① $A = \{1, 6, 12, 13, 9\}$

$B = \{3, 6, 12, 15\}$

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A $\{6, 12\}$ B $|\{1, 3, 6, 12, 13, 9, 15\}| = 7$

C $|\{1, 13, 9\}| = 3$

Problem Set 0

Please write your solutions in the LaTeX and Python templates provided. Aim for concise solutions; convoluted and obtuse descriptions might receive low marks, even when they are correct.

This assignment is meant to be an evaluation of your **individual** understanding coming into the course and should be completed **without collaboration** or outside help. You **may** ask for logistical help concerning LaTeX formatting and/or code submission.

Problem 0-1. Let $A = \left\{i + \binom{5}{i} \;\middle|\; i \in \mathbb{Z} \text{ and } 0 \le i \le 4\right\}$ and $B = \{3i \mid i \in \{1, 2, 4, 5\}\}$.
Evaluate: **(a)** $A \cap B$ **(b)** $|A \cup B|$ **(c)** $|A - B|$

Problem 0-2. Let X be the random variable representing the number of heads seen after flipping a fair coin three times. Let Y be the random variable representing the outcome of rolling two fair six-sided dice and multiplying their values. Please compute the following expected values.
Evaluate: **(a)** $\mathrm{E}[X]$ **(b)** $\mathrm{E}[Y]$ **(c)** $\mathrm{E}[X + Y]$

Problem 0-3. Let $A = 600/6$ and $B = 60 \bmod 42$. Are these statements True or False?
Evaluate: **(a)** $A \equiv B \pmod 2$ **(b)** $A \equiv B \pmod 3$ **(c)** $A \equiv B \pmod 4$

Problem 0-4. Prove **by induction** that $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$, for any integer $n \ge 1$.

Problem 0-5. Prove **by induction** that every connected undirected graph $G = (V, E)$ for which $|E| = |V| - 1$ is acyclic.

Problem 0-6. An **increasing subarray** of an integer array is any consecutive sequence of array integers whose values strictly increase. Write Python function `count_long_subarrays(A)` which accepts Python Tuple `A = (`$a_0, a_1, \ldots, a_{n-1}$`)` of $n > 0$ positive integers, and returns the number of longest increasing subarrays of `A`, i.e., the number of increasing subarrays with length at least as large as every other increasing subarray. For example, if `A = (1,3,4,2,7,5,6,9,8)`, your program should return `2` since the maximum length of any increasing subarray of `A` is three and there are two increasing subarrays with that length: specifically, subarrays `(1,3,4)` and `(5,6,9)`. You can download a code template containing some test cases from the website.

(SEE CODE)

Problem 0-3. Let $A = 600/6$ and $B = 60 \bmod 42$. Are these statements True or False?

Evaluate: **(a)** $A \equiv B \pmod 2$ **(b)** $A \equiv B \pmod 3$ **(c)** $A \equiv B \pmod 4$

$\boxed{A}$ $A \overset{?}{\equiv} B \pmod 2$

$100 \overset{?}{\equiv} 18 \pmod 2$

TRUE $0 \overset{\checkmark}{\equiv} 0 \pmod 2$

$\boxed{B}$ $100 \overset{?}{\equiv} 18 \pmod 3$

$1 \not\equiv 0 \pmod 3$

FALSE

$\boxed{C}$ $100 \equiv 18 \pmod 4$

FALSE $0 \not\equiv 2 \pmod 4$

Problem 0-2. Let X be the random variable representing the number of heads seen after flipping a fair coin three times. Let Y be the random variable representing the outcome of rolling two fair six-sided dice and multiplying their values. Please compute the following expected values.

Evaluate: **(a)** $\mathrm{E}[X]$ **(b)** $\mathrm{E}[Y]$ **(c)** $\mathrm{E}[X+Y]$

$\boxed{A}$ $X \sim \mathrm{Bin}(3, \frac{1}{2})$, $E(X) = 3 \cdot \frac{1}{2} = \frac{3}{2}$

$\boxed{B}$ Let $Y = Y_1 Y_2$ represent the outcomes of the two dice, respectively. Because $Y_1 \perp\!\!\!\perp Y_2$, $E(Y_1 Y_2) = E(Y_1)E(Y_2)$. Since $Y_1, Y_2 \sim \mathrm{DUnif}(\{1,2,3,4,5,6\})$,

$$E(Y_1)E(Y_2) = \left(\frac{1}{6}\sum_{y=1}^{6} y\right)^2 = \left(\frac{\cancel{6}(6+1)}{\cancel{6}\cdot 2}\right)^2 = \frac{49}{4} = E(Y)$$

$\boxed{C}$ By linearity of expectation: $E(X+Y) =$

$E(X) + E(Y) = \frac{3}{2} + \left(\frac{7}{2}\right)^2$

Problem 0-4. Prove **by induction** that $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$, for any integer $n \geq 1$.

Proof (by induction). Let $P(n) := \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$.

$\boxed{\text{BASE CASE}}$ $\sum_{i=1}^{1} i^3 = 1 \stackrel{\checkmark}{=} 1 = \left[\frac{1(\cancel{1+1})}{\cancel{2}}\right]^2$

$\boxed{\text{INDUCTIVE CASE}}$ Assume $P(n)$ to establish $P(n+1)$.

$$\sum_{i=1}^{n+1} i^3 = (n+1)^3 + \sum_{i=1}^{n} i^3 \stackrel{\text{I.H}}{=} (n+1)^3 + \left[\frac{n(n+1)}{2}\right]^2 =$$

$$\frac{4(n+1)(n+1)^2}{4} + \frac{n^2(n+1)^2}{4} = \frac{(n+1)^2}{4}(n^2+4n+4) =$$

$$= \frac{(n+1)^2}{2^2}(n+2)^2$$

$$\checkmark = \left[\frac{(n+1)((n+1)+1)}{2}\right]^2$$

$\square$

Problem 0-5. Prove **by induction** that every connected undirected graph $G = (V, E)$ for which $|E| = |V| - 1$ is acyclic.

Proof (by induction). Let $P(n) :=$ if G is an n-node undirected, connected graph with exactly $n-1$ edges, then G is acyclic. We proceed by inducting on n.

BASE CASE A graph with $n=1$ nodes and no edges is trivially acyclic.

INDUCTIVE CASE We assume $P(n)$ to establish $P(n) \Rightarrow P(n+1)$. Consider a connected, undirected graph G with $n+1$ vertices and n edges. By the Handshake Lemma, $\exists v$ with degree 1 since $2n/n+1 < 2$ (average degree). Inducing subgraph S (by removing v and its incident edge to S) gives S n-vertices and $n-1$ edges. The only way for G to have a cycle is if S has a cycle because the removed vertex v could not be a node incident to edges in a cycle in G since such nodes would have degree ≥ 2 and degree$(v) = 1$. Invoking inductive hypothesis, S cannot have cycle so G is acyclic. Thus, $P(n) \Rightarrow P(n+1)$. $\square$