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Problem Set 8

Problem 1. [25 points] Find Θ bounds for the following divide-and-conquer recurrences. Assume $T(1) = 1$ in all cases. Show your work.

(SEE NEXT 5 PAGES)

(a) [5 pts] $T(n) = 8T(\lfloor n/2 \rfloor) + n$

(b) [5 pts] $T(n) = 2T(\lfloor n/8 \rfloor + 1/n) + n$

(c) [5 pts] $T(n) = 7T(\lfloor n/20 \rfloor) + 2T(\lfloor n/8 \rfloor) + n$

(d) [5 pts] $T(n) = 2T(\lfloor n/4 \rfloor + 1) + n^{1/2}$

(e) [5 pts] $T(n) = 3T(\lfloor n/9 + n^{1/9} \rfloor) + 1$

Problem 2. [30 points] It is easy to misuse induction when working with asymptotic notation.

False Claim If

$T(n) = O(n) := \lim_{n \to \infty} \left| \frac{T(n)}{n} \right| < \infty$

$$T(1) = 1 \text{ and}$$
$$T(n) = 4T(n/2) + n$$

Then T(n) = O(n).

False Proof We show this by induction. Let $P(n)$ be the proposition that $T(n) = O(n)$.

Base Case: $P(1)$ is true because $T(1) = 1 = O(1)$.

Inductive Case: For $n \geq 1$, assume that $P(n-1), \ldots, P(1)$ are true. We then have that

$$T(n) = 4T(n/2) + n = 4O(n/2) + n = O(n)$$

And we are done.

(a) [5 pts] Identify the flaw in the above proof.

O(n) is a binary relation on two functions, one of which is f(n) = n, whereas the n in P(n) is for a particular integer.

1 $\boxed{A}$ $T(n) = 8T(\lfloor n/2 \rfloor) + n$

$$= 8\left(8T\left(\left\lfloor \frac{n}{4} \right\rfloor\right) + \frac{n}{2}\right) + n$$

$$= 8^2\, T\left(\left\lfloor \frac{n}{4} \right\rfloor\right) + \left(\frac{8}{2}\right)n + n = 64\, T\left(\left\lfloor \frac{n}{4} \right\rfloor\right) + 5n$$

$$= 8^2\left(8T\left(\left\lfloor \frac{n}{8} \right\rfloor\right) + \frac{n}{4}\right) + 5n$$

$$= 8^3\, T\left(\left\lfloor \frac{n}{8} \right\rfloor\right) + \left(\frac{8^2}{2^2}\right)n + 5n$$

$$= 512\, T\left(\left\lfloor \frac{n}{8} \right\rfloor\right) + 21n$$

$$= 8^k\, T\left(\left\lfloor \frac{n}{2^k} \right\rfloor\right) + n \cdot \sum_{i=0}^{k-1} 4^i$$

$$= 8^k\, T\left(\left\lfloor \frac{n}{2^k} \right\rfloor\right) + \left(\frac{1-4^k}{1-4}\right)n$$

$$= 2^{3k}\, T\left(\left\lfloor \frac{n}{2^k} \right\rfloor\right) + \left(\frac{1-2^{2k}}{-3}\right)n$$

Initial conditions of $T(1) = 1$ and $k = \log n$ give:

$$= n^3 \cancel{T(1)} - \frac{n}{3}\left(1 - n^2\right) = \frac{4}{3}n^3 - \frac{n}{3}$$

$$\Rightarrow \Theta(n^3)$$

(1) (B) $T(n) = 2T(\lfloor n/8 \rfloor + 1/n) + n$ Using Akra-Bazzi.

According to Master Method, $\Theta(n)$

First, $p = \frac{1}{3}$ such that $\sum_{i=0}^{k} a_i b_i^p$ where $a_i = 2$, $b_i = \frac{1}{8}$.

