Chapter 15

Kinetics of Particles: Impulse Momentum Method

15.1 Introduction:

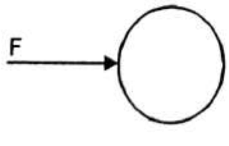
Having studied in the earlier chapters two different approaches, we shall in this chapter learn the third approach to the solution of kinetics of particles. This approach requires the use of Impulse Momentum Equation involving the parameters of forces, mass, velocity and time. Using this equation eliminates the determination of acceleration giving direct results in most of the cases.

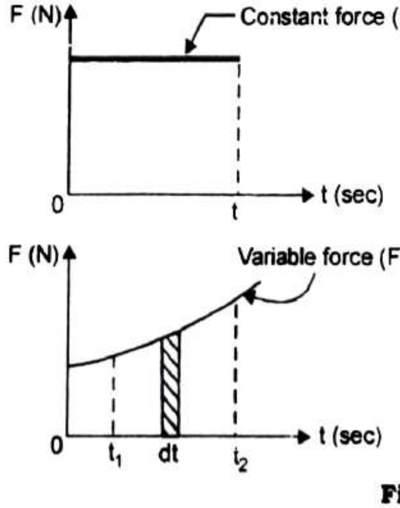
In the second part of this chapter we shall deduce the Conservation of Momentum Equation from the basic Impulse Momentum Equation and use it to solve problems involving a system of particles subject to internal action and reaction forces and not involving any external forces on the system (i.e. such a system where momentum is conserved).

Study of collision of particles forms the third part of this chapter. Here we shall learn the interesting phenomenon which takes place during a collision. The study reveals that there exists a certain relation between the velocities of the colliding particies before they collide to the velocities they acquire after impact.

15.2 Impulse:

Consider a particle acted upon by a force F for a duration of t sec.





This force is said to impart an impulse on the particle and the magnitude of this impulse is the product of the force and the duration for which it acts.

Fig. 15.1

If the force F is constant during the time it acts, then

$$\textbf{Impulse} = \mathbf{F} \times \mathbf{t} \qquad \text{............ [15.1 (a)]}$$

If the force F is variable, the impulse between the time interval t_1 and t_2 is

$$\textbf{Impulse} = \int_{t_1}^{t_2} \mathbf{F}\, \mathbf{dt} \qquad \text{............ [15.1 (b)]}$$

Impulse is a vector quantity and its unit is N.s

15.2.1 Impulsive Force:

A large force when acts for a very small time and which causes a considerable change in a particle's momentum is called an impulsive force. For example, when a moving particle coliides with another particle, the collision duration is very small, but the particles after collision have different magnitudes of velocities and in some cases even different directions of velocities, thereby indicating a considerable change in the momentum.

Other examples of impulsive forces are, when a bat hits a ball, the action and reaction forces at the contact point are impulsive forces which impart a new momentum to the ball. Also when a spring loaded toy gun releases the bullet, the spring force is an impulsive force in this case.

Impulsive forces are different from usual forces for the reason that the impulse generated by the impulsive forces is mainly due to the large force value, which acts for small time, whereas usual forces also generate impulse, where the duration (time) an equally important parameter, is large and equally contributes to the impulse generated.

15.3 Impulse Momentum Equation:

From Newton's Second Law Equation we have,

$$\Sigma F = ma$$

but $\quad a = dv/dt$

$$\therefore \quad \Sigma F = m\frac{dv}{dt}$$

$$\text{or} \quad \Sigma F = \frac{d(mv)}{dt} \qquad \text{............. make no change since } m \text{ is constant.}$$

Here the product mv is referred to as the *linear momentum* of the particle and may be defined as the amount of motion possessed by a moving body. It is a vector quantity and its SI units are kg.m/s or N.s

Now $\quad \Sigma F\, dt = d\,(mv)$

Integrating between the time interval t_1 to t_2 during which the velocity of the particle changes from v_1 to v_2.

