$\aleph_0$ Weekly Problem

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Problem

Show that each number in the sequence $49, 4489, 444889, \ldots$ is a perfect square.

Solution

Proof. The nth number in the sequence is given by $\left(\sum_{i=1}^{2n} 4 \cdot 10^{i-1}\right) + \left(\sum_{i=1}^{n} 4 \cdot 10^{i-1}\right) + 1$. We will show that for any $n \in \{1, 2, \ldots\}$, the nth term in the sequence equals a perfect square, namely, $\left(\left(\sum_{i=1}^{n} 6 \cdot 10^{i-1}\right) + 1\right)^2$.

$$\left(\left(\sum_{i=1}^{n} 6 \cdot 10^{i-1}\right) + 1\right)^2 = \left(\sum_{i=1}^{2n} 4 \cdot 10^{i-1}\right) + \left(\sum_{i=1}^{n} 4 \cdot 10^{i-1}\right) + 1$$

$$\left(\sum_{i=1}^{n} 6 \cdot 10^{i-1}\right)^2 + 2 \cdot \left(\sum_{i=1}^{n} 6 \cdot 10^{i-1}\right) + 1 = \sum_{i=1}^{2n} 4 \cdot 10^{i-1} + \sum_{i=1}^{n} 4 \cdot 10^{i-1} + 1$$

$$\left(\sum_{i=1}^{n} 6 \cdot 10^{i-1}\right)^2 + \sum_{i=1}^{n} 8 \cdot 10^{i-1} = \sum_{i=1}^{2n} 4 \cdot 10^{i-1}$$

$$36 \left(\sum_{i=1}^{n} 10^{i-1}\right)^2 + 8 \sum_{i=1}^{n} 10^{i-1} = 4 \sum_{i=1}^{2n} 10^{i-1}$$

$$9 \left(\sum_{i=1}^{n} 10^{i-1}\right)^2 + 2 \sum_{i=1}^{n} 10^{i-1} = \sum_{i=1}^{2n} 10^{i-1}$$

$$
\begin{aligned}
9\left(\sum_{i=1}^{n} 10^{i-1}\right)^2 + 2\sum_{i=1}^{n} 10^{i-1} &= 10^n \sum_{i=1}^{n} 10^{i-1} + \sum_{i=1}^{n} 10^{i-1} \\
9\left(\sum_{i=1}^{n} 10^{i-1}\right)^2 + \sum_{i=1}^{n} 10^{i-1} &= 10^n \sum_{i=1}^{n} 10^{i-1} \\
9\left(\sum_{i=1}^{n} 10^{i-1}\right) + 1 &= 10^n \\
9\left(\sum_{i=1}^{n} 10^{i-1}\right) &= 10^n - 1 = 9\left(\sum_{i=1}^{n} 10^{i-1}\right)
\end{aligned}
$$

By establishing the above equality, we have shown, for any index $n \in \{1, 2, \ldots\}$, that the nth number in the given sequence is a perfect square of the form $\left(\left(\sum_{i=1}^{n} 6 \cdot 10^{i-1}\right) + 1\right)^2$.

$\square$