MT 5103: Mathematics for Robotics
Fuzzy Logic: Introduction

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What is Fuzzy Logic?

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- Fuzzy logic is a mathematical language to express something.
- It has grammar, syntax, semantic like a language for communication.

- There are some other mathematical languages also known

  Relational algebra (operations on sets)

  Boolean algebra (operations on Boolean variables)

  Predicate logic (operations on well formed formulae, also called predicate propositions)

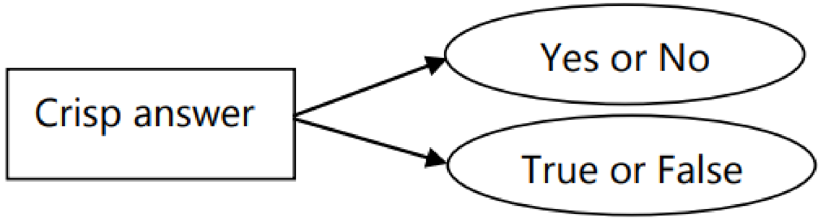
Main component of Fuzzy Logics is Fuzzy set

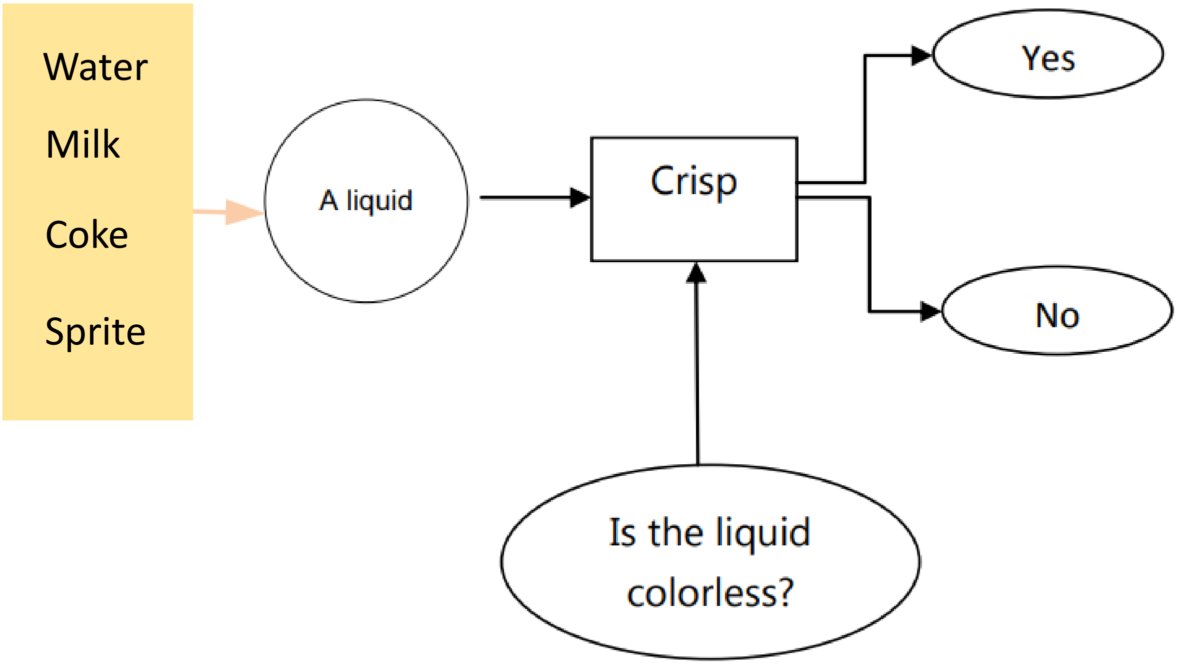
What is fuzzy?

- Dictionary meaning of fuzzy is not clear, noisy etc.
- Example: Is the picture on this slide is fuzzy?

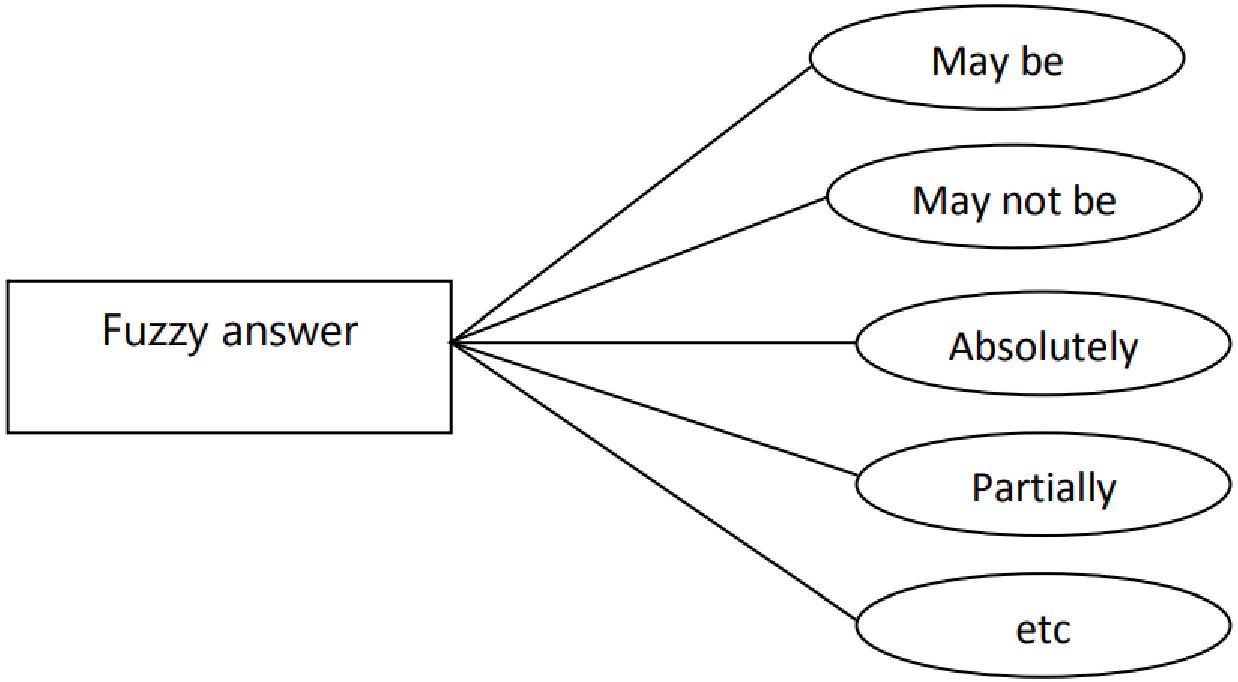
- Antonym of fuzzy is crisp

Fuzzy vs Crisp

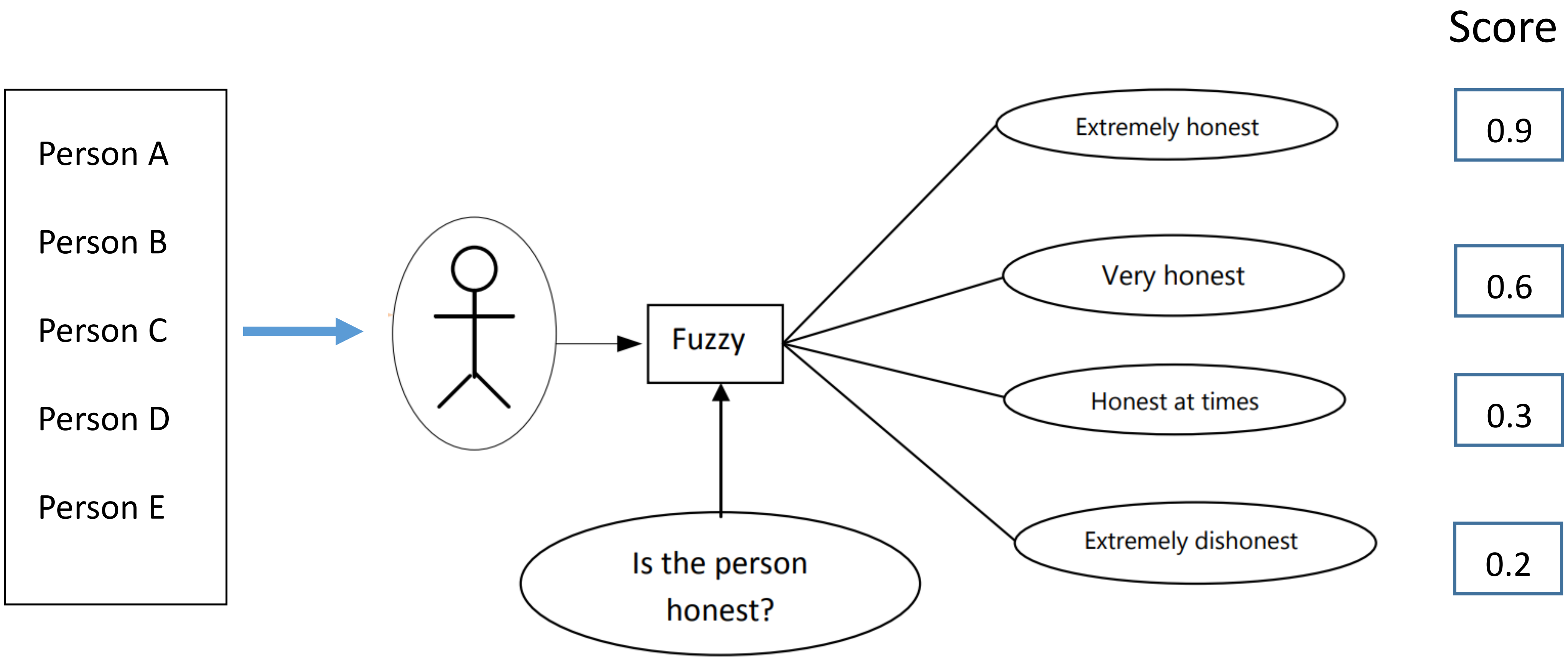




Fuzzy vs Crisp

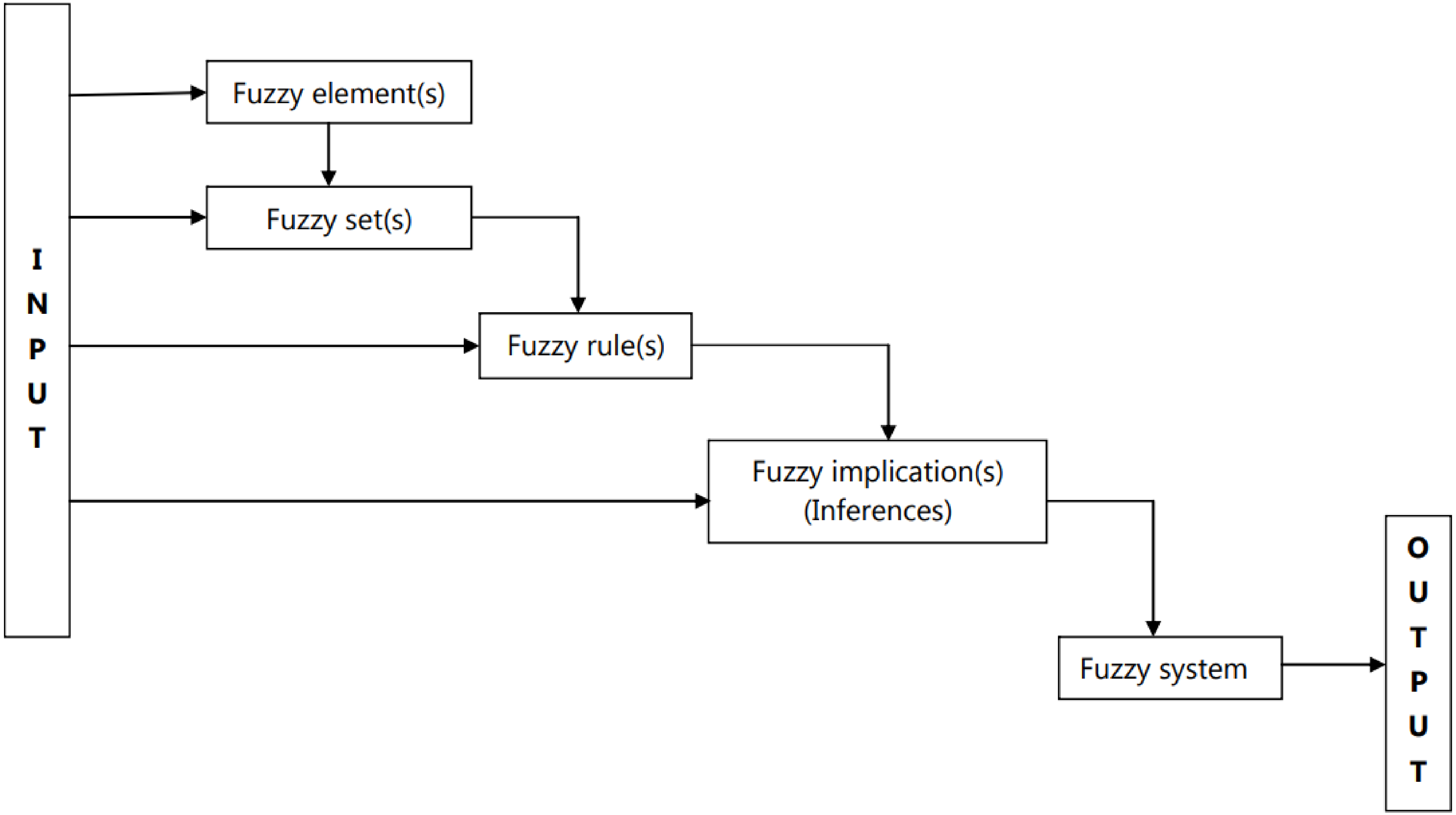


Example: Fuzzy Logic



World is Fuzzy !!!!

Concept of Fuzzy System

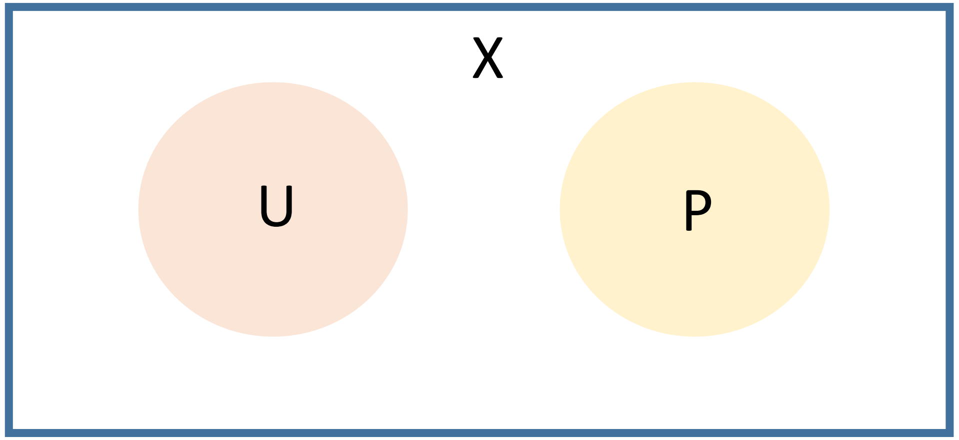


Crisp Set

X = Students in MU

U = Under Graduate Students = $\{U_1, U_{2,} \ldots, U_l\}$

P = Post Graduate Students = $\{P_{1,} P_2, \ldots, P_n\}$



Crisp Set: Sets with finite number of individuals, with solid boundary

Fuzzy Set

- **X** = All Post Graduate Students
- **S** = All good students

$$S = \{(S, \mu(s)) \mid s \in X\} \; and \; \mu(s) \; is \; a \; measurement \; of \; goodness \; of \; the \; students \; \boldsymbol{S}$$

Example: S = {(S_1, 0.8), (S_2, 0.7), (S_3, 0.2), (S_4, 0.9) }, etc.,

$\mu(s)$ = Degree of Membership and it lies between 0 and 1

Crisp vs Fuzzy

Crisp Set	Fuzzy Set
C ={S| S ϵ X}	F ={(S, μ(s))| S ϵ X} and μ (s) is the degree of S.
Collection of Elements	Collection of ordered Pairs
Inclusion of an element s ϵ X into S needs to have a strict boundary, i.e. Crisp	Inclusion of an element s ϵ X into F, is done with a degree of membership.

Crisp vs Fuzzy

A crisp set is a fuzzy set, but, a fuzzy set is not necessarily a crisp set.

Example:

$U = \{(U_1,1), (U_2,1), \ldots, (U_L,1\}$

$P = \{(P_1,0), (P_2,0), \ldots, (P_L,0\}$

Elements with degree of membership ($\mu(.)$) values 0 and 1 falls under crisp set.

Ex: Course evaluation (Crisp methods)

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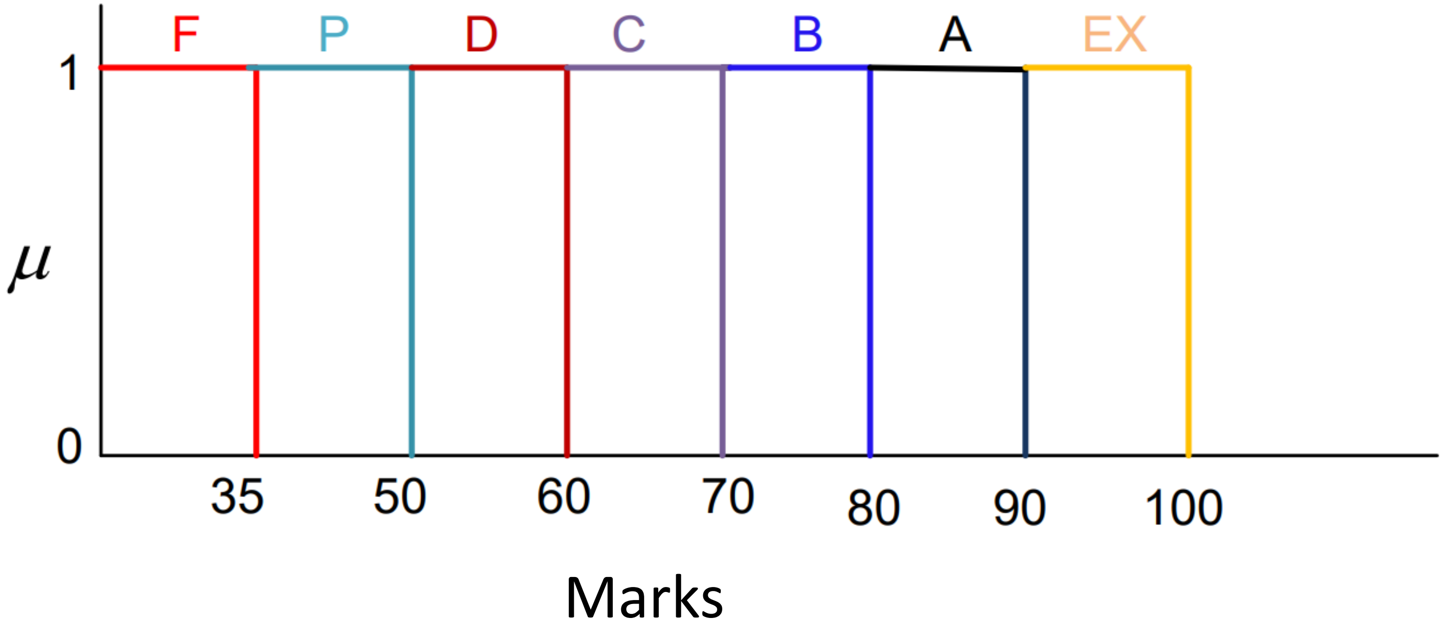
- A + : Marks ≥ 90
- A : 80 ≤ Marks < 90
- B : 70 ≤ Marks < 80
- C : 60 ≤ Marks < 70
- D : 50 ≤ Marks < 60
- P : 35 ≤ Marks < 50
- F : Marks ≤ 35



Ex: Course evaluation (Fuzzy way)

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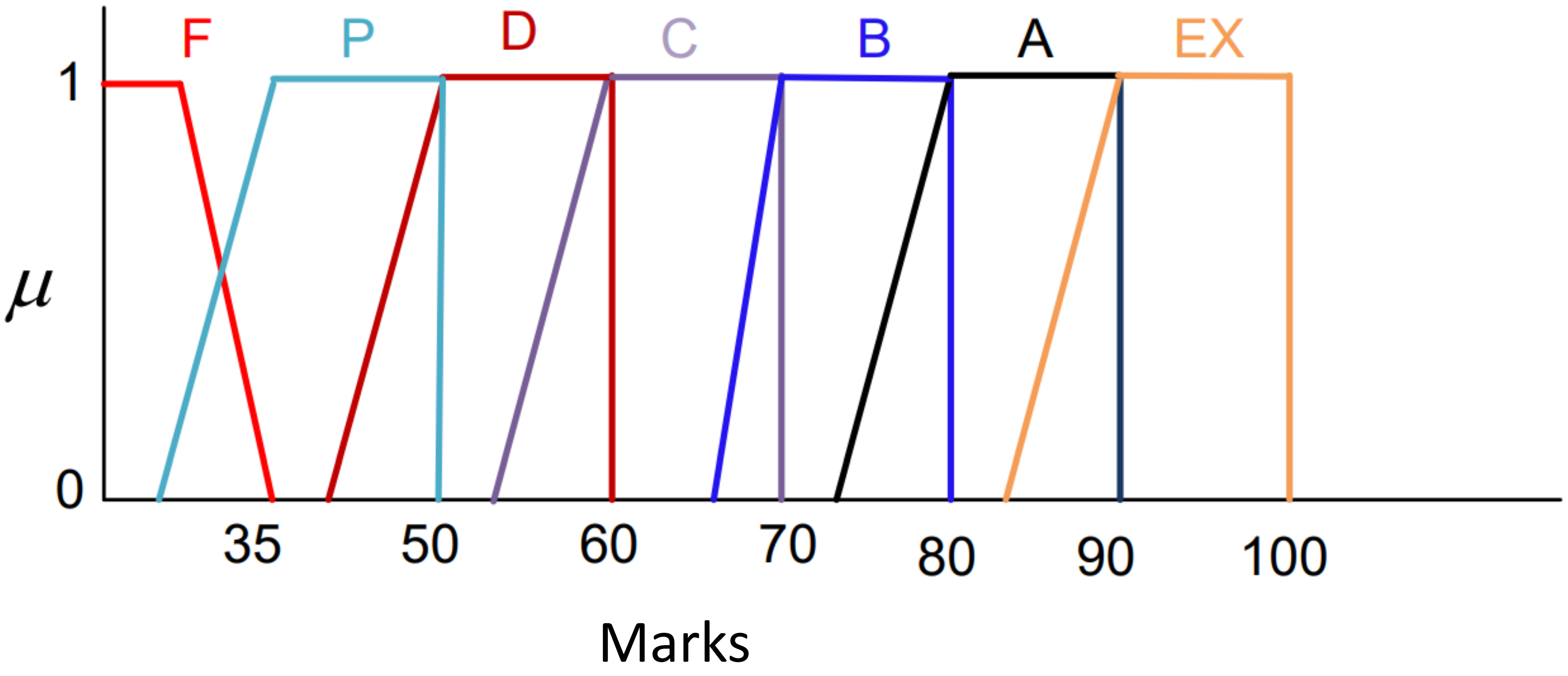


Some basic terminologies and notations

Membership function (and Fuzzy set):

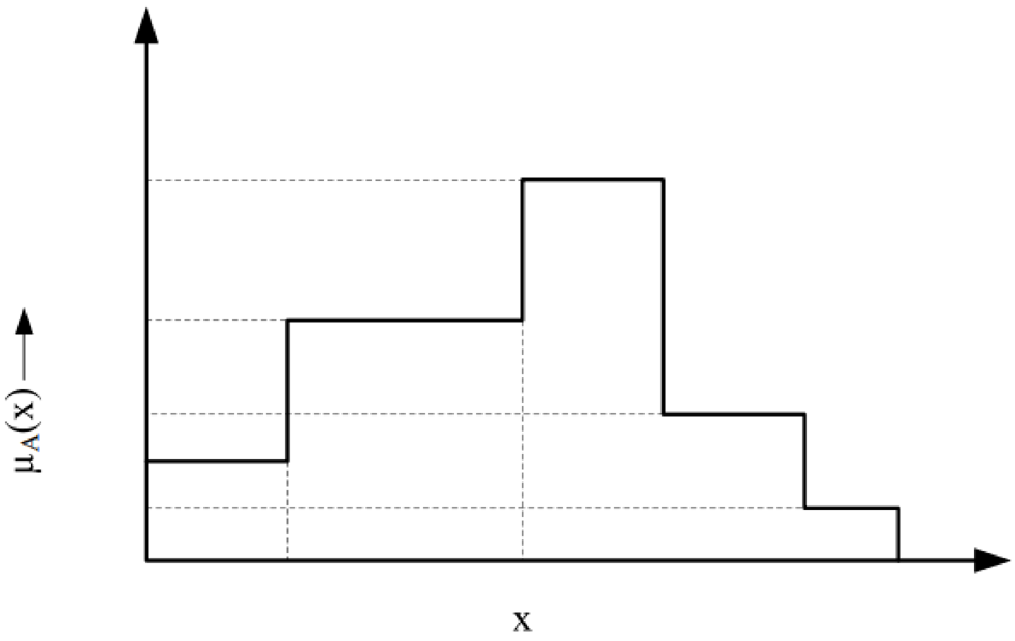
If ***X*** is a universe of discourse and $x \in \boldsymbol{X}$, then a fuzzy set *A* in ***X*** is defined as a set of ordered pairs, that is $A = \{(x, \mu_A(x)) | x \in \boldsymbol{X}\}$ where $\mu_A(x)$ is called the membership function for the fuzzy set A.

Note: $\mu_A(x)$ map each element of X onto a membership grade (or membership value) between 0 and 1 (both inclusive).

How (and who) decides $\mu_A(x)$ for a Fuzzy set ***A*** in ***X***

Membership function

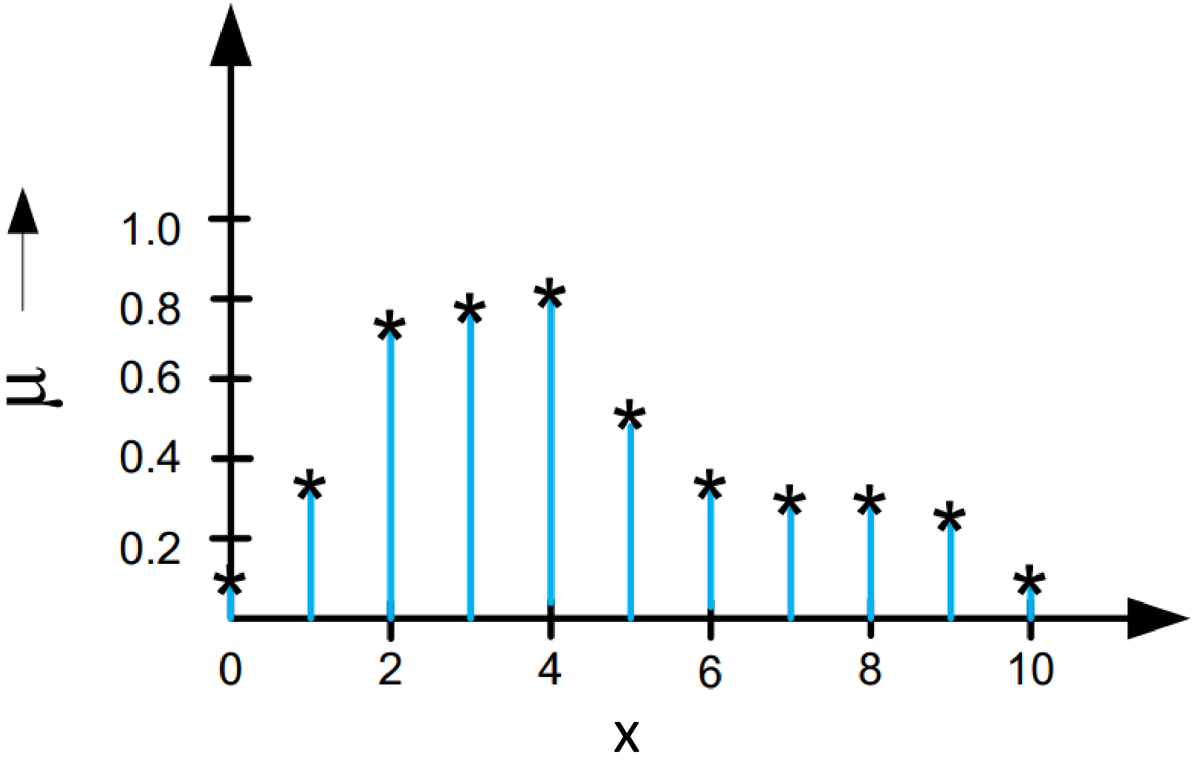
- The membership values may be of discrete values.



