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A COMPLETENESS THEOREM IN MODAL LOGIC[1]

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The present paper attempts to state and prove a completeness theorem for the system S5 of [1], supplemented by first-order quantifiers and the sign of equality. We assume that we possess a denumerably infinite list of individual variables $a, b, c, \ldots, x, y, z, \ldots, x_m, y_m, z_m, \ldots$ as well as a denumerably infinite list of n-adic predicate variables $P^n, Q^n, R^n, \ldots, P_m{}^n, Q_m{}^n, R_m{}^n, \ldots$; if $n=0$, an n-adic predicate variable is often called a "propositional variable." A formula $P^n(x_1, \ldots, x_n)$ is an n-adic prime formula; often the superscript will be omitted if such an omission does not sacrifice clarity. We adopt the primitive symbols $\wedge$, $\sim$, $\Box$, (x), $=$, respectively representing conjunction, negation, necessity, universal quantification, and identity; in terms of these and predicate variables we define the notion of a *well-formed formula*, or simply a *formula*, in the usual manner. Let A, B, C, etc. (with or without subscripts or accents) represent arbitrary formulas; sometimes we write these as $A(x_1, \ldots, x_n)$, etc., to call attention to certain variables. If a formula is given as $A(x)$, we define $A(y)$ as follows: *First*, if $A(x)$ contains any part $(y)B(y)$ containing x free, replace the variable y throughout that part by z, where z is the alphabetically earliest variable not occurring in $A(x)$. *Second, after the first replacements have been made*, replace all free occurrences of x by y. (On this definition, $(x)A(x) \supset A(y)$ *always holds*, without restrictions on substitution.) Analogous definitions are adopted when there is more than one variable involved. We define $A \vee B$ as $\sim(\sim A \wedge \sim B)$, $A \supset B$ as $\sim(A \wedge \sim B)$, $\Diamond A$ as $\sim\Box\sim A$, and $(\exists x)A(x)$ as $\sim(x)\sim A(x)$. For our formalization of S5 with quantifiers and equality we first take any formalization adequate for the classical first-order predicate calculus with equality, say that of Rosser [2] (pp. 101 and 163–4). We supplement this system by the following axiom schemes and rules of inference:[2]

A1: $\Box A \supset A$

A2: $\sim\Box A \supset \Box\sim\Box A$

A3: $\Box(A \supset B). \supset .\Box A \supset \Box B$

R1. If $\vdash A$ and $\vdash A \supset B$, $\vdash B$.

R2. If $\vdash A$, $\vdash \Box A$.

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[1] My thanks to the referee and to Professor H. B. Curry for their helpful comments on this paper and their careful reading of it. I must express an added debt of gratitude to Curry; without his constant encouragement of my research, publication of these results might have been delayed for years.

[2] See Prior [6] and the references given there.

We call the resulting system S5*$^{=}$; if equality is omitted, we call it S5*, and if quantifiers and equality are dropped, we call it S5.

Given a non-empty domain **D** and a formula A, we define a *complete assignment* for A in **D** as a *function* which to every free individual variable of A assigns an element of **D**, to every propositional variable of A assigns either **T** or **F**, and to every n-adic predicate variable of A assigns a set of ordered n-tuples of members of **D**. We define a *model* of A in **D** as an ordered pair (**G**, **K**), where **G** is a complete assignment for A in **D** and **K** is a set of complete assignments for A in **D** such that **G** ϵ **K** and such that every member of **K** agrees with **G** in its assignments for free individual variables of A (but not necessarily in its assignments for propositional and predicate variables of A). Let **H** be a member of **K** and B a subformula of A; we define **H** as assigning either **T** or **F** to B inductively, thus: If B is an n-adic prime formula $P(x_1, \ldots, x_n)$, and if ψ is the set of ordered n-tuples **H** assigns to P, and we assign elements of **D** $\alpha_1, \ldots, \alpha_n$ to $x_1, \ldots, x_n$ (the assignment must be consistent with **H**, but if some x_n is bound in A and hence not assigned an element, we make an arbitrary assignment), then B is assigned **T** if $(\alpha_1, \ldots, \alpha_n) \epsilon \psi$; otherwise, B is assigned **F**. Propositional variables are already, by hypothesis, assigned **T** or **F** by **H**. If B has the form $x = y$, it is assigned **T** if x and y are assigned the same element of **D**; otherwise it is assigned **F**. $\sim B$ is assigned **T**(**F**) if and only if B is assigned **F**(**T**). $B \wedge C$ is assigned **T** if B and C are both assigned **T**; otherwise it is assigned **F**. $(x)B(x)$ is assigned **T** if $B(x)$ is assigned **T** for every assignment of an element of **D** to x; otherwise, it is assigned **F**. $\Box B$ is assigned **T** if every member of **K** assigns **T** to B (subject to the stipulation that all members of **K** agree in their assignments to all free individual variables of B); otherwise, it is assigned **F**.

A is said to be *valid* in a model (**G**, **K**) of A in **D** if and only if A is assigned **T** by **G**. A is said to be *valid* in **D** if and only if A is valid in every model of A in **D**. A is said to be *satisfiable* in **D** if and only if there is some model of A in **D** in which A is valid. A is said to be *universally valid* if and only if A is valid in every non-empty domain.

