Homogeneous Functions

Ex. $f(x,y) = \dfrac{x^3+y^3}{x+y}$

$$f(xt, yt) = \frac{(xt)^3+(yt)^3}{xt+yt} = t^2\left(\frac{x^3+y^3}{x+y}\right) = t^2 f(x,y)$$

$\therefore$ $f(x,y)$ is homogeneous of degree 2.

* Euler's Theorem:

- $u = f(x,y)$

$$\therefore\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

$$\therefore\ x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

- $u = f(x,y,z)$

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + z^2\frac{\partial^2 u}{\partial z^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + 2yz\frac{\partial^2 u}{\partial y\partial y} + 2xz\frac{\partial^2 u}{\partial x\partial z} = n(n-1)u$$

- $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$$

- $$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)\,(g'(u) - 1) \qquad \left[g(u) = n \frac{f(u)}{f'(u)} \right]$$

Q1] $u = \sqrt{x} + \sqrt{y} + \sqrt{z}$

$u' = \sqrt{xt} + \sqrt{yt} + \sqrt{zt}$

$u' = t^{1/2} [\sqrt{x} + \sqrt{y} + \sqrt{z}]$

$u' = t^{1/2} u$

$\therefore$ This is a homogeneous func^n of degree $1/2$.

$\therefore$ By Euler's Theorem,

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = \frac{1}{2} u$$

Q2] $u = \sin^{-1}\left(\frac{x}{y}\right) + \cos^{-1}\left(\frac{y}{z}\right) - \log\left(\frac{z}{x}\right)$

$\rightarrow$

$u' = \sin^{-1}\left(\frac{xt}{yt}\right) + \cos^{-1}\left(\frac{yt}{zt}\right) - \log\left(\frac{zt}{xt}\right)$

$u' = t^{0} u$

$\therefore$ By Euler's Theorem,

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = nu = 0$$

Q3 $u = \dfrac{\sqrt{x} + \sqrt{y}}{x+y}$

$\rightarrow$

$$u' = \frac{\sqrt{xt} + \sqrt{yt}}{xt+yt} = \frac{1}{\sqrt{t}}\,u$$

By Euler's Theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\left(\frac{\sqrt{x} + \sqrt{y}}{x+y}\right)$$

Q4 $u = \dfrac{x^3y + y^3x}{3x}$ P.T. $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 6u$

$\rightarrow$

$$u = \frac{(xt)^3yt + (yt)^3xt}{3xt} = t^3 u$$

$\therefore$ u is Homogeneous func^n of degree 3.

$\therefore$ By Euler's Theorem,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u = 3.2u = 6u$$

Q5 $u = \dfrac{x^2y^3z}{x^2+y^2+z^2} + \sin^{-1}\left(\dfrac{xy+yz}{y^2+z^2}\right)$

$\rightarrow$ Here u is not Homogeneous,

$u = v + w$

$$v = \frac{x^2 y^3 z}{x^2+y^2+z^2} = f(x,y,z) \quad \& \quad w = \sin^{-1}\left(\frac{xy+yz}{y^2+z^2}\right) = g(x,y,z)$$

$$v' = \frac{(xt)^2 (yt)^3 (zt)}{(xt)^2+(yt)^2+(zt)^2} = t^4 v$$

V is a homogeneous func^n of degree 4.

By Euler's Theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = nv = 4v \quad \text{—— ①}$$

$$w' = \sin^{-1}\left(\frac{xyt^2 + yzt^2}{y^2t^2 + z^2t^2}\right) = t^0 w$$

W is a homogeneous func^n of degree 0.

$\therefore$ By Euler's Theorem,

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = 0 \quad \text{—— ②}$$

Adding ① & ② we get,

$$x\left(\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right) + y\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\right) + z\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial z}\right) = 4v + 0$$

$\because \; u = v + w$

$$\therefore \; x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 4\frac{x^2y^3z}{x^2+y^2+z^2}$$

Q6) $u = \dfrac{x^2 + xy}{y\sqrt{x}} + \dfrac{1}{x^7}\sin^{-1}\left(\dfrac{y^2 - xy}{x^2 - y^2}\right)$. solve $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} + x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$ at $x=1$, $y=2$

