underbrace{|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2}_{\text{sum of diagonals' lengths}^2} = \underbrace{2(|\vec{a}|^2+|\vec{b}|^2)}_{2\ \times\ \text{one diagonal's length}^2}$$

By the Law of Cosines:

$$|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2 = 2|\vec{a}|^2 + 2|\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta - 2|\vec{a}||\vec{b}|\cos(\pi-\theta)$$
$$= 2|\vec{a}|^2 + 2|\vec{b}|^2 - \cancel{2|\vec{a}||\vec{b}|(\cos\theta + \cos(\pi-\theta))}$$
$$= 2(|\vec{a}|^2+|\vec{b}|^2) \quad \square.$$