The basis of the informal analysis which motivated these definitions is that a proposition is necessary if and only if it is true in all "possible worlds." (It is not necessary for our present purposes to analyze the concept of a "possible world" any further.) Now let A be a formula with $P_1, \ldots, P_m$ as its propositional and predicate variables and $x_1, \ldots, x_n$ as its free individual variables. If we interpret every free individual variable as denoting a particular object and every propositional or predicate variable as denoting a particular proposition or predicate, then A becomes a "proposition" in the ordinary sense of the word. From an extensional point of view, an adequate semantic counterpart to this interpretation is given by the concept of a complete assignment for A in a domain **D**. In modal logic, however, we wish to know not only about the real world but about other conceivable worlds; P may be true in the real world but false in some imaginable one, and similarly for $P(x_1, \ldots, x_n)$. Thus we are led not to a single assignment but to a set **K** of assignments, all but one of which represent worlds which are conceivable but not actual; the assignment representing the actual world is singled out as **G**, and the pair (**G**, **K**) is said to form a model of A. Furthermore, since $x_1, \ldots, x_n$ represent individual objects, which remain the same in all worlds, we assume that all members of **K** agree in their assignments to individual variables. Clearly all the rules for assigning **T** or **F** to formulas now become valid when they are interpreted as representing an evaluation of the proposition corresponding to the formula as true or false in a given "world," whether real or possible. In particular, a proposition $\Box B$ is evaluated as true when and only when B holds in all conceivable worlds. A proposition can be said to be true if it holds in the actual world; this idea leads to our definition of validity in a model. In trying to construct a definition of universal logical validity, it seems plausible to assume not only that the universe of discourse may contain an arbitrary number of elements and that predicates may be assigned any given interpretations in the actual world, but also that any combination of possible worlds may be associated with the real world with respect to some group of predicates. In other words, it is plausible to assume that no further restrictions need be placed on **D**, **G**, and **K**, except the standard one that **D** be non-empty. This assumption leads directly to our definition of universal validity.

It is noteworthy that the theorems of this paper can be formalized in a metalanguage (such as Zermelo set theory) which is "extensional," both in the sense of possessing set-theoretic axioms of extensionality *and* in the sense of postulating no sentential connectives other than the truth-functions. Thus it is seen that at least a certain non-trivial portion of the semantics of modality is available to an extensionalist logician.

We shall now turn to our completeness proof. We base it on the concept of semantic tableaux introduced by Beth [4]. The present treatment is self-contained, although acquaintance with Beth's paper may facilitate comprehension.

We say that a formula B is *semantically entailed* by formulas $A_1, A_2, \ldots, A_n$ if and only if $A_1 \wedge A_2 \wedge \ldots A_n . \supset . B$ is universally valid; if $n=0$, this concept coincides with that of the universal validity of B.

A semantic tableau is a device for testing whether or not a given formula is semantically entailed by other given formulas. Clearly a necessary and sufficient condition that $A_1, \ldots, A_n$ should *not* entail B is that there should exist a model in which $A_1, \ldots, A_n$ are valid and B is not. We represent this situation by putting $A_1, \ldots, A_n$ in the left column of a tableau and B in the right column. Various other tableaux will be introduced later as a result of the rule Yr given below; these tableaux are called *auxiliary tableaux*, while the tableau initially introduced is the *main* tableau. Thus, in general, we are dealing not with a single tableau, but with a set of tableaux from which one member has been singled out as the main tableau. Indeed, as will be seen from rule Λr below, a construction may introduce a *system* of such sets, each set of the system being called an "alternative set." Given, then, a main tableau with $A_1, \ldots, A_n$ in the left column and B in the right column, we continue our construction by the following rules (which apply to any tableau of the set, main or auxiliary):[3]

Nl. If $\sim A$ appears in the left column of a tableau, put A in the right column of that tableau.

Nr. If $\sim A$ appears in the right column of a tableau, put A in the left column of the tableau.

Λl. If $A \wedge B$ appears in the left column of a tableau, put both A and B in the left column of the tableau.

Λr. If $A \wedge B$ appears in the right column of a tableau, there are two alternatives — either put A in the right column or put B in the right column. In this case we say that the tableau *splits* into two alternative tableaux. If the splitting tableau is the main tableau of a set of tableaux, the resulting alternative tableaux are main tableaux of two alternative sets; otherwise they are auxiliary tableaux of alternative sets.

Πl. If $(x)A(x)$ appears in the left column of a tableau, and a is a variable which occurs free in either column of any tableau of the set, then put $A(a)$ in the left column of the same tableau which contains $(x)A(x)$ in its left column.

Πr. If $(x)A(x)$ appears in the right column of a tableau, then we introduce a variable a which has not yet appeared in any tableau of the set, and we put $A(a)$ in the right column of the same tableau containing $(x)A(x)$ on the right.

Il. If $a=b$ (for some variables a and b) appears in the left column of a tableau, then in both columns of every tableau of the set we replace every formula $A(a, b)$ by $A(b, b)$.

Ir. No rule.

Yl. If $\Box A$ appears in the left column of a tableau, then we put A in the left column of every tableau of the set.

Yr. If $\Box A$ appears in the right column of a tableau, then we introduce a new auxiliary tableau which is started out by putting A in its right column.

In addition to these rules for construction of semantic tableaux, we add that if no free variable appears and none is introduced under Πr, then we introduce a free variable so that Πl can be applicable.

