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6.042/18.062J Mathematics for Computer Science
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Problem Set 5

Readings: Section 5.4 to 5.7 and 6.1-6.2.

Problem 1. [20 points] Recall that a tree is a connected acyclic graph. In particular, a single vertex is a tree. We define a *Splitting Binary Tree*, or *SBTree* for short, as either the lone vertex, or a tree with the following properties:

1. exactly one node of degree 2 (called the root).
2. every other node is of degree 3 or 1 (called internal nodes and leaves, respectively).

For the case of one single vertex (see above), that vertex is considered to be a leaf. It is easier to understand the definition visually, so an example is shown in Figure 1. An example of a tree which is not an SBTree is shown in Figure 2.

(a) [10 pts] Show if an SBTree has more than one vertex, then the induced subgraph obtained by removing the unique root consists of two disconnected SBTrees. You may assume that by removing the root you obtain two separate connected componenents, so all you need to prove is that those two components are SBTrees. (SEE BACK)

(b) [10 pts] Prove that two SBTrees with the same number of leaves must also have the same total number of nodes. *Hint: As a conjecture, guess an expression for the total number of nodes in terms of the number of leaves $N(l)$. Then use induction to prove that it holds for all trees with the same l* (SEE BACK)

Problem 2. [20 points]

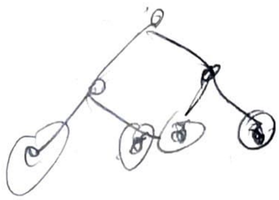
In "Die Hard: The Afterlife", the ghosts of Bruce and Sam have been sent by the evil Simon on another mission to save midtown Manhattan. They have been told that there is a bomb on a street corner that lies in Midtown Manhattan, which Simon defines as extending from 41st Street to 59th Street and from 3rd Avenue to 9th Avenue. Additionally, the code that they need to defuse the bomb is on another street corner. Simon, in a good mood, also tosses them two carrots:

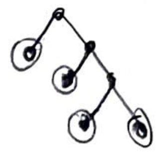
- He will have a helicopter initially lower them to the street corner where the bomb is.

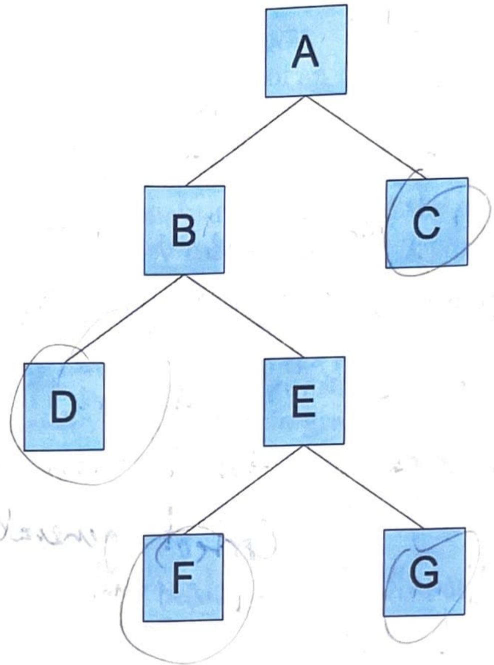
① A. Proof: Removing the root and its incident edges produces two connected components. The nodes incident to the root (but not the root) are SBTrees themselves since in the case they were leaves of the original SBTree, they are each single vertex SBTrees and in the case they were connected to subsequent nodes, after inducing the subgraph per above, they are degree 2 each ("new" roots) and all nodes connected to these "new" roots have degree 1 or 3 and form an acyclic connected graph (from original graph's properties) - this means each connected subcomponent is a SBTree. □