$$\int_{t_1}^{t_2} \sum F\,dt = \int_{v_1}^{v_2} d(mv)$$

$$= mv_2 - mv_1$$

$$\therefore \qquad mv_1 + \int_{t_1}^{t_2} \sum F\,dt = mv_2$$

$\int_{t_1}^{t_2} \sum F\,dt$ is the impulse on the particle, we therefore have

$$\mathbf{mv_1 + Impulse_{1\text{-}2} = mv_2} \qquad \text{..........} \quad [15.2]$$

Equation 15.2 is known as Impulse Momentum Equation. This equation gives rise to Principle of Impulse Momentum which may be stated as "*for a particle or a system of particles acted upon by forces during a time interval the total impulse acting on the system is equal to the difference between the final momentum and initial momentum during that period*".

15.4 Application of Impulse Momentum Equation:

Impulse Momentum Equation is a third equation other than the Newton's Second Law Equation and Work Energy Principle Equation, using which we can analyse the kinetics of particles.

Impulse Momentum Equation involves parameters viz. force, mass, velocity and time. Thus the velocity of the particle at the new position may be worked out knowing the forces acting on the particle and the duration for which they act.

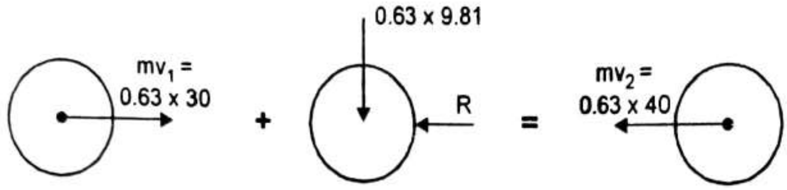
Also, if the initial and final velocities of the particle are known, the duration for which the particle is in motion can be found out.

This method therefore eliminates the calculation of acceleration and many times use of the kinematics relations, to give direct results. We will further deduce the Conservation of Momentum Equation from the Impulse Momentum Equation and apply it to a system of particles where the momentum is conserved.

To use the method of Impulse and Momentum, we need to draw three figures. The L.H.S. figure would show the initial momentum vector of the particle. The central figure shows the F.B.D. of the particle. The R.H.S. figure shows the final momentum vector of the particle. Drawing of these figures helps in writing the Impulse Momentum Equation. Since all terms in this equation are vector quantities, a proper sign convention for the direction need to be chosen before plugging in the values.

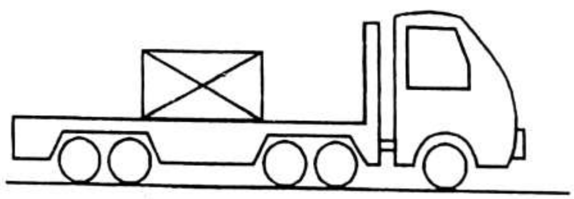
Ex. 15.1 A 630 gm cricket ball strikes the bat with a speed of 108 kmph and is hit back by the batsman with a speed of 144 kmph. If the ball was in contact with the bat for 72 milliseconds, determine the average impulsive force exerted on the ball.

Solution: Applying Impulsive Momentum Principle in the **x** direction to the ball just as it hits the bat and then rebounds back (i.e. during the 72 milliseconds contact period). Let R be the impulsive force received by the ball



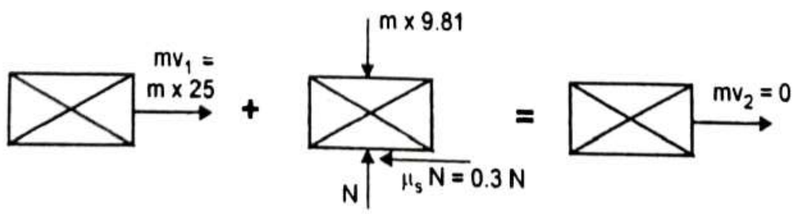
$$mv_1 + \text{Impulse}_{1\text{-}2} = mv_2$$
$$0.63 \times 30 - R \times 0.072 = -0.63 \times 40$$
$$R = 612.5 \text{ N}$$ .... **Ans.**

Ex. 15.2 A truck traveling at a constant speed of 90 kmph on a straight highway carries a package on its flat bed trailor. $\mu_s = 0.3$ and $\mu_k = 0.2$ between the package and the flat bed. If the truck suddenly wants to come to a halt determine the minimum time in which it can do so without the package slipping on the flat bed.