(Note that $|\frac{1}{n}| = O(\frac{n}{\log^2 n})$ and that $n = O(n^c)$ for $c \geq 1$.) Next, we evaluate:

$$\Theta\left(n^{\frac{1}{3}} + n^{\frac{1}{3}} \int_1^n \frac{u}{u^{1/3+1}} \, du\right) = \Theta\left(n^{\frac{1}{3}} + n^{\frac{1}{3}} \int_1^n u^{-\frac{1}{3}} \, du\right)$$

$$= \Theta\left(n^{\frac{1}{3}} + n^{\frac{1}{3}}\left(\left. \frac{3}{2} u^{\frac{2}{3}} \right|_1^n\right)\right) = \Theta\left(n^{\frac{1}{3}} + n^{\frac{1}{3}}\left(\frac{3}{2} n^{\frac{2}{3}} - \frac{3}{2}\right)\right)$$

$$= \Theta\left(n^{\frac{1}{3}} + \frac{3}{2} n - \frac{3}{2} n^{\frac{1}{3}}\right) = \Theta(n)$$

1) c) $T(n) = 7T(\lfloor n/20 \rfloor) + 2T(\lfloor n/8 \rfloor) + n$ Using Akra-Bazzi.

(All conditions met to use Akra-Bazzi.) First, find suitable p such that $\sum_{i=0}^{2} a_i b_i^p = 1 = 7\cdot\left(\frac{1}{20}\right)^p + 2\left(\frac{1}{8}\right)^p$.

But notice that actually finding p is needless:

$$\Theta\left(n^p\left(1 + \int_1^n \frac{u}{u^{1+p}}\, du\right)\right) =$$

$$\Theta\left(n^p + n^p \int_1^n u^{-p}\, du\right) =$$

$$\Theta\left(n^p + n^p\left(\frac{1}{1-p} u^{1-p} \Big|_1^n\right)\right) =$$

$$\Theta\left(n^p + n^p\left(\frac{1}{1-p} n^{1-p} - 1\right)\right) =$$

$$\Theta\left(\cancel{n^p} + \frac{n}{1-p} - \cancel{n^p}\right) = \Theta(n)$$

① D $$T(n) = 2T(\lfloor n/4 \rfloor + 1) + n^{1/2}$$

Using Akra-Bazzi.

According to the Master Method, $O(n^{1/2} \log n)$ ✓

Conditions to use Akra-Bazzi are met since $1 = O\left(\frac{n}{\log^2 n}\right)$ and $\left|\frac{d}{dn}(n^{1/2})\right| = O(n^c)$ for $c \geq 1 \in \mathbb{N}$. $\sum_{i=1}^{1} 2\left(\frac{1}{4}\right)^p = 1$ gives $p = \frac{1}{2}$. Evaluating

$\Theta\left(n^{1/2} + n^{1/2}\int_1^n \frac{u^{1/2}}{u^{3/2}}\, du\right) =$

$\Theta\left(n^{1/2} + n^{1/2}\int_1^n u^{-1}\, du\right) = \Theta\left(n^{\frac{1}{2}} + n^{\frac{1}{2}}(\ln n)\right) =$

$\Theta\left(n^{\frac{1}{2}} \log n\right)$

① E $T(n) = 3T(\lfloor n/9 + n^{1/9} \rfloor) + 1$

Using Akra-Bazzi!

$$3 \cdot \left(\frac{1}{9}\right)^p = 1 \implies p = \frac{1}{2}$$

Evaluate:

$$\Theta\left(n^{\frac{1}{2}} + n^{\frac{1}{2}} \int_1^n u^{-\frac{3}{2}}\, du\right) =$$

$$\Theta\left(n^{\frac{1}{2}} + n^{\frac{1}{2}} \left(-2u^{-\frac{1}{2}} \Big|_1^n\right)\right) = \Theta\left(n^{\frac{1}{2}} + n^{\frac{1}{2}}\left(-2n^{-\frac{1}{2}} + 2\right)\right)$$

$$= \Theta\left(n^{\frac{1}{2}} - 2 + 2n^{1/2}\right) = \Theta\left(n^{1/2}\right)$$

Checking Akra-Bazzi condition:

Does $n^{\frac{1}{9}} = O\left(\frac{n}{\log^2 n}\right)$?

