$\aleph_0$ Weekly Problem

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2024-08-31

Problem

Prove that if you add up the reciprocals of a sequence of consecutive positive integers, the numerator of the sum (in lowest terms) will always be odd. For example, $\frac{1}{7} + \frac{1}{8} + \frac{1}{9} = \frac{191}{504}$.

Solution

First we prove a lemma that will be helpful later:

Lemma. *In any sequence of consecutive positive integers $(a_1, a_2, \ldots, a_n)$, there is a distinct integer a_* with the highest power of 2 in its prime factorization.*

Proof. □

Now we can turn our attention to proving the original claim stated by the **Problem**:

Proof. Consider any sequence of positive integers $(a_1, a_2, \ldots, a_n)$. Let $N = \prod_{i=1}^{n} a_i$. Then the sum of the reciprocals of this sequence of positive integers can be expressed as:

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots \frac{1}{a_n} = \frac{\frac{N}{a_1} + \frac{N}{a_2} + \ldots + \frac{N}{a_n}}{N}.$$

Since $\operatorname{lcm}(a_1, a_2, \ldots, a_n) \cdot \gcd(a_1, a_2, \ldots, a_n) = N$, the numerator in the resultant fraction expressed in lowest terms, S, is:

$$S = \frac{N'}{a_1} + \frac{N'}{a_2} + \ldots + \frac{N'}{a_n},$$

where $N' = \frac{N}{\gcd(a_1, a_2, \ldots, a_n)}$.

By the Fundamental Theorem of Arithmetic, $\frac{N}{a_i}$ has a unique prime factorization, hence $\frac{N}{a_i} = 2^{m_{i,1}} 3^{m_{i,2}} 5^{m_{i,3}} \cdots$, for all $i \in \{1, 2, \ldots, n\}$. We can express N similarly: $N = 2^{M_1} 3^{M_2} 5^{M_3} \cdots$.

What remains to be shown is that there exists exactly *one* **odd** summand $\frac{N'}{a_i}$ for all $i \in \{1, 2, \ldots, n\}$, thereby resulting in **odd** S.

Consider $\frac{N}{a_*} = 2^{m_{*,1}} 3^{m_{*,2}} 5^{m_{*,3}} \cdots$ such that $m_{*,1} = \min\{m_{i,1} | i \in \{1, 2, \ldots, n\}\}$. Note that a_* is unique by the lemma established above. Because $2^{m_{*,1}}$ can be factored out of all terms $\frac{N}{a_i}$, it is a divisor of $\gcd(a_1, a_2, \ldots, a_n)$. It follows that $\frac{N'}{a_*}$ is the only odd summand of S.

$\square$