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Problem Set 7

Note that $\sum_{i=0}^{n} ix^i = x + 2x^2 + 3x^3 + \dots + nx^n = S$

$- \quad x^2 + 2x^3 + \dots + (n-1)x^n + nx^{n+1} = Sx$

$x + x^2 + x^3 + \dots + x^n - nx^{n+1} = (1-x)S$

$\Rightarrow \sum_{i=0}^{n} ix^i = \left[\frac{(1-x^n)}{(1-x)} - 1 - nx^{n+1}\right]\frac{1}{(1-x)}$

$= \frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}$

Problem 1. **[15 points]** Express

(CONTINUED ON BACK)

$$f(n) = \sum_{i=0}^{n} i^2 x^i$$

as a closed-form function of n.

Problem 2. **[20 points]**

(a) [5 pts] What is the product of the first n odd powers of two: $\prod_{k=1}^{n} 2^{2k-1}$? $= 2 \cdot 2^3 \cdot 2^5 \cdots 2^n$ (for odd n)

(SEE BACK)

(b) [5 pts] Find a closed expression for

$$\sum_{i=0}^{n}(3^i)\cdot\sum_{j=0}^{m}(3^j) = \sum_{i=0}^{n}\sum_{j=0}^{m}(3^i)(3^j) = \sum_{i=0}^{n}\sum_{j=0}^{m} 3^{i+j} = \left(\frac{1-3^{n+1}}{1-3}\right)\left(\frac{1-3^{m+1}}{1-3}\right) = \boxed{\frac{1-3^{n+1}-3^{m+1}+3^{n+m+2}}{4}}$$

(c) [5 pts] Find a closed expression for

$$\sum_{i=1}^{n}\left(ni + \frac{n(n+1)}{2}\right) = \sum_{i=1}^{n}\left(\sum_{i=1}^{n} i + \sum_{j=1}^{n} j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n}(i+j) = \sum_{i=1}^{n} ni + \sum_{i=1}^{n}\frac{n(n+1)}{2} = \boxed{n^2(n+1)}$$

$$= \frac{2n^2(n+1)}{2}$$

(d) [5 pts] Find a closed expression for

$$\prod_{i=1}^{n}\prod_{j=1}^{n} 2^i \cdot 3^j$$

(SEE BACK)

Problem 3. **[10 points]**

(a) [6 pts] Use integration to find upper and lower bounds that differ by at most 0.1 for the following sum. (You may need to add the first few terms explicitly and then use integrals to bound the sum of the remaining terms.)

$$\sum_{i=1}^{\infty} \frac{1}{(2i+1)^2}$$

(SEE BACK)

① $\frac{d}{dx}\left(\sum_{i=0}^{n} i x^{i}\right) = \sum_{i=0}^{n} i^{2} x^{i-1} = \frac{d}{dx}\left(\frac{x-(n+1)x \;+ nx}{(1-x)^2}\right)$ (FROM FRONT)

$$\Rightarrow \quad \sum_{i=0}^{n} i^2 x^i = \boxed{x\left[\frac{(-2n^2-2n+1)x^{n+1}+n^2x^{n+2}+(n+1)^2x^{n}-x-1}{(1-x)^3}\right]}$$

② (A) $\prod_{k=1}^{n} 2^{2k-1} \Leftrightarrow 2^{\sum_{k=1}^{n} 2k-1} = \boxed{2^{n^2}}$

(B) $\prod_{i=1}^{n}\prod_{j=1}^{n} 2^i 3^j = \begin{array}{l}(2\cdot 3)(2\cdot 3^2)\cdots(2\cdot 3^n)\\ \times(2^2\cdot 3)(2^2\cdot 3^2)\cdots(2^2 3^n)\\ \times(2^3 3)(2^3 3^2)\cdots(2^3 3^n)\end{array} = \prod_{i=1}^{n} 2^{in}\prod_{j=1}^{n} 3^{jn}$

$$= 2^{n(1+2+\dots+n)} \cdot 3^{n(1+2+\dots+n)} = \boxed{2^{\frac{n^2(n+1)}{2}} \cdot 3^{\frac{n^2(n+1)}{2}}}$$

