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Problem Set 2

Problem 1. [12 points] Define a *3-chain* to be a (not necessarily contiguous) subsequence of three integers, which is either monotonically increasing or monotonically decreasing. We will show here that any sequence of five distinct integers will contain a *3-chain*. Write the sequence as a_1, a_2, a_3, a_4, a_5. Note that a monotonically increasing sequences is one in which each term is greater than or equal to the previous term. Similarly, a monotonically decreasing sequence is one in which each term is less than or equal to the previous term. Lastly, a subsequence is a sequence derived from the original sequence by deleting some elements without changing the location of the remaining elements.

(a) [4 pts] Assume that $a_1 < a_2$. Show that if there is no *3-chain* in our sequence, then a_3 must be less than a_1. (Hint: consider a_4!) (SEE BACK)

(b) [2 pts] Using the previous part, show that if $a_1 < a_2$ and there is no *3-chain* in our sequence, then $a_3 < a_4 < a_2$. (SEE BACK)

(c) [2 pts] Assuming that $a_1 < a_2$ and $a_3 < a_4 < a_2$, show that any value of a_5 must result in a *3-chain*. (SEE BACK)

(d) [4 pts] Using the previous parts, prove by contradiction that any sequence of five distinct integers must contain a *3-chain*. (SEE BACK.)

Problem 2. [8 points]

Prove by either the Well Ordering Principle or induction that for all nonnegative integers, n:

$$\sum_{i=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2. \quad (1)$$

(SEE BACK OF LAST PAGE)

Problem 3. [25 points] The following problem is fairly tough until you hear a certain one-word clue. The solution is elegant but is slightly tricky, so don't hesitate to ask for hints!

NEEDED HINT

During 6.042, the students are sitting in an $n \times n$ grid. A sudden outbreak of beaver flu (a rare variant of bird flu that lasts forever; symptoms include yearning for problem sets and craving for ice cream study sessions) causes some students to get infected. Here is an example where $n = 6$ and infected students are marked $\times$.

(SEE BACK OF NEXT PAGE)

①A. Prove that $\nexists$ 3-chain in sequence $\Rightarrow a_3 < a_1$. We will prove the contrapositive: $a_3 > a_1 \Rightarrow \exists$ a 3-chain in sequence. To prove this implication we assume $a_1 < a_2$ (given) & $a_3 > a_1$. By casework we show a 3-chain always exists under these assumptions.

CASE 1: $a_3 > a_2 > a_1 \Rightarrow \exists$ a 3-chain in sequence

CASE 2: $a_2 > a_3 > a_1 \Rightarrow \exists$ a 3-chain in sequence since when $a_4 > a_3$, (a_1, a_3, a_4) is a 3-chain and when $a_4 < a_3$, (a_2, a_3, a_4) forms a 3-chain.

Thus, the contrapositive is proven. □

B. We will prove that if $\nexists$ a 3-chain sequence and $a_1 < a_2 \Rightarrow a_3 < a_4 < a_2$ by contradiction:

- $a_4 < a_3 < a_2$ violates $\nexists$ a 3-chain sequence (i.e., (a_2, a_3, a_4))
- $a_3 < a_2 < a_4$ violates $\nexists$ a 3-chain sequence (i.e., (a_1, a_2, a_4))
- Other permutations where $a_2 < a_3$ violate result from above: $\nexists$ a 3-chain in sequence $\Rightarrow a_3 < a_1 < a_2$.

Assuming $\neg(a_3 < a_4 < a_2)$ leads to contradiction in all cases so it must be true that $a_3 < a_4 < a_2$ while $a_1 < a_2$ & $\nexists$ a 3-chain in the sequence. □

C. We will show by casework that $(a_1 < a_2) \wedge (a_3 < a_4 < a_2) \Rightarrow \forall a_5$ $\exists$ 3-chain in sequence:

CASE 1: $a_1 < a_3 < a_4 < a_2 \Rightarrow \exists$ a 3-chain (a_1, a_3, a_4) in sequence irrespective of choice of a_5

CASE 2: $a_3 < a_1 < a_4 < a_2 \Rightarrow \exists$ a 3-chain in sequence because when $a_5 > a_4$, $a_3 < a_4 < a_5$ and when $a_5 < a_4$, $a_5 < a_4 < a_2$.

