② B. If each student shook hands with exactly 17 others the sum of degrees over all students (vertices) would be odd (111 · 17). However, handshakes (edges) are integers in count but (from TA) $\frac{111 \cdot 17}{2} = |E|$.

Ravi Dayabhai

Problem Set 4

CORRECT REASONING, but needed hint/proof of Lemma & missed proof by induction

Problem 1. [15 points] Let $G = (V, E)$ be a graph. A *matching* in G is a set $M \subset E$ such that no two edges in M are incident on a common vertex.

Let M_1, M_2 be two matchings of G. Consider the new graph $G' = (V, M_1 \cup M_2)$ (i.e. on the same vertex set, whose edges consist of all the edges that appear in either M_1 or M_2). Show that G' is bipartite. (SEE LAST PAGE)

Helpful definition: A *connected component* is a subgraph of a graph consisting of some vertex and every node and edge that is connected to that vertex.

Problem 2. [20 points] Let $G = (V, E)$ be a graph. Recall that the *degree* of a vertex $v \in V$, denoted d_v, is the number of vertices w such that there is an edge between v and w.

✓ (a) [10 pts] Prove that

$$2|E| = \sum_{v \in V} d_v.$$

"Handshake Lemma"

~~Proof~~: Each edge contributes two degrees (one degree for each spanning vertex)

✓ (b) [5 pts] At a 6.042 ice cream study session (where the ice cream is plentiful and it helps you study too) 111 students showed up. During the session, some students shook hands with each other (everybody being happy and content with the ice-cream and all). Turns out that the University of Chicago did another spectacular study here, and counted that each student shook hands with exactly 17 other students. Can you debunk this too?

✓ (c) [5 pts] And on a ~~more dull note~~, how many edges does K_n, the complete graph on n vertices, have? $\boxed{n(n-1)/2} = \binom{n}{2}$

Problem 3. [15 points] Two graphs are isomorphic if they are the same up to a relabeling of their vertices (see Definition 5.1.3 in the book). A property of a graph is said to be *preserved under isomorphism* if whenever G has that property, every graph isomorphic to G also has that property. For example, the property of having five vertices is preserved under isomorphism: if G has five vertices then every graph isomorphic to G also has five vertices.

✓ (a) [5 pts] Some properties of a simple graph, G, are described below. Which of these properties is *preserved under isomorphism*?

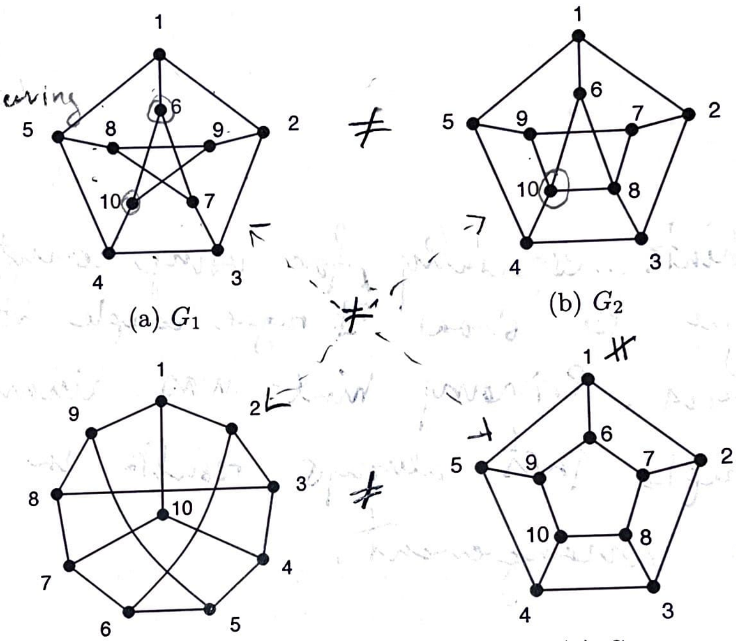
1. G has an even number of vertices.
2. None of the vertices of G is an even integer.
3. G has a vertex of degree 3.
4. G has exactly one vertex of degree 3.

