Homogeneous Functions

Ex. $f(x,y) = \dfrac{x^3+y^3}{x+y}$

$$f(xt, yt) = \frac{(xt)^3+(yt)^3}{xt+yt} = t^2\left(\frac{x^3+y^3}{x+y}\right) = t^2 f(x,y)$$

$\therefore$ $f(x,y)$ is homogeneous of degree 2.

* Euler's Theorem:

- $u = f(x,y)$

$$\therefore\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

$$\therefore\ x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

- $u = f(x,y,z)$

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + z^2\frac{\partial^2 u}{\partial z^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + 2yz\frac{\partial^2 u}{\partial y\partial y} + 2xz\frac{\partial^2 u}{\partial x\partial z} = n(n-1)u$$

- $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = n\dfrac{f(u)}{f'(u)}$

- $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)\,(g'(u) - 1)$ $\left[g(u) = n\frac{f(u)}{f'(u)}\right]$

Q1] $u = \sqrt{x} + \sqrt{y} + \sqrt{z}$

$u' = \sqrt{xt} + \sqrt{yt} + \sqrt{zt}$

$u' = t^{1/2}\left[\sqrt{x} + \sqrt{y} + \sqrt{z}\right]$

$u' = t^{1/2} u$

$\therefore$ This is a homogeneous func^n of degree $1/2$.

$\therefore$ By Euler's Theorem,

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{1}{2}u$

Q2] $u = \sin^{-1}\left(\frac{x}{y}\right) + \cos^{-1}\left(\frac{y}{z}\right) - \log\left(\frac{z}{x}\right)$

$\rightarrow$

$u' = \sin^{-1}\left(\frac{xt}{yt}\right) + \cos^{-1}\left(\frac{yt}{zt}\right) - \log\left(\frac{zt}{xt}\right)$

$u' = t^{0} u$

$\therefore$ By Euler's Theorem,

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu = 0$

Q3 $u = \dfrac{\sqrt{x} + \sqrt{y}}{x+y}$

$\rightarrow$

$$u' = \frac{\sqrt{xt} + \sqrt{yt}}{xt+yt} = \frac{1}{\sqrt{t}}u$$

By Euler's Theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\left(\frac{\sqrt{x} + \sqrt{y}}{x+y}\right)$$

Q4 $u = \dfrac{x^3y + y^3x}{3x}$ P.T. $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 6u$

$\rightarrow$

$$u = \frac{(xt)^3yt + (yt)^3xt}{3xt} = t^3u$$

$\therefore$ u is Homogeneous func^n of degree 3.

$\therefore$ By Euler's Theorem,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u = 3.2u = 6u$$

Q5 $u = \dfrac{x^2y^3z}{x^2+y^2+z^2} + \sin^{-1}\left(\dfrac{xy+yz}{y^2+z^2}\right)$

$\rightarrow$ Here u is not Homogeneous,

$u = v + w$

$$v = \frac{x^2y^3z}{x^2+y^2+z^2} = f(x,y,z) \quad \& \quad w = \sin^{-1}\left(\frac{xy+yz}{y^2+z^2}\right) = g(x,y,z)$$

$$v' = \frac{(xt)^2\,(yt)^3(zt)}{(xt)^2+(yt)^2+(zt)^2} = t^4 v$$

V is a homogeneous func^n of degree 4.

By Euler's Theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = nv = 4v \quad \text{—— ①}$$

$$w' = \sin^{-1}\left(\frac{xyt^2+yzt^2}{y^2t^2+z^2t^2}\right) = t^0 w$$

W is a homogeneous func^n of degree 0.

$\therefore$ By Euler's Theorem,

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = 0 \quad \text{—— ②}$$

Adding ① & ② we get,

$$x\left(\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\right) + y\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial y}\right) + z\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial z}\right) = 4v+0$$

$\because \; u = v+w$

$$\therefore \; x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 4\,\frac{x^2y^3z}{x^2+y^2+z^2}$$

Q6] $u = \dfrac{x^2+xy}{y\sqrt{x}} + \dfrac{1}{x^7}\sin^{-1}\left(\dfrac{y^2-xy}{x^2-y^2}\right)$. Solve $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} + x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$ at $x=1,\ y=2$

$\rightarrow$ $u = v + w$

$$v' = \frac{(xt)^2 + (xt)(yt)}{(yt)\sqrt{xt}} = \frac{t^2}{t^{3/2}}v = t^{1/2}v$$

$$\therefore\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}v \quad \text{—— ①}$$

$$\therefore\ x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = \frac{1}{2}\left(\frac{1}{2}-1\right)v = -\frac{1}{4}v \quad \text{—— ②}$$

$$w' = \frac{1}{(xt)^7}\sin^{-1}\left(\frac{(yt)^2 - (xt)(yt)}{(xt)^2 - (yt)^2}\right) = t^{-7}w$$

