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1A The equivalence classes are the n sets partitioning $\mathbb{Z}$ s.t each set consists of congruent integers modulo n.

1D. The equivalence classes are the connected components of G.

Problem Set 6

skipped due to time, but understood

Problem 1. **[20 points]** [15] For each of the following, either prove that it is an equivalence relation and state its equivalence classes, or give an example of why it is not an equivalence relation.

1B e.g. JSON R Danny $\wedge$ Danny R JSON is always false.

1C e.g. $(2 R 3) \wedge (3 R 4)$ does not imply $(2 R 4)$

(a) [5 pts] $R_n := \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \text{ s.t. } x \equiv y \pmod n\}$ R_n is an equivalence relation because congruence modulo n is reflexive, symmetric and transitive.

(b) [5 pts] $R :=$ ~~$\{(x, y) \in P \times P$ s.t. x is taller than $y\}$~~ where P is the set of all people in the world today. Not an equivalence relation because not symmetric ~~or reflexive~~

(c) [5 pts] $R := \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \text{ s.t. } gcd(x, y) = 1\}$ Not an equivalence relation because not transitive

(d) [5 pts] $R_G :=$ the set of $(x, y) \in V \times V$ such that V is the set of vertices of a graph G, and there is a path $x, v_1, \ldots, v_k, y$ from x to y along the edges of G. R_G is an equivalence relation since it is reflexive, symmetric, and transitive

Problem 2. **[20 points]** Every function has some subset of these properties:

injective surjective bijective

Determine the properties of the functions below, and briefly explain your reasoning.

(a) [5 pts] The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x \sin(x)$. surjective all elements in codomain are mapped to by f, but f is not injective

(b) [5 pts] The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 99x^{99}$. bijective = injective + surjective

(c) [5 pts] The function $f : \mathbb{R} \to \mathbb{R}$ defined by $\tan^{-1}(x)$. injective not surjective since range of f is $(-\frac{\pi}{2}, \frac{\pi}{2})$

(d) [5 pts] The function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) =$ the number of numbers that divide x. For example, $f(6) = 4$ because $1, 2, 3, 6$ all divide 6. *Note: We define here the set* $\mathbb{N}$ *to be the set of all positive integers* $(1, 2, \ldots)$. surjective

Problem 3. **[20 points]** In this problem we study partial orders (posets). Recall that a weak partial order $\preceq$ on a set X is reflexive ($x \preceq x$), anti-symmetric ($x \preceq y \wedge y \preceq x \to x = y$), and transitive ($x \preceq y \wedge y \preceq z \to x \preceq z$). Note that it may be the case that neither $x \preceq y$ nor $y \preceq x$. A chain is a list of *distinct* elements $x_1, \ldots, x_i$ in X for which $x_1 \preceq x_2 \preceq \cdots \preceq x_i$. An antichain is a subset S of X such that for all distinct $x, y \in S$, neither $x \preceq y$ nor $y \preceq x$.

③[A]. To show that relation $\precsim$ on set $\{1, 2, 3, \ldots, (n-1)(m-1)+1\}$ is a weak partial order such that $i \precsim j \Leftrightarrow i \leq j$ and $a_i \leq a_j$ for a given sequence of integers $(a_1, a_2, a_3, \ldots, a_{(n-1)(m-1)+1})$, we must show $\precsim$ is reflexive, antisymmetric, and transitive.

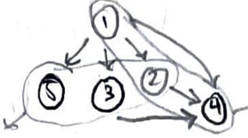
$\boxed{\text{REFLEXIVITY}}$ Relation $\precsim$ is reflexive iff $i \leq i$, which is true for all $i \in \{1, 2, 3, \ldots, (n-1)(m-1)+1\}$, and $a_i \leq a_i$, which is true for all integers a_i in the given sequence.

$\boxed{\text{ANTISYMMETRY}}$ Relation $\precsim$ is anti-symmetric iff for all distinct $i, j \in \{1, 2, 3, \ldots, (n-1)(m-1)+1\}$, $i \leq j \Rightarrow j \nleq i$ and $a_i \leq a_j \Rightarrow a_j \nleq a_i$. This is true since two distinct integers x, y are either $x < y$ ~~xor~~ $x > y$. If the former is true, the two implications are true. If the latter is true, the two implications are (vacuously) true.

