Physics Formulae Cheatsheet

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v0.1$^{\mathcal{N}}$ *Elemental Topaz* (ETP)

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0.1 Prerequisite Maths

This section covers all the prerequisite mathematics for the physics chapters to follow. It should hopefully serve as a handy reference, as well as a good learning guide.

The reader is encouraged to prove all the identities and theorems presented here before moving on. Don't use something you can't understand!

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0.1.1 Trigonometry

Tables

θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan(\theta)$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

φ	θ	$180° - \theta$	$180° + \theta$	$360° - \theta$
$\sin(\varphi)$	$\sin\theta$	$\sin\theta$	$-\sin\theta$	$-\sin\theta$
$\cos(\varphi)$	$\cos\theta$	$-\cos\theta$	$-\cos\theta$	$\cos\theta$
$\tan(\varphi)$	$\tan\theta$	$-\tan\theta$	$\tan\theta$	$-\tan\theta$

Laws and Identities

Pythagorean Identities:

$$cos^2\theta + sin^2\theta = 1 \tag{0.1.1}$$

Negative Angle Identities:

$$sin(-\theta) = -sin\,\theta \tag{0.1.2}$$

$$cos(-\theta) = cos\,\theta \tag{0.1.3}$$

$$tan(-\theta) = -tan\,\theta \tag{0.1.4}$$

Relationships:

$$sin(90° - \theta) = cos\,\theta \tag{0.1.5}$$

$$cos(90° - \theta) = sin\,\theta \tag{0.1.6}$$

$$tan(90^\circ - \theta) = \frac{1}{tan\,\theta} \tag{0.1.7}$$

Addition and Subtraction Identities:

$$sin(A+B) = sin\,A\,cos\,B + cos\,A\,sin\,B \tag{0.1.8}$$

$$sin(A-B) = sin\,A\,cos\,B - cos\,A\,sin\,B \tag{0.1.9}$$

$$cos(A+B) = cos\,A\,cos\,B - sin\,A\,sin\,B \tag{0.1.10}$$

$$cos(A-B) = cos\,A\,cos\,B + sin\,A\,sin\,B \tag{0.1.11}$$

Double-Angle Identities:

$$sin(2\theta) = 2\,sin\,\theta\,cos\,\theta \tag{0.1.12}$$

$$cos(2\theta) = cos^2\theta - sin^2\theta = 2\,cos^2\theta - 1 = 1 - 2\,sin^2\theta \tag{0.1.13}$$

0.1.2 Calculus

0.1.3 Linear Algebra

0.1.4 Complex Numbers

Introduction

The no-brainer:

$$i = \sqrt{-1} \tag{0.1.14}$$

Useful intuition:

Multiplying by i has the effect of rotating clockwise by 90° in the complex plane

Euler's formula:

$$e^{ix} = cos\,x + i\,sin\,x \tag{0.1.15}$$

Cartesian representation:

$$z = x + iy \tag{0.1.16}$$

Polar representation:

$$z = re^{i\theta} = r\,(cos\,\theta + i\,sin\,\theta) \tag{0.1.17}$$

Note that $r = |z|$ and that θ is called the argument of z, denoted by $arg(z)$.

Polar to Cartesian:

$$x = r\,cos\,\theta \tag{0.1.18}$$

$$y = r\,sin\,\theta \tag{0.1.19}$$

Cartesian to Polar:

$$r = |z| = \sqrt{x^2 + y^2} = \sqrt{\overline{z} \cdot z} \tag{0.1.20}$$

$$\theta = tan^{-1}(\frac{y}{x}) \tag{0.1.21}$$

Complex conjugate:

$$\overline{z} = x - iy = re^{-i\theta} = r(cos\,\theta - i\,sin\,\theta) \tag{0.1.22}$$

Note that $arg(\overline{z}) = -arg(z)$, and that $\because r^2 = \overline{z} \cdot z \quad \therefore \frac{1}{z} = r^{-1}\overline{z} = r^{-1}(cos\,\theta - i\,sin\,\theta)$.

