$\aleph_0$ Weekly Problem

Conrad Warren & Ravi Dayabhai

2024-07-14

Problem

Show that if every room in a house has an even number of doors, then the number of outside entrance doors must be even as well.

Solution

Let us represent the house as an undirected multi-graph $G = (V, E)$ on n nodes (i.e., $|V| = n$), where (without loss of generality) the vertices labeled $v_1, v_2, \ldots, v_{n-1}$ correspond to the rooms of the house and the nth node, v_n, represents the "outside." In this representation, each edge corresponds to a door; let O be the set of edges that are incident to v_n (i.e., the outside entrance doors) and $R = E \setminus O$ (i.e., the doors connecting two rooms).

Suppose, for sake of finding a contradiction, that a house consists of rooms all having an *even* number of doors, but the number of outside entrance doors, $|O|$, is *odd*.

Since each of $v_1, v_2, \ldots, v_{n-1}$ has an even degree, the sum of their degrees, $T = \sum_{i=1}^{n-1} \deg(v_i)$, will also be *even*. Then,

$$T = \sum_{i=1}^{n-1} \deg(v_i) = 2 \cdot |R| + |O|$$

because each edge in R contributes $+2$ to T, whereas each edge in O only contributes $+1$.

Notice that T will be *odd* as $2 \cdot |R|$ is even, $|O|$ is odd, and the sum of an odd and an even number is always odd. But this is a contradiction since we already established that T is even. Therefore, the number of outside entrance doors, $|O|$, must be *even*.