$\boxed{\text{TRANSITIVITY}}$ $\forall i, j, k \in \{1, 2, 3, \ldots, (n-1)(m-1)+1\}.\ (i \precsim j) \wedge (j \precsim k) \Rightarrow i \precsim k$ is true by transitivity of $\leq$. The same argument holds for $\forall a_i, a_j, a_k \in \{a_1, a_2, a_3, \ldots a_{(n-1)(m-1)+1}\}.\ (a_i \leq a_j) \wedge (a_j \leq a_k) \Rightarrow a_i \leq a_k$. □

③[B] We can use Dilworth's Lemma to show that any sequence of $(n-1)(m-1)+1$ integers must contain a non decreasing subsequence of size n or a decreasing sequence of size m by constructing a DAG D with $|V(D)| = (n-1)(m-1)+1$ vertices that are the given integers and edges defined by the relation $\vec{E}$ on the vertices such that $v_i \vec{E} v_j$ iff $a_i \leq a_j$ and $\nexists\, v_k \in V(D)$ such that $v_i \precsim v_k \precsim v_j$ (i.e., D is the Hasse diagram of this poset.) By Dilworth's Lemma, any DAG D with $(n-1)(m-1)+1$ vertices must have a chain of size greater than $n-1$ or an antichain of size at least $\frac{(n-1)(m-1)+1}{(n-1)}$ for all $n-1 > 0$. But chain & antichain in this context correspond to non-decreasing and decreasing subsequences, respectively, so we have proven the original statement. (Note that for case $n=1$, the degenerate sequence of one integer is a non-decreasing sequence of size 1, so the statement still holds.) Yes, but should've justified why $N-1$, which is because we prove something of the form $P \Rightarrow Q \vee R$ by showing $P \wedge \neg Q \Rightarrow R$.

using $N-1$ is negation of "... chain of greater than $N-1$" (as the bound) e.g. antichain (longest)



e.g. chain (longest)

nm − n − m + 2
"
number of vertices

m − 1

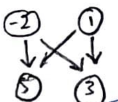
The aim of this problem is to show that any sequence of $(n-1)(m-1)+1$ integers either contains a non-decreasing subsequence of length n or a decreasing subsequence of length m. Note that the given sequence may be out of order, so, for instance, it may have the form $1, 5, 3, 2, 4$ if $n = m = 3$. In this case the longest non-decreasing and longest decreasing subsequences have length 3 (for instance, consider $1, 2, 4$ and $5, 3, 2$).

(a) [7 pts] Label the given sequence of $(n-1)(m-1)+1$ integers $a_1, a_2, \ldots, a_{(n-1)(m-1)+1}$. Show the following relation $\preceq$ on $\{1, 2, 3, \ldots, (n-1)(m-1)+1\}$ is a weak poset: $i \preceq j$ if and only if $i \leq j$ and $a_i \leq a_j$ (as integers). (SEE BACK OF PREVIOUS PAGE.)

For the next part, we will need to use Dilworth's theorem, as covered in lecture. Recall that Dilworth's theorem states that if $(X, \preceq)$ is any poset whose longest chain has length n, then X can be partitioned into n disjoint antichains.

(b) [7 pts] Show that in any sequence of $(n-1)(m-1)+1$ integers, either there is a non-decreasing subsequence of length n or a decreasing subsequence of length m. (SEE BACK OF PREVIOUS PAGE)

e.g.
1, −2, 5, 3



(c) [6 pts] Construct a sequence of $(n-1)(m-1)$ integers, for arbitrary n and m, that has no non-decreasing subsequence of length n and no decreasing subsequence of length m. Thus in general, the result you obtained in the previous part is best-possible. (SEE BACK)

14/20

Problem 4. [20 points] Louis Reasoner figures that, wonderful as the Beneš network may be, the butterfly network has a few advantages, namely: fewer switches, smaller diameter, and an easy way to route packets through it. So Louis designs an N-input/output network he modestly calls a *Reasoner-net* with the aim of combining the best features of both the butterfly and Beneš nets:

> The ith input switch in a Reasoner-net connects to two switches, a_i and b_i, and likewise, the jth output switch has two switches, y_j and z_j, connected to it. Then the Reasoner-net has an N-input Beneš network connected using the a_i switches as input switches and the y_j switches as its output switches. The Reasoner-net also has an N-input butterfly net connected using the b_i switches as inputs and the z_j switches as outputs.