B Proof: Let $N(\ell)$ be the number of nodes N as a function of a SBTree's number of leaves. We will show that $\forall \ell \in \mathbb{N}^*$ $N(\ell) = 2\ell - 1$ by induction on ℓ. The base case is trivially true as a single leaf SBTree $\Leftrightarrow$ single node SBTree, so $N(1) = 2(1) - 1 = 1$. To establish $N(\ell) \Rightarrow N(\ell+1)$, we assume $N(\ell)$ for some SBTree with ℓ leaves, T_ℓ. To add one additional leaf to T_ℓ, additional edges can only be made incident to the ℓ leaves or the root node. Choosing any of the ℓ leaves and connecting the selected leaf to exactly two new nodes is the only acceptable number of new incident edges to add (taking selected leaf of T_ℓ from degree 1 to 3) and yields net +1 leaf and +2 nodes, so $N(\ell+1) = N(\ell) + 2 = 2(\ell+1) - 1 = 2\ell + 1 = (2\ell - 1) + 2$ ✓. The unique root of T_ℓ has degree two, so the only acceptable number of new incident edges to add is 1; connecting T_ℓ via its root to a connected subcomponent consisting of two nodes connected by a single edge also results in +2 nodes, +1 leaf, net. (Edges cannot be added to the other nodes in T_ℓ as they already have max. degree 3 for a SBTree.) With $N(1)$, $N(\ell) \Rightarrow N(\ell+1)$ established, $N(\ell)$ is true for all $\ell \in \mathbb{N}$. □

* Adding another leaf requires adding two nodes:
① "new" root + leaf
② add 2 leaves to existing leaf







4 leaves
7 nodes

2 leaves
3 nodes

6 leaves
11 nodes

$\forall \ell$. number of leaves given by
? $N(\ell) = 2\ell - 1$

Figure 1: Splitting Binary Tree: Node A is the root, B and E are internal nodes, and C, D, F, and G are leaves. Notice how all internal nodes have degree 3.

5 leaves
9 nodes

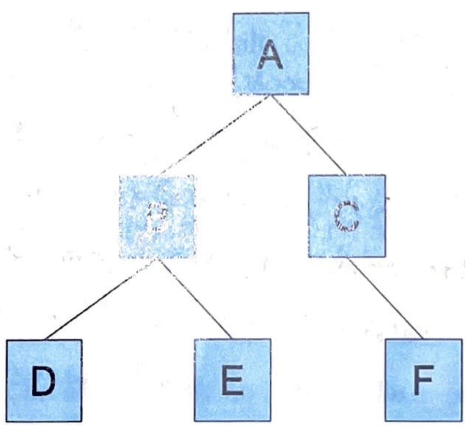


Figure 2: This is an example of a tree which is NOT a Splitting Binary Tree. Notice how both A and C have degree 2, when a BSTree can only have one such node.

② A) If N & M are both odd, then there are an odd number of vertices in the N×M grid. Suppose (for sake of contradiction) that the N×M grid is Hamiltonian. Then a Hamiltonian cycle can be found on the grid. This cycle would have an odd number of vertices. We can also show that any N×M undirected connected grid is bipartite by alternating assignment of each node (in a row; same holds column-wise) to subsets L(G) & R(G) such that there does not exist an edge e that connects two nodes in the same subset. We know a bipartite graph is 2-colorable. But a cycle with an odd number of vertices is **not** 2-colorable. Since a Hamiltonian cycle contains **all** vertices of a graph, we have reached a contradiction, thus no Hamiltonian cycle is found on an N×M grid for odd N, M. ☐

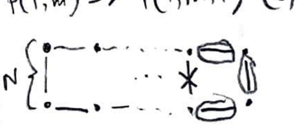
Correct general idea/reasoning.

② B) We will show that a Hamiltonian cycle exists on every N×M grid such that $N = 2n$ and $M = m+1$ for all $n, m \in \mathbb{Z}^+$. (The similar argument applies by symmetry when M is even and $N > 1$.) Let $P(n,m) :=$ $(2n) \times (m+1)$ grid is Hamiltonian. Proceeding by induction, to show $\forall n,m \in \mathbb{Z}^+.\ P(n,m)$, we must show that (1) $P(1,1)$ is true, (2) $P(1,m) \Rightarrow P(1,m+1)$ is true, and (3) $P(n,m) \Rightarrow P(n+1,m)$ is true. (1) is true since a Hamiltonian cycle is the subgraph that is the 2×2 graph itself. (2) is true since (assuming arbitrary m), a Hamiltonian cycle can be found by "connecting" the Hamiltonian cycle found in the $2\times(m+1)$ grid (assumed) to the two additional vertices in the grid that has dimensions $2\times((m+1)+1)$, per Figure (2). Lastly, assume $P(n,m)$ for arbitrary n, m. From (1) + (2), we know the additional component required to form a $2(n+1)\times(m+1)$ grid contains a Hamiltonian cycle and $P(n,m)$ means the subgraph of this $2(n+1)\times(m+1)$ graph that has dimensions $2n\times(m+1)$ contains a Hamiltonian cycle. These two components can be "connected" in similar fashion seen in Figure (3) to form a Hamiltonian cycle on the $2(n+1)\times(m+1)$ grid; thus $P(n,m) \Rightarrow P(n+1,m)$