A fuzzy set with discrete values of μ

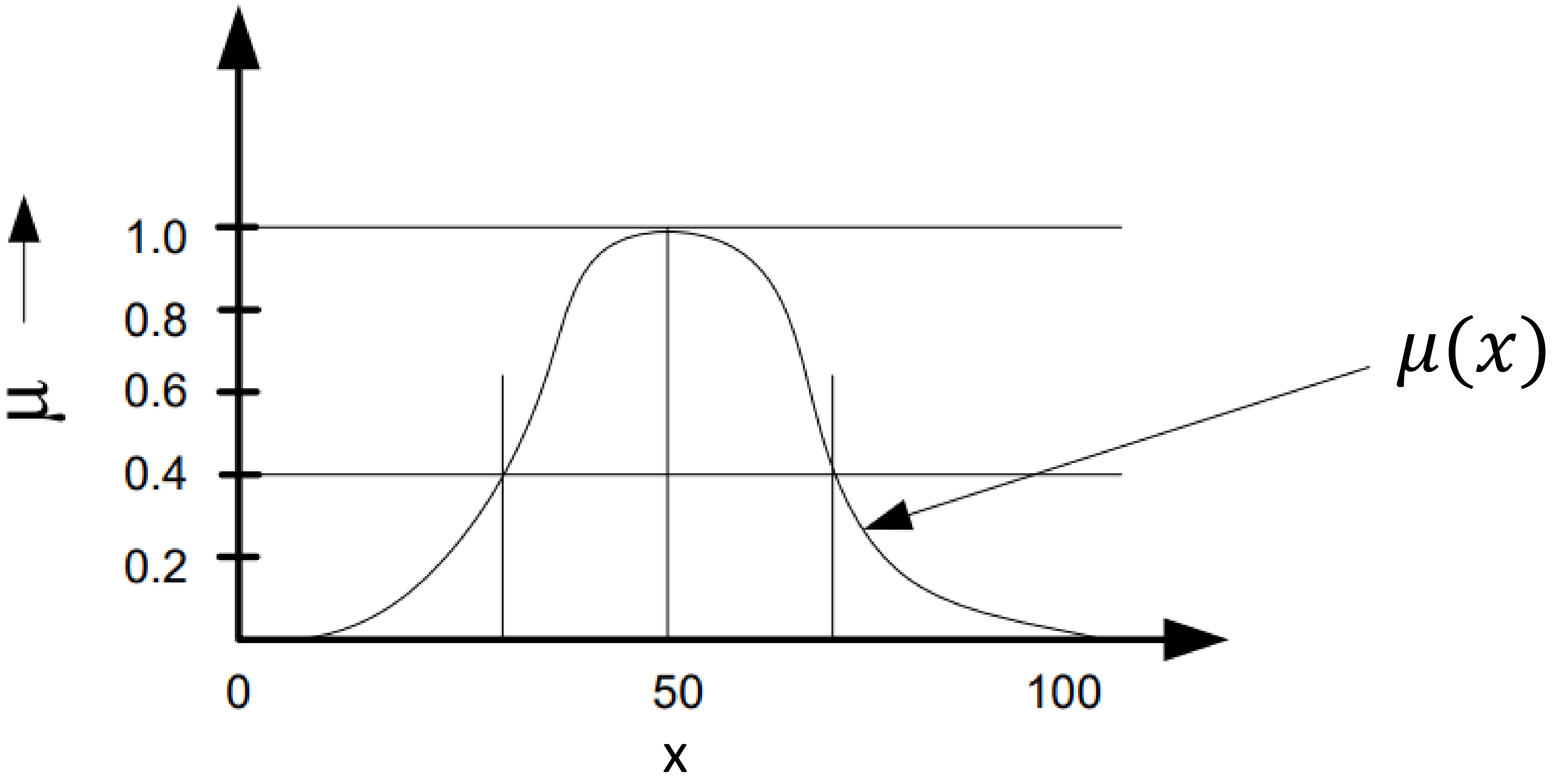
Membership function

Either elements and their membership values (both) also may be of discrete values.



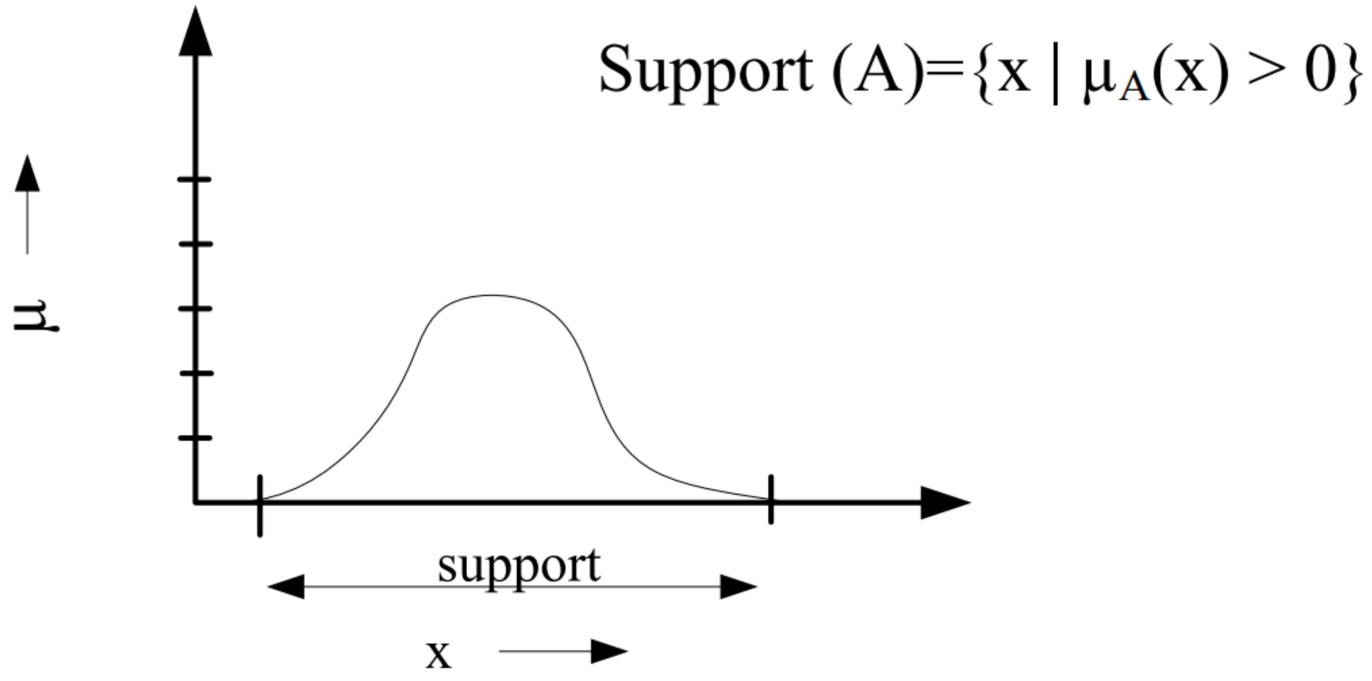
Membership function

- Membership function with continuous membership values



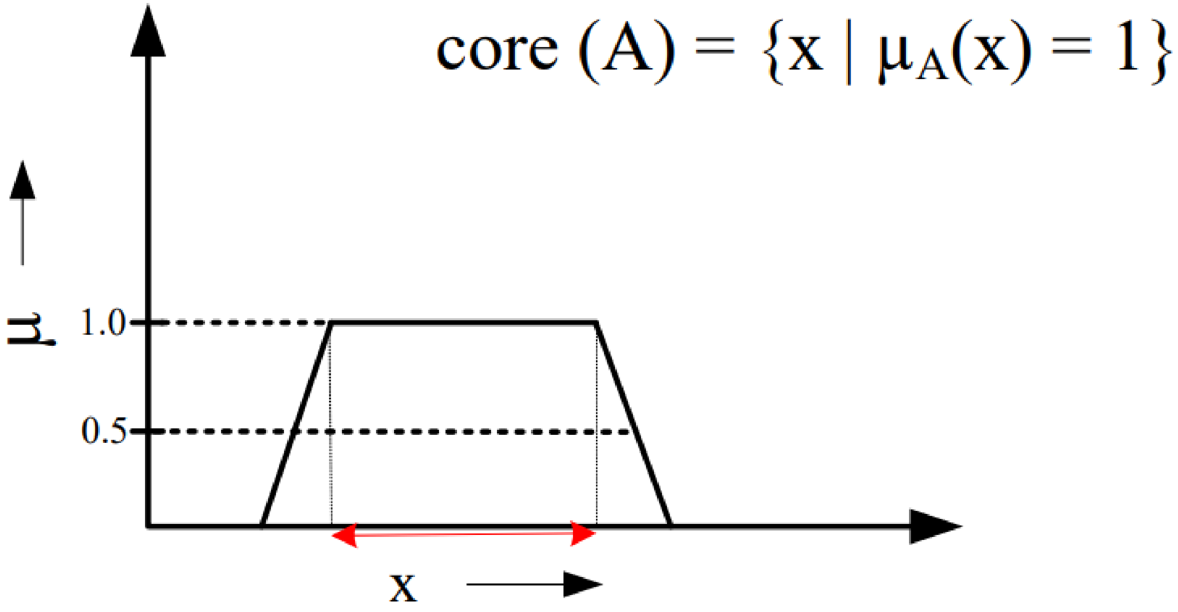
Fuzzy Terminologies: Support

- Support: The support of a fuzzy set ***A*** is the set of all points x ∈ ***X*** such that $\mu_A(x) > 0$



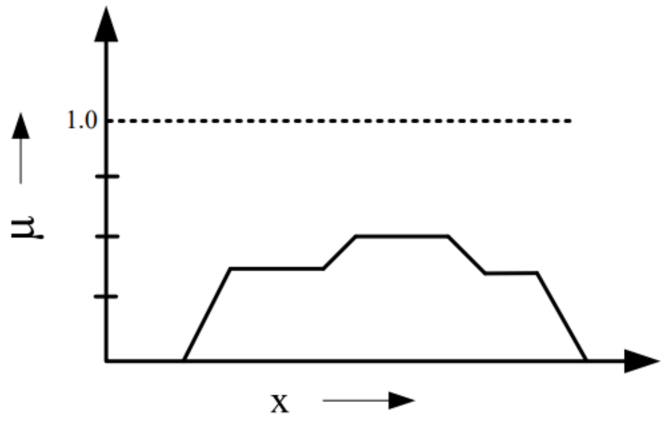
Fuzzy Terminologies: Core

- Core: The core of a fuzzy set ***A*** is the set of all points x in ***X*** such that $\mu_A(x) = 1$



Fuzzy Terminologies: Normality

- A fuzzy set ***A*** is a normal if its core is non-empty. In other words, we can always find a point $x \in X$ such that $\mu_A(x) = 1$.



Normality (A) = ?

Fuzzy Terminologies: Crossover Points

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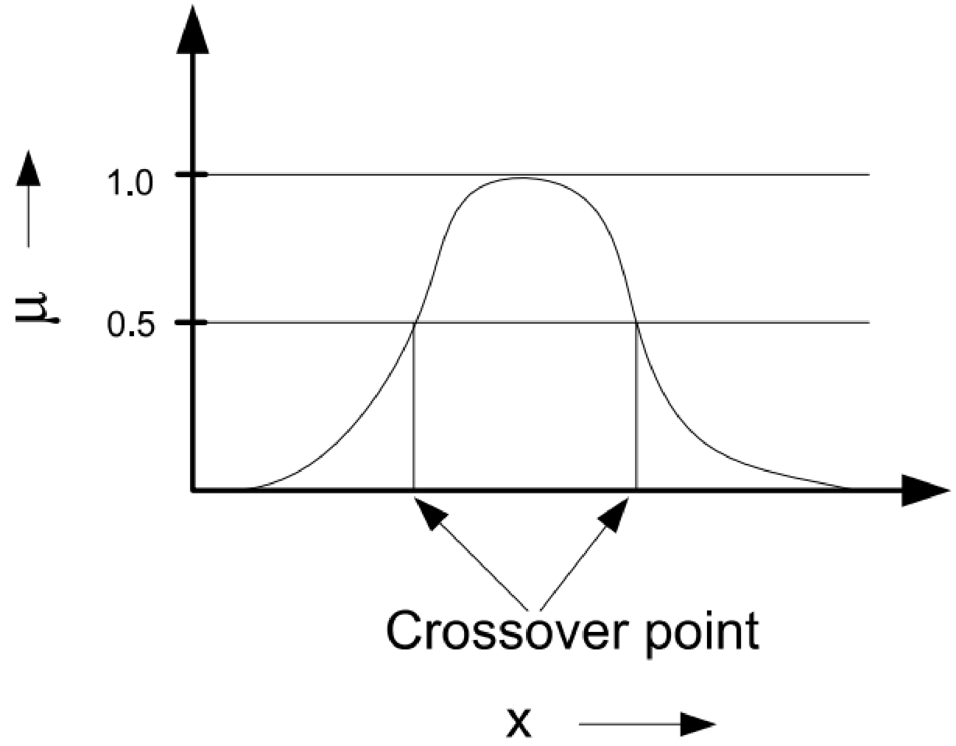
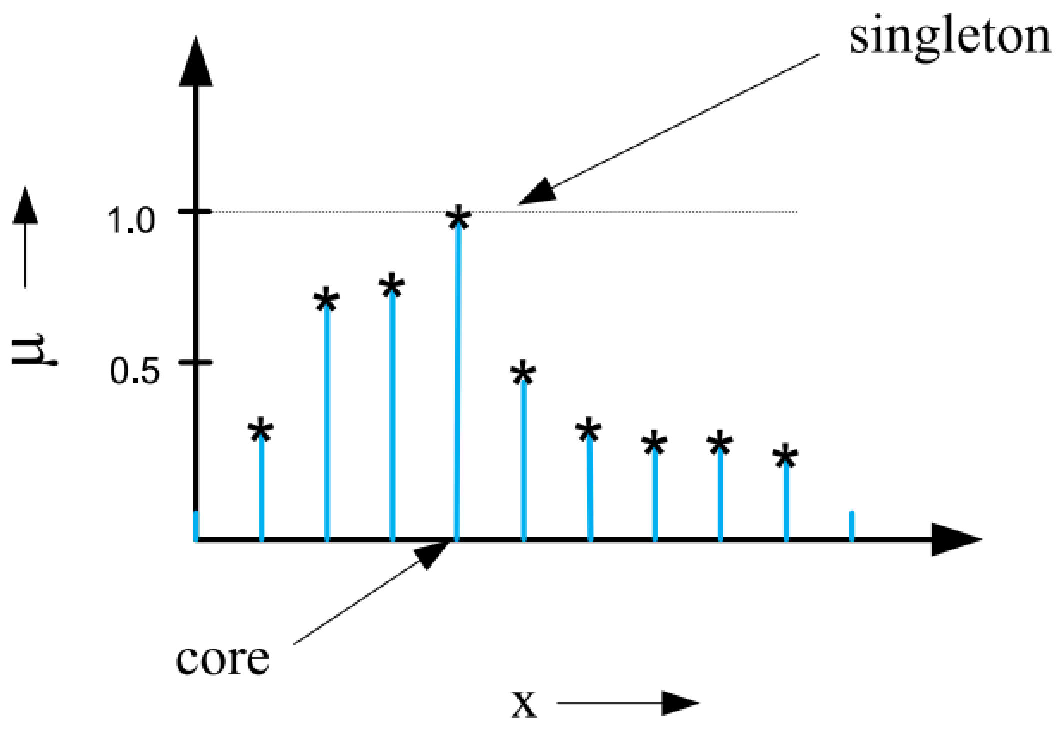
- A crossover point of a fuzzy set ***A*** is a point x ∈ ***X*** at which $\mu_A(x) = 0.5$. That is Crossover (**A**) = $\{x | \mu_A(x) = 0.5\}$.



Fuzzy Terminologies: Singleton

- A fuzzy set whose support is a single point in ***X*** with $\mu_A(x) = 1$ is called a fuzzy singleton. That is $|A| = \{ x \mid \mu_A(x) = 1\}$.



Fuzzy Terminologies:

- α – cut

  - The α-cut of a fuzzy set ***A*** is a crisp set that contains all the elements of universal set ***X*** whose membership grades in ***A*** are greater than or equal to the specified value of α

$$A_\alpha = \{x \mid \mu_A(x) \geq \alpha\}$$

- Strong α – cut

  - $A'_\alpha = \{x \mid \mu_A(x) > \alpha\}$

What is $A_1 = ?$

Fuzzy Terminologies: Bandwidth

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- For a normal and convex fuzzy set, the bandwidth (or width) is defined as the distance the two unique crossover points:

$$\text{Bandwidth}(A) = |\, x_1 - x_2 \,|$$

$$\text{where } \mu_A(x_1) = \mu_A(x_2) = 0.5$$

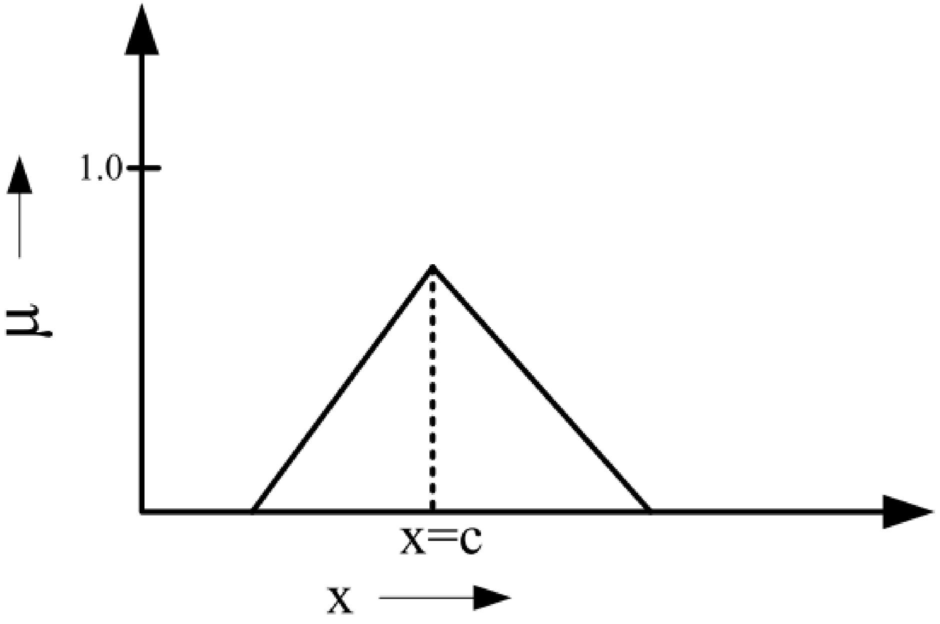
What if we have multiple cross over points?

We have to consider extreme points

Fuzzy terminologies: Symmetry

- A fuzzy set ***A*** is symmetric if its membership function around a certain point x = c, is symmetric.

  i.e., $\mu_A(x + c) = \mu_A(x - c)$ for all $x \in \boldsymbol{X}$.



Fuzzy Terminologies: Open and Closed

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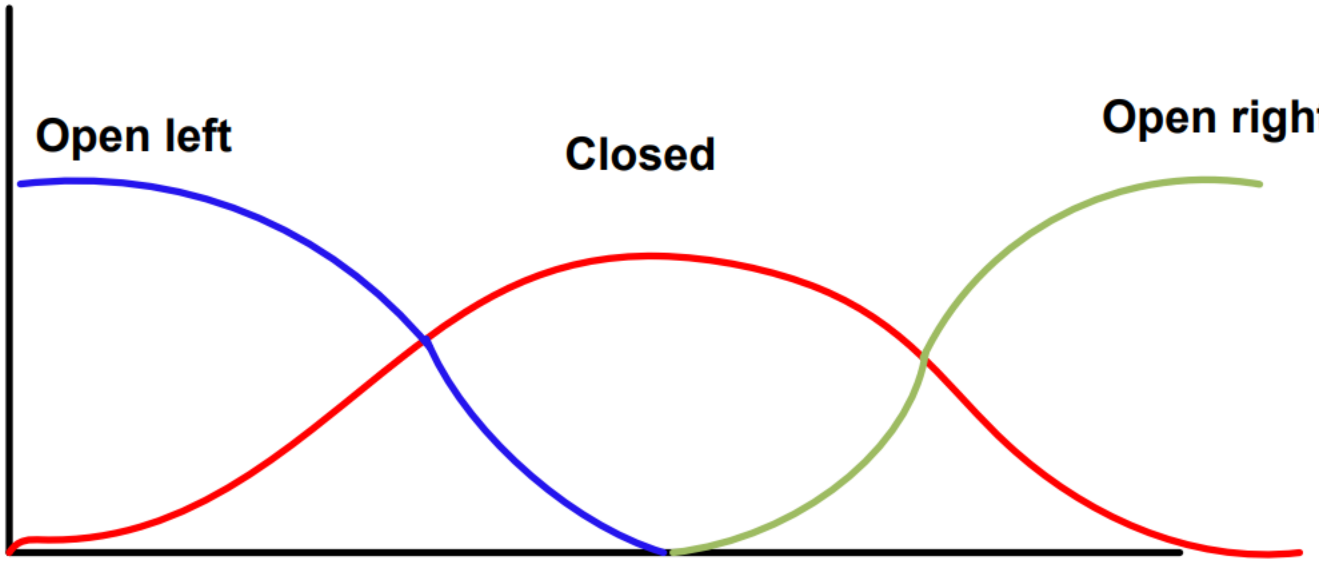
A fuzzy set A is

- Open left If $\lim_{x \to -\infty} \mu_A(x) = 1$ and $\lim_{x \to +\infty} \mu_A(x) = 0$
- Open right: If $\lim_{x \to -\infty} \mu_A(x) = 0$ and $\lim_{x \to +\infty} \mu_A(x) = 1$
- Closed If : $\lim_{x \to -\infty} \mu_A(x) = \lim_{x \to +\infty} \mu_A(x) = 0$

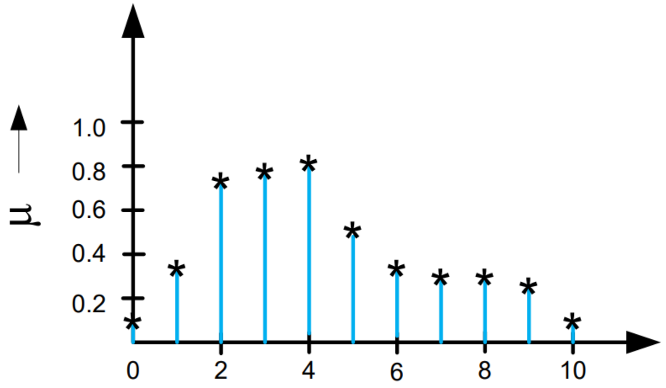


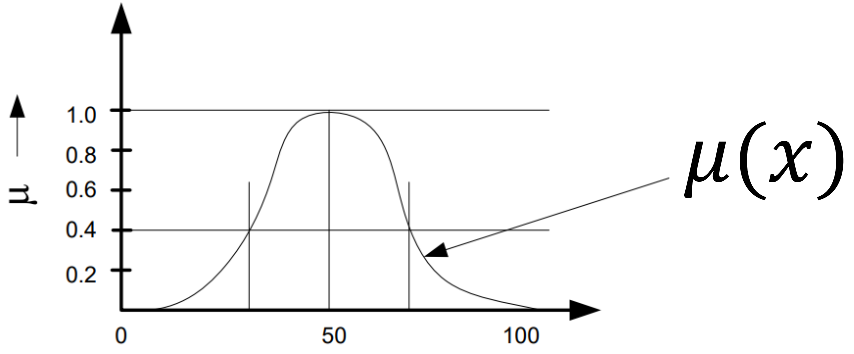
Fuzzy membership functions

- A fuzzy set is completely characterized by its membership function (sometimes abbreviated as MF and denoted as μ).
- Evidently, it would be important to learn how a membership function can be expressed (mathematically or otherwise).
- Quick Recap: A membership function can be on

(a) a discrete universe of discourse

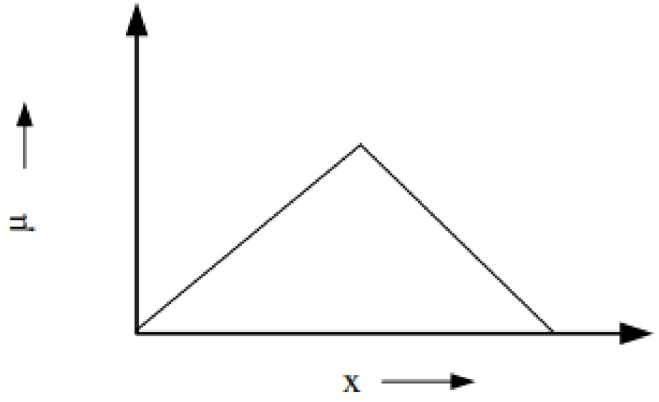
(b) a continuous universe of discourse.



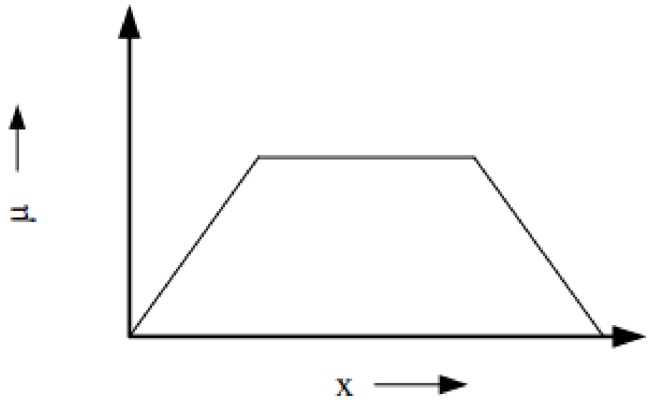


Fuzzy membership functions

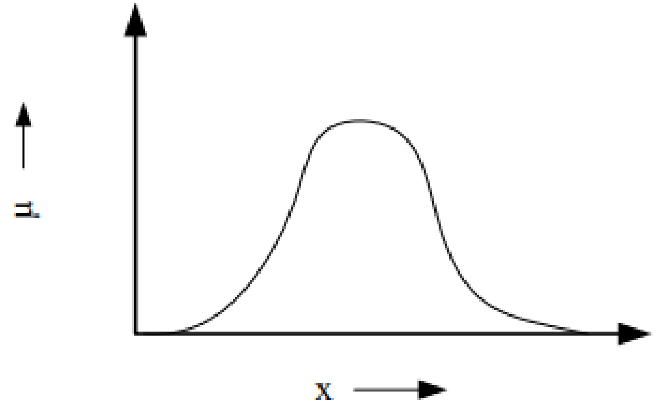
- Fuzzy membership functions



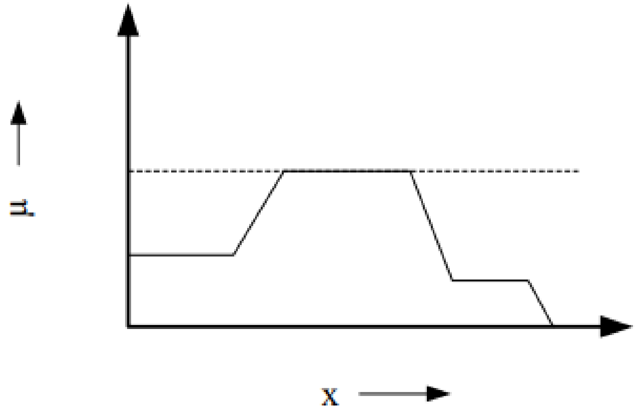
Triangle



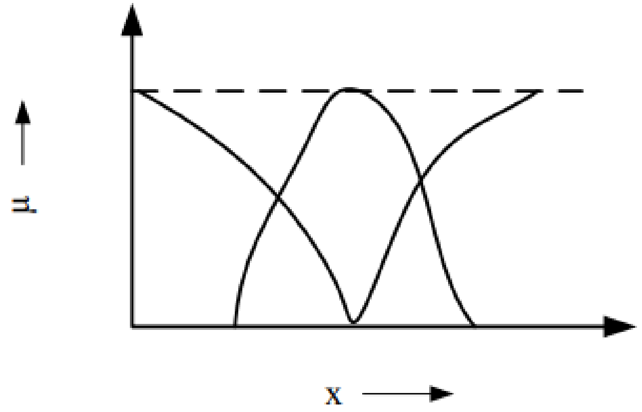
Trapezoidal



Curve



Non-Uniform



Uniform

Fuzzy MFs : Formulation and parameterization

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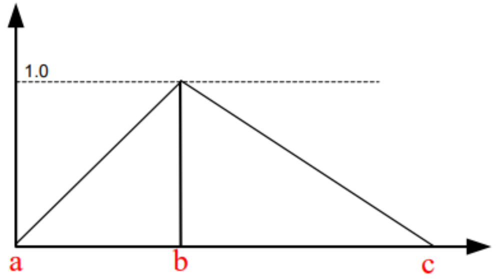
- In the following, we try to parameterize the different MFs on a continuous universe of discourse.
- Triangular MFs : A triangular MF is specified by three parameters {a, b, c} and can be formulated as follows.