In the Reasoner-net the minimum latency routing does not have minimum congestion. The *latency for min-congestion* (LMC) of a net is the best bound on latency achievable using routings that minimize congestion. Likewise, the *congestion for min-latency* (CML) is the best bound on congestion achievable using routings that minimize latency.

Fill in the following chart for the Reasoner-net and briefly explain your answers. same as ~~diameter~~ b/c Beneš network minimizes congestion at expense of latency (which is greater than butterfly's latency)

should be from smaller diameter butterfly net!

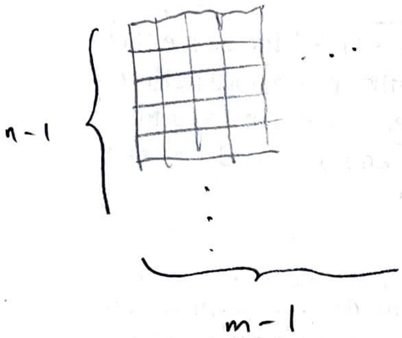
diameter	switch size(s)	# switches	congestion	LMC	CML
2 + ~~2~~ log N + 1	1×2, 2×2, 2×1	2N + N(log N + 1) + 2N log N	1		√N or √(N/2)

from I/O terminals to I/O switches

from "thicker" Beneš network

from butterfly network

③ 1c) We can construct such a sequence by populating an array w/ dimensions $(n-1)(m-1)$ per the figure below. Then populate the



$n-1$ rows with the same decreasing sequence of $m-1$ integers. The sequence is then read/ordered L→R per row starting at the top row and proceeding down.

e.g. $n=2$, $m=3$

16	13	10
16	13	10

$\Leftrightarrow (16, 13, 10, 16, 13, 10)$

$n = \log N$

Problem 5. [20 points] Let B_n denote the butterfly network with $N = 2^n$ inputs and N outputs, as defined in Notes 6.3.8. We will show that the congestion of B_n is exactly $\sqrt{N}$ when n is even.

Hints:

FACT 1. • For the butterfly network, there is a unique path from each input to each output, so the congestion is the maximum number of messages passing through a vertex for any matching of inputs to outputs.

FACT 2. ✓• If v is a vertex at level i of the butterfly network, there is a path from exactly 2^i input vertices to v and a path from v to exactly 2^{n-i} output vertices.

FACT 3. • At which level of the butterfly network must the congestion be worst? What is the congestion at the node whose binary representation is all 0s at that level of the network? level $\frac{n}{2}$

(a) [10 pts] Show that the congestion of B_n is at most $\sqrt{N}$ when n is even.

(b) [10 pts] Show that the congestion achieves $\sqrt{N}$ somewhere in the network and conclude that the congestion of B_n is exactly $\sqrt{N}$ when n is even.

$\boxed{A}$. From $\boxed{\text{FACT 1}}$, congestion is the maximum number of messages passing through a vertex for a routing. "Passing through a vertex" is the minimum of packets passed in and the packets passed out of a vertex. By $\boxed{\text{FACT 2}}$, congestion for the network B_n is maximized at level $i = \frac{n}{2}$ if a routing exists such that it contains the $2^{\frac{n}{2}}$ paths through a vertex at level $\frac{n}{2}$. This notional congestion would be $2^{\frac{n}{2}} = (2^n)^{\frac{1}{2}} = \sqrt{N}$. □.

$\boxed{B}$. We must show that such a routing exists that $\sqrt{N}$ congestion is achieved. WLOG, we can construct such a routing on butterfly network $B_{\log N}$ (for even power N). To do this let routing problem so vertex $((0,\dots,0), \frac{n}{2})$ has congestion $\sqrt{N}$. To do this let routing problem π be defined as permutation such that $\pi(0) = 0$, $\pi(\sqrt{N}) = 1$, $\pi(2\sqrt{N}) = 2, \dots, \pi((\sqrt{N}-1)\sqrt{N}) = \sqrt{N}-1$ with inputs that are not multiples of $\sqrt{N}$ permuted arbitrarily vis-a-vis remaining outputs. Then, by $\boxed{\text{FACT 3}}$, the vertex $(0,\dots,0,\frac{n}{2})$ will have congestion $\sqrt{N}$. since the routing, P_π, that optimally solves π has $\sqrt{N}$ packets passing thru vertex $(0,\dots,0,\frac{n}{2})$ from $\boxed{\text{FACT 1}}$. Thus, with $\boxed{A}$, congestion of $B_n = \sqrt{N}$ for even n.