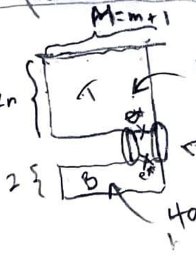
P(1,1) is true:

(1)

$P(1,m) \Rightarrow P(1,m+1)$ (2)



Add ⊖ edges, remove ✱ edges to form Hamiltonian cycle on $2\times((m+1)+1)$ grid



Hamiltonian subgraph from P(n,m) assumption

Hamiltonian subgraph from (1), (2)

This is the right idea/approach but need to prove that this join can happen!

The two cycles can be joined wherever $\exists e^*_A \in E_A$ that can be removed along with e^*_B to preserve Hamiltonian cycle property of visiting each vertex once!

- He promises that the code is placed only on a corner of a numbered street and a numbered avenue, so they don't have to search Broadway.

The map of midtown Manhattan is an example of an $N \times M$ (undirected) grid. In particular, midtown Manhattan is a 19×7 grid.

Bruce and Sam need to check all $19 \cdot 7 = 133$ street corners for the code. Once they are at a corner, they don't need any additional time to verify if the code is there. Once they find the code and return to the bomb, they can disarm it in 2 minutes (even, or especially, as the timer ticks down to 0). Also, they can run one block (in any of the four directions) in exactly 1 minute. They are given 135 minutes total in which to find the code and disarm the bomb, which means that they need to return to the bomb, code in hand, in 133 minutes.

Sam realizes that the map of NYC is actually a graph, and that they need to use a cool new 6.042 concept: A *Hamiltonian cycle* is a path that visits each vertex in a graph exactly once and ends at its starting point (so it is a cycle). A graph is *Hamiltonian* if it has a Hamiltonian cycle.

Hamiltonian graphs are really useful because you can visit each node and return to the starting point by taking only n steps, where n is the number of nodes – if a graph is not Hamiltonian, you would need at least $n+1$ steps to visit each of the n nodes and return to the starting point.

In general, we don't know how to efficiently determine whether a general graph is Hamiltonian or not. However, Sam is very excited because he thinks that he can show that Midtown Manhattan is Hamiltonian. If it is, Bruce and Sam can save the day! Will they make it?

(a) [10 pts] Show that they cannot do it – that is, more generally, show that if both N and M are odd, then the $N \times M$ grid is *not* Hamiltonian. *Hint: First show that any $N \times M$ 2-dimensional undirected grid is bipartite.*

(b) [10 pts] Suppose Simon defined Midtown in the more standard way as extending from 40th Street to 59th Street and from 3rd Avenue to 9th Avenue (that is suppose Midtown Manhattan was a 20×7 grid), and gave them another 7 minutes, = 142 minutes

140 nodes

1. Show that if either N is even and $M > 1$ or M is even and $N > 1$, then the $N \times M$ grid is Hamiltonian.
2. Explain why your proof breaks down when N and M are odd.
3. Would they survive? Does it depend on where the bomb is placed?

The base case immediately fails for $N=3, M=3$, so induction fails.

Yes. No, because 142 minutes $\geq$ (# edges traversed) $\times$ 1 $\frac{\text{edge}}{\text{min}}$ + 2 minutes.

(This is maximally 140 when every node can be visited.)

Problem 3. [20 points]

An n-node graph is said to be *tangled* if there is an edge leaving every set of $\lceil \frac{n}{3} \rceil$ or fewer vertices. As a special case, the graph consisting of a single node is considered tangled. (Recall that the notation $\lceil x \rceil$ refers to the smallest integer greater than or equal to x.)