Identities

In the following few sections; $z = x + i\,y$, $v = a + i\,b$ and $w = c + i\,d$ $(z, v, w \in \mathbb{C})$.

Magnitude:

$$\overline{z} \cdot z = (x + iy)(x - iy) = x^2 - (iy)^2 = x^2 + y^2$$

$$\therefore \quad \overline{z} \cdot z = |z|^2 \tag{0.1.23}$$

The real part:

$$Re(z) = \frac{1}{2}(z + \overline{z}) \tag{0.1.24}$$

The imaginary part:

$$Im(z) = \frac{1}{2i}(z - \overline{z}) \tag{0.1.25}$$

Operational properties of complex conjugates:

$$\overline{v \pm w} = \overline{v} \pm \overline{w} \tag{0.1.26}$$

$$\overline{v \cdot w} = \overline{v} \cdot \overline{w} \tag{0.1.27}$$

$$\overline{(\frac{v}{w})} = \frac{\overline{v}}{\overline{w}} \tag{0.1.28}$$

Operational properties of magnitude:

$$|v| = |\overline{v}| \tag{0.1.29}$$

$$|v \cdot w| = |v| \cdot |w| \tag{0.1.30}$$

$$\left|\frac{v}{w}\right| = \frac{|v|}{|w|} \tag{0.1.31}$$

Inequalities for Real and Imaginary parts:

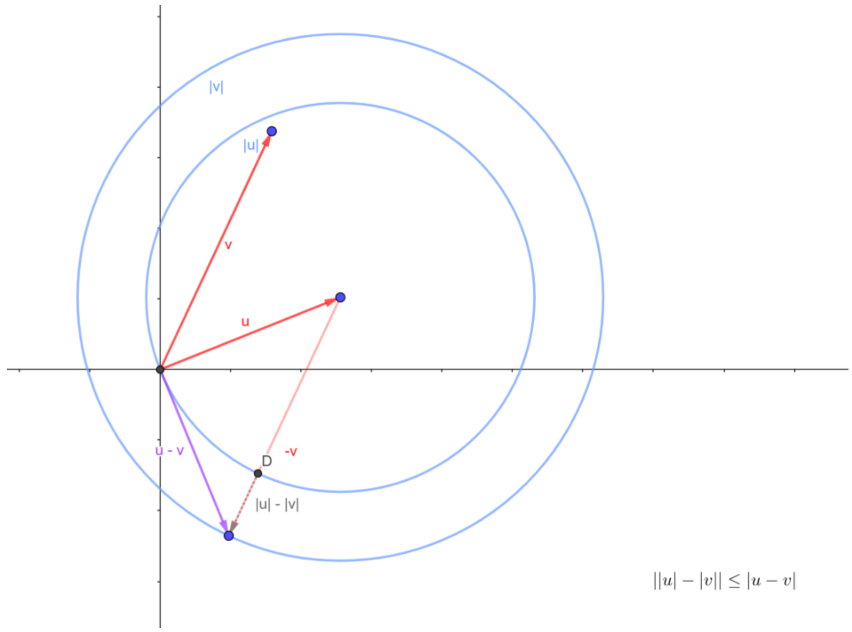
$$|Re(z)| \leq |z| \tag{0.1.32}$$

$$|Im(z)| \leq |z| \tag{0.1.33}$$

Triangle inequalities:

$$|v + w| \leq |v| + |w| \tag{0.1.34}$$

$$\Big||v| - |w|\Big| \leq \Big|v - w\Big| \tag{0.1.35}$$



These triangle inequalities are equal when the numbers are co-linear on the complex plane.