$$\lim_{n \to \infty} \left| \frac{n^{\frac{1}{9}} \log^2 n}{n} \right| \overset{?}{<} \infty$$

$$\lim_{n \to \infty} \left| \frac{\log^2 n}{n^{8/9}} \right| \overset{?}{<} \infty$$

$$\lim_{n \to \infty} \left| \frac{\log n}{n^{4/9}} \right|^2 \overset{?}{<} \infty$$

$$\Updownarrow$$

$$\lim_{n \to \infty} \left| \frac{\log n}{n^{4/9}} \right| \overset{?}{<} \infty \quad \checkmark$$

Note that $\frac{d}{dn}(\log n) < \frac{d}{dn}(n^{4/9})$, so $\left(\frac{\log n}{n^{4/9}}\right) \to 0$ as $n \to \infty$.

By the Master Method:

$$\left.\begin{aligned} a &= 3 \\ b &= 9 \\ d &= 0 \end{aligned}\right\} \quad \begin{aligned} &a > b^d \\ &(3 > 9^0) \end{aligned} \implies \Theta\left(n^{\log_9 3}\right) = \Theta\left(n^{1/2}\right) \checkmark$$

(b) [10 pts] A simple attempt to prove $T(n) \neq O(n)$ via induction ultimately fails. We assume for sake of contradiction that $T(n) = O(n)$. Then there exists positive integer n_0 and positive real number c such that for all $n \geq n_0$, $T(n) \leq cn$. We then define $P(n)$ as the proposition that $T(n) \leq cn$.

(SEE NEXT PAGE)

We then proceed with strong induction.

Base Case, $n = n_0$: $P(n_0)$ is true, by assumption.

Inductive Step: We assume $P(n_0)$, $P(n_0 + 1)$, ..., $P(n - 1)$ true.

Fill in the rest of this proof attempt, and explain why it doesn't work.

Note: As this problem was updated so late, the graders will be instructed to be exceedingly lenient when grading this.

(c) [5 pts] Using Akra-Bazzi theorem, find the correct asymptotic behavior of this recurrence.

(d) [10 pts] We have now seen several recurrences of the form $T(n) = aT(\lfloor n/b \rfloor) + n$. Some of them give a runtime that is $O(n)$, and some don't. Find the relationship between a and b that yields $T(n) = O(n)$, and prove that this is sufficient. (SEE PAGE AFTER NEXT)

Problem 3. [15 points] Define the sequence of numbers A_i by

$A_0 = 2$

$A_{n+1} = A_n/2 + 1/A_n$ (for $n \geq 1$)

This must be a typo otherwise A_1 is never defined!

Prove that $A_n \leq \sqrt{2} + 1/2^n$ for all $n \geq 0$.

Problem 4. [30 points] Find closed-form solutions to the following linear recurrences.

(a) [15 pts] $x_n = 4x_{n-1} - x_{n-2} - 6x_{n-3}$ $\quad (x_0 = 3, x_1 = 4, x_2 = 14)$

(SEE LAST SIX PAGES)

(b) [15 pts] $x_n = -x_{n-1} + 2x_{n-2} + n$ $\quad (x_0 = 5, x_1 = -4/9)$

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(2) B We must show $T(n) \leq cn$ given $P(n_0), P(n_0+1), \ldots, P(n-1)$; $T(n) = 4T(\frac{n}{2}) + n \leq cn$ for some $c \in \mathbb{R}^+$. In the case $n \geq 2n_0 \iff \frac{n}{2} \geq n_0 \Rightarrow T(n) \leq 4(\frac{cn}{2}) + n = 2cn + n$ (because $T(\frac{n}{2}) \leq \frac{cn}{2}$ from induction hypothesis). $2cn + n \not\leq cn$, so the induction does not establish the [false] assumption $T(n) = O(n)$, i.e., this assumption is never proved, so a contradiction is never reached.