3 ~~(2)~~. Proof. Suppose for sake of contradiction that there exists a mangled graph that is not connected. This graph has at least two connected components, and at least one connected component has at most $\lceil \frac{n}{2} \rceil$ vertices. But this means an edge is leaving this connected component of $\lceil \frac{n}{2} \rceil$ or fewer vertices. This contradicts the definition of a connected component, so the claim is established. □

(a) [7 pts] Find the error in the proof of the following claim.

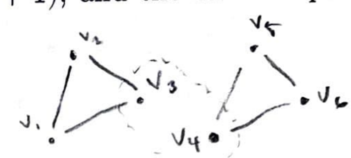
Claim. Every non-empty, tangled graph is connected.

Proof. The proof is by strong induction on the number of vertices in the graph. Let $P(n)$ be the proposition that if an n-node graph is tangled, then it is connected. In the base case, $P(1)$ is true because the graph consisting of a single node is defined to be tangled and is trivially connected.

In the inductive step, for $n \geq 1$ assume $P(1), \ldots, P(n)$ to prove $P(n+1)$. That is, we want to prove that if an $(n+1)$-node graph is tangled, then it is connected. Let G be a tangled, $(n+1)$-node graph. Arbitrarily partition G into two pieces so that the first piece contains exactly $\lceil \frac{n}{3} \rceil$ vertices, and the second piece contains all remaining vertices. Note that since $n \geq 1$, the graph G has a least two vertices, and so both pieces contain at least one vertex. By induction, each of these two pieces is connected. Since the graph G is tangled, there is an edge leaving the first piece, joining it to the second piece. Therefore, the entire graph is connected. This shows that $P(1), \ldots, P(n)$ imply $P(n+1)$, and the claim is proved by strong induction. □

not necessarily! see counterexample in $\boxed{B}$.

(b) [5 pts] Draw a tangled graph that is not connected.



(c) [8 pts] An n-node graph is said to be *mangled* if there is an edge leaving every set of $\lceil \frac{n}{2} \rceil$ or fewer vertices. Again, as a special case, the graph consisting of a single node is considered mangled. Prove the following claim. *Hint: Prove by contradiction.*

(SEE BACK OF PRIOR PAGE)

Claim. Every non-empty, mangled graph is connected.

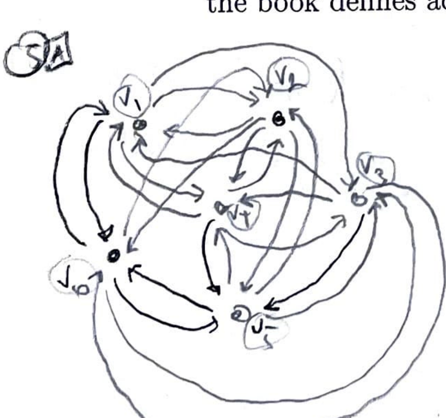
Problem 4. [15 points] (SEE BACK)

(a) [5 pts] Suppose that G is a simple, connected graph on n nodes. Show that G has exactly $n-1$ edges *iff* G is a tree.

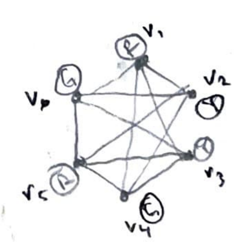
(b) [10 pts] Prove by induction that any connected graph has a spanning tree.

Problem 5. [15 points] The adjacency matrix of a graph is given below (Section 5.1.6 in the book defines adjacency matrices)

5A



$$\begin{array}{c|c} v_i & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$



④A **Proof**: To establish connected graph G has exactly $n-1$ edges $\iff$ G is a tree, we check both implications in the biconditional:

1. ✓ G is a tree $\Rightarrow$ G has exactly $n-1$ edges: Since G is connected it has at least $n-1$ edges, so we need to show that G has no more than $n-1$ edges. Suppose (for purposes of contradiction) that $G > n-1$ edges. Since G is connected, any additional edge between two vertices in G would create a cycle. But G is a tree and trees are acyclic, therefore G has exactly $n-1$ edges. ■