$\rightarrow$ $u = v + w$

$$v' = \frac{(xt)^2 + (xt)(yt)}{(yt)\sqrt{xt}} = \frac{t^2}{t^{3/2}}v = t^{1/2}v$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}v \quad \text{—— ①}$$

$$\therefore x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x \partial y} + y^2\frac{\partial^2 v}{\partial y^2} = \frac{1}{2}\left(\frac{1}{2} - 1\right)v = -\frac{1}{4}v \quad \text{—— ②}$$

$$w' = \frac{1}{(xt)^7}\sin^{-1}\left(\frac{(yt)^2 - (xt)(yt)}{(xt)^2 - (yt)^2}\right) = t^{-7}w$$

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = -7w \quad \text{—— ③}$$

$$x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x \partial y} + y^2\frac{\partial^2 w}{\partial y^2} = -7(-7-1) = 56w \quad \text{—— ④}$$

Adding ①, ②, ③ & ④

$$x^2\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 w}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial y}\right) + y^2\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\right) + x\left(\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right) + y\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\right) = -\frac{1}{4}v + 56w + \frac{1}{2}v - 7w$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{4}v + 49w$$

at $x = 1$ & $y = 2$

$$= \frac{3}{8} + 49\sin^{-1}\left(\frac{-2}{3}\right)$$

$\therefore v = \dfrac{3}{2}$ & $w = \sin^{-1}\left(\dfrac{-2}{3}\right)$

Q7] $u = \dfrac{x^2y^2z^2}{x^2+y^2+z^2} + \cos^{-1}\left(\dfrac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\right)$

$\rightarrow$ $u = v + w$

$v' = t^4 v$

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = 4v \quad \text{—— ①}$$

w is not homogeneous, $f(w) = \cos w = \dfrac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}} = h(x,y,z)$

$h(xt, yt, zt) = t^{1/2}\, h(x,y,z)$

$f(w) = \cos w$ is a homogeneous func^n of deg $1/2$.

By corollary 2,

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = \frac{1}{2}\frac{\cos w}{(-\sin w)} = -\frac{1}{2}\cot w \quad \text{—— ②}$$

Adding ① & ②,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 4v - \frac{1}{2}\cos w$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 4\left(\frac{x^2y^2z^2}{x^2+y^2+z^2}\right) - \frac{1}{2}\left(\cos^{-1}\left(\frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\right)\right)$$

Q8] $u = \operatorname{cosec}^{-1}\sqrt{\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$ P.T. $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y\dfrac{\partial^2 u}{\partial y^2} = \dfrac{\tan u}{144}(13 + \tan^2 u)$

$\rightarrow$ $f(u) = \operatorname{cosec}(u) = \sqrt{\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}} = h(x,y)$

$h(xt, yt) = t^{1/12}\, h(x,y)$

By corollary 3,

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{-1}{12} \tan u \left[\frac{-1}{12} (13 + \tan^2 u) \right]$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2}{\partial y^2} = \frac{1}{144} \tan u \left[13 + \tan^2 u \right]$$

$$g(u) = n \frac{f(u)}{f'(u)} = \frac{1}{12} \frac{\cancel{\operatorname{cosec} u}}{-\cancel{\operatorname{cosec} u} \cot u}$$

$$= \frac{-1}{12} \tan u$$

$$g'(u) - 1 = \frac{-1}{12} (\sec^2 u - 1)$$

$$= \frac{-1}{12} (1 + \tan^2 u) - 1 = \frac{-13}{12} - \frac{\tan^2 u}{12}$$

Q10] $x = e^u \tan v$, $y = e^u \sec v$, solve $\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \right) \left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} \right)$

$\rightarrow$ $y^2 - x^2 = e^{2u} \sec^2 v - e^{2u} \tan^2 v = e^{2u}$

$\therefore$ $u = \frac{1}{2} \log (y^2 - x^2)$

now, $\frac{x}{y} = \frac{e^u \tan v}{e^u \sec v} = \sin v$

$\therefore$ $v = \sin^{-1} \left(\frac{x}{y} \right)$

v is homogeneous funct of degree 0.

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 0$$

$$\therefore \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \right) \left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} \right) = 0$$

Q11] $u = \log \left(\frac{x^3 + y^3}{x^2 + y^2} \right)$

u is not homogeneous

$\therefore$ $f(u) = e^u = \frac{x^3 + y^3}{x^2 + y^2} = h(x, y)$

$$h(xt, yt) = \frac{t^3}{t^2} h(x,y) = t\, h(x,y)$$

$\therefore$ $f(u) = e^u$ is homogeneous func^n of degree 1.