(2) C Using Akra-Bazzi, we first find p such that $\sum_{i=1}^{1} a_i b_i^p = 1 \iff 4 \cdot (\frac{1}{2})^p = 1 \iff p = 2$. Then evaluating $\Theta(n^2 + n^2 \int_1^n \frac{u}{u^3} du) = \Theta(n^2 + n^2(-u^{-1}\Big|_1^n)) = \Theta(n + n(-\frac{1}{n} - (-1))) = \Theta(n^2 - n^1 + n^2) = \Theta(n^2)$,

also confirmed by Master Method: $\Theta(n^{\log_2 4}) = \Theta(n^2)$

② $\boxed{P}$ Recurrence of the form $aT(\lfloor \frac{n}{b} \rfloor) + n$ is $O(n)$ when $a < b$. (Also given by Master Method)

(AKRA-BAZZI)

$a \cdot b^p = 1$, $p < 1$ when $a < b$.

$$\Theta\left(n^p + n^p \int_1^n \frac{u}{u^{1+p}}\, du\right) \approx \Theta\left(n + n\int_1^n u\, du\right)$$

(if $p < 1$)

$$= \Theta\left(n^p + n^p\left(\frac{1}{1-p} u^{1-p} \Big|_1^n\right)\right) =$$

$$\Theta\left(n^p + n^p\left(\frac{1}{1-p} n^{1-p} - 1\right)\right) =$$

$$\Theta\left(\cancel{n^p} + \frac{n}{1-p} - \cancel{n^p}\right) = \Theta(n)$$

Problem 3. [15 points] Define the sequence of numbers A_i by

$A_0 = 2$

$A_{n+1} = A_n/2 + 1/A_n$ (for $n \geq 1$) $\longrightarrow$ gives $A_2, A_3, A_4, \ldots$

Prove that $A_n \leq \sqrt{2} + 1/2^n$ for all $n \geq 0$.

Proof. (by induction) Let $P(n) := A_n \leq \sqrt{2} + \frac{1}{2^n}$ for all $n \geq n_0$, where $n_0 = 0$.

$\boxed{\text{BASE CASE}}$ $P(0)$ is true because $2 \leq \sqrt{2} + 1$.

$\boxed{\text{INDUCTIVE CASE}}$

? Notice that $\forall x \in \mathbb{R}^+$, $\frac{x}{2} + \frac{1}{x} \geq \sqrt{2}$.

So if we can show $A_n > 0$ for all $n \geq 0$, then (along with induction hypothesis):

$$A_{n+1} = \frac{A_n}{2} + \frac{1}{A_n}$$

$A_n \leq \sqrt{2} + \frac{1}{2^n}$

$A_n \geq \sqrt{2}$ from Lemma

$$\leq \frac{\left(\sqrt{2} + \frac{1}{2^n}\right)}{2} + \frac{1}{\sqrt{2}}$$

(biggest this term could be)

$$A_{n+1} \leq \sqrt{2} + \frac{1}{2^{n+1}}$$

4 (a) [15 pts] $x_n = 4x_{n-1} - x_{n-2} - 6x_{n-3} \quad (x_0 = 3, x_1 = 4, x_2 = 14)$

$$f_n = 4(f_{n-1}) - 1(f_{n-2}) - 6(f_{n-3})$$

Let $F(x)$ be generating function for this sequence: $F(x) := f_0 + f_1 x + f_2 x^2 + \ldots$

NEED to GET RID OF x^3 and higher order terms...

$$\begin{array}{r|l}
F(x) & f_0 + f_1 x + f_2 x^{2} + f_3 x^3 + f_4 x^4 \ldots \\
+6x^3(F(x)) & \qquad\qquad\qquad\qquad 6f_0 x^3 + 6f_1 x^4 \ldots \\
+x^2(F(x)) & \qquad\qquad\quad f_0 x^2 + f_1 x^3 + f_2 x^4 \ldots \\
-4x(F(x)) & \quad -4f_0 x - 4f_1 x^2 - 4f_2 x^3 - 4f_3{}^4 \ldots \\
\hline
 & f_0 + (f_1 - 4f_0)x + (f_2 + f_0 - 4f_1)x^2 + ☺ x \ldots
\end{array}$$

$$F(x)(6x^3 + x^2 - 4x) = 3 + (4 - 12)x + (14 + 3 - 16)x^2$$

$$= \frac{3 - 8x + x^2}{6x^3 + x^2 - 4x}$$

Don't really understand where to go from here with this generating function :(

REVERTING TO PRESCRIBED "GUESS-AND-CHECK"...