Operations

Cartesian form

Addition and subtraction:

$$v \pm w = (a \pm c) + i(b \pm d) \tag{0.1.36}$$

Multiplication:

$$vw = (ac - bd) + i\,(ad + bc) \tag{0.1.37}$$

Division:

$$\frac{v}{w} = \frac{a + i\,b}{c + i\,d} \cdot \frac{c - i\,d}{c - i\,d} = \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + i\frac{bc - ad}{c^2 + d^2} \tag{0.1.38}$$

Polar form

Suppose that $v = re^{i\theta} = r(cos\,\theta + i\,sin\,\theta)$ and that $w = se^{i\psi} = s(cos\,\psi + i\,sin\,\psi)$.

Multiplication:

$$vw = rse^{i(\theta+\psi)} = rs(cos(\theta + \psi) + i\,sin(\theta + \psi)) \tag{0.1.39}$$

In other words:

$$|v \cdot w| = |v| \cdot |w| \quad and \quad arg(v \cdot w) = arg(v) + arg(w)$$

$\therefore$ Multiplying v by w has geometric meaning of scaling v by $|w|$ and rotating it by $arg(w)$

Division:
Note that $\frac{1}{w} = s^{-1}(cos(-\psi) + i\, sin(-\psi))$

$$\frac{v}{w} = v \cdot \frac{1}{w} = rs^{-1}(cos(\theta - \psi) + i\, sin(\theta - \psi)) \tag{0.1.40}$$

In other words:

$$|\frac{v}{w}| = \frac{|v|}{|w|} \quad and \quad arg(\frac{v}{w}) = arg(v) - arg(w)$$

$\therefore$ Dividing v by w has geometric meaning of scaling v by $-|w|$ and rotating it by $-arg(w)$

Theorems

De Moivre's Theorem: Let $z = r(cos\,\theta + i\, sin\,\theta) \in \mathbb{C}$, then for any $n \in \mathbb{Z}$,

$$z^n = r^n(cos(n\theta) + i\, sin(n\theta)) \tag{0.1.41}$$

Roots of Unity

The square roots of unity are numbers that square themselves to become 1. Similarly, the cube roots of unity are (complex) numbers that cube themselves to become 1.

Let $z = r(cos\,\theta + i\, sin\,\theta) \in \mathbb{C}$ be a cube root of unity.

Then, by De Moivre's Theorem, we have:

$$z^3 = r^3(cos(3\theta) + i\, sin(3\theta)) = 1$$

$$\because 1 = cos\,0° + i\, sin\,0° = cos\,360° + i\, sin\,360° = cos\,720° + i\, sin\,720°$$

$$\therefore (r = 1,\ \theta = 0°)\ or\ (r = 1,\ \theta = 120°)\ or\ (r = 1,\ \theta = 240°)$$

$$\therefore z = 1,\ z = -\frac{1}{2} + i\frac{\sqrt{3}}{2},\ or\ z = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

$\mathbf{n^{th}}$ Roots of Unity

We know:

$$1 = cos(360°k) + i\, sin(360°k), \quad k = 0, 1, \ldots n - 1$$

Let $z_k = r_k(cos\,\theta_k + i\, sin\,\theta_k) \in \mathbb{C}$ be a n^{th} root of unity, and let

$$\omega = cos\left(\frac{360°}{n}\right) + i\, sin\left(\frac{360°}{n}\right)$$

Then

$$z_k^n = r_k^n(cos(n\theta_k) + i\,sin(n\theta_k)) = 1$$

$$\therefore z_k = cos\left(\frac{360°k}{n}\right) + i\,sin\left(\frac{360°k}{n}\right), \quad k = 0, 1, \ldots n-1 \tag{0.1.42}$$

Or

$$z_k = \left(cos\left(\frac{360°}{n}\right) + i\,sin\left(\frac{360°}{n}\right)\right)^k = \omega^k, \quad k = 0, 1, \ldots n-1 \tag{0.1.43}$$

Properties of Roots of Unity

$$x^n - 1 = (x-1)(x-\omega)(x-\omega^2)\ldots(x-\omega^{n-1}) \tag{0.1.44}$$

$$x^n - 1 = (x-1)(1 + x + x^2 + \cdots + x^{n-1}) \tag{0.1.45}$$

$$\overline{\omega^k} = \overline{\omega}^k = \frac{1}{\omega^k} = \frac{\omega^n}{\omega^k} = \omega^{n-k}, \quad k = 0, 1, \ldots n-1 \tag{0.1.46}$$

