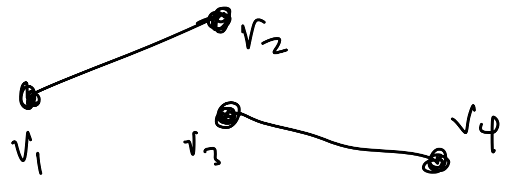
① Consider the graph $G$ consisting of vertices $V_G = \{v_1, v_2, v_3, v_4\}$ and edges $E_G = \{v_1 - v_2, v_3 - v_4\}$. This graph is <u>not</u> connected but has all positive degree vertices.




6.042/18.062J Mathematics for Computer Science — October 6, 2010
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# Problems for Recitation 8

## 1 Build-up error

Recall a graph is ***connected*** iff there is a path between every pair of its vertices.

**False Claim.** *If every vertex in a graph has positive degree, then the graph is connected.*

1. Prove that this Claim is indeed false by providing a counterexample.

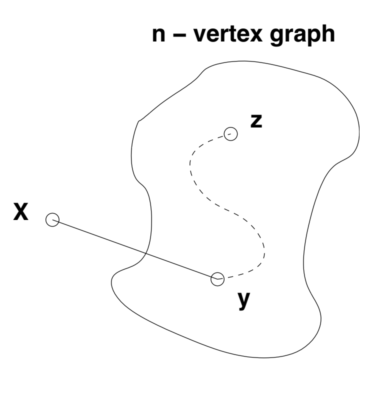
2. Since the Claim is false, there must be a logical mistake in the following bogus proof. Pinpoint the *first* logical mistake (unjustified step) in the proof.

   *Proof.* We prove the Claim above by induction. Let $P(n)$ be the proposition that if every vertex in an $n$-vertex graph has positive degree, then the graph is connected.

   **Base cases**: $(n \leq 2)$. In a graph with 1 vertex, that vertex cannot have positive degree, so $P(1)$ holds vacuously.

   $P(2)$ holds because there is only one graph with two vertices of positive degree, namely, the graph with an edge between the vertices, and this graph is connected.

   **Inductive step**: We must show that $P(n)$ implies $P(n+1)$ for all $n \geq 2$. Consider an $n$-vertex graph in which every vertex has positive degree. By the assumption $P(n)$, this graph is connected; that is, there is a path between every pair of vertices. Now we add one more vertex $x$ to obtain an $(n+1)$-vertex graph:




   All that remains is to check that there is a path from $x$ to every other vertex $z$. Since $x$ has positive degree, there is an edge from $x$ to some other vertex, $y$. Thus, we can

This only proves $P(n+1)$ for $(n+1)$-vertex graphs w/ all positive degree vertices constructed by adding a vertex and edge to a $n$-vertex graph. Not all $(n+1)$-vertex graphs w/ all positive degree vertices can be constructed this way, so it can't be shown all $(n+1)$-vertex graphs are connected.



obtain a path from $x$ to $z$ by going from $x$ to $y$ and then following the path from $y$ to $z$. This proves $P(n+1)$.

By the principle of induction, $P(n)$ is true for all $n \geq 0$, which proves the Claim.

□

## 2 The Grow Algorithm

Yesterday in lecture, we saw the following algorithm for constructing a minimum-weight spanning tree (MST) from an edge-weighted $N$-vertex graph $G$.

ALG-GROW:

1. Label the edges of the graph $e_1, e_2, \ldots, e_t$ so that $wt(e_1) \leq wt(e_2) \ldots \leq wt(e_t)$.
2. Let $S$ be the empty set.
3. For $i = 1 \ldots t$, if $S \cup \{e_i\}$ does not contain a cycle, then extend $S$ with the edge $e_i$.
4. Output $S$.

### 2.1 Analysis of ALG-GROW

In this problem you may assume the following lemma from the problem set:

**Lemma 1.** *Suppose that $T = (V, E)$ is a simple, connected graph. Then $T$ is a tree iff $|E| = |V| - 1$.*

In this exercise you will prove the following theorem.