Solution: As the truck driver applies the brakes, the package kept on it tends to slip forward. However the static frictional force prevents the package from slipping. Since the truck has to come to a halt in a minimum possible time implies that the static frictional force reaches its maximum value i.e. $\mu_s N$.

Let us analyse the kinetics of only the package. We shall draw three figures of the package. The L.H.S. and R.H.S. figures represent the initial and final momentum, while the central figure represents the FBD and is use to calculate the impulse, since Impulse = Force × time



Applying Impulse Momentum Equation in the **x** direction $\rightarrow$ + ve

$$mv_1 + \text{Impulse}_{1\text{-}2} = mv_2$$

The only force in the **x** direction is 0.3 N

$\therefore$ Impulse = Force × time = (0.3 N) × t

also N = m × 9.81 $\therefore$ Impulse = [0.3 (m × 9.81)] × t

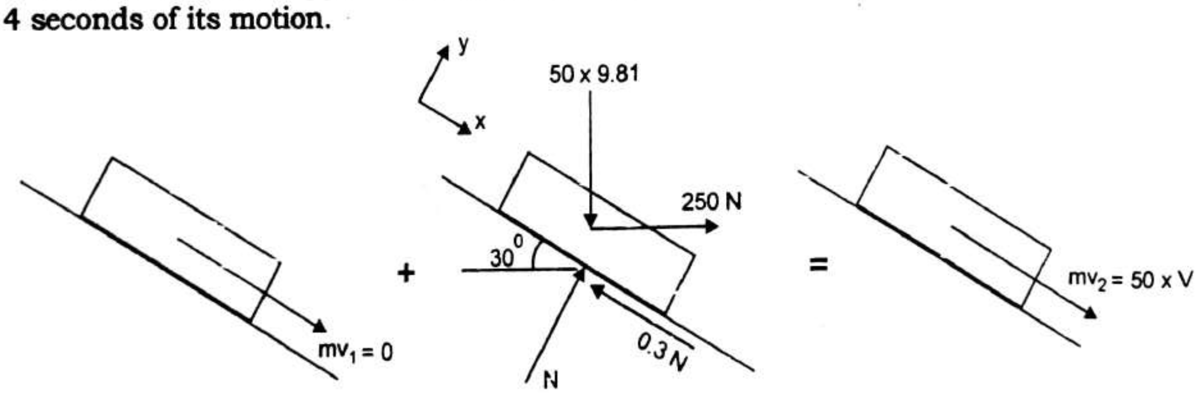
Substituting in the above equation, we get

$$m \times 25 + [-0.3 \times (m \times 9.81) \times t] = 0$$

or t = 8.495 sec .... **Ans.**

Ex. 15.3 A block of mass of 50 kg is placed on a plane inclined at 30° with the horizontal. A horizontal force of 250 N acts on the block tending to move the block down the plane. Determine its velocity 4 sec after starting from rest. Take $\mu_k = 0.3$.

Solution: We shall apply the Impulse Momentum Equation to the block for the first 4 seconds of its motion.



Applying Impulse Momentum Equation in the y direction

$$mv_1 + \text{Impulse}_{1\text{-}2} = mv_2$$

Forces in the y direction are the weight component, the component of 250 N and the normal reaction.

$\therefore$ $0 + [-50 \times 9.81 \cos 30 + 250 \sin 30 + N] \times 4 = 0$

or N = 300 Newton

Applying Impulse Momentum Equation in the **x** direction

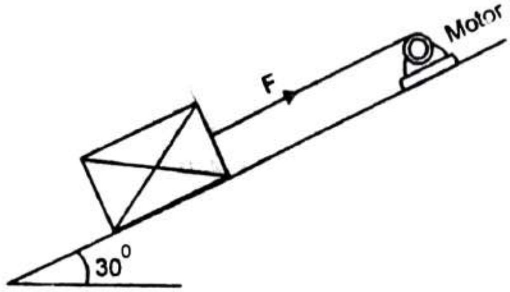
$$mv_1 + \text{Impulse}_{1\text{-}2} = mv_2$$

Forces in the x direction are the weight component, the component of 250 N and the frictional force.