[3] The names of these rules were suggested by those for the inferential rules of Curry [8].

A tableau is said to be *closed* if and only if either a formula occurs in both of its columns or $a=a$, for some variable a, occurs in its right column. A set of tableaux is closed when and only when at least one of its members (either main or auxiliary) is closed. Because of Λr, a construction beginning with $A_1, \ldots, A_n$ in a left column and B in a right column may split into alternative sets; in this case we say that the construction is closed if and only if all its alternative sets are closed.

THEOREM 1. B is semantically entailed by $A_1, \ldots, A_n$ if and only if the construction beginning with $A_1, \ldots, A_n$ in a left column and B in a right column is closed.

PROOF. The theorem follows from the following two lemmas.

LEMMA 1. If a construction beginning with $A_1, \ldots, A_n$ on the left and B on the right is closed, then B is semantically entailed by $A_1, \ldots, A_n$.

PROOF. Assume for *reductio ad absurdum* that B is not semantically entailed by $A_1, \ldots, A_n$. Then there is a non-empty domain $\mathbf{D}$ and model $(\mathbf{G}, \mathbf{K})$ of $A_1 \wedge A_2 \wedge \ldots A_n . \supset .B$ (or of B, if $n=0$) in $\mathbf{D}$ such that $A_1 \wedge A_2 \wedge \ldots A_n . \supset .B$ (or B) is not valid in $(\mathbf{G}, \mathbf{K})$. We shall show that every statement on the left (right) side of the main tableau of the construction beginning with $A_1, \ldots, A_n$ on the left and B on the right is assigned $\mathbf{T}(\mathbf{F})$ by $\mathbf{G}$. Further, we shall show that every auxiliary tableau corresponds in the same manner to some member of $\mathbf{K}$.

Since $A_1 \wedge \ldots A_n . \supset .B$ (or B) is not valid in $(\mathbf{G}, \mathbf{K})$, it is assigned $\mathbf{F}$ by $\mathbf{G}$. By the valuation rules for "$\wedge$" and "$\sim$" and the definition of "$\supset$", $A_1, \ldots, A_n$ are all assigned $\mathbf{T}$ and B is assigned $\mathbf{F}$ by $\mathbf{G}$. By the valuation rules for $\sim$, if $\sim C$ is assigned $\mathbf{T}$, C must be assigned $\mathbf{F}$; this fact validates Nl. Similarly we can validate Nr and Λl. If $C \wedge D$ is assigned $\mathbf{F}$, either C or D must be assigned $\mathbf{F}$; Λr thus correctly instructs us to consider these two alternative possibilities. If $(x)A(x)$ is assigned $\mathbf{T}$, every element of $\mathbf{D}$ must be assigned $\mathbf{T}$; hence Πl is valid. If $(x)A(x)$ is assigned $\mathbf{F}$, then there is a member a of $\mathbf{D}$ such that $A(a)$ is assigned $\mathbf{F}$; hence Πr is valid. If $a=b$ is assigned $\mathbf{T}$, then a and b are assigned the same member of $\mathbf{D}$; hence the substitution described in Il is valid. If $\Box A$ is assigned $\mathbf{T}$, then every member of $\mathbf{K}$ assigns $\mathbf{T}$ to A; hence Yl is valid. If $\Box A$ is assigned $\mathbf{F}$, there exists a member of $\mathbf{K}$ which assigns $\mathbf{F}$ to A; hence Yr is justified. Finally, our stipulation that at least one free variable should be introduced corresponds to the restriction that $\mathbf{D}$ be non-empty.

Since the construction is closed, every alternative set contains a tableau which either has a formula in both columns or has $a=a$ on the right. This, however, means that some member of $\mathbf{K}$ must either assign both $\mathbf{T}$ and $\mathbf{F}$ to some formula or must assign $\mathbf{F}$ to $a=a$. Since our valuation rules make both of these alternatives impossible, it follows that the domain $\mathbf{D}$ and model $(\mathbf{G}, \mathbf{K})$ cannot exist. Q.E.D.

LEMMA 2. If the construction beginning with $A_1, \ldots, A_n$ on the left and B on the right is not closed, then B is not semantically entailed by $A_1, \ldots, A_n$.

PROOF. Since the construction is not closed, there exists a set of tableaux, one of the construction's alternative sets, which is not closed. We shall choose such a set and ignore all other alternative sets. Let **D** be the set of all free variables occurring in our set of tableaux (and not eliminated by an application of Il). For every tableau in the set, we define an assignment for $A_1 \wedge \ldots A_n. \supset .B$ as follows: Every free variable which is not eliminated by Il, is assigned to itself; a free variable eliminated by Il is assigned the variable which replaces it. A propositional variable is assigned **T** if it appears on the left in the tableau; otherwise it is assigned **F**. A predicate variable P^n is assigned the set of all ordered n-tuples $(x_1, \ldots, x_n)$ such that $P^n(x_1, \ldots, x_n)$ appears on the left of the tableau. We now have a set **K** of complete assignments corresponding to our set of tableaux; if **G** is the assignment corresponding to the main tableau of the set, (**G**, **K**) is a model of $A_1 \wedge \ldots A_n. \supset .B$ in **D**.