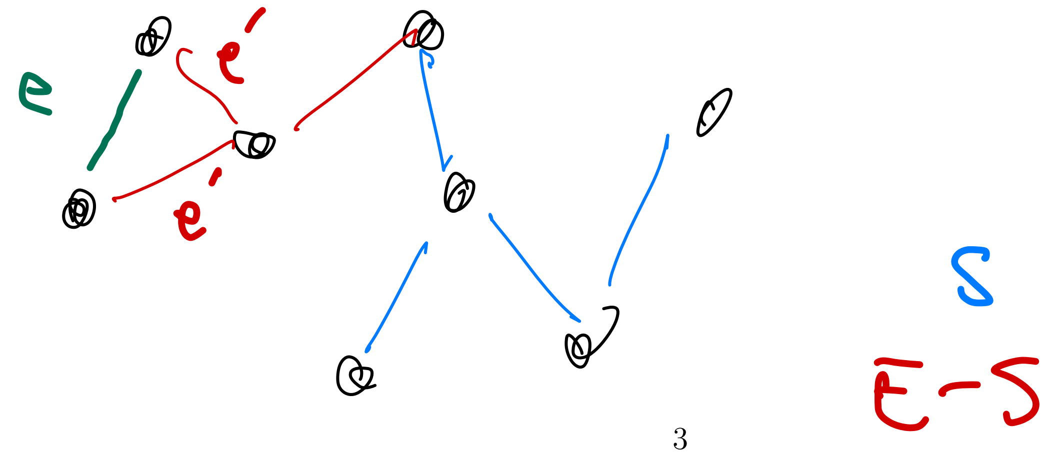
**Theorem.** *For any connected, weighted graph $G$, ALG-GROW produces an MST of $G$.*

(a) Prove the following lemma.

**Lemma 2.** *Let $T = (V, E)$ be a tree and let $e$ be an edge not in $E$. Then, $G = (V, E \cup \{e\})$ contains a cycle.*

(Hint: Suppose $G$ does *not* contain a cycle. Is $G$ a tree?)

(b) Prove the following lemma.






**Lemma 3.** *Let $T = (V, E)$ be a spanning tree of $G$ and let $e$ be an edge not in $E$. Then there exists an edge $e' \neq e$ in $E$ such that $T^* = (V, E - \{e'\} \cup \{e\})$ is a spanning tree of $G$.*

(Hint: Adding $e$ to $E$ introduces a cycle in $(V, E \cup \{e\})$.)

**(c)** Prove the following lemma.

**Lemma 4.** *Let $T = (V, E)$ be a spanning tree of $G$, let $e$ be an edge not in $E$ and let $S \subseteq E$ such that $S \cup \{e\}$ does not contain a cycle. Then there exists an edge $e' \neq e$ in $E - S$ such that $T^* = (V, E - \{e'\} \cup \{e\})$ is a spanning tree of $G$.*

(Hint: Modify your proof to part (b). Of all possible edges $e' \neq e$ that can be removed to construct $T^*$, at least one is not in $S$.)

Read thru & understood solution but did not formulate argument myself :(

**(d)** Prove the following lemma.

**Lemma 5.** *Define $S_m$ to be the set consisting of the first $m$ edges selected by ALG-GROW from a connected graph $G$. Let $P(m)$ be the predicate that if $m \leq |V|$ then $S_m \subseteq E$ for some MST $T = (V, E)$ of $G$. Then $\forall m \,.\, P(m)$.*

(Hint: Use induction. There are two cases: $m + 1 > |V|$ and $m + 1 \leq |V|$. In the second case, there are two subcases.)

**(e)** Prove the theorem. (Hint: Lemma 5 says there exists an MST $T = (V, E)$ for $G$ such that $S \subseteq E$. Use contradiction to rule out the case in which $S$ is a proper subset of $E$.)

- ALG-GROW produces $S$ and at some point, $m$, $S = E$
- Need to show ALG-GROW "keeps going" until $S = E$, i.e., it doesn't terminate such that $S \subset E$.

(6) **(a)** Prove the following lemma.

**Lemma 2.** *Let $T = (V, E)$ be a tree and let $e$ be an edge not in $E$. Then, $G = (V, E \cup \{e\})$ contains a cycle.*

(Hint: Suppose $G$ does *not* contain a cycle. Is $G$ a tree?)

Proof: Keeping vertex set V, the addition of edge e must span two vertices in V. But there existed a path from one of these vertices to the other already (T is a tree; trees are connected graphs), so introduction of e creates a cycle since there no longer exists one, distinct path between these two vertices (another property of any two vertices in a tree).

2 $\boxed{B}$

**Lemma 3.** *Let $T = (V, E)$ be a spanning tree of $G$ and let $e$ be an edge not in $E$. Then there exists an edge $e' \neq e$ in $E$ such that $T^* = (V, E - \{e'\} \cup \{e\})$ is a spanning tree of $G$.*

(Hint: Adding $e$ to $E$ introduces a cycle in $(V, E \cup \{e\})$.)