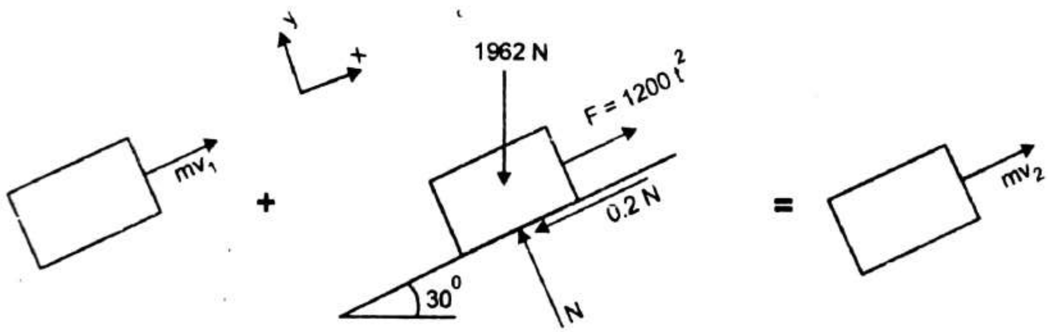
$$\therefore \quad 0 + [\,50\times 9.81 \sin 30 + 250 \cos 30 - 0.3 \times 300\,] \times 4 = 50\,v$$

or $\quad v = 29.74$ m/s .... **Ans.**

Ex. 15.4 A 200 kg package is being pulled up by a cable powered by a motor. For a short time the force in the cable is $F = 1200\,t^2$ N. If the package has an initial velocity of 3 m/s at $t = 0$ find its velocity at $t = 2$ sec. Take $\mu_k = 0.2$ between package and incline.



Solution:



This is a case of variable force for which Impulse = $\int F\,dt$

Applying Impulse Momentum Equation in the x direction

$$mv_1 + \text{Impulse}_{1\text{-}2} = mv_2$$

$$200 \times 3 + \int_0^2 \left[1200\,t^2 - 1962 \sin 30 - 0.2(1962 \cos 30)\right] dt = 200 \times v$$

$$600 + \left[1200\frac{t^3}{3} - 1321\,t\right]_0^2 = 200\,v$$

$$\therefore \quad v = 5.79 \text{ m/s}$$ .... **Ans.**

15.5 Conservation of Momentum Equation:

Figure shows a man (B) standing at the end of a boat (A). The system of man and boat is initially at rest i.e. $v_A = 0$ and $v_B = 0$

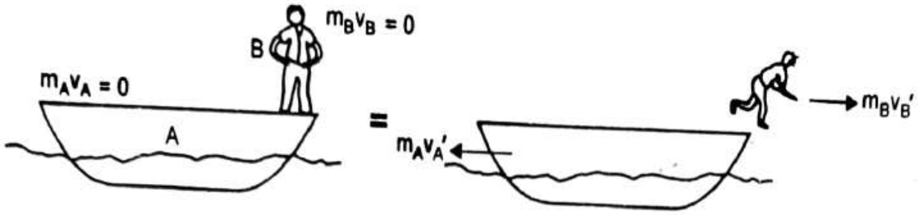


Fig. 15.2

If the man jumps off horizontally in the water with a velocity v_B', he induces a backward motion to the boat, which now starts moving with a velocity v_A'.

If the water resistance is neglected the question is, how is the motion of the boat induced?

To jump off from the boat, the man exerts an impulsive force through his feet on the surface of the boat. This induces a reaction impulsive force to the man, which throws him in the water. Also since the water resistance is small, it may be neglected. Thus the impulsive force exerted by the man is the cause of the backward motion of boat.