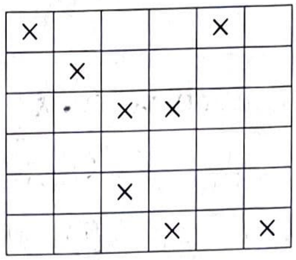
CASE 3: $a_3 < a_4 < a_1 < a_2 \Rightarrow \exists$ a 3-chain in sequence because when $a_5 > a_4$, $a_3 < a_4 < a_5$ and when $a_5 < a_4$, $a_5 < a_4 < a_2$. □

D. We will prove by contradiction that $\forall a_1, a_2, a_3, a_4, a_5.\ a_1 \neq a_2 \neq a_3 \neq a_4 \neq a_5 \Rightarrow$ 3-chain exists in sequence. First, assume a sequence length 5 where no 3-chain exists. Two cases exist: CASE 1: $a_1 < a_2 \Rightarrow$ a 3-chain must exist (from results of A., B., C. above (a contradiction))

CASE 2: $a_1 > a_2 \Rightarrow$ a 3-chain must exist (from results above but reversing signs (a contradiction)).

Both cases yield the same contradiction, therefore the original assumption is proved false. □

(X = infected)

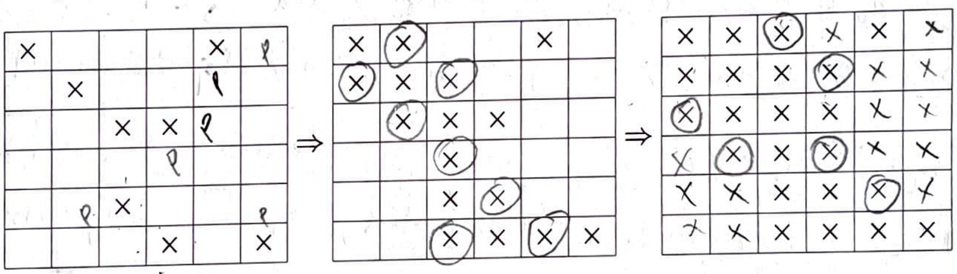


Now the infection begins to spread every minute (in discrete time-steps). Two students are considered *adjacent* if they share an edge (i.e., front, back, left or right, but NOT diagonal); thus, each student is adjacent to 2, 3 or 4 others. A student is infected in the next time step if either

- the student was previously infected (since beaver flu lasts forever), or
- the student is adjacent to *at least two* already-infected students.

In the example, the infection spreads as shown below.

this example will eventually be filled



In this example, over the next few time-steps, all the students in class become infected.

Theorem. *If fewer than n students in class are initially infected, the whole class will never be completely infected.*

Prove this theorem.

Hint: When one wants to understand how a system such as the above "evolves" over time, it is usually a good strategy to (1) identify an appropriate property of the system at the initial stage, and (2) prove, by induction on the number of time-steps, that the property is preserved at every time-step. So look for a property (of the set of infected students) that remains invariant as time proceeds.

"perimeter"

If you are stuck, ask your recitation instructor for the one-word clue and even more hints!

Problem 4. [10 points] Find the flaw in the following *bogus* proof that $a^n = 1$ for all nonnegative integers n, whenever a is a nonzero real number.

Proof. The *bogus* proof is by induction on n, with hypothesis

$$P(n) ::= \forall k \leq n.\, a^k = 1,$$

(5) **Proof**: We will give a proof by induction that $G_n = 3^n - 2^n, \forall n \in \mathbb{N}$ where $n \geq 2$ and $G_n := 5G_{n-1} - 6G_{n-2}$ and $G_0 = 0, G_1 = 1$.