We now show by induction on the number of symbols of C that any formula C occurring on the left (right) side of a tableau is assigned **T**(**F**) by the corresponding assignment function. Clearly this is true for prime formulas (including propositional variables) occurring on the left. If they occur on the right, then since the tableau is not closed, they cannot occur on the left; hence they are assigned **F**. If an equality formula $a=b$ occurs on the left, then by Il, it is replaced by $b=b$; this latter formula must be assigned **T**. If $a=b$ occurs on the right, since the tableau is not closed, a and b must be distinct variables which remain distinct after all replacements by Il. Hence, by the given assignment for free variables, a and b are assigned to distinct objects, and $a=b$ is thus assigned **F**. If $\sim C$ appears on the left, by Nl C appears on the right; hence, by the hypothesis of the induction, C is assigned **F**. By our valuation rules, $\sim C$ is assigned **T**. Similarly we can treat the cases where $\sim C$ appears on the right or $C_1 \wedge C_2$ or $(x)C(x)$ appears on either side. If $\Box C$ appears on the left, by Yl C appears in the left column of every tableau of the set; hence, by the hypothesis of the induction, C is assigned **T** by every member of **K**. Hence, by our valuation rules, $\Box C$ is assigned **T**. Finally, if $\Box C$ occurs on the right, by Yr C appears on the right in some tableau of the set; hence, by the hypothesis of the induction, C is assigned **F** in some member of **K**. Hence, by our valuation rules, $\Box C$ is assigned **F**.

Since $A_1, \ldots, A_n$ occur on the left column of the main tableau, they are assigned **T** by **G**; similarly **G** assigns **F** to B. By our valuation rules, **G** assigns **F** to $A_1 \wedge \ldots A_n. \supset .B$; hence $A_1 \wedge \ldots A_n. \supset .B$ is not valid in (**G**, **K**). Therefore B is not semantically entailed by $A_1, \ldots, A_n$. Q.E.D.

THEOREM 2. If a formula is satisfiable in a non-empty domain, it is valid in a model (**G**, **K**) in a domain **D**, where **D** and **K** are both either finite or denumerable. If a formula is valid in every finite (non-empty) or denumerable domain, it is universally valid.

PROOF. The second sentence of the theorem follows easily from the first. If a formula B is satisfiable in some non-empty domain, $\sim B$ is not universally valid; hence, by Theorem 1, the construction started by putting $\sim B$ in a right column is not closed. Hence the proof of Lemma 2 constructs a particular domain **D** and model (**G**, **K**) in which $\sim B$ is not valid, *i.e.*, in which B is valid. Clearly, however, **D** and **K** are both finite or denumerable, as can be seen from the rules by which tableaux are constructed.

THEOREM 3. If a formula not containing the sign of equality is satisfiable in some non-empty domain, it is valid in a model (**G**, **K**) in a domain **D**, where **K** is finite or denumerable and **D** is denumerable. Further, if it is valid in every denumerable domain, it is universally valid.

PROOF. This theorem follows easily from Theorem 2 and the following lemma:

LEMMA 3. If a formula A not containing the sign of equality is valid in a model (**G**, **K**) in a non-empty domain **D**, and **D** is a subset of **D**$'$, then A is valid in a model (**G**$'$, **K**$'$) in **D**$'$, where **K** and **K**$'$ are equinumerous.

PROOF. Since **D** is non-empty, let a be an element of **D**. For every assignment **H** ϵ **K** we define **H**$'$, an assignment for A in **D**$'$, thus. For all free individual variables and propositional variables, **H**$'$ makes the same assignments as **H**. If **H** assigns to a predicate variable P^n a set **S** of ordered n-tuples of elements of **D**, **H**$'$ assigns to P^n a set **S**$'$ containing all n-tuples of **S**, plus any n-tuple obtained by replacing a in some of its occurrences by any elements contained in **D**$'$ but not in **D**. **K**$'$ is then obtained from **K** by replacing every **H** ϵ **K** by **H**$'$. It is now easy to prove that A is valid in (**G**$'$, **K**$'$).

THEOREM 4. If, in Theorems 2 and 3, the formula in question does not contain "$\Box$", then **K** can be stipulated to be the unit set of **G**.

PROOF. This can easily be proved by analysis of the construction of **K** in Theorems 2 and 3.

Clearly Theorems 2 and 3 are the modal analogues of the Löwenheim-Skolem Theorem. Theorem 4 states (roughly) that if modality is not present, the usual form of the Löwenheim-Skolem Theorem can be obtained. Furthermore, we can extend our versions of the Löwenheim-Skolem Theorem to joint satisfiability of infinitely many formulas, if we allow a construction to start by putting infinitely many formulas in either or both columns.

Although, as Beth has shown in [4], a tableau construction may proceed indefinitely, and thus it may conceivably introduce infinitely many variables, formulas, and tableaux, it is equally clear that if we start a construction with finitely many formulas in both columns of a main tableau, after any finite number of applications of the rules only finitely many formulas, variables, and tableaux are introduced. We call the stage in which $A_1, \ldots, A_n$ are put in the left column and B in the right column the initial stage of the construction; the stage at which the mth rule has been applied is the $m+1$th stage.