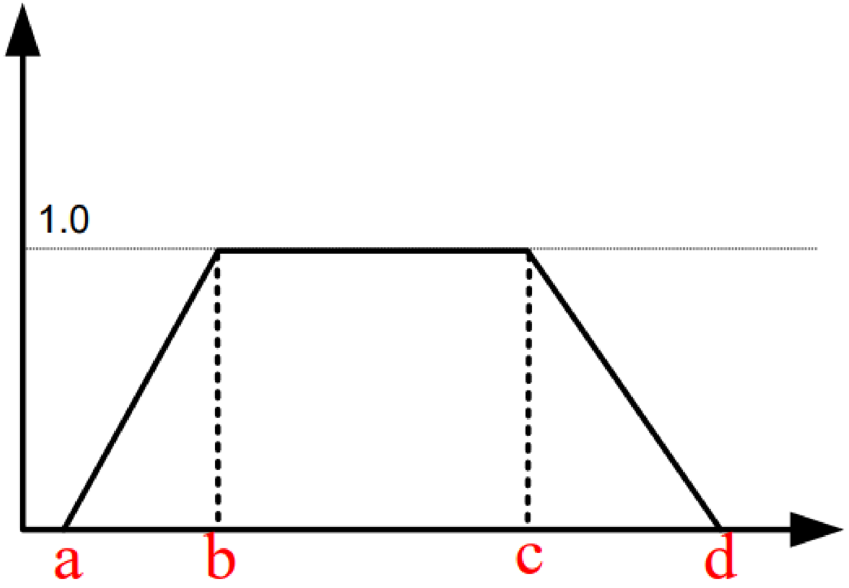
$$triangle(x; a, b, c) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ \frac{c-x}{c-b} & \text{if } b \leq x \leq c \\ 0 & \text{if } c \leq x \end{cases}$$



Fuzzy MFs: Trapezoidal

- A trapezoidal MF is specified by four parameters {a, b, c, d} and can be defined as follows:

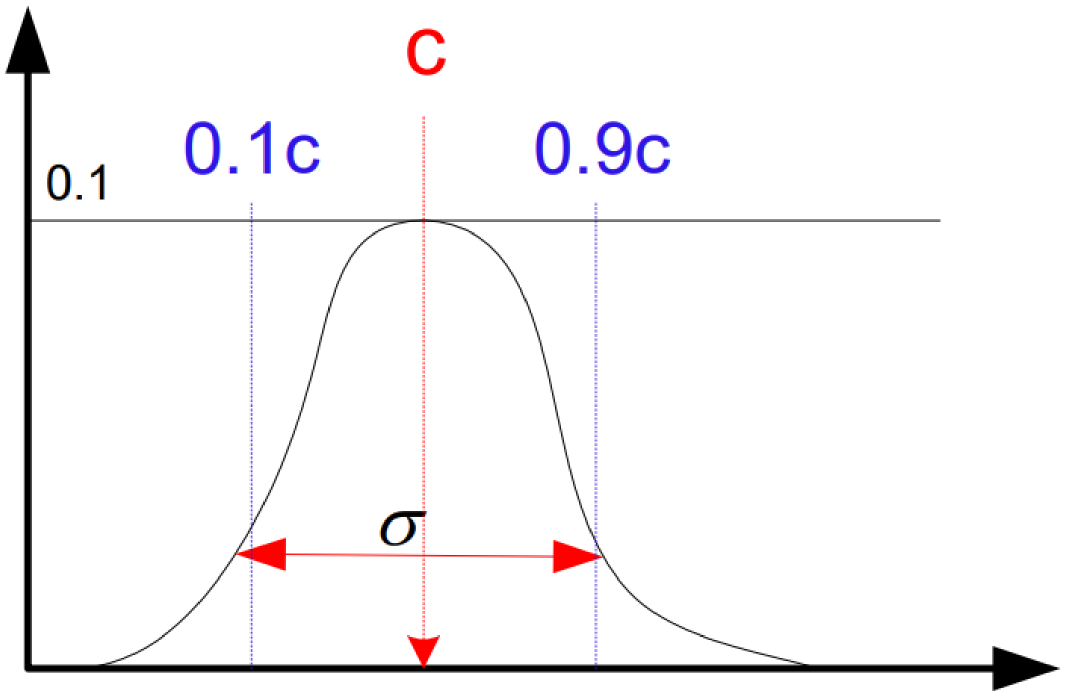
$$trapeziod(x; a, b, c, d) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } b \leq x \leq c \\ \frac{d-x}{d-c} & \text{if } c \leq x \leq d \\ 0 & \text{if } d \leq x \end{cases}$$



Fuzzy MFs: Gaussian

- A Gaussian MF is specified by two parameters {c, σ} and can be defined as below:

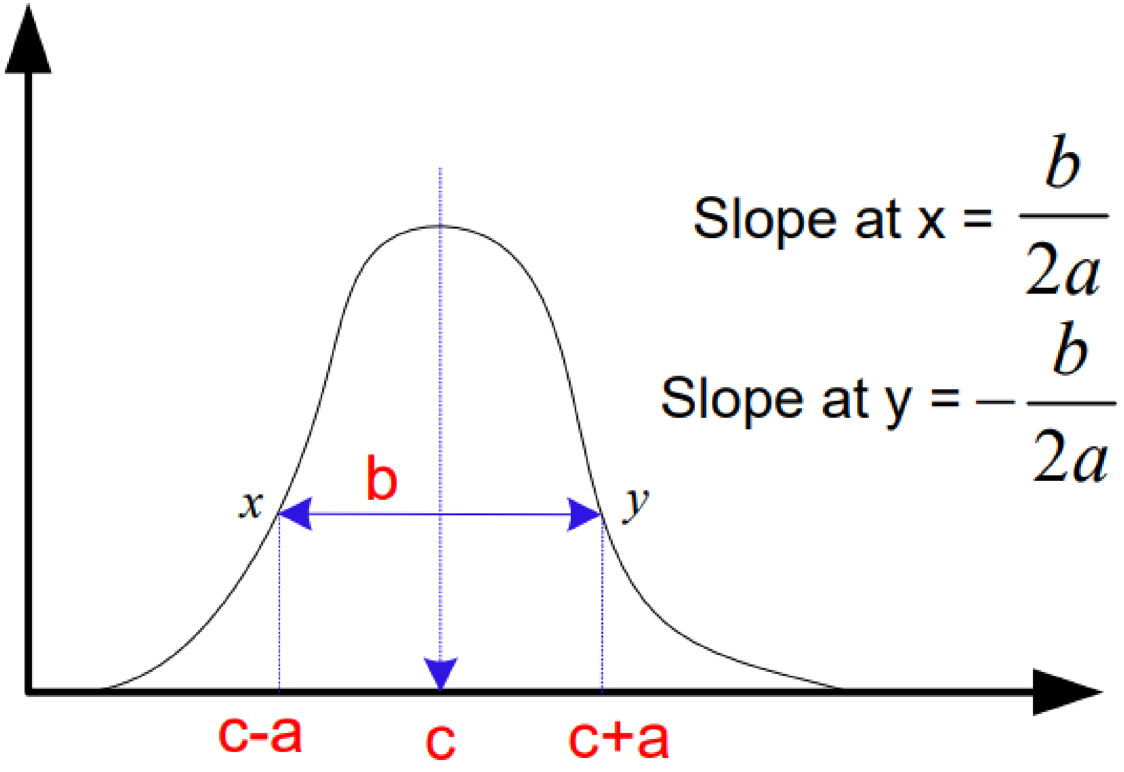
$$gaussian(x: c, \sigma) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}$$



Fuzzy MFs: Generalized bell

- It is also called ***Cauchy MF***. A generalized bell MF is specified by three parameters {a, b, c} and is defined as:

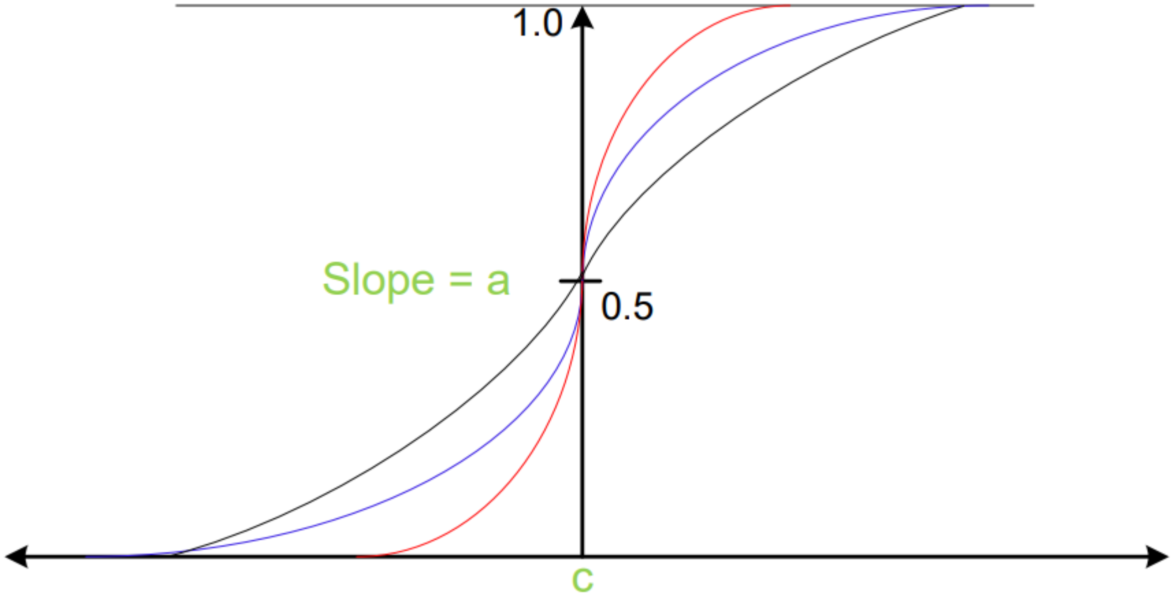
$$bell\,(x\colon a,b,c) = \frac{1}{1+\left|\frac{x-c}{a}\right|^{2b}}$$



Fuzzy MFs: Sigmoidal MFs

- Parameters: {a, c} ; where c = crossover point and a = slope at c;

$$sigmoidal\,(x{:}\,a, c) = \frac{1}{1 + e^{-\frac{a}{x-c}}}$$



Fuzzy set operations: Union

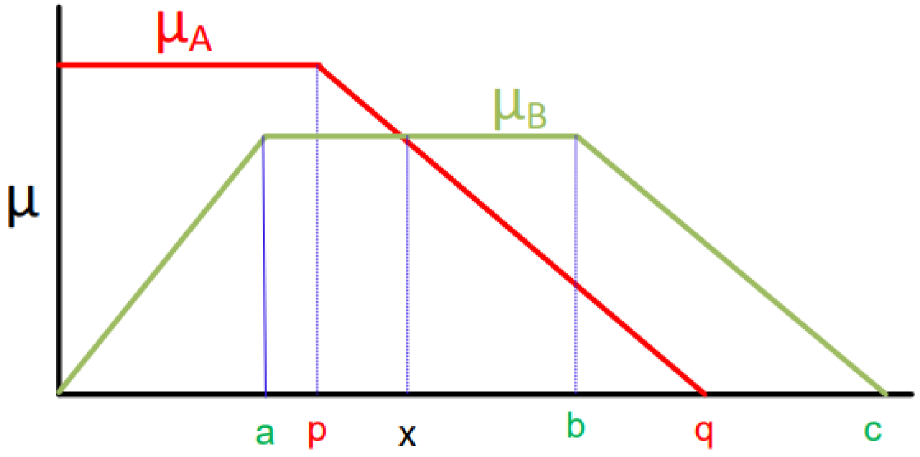
- Union (A ∪ B):

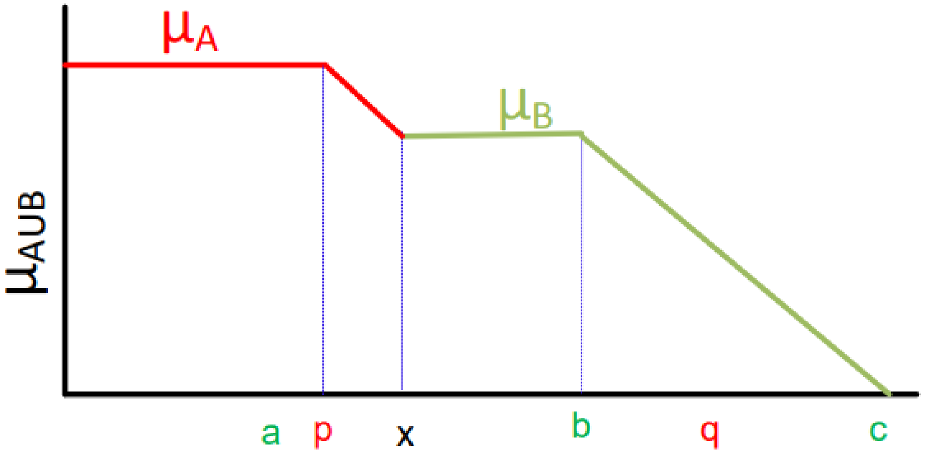
$$\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$$

Example: $A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$

$B = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5)\};$

$A \cup B = \{(x_1, 0.5), (x_2, 0.3), (x_3, 0.5)\}$





Fuzzy set operations: Intersection

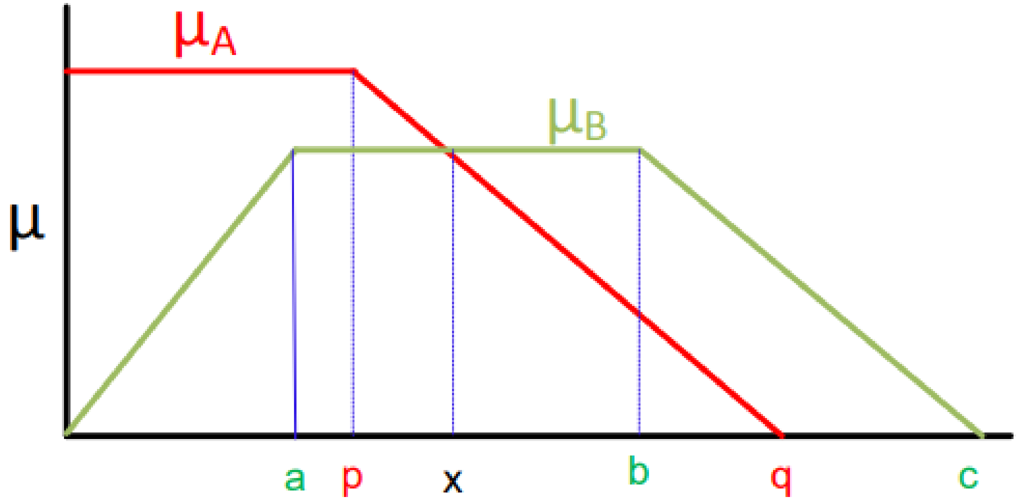
- Intersection ($A \cap B$):

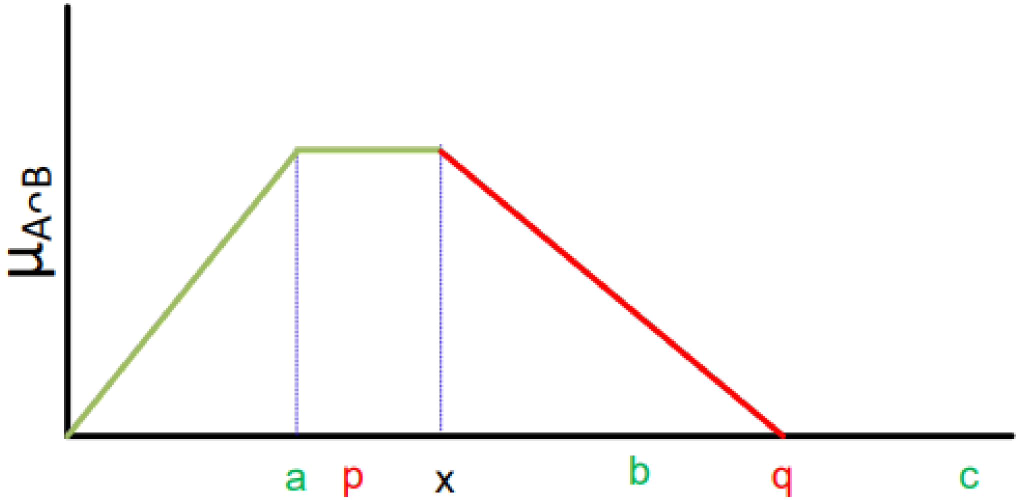
$$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$$

Example: $A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$

$B = \{(x1, 0.2), (x2, 0.3), (x3, 0.5)\}$

$A \cap B = \{(x_1, 0.2), (x_2, 0.1), (x_3, 0.4)\}$





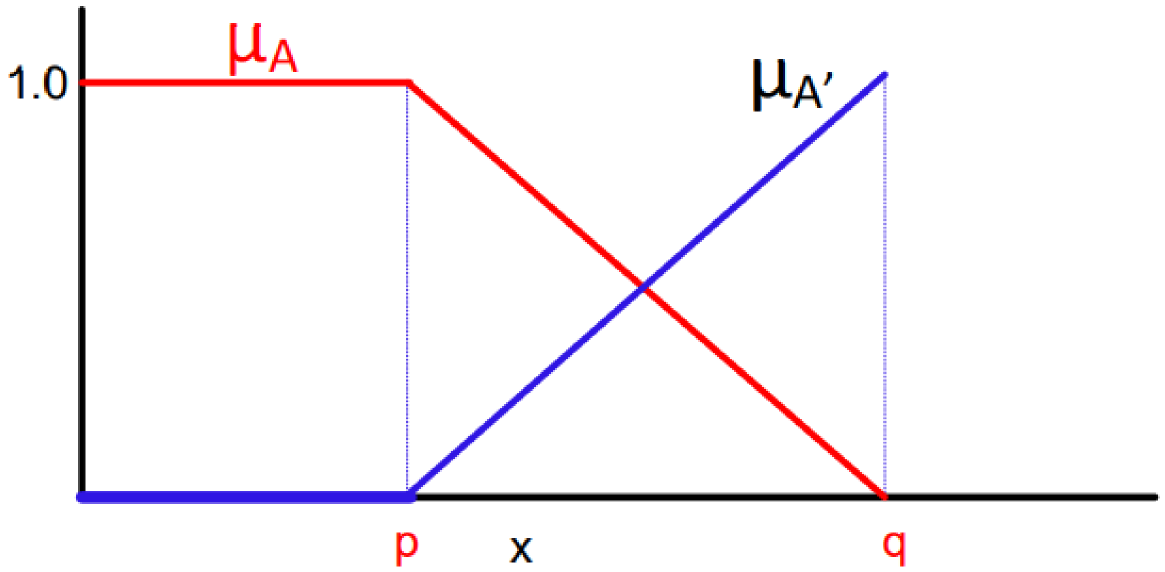
Fuzzy set operations: Complement

- Complement (A^C):

$$\mu_A^{\ C}(x) = 1-\mu_A(x)$$

- Example: $A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$

  $A^C = \{(x_1, 0.5), (x_2, 0.9), (x_3, 0.6)\}$



Fuzzy set operations: Products

- Algebraic product or Vector product

$$(A \bullet B): \mu_{A \bullet B}(x) = \mu_A(x) \bullet \mu_B(x)$$

Example: $A = \{(x_1, 0.5), (x_2, 0.2), (x_3, 0.4)\}$

$B = \{(x_1, 0.1), (x_2, 0.3), (x_3, 0.5)\};$

$A \bullet B = \{(x_1, 0.05), (x_2, 0.06), (x_3, 0.02)\}$

- Scalar product ($\alpha \times A$):

$$\mu\alpha_A(x) = \alpha \cdot \mu_A(x)$$

- Sum ($A + B$):

$$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)$$

Fuzzy set operations

- Difference ($A - B = A \cap B^C$):

  $$\mu_{A-B}(x) = \mu_{A \cap BC}(x)$$

- Disjunctive sum: $A \oplus B = (A^C \cap B) \cup (A \cap B^C))$

- Bounded Sum: $|A(x) \oplus B(x)|$

  $$\mu_{|A(x) \oplus B(x)|} = \min\{1, \mu_A(x) + \mu_B(x)\}$$

- Bounded Difference: $|A(x)\ B(x)|$

  $$\mu_{|A(x)\ B(x)|} = \max\{0, \mu_A(x) + \mu_B(x) - 1\}$$

Fuzzy set operations: Equality and Power

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- Equality (A = B):

$$\mu_A(x) = \mu_B(x)$$

- Power of a fuzzy set A α:

$$\mu_{A\alpha}(x) = \{\mu_A(x)\}\,\alpha$$

If $\alpha < 1$, then it is called dilation

If $\alpha > 1$, then it is called concentration

Generation of MFs

- Given a membership function of a fuzzy set representing a linguistic hedges, we can derive many more MFs representing several other linguistic hedges using the concept of Concentration and Dilation.
- **Concentration:**

$$\boldsymbol{A}^k = [\mu_A(x)]^k \; ; \; k > 1$$

- **Dilation:**

$$\boldsymbol{A}^k = [\mu_A(x)]^k \; ; \; k < 1$$

For Example: Old we can have : old, very old, very very old, extremely old etc.
Very old = $(\text{old})^2$ and so on
Or, less old = $(\text{old})^{0.5}$

Basic fuzzy set operations: Cartesian product

Caretsian Product ($A \times B$):

$$\mu_{A\times B}(x, y) = min\{\mu_A(x), \mu_B(y)\}$$

Example 3:

A(x) = $\{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5), (x_4, 0.6)\}$

B(y) = $\{(y_1, 0.8), (y_2, 0.6), (y_3, 0.3)\}$

$$A \times B = \min\{\mu_A(x), \mu_B(y)\} = \begin{array}{cc} & \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} & \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.3 & 0.3 & 0.3 \\ 0.5 & 0.5 & 0.3 \\ 0.6 & 0.6 & 0.3 \end{bmatrix} \end{array}$$

Properties of fuzzy sets

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Commutativity :

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

Associativity :

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributivity :

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Idempotence :

$$A \cup A = A$$
$$A \cap A = \emptyset$$
$$A \cup \emptyset = A$$
$$A \cap \emptyset = \emptyset$$

Transitivity :

If $A \subseteq B, B \subseteq C$ then $A \subseteq C$

Involution :

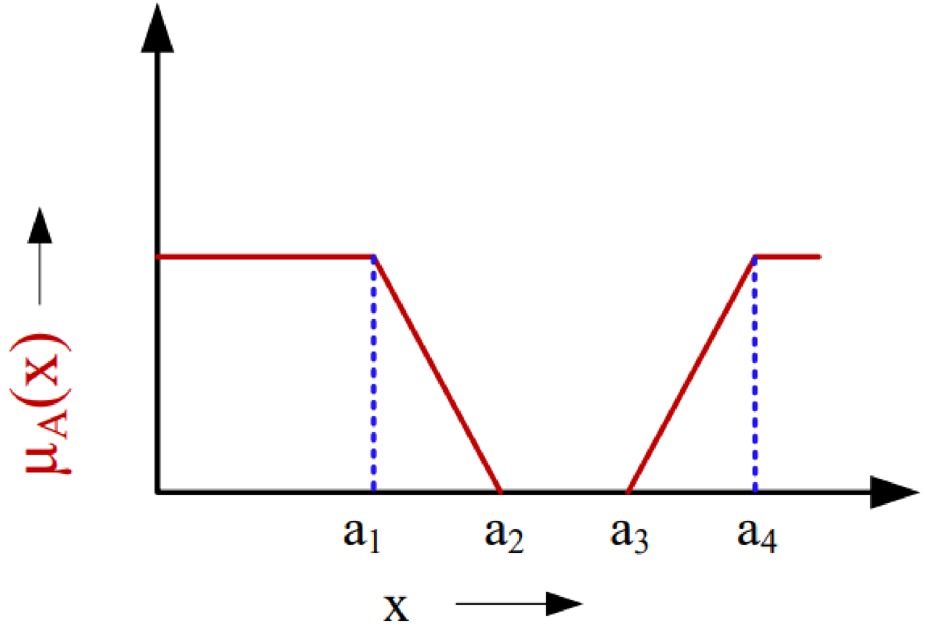
$$(A^c)^c = A$$

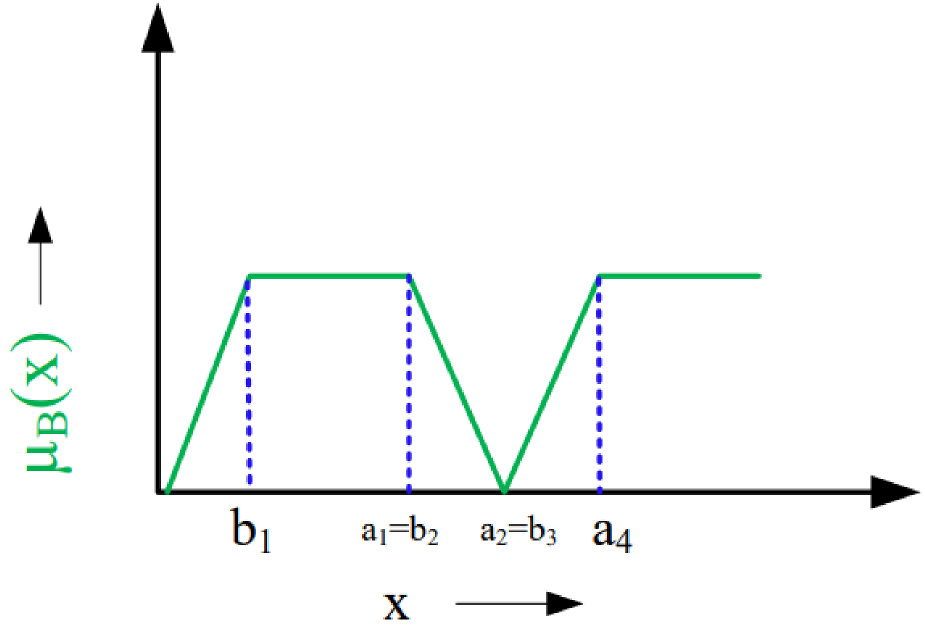
De Morgan's law :

$$(A \cap B)^c = A^c \cup B^c$$
$$(A \cup B)^c = A^c \cap B^c$$

Example 1: Fuzzy Set Operations

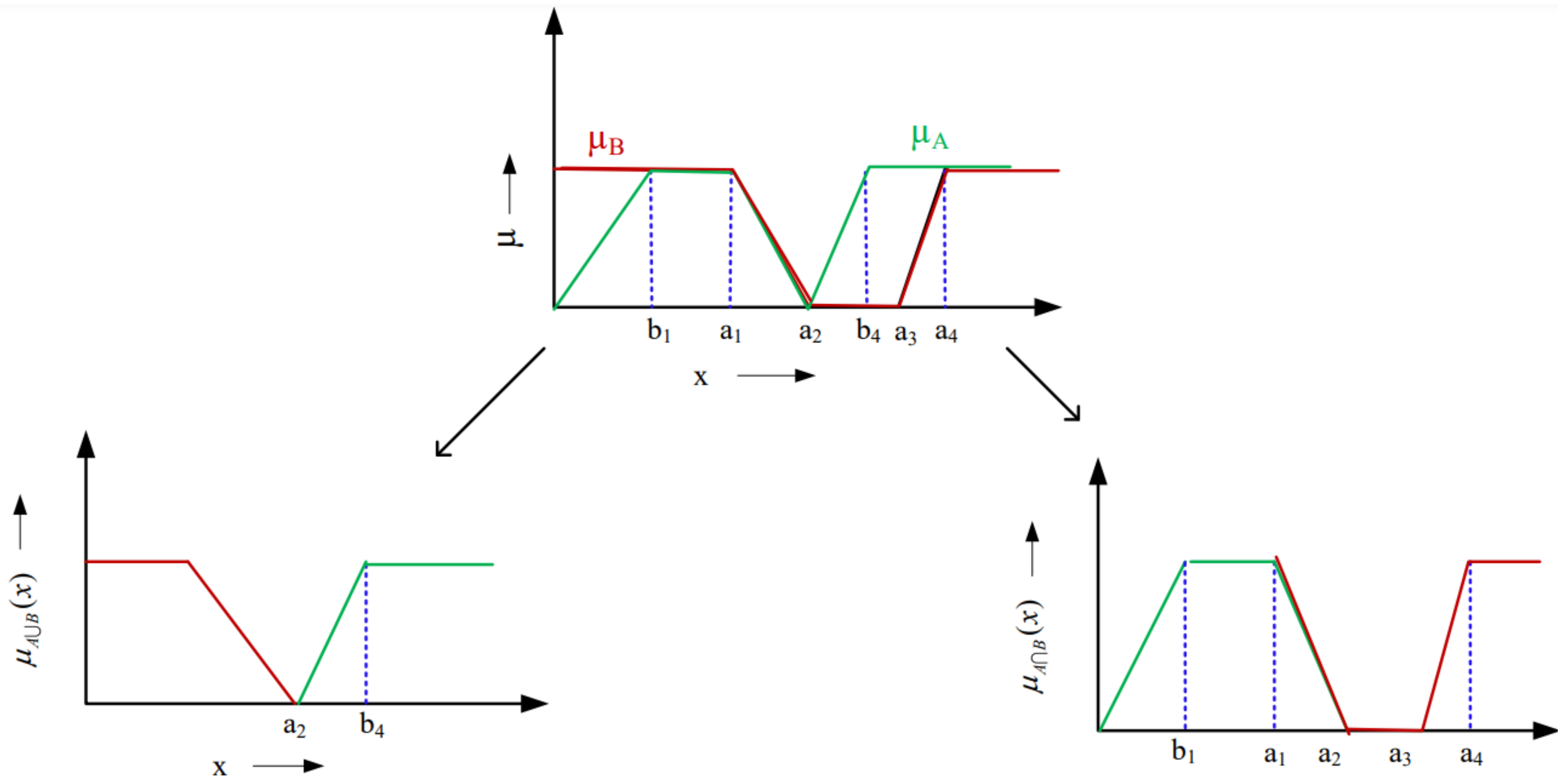
- Let A and B are two fuzzy sets defined over a universe of discourse X with membership functions $\mu_A(x)$ and $\mu_B(x)$, respectively. Two MFs $\mu_A(x)$ and $\mu_B(x)$ are shown graphically.