By expanding the RHS constant term in 0.1.44:

$$\prod_{k=0}^{n-1} w^k = (-1)^{n+1} \tag{0.1.47}$$

$$\textit{I.e. The product of all roots} = \begin{cases} -1, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases}$$

From 0.1.45:

$$\sum_{k=0}^{n-1} w^k = 0 \tag{0.1.48}$$

By 0.1.44 and 0.1.45:

$$1 + x + x^2 + \cdots + x^{n-1} = (x-\omega)(x-\omega^2)\ldots(x-\omega^{n-1}) \tag{0.1.49}$$

Roots of Any Complex Number

Let

$$u = r(cos(360°k + \psi) + i\,sin(360°k + \psi)) \in \mathbb{C}, \quad k = 0, 1, 2, \ldots n-1$$

with $0° \leq \psi \leq 360°$; and let $z_k = r_k(cos\,\theta_k + i\,sin\,\theta_k) \in \mathbb{C}$ be a n^{th} root of u, such that

$$z_k^n = r_k^n(cos(n\theta_k) + i\,sin(n\theta_k)) = u$$

$$\therefore r_k = r^{\frac{1}{n}}, \quad \theta_k = \frac{360°k + \psi}{n} = \frac{360°k}{n} + \frac{\psi}{n}, \quad k = 0, 1, 2, \ldots n-1$$

Now define

$$v = r^{\frac{1}{n}} \left(cos\left(\frac{\psi}{n}\right) + i\, sin\left(\frac{\psi}{n}\right)\right)$$

Furthermore, let ω^k be the n^{th} roots of unity for $k = 0, 1, 2, \ldots n-1$. Then

$$z_k = r^{\frac{1}{n}} \left(cos\left(\frac{360°k}{n} + \frac{\psi}{n}\right) + i\, sin\left(\frac{360°k}{n} + \frac{\psi}{n}\right)\right) \tag{0.1.50}$$

Or

$$z_k = r^{\frac{1}{n}} \left(cos\left(\frac{\psi}{n}\right) + i\, sin\left(\frac{\psi}{n}\right)\right) \left(cos\left(\frac{360°k}{n}\right) + i\, sin\left(\frac{360°k}{n}\right)\right) = vw^k \tag{0.1.51}$$

Refer to Appendix A for a proof of the factorisation used in 0.1.51.

Chapter 1

Classical Mechanics

All units used in this chapter are by default SI units. Take care to note that angles are in radians, not degrees.

1.1 Simple Harmonic Motion

Frequency and time period:

$$f = \frac{1}{T} \tag{1.1.1}$$

Hooke's law:

$$F = -kx \tag{1.1.2}$$

Elastic potential energy:

$$PE_e = \frac{1}{2}kx^2 \tag{1.1.3}$$

Sum of energies in ideal Hookean system is constant:

$$\frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}kA^2 \tag{1.1.4}$$

Derivatives of position:

$$|v| = \sqrt{(A^2 - x^2)\frac{k}{m}} \tag{1.1.5}$$

$$a = -\frac{k}{m}x \tag{1.1.6}$$

Reference circle:

$$v_x = -|v_{max}|sin(\theta) \tag{1.1.7}$$

$$T = \frac{2\pi A}{|v_{max}|} \tag{1.1.8}$$

However, $v_{max} = A\sqrt{\frac{k}{m}}$, and hence $w = \sqrt{\frac{k}{m}}$, therefore:

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{1.1.9}$$

Simple pendulum:

$$T = 2\pi\sqrt{\frac{L}{g}} \tag{1.1.10}$$

1.2 Density and Elasticity

Density:

$$\rho = \frac{m}{V} \tag{1.2.1}$$

Stress:

$$\sigma = \frac{F}{A} \tag{1.2.2}$$

Strain:

$$Strain = \frac{Change\ in\ dimension}{Original\ dimension} \tag{1.2.3}$$

Young's Modulus / Modulus of elasticity:

$$Y = \frac{Stress}{Strain} \tag{1.2.4}$$

Bulk Modulus:

$$B = -\frac{\Delta P}{\Delta V/V_0} = -\frac{V_0 \Delta P}{\Delta V} \tag{1.2.5}$$

Shear Modulus:

$$S = \frac{F/A}{\Delta L/L_0} = \frac{F L_0}{A \Delta L} \approx \frac{F}{A\gamma} \tag{1.2.6}$$

Chapter 2

Electromagnetism

2.1 Electric Potential and Capacitance

Electric potential energy:

$$PE_E = k_0 \frac{q_1 q_2}{r} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r} \tag{2.1.1}$$

Absolute potential / Electric potential is electric potential energy per unit charge:

$$V = k_0 \frac{q}{r} = \frac{1}{4\pi\epsilon_0} \frac{q}{r} \tag{2.1.2}$$

The potential difference across A and B is the work done against electrical forces in carrying a unity positive test-charge from A to B:

$$V = \frac{W}{q} \tag{2.1.3}$$

Potential difference in parallel plates:

$$V = E_x x = Ed \tag{2.1.4}$$

A capacitor is a device that stores charge. The formula for capacitance is:

$$C = \frac{q}{V} \tag{2.1.5}$$

Parallel-Plate Capacitor:

$$C = K\epsilon_0 \frac{A}{d} = \epsilon \frac{A}{d} \tag{2.1.6}$$

Capacitors in parallel:

$$q = q_1 + q_2 + q_3 + \dots \tag{2.1.7}$$

$$C = C_1 + C_2 + C_3 + \dots \tag{2.1.8}$$

Capacitors in series:

$$q = q_1 = q_2 = q_3 = \dots \tag{2.1.9}$$

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots \tag{2.1.10}$$

Energy stored in capacitor:

$$PE_E = \frac{1}{2}qV = \frac{1}{2}CV^2 = \frac{1}{2}\frac{q^2}{C} \tag{2.1.11}$$

2.2 Equivalence Resistance and Simple Circuits

Resistors in series:

$$R = R_1 + R_2 + R_3 + \ldots \tag{2.2.1}$$

Resistors in parallel:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots \tag{2.2.2}$$

2.3 Kirchhoff's Laws

Kirchhoff's Node (or Junction) Rule: The sum of all the currents coming into a node must equal the sum of all the currents leaving that node.

Kirchhoff's Loop (of Circuit) Rule: As one traces around any closed loop in the circuit, the algebraic sum of the potential changes encountered is zero.

2.4 Forces in Magnetic Fields

2.5 Reflection of Light

Law of Reflection:

$$\theta_i = \theta_r \tag{2.5.1}$$

Mirror equation:

$$\frac{1}{s_o} + \frac{1}{s_i} = -\frac{2}{R} = \frac{1}{f} \tag{2.5.2}$$

The size of the Image:

$$M_T = \frac{y_i}{y_o} = -\frac{s_i}{s_o} \tag{2.5.3}$$

2.6 Refraction of Light

Index of Refraction:

$$n = \frac{c}{v} \tag{2.6.1}$$

Snell's Law:

$$n_i sin\theta_i = n_t sin\theta_t \tag{2.6.2}$$

Prisms: A prism can be used to disperse light into its various colours, because the index of refraction of a material varies with wavelength, causing colours of light to refract differently. In general, red is refracted least and blue is refracted most.

2.7 Thin Lens

Lenses in contact:

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} \tag{2.7.1}$$

Chapter 3

Relativity

Chapter 4

Quantum Mechanics

Appendix A

Selected Proofs

A.1 Prerequisite Maths

Factorisation of $cos(A + B) + i\, sin(A + B)$