We define the characteristic formula of a given tableau at a particular stage as $A_1 \wedge \ldots A_m \wedge \sim B_1 \wedge \ldots \sim B_n$, where $A_1, \ldots, A_m$ $(B_1, \ldots, B_n)$ are the formulas found on the left (right) side of the tableau at this stage. We then define the characteristic formula of any one of the alternative sets at a given stage as $(\exists a_1)(\exists a_2)\ldots(\exists a_p)(A \wedge \Diamond B_1 \wedge \ldots \Diamond B_q)$, where A is the characteristic formula of the main tableau of the set, $B_1, \ldots, B_q$ are the characteristic formulas of the auxiliary tableaux of the set, and $a_1, \ldots, a_p$ are the free variables of $A \wedge \Diamond B_1 \wedge \ldots \Diamond B_q$. Finally we define the characteristic formula of a stage as $D_1 \vee \ldots D_x$, where $D_1, \ldots D_x$ are the characteristic formulas of the alternative sets of the stage. Clearly the characteristic formula of the initial stage is $(\exists a_1)(\exists a_2)\ldots(\exists a_p)(A_1 \wedge \ldots A_y \wedge \sim B)$ where $A_1, \ldots, A_y$ are the formulas put on the left and B is the formula put on the right; if $y=0$, the characteristic formula is simply $(\exists a_1) \ldots (\exists a_p) \sim B$.

LEMMA 4. If A is the characteristic formula of the initial stage of a construction, and B is the characteristic formula of any stage of the construction, then $\vdash A \supset B$ in S5*=.

PROOF. We shall show that the characteristic formula of the nth stage of a construction implies the characteristic formula of the $n+1$th stage. From this fact our lemma follows easily, using the transitivity of implication. Let A, then, be the characteristic formula of a stage and B be the characteristic formula of the following stage; we are to show that $\vdash A \supset B$ in S5*=. In general A will be an alternation $A_1 \vee \ldots A_m$, representing several alternative sets. The rule by which B is obtained from A will affect only one of these alternative sets; *i.e.*, it will change A_x $(1 \leqq x \leqq m)$ to $A_{x'}$ and leave the other components unchanged. Since $\vdash A_x \supset A_{x'} :\supset: A_1 \vee \ldots A_x \vee \ldots A_m . \supset . A_1 \vee \ldots A_{x'} \vee \ldots A_m$ is valid in S5, it is sufficient to show that $\vdash A_x \supset A_{x'}$ and thus ignore the other alternative sets. If A_x is $(\exists a_1)\ldots(\exists a_p)B$ and $A_{x'}$ is $(\exists a_1)\ldots(\exists a_n)B'$, clearly it suffices to show $\vdash B \supset B'$ in order to show $\vdash A_x \supset A_{x'}$. Since every rule except Il, Yl, and Yr applies to only one tableau of a set, it is sufficient in all cases but these three to consider the characteristic formula of this tableau alone and ignore the rest of the set. Let the rule being considered transform a tableau with characteristic formula C to one with characteristic formula C'. If the tableau is auxiliary we actually are required to prove $\vdash \Diamond C \supset \Diamond C'$, but this can come from $\vdash C \supset C'$ by R2, $\vdash \Box(C \supset C') . \supset . \Diamond C \supset \Diamond C'$, and Rl. Let C then be $D_1 \wedge \ldots D_p$, generally the rule will operate on a single-formula D_y $(1 \leqq y \leqq p)$ so as to change $D_1 \wedge \ldots D_y \wedge \ldots D_p$ $(=C)$ into $D_1 \wedge \ldots D_y \wedge \ldots D_p \wedge E$ $(=C')$. Clearly in order to prove $\vdash D_1 \wedge \ldots D_y \wedge \ldots D_p . \supset . D_1 \wedge \ldots D_y \wedge \ldots D_p \wedge E$ it is sufficient to prove $\vdash D_y \supset E$. Bearing these preliminary remarks in mind, we consider the following cases:

Case Nl. This case is justified by $\vdash \sim A \supset \sim A$.

Case Nr, justified by $\vdash \sim\sim A \supset A$.

Case Λl, justified by $\vdash A \wedge B . \supset . A \wedge B$.

Case Λr. Let the characteristic formula of the set to which Λr is being applied be either $(\exists a_1) \ldots (\exists a_p)(C \wedge \Diamond(D \wedge \sim(A \wedge B)))$ or $(\exists a_1) \ldots (\exists a_p)$ $(C \wedge D \wedge \sim(A \wedge B))$, where $A \wedge B$ is the formula to which Λr is applied; the first formula is applicable if Λr is applied to an auxiliary tableau, the second if Λr is applied to a main tableau. Assume the characteristic formula to be as in the first case: Then we have $\vdash D \wedge \sim(A \wedge B) : \supset : D \wedge \sim(A \wedge B) \wedge \sim A . \vee . D \wedge \sim(A \wedge B) \wedge \sim B$. Hence we have $\vdash \Diamond(D \wedge \sim(A \wedge B)) . \supset . \Diamond(D \wedge \sim(A \wedge B) \wedge \sim A . \vee . D \wedge \sim(A \wedge B) \wedge \sim B)$. By a well-known theorem of S5, we have $\vdash \Diamond(D \wedge \sim(A \wedge B) \wedge \sim A . \vee . D \wedge \sim(A \wedge B) \wedge \sim B) . \supset . \Diamond(D \wedge \sim(A \wedge B) \wedge \sim A) \vee \Diamond(D \wedge \sim(A \wedge B) \wedge \sim B)$; hence we have $\vdash \Diamond(D \wedge \sim(A \wedge B)) . \supset . \Diamond(D \wedge \sim(A \wedge B) \wedge \sim A) \vee \Diamond(D \wedge \sim(A \wedge B) \wedge \sim B)$. From this we get $\vdash C \wedge \Diamond(D \wedge \sim(A \wedge B)) : \supset : C . \wedge . \Diamond(D \wedge \sim(A \wedge B) \wedge \sim A) \vee \Diamond(D \wedge \sim(A \wedge B) \wedge \sim B)$; since $\vdash C . \wedge . \Diamond(D \wedge \sim(A \wedge B) \wedge \sim A) \vee \Diamond(D \wedge \sim(A \wedge B) \wedge \sim B) : \supset : C \wedge \Diamond(D \wedge \sim(A \wedge B) \wedge \sim A) . \vee . C \wedge \Diamond(D \wedge \sim(A \wedge B) \wedge \sim B)$, we obtain $\vdash C \wedge \Diamond(D \wedge \sim(A \wedge B)) : \supset : C \wedge \Diamond(D \wedge \sim(A \wedge B) \wedge \sim A) . \vee . C \wedge \Diamond(D \wedge \sim(A \wedge B) \wedge \sim B)$. Attaching existential quantifiers, we have the desired result. Similarly for the second case.

