$\aleph_0$ Weekly Problem

Ravi Dayabhai   Conrad Warren

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Problem

The product of three positive reals is 1. Their sum is greater than the sum of their reciprocals. Prove that exactly one of these numbers is strictly greater than 1.

Solution

Proof. Let a, b, and c be the three positive reals. We are given that $abc = 1$ and $a+b+c > \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. We want to show that exactly one of a, b, or c is strictly greater than 1. To do that we will show:

1. **At least one of a, b, or c is strictly greater than 1.**

   Assume, for the sake of contradiction, that $a \leq 1$, $b \leq 1$, and $c \leq 1$. Then $\frac{1}{a} \geq 1$, $\frac{1}{b} \geq 1$, and $\frac{1}{c} \geq 1$. Therefore,

   $$a+b+c \leq 3 \quad \text{and} \quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 3.$$

   This contradicts the given condition $a+b+c > \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. Hence, at least one of a, b, or c must be strictly greater than 1.

2. **At most one of a, b, or c is strictly greater than 1.**

   Assume, for the sake of contradiction, that more than one of a, b, or c is strictly greater than 1. Immediately a contradiction is reached if all three of a, b, and c are strictly greater than 1, since their product would be greater than 1.

   For the case when two of a, b, and c are strictly greater than 1, we can assume without loss of generality that $a > 1$ and $b > 1$. Therefore, it must be the case that $c = \frac{1}{ab} < 1$. Substituting c in the given condition $a+b+c > \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ gives $a+b+\frac{1}{ab} > \frac{1}{a} + \frac{1}{b} + ab$. Rearranging the inequality, we get

$$
\begin{aligned}
a + b + \frac{1}{ab} - \frac{1}{a} - \frac{1}{b} - ab &> 0 \\
\left(a - \frac{1}{a}\right) + \left(b - \frac{1}{b}\right) + \left(\frac{1}{ab} - ab\right) &> 0 \\
\frac{a^2b - b}{ab} + \frac{b^2a - a}{ab} + \frac{1 - a^2b^2}{ab} &> 0 \\
\frac{a^2b + b^2a - a - b}{ab} &> -\frac{1 - a^2b^2}{ab} \\
ab(a+b) - (a+b) &> -(1 - ab)(1 + ab) \\
(a+b)(ab-1) &> (ab-1)(1+ab) \\
a + b &> ab + 1 \\
0 &> ab - a - b + 1.
\end{aligned}
$$

This is the contradiction we desire, because $ab - a - b + 1 = (a-1)(b-1) \geq 0$. Therefore, it is also not possible for two of a, b, and c to be strictly greater than 1. Since it is not possible for two or three of a, b, and c to be strictly greater than 1, then at most one of a, b, or c is strictly greater than 1.

Combining the results of parts 1 and 2, we conclude that exactly one of a, b, or c is strictly greater than 1.

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