BASE CASE: $n = 2$. $G_2 = 5G_1 - 6G_0 \stackrel{?}{=} 3^2 - 2^2$

(n=0,1 cases were given!)

$5 \stackrel{\checkmark}{=} 5$.

INDUCTIVE STEP: ($\forall n \in \mathbb{N}; n \geq 2$) Assume for induction that $5G_{n-1} - 6G_{n-2} = 3^n - 2^n$. Then we show $5G_n - 6G_{n-1} = 3^{(n+1)} + 2^{(n+1)} \iff$

$$5(3^n) - 5(2^n) - 2(3^n) + 3(2^n) = 3^n(3) - (2)(2^n) \stackrel{\checkmark}{=} 3^{n+1} - 2^{n+1}$$

(8) **Proof**: We will give a proof by induction that the total # of infected students, $T < n^2$ when m_0 (the initial number of infected) $< n$. To do that, notice that the total count of boundaries (b) when $T = n^2$ is $4n \iff b < 4n \Rightarrow T < n^2$. (A "boundary" is an edge of grid enclosing a contiguous group of infected students; a "contiguous group" are all infected students that can be 4-way reached. We will show $\forall t: b_t < 4n$ by induction on time steps t:

$\boxed{\text{BASE CASE}}$ $b_0 < 4n$ because $b_0 \leq 4(m_0)$ and $m_0 < n$.

$\boxed{\text{INDUCTIVE CASE}}$ $b_t < 4n \Rightarrow b_{t+1} < 4n$ is true by LEMMA 1 & 2.

Lemma 1: $b_{t+1} - b_t > 0$ only when new students are infected from $t \to t+1$

Lemma 2: When a new student is infected, b_t can only stay the same or decrease.

Proof (Lemma 2): Consider cases when a student gets infected: from $t \to t+1$

$\boxed{\text{CASE 1}}$ Spanned on 2 sides $\Rightarrow$ boundary count for spanning & newly infected student stays same.

$\boxed{\text{CASE 2}}$ Spanned on 3 sides $\Rightarrow$ " " reduced by 2

$\boxed{\text{CASE 3}}$ Spanned on 4 sides $\Rightarrow$ " " reduced by 4.

Hence, $\forall t$ $b_t < 4n \Rightarrow T < n^2$. $\square$

NEEDED HINTS.

(6) A. **Proof**: By casework, no move changes relative ordering.

$\boxed{\text{CASE 1}}$ Tile can only move to right or left. The tile when moved effectively swiped places with the space, not another tile, so ordering is maintained

$\boxed{\text{CASE 2}}$ Similar to above: regardless if moving left or right, the tile changes places with no other tile and therefore ordering is preserved.

The flaw in the proof by induction is that $P(n) \not\Rightarrow P(n+1)$ for $n = 0$: $a^1 = \frac{1 \cdot 1}{a^{-1}} = a$, and a does not necessarily equal 1!

where k is a nonnegative integer valued variable.

Base Case: $P(0)$ is equivalent to $a^0 = 1$, which is true by definition of a^0. (By convention, this holds even if $a = 0$.)

Inductive Step: By induction hypothesis, $a^k = 1$ for all $k \in \mathbb{N}$ such that $k \leq n$. But then

$$a^{n+1} = \frac{a^n \cdot a^n}{a^{n-1}} = \frac{1 \cdot 1}{1} = 1,$$

which implies that $P(n+1)$ holds. It follows by induction that $P(n)$ holds for all $n \in \mathbb{N}$, and in particular, $a^n = 1$ holds for all $n \in \mathbb{N}$. □

Problem 5. [10 points] Let the sequence $G_0, G_1, G_2, \ldots$ be defined recursively as follows: $G_0 = 0$, $G_1 = 1$, and $G_n = 5G_{n-1} - 6G_{n-2}$, for every $n \in \mathbb{N}, n \geq 2$.