Case Πl. Justified by $\vdash (x)A(x) \supset A(a)$.

Case Πr. Let the characteristic formula of the set to which we apply Πr be $(\exists a_1) \ldots (\exists a_p)(D \wedge \Diamond(E \wedge \sim(x)A(x)))$ or $(\exists a_1) \ldots (\exists a_p)(D \wedge E \wedge \sim(x) A(x))$. We shall consider the first case alone. Let b be a variable not occurring in D, E, or $(x)A(x)$. We have $\vdash E \wedge \sim(x)A(x) . \supset . (\exists b)(E \wedge \sim(x)A(x) \wedge \sim A(b))$. From this we obtain $\vdash \Diamond(E \wedge \sim(x)A(x)) . \supset . \Diamond(\exists b)(E \wedge \sim(x)A(x) \wedge \sim A(b))$. By a theorem of Prior [6], we have in S5*
$\vdash \Diamond(\exists b)(E \wedge \sim(x)A(x) \wedge \sim A(b)) . \supset . (\exists b)\Diamond(E \wedge \sim(x)A(x) \wedge \sim A(b))$; from this it follows that $\vdash \Diamond(E \wedge \sim(x)A(x)) \supset (\exists b)\Diamond(E \wedge \sim(x)A(x) \wedge \sim A(b))$. From this we obtain easily $\vdash (\exists a_1) \ldots (\exists a_p)(D \wedge \Diamond(E \wedge \sim(x)A(x))) . \supset . (\exists a_1) \ldots (\exists a_p)(\exists b)(D \wedge \Diamond(E \wedge \sim(x)A(x) \wedge \sim A(b)))$.

Case Il. The characteristic formula of the set is $(\exists a_1) \ldots (\exists a_x) \ldots (\exists a_y) \ldots (\exists a_p)(D \wedge \Diamond(E \wedge a_x = a_y))$ or $(\exists a_1) \ldots (\exists a_x) \ldots (\exists a_y) \ldots (\exists a_p)(D \wedge E \wedge a_x = a_y)$; we consider the first case alone. Clearly $\vdash \Diamond(E \wedge a_x = a_y) \supset \Diamond(a_x = a_y)$. By a theorem of Quine [5] p. 80, formula (52), we have $\vdash a_x = a_y \supset \Box(a_x = a_y)$;

hence by a theorem of Prior [6], section 3, we have $\vdash\Diamond(a_x=a_y)\supset a_x=a_y$, and hence $\vdash\Diamond(E\wedge a_x=a_y)\supset a_x=a_y$. From this we obtain easily $\vdash(\exists a_1)\ldots(\exists a_x)\ldots(\exists a_y)\ldots(\exists a_p)(D\wedge\Diamond(E\wedge a_x=a_y)).\supset.(\exists a_1)\ldots(\exists a_x)\ldots(\exists a_y)\ldots(\exists a_p)(a_x=a_y\wedge D\wedge\Diamond(E\wedge a_x=a_y))$. Since we have $\vdash(\exists a_1)\ldots(\exists a_x)\ldots(\exists a_y)\ldots(\exists a_p)(a_x=a_y\wedge D\wedge\Diamond(E\wedge a_x=a_y)).\supset.(\exists a_1)\ldots(\exists a_r)\ldots(\exists a_y)\ldots(\exists a_p)E'$, where E' is the result of replacing a_x by a_y (after making any necessary changes in the bound variables) in $D\wedge\Diamond(E\wedge a_x=a_y)$, our result is proved.

Case Yl. Let the characteristic formula of the set involved be $(\exists a_1)\ldots(\exists a_p)(D\wedge\Diamond(E\wedge\Box A))$ or $(\exists a_1)\ldots(\exists a_p)(D\wedge E\wedge\Box A)$; as usual we shall discuss the first case. We have $\vdash\Diamond(E\wedge\Box A)\supset\Diamond\Box A$; and since in S5 $\vdash\Diamond\Box A\supset\Box A$, $\vdash\Diamond(E\wedge\Box A)\supset\Box A$. Further we have $\vdash\Box A:\supset C.\supset.A\wedge C$; this justifies putting C in the left column of any main tableau. Similarly $\vdash\Box A.\supset.\Diamond C\supset\Diamond(C\wedge A)$ justifies putting A in the left column of an auxiliary tableau.