The interesting part here is that the impulsive force exerted by the man on the boat or the reaction impulsive force of the boat on the man are nothing but action and reaction forces acting for the same time interval. Due to this the net impulse in the direction of motion is zero. The Impulse Momentum Equation thereby reduces to Conservation of Momentum Equation . Applying this concept to the above example, we have

$$mv_1 + \text{Impulse}_{1\text{-}2} = mv_2$$

i.e. $$(m_A v_A + m_B v_B) + \text{Impulse}_{1\text{-}2} = (m_A v'_A + m_B v'_B)$$

since $$m_A v_A = m_B v_B = 0$$

and $\text{Impulse}_{1\text{-}2} = 0$, we get

$$0 = m_A v_A' + m_B v_B'$$

$$-m_A v_A' = m_B v_B'$$

This relation indicates that knowing the masses of A and B and velocity of man (v_B') as he jumps, we can find the velocity of boat (v_A'). The negative sign attached to the magnitude of the velocity tells us that the boat moves in the opposite direction to that of man.

The conservation of momentum phenomenon takes place in other similar situations such as when a bullet is fired from the gun and thereby the gun recoils backwards, or during the collision of two particles, the particles move with different velocities after the collision. In general we may say "*for dynamic situations involving a system of particles, if the net impulse is zero, the momentum of the system is conserved*". The equation of Conservation of Momentum is therefore expressed as.

Initial Momentum = Final Momentum ...... [15.3]

Ex. 15.5 A man of mass 70 kg and a boy of mass 30 kg dive off the end of a boat, of mass 150 kg, with a horizontal velocity of 2 m/s relative to the boat. If initially the boat is at rest, find its velocity just after

1) both the man and boy dive off simultaneously
2) the man dives first followed by the boy.

Solution: *Case (1) both the person dive simultaneously*

Let the boat move backwards as the man and boy jump forward.
From Relative Motion Equation we can write

$$v_{man/boat} = v_{man} - v_{boat} \qquad \rightarrow +\text{ve}$$

$$2 = v_{man} - (-v_{boat})$$

$$\therefore \quad v_{man} = 2 - v_{boat}$$

also

$$v_{boy/boat} = v_{boy} - v_{boat}$$

$$2 = v_{boy} - (-v_{boat})$$

$$\therefore \quad v_{boy} = 2 - v_{boat}$$

Applying Conservation of Momentum Equation to the system of boy, man and boat

Initial momentum = Final Momentum

$$0 = (m \times v)_{boy} + (m \times v)_{man} + (m \times v)_{boat} \qquad \rightarrow +\text{ve}$$

$$0 = 30\,(2 - v_{boat}) + 70\,(2 - v_{boat}) + 150\,(-v_{boat})$$

$$\therefore \quad v_{boat} = 0.8 \text{ m/s (backwards)}$$ .... **Ans.**

Case (2) The man dives first followed by the boy

Let the man jump off the boat, the boy still being on the boat.

Applying Conservation of Momentum Equation

Initial Momentum = Final Momentum ($\rightarrow$ +ve)

$$0 = (m \times v)_{man} + (m \times v)_{boat}$$

$$0 = 70 \times (2 - v_{boat}) + 180\,(-v_{boat})$$

$$\therefore \quad v_{boat} = 0.56 \text{ m/s (backwards)}$$

Now the boy jumps forwards from the boat which is moving backwards with velocity of 0.56 m/s.

