$\aleph_0$ Weekly Problem

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Problem

Show that if n has an odd divisor greater than 1, then $2^n + 1$ is not prime.

Solution

Proof. We assume n has an odd divisor greater than 1 and show that $2^n + 1$ is not prime.

By the Fundamental Theorem of Arithmetic, n has a unique prime factorization $n = 2^{k_0}3^{k_1}5^{k_2}\cdots$. Let $P_{\text{odd}} = 3^{k_1}5^{k_2}\cdots > 1$ and $b = 2^{2^{k_0}}$, then $2^n = b^{P_{\text{odd}}}$. Notice that P_{odd} is odd. Therefore, $2^n + 1 = b^{P_{\text{odd}}} + 1 \equiv -1 + 1 \equiv 0 \pmod{b+1}$. Hence, if n has an odd divisor greater than 1, then $2^n + 1$ is not prime.

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