Case Yr. Let the characteristic formula of the set to which Yr is applied be $(\exists a_1)\ldots(\exists a_p)(D\wedge\Diamond(E\wedge\sim\Box A))$ or $(\exists a_1)\ldots(\exists a_p)(D\wedge E\wedge\sim\Box A)$; as usual, we consider the first case. We have $\vdash\Diamond(E\wedge\sim\Box A)\supset\Diamond\sim\Box A$; further since in S5 we have $\vdash\Diamond\sim\Box A\supset\Diamond\sim A$, we have $\vdash D\wedge\Diamond(E\wedge\sim\Box A).\supset D\wedge\Diamond(E\wedge\sim\Box A)\wedge\Diamond\sim A$. But $\Diamond\sim A$ is the characteristic formula of the new tableau introduced by Yr; hence our result follows, attaching the existential quantifiers. Q.E.D.

THEOREM 5. If A is universally valid, then $\vdash A$ in S5*=.

PROOF. Since A is universally valid, by Theorem 1 the tableau construction beginning with A in a right column is closed. Let B be the characteristic formula of the earliest stage at which the closure is provable (i.e., the earliest stage at which every alternative set contains a tableau with either a formula in both columns or a formula $a=a$ on the right). Then by Lemma 4, $\vdash(\exists a_1)\ldots(\exists a_p)\sim A.\supset.B$; we shall prove $\vdash\sim B$, from which $\vdash\sim(\exists a_1)\ldots(\exists a_p)\sim A$, and hence ultimately $\vdash A$, follows easily. In general B will be of the form $C_1\vee\ldots C_n$, where the C_x's represent alternative sets; in order to prove $\vdash\sim B$ it is sufficient to prove $\vdash\sim C_x$ for every x, $1\leqq x\leqq n$. Again C_x is of the form $D_0\wedge\Diamond D_1\wedge\ldots\Diamond D_m$; since the set is closed, there exists a tableau of the set, represented by $D_y(0\leqq y\leqq m)$, which is closed. Since, using R2, $\vdash\sim D_y$ implies $\vdash\sim\Diamond D_y$, it is clearly sufficient to prove $\vdash\sim D_y$ in order to obtain $\vdash\sim C_x$. By the definitions of closure and characteristic formula, since the tableau corresponding to D_y is closed, D_y must contain either two conjunction terms E and $\sim E$ or a conjunction term $\sim a=a$. Either case suffices to prove $\vdash\sim D_y$. Q.E.D.

Theorem 5 is our completeness theorem for the system S5*=; if equality (equality and quantification) is (are) dropped, the proofs of Lemma 4 and Theorem 5 still hold for S5* (S5). Combining Theorems 2, 3, and 5, we obtain the following corollaries:

COROLLARY 1. If a formula A of S5*= is valid in every finite (non-empty) or denumerable domain, then $\vdash A$ in S5*=.

COROLLARY 2. If a formula A of S5* is valid in every denumerable domain (or by isomorphism of equinumerous domains, in a single denumerable domain), then $\vdash A$ in S5*.

We shall now prove a consistency theorem for S5*=, the converse of our Theorem 5.

THEOREM 6. If $\vdash A$ in S5*=, A is universally valid.

PROOF. Construction of appropriate semantic tableaux will verify that every axiom of S5*= is universally valid. The valuation rules for "$\supset$" suffice to show that if A and $A \supset B$ are universally valid, so is B; this validates Rl. If A is universally valid, by Theorem 1 a tableau construction beginning with A on the right eventually closes. If we begin a construction with $\Box A$ on the right, Yr instructs us to put A on the right in a new tableau. Since this construction closes, so does the construction beginning with $\Box A$ on the right; hence, by Theorem 1, $\Box A$ is universally valid. This validates R2 for universal validity.

THEOREM 7. $\vdash A$ in S5*= if and only if A is universally valid.

For the propositional calculus, it is customary to determine universal validity by means of truth tables. Although the semantic tableaux already give a convenient decision procedure for S5, it will be instructive to construct analogous truth tables for S5. An ordinary classical truth table is a set of possible valuations of the propositional variables; each set of possible valuations for each propositional variable is determined by a row of the table. We then evaluate a formula using the usual rules. For S5 truth tables we adopt a similar definition, except that in any table some (but not all) rows may be omitted. Thus a formula has many truth tables, depending on how many rows are omitted. We evaluate "$\wedge$" and "$\sim$" according to the usual methods. In any truth table $\Box A$ is assigned **T** in every row if A is assigned **T** in every row; otherwise $\Box A$ is assigned **F** in every row. A formula B is a tautology of S5 if and only if it is assigned **T** in every row of each of its tables. Clearly a truth table for B corresponds to a set **K** of assignments for B, since by hypothesis B contains only propositional variables to be assigned **T** or **F**. If we pick out a particular row of a table as its designated row and let the corresponding assignment be **G**, we obtain a model (**G**, **K**) for B. (In the propositional calculus, reference to a domain **D** is unnecessary.) Using these observations it is easy to prove that for formulas of S5 our notion of tautology coincides with our notion of universal validity.

THEOREM 8. $\vdash A$ in S5 if and only if A is a tautology of S5.

PROOF. Show the equivalence of tautologyhood and universal validity, and use Theorem 7 restricted to S5.