③ (A) Trying with just one term made explicit for strictly decreasing function $f(x) = \frac{1}{(2x+1)^2}$: $f(n) + \int_1^n f(x)\,dx \le \sum_{x=1}^{n} f(i) \le f(1) + \int_1^n f(x)\,dx$. Plugging in $\int_1^n \frac{dx}{(2x+1)^2} = -\frac{1}{4n+2} + \frac{1}{6}$, $\frac{1}{(2n+1)^2} - \frac{1}{4n+2} + \frac{1}{6} \le \sum_{i=1}^{n} f(i) \le \frac{1}{9} - \frac{1}{4n+2} + \frac{1}{6}$ which gives (as $n \to \infty$) $0 \le \sum_{i=1}^{n} f(i) \le \frac{1}{9}$, but we want bounds at most $\frac{1}{10}$.

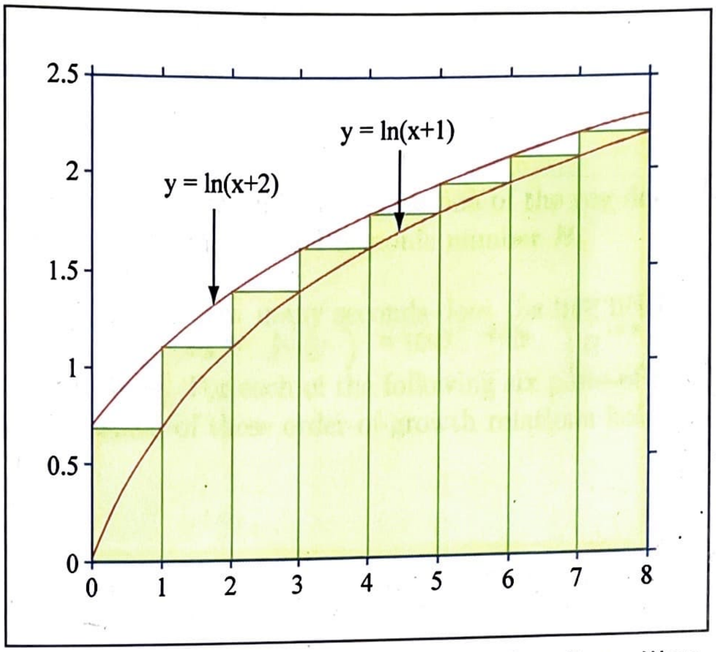
Using two explicit terms gives tighter bounds:

$$f(1) + f(n) + \int_2^n \frac{dx}{(2x+1)^2} \le \sum_{i=1}^{n} f(i) \le f(1) + f(2) + \int_2^n \frac{dx}{(2x+1)^2},$$

Evaluating $\int_2^n \frac{dx}{(2x+1)^2}$ and explicit terms gives:

$$\cancel{\tfrac{1}{9}} + \cancel{\tfrac{1}{(2n+1)^2}} - \cancel{\tfrac{1}{4n+2}} + \cancel{\tfrac{1}{10}} \le \sum_{i=1}^{x} f(i) \le \cancel{\tfrac{1}{9}} + \frac{1}{25} - \cancel{\tfrac{1}{4n+2}} + \cancel{\tfrac{1}{10}}$$

$(n \to \infty)$ $\boxed{0 \le \sum_{i=1}^{n} f(i) \le \frac{1}{25}}$.



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(b) [4 pts] Assume n is an integer larger than 1. Which of the following inequalities, if any, hold. You may find the graph helpful.

