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6.042/18.062J Mathematics for Computer Science
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Problem Set 1

Problem 1. [24 points] (SEE BACK)

Translate the following sentences from English to predicate logic. The domain that you are working over is X, the set of people. You may use the functions $S(x)$, meaning that "x has been a student of 6.042," $A(x)$, meaning that "x has gotten an 'A' in 6.042," $T(x)$, meaning that "x is a TA of 6.042," and $E(x, y)$, meaning that "x and y are the same person."

(a) [6 pts] There are people who have taken 6.042 and have gotten A's in 6.042

(b) [6 pts] All people who are 6.042 TA's and have taken 6.042 got A's in 6.042

(c) [6 pts] There are no people who are 6.042 TA's who did not get A's in 6.042.

(d) [6 pts] There are at least three people who are TA's in 6.042 and have not taken 6.042

Problem 2. [24 points] (SEE BACK)

Use a truth table to prove or disprove the following statements:

(a) [12 pts]

$$\neg(P \vee (Q \wedge R)) = (\neg P) \wedge (\neg Q \vee \neg R)$$

(b) [12 pts]

$$\neg(P \wedge (Q \vee R)) = \neg P \vee (\neg Q \vee \neg R)$$

Problem 3. [24 points]

The binary logical connectives $\wedge$ (*and*), $\vee$ (*or*), and $\Rightarrow$ (*implies*) appear often in not only computer programs, but also everyday speech. In computer chip designs, however, it is considerably easier to construct these out of another operation, nand, which is simpler to represent in a circuit. Here is the truth table for nand:

P	Q	P nand Q
true	true	false
true	false	true
false	true	true
false	false	true

1) A. $\exists x, y \in X.\ S(x) \wedge S(y) \wedge A(x) \wedge A(y) \wedge \neg E(x,y)$.

B. $\forall x \in X.\ S(x) \wedge T(x) \Rightarrow A(x)$.

C. $\forall x \in X.\ T(x) \Rightarrow A(x) \iff \neg \exists x \in X.\ T(x) \wedge \neg A(x)$.

D. $\exists x, y, z \in X.\ T(x) \wedge T(y) \wedge T(z) \wedge \neg S(x) \wedge \neg S(y) \wedge \neg S(z) \wedge \neg E(x,y) \wedge \neg E(y,z) \wedge \neg E(x,z)$.

2) A. (expect to be TRUE by De Morgan's Law)

P	Q	R	$\neg(P \vee (Q \wedge R))$	$\stackrel{?}{=}$	$(\neg P) \wedge (\neg Q \vee \neg R)$
T	T	T	F	✓	F
T	T	F	F	✓	F
T	F	T	F	✓	F
T	F	F	F	✓	F
F	T	T	F	✓	F
F	T	F	T	✓	T
F	F	T	T	✓	T
F	F	F	T	✓	T

The statements are equivalent under all truth assignments $\iff$ The statements are equivalent.

B. (expect to be FALSE by incorrect distribution of negation)

P	Q	R	$\neg(P \wedge (Q \vee R))$	$\stackrel{?}{=}$	$(\neg P) \vee (\neg Q \vee \neg R)$
T	T	T	F	✓	F
T	T	F	F	X	T
T	F	T	F	X	T
T	F	F	T	✓	T
F	T	T	T	✓	T
F	T	F	T	✓	T
F	F	T	T	✓	T
F	F	F	T	✓	T

The statements are NOT equivalent because there exists at least one truth assignment where the statements do not yield the same truth value

4) First, divide coins into two groups of 6 and weigh both groups; throw out all coins in heavier group. Take remaining 6 coins and repeat the process, leaving you with 3 coins. Pick any two and weigh:

1. if one is lighter than the other, you found "fake" coin
2. if both are equal weight, the unweighed coin is "fake"

(a) [12 pts] For each of the following expressions, find an equivalent expression using only **nand** and ¬ (*not*), as well as grouping parentheses to specify the order in which the operations apply. You may use A, B, and the operators any number of times.

(i) $A \wedge B$ $\boxed{\neg (A \text{ nand } B)}$

(ii) $A \vee B \Leftrightarrow \neg[(\neg A) \wedge (\neg B)] \Leftrightarrow \boxed{(\neg A) \text{ nand } (\neg B)}$

(iii) $A \Rightarrow B \Leftrightarrow (\neg A) \vee B \Leftrightarrow \neg[A \wedge (\neg B)] \Leftrightarrow \boxed{(A \text{ nand } (\neg B))}$

(b) [4 pts] It is actually possible to express each of the above using only **nand**, without needing to use ¬. Find an equivalent expression for $(\neg A)$ using only **nand** and grouping parentheses. $\boxed{(A) \text{ nand } (A)} \Leftrightarrow \neg A$

(c) [8 pts] The constants `true` and `false` themselves may be expressed using only **nand**. Construct an expression using an arbitrary statement A and **nand** that evaluates to `true` regardless of whether A is `true` or `false`. Construct a second expression that always evaluates to `false`. Do not use the constants `true` and `false` themselves in your statements. (SEE BELOW)