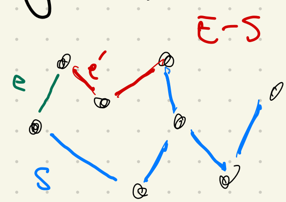
Proof: If $e \notin E$ and all vertices are spanned by $T$, then the two vertices spanned by $e$ are connected by a unique path and adding $e$ to $E$ introduces a cycle in $(V, E \cup \{e\})$. But this means $\exists e'$ on the original path that can be removed from $E$ such that this subgraph of $G$ is $T^*$ (all vertices still spanned and $|E - \{e'\} \cup \{e\}| = |V| - 1$, so $T^*$ is a spanning tree).

2 $\boxed{C}$

**(c)** Prove the following lemma.

**Lemma 4.** *Let $T = (V, E)$ be a spanning tree of $G$, let $e$ be an edge not in $E$ and let $S \subseteq E$ such that $S \cup \{e\}$ does not contain a cycle. Then there exists an edge $e' \neq e$ in $E - S$ such that $T^* = (V, E - \{e'\} \cup \{e\})$ is a spanning tree of $G$.*

(Hint: Modify your proof to part (b). Of all possible edges $e' \neq e$ that can be removed to construct $T^*$, at least one is not in $S$.)




We know from $\boxed{B}$ that $E \cup \{e\}$ creates a cycle and that $S \cup \{e\}$ does not contain a cycle. Thus, $\exists e' \in E - S$ that can be removed from $E$ when $e$ is added such that $T^*$ (from above) is constructed since $E - S$ must contain and edge forming the cycle introduced by $e$.

2 10

(d) Prove the following lemma.

**Lemma 5.** *Define $S_m$ to be the set consisting of the first $m$ edges selected by ALG-GROW from a connected graph $G$. Let $P(m)$ be the predicate that if $m \leq |V|$ then $S_m \subseteq E$ for some MST $T = (V, E)$ of $G$. Then $\forall m \,.\, P(m)$.*

(Hint: Use induction. There are two cases: $m + 1 > |V|$ and $m + 1 \leq |V|$. In the second case, there are two subcases.)

Had to peek at answer!

Proof (by inducting on m). Let P(m) be the inductive hypothesis.

$\boxed{\text{BASE CASE}}$ P(0) is trivially true since $\emptyset$ is a subset of all sets.

$\boxed{\text{INDUCTIVE CASE}}$ We assume P(m) for the purposes of establishing P(m+1), i.e., $P(m) \Rightarrow P(m+1)$.

- For $m \geq |V|$, $P(m) \Rightarrow P(m+1)$ since $P(m+1)$ is vacuously true because $m \geq |V| \Rightarrow m+1 \nleq |V|$.
- For $m < |V|$, there are two cases to consider regarding the $(m+1)^{st}$ edge e:
  - A. $e \in E$ (from inductive hypothesis) $\Rightarrow S_m \cup \{e\} \subseteq E$, so $P(m+1)$ is true.
  - B. $e \notin E$. We can show by LEMMA 4 that $\exists e' \in E - S_m$ such that a new spanning tree $T^* = (V, E - \{e'\} \cup \{e\})$ can be constructed. To show $T^*$ is a minimum weight spanning tree, note that the ALG-GROW algorithm considers edges in increasing order of weights and $e' \in E - S_m$, so weight of $e \leq$ weight of e.

② $\boxed{E}$

**Theorem.** *For any connected, weighted graph $G$, ALG-GROW produces an MST of $G$.*

**(e)** Prove the theorem. (Hint: Lemma 5 says there exists an MST $T = (V, E)$ for $G$ such that $S \subseteq E$. Use contradiction to rule out the case in which $S$ is a proper subset of $E$.)

Proof (by contradiction): Suppose, for sake of contradiction, that ALG-GROW produced $S \neq E$ (i.e., $S \subset E$ where $T = (V, E)$ is some MST of $G$). All spanning trees of $G$ have $|E| = |V| - 1$ by LEMMA 1, so $|S| < |E| < |V| \Rightarrow |S| \leq |V| - 2 \Rightarrow \exists v^* \in V$ that is not spanned by $S$. Since ALG-GROW stopped, $\forall e_i \in E - S$ $\{e_i\} \cup S$ would've created a cycle (otherwise ALG-GROW would continue). But we have a contradiction since $\exists e^* \in E - S$ where $S \cup \{e^*\}$ would span $v^*$ and not create a cycle. Therefore, it cannot be the case $S \neq E \Leftrightarrow S = E$, so the algorithm produces a MST in its entirety over $G$. $\square$