1 Problem 1

An undirected graph G has ***width*** w if the vertices can be arranged in a sequence

$$v_1, v_2, v_3, \ldots, v_n$$

such that each vertex v_i is joined by an edge to at most w preceding vertices. (Vertex v_j *precedes* v_i if $j < i$.) Use induction to prove that every graph with width at most w is $(w+1)$-colorable. (Recall that a graph is k-colorable iff every vertex can be assigned one of k colors so that adjacent vertices get different colors.)

Proof (by induction). We will complete this proof by induction by inducting on the number of vertices n and maximal width w of a graph with the inductive hypothesis $P(n, w)$ defined as: G is a n-node undirected graph with maximal width $w \implies G$ is $(w+1)$-colorable.

Base case. $P(1, 0)$ is trivially true since a single node graph can be colored with one color.

Inductive cases. To establish $P(n, w)$, there are two more implications to prove:

1. $P(n, 0) \implies P(n+1, 0)$
2. $P(n, w) \implies P(n, w+1)$

Note that these implications must hold $\forall n.\ n \geq 1\ \forall w.\ 0 \leq w < n$ since the greatest maximal width, $w^* = n-1$, is the highest degree any node in an n-node graph can have. Thus, w is bounded by n since no sequence (as described in the problem statement) of vertices produces $w \geq n$ – the greatest maximal width w^* can be found by any arbitrary sequence of vertices with the highest degree vertex at the end of the sequence. When $w < w + 1 \leq w^* < n$ is violated, $P(n, w)$ is vacuously true.

1. $P(n, 0) \implies P(n+1, 0)$ is true since all graphs without any edge can be colored by $w + 1 = 0 + 1 = 1$ color.
2. $P(n, w) \implies P(n, w+1)$ is true (when $w < w+1 \leq w^* < n$) because the extra available color when an edge is added between the highest degree node in G to another vertex which it was not connected to before (read: the only way to increase the maximal width of a graph by 1) can be used to color either endpoint in the case these two vertices are colored the same.

□

2 Problem 2

A ***planar graph*** is a graph that can be drawn without any edges crossing.

1. First, show that any subgraph of a planar graph is planar.

   Solution. A subgraph of a graph contains a subset of the edges found in the graph. Since the graph is planar, all of its edges do not cross with any of the other edges. Every element of a subset of these edges do not cross either with the members either, so the subgraph is also planar.

2. Also, any planar graph has a node of degree at most 5. Now prove by induction that any planar graph can be colored in at most 6 colors.

   Proof (by induction). We will complete this proof by induction by inducting on the number of vertices n that comprise a planar graph G. The inductive hypothesis, $P(n)$, is that such a graph G is 6-colorable.

   Base case. $P(1)$ is trivially true since *any* one-node graph can be colored in as few as 1 color.

   Inductive case. We will show $P(n) \implies P(n+1)$. Let G' be a planar graph consisting of $n+1$ nodes. Removing a node with degree ≤ 5 (which is guaranteed to exist from the fact in the problem statement) and its incident edges yields an induced subgraph G which is planar (from the fact proven in the prior question) and 6-colorable (assuming $P(n)$). The removed vertex is connected to nodes representing at most 5 different colors. Using a sixth color to color this removed vertex results in G' being 6-colorable.

   $\square$