$\aleph_0$ Weekly Problem

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Problem

In a group of nine people, one person knows two of the others, two people each know four others, four each know five others, and the remaining two each know six others. Show that there are three people who all know each other.

Solution

Proof. We will proceed by casework, modeling the problem as an undirected graph on nine vertices (corresponding the individuals) wherein any edge represents the "knows" relation between the two individuals corresponding to the edge's incident vertices. We will show there exists a clique of size three on this undirected graph[1].

Let vertices A_1 and A_2 represent the two individuals who know six others (read: A-vertices). The remaining seven vertices (corresponding to the individuals who know fewer than six others) are labeled $B_1, B_2, \ldots, B_7$ (read: B-vertices).

Case 1 (The edge (A_1, A_2) exists.). *When edge* (A_1, A_2) *exists, there are a total of* 10 *remaining edges that are incident to an A-node and a B-node. By the Pigeonhole Principle, there exists a vertex B_* that must neighbor both A_1 and A_2. A clique of size three exists since there are edges connecting all pairs taken from $\{A_1, A_2, B_*\}$. For example, in Fig. 1, B_* can be B_3, B_4, or B_5.*

[1] The "knows" relation is assumed to be symmetric because the claim we intend to prove does not always hold for a directed graph.

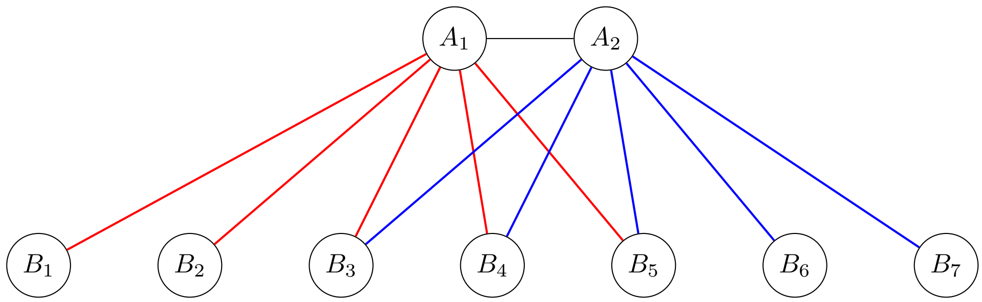

Figure 1: Edge (A_1, A_2) exists.

Case 2 (The edge (A_1, A_2) does not exist.)**.** *There are two sub-cases to consider when edge (A_1, A_2) does not exist.*

Case 2.1 (The edge (A_1, A_2) does not exist and A-nodes share the same set of neighbors.)**.** *Without loss of generality, let vertex B_7 be the node not connected to either of the A-vertices (see Fig. 2). For all $(u, v) \in \{B_1, B_2, \ldots, B_6\}$, if edge (u, v) exists, a clique of size three exists (namely, among either A-vertex, u, and v).*

By the Pigeonhole Principle, there exists $B_* \in \{B_1, B_2, \ldots, B_6\}$ that has degree four, as the problem states two nodes have degree four. Since there exists two edges that are both incident to B_* and neither of which is incident to an A-node, at least one of these edges must connect to a vertex in $\{B_1, B_2, \ldots, B_6\} \setminus \{B_*\}$ (again, by the Pigeonhole Principle). This results in some edge $(u, v) \in \{B_1, B_2, \ldots, B_6\} \setminus \{B_*\} \subset \{B_1, B_2, \ldots, B_6\}$ existing and, therefore, a clique of size three existing.

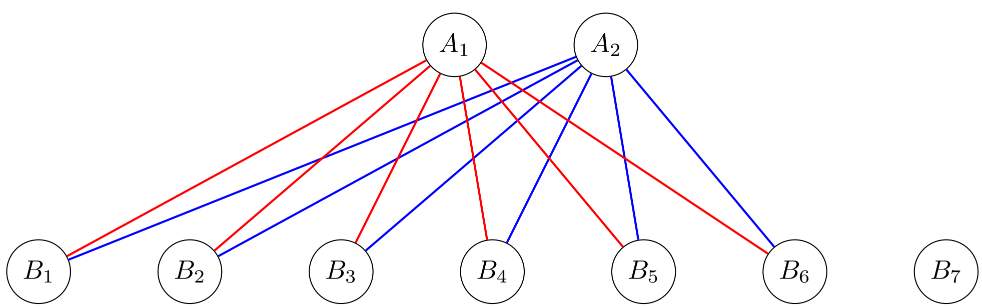

Figure 2: The edge (A_1, A_2) does not exist and A-nodes share the same set of neighbors.

Case 2.2 (The edge (A_1, A_2) does not exist and A-nodes do not share the same set of neighbors.)**.** *Without loss of generality, let node A_1 connect to six B-vertices $\{B_1, B_2, \ldots, B_6\}$. Since A_2 cannot connect to the same set of B-vertices, B_7 must neighbor A_2, as do five vertices in the set of B-vertices adja-*

cent to A_1 due to the Pigeonhole Principle; without loss of generality, let this subset of the neighbors of vertex A_1 be $\{B_2, B_3, \ldots, B_6\}$ (see Fig. 3).

Since two B-vertices must have degree four, at least one must be in $\{B_1, B_2, \ldots, B_6\}$ by the Pigeonhole Principle. But having one vertex $B_* \in \{B_1, B_2, \ldots, B_6\}$ with degree four necessarily means there exists an edge between B_* and a vertex $B_{**} \in \{B_1, B_2, \ldots, B_6\} \setminus \{B_*\}$ by the Pigeonhole Principle. This edge creates a clique of size three, consisting of the vertices B_*, B_{**}, and a common neighbor found in $\{A_1, A_2\}$.

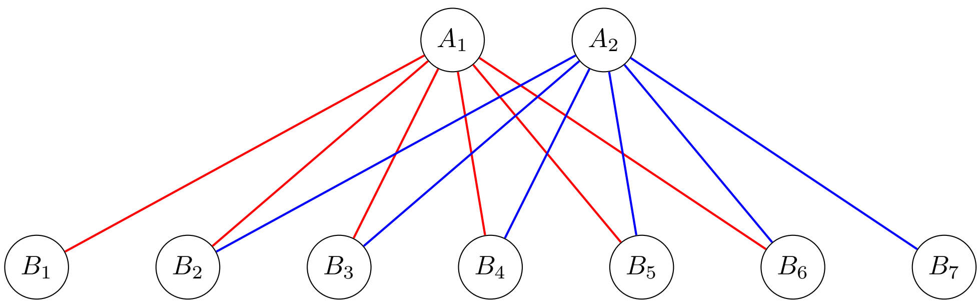

Figure 3: The edge (A_1, A_2) does not exist and A-nodes do not share the same set of neighbors.

We have shown via casework that there always exists a group of three individuals that all know each other by demonstrating that a clique of size three exists in the undirected graph modeling the set of relationships described in the problem statement.

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