Applying Conservation of Momentum Equation to the system of the boy and boat

Initial Momentum = Final Momentum ($\rightarrow$ + ve)

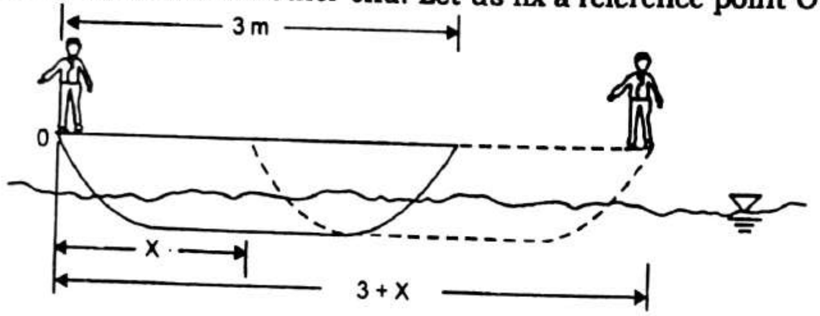
$$(m \times v)_{boat} = (m \times v)_{boy} + (m \times v)_{boat}$$

$$180 \times (-0.56) = 30\,(2 - v_{boat}) + 150\,(-v_{boat})$$

$\therefore$ $v_{boat} = 0.893$ m/s (backwards) ...... **Ans.**

Ex. 15.6 A man of weight 800 N is standing on one end of a boat of weight 2400 N and 3 m long. He then walks to the other end of the boat. What is the corresponding displacement of the boat? Neglect water resistance to motion.

Solution: Let us initially assume that the boat is displaced in the forward direction by x meters as the man reaches the other end. Let us fix a reference point O as shown.



Applying Conservation of Momentum Equation in the x direction to the system of man and boat.

Initial momentum = Final Momentum ($\rightarrow$ + ve)

$$0 = (m \times v)_{man} + (m \times v)_{boat}$$

$$0 = \frac{800}{g} \times v_{man} + \frac{2400}{g} v_{boat}$$

$$0 = 800\left(\frac{dx}{dt}\right)_{man} + 2400 \times \left(\frac{dx}{dt}\right)_{boat} \quad \text{......... since } v = \frac{dx}{dt}$$

$$0 = 800\, dx_{man} + 2400\, dx_{boat}$$

Integrating with proper limits

$$0 = 800 \int_0^{(3+x)} dx_{man} + 2400 \int_0^{x} dx_{boat}$$

$$0 = 800\,[x_{man}]_0^{3+x} + 2400\,[x_{boat}]_0^{x}$$

$$0 = 800\,[(3 + x) - 0] + 2400\,[x - 0]$$

$\therefore$ $x = -0.75$ m

or $x = 0.75$ m (backwards) .... **Ans.**

Exercise 15.1

P1. A bullet of mass 30 gm moving with a velocity of 226 m/s strikes a wooden log and penetrates through a distance of 185 mm. Calculate the average retarding force offered by the log in stopping the bullet. Use Impulse Momentum Equation only.

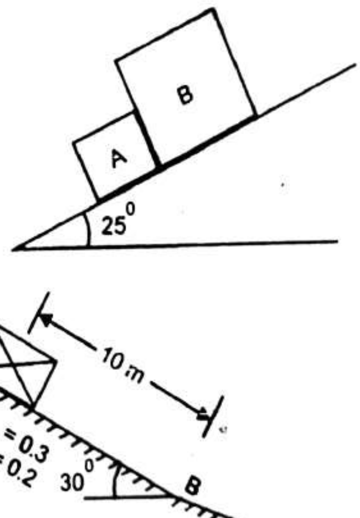
P2. A dish slides 1200 mm on a level table before coming to rest. If $\mu_k = 0.18$ between the dish and table, what was the time of travel. Use Impulse Momentum Equation only

P3. A car at 72 kmph applies brakes and comes to rest in 8 sec. Find the minimum coefficient of friction between wheel and road. Solve by Impulse Momentum Equation.

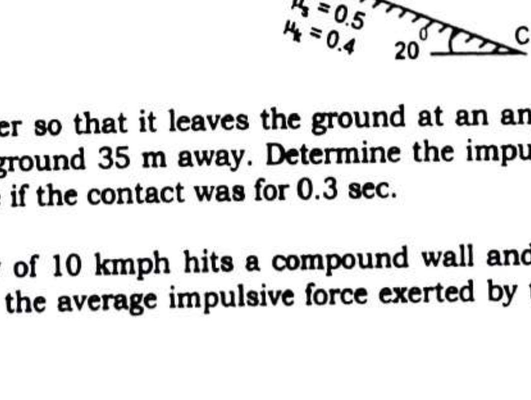
P4. A particle of mass 5 kg is acted upon by a force defined by a relation $F = 60\,t + 40$ N. Find its velocity after 5 sec if it starts from rest and moves in a straight path.

