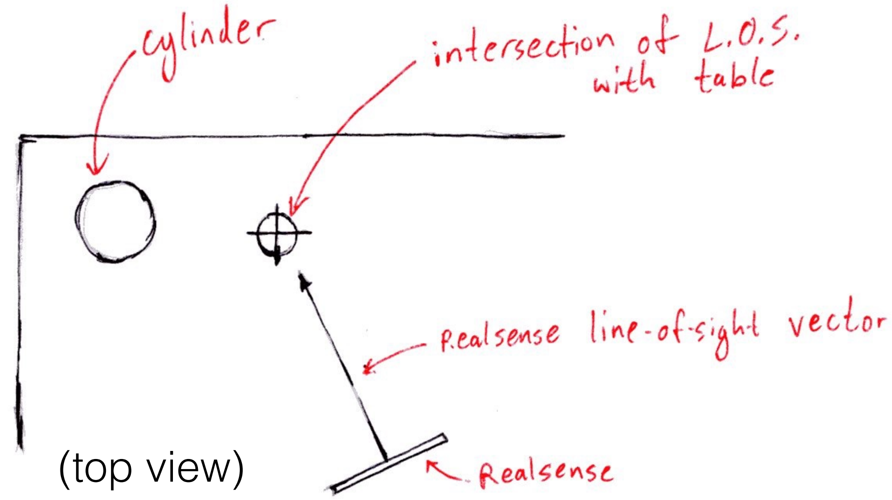
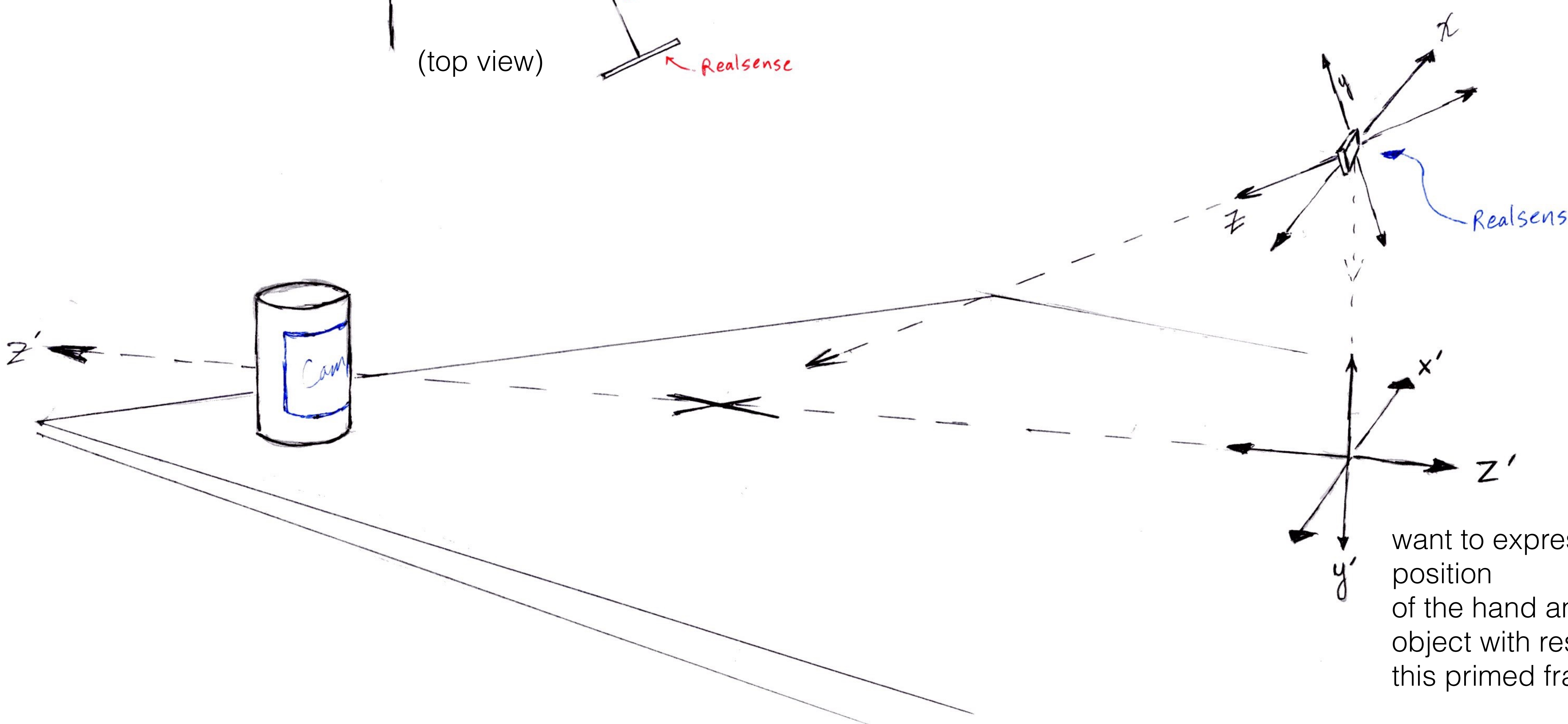
Transforming to Worldspace