Prove that for all $n \in \mathbb{N}$, $G_n = 3^n - 2^n$. (SEE BACK OF PREV. PAGE)

Problem 6. [20 points]

In the 15-puzzle, there are 15 lettered tiles and a blank square arranged in a 4×4 grid. Any lettered tile adjacent to the blank square can be slid into the blank. For example, a sequence of two moves is illustrated below:

A	B	C	D
E	F	G	H
I	J	K	L
M	O	N	

→

A	B	C	D
E	F	G	H
I	J	K	L
M	O		**N**

→

A	B	C	D
E	F	G	H
I	J		L
M	O	**K**	N

In the leftmost configuration shown above, the O and N tiles are out of order. Using only legal moves, is it possible to swap the N and the O, while leaving all the other tiles in their original position and the blank in the bottom right corner? In this problem, you will prove the answer is "no".

Theorem. *No sequence of moves transforms the board below on the left into the board below on the right.*

$i: 0$
P_r: even
P_b: even

A	B	C	D
E	F	G	H
I	J	K	L
M	O	N	

A	B	C	D
E	F	G	H
I	J	K	L
M	N	O	

$i: 1$
P_r: odd
P_b: odd

(a) [2 pts] We define the "order" of the tiles in a board to be the sequence of tiles on the board reading from the top row to the bottom row and from left to right within a row. For example, in the right board depicted in the above theorem, the order of the tiles is A, B, C, D, E, etc.

Can a row move change the order of the tiles? Prove your answer. (SEE BACK OF PREV. PAGE)

⑥ B. A column move changes the relative positions for 3 pairs of tiles: for tile at order s, $\{(s, s \pm k) \mid 1 \le k \le 3\}$ for $k \in \mathbb{N}$. order pairs switch: (+ for ↓ move, − for ↑ move) in a given state where tile with order s can make a column move.

Proof: Let i be index position for tile L_i. Iff L_i can make a column move, the open space it can move to is at index $i \pm 4$. All tiles L_k for $k > i+4$ (for ↓ move; $k > i$ for ↑ move) preserve relative ordering, as do tiles L_m for $m < i$ (for ↓ move, $m < i-4$ for ↑ move). Hence, the only tiles affected are L_n where $i-4 < n < i$ (for ↑ move; $i < n < i+4$ for ↓ move) ⇒ 3 pairs irrespective of ↑ or ↓ move.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

⑥ C. A row move has no effect on the # of inversions because a row move does not change relative ordering of any of the tiles. (by ⑥A.)

⑥ D. A column move switches the parity of # of inversions (i.e. from even to odd, v.v.)

Proof: By cases, we show the affect of 1 column move on parity (p) from time step $t \to t+1$ (i.e., $P_t \to P_{t+1} = \neg(P_t)$). By ⑥B., we know relative ordering of 3 pairs changes for any column move. This means the # of inversions contributing to total # of inversions by these 3 pairs (i_t) proceeds to i_{t+1} to TABLE 1. In all cases, the overall # of inversions at P_{t_p} is increased or decreased by an odd number of inversions.

TABLE 1.

i_t	i_{t+1}	Δ
3	0	−3
2	1	−1
1	2	1
0	3	3

$\boxed{\text{CASE 1}}$ P_t is even ⇒ P_{t+1} is odd

$\boxed{\text{CASE 2}}$ P_t is odd ⇒ P_{t+1} is even. □.

