

# №<sub>0</sub> Weekly Problem

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2024-08-31

## Problem

Prove that if you add up the reciprocals of a sequence of consecutive positive integers, the numerator of the sum (in lowest terms) will always be odd. For example,  $\frac{1}{7} + \frac{1}{8} + \frac{1}{9} = \frac{191}{504}$ .

## Solution

First, we prove the following lemma that will be helpful later.

**Lemma.** *In any sequence of consecutive positive integers  $(a_1, a_2, \dots, a_n)$ , there is a unique integer  $a_*$  with the highest power of 2 in its prime factorization.*

*Proof.* Suppose, for the sake of finding a contradiction, that there exists at least two positive integers in the consecutive positive integer sequence  $(a_1, a_2, \dots, a_n)$  sharing a maximal power of 2 in their prime factorizations.

Without loss of generality, let  $a_j$  and  $a_k$  (with  $j < k$ ) have the same, highest power of 2 in their prime factorizations (say,  $2^m$ ) and all intervening  $a_l$  (for all  $l$  such that  $j < l < k$ ) having a smaller power of 2 in its prime factorization. Then,  $a_j = 2^m d$  such that  $d$  is odd (since it is the product of only odd integers) and  $a_k = 2^m(d+1)$ .

$d+1$  is even and factoring out a 2 from  $d+1$  gives  $a_k = 2^{m+1} \left(\frac{d+1}{2}\right)$ , where  $\frac{d+1}{2}$  is an integer. Herein lies a contradiction — it is assumed that  $a_j$  and  $a_k$  have the same, highest power of 2 among the prime factorizations of all  $a_i$  in the sequence.

Reaching a contradiction, we have proven the lemma.  $\square$

Next, we turn our attention to proving the original claim stated by the **Problem**.

**Claim.** *If you add up the reciprocals of a sequence of consecutive positive integers, the numerator of the sum (in lowest terms) will always be odd.*

*Proof.* Consider any sequence of consecutive positive integers  $(a_1, a_2, \dots, a_n)$ . Let  $N = \prod_{i=1}^n a_i$ . Then,

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} = \frac{\frac{N}{a_1} + \frac{N}{a_2} + \dots + \frac{N}{a_n}}{N}.$$

Letting  $\text{lcm}(1, a_1, a_2, \dots, a_n) = N'$ , the numerator,  $S$ , in the resultant fraction expressed in lowest terms is:

$$S = \frac{N'}{a_1} + \frac{N'}{a_2} + \dots + \frac{N'}{a_n}.$$

By the Fundamental Theorem of Arithmetic, each  $\frac{N}{a_i}$  has a unique prime factorization; hence  $\frac{N}{a_i} = 2^{m_{i,1}} 3^{m_{i,2}} 5^{m_{i,3}} \dots$  for all  $i \in \{1, 2, \dots, n\}$ . What remains to be shown is that there exists exactly one odd summand (of  $S$ ),  $\frac{N'}{a_*}$ , implying  $S$  is odd.

Consider  $\frac{N}{a_*} = 2^{m_{*,1}} 3^{m_{*,2}} 5^{m_{*,3}} \dots$ , where  $a_*$  is the member of the sequence with the highest power of 2 in its prime factorization. Note that  $a_*$  is unique by the lemma established above. This implies a unique  $m_{*,1} = \min\{m_{i,1} \mid i \in \{1, 2, \dots, n\}\}$ . Because  $2^{m_{*,1}}$  can be factored out of all  $\frac{N}{a_i}$ , it follows that  $\frac{N'}{a_*}$  is the only odd summand of  $S$ , giving odd  $S$ .

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