Guessing the form: $f(n) = x^n$ for recurrence $f(n) = 4f(n-1) - f(n-2) - 6f(n-3)$

gives $x^n = 4x^{n-1} - x^{n-2} - 6x^{n-3}$

$$x^3 - 4x^2 + x + 6 = 0$$

$$(x-3)(x-2)(x+1) = 0$$

which has roots $3, 2, -1$ and the linear combo thereof gives solution:

$$f(n) = r(3^n) + s(2^n) + t(-1^n)$$

Solving $\begin{cases} 3 = r + s + t \\ 4 = 3r + 2s - t \\ 14 = 9r + 4s + t \end{cases}$ gives closed-form of $f(n)$

$$f(n) = (3^n) + (2^n) + (-1)^n$$

(SEE NEXT PAGE FOR GAUSS-JORDAN ELIMINATION)

Solving $\begin{cases} 3 = r + s + t \\ 4 = 3r + 2s - t \\ 14 = 9r + 4s + t \end{cases}$ gives closed-form of linear-recurrence $f(n)$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 3 & 2 & -1 & 4 \\ 9 & 4 & 1 & 14 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & -4 & -5 \\ 9 & 4 & 1 & 14 \end{array}\right] \Rightarrow \quad \cdots$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & -1 & -4 & -5 \\ 9 & 4 & 1 & 14 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & -1 & -4 & -5 \\ 0 & 4 & 28 & 32 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 12 & 12 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & -4 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

4 **(b)** [15 pts] $x_n = -x_{n-1} + 2x_{n-2} + n \quad (x_0 = 5, x_1 = -4/9)$

First solve for homogeneous recurrence:

$f(n) = -f(n-1) + 2f(n-2)$

The characteristic equation for the homogeneous recurrence is:

$$x^n = -x^{n-1} + 2x^{n-2}$$

$$\Updownarrow$$

$$x^2 = -x + 2$$

$$\Updownarrow$$

$$x^2 + x - 2, \iff (x-1)(x+2)$$

Two roots:

$1, -2$

$\Uparrow$

Then solve inhomogeneous part:

TRY POLYNOMIAL OF DEGREE 1:

$$an+b = -(a(n-1)+b) + 2(a(n-2)+b) + n$$

$$\cancel{an+b} = \cancel{-an} + a \cancel{-b} + 2\cancel{an} - 4a + \cancel{2b} + n$$

$$0 = a - 4a + n$$

$$0 = -3a + n \leftarrow$$ No solution $\forall n$.

TRY POLYNOMIAL OF DEGREE 2:

$$\cancel{an^2} + \cancel{bn} + \cancel{c} = -(a(n-1)^2 + b(n-1) + c)$$

$$+ 2(a(n-2)^2 + b(n-2) + c) + n$$

$$= -(a(n^2-2n+1) + bn - b + c)$$

$$+ 2(a(n^2-4n+4) + bn - 2b + c) + n$$

$$= \cancel{-an^2} + 2an - a \cancel{-bn} + b \cancel{-c}$$

$$\cancel{2an^2} - 8an + 8a \cancel{+2bn} - 4b + \cancel{2c} + n$$

$$0 = -6an + 7a - 3b + n$$

$$0 = n(-6a+1) + 7a - 3b$$

$$a = \frac{1}{6},\ b = \frac{7}{18} \quad \forall n.$$

ADD HOMOGENEOUS & PARTICULAR SOLUTIONS:

$$f(n) = s(1)^n + t(-2)^n + \frac{1}{6}n^2 + \frac{7}{18}n$$

USE BOUNDARY CONDITIONS:

$$f(0) = 5 = s + t$$

$$f(1) = -\frac{4}{9} = s - 2t + \frac{5}{9}$$

solving $\begin{cases} 5 = s + t \\ -1 = s - 2t \end{cases}$ gives closed-form of linear recurrence $f(n)$

Using elimination:

$$6 = 3t$$

$$t = 2, \quad s = 3, \quad \text{so}$$

$$f(n) = 3 + 2(-2)^n + \frac{1}{6}n^2 + \frac{7}{18}n$$