P5. A man of mass 70 kg sits in a stationary boat A and holds a rope tied to another stationary boat B. The man now pulls the rope with a jerk due to which boat B moves with a relative velocity of -9 m/s towards boat A. Find the new true velocities of the boats. If the jerk lasted for 0.4 sec find the average tension in the rope during this period. Take mass of both the boats to be 200 kg each and neglect water resistance.

P6. Two packages A (10 kg) and B (6 kg) are in contact with each other and released from rest at t = 0, down the incline. Between package A and incline $\mu_s = 0.4$ and $\mu_k = 0.3$ and for package B $\mu_s = 0.25$ and $\mu_k = 0.2$. Determine the velocity of the two packages at t = 5 sec. Also find the contact force between the two packages during motion.



P7. A crate is released from rest from point A. It travels down the two inclined planes and finally comes to a halt at C. Determine the time for which the crate was in motion.



P8. A 380 gm football is kicked by a player so that it leaves the ground at an angle of 40° with the horizontal and lands on the ground 35 m away. Determine the impulse given to the ball. Also find the impulsive force if the contact was for 0.3 sec.

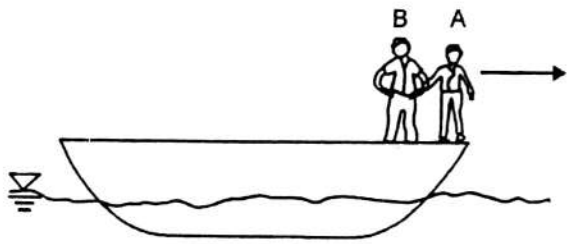
P9. A 1500 kg car moving with a velocity of 10 kmph hits a compound wall and is brought to rest in 400 milliseconds. What is the average impulsive force exerted by the wall on the car bumper?

P10. A 1200 kg automobile is traveling at 90 kmph when brakes are fully applied, causing all four wheels to skid. If μ_k is 0.1, find the time required for the automobile to come to halt.

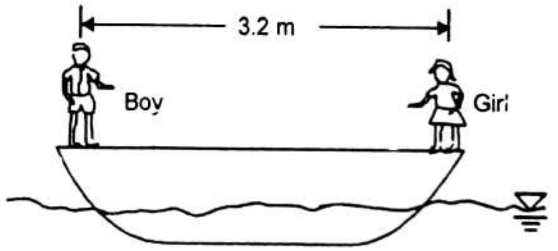
P11. A bullet of mass 1 gram has a velocity of 1000 m/s as it enters a fixed block of wood. It comes to rest in 2 milliseconds after entering the block. Determine the average force that acted on the bullet and the distance penetrated by it.

P12. A 30 kg boy is stationary on a 50 kg boat which is initially at rest. If the boy jumps off the boat horizontally with a speed of 3 m/s relative to the boat, determine the speed of the boat. Neglect water resistance.

P13. Two men A (500 N) and B (700 N) lined up at one end of the boat (3000 N) and dived horizontally off the boat in succession with a velocity of 4 m/s relative to the boat. Find the velocity of the boat after the second man B had dived. Neglect water resistance.



P14. A boy (50 kg) and a girl (40 kg) stood on the two ends of a boat (70 kg). If they exchange their positions, determine the displacement of the boat. Neglect water friction.