Example (Cont..)

- The plots of union A ∪ B and intersection A ∩ B are shown in the following.

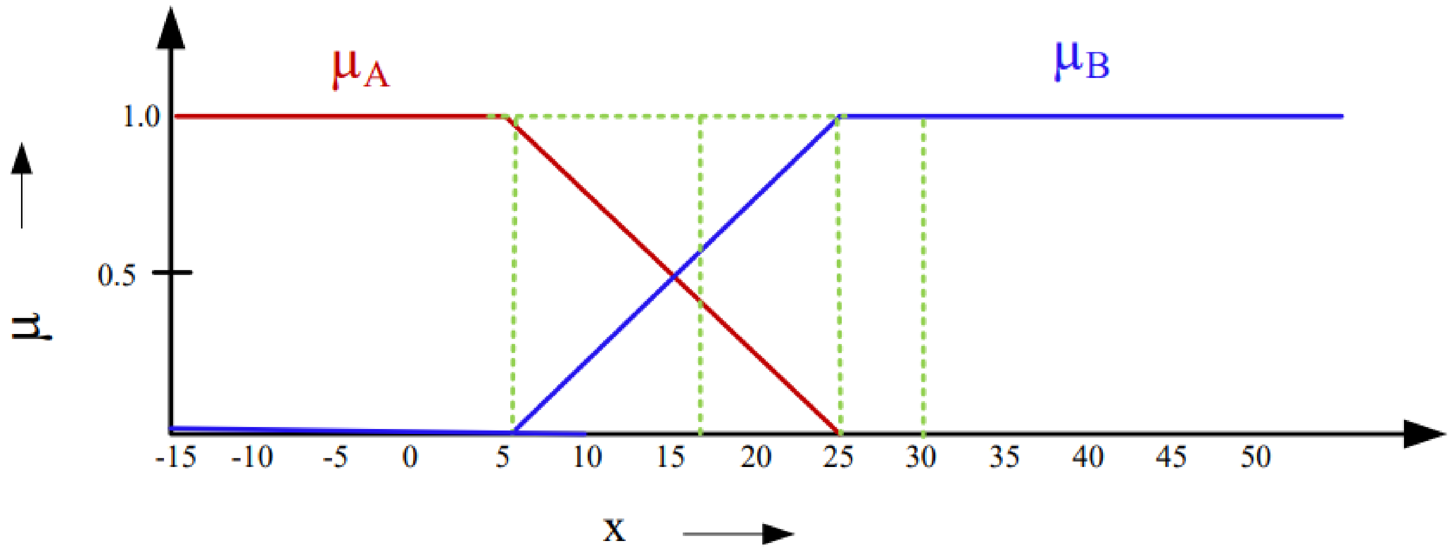


Example 2: A real-life example

- Two fuzzy sets A and B with membership functions $\mu_A(x)$ and $\mu_B(x)$, respectively defined as below.

  A = **Cold climate** with $\mu_A(x)$ as the MF.

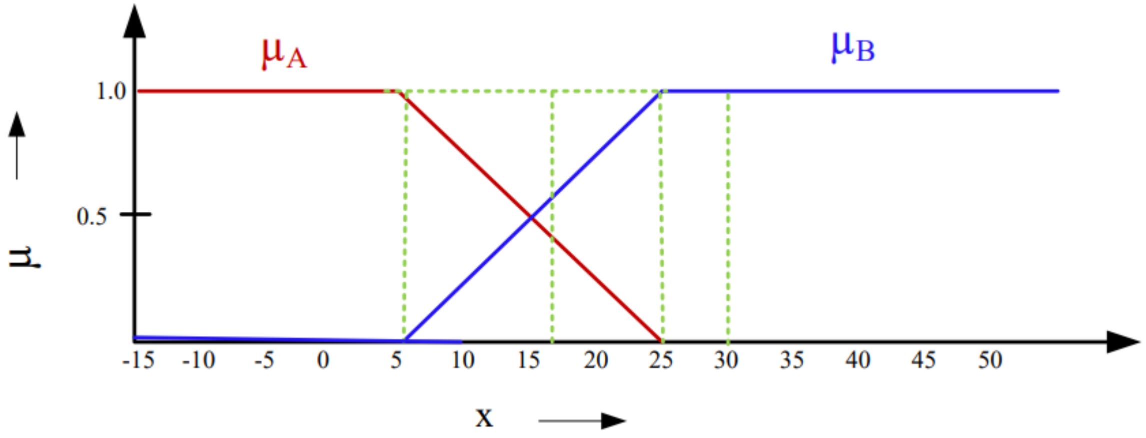
  B = **Hot climate** with $\mu_B(x)$ as the M.F.

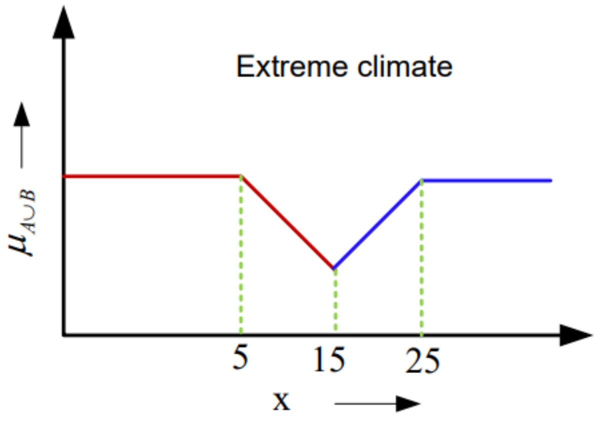


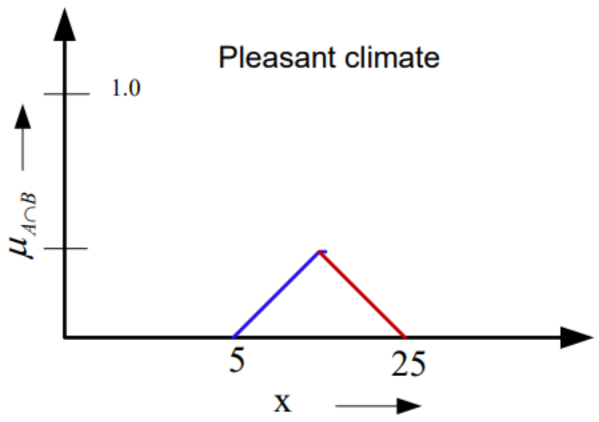
Ex2 (cont..)

Find fuzzy sets of

- Not cold climate
- Not hold climate
- Extreme climate
- Pleasant climate







Fuzzy Relation

- To understand the fuzzy relations, it is better to discuss first crisp relation. Suppose, A and B are two (crisp) sets.

The Cartesian product denoted as $A \times B$ is a collection of order pairs, such that $A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$

Ex 1: Consider the two crisp sets A and B as given below.

$A = \{1, 2, 3, 4\}$; $B = \{3, 5, 7\}$.

$A \times B = \{(1, 3),(1, 5),(1, 7),(2, 3),(2, 5),(2, 7),(3, 3),(3, 5), (3, 7),(4, 3),(4, 5),(4, 7)\}$

Ex1 (Cont..)

A × B = {(1, 3),(1, 5),(1, 7),(2, 3),(2, 5),(2, 7),(3, 3),(3, 5), (3, 7),(4, 3),(4, 5),(4, 7)}

Let us define a relation R as R = {(a, b)|b = a + 1,(a, b) ∈ A × B}

Then, R = {(2, 3),(4, 5)}

We can represent the relation R in a matrix form as follows.

$$R = \begin{array}{c c} & \begin{array}{ccc} 3 & 5 & 7 \end{array} \ \text{B} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ \text{A} & \end{array}$$

Crisp Set Operations

- Let, $R(x, y)$ and $S(x, y)$ are the two relations define over two crisp sets $x \in A$ and $y \in B$

Union: $R(x, y) \cup S(x, y) = \max(R(x, y), S(x, y))$;

Intersection: $R(x, y) \cap S(x, y) = \min(R(x, y), S(x, y))$;

Complement: $R^c(x, y) = 1 - R(x, y)$

Ex:

- Suspose, R(x, y) and S(x, y) are the two relations define over two crisp sets x ∈ A and y ∈ B

$$R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}; s = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Find:

  a) R ∪ S

  b) R ∩ S

  c) R^c

Composition of two crisp relations

- Given R is a relation on X, Y and S is another relation on Y, Z.

  Then, $R \circ S$ is called a composition of relation on X and Z which is defined as

$$R \circ S = \{(x, z) \mid (x, y) \in R \text{ and } (y, z) \in S \text{ and } \forall y \in Y\}$$

Max – Min Composition is used to find $R \circ S$

It is defined as $T = R \circ S$;

$$T(x, z) = \max\{\min\{R(x, y), S(y, z) \text{ and } \forall y \in Y\}\}$$

Ex:

- Given $X = \{1, 3, 5\}$; $Y = \{1, 3, 5\}$; R and S is on $X \times Y$.

  $R = \{(x, y) \mid y = x + 2\};$

  $S = \{(x, y) \mid x < y\}$

$R = \{(1, 3),(3, 5)\}$; $S = \{(1, 3),(1, 5),(3, 5)\}$

$R \circ S = \max\{\min\{R(x, y), S(y, z) \text{ and } \forall y \in Y\}\}$

Using max-min composition, $R \circ S = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Fuzzy Cartesian product

- A is a fuzzy set on the universe of discourse X with $\mu A(x) | x \in X$
- B is a fuzzy set on the universe of discourse Y with $\mu B(y) | y \in Y$
- Then, $R = A \times B \subset X \times Y$; where R has its membership function given by

$$\mu_R(x, y) = \mu_{A\times B}(x, y) = \min\{\mu_A(x), \mu_B(y)\}$$

- Example : A = {(a1, 0.2),(a2, 0.7),(a3, 0.4)} and B = {(b1, 0.5),(b2, 0.6)}

$$R = A \times B = \begin{bmatrix} 0.2 & 0.2 \\ 0.5 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$$

Operations on Fuzzy Relations:

Let, R and S be two fuzzy relations on A × B.

- Union: $\mu_{R \cup S}(a, b) = \max\{\mu_R(a, b), \mu_S(a, b)\}$

- Intersection: $\mu_{R \cap S}(a, b) = \min\{\mu_R(a, b), \mu_S(a, b)\}$

- Complement: $\mu_{R^c}(a, b) = 1 - \mu_R(a, b)$

- Composition $T = R \circ S$

$$\mu R \circ S = \max_{y \in Y} \{\min(\mu_R(x, y), \mu_S(y, z))\}$$

Example:

$X = (x_1, x_2, x_3)$; $Y = (y_1, y_2)$; $Z = (z_1, z_2, z_3)$;

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\!\!\begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0.9 \\ 0.8 & 0.6 \end{bmatrix} \end{array}$$

$$S = \begin{array}{c} \\ y_1 \\ y_2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.6 & 0.4 & 0.7 \\ 0.5 & 0.8 & 0.9 \end{bmatrix} \end{array}$$

$$R \circ S = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.5 & 0.4 & 0.5 \\ 0.5 & 0.8 & 0.9 \\ 0.6 & 0.6 & 0.7 \end{bmatrix} \end{array}$$

$\mu_{R \circ S}(x_1, y_1) = max\{min(x_1, y_1), min(y_1, z_1), min(x_1, y_2), min(y_2, z_1)\}$
$= max\{min(0.5, 0.6), min(0.1, 0.5)\} = max\{0.5, 0.1\} = 0.5$ and so on.

Fuzzy relation : Example

- Let, R = x is relevant to y;

  S = y is relevant to z;

  are two fuzzy relations defined on X × Y and Y × Z, respectively,

  where X = {1, 2, 3} ,Y = {α, β, γ, δ} and Z = {a, b}.

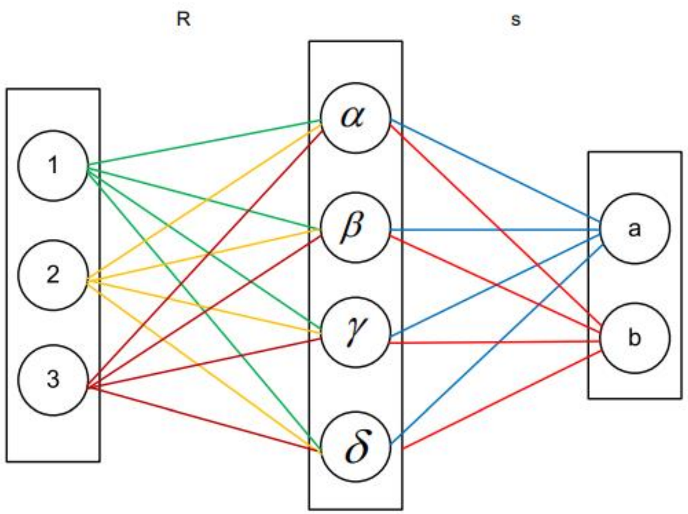
  Assume that R and S can be expressed with the following relation matrices :

$$R = \begin{array}{c c} & \begin{array}{cccc} \alpha & \beta & \gamma & \delta \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0.1 & 0.3 & 0.5 & 0.7 \\ 0.4 & 0.2 & 0.8 & 0.9 \\ 0.6 & 0.8 & 0.3 & 0.2 \end{bmatrix} \end{array} \qquad S = \begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} \alpha \\ \beta \\ \gamma \\ \delta \end{array} & \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.3 \\ 0.5 & 0.6 \\ 0.7 & 0.2 \end{bmatrix} \end{array}$$

Fuzzy relation : Example

- Now, we want to find R ◦ S, which can be interpreted as a derived fuzzy relation x is relevant to z.
- Suppose, we are only interested in the degree of relevance between 2 ∈ X and a ∈ Z. Then, using max-min composition,

$$\mu_{R \circ S}(2, a) = max\{(0.4 \wedge 0.9), (0.2 \wedge 0.2), (0.8 \wedge 0.5), (0.9 \wedge 0.7)\}$$
$$= max\{0.4, 0.2, 0.5, 0.7\} = 0.7$$



$$R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} \alpha & \beta & \gamma & \delta \end{array} \\ \begin{bmatrix} 0.1 & 0.3 & 0.5 & 0.7 \\ 0.4 & 0.2 & 0.8 & 0.9 \\ 0.6 & 0.8 & 0.3 & 0.2 \end{bmatrix} \end{array}$$

$$S = \begin{array}{c} \\ \alpha \\ \beta \\ \gamma \\ \delta \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.3 \\ 0.5 & 0.6 \\ 0.7 & 0.2 \end{bmatrix} \end{array}$$

Fuzzy relation : Example

- Consider the following two sets P and D, which represent a set of paddy plants and a set of plant diseases.

  P = {P1, P2, P3, P4} a set of four varieties of paddy plants.

  D = {D1, D2, D3, D4} of the four various diseases affecting the plants.

In addition, also consider another set S = {S1, S2, S3, S4} be the common symptoms of the diseases.

Let, R be a relation on P × D, representing which plant is susceptible to which diseases; and S between Disease and Symptoms

$$R = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{c} \begin{array}{cccc} D_1 & D_2 & D_3 & D_4 \end{array} \\ \begin{bmatrix} 0.6 & 0.6 & 0.9 & 0.8 \\ 0.1 & 0.2 & 0.9 & 0.8 \\ 0.9 & 0.3 & 0.4 & 0.8 \\ 0.9 & 0.8 & 0.4 & 0.2 \end{bmatrix} \end{array} \qquad S = \begin{array}{c} \\ D_1 \\ D_2 \\ D_3 \\ D_4 \end{array} \begin{array}{c} \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{bmatrix} 0.1 & 0.2 & 0.7 & 0.9 \\ 1.0 & 1.0 & 0.4 & 0.6 \\ 0.0 & 0.0 & 0.5 & 0.9 \\ 0.9 & 1.0 & 0.8 & 0.2 \end{bmatrix} \end{array}$$

Example (Cont..)

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- The association of plants with the different symptoms of the disease using max-min composition.

$$R \circ S = \begin{array}{c c} & \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{array}{c} P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} & \begin{bmatrix} 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.7 & 0.9 \end{bmatrix} \end{array}$$

2D membership function : An example

Let, $X = R^+ = y$ (the positive real line)
and $R = X \times Y$ = ”y is much greater than x”

The membership function of $\mu_R(x, y)$ is defined as

$$\mu_R(x, y) = \begin{cases} \frac{(y-x)}{4} & if \quad y > x \\ 0 & if \quad y \leq x \end{cases}$$

Suppose, $X = \{3, 4, 5\}$ and $Y = \{3, 4, 5, 6, 7\}$, then

$$R = \begin{array}{c} \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 0.25 & 0.5 & 0.75 & 1.0 \\ 0 & 0 & 0.25 & 0.5 & 0.75 \\ 0 & 0 & 0 & 0.25 & 0.5 \end{bmatrix} \end{array}$$

Example:

- How to represent the following?

If x is A or y is B then z is C;

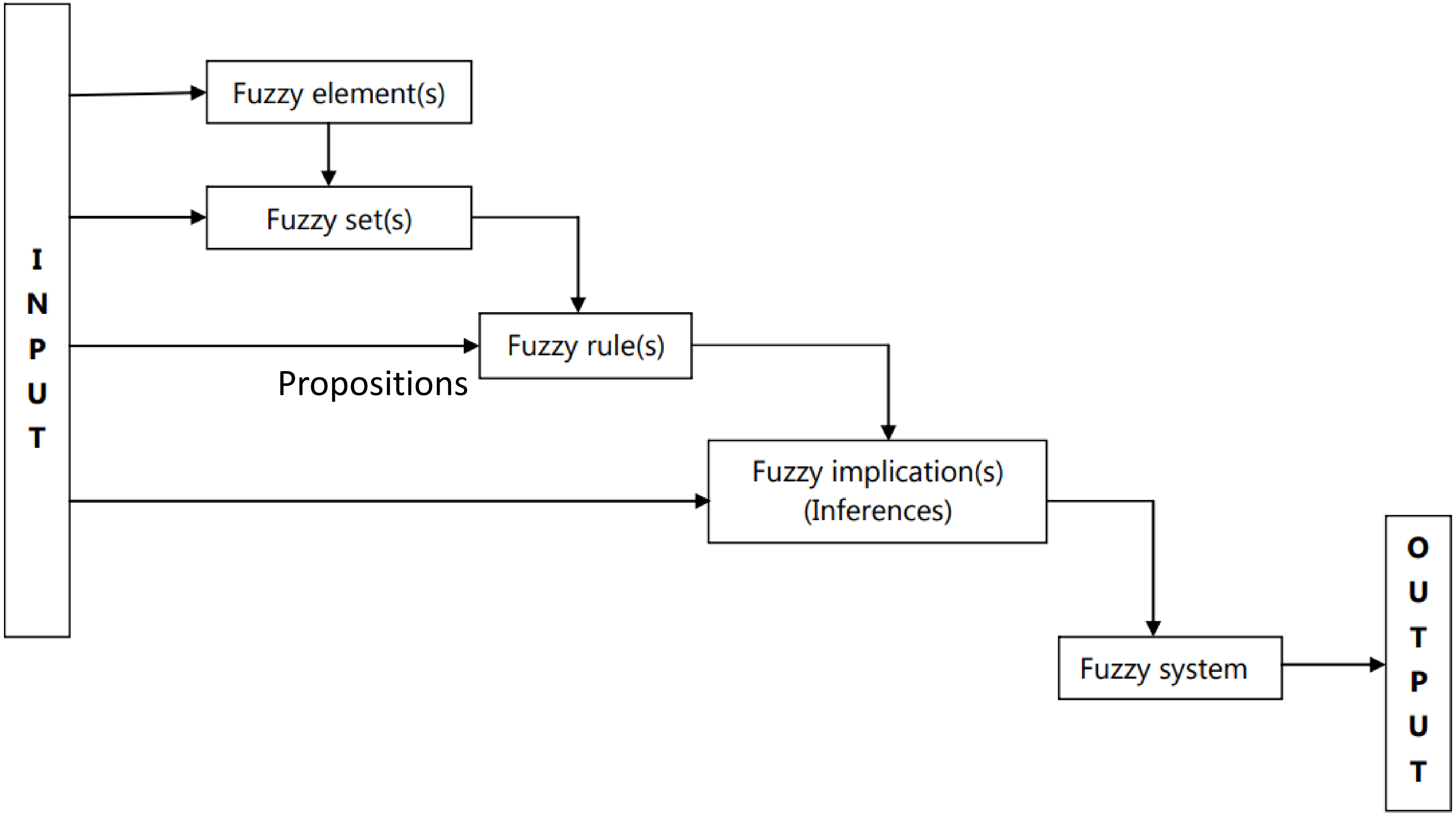
Given that

R1: If x is A then z is c [R1 ∈ A × C]

R2: If y is B then z is C [R2 ∈ B × C]

Hint: You have given two relations R1 and R2.

Fuzzy System



Fuzzy Propositions

- What is a proposition?

  - a statement or assertion that expresses a judgement or opinion

- Binary or Two Valued Proposition (Crisp Logic and Boolean Logic)

- Fuzzy or Multi-valued logic Proposition
- The basic assumption upon which crisp logic is based - that every proposition is either TRUE or FALSE.
- The classical two-valued logic can be extended to multi-valued logic.
  Ex: Three valued logic to denote true(1), false(0) and indeterminacy (1/2)

Three-valued logic

Fuzzy connectives defined for such a three-valued logic better can be stated as follows:

Symbol	Connective	Usage	Definition
$\neg$	NOT	$\neg$P	$1 - T(P)$
$\vee$	OR	$P \vee Q$	max{T(P), T(Q) }
$\wedge$	AND	$P \wedge Q$	min{ T(P),T(Q) }
$\Longrightarrow$	IMPLICATION	$(P \Longrightarrow Q)$ or $(\neg P \vee Q)$	max{(1 - T(P)), T(Q) }
$=$	EQUALITY	$(P = Q)$ or $[(P \Longrightarrow Q) \wedge (Q \Longrightarrow P)]$	$1 - \lvert T(P) - T(Q) \rvert$

Three valued Logic

- Different operations with three-valued logic can be extended as shown in the following truth table:

a	b	$\wedge$	$\vee$	$\neg$a	$\Longrightarrow$	$=$
0	0	0	0	1	1	1
0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	1	$\frac{1}{2}$
0	1	0	1	1	1	0
$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
$\frac{1}{2}$	1	$\frac{1}{2}$	1	$\frac{1}{2}$	1	$\frac{1}{2}$
1	0	0	1	1	0	0
1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$
1	1	1	1	1	1	1

AND (∧), OR (∨), NOT (¬), IMPLICATION (⇒) and EQUAL (=)

Fuzzy Proposition

- Example 1:

  P : Ram is honest

1) T(P) = 0.0 : Absolutely false
2) T(P) = 0.2 : Partially false
3) T(P) = 0.4 : May be false or not false
4) T(P) = 0.6 : May be true or not true
5) T(P) = 0.8 : Partially true
6) T(P) = 1.0 : Absolutely true.

Fuzzy Proposition: Example 2:

P : Mary is efficient ; $T(P) = 0.8$;

Q : Ram is efficient ; $T(Q) = 0.6$

1)Mary is not efficient.

$$T(\neg P) = 1 - T(P) = 0.2$$

2) Mary is efficient and so is Ram.

$$T(P \wedge Q) = \min\{T(P), T(Q)\} = 0.6$$

3) Either Mary or Ram is efficient

$$T(P \vee Q) = \max\{T(P), T(Q)\} = 0.8$$

4) If Mary is efficient then so is Ram

$$T(P \Rightarrow Q) = \max\{1 - T(P), T(Q)\} = 0.6$$

Quick Recap

- Fuzzy proposition vs. Crisp proposition

- The fundamental difference between crisp (classical) proposition and fuzzy propositions is in the range of their truth values.
- While each classical proposition is required to be either true or false, the truth or falsity of fuzzy proposition is a matter of degree.
- The degree of truth of each fuzzy proposition is expressed by a value in the interval [0,1] both inclusive.

Canonical representation of Fuzzy proposition

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- Suppose, X is a universe of discourse of five persons. Intelligent of $x \in X$ is a fuzzy set as defined below.

  Intelligent: {(x1, 0.3),(x2, 0.4),(x3, 0.1),(x4, 0.6),(x5, 0.9)}

- We define a fuzzy proposition as follows:

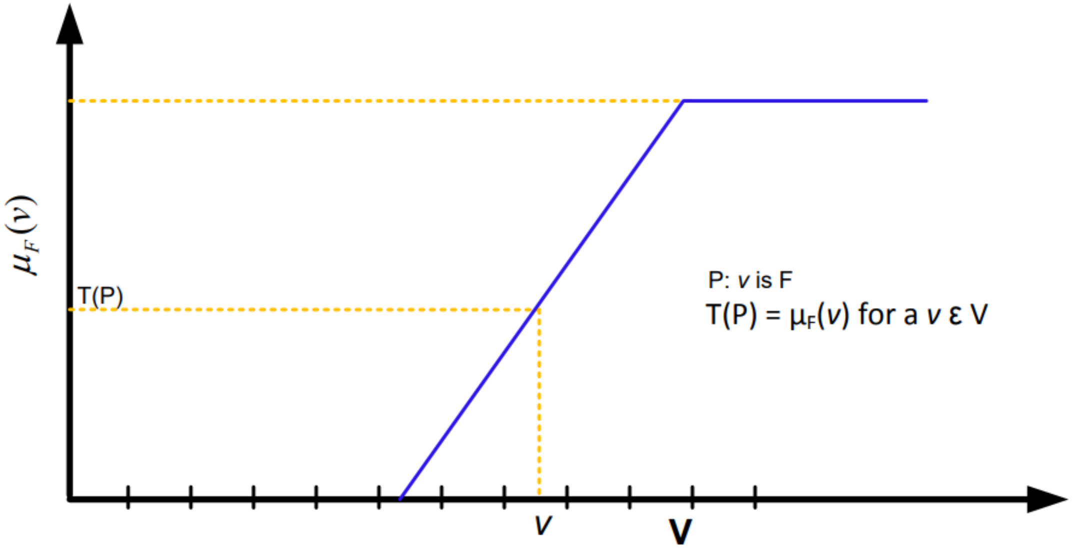
  P : x is intelligent   T(P) = ?