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = -7w \quad \text{—— ③}$$

$$x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x\partial y} + y^2\frac{\partial^2 w}{\partial y^2} = -7(-7-1) = 56w \quad \text{—— ④}$$

Adding ①, ②, ③ & ④

$$x^2\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 w}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 v}{\partial x\partial y} + \frac{\partial^2 w}{\partial x\partial y}\right) + y^2\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\right) + x\left(\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right) + y\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\right) = -\frac{1}{4}v + 56w + \frac{1}{2}v - 7w$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{4}v + 49w$$

at $x=1$ & $y=2$

$$= \frac{3}{8} + 49\sin^{-1}\left(\frac{-2}{3}\right)$$

$\therefore\ v = \dfrac{3}{2}$ & $w = \sin^{-1}\left(\dfrac{-2}{3}\right)$

Q7] $u = \dfrac{x^2y^2z^2}{x^2+y^2+z^2} + \cos^{-1}\left(\dfrac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\right)$

$\rightarrow$ $u = v + w$

$v' = t^4 v$

$x\dfrac{\partial v}{\partial x} + y\dfrac{\partial v}{\partial y} + z\dfrac{\partial v}{\partial z} = 4v$ —— ①

W is not homogeneous, $f(w) = \cos w = \dfrac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}} = h(x,y,z)$

$h(xt, yt, zt) = t^{1/2}\, h(x,y,z)$

$f(w) = \cos w$ is a homogeneous func^n of deg $1/2$.

By corollary 2,

$x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} + z\dfrac{\partial w}{\partial z} = \dfrac{1}{2}\dfrac{\cos w}{(-\sin w)} = -\dfrac{1}{2}\cot w$ —— ②

Adding ① & ②,

$x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 4v - \dfrac{1}{2}\cos w$

$x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 4\left(\dfrac{x^2y^2z^2}{x^2+y^2+z^2}\right) - \dfrac{1}{2}\left(\cos^{-1}\left(\dfrac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\right)\right)$

Q8] $u = \operatorname{cosec}^{-1}\sqrt{\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$ P.T. $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y\dfrac{\partial^2 u}{\partial y^2} = \dfrac{\tan u}{144}(13 + \tan^2 u)$

$\rightarrow$ $f(u) = \operatorname{cosec}(u) = \sqrt{\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}} = h(x,y)$

$h(xt, yt) = t^{1/12}\, h(x,y)$

By corollary 3,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{-1}{12}\tan u\left[-\frac{1}{12}(13+\tan^2 u)\right]$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2}{\partial y^2}=\frac{1}{144}\tan u\,[13+\tan^2 u]$$

$$g(u)=n\frac{f(u)}{f'(u)}=\frac{1}{12}\,\frac{\cancel{\operatorname{cosec} u}}{-\cancel{\operatorname{cosec} u}\cot u}$$

$$=\frac{-1}{12}\tan u$$

$$g'(u)-1=\frac{-1}{12}(\sec^2 u-1)$$

$$=\frac{-1}{12}(1+\tan^2 u)-1=\frac{-13}{12}-\frac{\tan^2 u}{12}$$

Q10] $x=e^u\tan v$, $y=e^u\sec v$, solve $\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}\right)$

$\rightarrow$ $y^2-x^2=e^{2u}\sec^2 v-e^{2u}\tan^2 v=e^{2u}$

$\therefore$ $u=\frac{1}{2}\log(y^2-x^2)$

now, $\frac{x}{y}=\frac{e^u\tan v}{e^u\sec v}=\sin v$

$\therefore$ $v=\sin^{-1}\left(\frac{x}{y}\right)$

v is homogeneous funct of degree 0.

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=0$$

$$\therefore\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}\right)=0$$

Q11] $u=\log\left(\frac{x^3+y^3}{x^2+y^2}\right)$

u is not homogeneous

$\therefore$ $f(u)=e^u=\frac{x^3+y^3}{x^2+y^2}=h(x,y)$

$$h(xt, yt) = \frac{t^3}{t^2} h(x,y) = t\, h(x,y)$$

$\therefore$ $f(u) = e^u$ is homogeneous func^n of degree 1.

$\therefore$ By Euler Theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = 1\,\frac{\cancel{e^u}}{\cancel{e^u}} = 1.$$