$\therefore$ By Euler Theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = 1\,\frac{\cancel{e^u}}{\cancel{e^u}} = 1.$$

* Maxima & Minima

Step I: $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$

from step I we'll get $(a,b) \Rightarrow$ stationary points

Step II: Find $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$ at (a,b)

Step III: If $rt - s^2 > 0$ & $r < 0$, $t < 0$ $\Rightarrow$ maximum

If $rt - s^2 > 0$ & $r > 0$, $t > 0$ $\Rightarrow$ minimum

If $rt - s^2 < 0$ $\Rightarrow$ neither maxima nor minima (saddle point)

If $rt - s^2 = 0$ $\Rightarrow$ test fails.

Q] Determine maxima & minima of $x^4 + y^4 - 2x^2 + 4xy - 2y^2$

→ $f(x) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

$\frac{\partial f}{\partial x} = 4x^3 - 4x + 4y = 0 \Rightarrow x^3 - x + y = 0$ — ①

$\frac{\partial f}{\partial y} = 4y^3 - 4y + 4x = 0 \Rightarrow y^3 - y + x = 0$ — ②

Adding ① & ② we get,

$x^3 + y^3 = 0$

$(x+y)(x^2 - xy + y^2) = 0$

$\therefore x = -y$

Subtituting this in eqn ①

$x^3 - x - x = 0$

$x^3 - 2x = 0$

$x(x^2 - 2) = 0$

$x = 0$ or $x = \pm\sqrt{2}$

If $x = 0$, $y = 0$

If $x = \pm\sqrt{2}$, $y = \mp\sqrt{2}$

$\therefore$ Stationary points are $(0,0)$, $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$

$r = f_{xx} = 12x^2 - 4$

$s = f_{xy} = 4$

$t = f_{yy} = 12y^2 - 4$

Points	r	s	t	$rt-s^2$	Conclusion
$0,0$	-4	4	-4	0	Test fails
$\sqrt{2},-\sqrt{2}$	20	4	20	384	Minimum
$-\sqrt{2},\sqrt{2}$	20	4	20	384	Minimum

Minimum value at $f(\sqrt{2},-\sqrt{2})$ & $f(-\sqrt{2},\sqrt{2})$ is

$f(\sqrt{2},-\sqrt{2}) = f(-\sqrt{2},\sqrt{2}) = -8.$

Q] Divide 90 into three parts such that the sum of their products taken two at a time is max.

→ $x+y+z=90$

$f(x,y) = xy+yz+zx$

$= xy + y(90-x-y) + x(90-x-y)$

$= 90x + 90y - xy - x^2 - y^2$ — ①

$f_x = 90 - y - 2x = 0 \Rightarrow 2x+y=90$

$f_y = 90 - x - 2y = 0 \Rightarrow x+2y=90$

$\therefore x=30, y=30$

$\therefore (30, 30)$ stationary point

$r = f_{xx} = -2$ $\qquad \therefore z = 90-x-y = 30$

$s = f_{xy} = -1$ $\qquad \therefore x=y=z=30$

$t = f_{yy} = -2$

$\therefore rt-s^2 = 4>0$

$r<0, t<0 \Rightarrow$ maxima

$$\frac{dy}{dx} = \frac{y^3}{e^{2x} + y^2}$$

$$e^{2x}\frac{dy}{dx} + y^2\frac{dy}{dx} = y^3$$

Multiplying $2e^{-2x}$ on both sides

$$2\frac{dy}{dx} + 2y^2\frac{dy}{dx}e^{-2x} = 2y^3e^{-2x}$$

Dividing by y

$$\frac{2}{y}\frac{dy}{dx} + 2y\frac{dy}{dx}e^{-2x} = 2y^2e^{-2x}$$

$$\int\frac{2}{y}\frac{dy}{dx} + \int\left(2y\frac{dy}{dx}e^{-2x} - 2y^2e^{-2x}\right) = 0$$

$$2\ln y + e^{-2x}y^2 = c$$

$f(x) = e^{2x}$

$f'(x) = -2e^{2x}$

$\therefore \int f(x) + f'(x) = f(x) + C$

↳ formula

$\int y = \frac{y^2}{2}$

$\therefore 2\int y = y^2$