- The canonical form of fuzzy proposition of this type, P is expressed by the sentence P : v is F .
- Predicate in terms of fuzzy set.

  P : v is F ; where v is an element that takes values v from some universal set V and F is a fuzzy set on V that represents a fuzzy predicate.

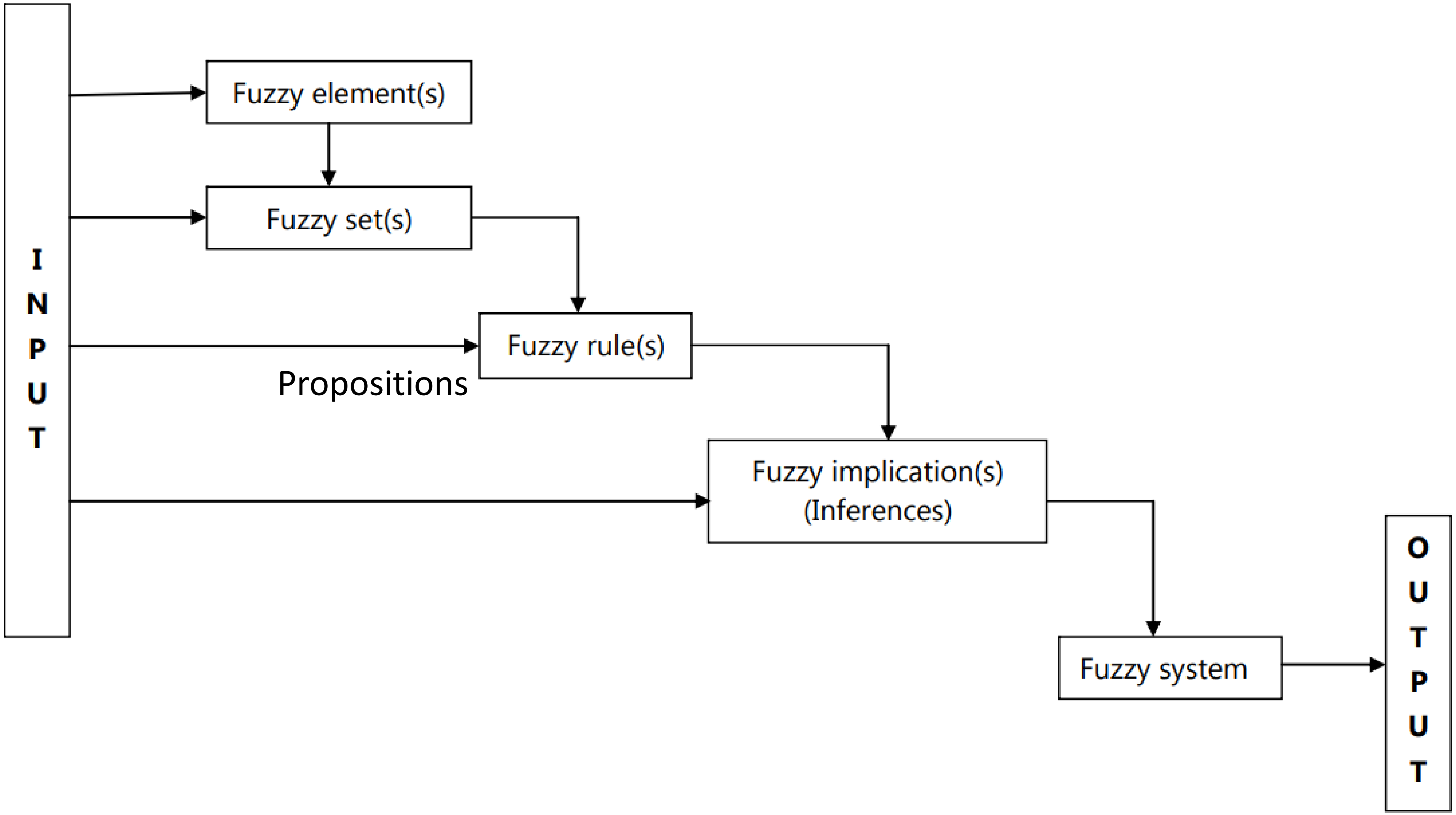
Graphical interpretation of fuzzy proposition

- Given a Fuzzy set, a particular element v, this element belongs to F with membership grade μ_F (v)



For a given value *v* of variable V in proposition P, T(P) denotes the degree of truth of proposition P.

Concept of Fuzzy System



Fuzzy Implication

- A fuzzy implication (also known as fuzzy If-Then rule, fuzzy rule, or fuzzy conditional statement) assumes the form :

If x is A then y is B

where, A and B are two linguistic variables defined by fuzzy sets A and B on the universe of discourses X and Y, respectively.

- Often, x is A is called the antecedent or premise, while y is B is called the consequence or conclusion.

Fuzzy implication : Example 1

- If pressure is High then temperature is High

- If mango is Yellow then mango is Sweet else mango is Sour

  (if then else – another structure)

- If road is Good then driving is Smooth else traffic is High

- The fuzzy implication is denoted as $R : A \to B$

  (Notation: If x is A then y is B)

- In essence, it represents a binary fuzzy relation R on the (Cartesian) product of $A \times B$

Fuzzy implication : Example 2

- Suppose, P and T are two universes of discourses representing pressure and temperature, respectively as follows.

- P = { 1,2,3,4} and T ={ 20, 25, 30, 35, 40, 45, 50 }

- Let the linguistic variable High temperature and Low pressure are given as

T_{HIGH} = {(20, 0.2),(25, 0.4),(30, 0.6),(35, 0.6),(40, 0.7),(45, 0.8),(50, 0.8)}

P_{LOW} = (1, 0.8),(2, 0.8),(3, 0.6),(4, 0.4)

Fuzzy implication : Example 2

- Then the fuzzy implication If temperature is High then pressure is Low can be defined as

T_{HIGH} = {(20, 0.2),(25, 0.4),(30, 0.6),(35, 0.6),(40, 0.7),(45, 0.8),(50, 0.8)}
P_{LOW} = (1, 0.8),(2, 0.8),(3, 0.6),(4, 0.4)

$R : T_{HIGH} \rightarrow P_{LOW}$

$$R = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 20 \\ 25 \\ 30 \\ 35 \\ 40 \\ 45 \\ 50 \end{array} & \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 \\ 0.4 & 0.4 & 0.4 & 0.4 \\ 0.6 & 0.6 & 0.6 & 0.4 \\ 0.6 & 0.6 & 0.6 & 0.4 \\ 0.7 & 0.7 & 0.6 & 0.4 \\ 0.8 & 0.8 & 0.6 & 0.4 \\ 0.8 & 0.8 & 0.6 & 0.4 \end{bmatrix} \end{array}$$

Note : If temperature is 40 then what about pressure?

Interpretation of fuzzy rules

- In general, there are two ways to interpret the fuzzy rule A → B as

a) A coupled with B

b) A entails B

For the case of A couple with B

$R : A \rightarrow B = A \times B = \int_{X \times Y} \mu_A(x) * \mu_B(y) | (x,y)$; where $*$ is called a T-norm operator.

Interpretation as A coupled with B

- In the following, few implications of $R : A \rightarrow B$

- **Min operator:** $R_m = A \times B = \int_{X \times Y} \mu_A(x) \wedge \mu_B(y) | _{(x,y)}$ or $f_{min}(a, b) = a \wedge b$

  [Mamdani rule]

- **Algebric product operator:**

  $R_{ap} = A \times B = \int_{X \times Y} \mu_A(x).\mu_B(y) | _{(x,y)}$ or $f_{ap}(a, b) = ab$

  [Larsen rule]

Interpretation as A coupled with B

- **Bounded product operator:**

$$R_{bp} = A \times B = \int_{X \times Y} \mu_A(x) \Theta\, \mu_B(y) |_{(x,y)} =$$

$$\int_{X \times Y} 0 \vee (\mu_A(x) + \mu_B(y) - 1) |_{(x,y)} \text{ or } f_{bp} = 0 \vee (a + b - 1)$$

Drastic product operator

$$R_{dp} = A \times B = \int_{X \times Y} \mu_A(x) \hat{\bullet} \mu_B(y) |_{(x,y)}$$

$$\text{or } f_{dp}(a,b) = \begin{cases} a & \text{if} \quad b = 1 \\ b & \text{if} \quad a = 1 \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

Interpretation of A entails B

- There are three main ways to interpret such implication:

Material implication : R : A → B = $\bar{A}$ ∪ B

Propositional calculus : R : A → B = $\bar{A}$ ∪ (A ∩ B)

Extended propositional calculus : R : A → B = ($\bar{A}$ ∩ $\bar{B}$) ∪ B

Interpretation of A entails B

- With the above mentioned implications, there are a number of fuzzy implication functions that are popularly followed in fuzzy rule-based system.

Zadeh's arithmetic rule :

$$R_{za} = \bar{A} \cup B = \int_{X \times Y} 1 \wedge (1 - \mu_A(x) + \mu_B(y))|_{(x,y)}$$

or

$$f_{za}(a, b) = 1 \wedge (1 - a + b)$$

Zadeh's max-min rule :

$$R_{mm} = \bar{A} \cup (A \cap B) = \int_{X \times Y} (1 - \mu_A(x)) \vee (\mu_A(x) \wedge \mu_B(y))|_{(x,y)}$$

or

$$f_{mm}(a, b) = (1 - a) \vee (a \wedge b)$$

Interpretation of A entails B

Boolean fuzzy rule

$$R_{bf} = \bar{A} \cup B = \int_{X \times Y} (1 - \mu_A(x)) \vee \mu_B(x)|_{(x,y)}$$

or

$$f_{bf}(a, b) = (1 - a) \vee b;$$

Goguen's fuzzy rule:

$$R_{gf} = \int_{X \times Y} \mu_A(x) * \mu_B(y)|_{(x,y)} \text{ where } a * b = \begin{cases} 1 & if \quad a \leq b \\ \frac{b}{a} & if \quad a > b \end{cases}$$

Example: Zadeh's Max-Min rule

- **If x is A then y is B** with the implication of Zadeh's max-min rule can be written equivalently as :

$$R_{mm} = (A \times B) \cup (\bar{A} \times Y)$$

Here, Y is the universe of discourse with membership values for all $y \in Y$ is 1, that is , $\mu_Y(y) = 1 \forall y \in Y$.

Suppose X = {a, b, c, d} and Y = {1, 2, 3, 4}

and A = {(a, 0.0),(b, 0.8),(c, 0.6),(d, 1.0)}

B = {(1, 0.2),(2, 1.0),(3, 0.8),(4, 0.0)} are two fuzzy sets.

Determine $R_{mm} = (A \times B) \cup (\bar{A} \times Y)$

Example: Zadeh's min-max rule:

- $R_{mm} = (A \times B) \cup (\bar{A} \times Y)$

X = {a, b, c, d} and Y = {1, 2, 3, 4}
A = {(a, 0.0),(b, 0.8),(c, 0.6),(d, 1.0)}
B = {(1, 0.2),(2, 1.0),(3, 0.8),(4, 0.0)} are two fuzzy sets.

$$A \times B = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0.8 & 0 \\ 0.2 & 0.6 & 0.6 & 0 \\ 0.2 & 1.0 & 0.8 & 0 \end{bmatrix} \end{array}$$

$$\bar{A} \times Y = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.2 & 0.2 & 0.2 & 0.2 \\ 0.4 & 0.4 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$R_{mm} = (A \times B) \cup (\bar{A} \times Y) = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.2 & 0.8 & 0.8 & 0.2 \\ 0.4 & 0.6 & 0.6 & 0.4 \\ 0.2 & 1.0 & 0.8 & 0 \end{bmatrix} \end{array}$$

Example:

- **If** x is A **then** y is B **else** y is C

Relation R is give by : ? $(A\ X\ B) \cup (\bar{A}\ X\ C)$

$$X = \{a, b, c, d\}$$

$$Y = \{1, 2, 3, 4\}$$

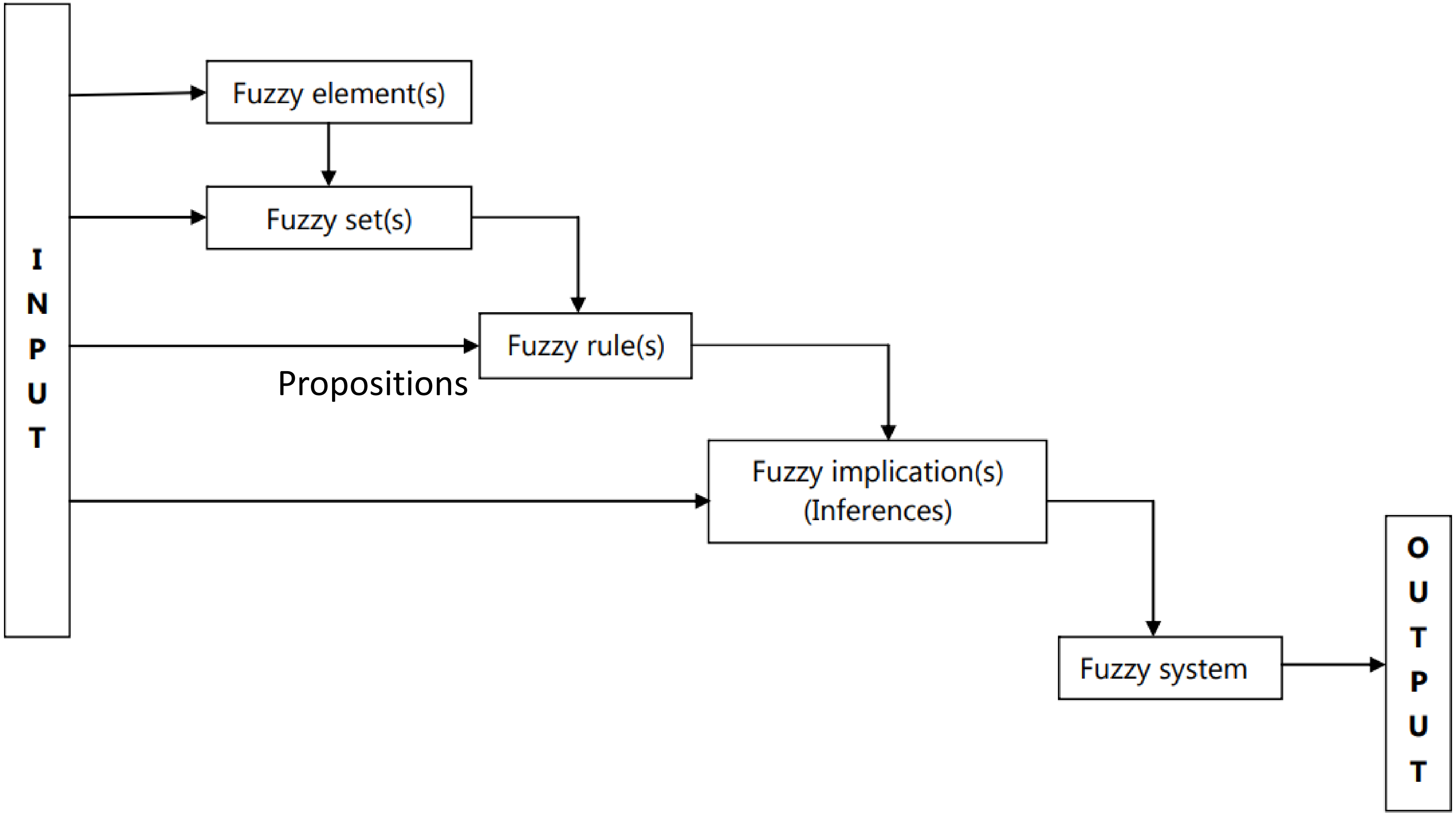
$$A = \{(a, 0.0), (b, 0.8), (c, 0.6), (d, 1.0)\}$$

$$B = \{(1, 0.2), (2, 1.0), (3, 0.8), (4, 0.0)\}$$

$$C = \{(1, 0), (2, 0.4), (3, 1.0), (4, 0.8)\}$$

$$R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0.4 & 1.0 & 0.8 \\ 0.2 & 0.8 & 0.8 & 0.2 \\ 0.2 & 0.6 & 0.6 & 0.4 \\ 0.2 & 1.0 & 0.8 & 0 \end{bmatrix} \end{array}$$

Concept of Fuzzy System



Fuzzy Inference:

- Inference ?

  - If two propositions are given, then defining a new proposition

Let us understand – some operations

Ex: a) if it is raining, then the ground is wet
b) it is raining

Inference : Ground is wet

i) If P, then Q
ii) P
Therefore, Q

Modus Ponens - if A implies B and A is true, then B must also be true.

Fuzzy Inference:

- Ex: a) if it is raining, then the ground is wet

  b) ground is not wet

  Inference: It is not raining

i) If P, then Q

ii) ~ Q

Therefore, ~ P

Modus Tollens - if A implies B and B is false, then A must also be false.

Chain Rule - $P \Rightarrow Q$, $Q \Rightarrow R$, We can infer $P \Rightarrow R$

Fuzzy Inference

- Generalized Modus Ponens (GMP)

  If x is A Then y is B

  x is A

  Inference: y is B

- Generalized Modus Tollens (GMT)

  If x is A Then y is B

  x is B'

  Inference: y is A'

Fuzzy inferring procedures

- If A, B, A’ and B’ are fuzzy sets.
- To compute the membership function A’ and B’ the max-min composition of fuzzy sets B’ and A’, respectively with R(x, y) (which is the known implication relation) is to be used.

- Thus

$B' = A' \circ R(x, y) \qquad \mu_B(y) = \max[\min(\mu_{A'}(x), \mu_R(x, y))]$

$A' = B' \circ R(x, y) \qquad \mu_A(x) = \max[\min(\mu_{B'}(y), \mu_R(x, y))]$

Example: Generalized Modus Tollens

- Modus Tollens (GMT)
- P: If x is A Then y is B
- Q: y is B’
- Inference: x is A’

Ex: Let sets of variables x and y be X = {x1, x2, x3} and y = {y1, y2}, respectively.

- Assume that a proposition If x is A Then y is B given where A = {(x1, 0.5),(x2, 1.0),(x3, 0.6)} and B = {(y1, 0.6),(y2, 0.4)}
- Assume now that a fact expressed by a proposition y is B is given where B’ = {(y1, 0.9),(y2, 0.7)}.
- From the above, we are to conclude that x is A’.

Example: Generalized Modus Tollens

We first calculate $R(x,y) = (A \times B) \cup (\overline{A} \times y)$

$$R(x,y) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} \end{array}$$

Next, we calculate $A' = B' \circ R(x,y)$

$$A' = \begin{bmatrix} 0.9 & 0.7 \end{bmatrix} \circ \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} \end{array} = \begin{bmatrix} 0.5 & 0.9 & 0.6 \end{bmatrix}$$

Hence, we calculate that x is A' where
$A' = [(x_1, 0.5), (x_2, 0.9), (x_3, 0.6)]$

Example

- Apply the fuzzy GMP rule to deduce Rotation is quite slow
- Given that :

  If temperature is High then rotation is Slow.

  Temperature is Very High

Let, X = {30, 40, 50, 60, 70, 80, 90, 100} be the set of temperatures.

Y = {10, 20, 30, 40, 50, 60} be the set of rotations per minute.

Example (Cont…)

- The fuzzy set High(H), Very High (VH), Slow(S) and Quite Slow (QS) are given below.

$H = \{(70, 1),(80, 1),(90, 0.3)\}$

$VH = \{(90, 0.9),(100, 1)\}$

$S = \{(30, 0.8),(40, 1.0),(50, 0.6)\}$

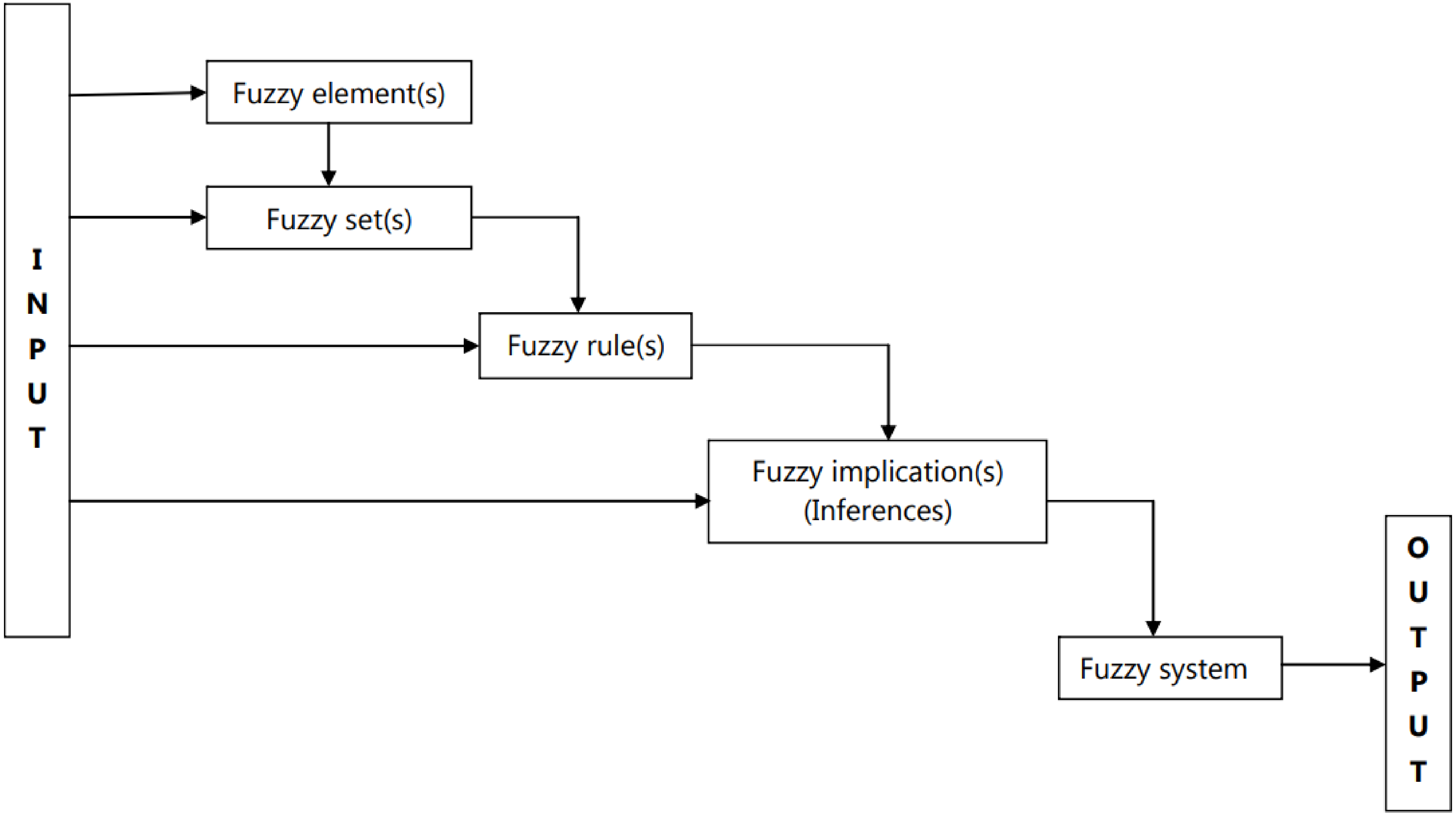
$QS = \{(10, 1),(20, 0.8)\}$

If temperature is High then the rotation is Slow. **R = (H × S) ∪ ($\overline{\mathrm{H}}$ × Y)**

Temperature is Very High

Thus, to deduce ”rotation is Quite Slow”, we make use the composition rule $QS = VH \circ R(x, y)$

Defuzzification



What is defuzzification?

- Defuzzification means the fuzzy to crisp conversion

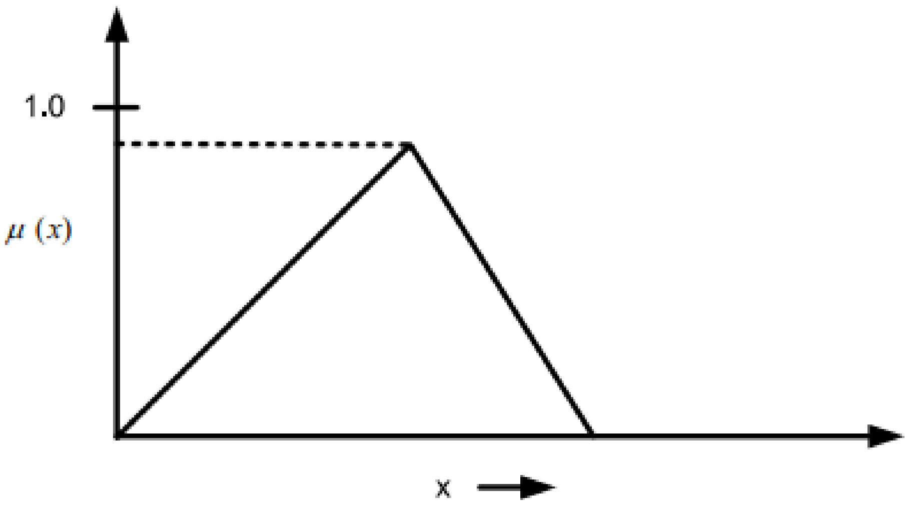
Example 1: Suppose, T_{HIGH} denotes a fuzzy set representing temperature is High.

T_{HIGH} is given as follows.
T_{HIGH} = (15,0.1), (20, 0.4), (25,0.45), (30,0.55), (35,0.65), (40,0.7), (45,0.85),(50,0.9)

What is the crisp value that implies for the high temperature

Example 2: Fuzzy to crisp

As an another example, let us consider a fuzzy set whose membership finction is shown in the following figure.



What is the crisp value of the fuzzy set in this case?

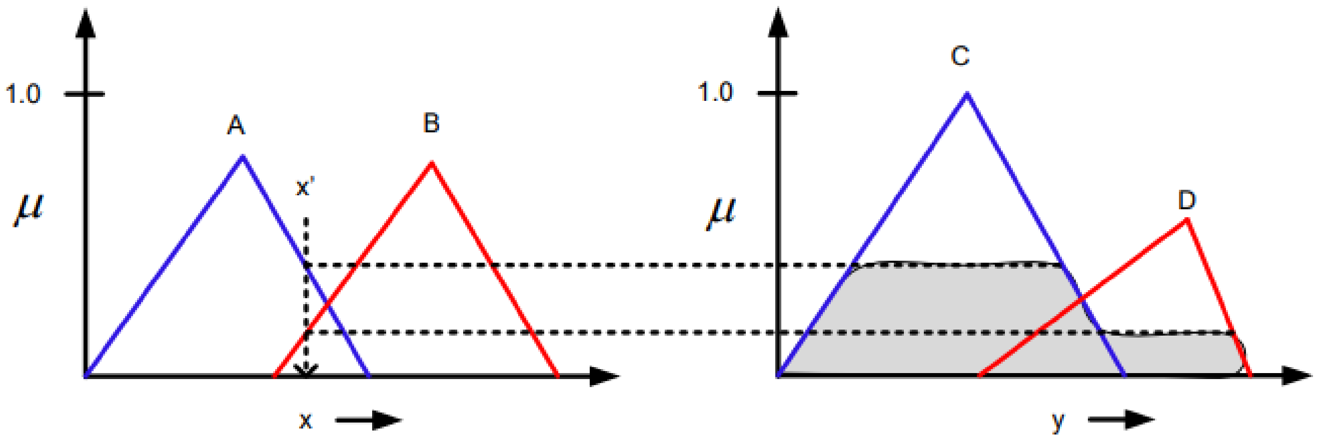
Example 2: Fuzzy to crisp

Now, consider the following two rules in the fuzzy rule base.

R1: If *x* **is** *A* then *y* **is** *C*

R2: If *x* **is** *B* then *y* **is** *D*

A pictorial representation of the above rule base is shown in the following figures.



What is the crisp value that can be inferred from the above rules given an input say x ?

Why defuzzification?