$\boxed{\text{1, 3, 4 are preserved under isomorphism}}$

(b) [10 pts] Determine which among the four graphs pictured in the Figures are isomorphic. If two of these graphs are isomorphic, describe an isomorphism between them. If they are not, give a property that is preserved under isomorphism such that one graph has the property, but the other does not. For at least one of the properties you choose, *prove* that it is indeed preserved under isomorphism (you only need prove one of them).

G_1 and G_3 are isomorphic: $\exists$ an edge-preserving bijection $f: V(G_1) \to V(G_3)$ $\forall u, v \in V(G_1)$:

$f(1) = 8$, $f(2) = 7$, $f(3) = 10$, $f(4) = 4$, $f(5) = 3$, $f(6) = 9$, $f(7) = 1$, $f(8) = 2$, $f(9) = 6$, $f(10) = 5$



(a) G_1 (b) G_2 (c) G_3 (d) G_4

G_2 not isomorphic to G_4 because $\exists$ degree 4 node in G_2 but not G_4. Same reason for non-isomorphism b/w G_2 and G_3 and G_2 and G_1

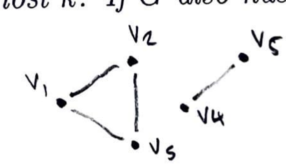
G_1 not isomorphic to G_4 because $\exists$ a path of leng[th] 3 between all pairs of adjacent nodes in G[$_4$] but this is not true for G_1. Same reason for non-isomorphism b/w G_3 and G_4.

Figure 1: Which graphs are isomorphic?

Problem 4. [15 points] Recall that a ***coloring*** of a simple graph is an assignment of a color to each vertex such that no two adjacent vertices have the same color. A ***k-coloring*** is a coloring that uses at most k colors.

False Claim. *Let G be a (simple) graph with maximum degree at most k. If G also has a vertex of degree less than k, then G is k-colorable.*

(a) [5 pts] Give a counterexample to the False Claim when $k = 2$.



$G := (V, E)$ where $V = \{v_1, v_{...}$ $v_4, v_5\}$ and $E = \{v_1 - v_2, v_2 -$ $v_3 - v_1, v_4 - v_5\}$.

(b) [10 pts] Consider the following proof of the False Claim:

⑤ This claim is ~~false~~ false, so we proceed by proving the negation of this claim, i.e., for all $n \geq 3$, for all boys' & girls' preference sets, there exists a dating arrangement that is not stable. To construct such an arrangement, we can find a boy-girl pair such that neither is each other's least-favored partner. Partnering each with their least favorite partner (and randomly pairing the others) guarantees a rogue couple $\Leftrightarrow$ not stable arrangement. Now to show that such a boy-girl pair exists for all preference sets for all $n \geq 3$, notice that there are n^2 potential boy-girl pairings and at most $2n$ pairings such that at least one partner is the other's least favorite. For $n \geq 3$, $n^2 > 2n$, so there always exists a pairing that yields the desired conditions for the boy-girl pair needed to construct an unstable arrangement. □

Needed hints... especially for using counting as proof technique to show $\exists$ rogue couple $\forall n \geq 3$ $\forall$ set of preferences. Primary hint was always finding B-G couple that always results in at least one unstable arrangement.

Proof. Proof by induction on the number n of vertices:

(n=2, k=1) G_n is not 1-colorable!.

Induction hypothesis: $P(n)$ is defined to be: Let G be a graph with n vertices and maximum degree a most k. If G also has a vertex of degree less than k, then G is k-colorable.

Base case: (n=1) G has only one vertex and so is 1-colorable. So $P(1)$ holds.

Inductive step:

We may assume $P(n)$. To prove $P(n+1)$, let G_{n+1} be a graph with $n+1$ vertices and maximum degree at most k. Also, suppose G_{n+1} has a vertex, v, of degree less than k. We need only prove that G_{n+1} is k-colorable.

These vertices may not exist!

