$\aleph_0$ Weekly Problem

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Problem

Prove that if you add up the reciprocals of a sequence of consecutive positive integers, the numerator of the sum (in lowest terms) will always be odd. For example, $\frac{1}{7} + \frac{1}{8} + \frac{1}{9} = \frac{191}{504}$.

Solution

First, we prove the following lemma that will be helpful later.

Lemma. *In any sequence of consecutive positive integers $(a_1, a_2, \ldots, a_n)$, there is a unique integer a_* with the highest power of 2 in its prime factorization.*

Proof. Suppose, for the sake of finding a contradiction, that there exists at least two positive integers in the consecutive positive integer sequence $(a_1, a_2, \ldots, a_n)$ sharing a maximal power of 2 in their prime factorizations.

Without loss of generality, let a_j and a_k (with $j < k$) have the same, highest power of 2 in their prime factorizations (say, 2^m) and all intervening a_l (for all l such that $j < l < k$) having a smaller power of 2 in its prime factorization. Then, $a_j = 2^m d$ such that d is odd (since it is the product of only odd integers) and $a_k = 2^m(d+1)$.

$d+1$ is even and factoring out a 2 from $d+1$ gives $a_k = 2^{m+1}\left(\frac{d+1}{2}\right)$, where $\frac{d+1}{2}$ is an integer. Herein lies a contradiction — it is assumed that a_j and a_k have the same, highest power of 2 among the prime factorizations of all a_i in the sequence.

Reaching a contradiction, we have proven the lemma. $\square$

Next, we turn our attention to proving the original claim stated by the **Problem**.

Claim. *If you add up the reciprocals of a sequence of consecutive positive integers, the numerator of the sum (in lowest terms) will always be odd.*

Proof. Consider any sequence of consecutive positive integers $(a_1, a_2, \ldots, a_n)$. Let $N = \prod_{i=1}^{n} a_i$. Then,

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots \frac{1}{a_n} = \frac{\frac{N}{a_1} + \frac{N}{a_2} + \dots + \frac{N}{a_n}}{N}.$$

Letting $\operatorname{lcm}(1, a_1, a_2, \dots, a_n) = N'$, the numerator, S, in the resultant fraction is:

$$S = \frac{N'}{a_1} + \frac{N'}{a_2} + \dots + \frac{N'}{a_n}.$$

Showing S has an odd numerator is sufficient to prove the claim. By the Fundamental Theorem of Arithmetic, each $\frac{N}{a_i}$ has a unique prime factorization; hence $\frac{N}{a_i} = 2^{m_{i,1}} 3^{m_{i,2}} 5^{m_{i,3}} \cdots$ for all $i \in \{1, 2, \dots, n\}$. What remains to be shown is that there exists exactly one odd summand (of S), $\frac{N'}{a_*}$, implying S is odd.

Consider $\frac{N}{a_*} = 2^{m_{*,1}} 3^{m_{*,2}} 5^{m_{*,3}} \cdots$, where a_* is the member of the sequence with the highest power of 2 in its prime factorization. Note that a_* is unique by the lemma established above. This implies a unique $m_{*,1} = \min\{m_{i,1} \mid i \in \{1, 2, \dots, n\}\}$. Because $2^{m_{*,1}}$ can be factored out of all $\frac{N}{a_i}$, it follows that $\frac{N'}{a_*}$ is the only odd summand of S, giving odd S.

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