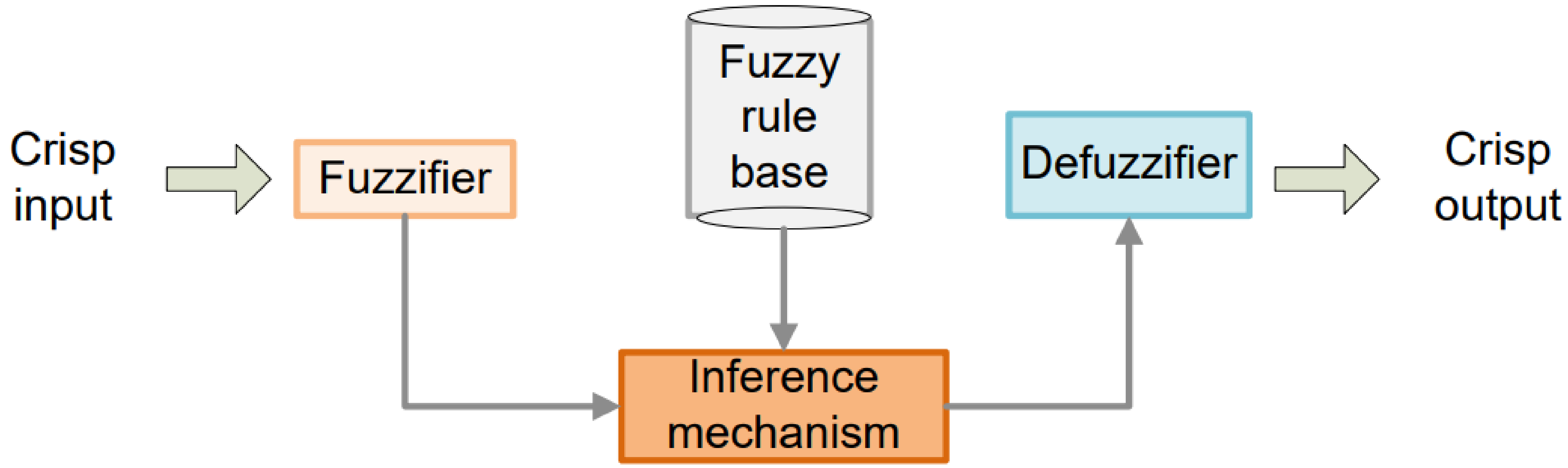
- The fuzzy results generated can not be used in an application, where decision has to be taken only on crisp values.

- Example: If T_{HIGH} then rotate R_{FIRST} .

- Here, may be input T_{HIGH} is fuzzy, but action rotate should be based on the crisp value of R_{FIRST}.

Generic structure of a Fuzzy system

Following figures shows a general fraework of a fuzzy system.



Defuzzification methods

- A number of defuzzification methods are known.
- Lambda-cut method
- Weighted average method
- Maxima methods
- Centroid methods

Lambda-cut method

- Lambda-cut method is applicable to derive crisp value of a fuzzy set or relation.
- In literature, Lambda-cut method is also alternatively termed as Alpha-cut method.

- In this method a fuzzy set A is transformed into a crisp set A_λ for a given value of λ $(0 \leq \lambda \leq 1)$

  (or) In other-words, $A_\lambda = \{x \mid \mu_A(x) \geq \lambda\}$

Lambda-cut for a fuzzy set : Example

- $A1 = \{(x1, 0.9),(x2, 0.5),(x3, 0.2),(x4, 0.3)\}$

- Then $A_{0.6} = \{(x1, 1),(x2, 0),(x3, 0),(x4, 0)\} = \{x1\}$

- $A2 = \{(x1, 0.1),(x2, 0.5),(x3, 0.8),(x4, 0.7)\}$

$A_{0.2} = \{(x1, 0),(x2, 1),(x3, 1),(x4, 1)\} = \{x2, x3, x4\}$

Crisp set can be null or can have one or more elements in the set

Lambda-cut sets : Example

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Two fuzzy sets P and Q are defined on x as follows.

$\mu(x)$	x_1	x_2	x_3	x_4	x_5
P	0.1	0.2	0.7	0.5	0.4
Q	0.9	0.6	0.3	0.2	0.8

Find the following :

(a) $P_{0.2}, Q_{0.3}$

(b) $(P \cup Q)_{0.6}$

(c) $(P \cup \overline{P})_{0.8}$

(d) $(P \cap Q)_{0.4}$

Lambda-cut for a fuzzy relation

The Lambda-cut method for a fuzzy set can also be extended to fuzzy relation also.

Example: For a fuzzy relation R

$$R = \begin{bmatrix} 1 & 0.2 & 0.3 \\ 0.5 & 0.9 & 0.6 \\ 0.4 & 0.8 & 0.7 \end{bmatrix}$$

We are to find λ-cut relations for the following values of λ = 0, 0.2, 0.9, 0.5

$$R_0 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } R_{0.2} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$R_{0.9} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } R_{0.5} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

All elements greater than or equal to λ should be 1

Properties of λ-cut sets

- If A and B are two fuzzy sets, defined with the same universe of discourse, then
- $(A \cup B)_\lambda = A_\lambda \cup B_\lambda$
- $(A \cap B)_\lambda = A_\lambda \cap B_\lambda$
- $\overline{A_\lambda} \neq \bar{A}_\lambda$ except for value of $\lambda = 0.5$
- For any $\lambda \leq \alpha$, where α varies between 0 and 1, it is true that

  $A_\alpha \subseteq A_\lambda$,

Properties of λ-cut relations

- If R and S are two fuzzy relations, defined with the same fuzzy sets over the same universe of discourses, then
- $(R \cup S)_\lambda = R_\lambda \cup S_\lambda$
- $(R \cap S)_\lambda = R_\lambda \cap S_\lambda$
- $\overline{R_\lambda} \neq \bar{R}_\lambda$
- For $\lambda \leq \alpha$, where α between 0 and 1 , then $R_\alpha \subseteq R_\lambda$

Lambda-cut method converts a fuzzy set (or a fuzzy relation) into crisp set (or relation).

Output of a fuzzy System

The output of a fuzzy system can be a single fuzzy set or union of two or more fuzzy sets.

To understand the second concept, let us consider a fuzzy system with *n*-rules.

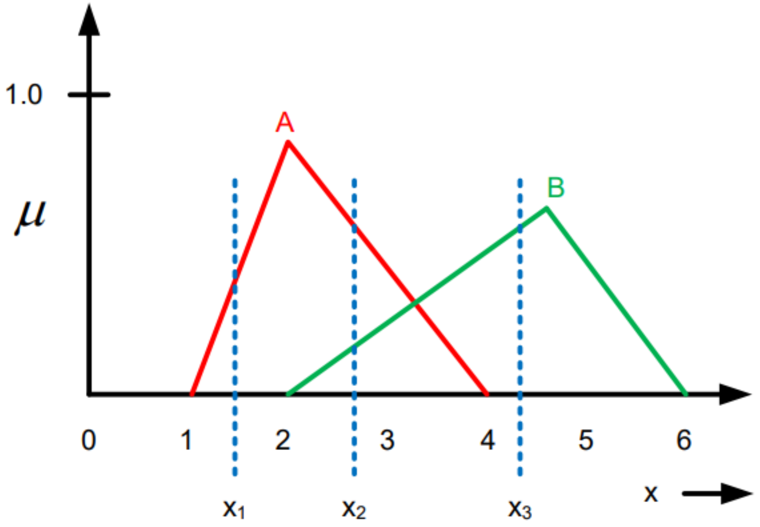
R_1: If x is A_1 then y is B_1
R_2: If x is A_2 then y is B_2
....
....
R_n: If x is A_n then y is B_n

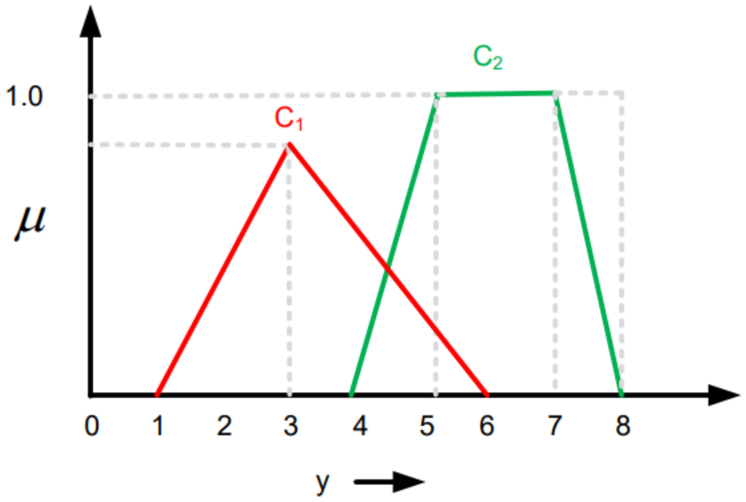
In this case, the output y for a given input $x = x_1$ is possibly $B = B_1 \cup B_2 \cup .....B_n$

Output fuzzy set : Illustration

- Suppose, two rules R_1 and R_2 are given as follows:
- R_1: If x is A_1 then y is C_1
- R_2: If x is A_2 then y is C_2

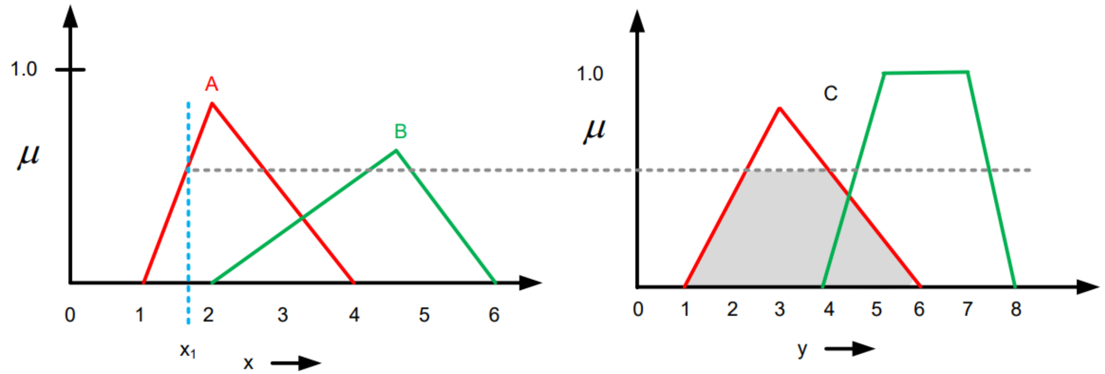
  Here, the output fuzzy set $C = C_1 \cup C_2$.





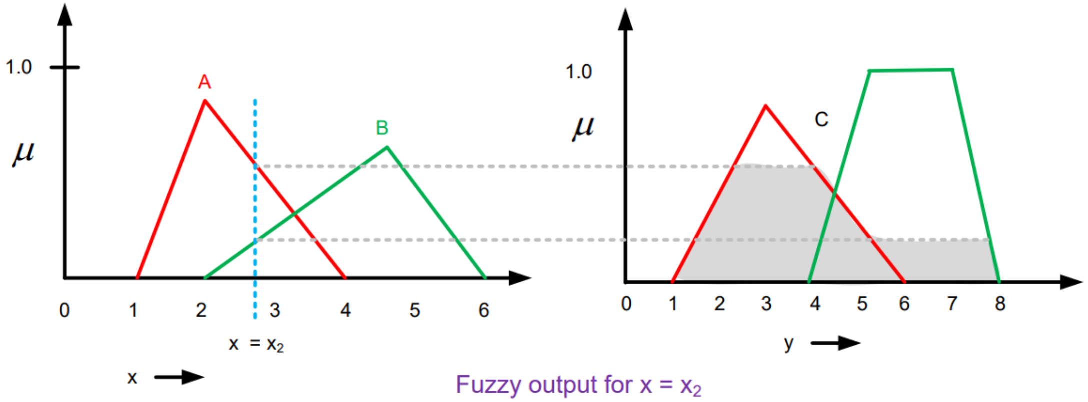
Output fuzzy set : Illustration

The fuzzy output for $x = x_1$ is shown below.



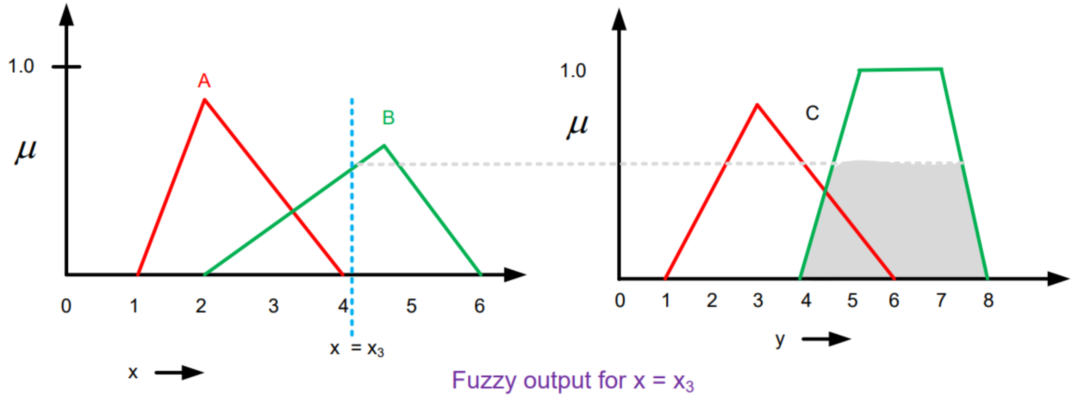
Fuzzy output for $x = x_1$

The fuzzy output for $x = x_2$ is shown below.



Fuzzy output for $x = x_2$

The fuzzy output for $x = x_3$ is shown below.



Fuzzy output for $x = x_3$

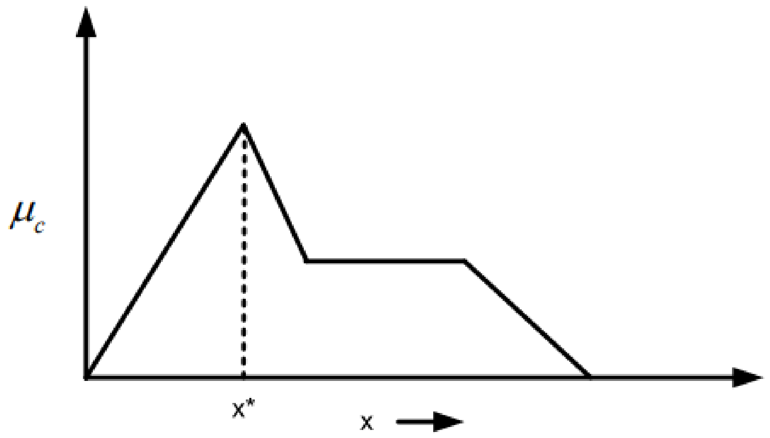
Defuzzification :Maxima Method

- Height method
- First of maxima (FoM)
- Last of maxima (LoM)
- Mean of maxima(MoM)

Maxima Method : Height Method

This method is based on Max-membership principle, and defined as follows.

$$\mu_C(x^*) \geq \mu_C(x) \text{ for all } x \in X$$

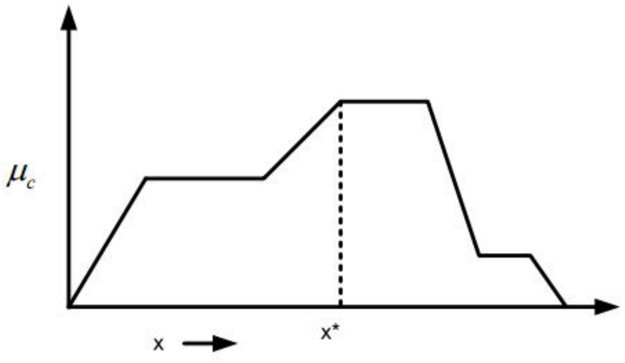


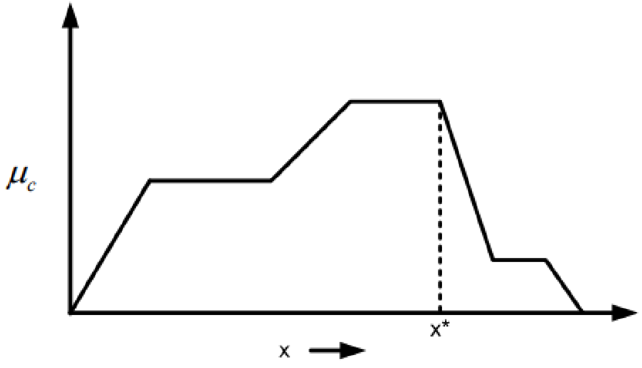
Note:

1. Here, x^* is the height of the output fuzzy set C.
2. This method is applicable when height is unique.

Maxima Method:

FoM: First of Maxima : $x^* = min\{x|C(x) = max_w C\{w\}\}$ LoM : Last of Maxima : $x^* = max\{x|C(x) = max_w C\{w\}\}$





Mean of Maxima: $$x^* = \frac{\sum_{x_i \in M}(x_i)}{|M|}$$

where, $M = \{x_i|\mu(x_i) = h(C)\}$ where $h(C)$ is the height of the fuzzy set C

MoM : Example 1

Suppose, a fuzzy set **Young** is defined as follows:

Young = {(15,0.5), (20,0.8), (25,0.8), (30,0.5), (35,0.3) }

Then the crisp value of **Young** using MoM method is

$$x^* = \frac{20+25}{2} = 22.5$$

Thus, a person of 22.5 years old is treated as young!

Defuzzification: Centroid methods

- Center of gravity method (CoG)
- Center of sum method (CoS)
- Center of area method (CoA)

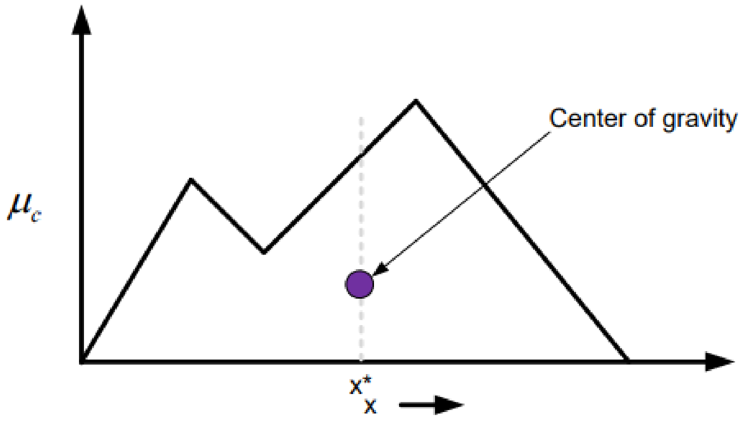
Centroid method : CoG

The basic principle in CoG method is to find the point x^* where a vertical line would slice the aggregate into two equal masses.

Mathematically, the CoG can be expressed as follows :

$$x^* = \frac{\int x.\mu_C(x)dx}{\int \mu_C(x)dx}$$

Graphically,



x^* is the x-coordinate of center of gravity.

$\int \mu_C(x)dx$ denotes the area of the region bounded by the curve μ_C.

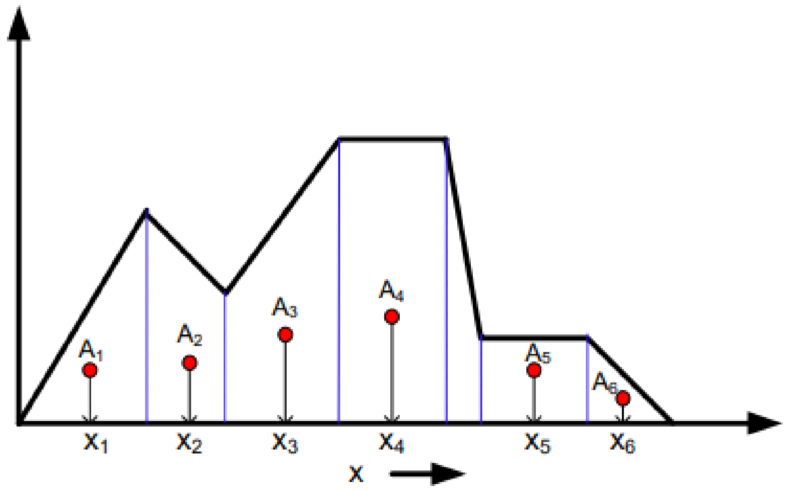
If μ_C is defined with a discrete membership function, then CoG can be stated as :

$$x^* = \frac{\sum_{i=1}^{n} x_i.\mu_C(x_i)}{\sum_{i=1}^{n} \mu_C(x_i)} \ ;$$

Here, x_i is a sample element and n represents the number of samples in fuzzy set C.

CoG : A geometrical method of calculation

Divide the entire region into a number of small regular regions (e.g. triangles, trapizoid etc.)

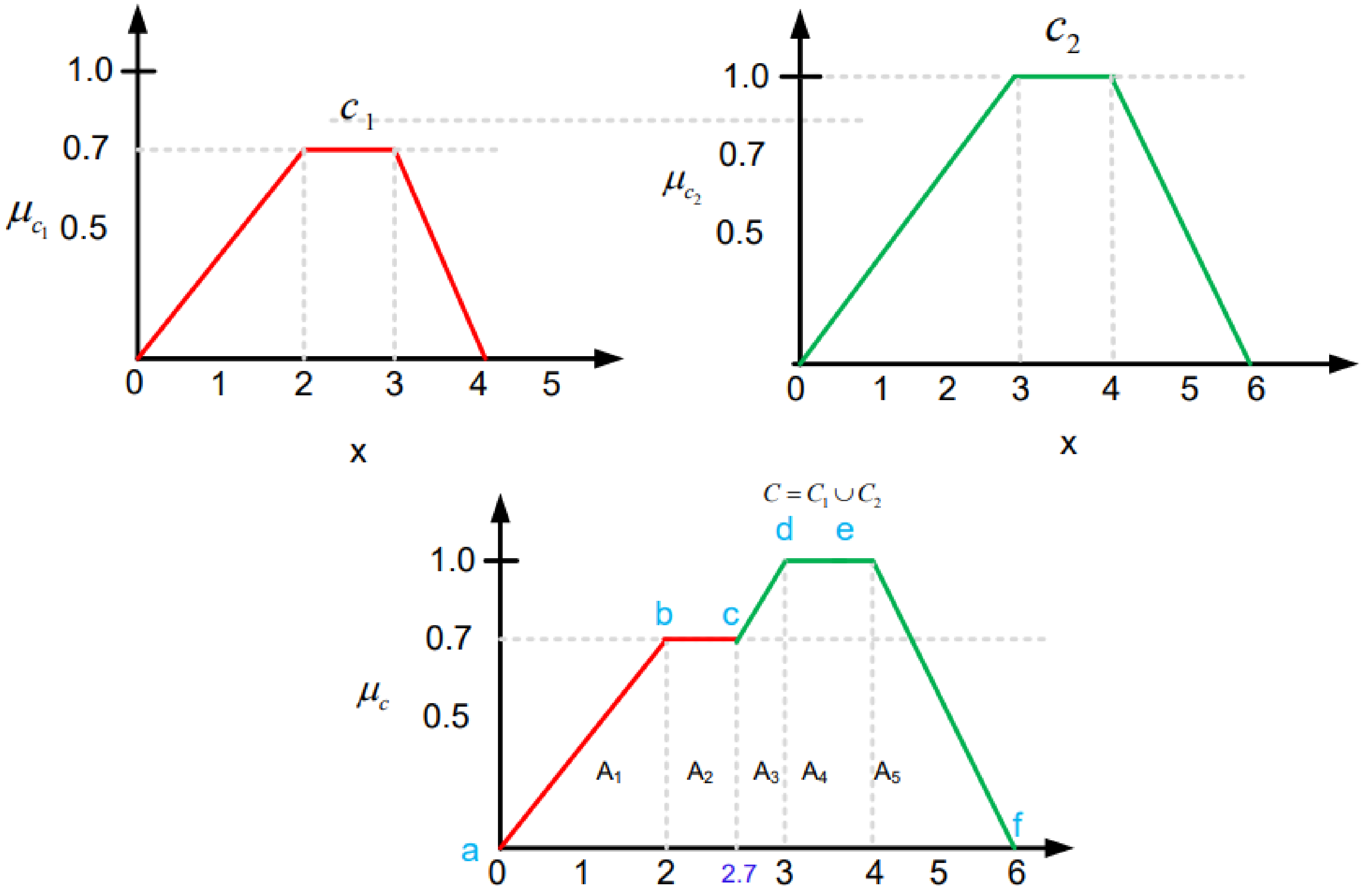


Let A_i and x_i denotes the area and c.g. of the i-th portion.

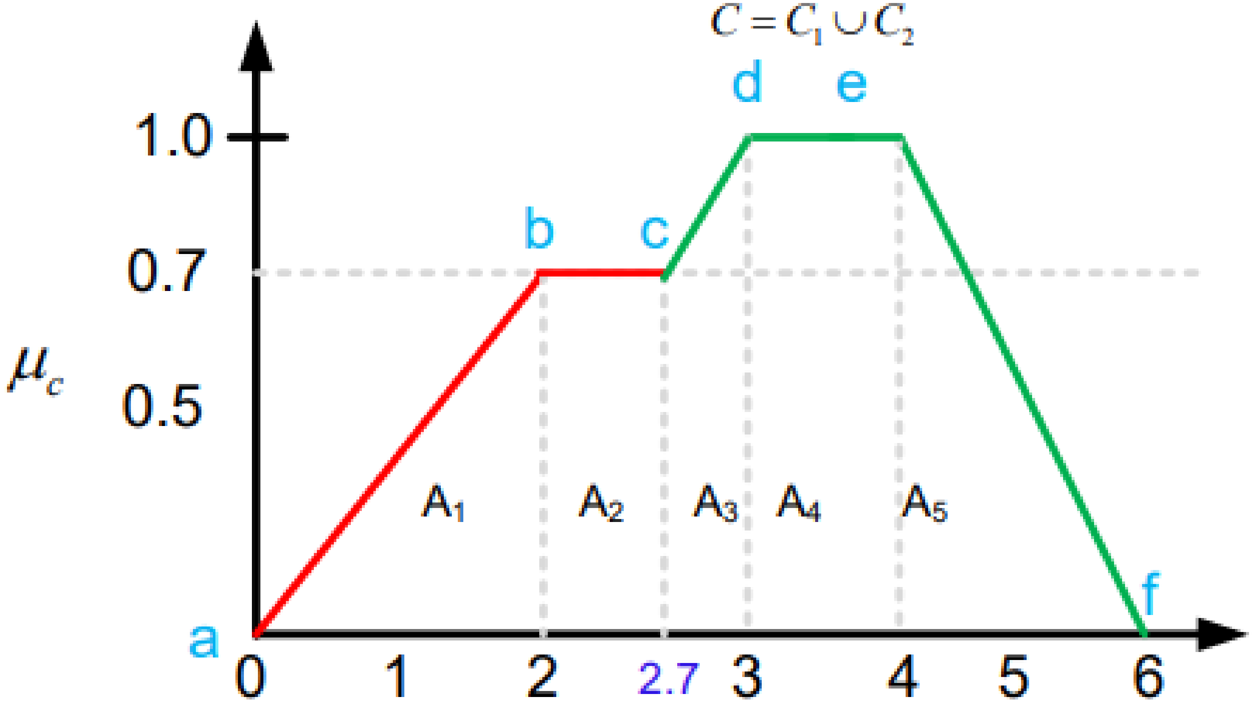
Then x^* according to CoG is

$$x^* = \frac{\sum_{i=1}^{n} x_i.(A_i)}{\sum_{i=1}^{n} A_i}$$

CoG: An example of integral method of calculation



Example (Cont...)



$$\mu_c(x) = \begin{cases} 0.35x & 0 \leq x < 2 \\ 0.7 & 2 \leq x < 2.7 \\ x - 2 & 2.7 \leq x < 3 \\ 1 & 3 \leq x < 4 \\ (-0.5x + 3) & 4 \leq x \leq 6 \end{cases}$$

For $A_1 : y - 0 = \frac{0.7}{2}(x - 0)$, or $y = 0.35x$

For $A_2 : y = 0.7$

For $A_3 : y - 0 = \frac{1-0}{3-2}(x - 2)$, or $y = x - 2$

For, $A_4 : y = 1$

For, $A_5 : y - 1 = \frac{0-1}{6-4}(x - 4)$, or $y = -0.5x + 3$

Example (Cont...)

