$\aleph_0$ Weekly Problem

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Problem

Show that there do not exist any equilateral triangles in the plane whose vertices are lattice points (integer coordinates).

Solution

Proof. We will attempt to construct an equilateral triangle in the plane whose vertices are all lattice points (i.e., having integer coordinates), but show that this is impossible, thereby, demonstrating no such triangle can exist.

We start by fixing one vertex v_1 at point $(0, 0)$, the second v_2 at point $(\frac{\sqrt{3}r}{2}, 0)$, and the third v_3 at $(\frac{\sqrt{3}r}{2}, \frac{3}{2}r)$, yielding an equilateral triangle having side lengths equal to $\sqrt{3}r$ (notionally circumscribed by a circle centered at $(\frac{\sqrt{3}}{2}r, \frac{1}{2}r)$ having radius r).

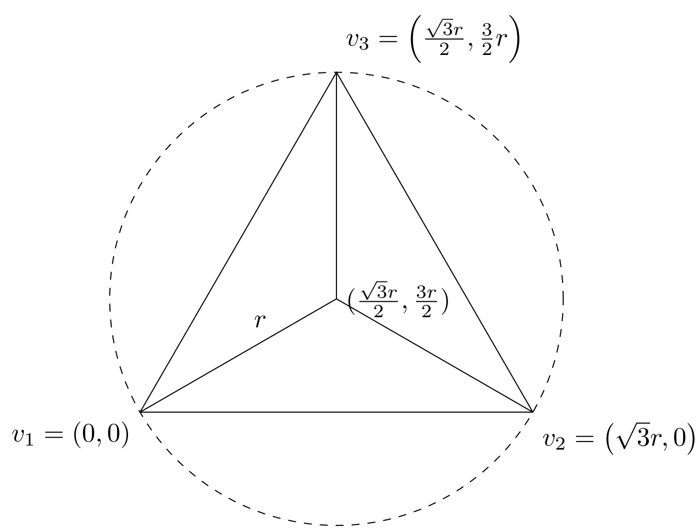

Figure 1: Equilateral triangle with side length $\sqrt{3}r$

Note that we can scale the triangle by r and rotate the triangle about v_1 by some angle θ between the v_1–v_2 side of the triangle and the x-axis, as seen in figure **??**.

Treating v_2 and v_3 as vectors allows us to perform any scaling and rotation. With v_1 fixed, we can express the new coordinates of the triangle's vertices under any scaling and rotation as:

$$\begin{aligned}
v_1' &= v_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
v_2' &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sqrt{3}r \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{3}r\cos\theta \\ \sqrt{3}r\sin\theta \end{bmatrix} \\
v_3' &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}r}{2} \\ \frac{3r}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}r}{2}\cos\theta - \frac{3r}{2}\sin\theta \\ \frac{\sqrt{3}r}{2}\sin\theta + \frac{3r}{2}\cos\theta \end{bmatrix}
\end{aligned}$$

If we let $A = \sqrt{3}r\cos\theta$ and $B = \sqrt{3}r\sin\theta$, we get:

$$\begin{aligned}
v_2' &= \begin{bmatrix} A \\ B \end{bmatrix} \\
v_3' &= \begin{bmatrix} \frac{A}{2} - \frac{B\sqrt{3}}{2} \\ \frac{B}{2} - \frac{A\sqrt{3}}{2} \end{bmatrix}
\end{aligned}$$

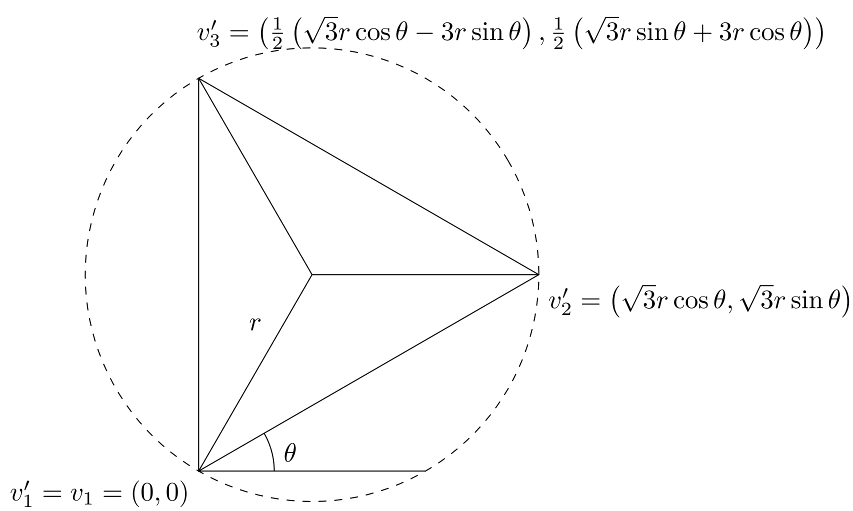

Figure 2: Equilateral triangle rotated around point $(0,0)$

But now we clearly see that not all four coordinates of v_2' and v_3' can be simultaneously integer-valued, since integer-valued coordinates of v_2' (i.e., A and

B) implies non-integer values for the coordinates of v_3': an irrational number (e.g., $\frac{B\sqrt{3}}{2}$) subtracted from a rational number (e.g., $\frac{A}{2}$) produces an irrational (e.g., $\frac{A}{2} - \frac{B\sqrt{3}}{2}$).

Since the choice of fixing v_1 was arbitrary, the above holds for any choice of integer-valued coordinates for v_1 (any non-integer valued coordinate for v_1 immediately violates the construction) because the addition of an integer-valued offset vector to each vertex doesn't change the fact that an irrational term will exist in at least one of the coordinates of v_2' or v_3'.

$\square$