⑥ E. Proof: Let $P(t) \equiv p_b^{(t)} \neq p_p^{(t)}$, $\forall t.\ t \in \mathbb{N}$ where p_b & p_p are the parity of of the row containing blank square & the parity of # of inversions in puzzle. By induction we prove $P(t)\ \forall t\ t \in \mathbb{N}$:

☑ Base case: $P(0) \Leftrightarrow$ even ≠ odd

☑ Inductive case: We only need to consider when the next move $(t+1)$ is a column move since row moves change neither p_b nor p_p (from ⑥c.). Assume $P(t)$. We establish $P(t) \Rightarrow P(t+1)$ if we show $P(t+1)$. By casework we can show $P(t+1)$: when $p_b^{(t)}$ is even, $p_p^{(t)}$ is odd and a column move $(t+1)$ makes $p_b^{(t+1)}$ odd, $p_p^{(t+1)}$ even. Similar reasoning for $p_b^{(t)}$ odd, $p_p^{(t)}$ even holds. With base cases, $P(t)\ \forall t \in \mathbb{N}$ is established. □

(b) [2 pts] How many pairs of tiles will have their relative order changed by a column move? More formally, for how many pairs of letters L_1 and L_2 will L_1 appear earlier in the order of the tiles than L_2 before the column move and later in the order after the column move? Prove your answer correct. (SEE BACK OF PREV. PAGE)

(c) [2 pts] We define an *inversion* to be a pair of letters L_1 and L_2 for which L_1 precedes L_2 in the alphabet, but L_1 appears after L_2 in the order of the tiles. For example, consider the following configuration:

A	B	C	E
D	H	G	F
I	J	K	L
M	N	O	

(SEE BACK OF PREV PAGE)

There are exactly four inversions in the above configuration: E and D, H and G, H and F, and G and F.

What effect does a row move have on the parity of the number of inversions? Prove your answer.

(d) [4 pts] What effect does a column move have on the parity of the number of inversions? Prove your answer.

(e) [8 pts] The previous problem part implies that we must make an *odd* number of column moves in order to exchange just one pair of tiles (N and O, say). But this is problematic, because each column move also knocks the blank square up or down one row. So after an *odd* number of column moves, the blank can not possibly be back in the last row, where it belongs! Now we can bundle up all these observations and state an *invariant*, a property of the puzzle that never changes, no matter how you slide the tiles around.

Lemma. *In every configuration reachable from the position shown below, the parity of the number of inversions is different from the parity of the row containing the blank square.*

puzzle parity NEVER* equal to empty square's row parity → INVARIANT

row 1	A	B	C	D
row 2	E	F	G	H
row 3	I	J	K	L
row 4	M	O	N	

(SEE BACK OF PREV. PAGE)

Prove this lemma.

(f) [2 pts] Prove the theorem that we originally set out to prove. (SEE BACK)

Problem 7. **[15 points]** There are two types of creature on planet Char, Z-lings and B-lings. Furthermore, every creature belongs to a particular generation. The creatures in each generation reproduce according to certain rules and then die off. The subsequent generation consists entirely of their offspring.

④ ~~7~~. Proof: If a board is ordered, the parity of the row containing the blank is the same as parity of the puzzle. (This is trivially true because 4 and 0 (parity of puzzle ≡ parity of # of inversions).) Let $p_b^{(t)}, p_p^{(t)}$ be these two values after t moves, resp. and we show $\boxed{1}$ that this assertion's contrapositive & $\boxed{2}$ that $p_b^{(t)} \neq p_p^{(t)}$ $\forall t \in \mathbb{N}$, thus the theorem is proved because $\boxed{1}$ true assertion $\Leftrightarrow$ contrapositive and $\boxed{2}$ from ④E. ∎.

⑦ Proof. Start by making stronger statement $P(t) ::= Z_t \le B_t, \forall t \in \mathbb{N}$ given $Z_0 = 200$, $B_0 = 800$. We will prove by induction (on t) that this stronger statement holds $\Rightarrow \forall t.\ t \in \mathbb{N}\ Z_t \le 2B_t$.