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$$\mu_c(x) = \begin{cases} 0.35x & 0 \le x < 2 \\ 0.7 & 2 \le x < 2.7 \\ x-2 & 2.7 \le x < 3 \\ 1 & 3 \le x < 4 \\ (-0.5x+3) & 4 \le x \le 6 \end{cases}$$

For A_1 : $y - 0 = \frac{0.7}{2}(x-0)$, or $y = 0.35x$

For A_2 : $y = 0.7$

For A_3 : $y - 0 = \frac{1-0}{3-2}(x-2)$, or $y = x - 2$

For, A_4 : $y = 1$

For, A_5 : $y - 1 = \frac{0-1}{6-4}(x-4)$, or $y = -0.5x + 3$

Thus, $x^* = \frac{\int x.\mu_c(x)dx}{\int \mu_c(x)dx} = \frac{N}{D}$

$N = \int_0^2 0.35x^2dx + \int_2^{2.7} 0.7x^2dx + \int_{2.7}^3 (x^2 - 2x)dx + \int_3^4 xdx + \int_4^6 (-0.5x^2 + 3x)dx$

$= 10.98$

$D = \int_0^2 0.35xdx + \int_2^{2.7} 0.7xdx + \int_{2.7}^3 (x-2)dx + \int_3^4 dx + \int_4^6 (-0.5x+3)dx$

$= 3.445$

Thus, $x^* = \frac{10.98}{3.445} = 3.187$

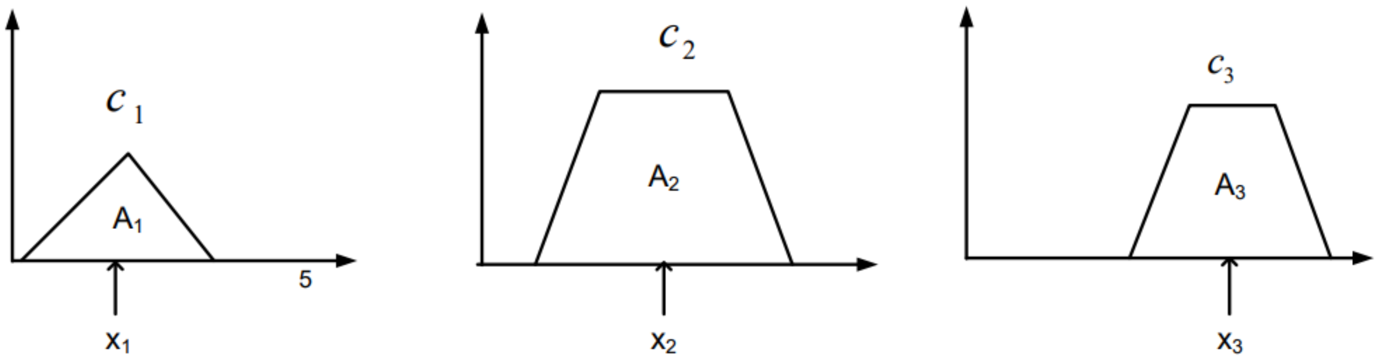
Centroid method : Center of Sum (CoS)

If the output fuzzy set $C = C_1 \cup C_2 \cup ....C_n$, then the crisp value according to CoS is defined as

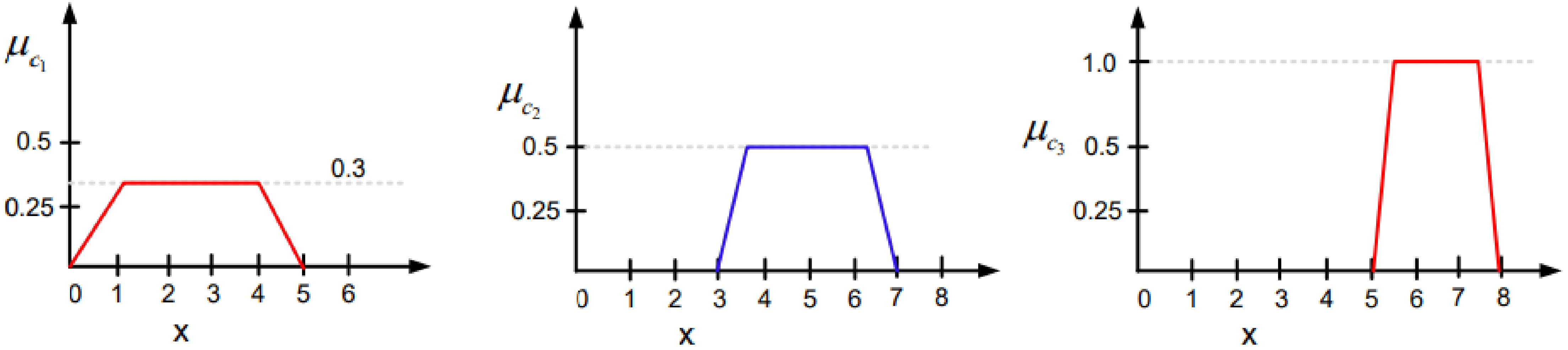
$$x^* = \frac{\sum_{i=1}^{n} x_i.A_{c_i}}{\sum_{i=1}^{n} A_{c_i}}$$

Here, A_{c_i} denotes the area of the region bounded by the fuzzy set C_i and x_i is the geometric center of the area A_{c_i}.

Graphically,



CoS: Example



$$A_{c_1} = \frac{1}{2} \times 0.3 \times (3+5), x_1 = 2.5$$

$$A_{c_2} = \frac{1}{2} \times 0.5 \times (4+2), x_2 = 5$$

$$A_{c_3} = \frac{1}{2} \times 1 \times (3+1), x_3 = 6.5$$

$$\text{Thus, } x^* = \frac{\frac{1}{2}\times0.3\times(3+5)\times2.5+\frac{1}{2}\times0.5\times(4+2)\times5+\frac{1}{2}\times1\times(3+1)\times6.5}{\frac{1}{2}\times0.3\times(3+5+\frac{1}{2}\times0.5\times(4+2)+\frac{1}{2}\times1\times(3+1)} = 5.00$$

Note: The crisp value of C = C1 ∪ C2 ∪ C3 using CoG method can be found to be calculated as x ∗ = 4.9 (Smaller Why?)

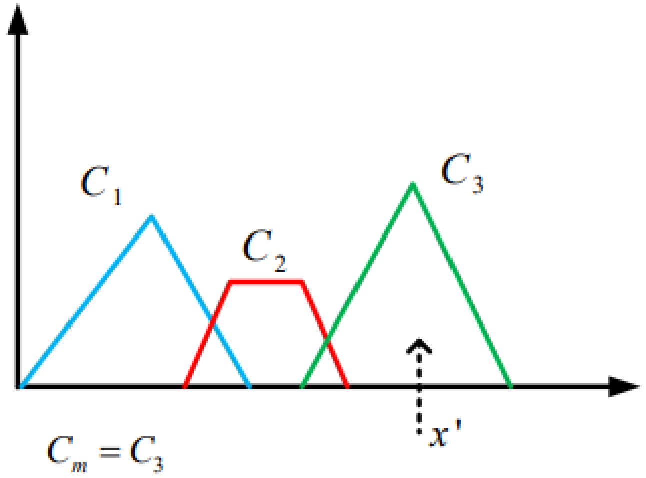
Centroid method: Center of largest area

If the fuzzy set has two subregions, then the center of gravity of the subregion with the largest area can be used to calculate the defuzzified value.

Mathematically, $x^* = \frac{\int \mu_{cm}(x).x' \, dx}{\int \mu_{cm}(x) dx}$;

Here, C_m is the region with largest area, x' is the center of gravity of C_m.

Graphically,



Weighted average method

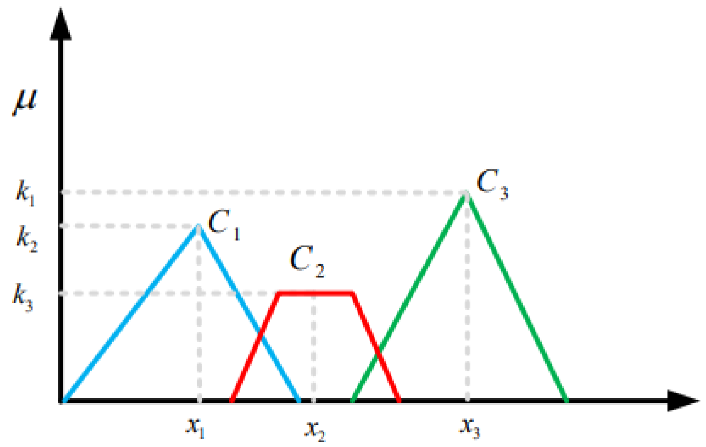
This method is also alternatively called ”Sugeno defuzzification” method.

The method can be used only for symmetrical output membership functions.

The crisp value accroding to this method is

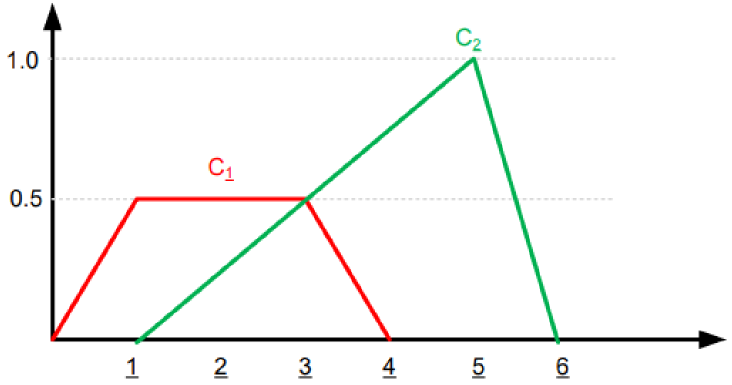
$$x^* = \frac{\sum_{i=1}^{n} \mu_{C_i}(x_i).(x_i)}{\sum_{i=1}^{n} \mu_{C_i}(x_i)}$$

where, $C_1, C_2, \ldots C_n$ are the output fuzzy sets and (x_i) is the value where middle of the fuzzy set C_i is observed.

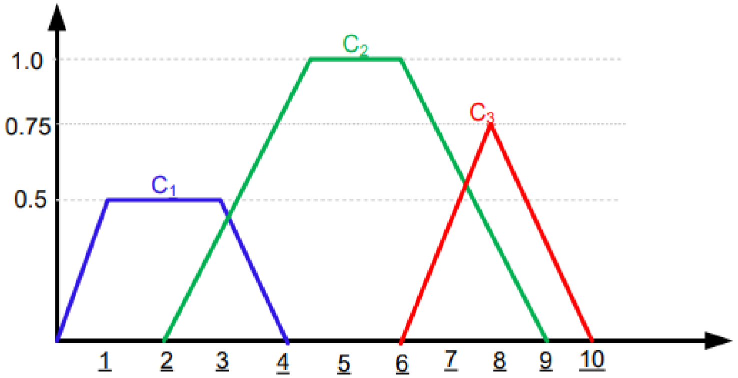


Exercise

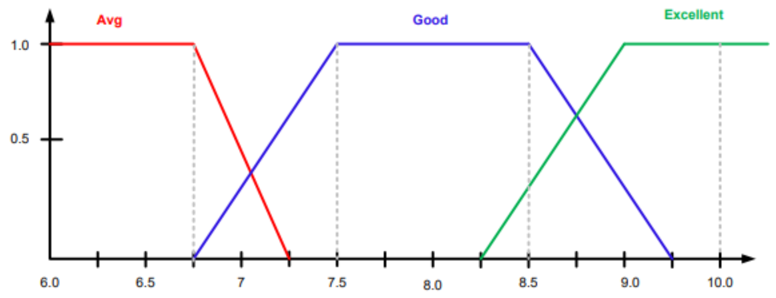
Find the crisp value of the following using all defuzzified methods.



Find the crisp value of the following using all defuzzified methods.



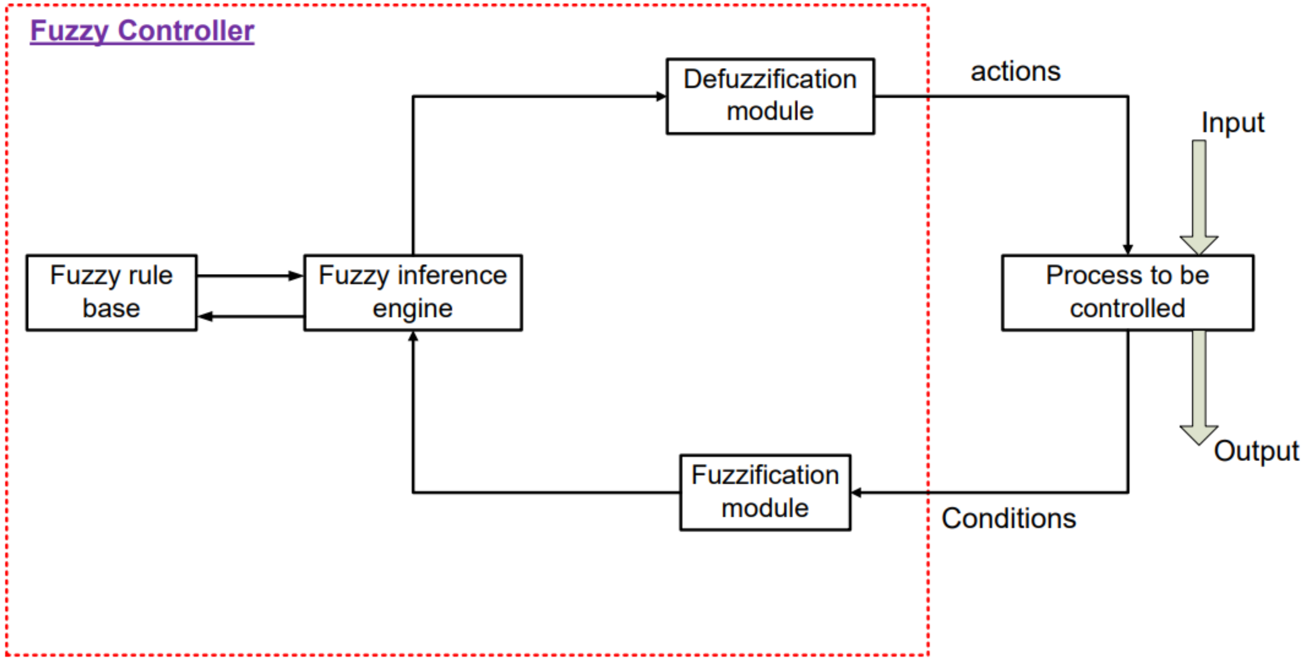
The membership function defining a student as Average, Good, and Excellent denoted by respective membership functions are as shown below.



Find the crisp value of "Good Student"

Fuzzy Systems : Fuzzy Logic Controller

A general scheme of a fuzzy controller is shown in the following figure.



We use FLC where an exact mathematical formulation of the problem is not possible or very difficult. These difficulties are due to non-linearities, time-varying nature of the process, large unpredictable environment disturbances etc.

Fuzzy Systems : Fuzzy Logic Controller

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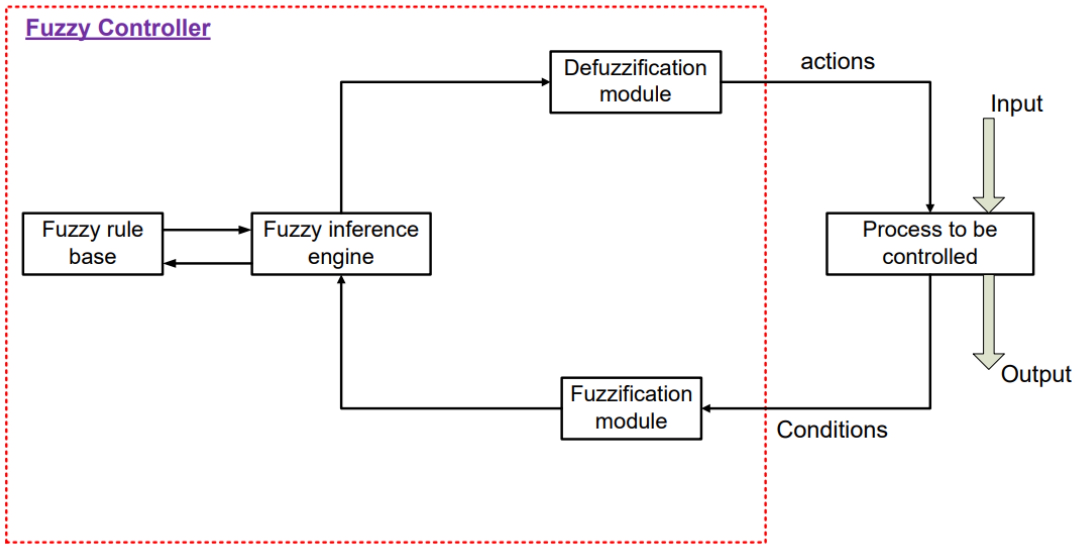
As shown in Figure, a fuzzy controller operates by repeating a cycle of the following four steps

- Measurements (inputs) are taken of all variables that represent relevant condition of controller process.
- These measurements are converted into appropriate fuzzy sets to express measurements uncertainties. This step is called fuzzification
- The fuzzified measurements are then used by the inference engine to evaluate the control rules stored in the fuzzy rule base. The result of this evaluation is a fuzzy set (or several fuzzy sets) defined on the universe of possible actions.
- This output fuzzy set is then converted into a single (crisp) value (or a vector of values). This is the final step called defuzzification. The defuzzified values represent actions to be taken by the fuzzy controller.

A general scheme of a fuzzy controller is shown in the following figure.



Fuzzy Systems : Fuzzy Logic Controller

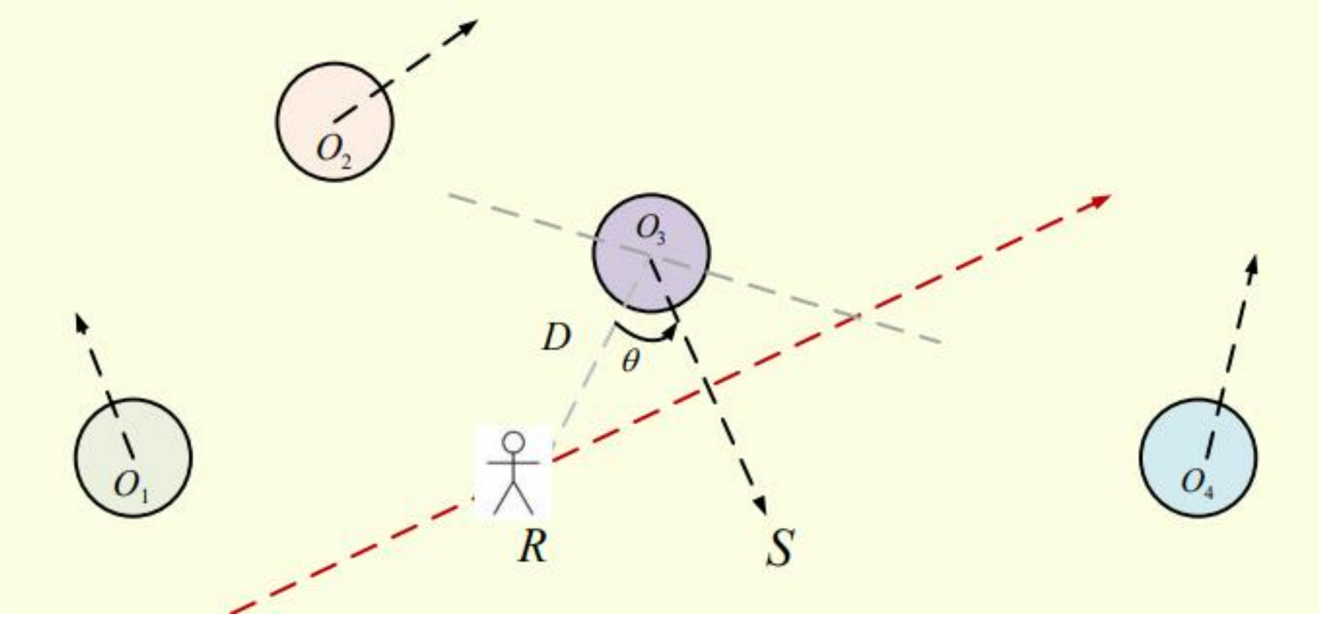
There are two approaches of FLC known.

- Mamdani approach
- Takagi and sugeno's approach

- Mamdani approach follows linguistic fuzzy modeling and characterized by its high interpretability and low accuracy.
- Takagi and Sugeno's approach follows precise fuzzy modeling and obtains high accuracy but at the cost of low interpretably.

Mamdani approach : Mobile Robot

- Consider the control of navigation of a mobile robot in the pressure of a number of moving objects.
- To make the problem simple, consider only four moving objects, each of equal size and moving with the same speed.



Mamdani approach : Mobile Robot

- We consider two parameters : D, the distance from the robot to an object and θ the angle of motion of an object with respect to the robot.
- The value of these parameters with respect to the most critical object will decide an output called deviation (δ).
- We assume the range of values of D is [0.1, ... 2.2] in meter and

  θ is [-90, ... , 0, ... 90] in degree.

After identifying the relevant input and output variables of the controller and their range of values, the Mamdani approach is to select some meaningful states called ”linguistic states” for each variable and express them by appropriate fuzzy sets.

Linguistic States

- For the current example, we consider the following linguistic states for the three parameters.
- Distance is represented using four linguistic states:

  1 VN : Very Near

  2 NR : Near

  3 VF : Very Far

  4 FR : Far
- Angle (for both angular direction (θ) and deviation (δ)) are represented using five linguistic states:

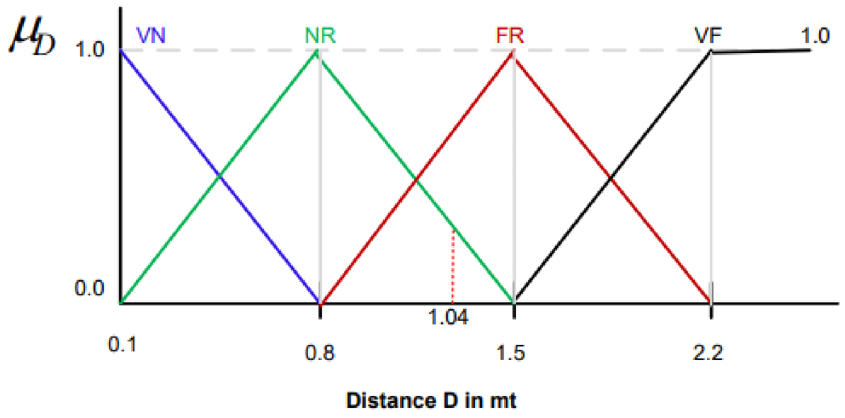
  1 LT : Left   2 AL : Ahead Left

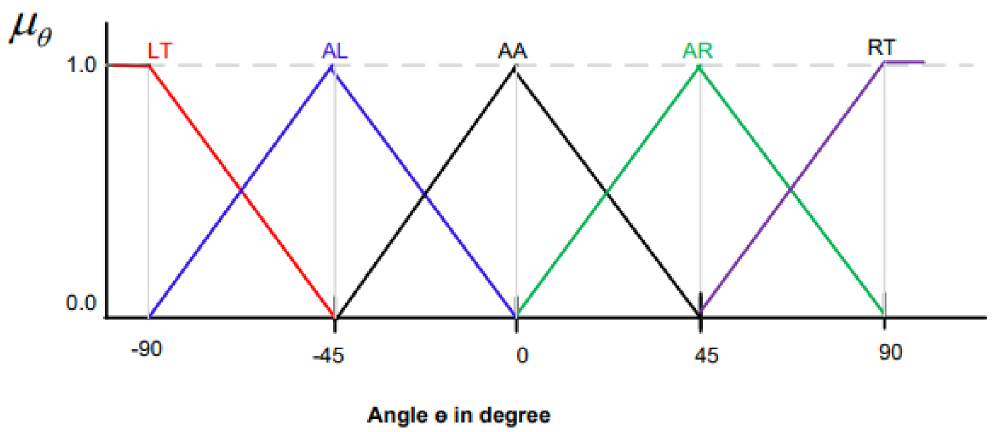
  3 AA: Ahead   4 AR : Ahead Right

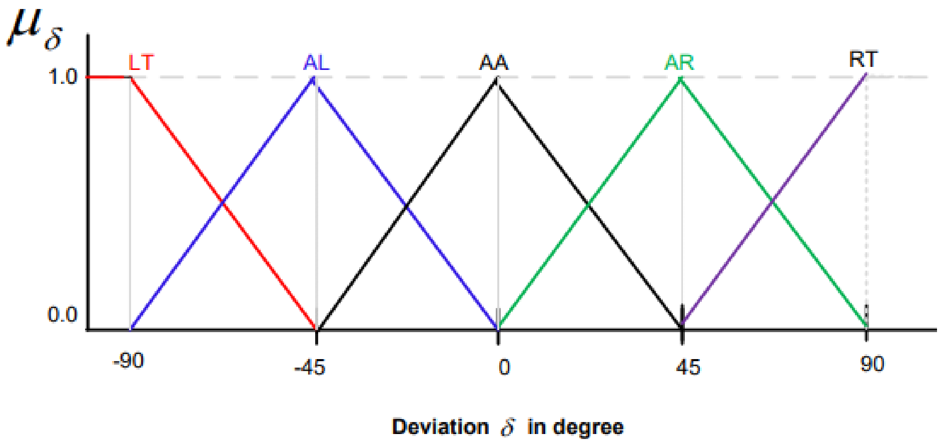
  5 RT : Right

Linguistic States

- Three different fuzzy sets for the three different parameters are given below







Fuzzy rule base

- Once the fuzzy sets of all parameters are worked out, our next step in FLC design is to decide fuzzy rule base of the FLC.
- The rule base for the FLC of mobile robot is shown in the form of a table below.

$$
\begin{array}{c c}
 & \begin{array}{ccccc} LT & AL & AA & AR & RT \end{array} \\
\begin{array}{c} VN \\ NR \\ FR \\ VF \end{array} & \begin{bmatrix} AA & AR & AL & AL & AA \\ AA & AA & RT & AA & AA \\ AA & AA & AR & AA & AA \\ AA & AA & AA & AA & AA \end{bmatrix}
\end{array}
$$

Fuzzy rule base for the mobile robot

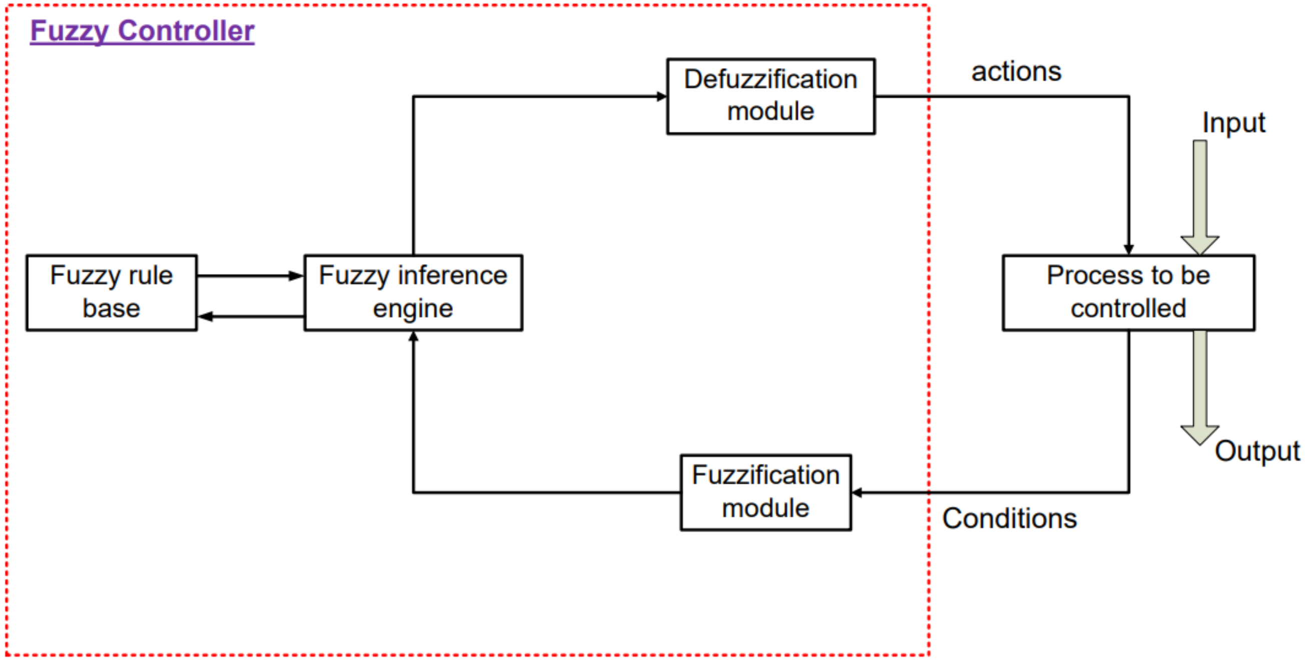
- Note that this rule base defines 20 rules for all possible instances. These rules are simple rules and take in the following forms.