1. $\sum_{i=1}^{n} \ln(i+1) \leq \int_{0}^{n} \ln(x+2)dx$ TRUE

2. $\sum_{i=1}^{n} \ln(i+1) \leq \ln 2 + \int_{1}^{n} \ln(x+1)dx$ FALSE

Problem 4. [15 points] There is a bug on the edge of a 1-meter rug. The bug wants to cross to the other side of the rug. It crawls at 1 cm per second. However, at the end of each second, a malicious first-grader named Mildred Anderson *stretches* the rug by 1 meter. Assume that her action is instantaneous and the rug stretches uniformly. Thus, here's what happens in the first few seconds:

- The bug walks 1 cm in the first second, so 99 cm remain ahead.
- Mildred stretches the rug by 1 meter, which doubles its length. So now there are 2 cm behind the bug and 198 cm ahead.
- The bug walks another 1 cm in the next second, leaving 3 cm behind and 197 cm ahead.
- Then Mildred strikes, stretching the rug from 2 meters to 3 meters. So there are now $3 \cdot (3/2) = 4.5$ cm behind the bug and $197 \cdot (3/2) = 295.5$ cm ahead.
- The bug walks another 1 cm in the third second, and so on.

Your job is to determine this poor bug's fate.

(a) [5 pts] During second i, what *fraction* of the rug does the bug cross? $\boxed{\frac{1}{100i}}$

$$\boxed{\frac{1}{100}\sum_{i=1}^{n}\frac{1}{i} = \frac{H_n}{100}}$$

(b) [5 pts] Over the first n seconds, what fraction of the rug does the bug cross altogether? Express your answer in terms of the Harmonic number H_n.

(c) [5 pts] Approximately how many seconds does the bug need to cross the entire rug?

$H_n \approx \ln(n) = 100 \Rightarrow \boxed{e^{100}}$

Problem 5. [20 points] For each of the following six ~~pairs of functions~~ f and g (parts (a) through (f)), state which of these order-of-growth relations hold (more than one may hold, or none may hold):

$$f = o(g) \qquad f = O(g) \qquad f = \omega(g) \qquad f = \Omega(g) \qquad f = \Theta(g) \qquad f \sim g$$

(SEE BACK.)

(a)	$f(n) = \log_2 n$	$g(n) = \log_{10} n$
(b)	$f(n) = 2^n$	$g(n) = 10^n$
(c)	$f(n) = 0$	$g(n) = 17$
(d)	$f(n) = 1 + \cos\left(\frac{\pi n}{2}\right)$	$g(n) = 1 + \sin\left(\frac{\pi n}{2}\right)$
(e)	$f(n) = 1.0000000001^n$	$g(n) = n^{10000000000}$

Problem 6. [15 points] This problem continues the study of the asymptotics of factorials.

(a) [5 pts]

Either prove or disprove each of the following statements.

- $n! = O((n+1)!)$ $\boxed{\text{True}}$ because $\lim_{n\to\infty}\left|\frac{n!}{(n+1)!}\right| = 0 < \infty$.
- $n! = \Omega((n+1)!)$ $\boxed{\text{False}}$ because " $0 \neq 0$.
- $n! = \Theta((n+1)!)$ $\boxed{\text{False}}$ because second point is false.
- $n! = \omega((n+1)!)$ $\boxed{\text{False}}$ because $\lim_{n\to\infty}\left|\frac{n!}{(n+1)!}\right| = 0 \neq \infty$
- $n! = o((n+1)!)$ $\boxed{\text{True}}$; see first point.

(b) [5 pts] Show that $n! = \omega\left(\left(\frac{n}{3}\right)^{n+e}\right)$. (SEE BACK.)

(c) [5 pts] Show that $n! = \Omega(2^n)$ (SEE NEXT PAGE)

5 A. $f(n) = \log_2 n$, $g(n) = \log_{10} n$. $\lim_{n\to\infty} \left|\frac{\log_2 n}{\log_{10} n}\right| = c$ where $2^c = 10$, so $\boxed{f = O(g),\ f = \Omega(g),\ f = \Theta(g)}$

B. $f(n) = 2^n$, $g(n) = 10^n$. $\lim_{n\to\infty} \left|\frac{2^n}{10^n}\right| = \lim_{n\to\infty} \left(\frac{1}{5}\right)^n = 0$, so $\boxed{f = O(g),\ f = o(g)}$