Alternative proof. The proof just given, when written in detail, is completely finitary and rigorous. Nevertheless it may be argued that we should give our proof a more familiar form, say analogous to the treatments of ordinary tautology by Kalmár's method in Rosser [2] (pp. 70–74) and Kleene [1] (Sections 29, 30). We shall outline such a proof, although details will not be given. We prove two lemmas. First define the characteristic formula of a row as $P_1 \wedge \ldots P_m \wedge \sim Q_1 \wedge \ldots \sim Q_n$, where the P_x's(Q_x's) are the propositional variables assigned **T** (**F**) by the row. The characteristic formula of a table with designated row is $A_0 \wedge \Diamond A_1 \wedge \ldots \Diamond A_p \wedge \sim\Diamond B_1 \wedge \ldots \sim\Diamond B_q$, where A_0 is the characteristic formula of the designated row of the table, $A_1, \ldots, A_p$ are the characteristic formulas of the other rows of the table, and $B_1, \ldots, B_q$ are the characteristic formulas of the rows omitted from the table. Let C be the characteristic formula of a table with designated row and let D be the formula evaluated by the table. The first lemma is that if the designated row assigns **T**(**F**) to D, $\vdash C \supset D$ ($\vdash C \supset \sim D$). (In particular if D is a tautology, $\vdash C \supset D$ for every possible C.) This lemma can be proved by induction on the number of symbols in D. The second lemma is that for any D, the alternation of the characteristic formulas of all possible tables with designated rows for D is a theorem of S5. From these two lemmas the completeness part of our theorem follows easily. The consistency part is proved by observing that all axioms of S5 are tautologies and that R1 and R2 yield only tautologies when applied to tautologies.[4]

Thus far, we have not permitted quantification on propositional variables. We now define a system S5 with propositional quantifiers, containing propositional quantifiers and the following axiom schemes (besides those of S5):

(4) $(P)A(P) \supset A(Q)$, subject to the usual restrictions on substitution.

(5) $(P)(A(P) \supset B(P)) . \supset .(P)A(P) \supset (P)B(P)$.

(6) $A \supset (P)A$, if P is not free in A.

(7) $(\exists P_1)\ldots(\exists P_n)A$, where A is the characteristic formula of an S5 truth table with designated row (as defined in Theorem 8), and $P_1, \ldots, P_n$ are its free propositional variables.

(7) is a greatly strengthened version of the B9 of [1]; the reader should test its plausibility by actual examples. We further agree that any universal propositional quantification of an axiom is an axiom. The rules of inference are R1 and R2.

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[4] In earlier work I carried out this alternative proof in detail, before acquaintance with Beth's paper led me to generalize the truth tables to semantic tableaux and a completeness theorem.

THEOREM 9. Let A be a formula of S5 unprovable in S5 and let $P_1, \ldots, P_n$ be its free propositional variables. Then if $(P_1)\ldots(P_n)A$ is added to S5 with propositional quantifiers, the resulting system is inconsistent.

PROOF. Since A is not provable in S5, by Theorem 8 some truth table contains a row in which A is assigned **F**. Let B be the characteristic formula of the table, with some **F**-row of A as the designated row. By the first lemma in the alternative proof of Theorem 8 (or by Theorem 8 itself), $\vdash B \supset \sim A$ in S5, Then in S5 with propositional quantifiers, $\vdash (P_1, \ldots, P_n)(B \supset \sim A)$, and hence $\vdash (\exists P_1)\ldots(\exists P_n)B \supset (\exists P_1)\ldots(\exists P_n)\sim A$. Since $(\exists P_1)\ldots(\exists P_n)B$ is an instance of axiom scheme (7), we have $\vdash (\exists P_1)\ldots(\exists P_n)\sim A$, contradicting $(P_1)\ldots(P_n)A$.

Theorem 9 is a completeness result for S5 analogous to Corollary 2, p. 134 of Kleene [7].[5]

Theorem 9 can be reformulated as stating that if a formula A of S5 with propositional quantifiers is unprovable in that system, then adding the closure of A renders the system inconsistent, *as long as A does not itself contain propositional quantifiers.* [(*Added December 19, 1958*) The italicized restriction can be removed by extending the tableau constuction to formulas with propositional quantifiers. In this way we could obtain a completeness theorem for S5 with propositional quantifiers. The details are not given here.]

The completeness theorem given in the present paper is based on the system S5. It is well known that many alternative modal systems exist; five distinct systems are proposed in [1] alone. Further, if modal logic is extended to admit quantification and identity, there are other controversial laws such as $(x)\Box A(x) \supset \Box(x)A(x)$ and $(a, b)(a=b \supset \Box a=b)$. Some of these systems, alternative to S5*=, lead to alternative notions of completeness; and any comparison of them for "acceptability" can be based on an examination of these alternative semantical notions. The details of such considerations will appear in a sequel to the present paper.

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[5] An alternative formulation of Theorem 9, avoiding the machinery of propositional quantifiers, can be obtained in S5, with a postulated substitution rule for propositional variables. In such a system we let all formulas of the form (7), with the existential quantifiers replaced by a single negation sign, be postulated as directly refutable (see [3]). Then the resulting system is complete, in the sense that every formula is either provable or refutable; hence if we add an unprovable formula to the system, we obtain inconsistency in the sense of [3].

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