To do this, first remove the vertex v to produce a graph, G_n, with n vertices. Removing v reduces the degree of all vertices adjacent to v by 1. So in G_n, each of these vertices has degree less than k. Also the maximum degree of G_n remains at most k. So G_n satisfies the conditions of the induction hypothesis $P(n)$. We conclude that G_n is k-colorable. X

Now a k-coloring of G_n gives a coloring of all the vertices of G_{n+1}, except for v. Since v has degree less than k, there will be fewer than k colors assigned to the nodes adjacent to v. So among the k possible colors, there will be a color not used to color these adjacent nodes, and this color can be assigned to v to form a k-coloring of G_{n+1}. □

Identify the exact sentence where the proof goes wrong.

Problem 5. [15 points] Prove or disprove the following claim: for some $n \geq 3$ (n boys and n girls, for a total of $2n$ people), there exists a set of boys' and girls' preferences such that every dating arrangement is stable. (SEE BACK OF PRIOR PAGE)

Problem 6. [20 points]

Let $(s_1, s_2, ..., s_n)$ be an arbitrarily distributed sequence of the number $1, 2, ..., n-1, n$. For instance, for $n = 5$, one arbitrary sequence could be $(5, 3, 4, 2, 1)$.

Define the graph G=(V,E) as follows:

1. $V = \{v_1, v_2, ..., v_n\}$
2. $e = (v_i, v_j) \in E$ if either:
   (a) $j = i + 1$, for $1 \leq i \leq n-1$
   (b) $i = s_k$, and $j = s_{k+1}$ for $1 <= k <= n-1$

(SEE BACK)

(a) [10 pts] Prove that this graph is 4-colorable for any $(s_1, s_2, ..., s_n)$.

Hint: First show that a line graph is 2-colorable. Note that a line graph is defined as follows: The n-node graph containing $n-1$ edges in sequence is known as the line graph L_n.

(b) [10 pts] Suppose $(s_1, s_2, ..., s_n) = (1, a_1, 3, a_2, 5, a_3, ...)$ where $a_1, a2....$ is an arbitrary distributed sequence of the even numbers in $1, ..., n-1$. Prove that the resulting graph is 2-colorable.

⑥ A. **Proof**: First we start by showing a line graph can be colored by 2 colors by induction on the number of edges. The base case is true by assigning one endpoint one color and the other of the lone edge another. The inductive hypothesis, $P(n-1)$, is that an n-vertex ($n-1$ edge) line graph is 2-colorable. Assuming $P(n-1)$, $P(n)$ is true because the next edge endpoint can be colored the same as the prior edge's start point. Thus, $P(n-1) \Rightarrow P(n)$ is shown and we have proven any line graph is 2-colorable.

The graph in question consists of union of two sets of edges: those forming a line subgraph on vertex labels (i.e., $\{v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n\}$) and those forming a line subgraph on vertex labels where sequence of labels is given by S_n. ① WLOG, choose 2 colors to color the first line subgraph and 2 different colors to color the second, respectively. ② Order the vertices by label ascending (i.e., $v_1, v_2, \ldots, v_n$) to create ordered set of the second line subgraph's colors.

EXAMPLE: $S_n = (2,4,6,8,5,1,3,7)$, colors $= \{(R,G),(B,Y)\}$

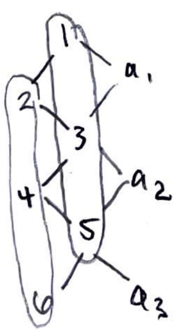
1st line subgraph	v_1,	v_2,	v_3,	v_4,	v_5,	v_6,	v_7,	v_8
1st coloring	R	G	R	G	R	G	R	G
2nd line subgraph	v_2,	v_4,	v_6,	v_8,	v_5,	v_1,	v_3,	v_7
2nd coloring	B	Y	B	Y	B	Y	B	Y

①

ordered vertices:	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
1st colorings:	R	G	R	G	R	G	R	G
② 2nd	Y	B	B	Y	B	B	Y	Y
③ FINAL COLORING:	Y	B	R	Y	B	G	Y	G

③ Whenever this sequence repeats a color, switch the repeated color with the corresponding color from the other coloring. The final result has at most 4 distinct colors and satisfies both line graph colorings.

