

## 1 Team Problem: A Mystery

A certain cabal within the 6.042 course staff is plotting to make the final exam *ridiculously hard*. ("Problem 1. Prove that the axioms of mathematics are complete and consistent. Express your answer in Mayan hieroglyphics.") The only way to stop their evil plan is to determine exactly who is in the cabal. The course staff consists of nine people:

$$\{\text{Oscar, Stav, Darren, Patrice, David, Nick, Martyn, Tom}\}$$

The cabal is a subset of the nine people. A membership roster has been found and appears below, but it is deviously encrypted in logic notation. The predicate *iscabal* indicates who is in the cabal; that is, *incabal*(*x*) is true if and only if *x* is a member. Translate each statement below into English and deduce who is in the cabal.

$$(i) \exists x \exists y \exists z (x \neq y \wedge x \neq z \wedge y \neq z \wedge incabal(x) \wedge incabal(y) \wedge incabal(z))$$

There are at least three people in the cabal.

$$(ii) \neg(incabal(\text{Stav}) \wedge incabal(\text{David}))$$

At least one of Stav or David are *not* in the cabal.



Stav and David cannot both be in the cabal.

$$(iii) (incabal(\text{Martyna}) \vee incabal(\text{Patrice})) \implies \forall x incabal(x)$$

If at least one person isn't in the cabal, then Martyna and Patrice are *textitnot* in the cabal.

$$(iv) incabal(\text{Stav}) \implies incabal(\text{David})$$

If Stav is in the cabal, then David is in the cabal, too.



If David is *not* in the cabal, then Stav isn't either.

$$(v) incabal(\text{Darren}) \implies incabal(\text{Martyna})$$

If Darren is in the cabal, then Martyna is in the cabal, too.



If Martyna is *not* in the cabal, then Darren is *not* in the cabal.

$$(vi) (incabal(\text{Oscar}) \vee incabal(\text{Nick})) \implies \neg incabal(\text{Tom})$$

If Oscar is in the cabal, then Tom isn't *and* if Nick is in the cabal, then Tom isn't.

$\Updownarrow$

If Tom is in the cabal, then neither Oscar nor Nick are in the cabal.

(vii)  $(\text{incabal}(\text{Oscar}) \vee \text{incabal}(\text{David})) \implies \neg \text{incabal}(\text{Marten})$

If Oscar is in the cabal, then Marten isn't *and* if David is in the cabal, then Marten isn't.

$\Updownarrow$

If Marten is in the cabal, then neither Oscar nor David are in the cabal.

**Solution.** From (ii), we know that at least one of Stav or David are not cabal members, which implies (by (iii)) that Martyna and Patrice are not cabal members. From (iv), we know Stav can't be in the cabal otherwise David would be, too, yielding a contradiction vis-a-vis (ii). The contrapositive of (v) exculpates Darren since we already know Martyna is not a cabal member.

At this stage 5 members are left: {Oscar, Nick, Marten, Tom, David}. If we assume Oscar is in the cabal, then by (vi) and (vii), the cabal must be {Oscar, Nick, David}.

We can check that this is the only logical configuration of the cabal by considering the remaining cases concerning David's and Oscar's memberships:

**Case 1.** (Neither David nor Oscar are in the cabal) By necessity, from (i), the cabal would be {Nick, Martin, Tom}. Thus, this case cannot be since the contrapositive of (vi) is contradicted by having Nick and Tom in the cabal together.

**Case 2.** (David is in the cabal but Oscar is not) Again, a contradiction arises in this case because at least one of Marten or Tom must be included by (ii). If Tom were to be included, Nick would be out (by (vi)) forcing Marten to be included (which cannot be true by the contrapositive of (vii) ruling out David – something we assumed for this case).

**Case 3.** (Oscar is in the cabal but David is not) This case does not produce a valid cabal because the inclusion of Oscar (and David, by assumption), yields a contradiction to (ii) since Oscar's inclusion only leaves Nick as co-conspirator.