$\aleph_0$ Weekly Problem

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Problem

Each of the numbers 1 to 10^6 is repeatedly replaced by its digital sum until we reach 10^6 one-digit numbers. Will these have more 1's or 2's?

Solution

Repeatedly taking the digital sum of a positive integer k is equivalent to taking k's digital root, which equals k (mod 9) unless $k \equiv 0$ (mod 9), in which case the digital root is 9.

Since $10^6 \equiv 1$ (mod 9) and there are as many integers congruent to 1 as to 2 modulo 9 in the range 1 to $10^6 - 1$ (inclusive), there is one more 1 than 2 in the final sequence of one-digit numbers. Hence, there are more 1's than 2's.