April 7th, 2016

The Goal, pictorially:





want to express the position
of the hand and the object with respect to this primed frame

The goal, symbolically:

$$\hat{\mathbf{x}}' = \hat{\mathbf{x}}$$

$$\hat{\mathbf{y}}' = -\hat{\mathbf{n}}$$

$$\hat{\mathbf{z}}' = -\hat{\mathbf{n}} \times \hat{\mathbf{x}}$$

$$\mathcal{O}' = \mathcal{O} + (\text{camera height}) \cdot \hat{\mathbf{n}}$$

Starting with the general affine transformation,

$$
\begin{aligned}
\hat{\mathbf{x}}' &= M_{0,0}\hat{\mathbf{x}} + M_{0,1}\hat{\mathbf{y}} + M_{0,2}\hat{\mathbf{z}} \\
\hat{\mathbf{y}}' &= M_{1,0}\hat{\mathbf{x}} + M_{1,1}\hat{\mathbf{y}} + M_{1,2}\hat{\mathbf{z}} \\
\hat{\mathbf{z}}' &= M_{2,0}\hat{\mathbf{x}} + M_{2,1}\hat{\mathbf{y}} + M_{2,2}\hat{\mathbf{z}} \\
\mathcal{O}' &= \mathcal{O} + M_{3,0}\hat{\mathbf{x}} + M_{3,1}\hat{\mathbf{y}} + M_{3,2}\hat{\mathbf{z}}
\end{aligned}
$$

reference: http://www.uio.no/studier/emner/matnat/ifi/INF3320/h03/undervisningsmateriale/lecture3.pdf

Or, in matrix form, with the entries filled in:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ n_x & n_y & n_z & 0 \\ (-\hat{\mathbf{n}} \times \hat{\mathbf{x}})_x & (-\hat{\mathbf{n}} \times \hat{\mathbf{x}})_y & (-\hat{\mathbf{n}} \times \hat{\mathbf{x}})_z & 0 \\ t_x & t_y & t_z & 1 \end{pmatrix}$$

This is the change of basis matrix

So the same point in space can be referred to in either of the 2 coordinate systems

$$\begin{pmatrix} p_0 & p_1 & p_2 & 1 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \\ \mathcal{O} \end{pmatrix} = \begin{pmatrix} p'_0 & p'_1 & p'_2 & 1 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}}' \\ \hat{\mathbf{y}}' \\ \hat{\mathbf{z}}' \\ \mathcal{O}' \end{pmatrix}$$

$$= \begin{pmatrix} p'_0 & p'_1 & p'_2 & 1 \end{pmatrix} M \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \\ \mathcal{O} \end{pmatrix}$$

Standard basis frame,
of the camera

New basis,
in the plane,
below the camera

Which means that

$$p_i = M^T p'_i$$

i.e., $$\boxed{p'_i = \boxed{(M^T)^{-1}} p_i}$$

the part we want

…the world space transform

$$(M^T)^{-1} = \begin{pmatrix} 1 & n_x & (-\hat{\mathbf{n}} \times \hat{\mathbf{x}})_x & t_x \\ 0 & n_y & (-\hat{\mathbf{n}} \times \hat{\mathbf{x}})_y & t_y \\ 0 & n_z & (-\hat{\mathbf{n}} \times \hat{\mathbf{x}})_z & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1}$$