- Rule 1: If (distance is VN) and (angle is LT) Then (deviation is AA)
- . . . . . .
- …………
- Rule 13: If (distance is FR) and (angle is AA) Then (deviation is AR)
- . . . . . .
- …………
- Rule 20: Rule 1: If (distance is VF) and (angle is RT) Then (deviation is AA)

$$
\begin{array}{cc}
 & \begin{array}{ccccc} LT & AL & AA & AR & RT \end{array} \\
\begin{array}{c} VN \\ NR \\ FR \\ VF \end{array} &
\begin{bmatrix}
AA & AR & AL & AL & AA \\
AA & AA & RT & AA & AA \\
AA & AA & AR & AA & AA \\
AA & AA & AA & AA & AA
\end{bmatrix}
\end{array}
$$

Fuzzy Systems : Fuzzy Logic Controller

A general scheme of a fuzzy controller is shown in the following figure.



Fuzzification of inputs

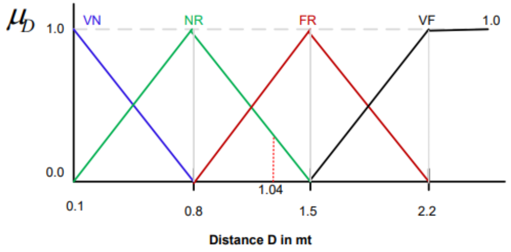
- The next step is the fuzzification of inputs.
- Let us consider, at a specific instance, the object O_3 is at a distance D = 1.04 m and angle $\theta = 30^0$
- For this input, we are to decide the deviation δ of the robot as output.
- From the given fuzzy sets and input parameters values, we say that the distance D = 1.04m may be called as either NR (near) or FR (far).
- Similarly, the input angle $\theta = 30^0$ can be declared as either AA (ahead) or AR(ahead right).

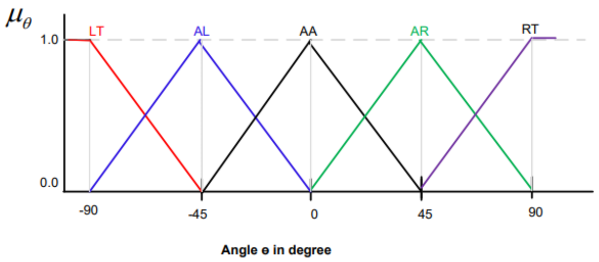
Fuzzification of inputs

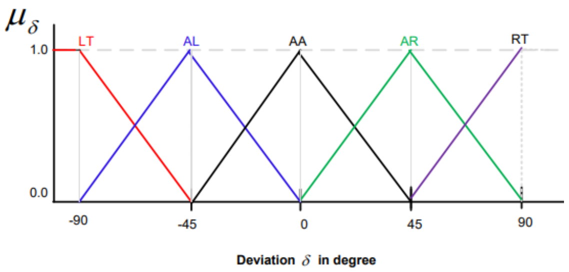
- We can determine the membership values corresponding to these values, which is as follows.

$x = 1.04m$; $\mu_{NR}(x) = 0.6571$ and $\mu FR(x) = 0.3429$

$y = 30^0$; $\mu_{AA}(y) = 0.3333$ $\mu_{AR}(y) = 0.6667$







Rule strength computation

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There are many rules in the rule base and all rules may not be applicable. For the given x = 1.04 and θ = 30º , only following four rules out of 20 rules are usable/firable.

- R1: If (distance is NR) and (angle is AA) Then (deviation is RT)
- R2: If (distance is NR) and (angle is AR) Then (deviation is AA)
- R3: If (distance is FR) and (angle is AA) Then (deviation is AR)
- R4: If (distance is FR) and (angle is AR) Then (deviation is AA)

$$\begin{array}{c c} & \begin{array}{ccccc} LT & AL & AA & AR & RT \end{array} \\ \begin{array}{c} VN \\ NR \\ FR \\ VF \end{array} & \begin{bmatrix} AA & AR & AL & AL & AA \\ AA & AA & RT & AA & AA \\ AA & AA & AR & AA & AA \\ AA & AA & AA & AA & AA \end{bmatrix} \end{array}$$

Rule strength computation

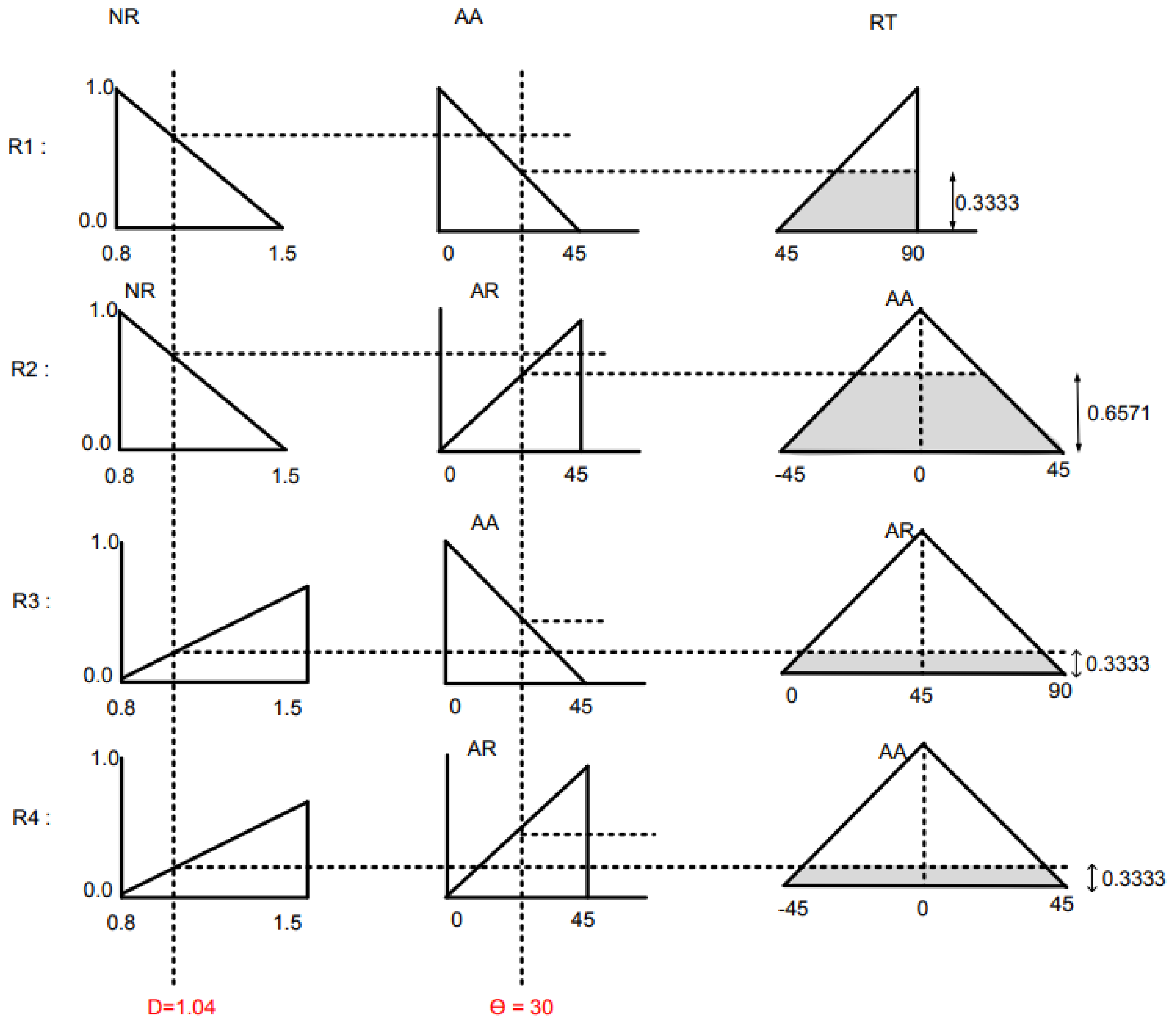
- The strength (also called α values) of the firable rules are calculated as follows.
- $\alpha(R1) = \min(\mu_{NR}(x), \mu_{AA}(y)) = \min(0.6571, 0.3333) = 0.3333$
- $\alpha(R2) = \min(\mu_{NR}(x), \mu_{AR}(y)) = \min(0.6571, 0.6667) = 0.6571$
- $\alpha(R3) = \min(\mu_{FR}(x), \mu_{AA}(y)) = \min(0.3429, 0.3333) = 0.3333$
- $\alpha(R4) = \min(\mu_{FR}(x), \mu_{AR}(y)) = \min(0.3429, 0.6667) = 0.3429$

- In practice, all rules which are above certain threshold value of rule strength are selected for the output computation.

If α (Lambda Cut) is 0.5, Then ?

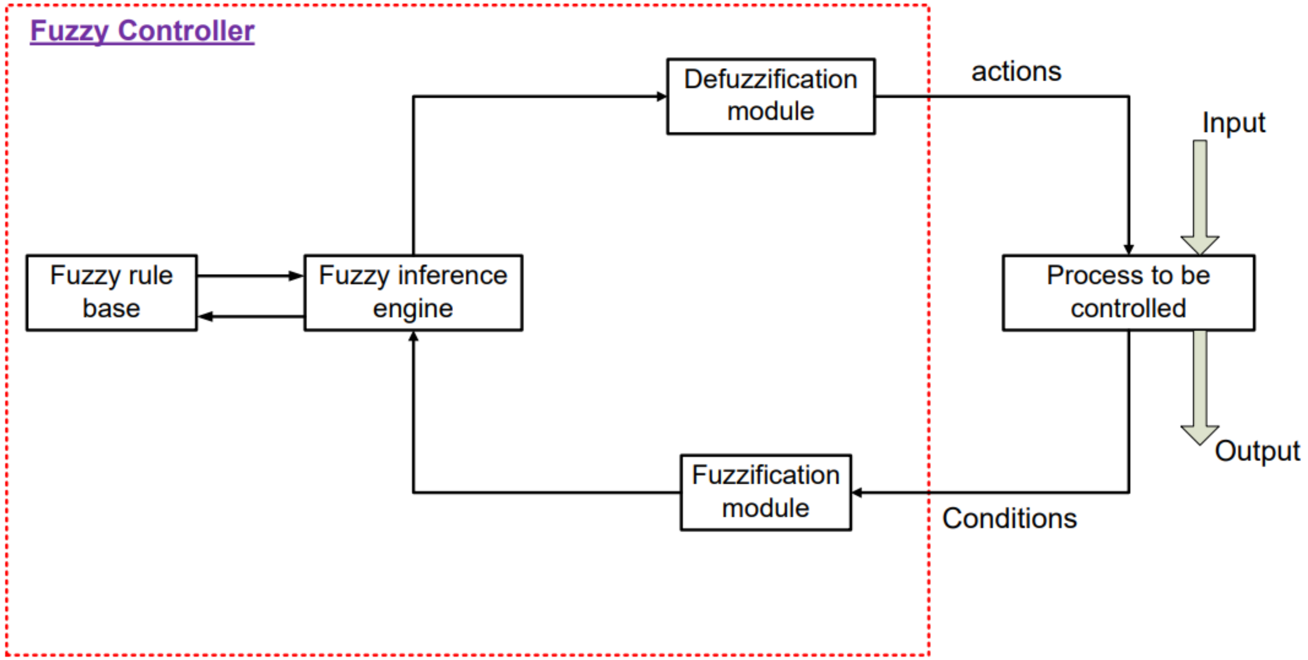
Fuzzy output

- For four rules, we find the following results.

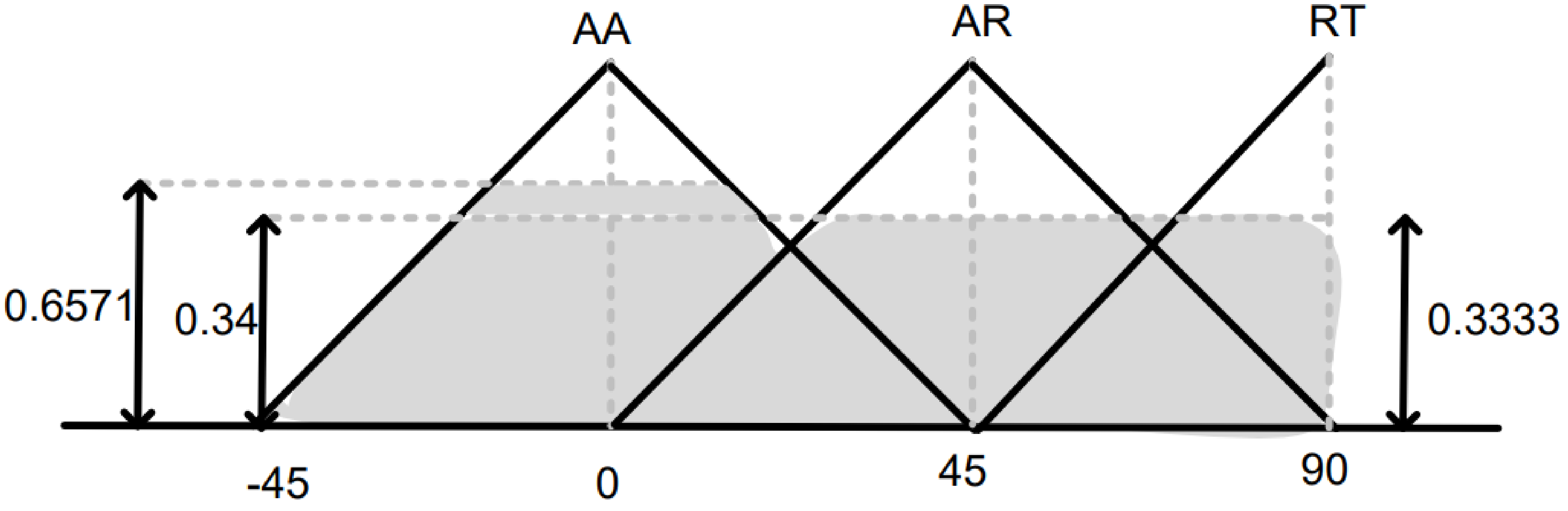


Fuzzy Systems : Fuzzy Logic Controller

A general scheme of a fuzzy controller is shown in the following figure.



Defuzzification

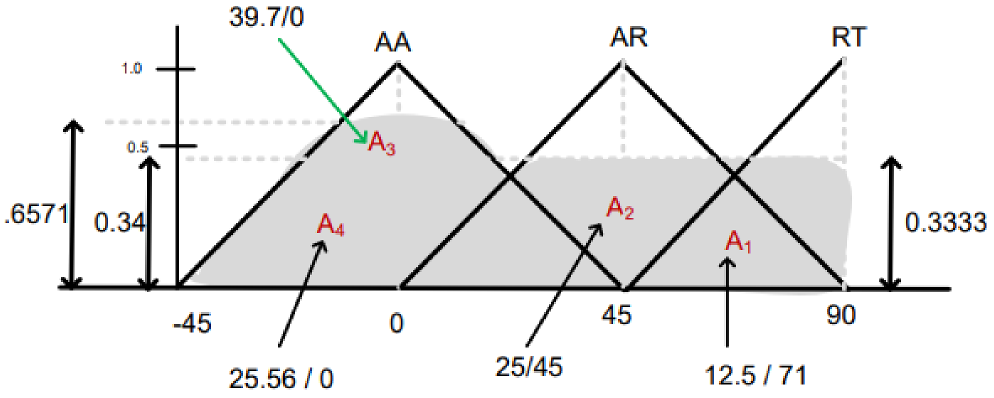


Aggregation of all results

Defuzzification: Output

From the combined fuzzified output for all four fired rules, we get the crisp value using Center of Sum method as follows.

$$V = \frac{12.5 \times 71 + 25 \times 45 + 25.56 \times 0 + 25.56 \times 0}{12.5 + 39.79 + 25 + 25.56} = 19.59$$



Conclusion : Therefore, the robot should deviate by 19.58089 degree towards the right with respect to the line joining to the move of direction to avoid collision with the obstacle O_3.

Takagi and Sugeno’s approach: Mobile Robot

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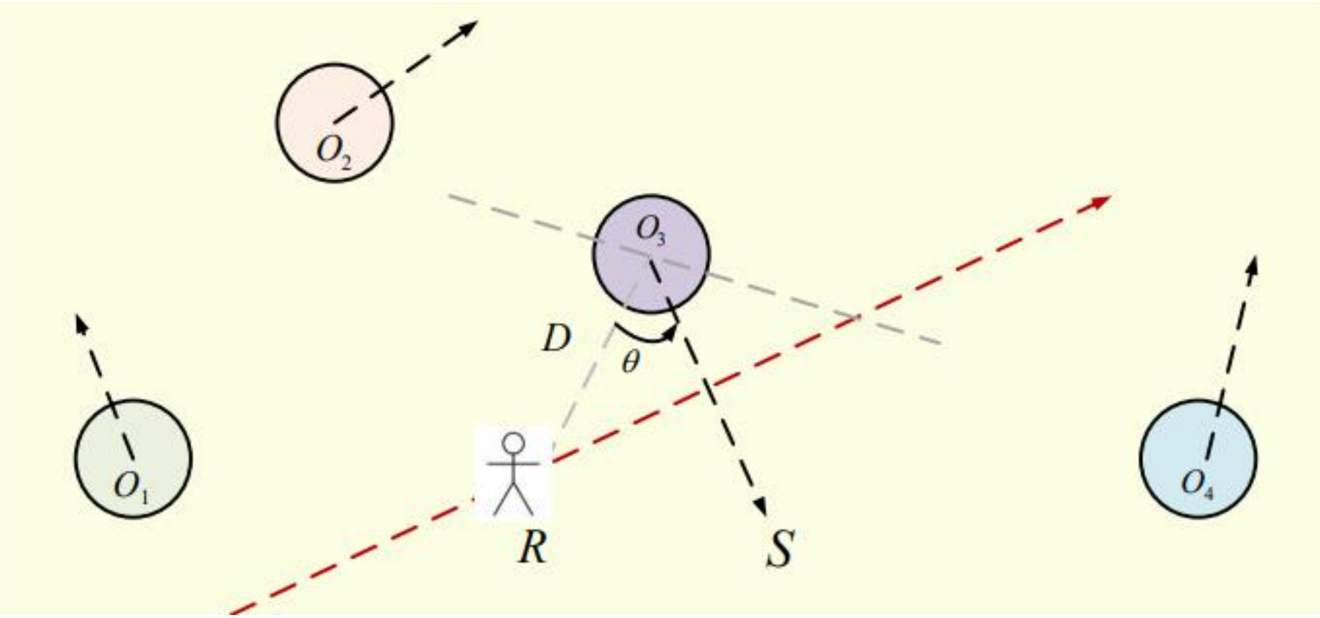
- Consider the control of navigation of a mobile robot in the pressure of a number of moving objects.
- To make the problem simple, consider only four moving objects, each of equal size and moving with the same speed.



Takagi and Sugeno's approach

- In this approach, a rule is composed of fuzzy antecedent and functional consequent parts.

- Any i-th rule, in this approach is represented by If (x_1 is A_1^i) and (x_2 is A_2^i) ...... and (x_n is A_n^i)

- Then, $y^i = a_0^i + a_1^i x_1 + a_2^i x_2 + \ldots\ldots + a_n^i x_n$ where, $a_0, a_1, a_2, \ldots a_n$ are the coefficients.

Deciding the coefficients are hard part and are determined based on experience

Takagi and Sugeno's approach

- The weight of i-th rule can be determined for a set of inputs $x_1, x_2, \ldots. x_n$ as follows.

$$w^i = \mu^i_{A_1}(x_1) \times \mu^i_{A_2}(x_2) \times \ldots.. \times \mu^i_{A_n}(x_n)$$

where $A_1, A_2, \ldots. , A_n$ indicates membership function distributions of the linguistic hedges used to represent the input variables and μ denotes membership function value.

- The combined action then can be obtained as

$$y = \frac{\sum_i^k w^i y^i}{\sum_i^k w^i}$$

where k denotes the total number of rules.

Illustration:

- Sensorial inputs for our problem is θ and D.

- These two inputs have the following linguistic states :

θ : L(low), M(Medium), H(High)

D : NR(Near), FR (Far), VF(Very Far)

Note: The rule base of such as system is decided by a maximum of

$3 \times 3 = 9$ feasible rules.

Illustration

- The output of any i^{th} rule can be expressed by the following.

$$y^i = f(\theta, D) = a_j^i\theta + b_k^i D;$$

where, j,k = 1,2,3.

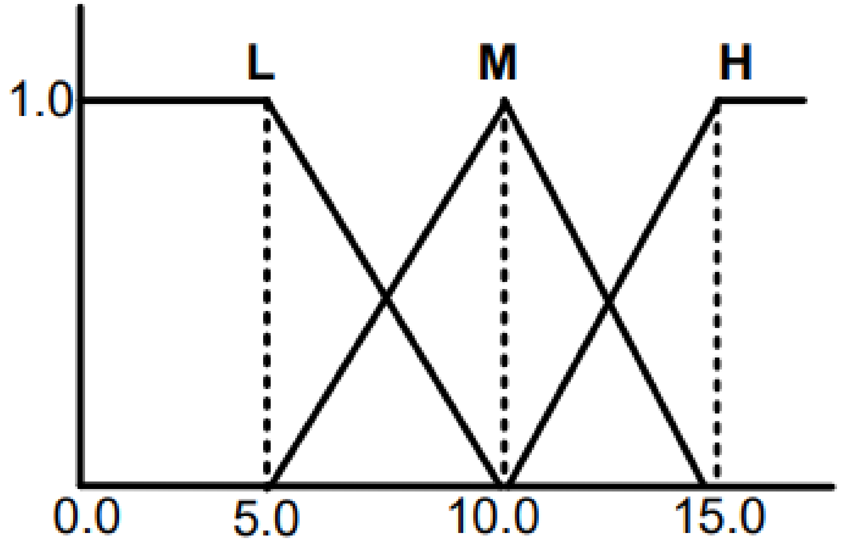
Ex: a_1^i = 1, a_2^i = 2, a_3^i = 3 if θ = L, M and H, respectively.

b_1^i = 1, b_2^i = 2, b_3^i = 3 if D = NR, FR, and VF, respectively.

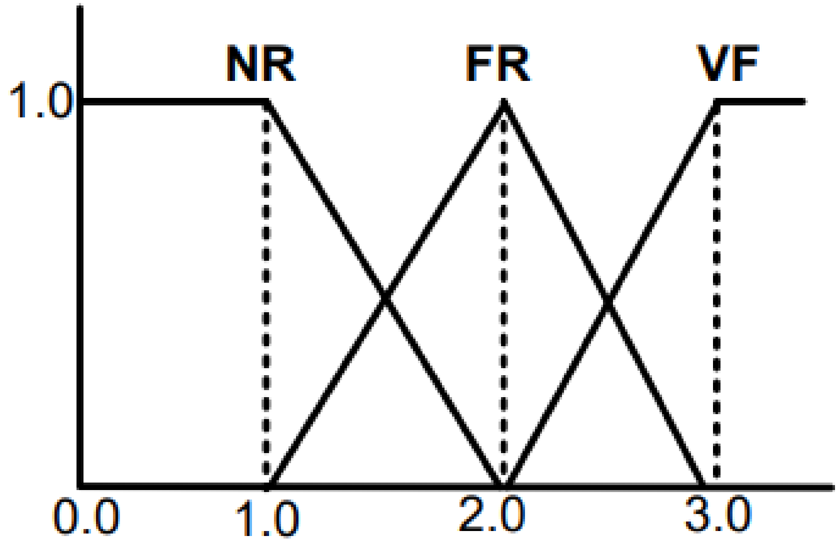
Now, We have to calculate the output of FLC for $\theta = 6^0$ and D = 2.2

Illustration

Distributions for $\boldsymbol{\theta}$ and $\boldsymbol{D}$ are as follows



$\theta \rightarrow$



$D \rightarrow$

Solution

a) The input θ = 6.0 can be called either L or M. Similarly, the input D = 2.2 can be declared either FR or VF.

b) Using the membership function graph, we have the following.

$\mu_L(\theta) = 0.8 \quad \mu_M(\theta) = 0.2$

$\mu_{FR}(D) = 0.8 \quad \mu_{VF}(D) = 0.2$

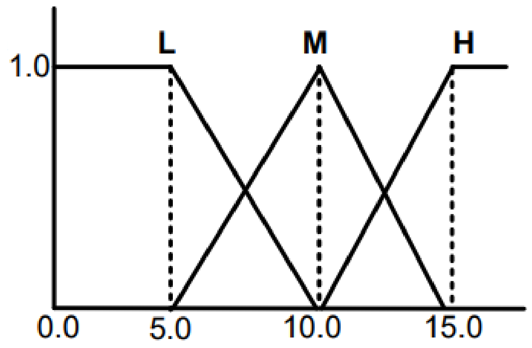
c) For the input set, following four rules can be fired out of all 9 rules.

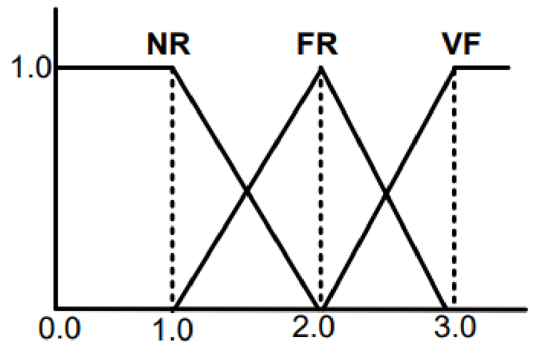
R1: θ is L and D is FR

R2: θ is L and D is VF

R3: θ is M and D is FR

R4: θ is M and D is VF





Solution:

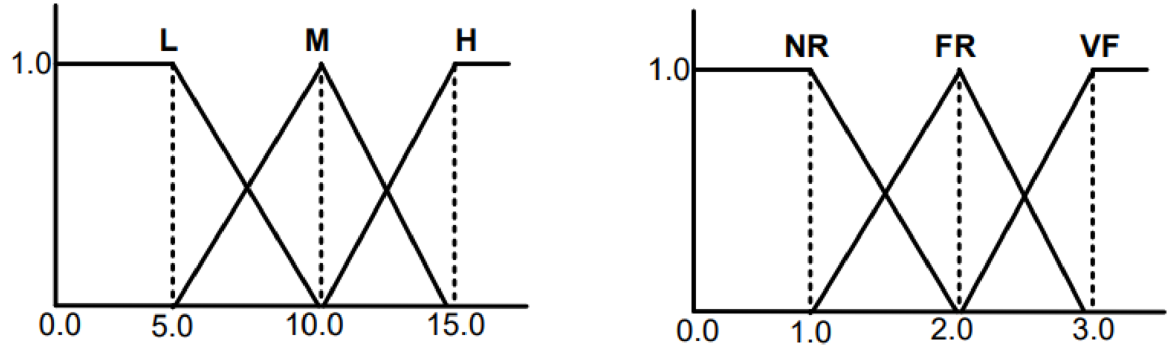
d) Now, the weights for each of the above rules can be determined as follows.

R1: $w^1 = \mu_L \times \mu_{FR} = 0.8 \times 0.8 = 0.6$

R2: $w^2 = \mu_L \times \mu_{VF} = 0.8 \times 0.2 = 0.16$

R3: $w^3 = \mu_M \times \mu_{FR} = 0.2 \times 0.8 = 0.16$

R4: $w^4 = \mu_M \times \mu_{VF} = 0.2 \times 0.2 = 0.6$



e) The functional consequent values for each rules can be calculated as below.

$y^1 = \theta + 2D = 6.0 + 2 \times 2.2 = 10.4$

$y^2 = \theta + 3D = 6.0 + 3 \times 2.2 = 12.6$

$y^3 = 2\theta + 2D = 2 \times 6.0 + 2 \times 2.2 = 16.4$

$y^4 = 2\theta + 3D = 2 \times 6.0 + 3 \times 2.2 = 18.6$

R1: θ is L and D is FR

R2: θ is L and D is VF

R3: θ is M and D is FR

R4: θ is M and D is VF

$y^i = f(\theta, D) = a_j^i\theta + b_k^i D;$

Ex: $a_1^i = 1$, $a_2^i = 2$, $a_3^i = 3$ if θ = L, M and H, respectively.

$b_1^i = 1$, $b_2^i = 2$, $b_3^i = 3$ if D = NR, FR, and VF, respectively.

Solution:

f) Therefore, the output y of the controller can be determined as follows.

$$\delta = \frac{w^1y^1+w^2y^2+w^3y^3+w^4y^4}{w^1+w^2+w^3+w^4} = 12.04$$

$$y = \frac{\sum_i^k w^i y^i}{\sum_i^k w^i}$$