Problem 4. [10 points] You have 12 coins and a balance scale, one of which is fake. All the real coins weigh the same, but the fake coin weighs less than the rest. All the coins visually appear the same, and the difference in weight is imperceptible to your senses. In at most 3 weighings, give a strategy that detects the fake coin. (Note: the scale in this problem is a scale with two dishes, which tips toward the side that is heavier. For clarification, do an image search for "balance scale"). (SEE BACK OF PREV. PAGE)

Problem 5. [6 points] Prove the following statement by proving its contrapositive: if r is irrational, then $r^{1/5}$ is irrational. (Be sure to *state* the contrapositive explicitly.) (SEE BACK)

Problem 6. [12 points] Suppose that $w^2 + x^2 + y^2 = z^2$, where w, x, y, and z always denote positive integers. (Hint: It may be helpful to represent even integers as $2i$ and odd integers as $2j + 1$, where i and j are integers)

Prove the proposition: z is even if and only if w, x, and y are even. Do this by considering all the cases of w, x, y being odd or even. (SEE BACK)

3 c. $(A) \vee (\neg A)$ always is TRUE. LOGICALLY CORRECT, but complicated. :)

$\Updownarrow$

$(A) \vee (A \text{ nand } A)$ TRUE: $(A \text{ nand } (A \text{ nand } A))$

$\Updownarrow$

$\boxed{(A \text{ nand } A) \text{ nand } [(A \text{ nand } A) \text{ nand } (A \text{ nand } A)]}$ TRUE Negating the derived expression always gives FALSE.

$(A \text{ nand } (A \text{ nand } A)] \text{ nand } (A \text{ nand } (A \text{ nand } A))$

FALSE: $(A \text{ nand } (A \text{ nand } A))$

$\boxed{[(A \text{ nand } A) \text{ nand } ((A \text{ nand } A) \text{ nand } (A \text{ nand } A))] \text{ nand } [\quad " \quad]}$ FALSE

(Alternatively, derive from $(A) \wedge (\neg A)$.)

5 Prove: r is irrational $\Rightarrow r^{\frac{1}{5}}$ is irrational.

Proof: We will prove the contrapositive: $r^{\frac{1}{5}} \in \mathbb{Q} \Rightarrow r \in \mathbb{Q}$.
If $r^{\frac{1}{5}}$ is rational, it can be expressed as p/q where $p, q \in \mathbb{Z}$ sharing no common factor (other than 1). $r = (r^{\frac{1}{5}})^5 = \frac{p^5}{q^5}$, which is rational by definition since any integer raised to a positive integer power is still an integer. With the contrapositive proved, so too is the original implication. □

6 Prove: $2|z \Leftrightarrow (2|w) \wedge (2|x) \wedge (2|y)$

Proof: We will prove this proposition by case work on $\neg(2|w \wedge 2|y \wedge 2|x) \Rightarrow \neg 2|z$.
Either 1 w, y, x are all odd or 2 exactly one of w, y, x are odd shows that z is odd (i.e., not divisible by 2) since 1 yield's a sum of 3 odd positive integers (which is odd) and 2 yields an odd sum, too ("even" + "odd" + "even" = "odd"). (Note that these results are justified by the fact that an even positive integer squared is always even, and an odd positive integer squared is always odd.) We are left with 2 cases: 3 exactly two of w, y, x are odd and 4 all w, y, x are even (i.e., none are odd). Case 3 is invalidated by showing that $2|z \Leftarrow$ exactly one of w, y, x is even yields a contradiction: let z, w, y, x be represented as $2m, 2i, 2j+1, 2k+1$, respectively* where $m, i \in \mathbb{Z}^+$ and $j, k \in \mathbb{N}$ to be consitent with fact that $w, x, y, z \in \mathbb{Z}^+$. (This is the formalization of the assumption we wish to prove false, i.e., $2|z \Leftarrow$ exactly one of x, y, x is even.)

$$z^2 = w^2 + y^2 + x^2 \Leftrightarrow \begin{aligned} (2m^2) &= (2i)^2 + (2j+1)^2 + (2k+1)^2 \\ 4m^2 &= 4(i^2+j^2+k^2) + 4(j+k) + 2 \\ m^2 &= (i^2+j^2+k^2) + j + k + 1/2 \end{aligned}$$

... but m^2 must be a positive integer! Thus, $2|n \not\Leftrightarrow$ exactly one of w, y, z is even, invalidating candidacy for 3. 1, 2, 3 comprise the proof for the contrapositive of $2|z \Rightarrow (2|w) \wedge (2|y) \wedge (2|x)$, so we only have to show $(2|w) \wedge (2|y) \wedge (2|x) \Rightarrow 2|z$ to establish the proof of original biconditional. Let $w = 2i$, $y = 2j$, $x = 2k$ such that $i, j, k \in \mathbb{Z}^+$. Then $z^2 = (2i)^2 + (2j)^2 + (2k)^2 = 4(i^2+j^2+k^2) \Leftrightarrow \left(\frac{z}{2}\right)^2 = i^2+j^2+k^2$. Since $z \in \mathbb{Z}^+$ and $i^2+j^2+k^2 \geq 3$ and $i^2+j^2+k^2 \in \mathbb{Z}^+$, then $2|z$ and implications in both directions are established, proving the biconditional. □

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*we could choose any of w, y, x to be $2i$ WLOG