2. ✓ G has exactly $n-1$ edges $\Rightarrow$ G is a tree: To show G is a tree, we need to show G is acyclic (since we already know G is connected). Suppose (for purposes of contradiction) that S, a subgraph of G, contained a cycle (so, by definition, G was cyclical). S would have k nodes and at least k edges since a cycle has the same number of edges v nodes. By induction, adding back each vertex in G not in S would add back at least 1 edge (invariant: # of edges $\geq$ # of nodes) because G is connected. However, this contradicts the antecedent of our claim that G has exactly $n-1$ edges, thus G is acyclic and therefore a tree. ■ □

④B ✓ **Proof** (by induction.) Let $P(n) :=$ connected graph G with n vertices has a spanning tree. The $\boxed{\text{BASE CASE}}$ $n=1$ is trivially true. Because all $(n+1)$-node connected graphs can be built up by adding a vertex to a n-node connected subgraph, we may assume $P(n)$ to establish $P(n+1)$, the $\boxed{\text{INDUCTIVE CASE}}$. Take an arbitrary n-node connected graph G_n. An additional vertex u may be connected to G_n to form connected graph G_{n+1}. G_{n+1}'s spanning tree consists of G_n's spanning tree $T_n = (V_{T,n}, E_{T,n})$ along with new node u and any one of the edges incident to u, i.e., $T_{n+1} = (V_{T,n} \cup u, E_{T,n} \cup e)$ where e is an edge incident to T_n and u. This construction shows $P(n+1)$ is true, thus $P(n)$ is true for all $n \geq 1$. □

(SEE PRIOR PAGE)

(a) [4 pts] Draw the graph defined by this adjacency matrix. Label the vertices of your graph 1, 2, . . . , 6 so that vertex i corresponds to row and column i of the matrix.

(b) [4 pts] In a graph, we define the *distance* between to vertices to be the length of the shortest path between them. We define the *diameter* of a graph to be the largest distance between any two nodes. What is the diameter of this graph? Explain why. (SEE BACK)

(c) [3 pts] Find a cycle in this graph of maximum length and explain why it has maximum length. $v_1 \to v_2 \to v_4 \to v_5 \to v_3 \to v_6 \to v_1$, length 6 is max length because $\exists$ a Hamiltonian cycle on this graph.

(d) [4 pts] Give a coloring of the vertices that uses the minimum number of colors. Prove that this a minimum coloring. 3 colors can color this graph; see back for proof.

Problem 6. [10 points] Let G be a graph. In this problem we show every vertex of odd degree is connected to at least one other vertex of odd degree in G.

longest "trail"

(a) [6 pts] Let v be an odd degree node. Consider the longest walk starting at v that does not repeat any edges (though it may omit some). Let w be the final node of that walk. Show that $w \neq v$. (SEE BELOW.)

(b) [4 pts] Show that w must also have odd degree.

6 a. Proof. For the sake of contradiction suppose $w = v$. Then, the longest walk not repeating edges is a closed walk. A closed walk to/from v consists of $2k$ edges incident on v (one "going", one "coming") for k times the walk returns to v. Suppose the longest non-edge-repeating closed walk returns to v k^* times. But degree of $v \geq 2k^* + 1$ since v has odd degree. This means a longer walk not repeating edges exists — by including at least one edge from v not in the k^* closed walk — a contradiction to the k^* closed walk being the longest walk from v not repeating an edge. Thus, $w \neq v$. ∎

b Proof. Let $2 \mid \deg(w)$ for contradiction's sake. All edges incident to w must be in the longest walk not repeating any edges (otherwise a longer such walk would exist ending at another node not w). Since all edges incident to w are in the desired walk and w has even degree, the only way for a non-edge-repeating walk to end at w is to start at w, i.e., $w = v$. From above, $w \neq v$, and a contradiction is reached; w must have odd degree. ∎

④ R. The diameter of this graph is $\boxed{2}$ as any pair of nodes connected by at least 1 walk (in both directions), i.e., A^2 has no entries that are 0. Since 2 is the smallest integer such that all entries are positive, the entries where $A^2 - A = A^2$ are paths between the corresponding vertices.

⑤ R. Proof: We can show that this graph is 3-colorable by construction (see pg. 4), so we must show that it isn't two-colorable. (It's not one colorable since the graph is not empty.). The graph is not 2-colorable because each vertex is connected to 4 other vertices, and at least one pair of those other vertices is connected.