<u>Base case</u>: $P(0)$ is true because $Z_0 \le B_0$, i.e., $200 \le 800$

<u>Inductive case</u>: We will prove $P(t) \Rightarrow P(t+1)$ by case work:

$$Z_t \le B_t \Rightarrow \begin{cases} B_{t+1} = Z_t + 2\left\lfloor \frac{B_t - Z_t}{2} \right\rfloor \\ Z_{t+1} = Z_t + \left\lfloor \frac{B_t - Z_t}{2} \right\rfloor \end{cases} \Rightarrow \frac{Z_{t+1}}{B_{t+1}} \le 1$$ because

$\boxed{\text{CASE 1.1}}$

when $\underbrace{(B_t - Z_t) \bmod 2 = 0}_{\text{includes cases where } Z_t = B_t} \Rightarrow \dfrac{Z_t + \frac{1}{2}(B_t - Z_t)}{Z_t + (B_t - Z_t)} = \begin{cases} 1 & \text{when } B_t = Z_t \\ \vdots & \\ \frac{1}{2} & \text{when } Z_t = 0 \end{cases}$

$\boxed{\text{CASE 1.2}}$

$\underbrace{(B_t - Z_t) \bmod 2 = 1}$ $\Rightarrow \dfrac{Z_t + \frac{1}{2}(B_t - Z_t - 1)}{Z_t + (B_t - Z_t - 1)} = \begin{cases} 1 & \text{when } Z_t = B_t - 1 \\ \vdots & \\ \frac{1}{2} & \text{when } Z_t = 0 \end{cases}$

$\Downarrow$

$Z_t \le B_t - 1$ because only way to have odd $(B_t - Z_t)$ is for $\boxed{1}$ B_t to be odd and Z_t to be even less than B_t ($\Rightarrow B_t \ge 1$) or $\boxed{2}$ B_t to be even and Z_t to be odd less than B_t ($\Rightarrow B_t \ge 2$, since $B_t = 0 \Rightarrow Z_t = 0 \Rightarrow (B_t - Z_t) \bmod 2 \neq 1$.)

Thus, $P(0) \wedge (P(t) \Rightarrow P(t+1)) \Rightarrow P(t), \forall t.\ t \in \mathbb{N}$. ∎

The creatures of Char pair with a mate in order to reproduce. First, as many Z-B pairs as possible are formed. The remaining creatures form Z-Z pairs or B-B pairs, depending on whether there is an excess of Z-lings or of B-lings. If there are an odd number of creatures, then one in the majority species dies without reproducing. The number and type of offspring is determined by the types of the parents

- If both parents are Z-lings, then they have three Z-ling offspring.
- If both parents are B-lings, then they have two B-ling offspring and one Z-ling offspring.
- If there is one parent of each type, then they have one offspring of each type.

There are 200 Z-lings and 800 B-lings in the first generation. Use induction to prove that the number of Z-lings will always be at most twice the number of B-lings.

Hint: You may want to use a stronger hypothesis for the induction.

② Prove $\sum_{i=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$, $\forall n \in \mathbb{N}$.

Proof: We will prove this theorem by contradiction and use of the Well Ordering Principle. If theorem is FALSE $\Leftrightarrow \exists c \in C$ where c is the smallest counter example in set $C = \{n \in \mathbb{N} \mid 1^3 + 2^3 + \dots + n^3 \neq \left(\frac{n(n+1)}{2}\right)^2\}$.

$c > 0$ since theorem is true for $n=0 \Rightarrow (c-1) \in \mathbb{N}$ where theorem is true. If we can show that the the theorem holds for c, we have a contradiction since c cannot be in C:

$c^3 + \sum_{i=0}^{c-1} i^3 \overset{?}{=} \left(\frac{c(c+1)}{2}\right)^2$. This equality holds (see steps below), so we have our desired contradiction, which means the theorem is proved by the Well Ordering Principle: $c^3 + \sum_{i=0}^{c-1} i^3 \overset{?}{=} \left(\frac{c(c+1)}{2}\right)^2 \Leftrightarrow c^3 + \left(\frac{(c-1)((c-1)+1)}{2}\right)^2 \overset{?}{=} \left(\frac{c^2+c}{2}\right)^2$

$\Leftrightarrow c^4 + 2c^3 + c^2 \overset{\checkmark}{=} c^4 + 2c^3 + c^2$. $\square$