⑥ B. **Proof**: Using the same procedure from above always yields this graph colored by 2 colors because, WLOG, an even-labelled vertex only connects to an odd-labelled vertex and an odd-labelled vertex only connects to even-labelled vertices, so one color can color even vertices and another can color the odd vertices, i.e., a bijection exists between $\{2,4,6,\ldots\}$ and $\{a_1, a_2, a_3, \ldots\}$

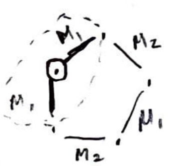


NICE, DUDE. ü

6.042J / 18.062J Mathematics for Computer Science
Fall 2010

① Proof: We will show that $G' = (V, M_1 \cup M_2)$ has no odd length cycles and that if a graph has no odd length cycles, it is bipartite to prove G' is a bipartite graph. (Treat these statements as Lemma 1 & 2, respectively.)

- Lemma 1 is proven by contradiction supposing G' has at least one odd length cycle. WLOG (by symmetry), choose a vertex along this odd length cycle and assign edges comprising this cycle to M_1 & M_2 in alternating fashion (since a vertex can only appear in an edge in a matching, at most, once and each vertex in a cycle is at least degree 2). Because the cycle has odd number of edges, the chosen vertex appears in two edges in the same matching – a contradiction of the definition of a mapping.



- Lemma 2 considers only the connected components of graph G' and showing these connected components are bipartite since unconnected components are trivially bipartite and if all connected components are bipartite, then the graph is bipartite. For each connected component let w be a vertex in said connected component and the distance between w and another vertex v be $d(w,v) :=$ number of edges along shortest path between w & v. Connected component C is bipartite iff

Needed lots of help w/ this proof.

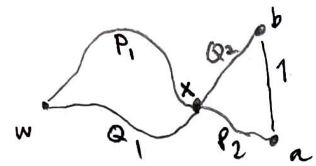
$$\begin{cases} L(C) = \{v \in V(C) \mid d(w,v) \text{ is even}\} \\ R(C) = \{v \in V(C) \mid d(w,v) \text{ is odd}\} \\ L(C) \cap R(C) = \emptyset \\ L(C) \cup R(C) = V(C) \\ \forall e \in E(C).\ e \text{ has one vertex in } L(C) \text{ and one in } R(C) \end{cases}$$

(Argument proceeds in same way if definit[ions] for $L(C)$ & $R(C)$ are switched.)

WLOG, suppose $a, b \in V(C)$ and $a, b \in L(G)$ and a, b are adjacent for sake of demonstrating a contradiction. (CONTINUED ON BACK)

(① CONTINUED)

$a \neq b \neq w$. because if $a = w$, $d(a,w) = 0$ but $d(w,b) = 1$ which is inconsistent with $b \in L(G)$. (Similar reasoning shows $b \neq w$ and $a \neq b$ because they are adjacent vertices, i.e., $d(a,b) = 1$.) Let x be the last common vertex



common to path P and Q where $d(w,a) = |P|$ and $d(w,b) = |Q|$. Let P and Q be subdivided by x such that $|P| = |P_1| + |P_2|$ and $|Q| = |Q_1| + |Q_2|$ where P_1, Q_1 are paths from w to x and P_2, Q_2 are paths from w to a & b, respectively. Immediately, $|P_1| = |Q_1|$ (otherwise a shorter path to a or b would exist). Since, $|P|$ & $|Q|$ are even, $|P_1| = |Q_1|$, then $|Q_2|$ & $|P_2|$ must have the same parity. However, this would imply the cycle formed by vertices $\{x, a, b\}$ and edges $\{a - b\} \cup Q_2 \cup P_2$ is odd length, i.e., $|Q_2| + |P_2| + 1$ is odd. We assumed G' has no odd length cycle, hence we have reached a contradiction, so $\nexists\, a, b \in L(R) \mid (a-b) \in E(C)$. Thus G' is bipartite. □