$\aleph_0$ Weekly Problem

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Problem

Prove that if you add up the reciprocals of a sequence of consecutive positive integers, the numerator of the sum (in lowest terms) will always be odd. For example, $\frac{1}{7} + \frac{1}{8} + \frac{1}{9} = \frac{191}{504}$.

Solution

First, we prove a lemma that will be helpful later:

Lemma. *In any sequence of consecutive positive integers $(a_1, a_2, \ldots, a_n)$, there is a distinct integer a_* with the highest power of 2 in its prime factorization.*

Proof. Suppose, for the sake of finding a contradiction, that there exists at least two positive integers in the consecutive positive integer sequence $(a_1, a_2, \ldots, a_n)$ sharing a maximal power of 2 in their prime factorizations. Without loss of generality, let a_j and a_k (and $j < k$, without loss of generality) both have the same highest power of 2 in their prime factorizations, say $2^{m_\dagger}$, and no intervening a_l (i.e., $j < l < k$) having a higher power of 2 in its prime factorization. Then, $a_j = 2^{m_\dagger} d$ such that d is odd (since it is the product of all odd primes dividing a_j) and $a_k = 2^m(d+1)$. But $d+1$ is even and factoring out a 2 from $d+1$ gives $a_k = 2^{m_\dagger+1}\left(\frac{d+1}{2}\right)$. This results in a contradiction since it is assumed that both a_j and a_k have the same, highest power of 2 in the prime factorizations among all a_i. Having found a contradiction, we have proven the lemma. $\square$

Next, we turn our attention to proving the original claim stated by the **Problem**:

Proof. Consider any sequence of positive integers $(a_1, a_2, \ldots, a_n)$. Let $N = \prod_{i=1}^{n} a_i$. Then the sum of the reciprocals of this sequence of positive integers can be expressed as:

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots \frac{1}{a_n} = \frac{\frac{N}{a_1} + \frac{N}{a_2} + \ldots + \frac{N}{a_n}}{N}.$$

Letting $\text{lcm}(a_1, a_2, \ldots, a_n) = N'$, the numerator in the resultant fraction expressed in lowest terms, S, is:

$$S = \frac{N'}{a_1} + \frac{N'}{a_2} + \ldots + \frac{N'}{a_n}.$$

By the Fundamental Theorem of Arithmetic, $\frac{N}{a_i}$ has a unique prime factorization, hence $\frac{N}{a_i} = 2^{m_{i,1}} 3^{m_{i,2}} 5^{m_{i,3}} \cdots$, for all $i \in \{1, 2, \ldots, n\}$. We can express N similarly: $N = 2^{M_1} 3^{M_2} 5^{M_3} \cdots$.

What remains to be shown is that there exists exactly one odd summand (of S) $\frac{N'}{a_i}$ for all $i \in \{1, 2, \ldots, n\}$, thereby resulting in odd S.

Consider $\frac{N}{a_*} = 2^{m_{*,1}} 3^{m_{*,2}} 5^{m_{*,3}} \cdots$ such that $m_{*,1} = \min\{m_{i,1} | i \in \{1, 2, \ldots, n\}\}$. Note that a_* is unique by the lemma established above. Because $2^{m_{*,1}}$ can be factored out of all $\frac{N}{a_i}$, it follows that $\frac{N'}{a_*}$ is the only odd summand of S, yielding odd S.

$\square$