C. $f(n) = 0$, $g(n) = 17$. $\lim_{n\to\infty} \left|\frac{0}{17}\right| = 0$, so $\boxed{f = O(g),\ f = o(g)}$

D. $f(n) = 1 + \cos\left(\frac{\pi n}{2}\right)$, $g(n) = 1 + \sin\left(\frac{\pi n}{2}\right)$. $\lim_{n\to\infty} \left|\frac{1+\cos\left(\frac{\pi n}{2}\right)}{1+\sin\left(\frac{\pi n}{2}\right)}\right|$ does not exist, but $\limsup_{n\to\infty} \left|\frac{1+\cos\left(\frac{\pi n}{2}\right)}{1+\sin\left(\frac{\pi n}{2}\right)}\right| = 2$ so $\boxed{f = O(g)}$. But $\limsup_{n\to\infty} \left|\frac{g(n)}{f(n)}\right| = 2$ so $\boxed{g = O(f)}$. Taken together $\boxed{f = \Theta(g)}$.

None of the relations hold.

DUMB MISTAKE! Ratio of $f(x)/g(x)$ in the limit superior is ∞ so not O relation in either case!

E. Exponentials beat polynomials in the limit, so $\boxed{f = \Omega(g),\ f = \omega(g)}$

Yes, but lazy explanation! 18/

requires repeated application of L'Hopital's rule until $g(n)$ is fully "unpacked"

6 A. $\prod_{i=1}^{n} i = n! \Rightarrow \sum_{i=1}^{n} \ln(i)$ which can be bounded with integration

bounds: $\cancel{\ln(1)} + \int_1^n \ln x \, dx \le \sum_{i=1}^{n} \ln(i) \le \ln(n) + \int_1^n \ln x \, dx \iff$

$$n\ln(n) - (n-1) \le \sum_{i=1}^{n} \ln(i) \le \ln(n) + n\ln(n) - (n-1) \iff$$

$$n^n \cdot \frac{1}{e^{n-1}} \le n! \le n \cdot n^n \cdot \frac{1}{e^{n-1}} \iff$$

$$n^n \cdot e^{-n} \cdot e \le n! \le n^{n+1} \cdot e^{-n} \cdot e \iff$$

$$h(n) = e\left(\frac{n}{e}\right)^n \le n! \le ne\left(\frac{n}{e}\right)^n$$

If we can show $\lim_{n\to\infty} \left|\frac{e\left(\frac{n}{e}\right)^n}{\left(\frac{n}{3}\right)^{n+e}}\right| > 0 = \lim_{n\to\infty} \left|\frac{e}{\left(\frac{n}{3}\right)^e} \cdot \frac{\left(\frac{n}{e}\right)^n}{\left(\frac{n}{3}\right)^n}\right| = \lim_{n\to\infty} \frac{e}{\left(\frac{n}{3}\right)^e}\left(\frac{3}{e}\right)^n =$

$\frac{e}{\left(\frac{1}{3}\right)^e} \cdot \lim_{n\to\infty} \frac{\left(\frac{3}{e}\right)^n}{n^e} = \infty$, then $n!$ is strictly lower bounded (asymptotically) by $\left(\frac{n}{3}\right)^{n+e}$ by transitivity since the lower integration bound of $n!$ is strictly lower bounded by $\left(\frac{n}{3}\right)^{n+e}$, i.e., $h = \omega\left(\left(\frac{n}{3}\right)^{n+e}\right) \Rightarrow n! = \omega\left(\left(\frac{n}{3}\right)^{n+e}\right)$

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5 k. To show $n! = \Omega(2^n)$, we need to show $\lim_{n\to\infty} \left|\frac{n!}{2^n}\right| > 0$. Using the result from e, if we can show that the lower integration bound on $n!$ is greater than 2^n in the limit, we know $n!$ will be greater than 2^n in the limit. So, we must show $\lim_{n\to\infty} \left|\frac{e\left(\frac{n}{e}\right)^n}{2^n}\right| > 0$.

$$e \cdot \lim_{n\to\infty} \frac{\left(\frac{n}{e}\right)^n}{2^n} = e \cdot \lim_{n\to\infty} \left(\frac{n}{2e}\right)^n = \infty > 0.$$