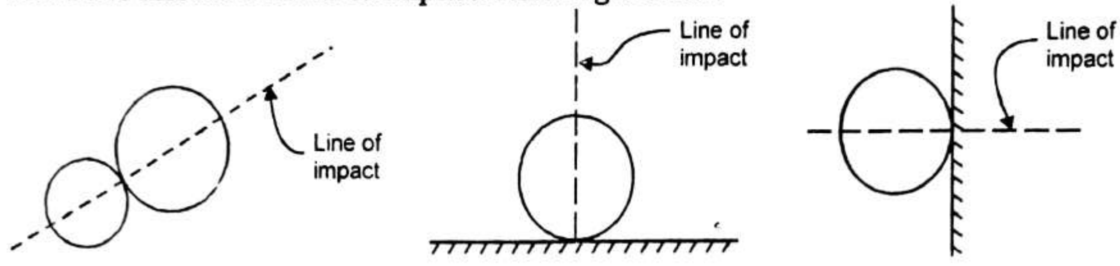
P15. A 30 kg boy is standing on a 50 kg boat A which is initially at rest. If the boy jumps off horizontally with a speed of 2 m/s relative to the boat A on another identical stationary boat B, determine the speed of both the boats A and B. Neglect water resistance.

15.6 Impact

A collision of two bodies which takes place during a very small time interval and during which the colliding bodies exert relatively larger forces on each other is known as Impact.

15.6.1 Line of Impact:

When two bodies collide, the line joining the common normals of the colliding bodies is known as Line of Impact. Refer figure 15.3

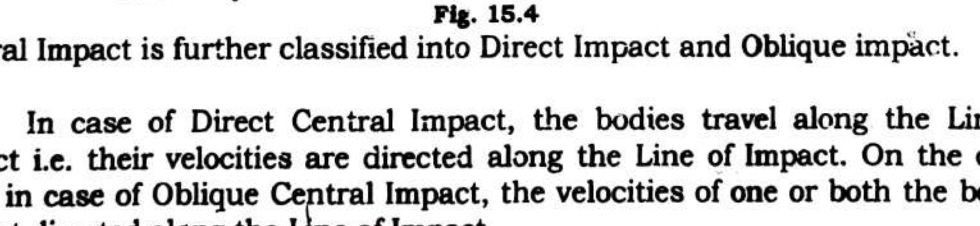


(a) Two strikers collide (b) Ball strikes the ground (c) Ball hits a wall

Fig. 15.3

15.6.2 Types of Impact:

Impacts are basically divided into two categories viz. a Central Impact and an Eccentric Impact. When the mass centres of the colliding bodies lie on the Line of Impact, the Impact is said to be a Central Impact. When the mass centres of the colliding bodies are not located on the Line of Impact, we call such Impact to be an Eccentric Impact.



Central Impact Eccentric Impact

Fig. 15.4

Central Impact is further classified into Direct Impact and Oblique impact.

In case of Direct Central Impact, the bodies travel along the Line of Impact i.e. their velocities are directed along the Line of Impact. On the other hand in case of Oblique Central Impact, the velocities of one or both the bodies are not directed along the Line of Impact.

Direct Central Impact Oblique Central Impact

Fig. 15.5

15.7 Direct Central Impact:

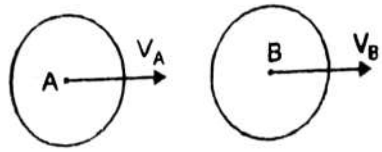
Let us now study the phenomenon of a Direct Central Impact. What really happens during an impact is quite interesting. Let us understand stage by stage as to what happens during a direct central impact.

1) Fig. (a) shows two particles A and B with velocities v_A and v_B. If v_A is grater than v_B, the impact will soon take place.

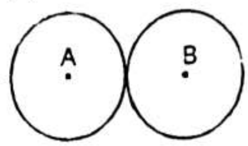
2) The period of impact is made of period of deformation and period of restitution.

3) During the period of deformation Fig (b), the two particles exert large impulsive force on each other. The deformation of both the particles continue till maximum deformation. At this stage both the particles are said to have momentarily united and move with common velocity v. Fig (c).

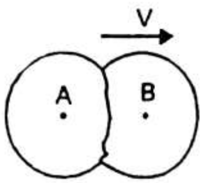
4) Now the period of restitution begins Fig (c). The two particles restore their shape during this period. Sometimes permanent deformations are also set in the particles. During this period the impulsive force exerted by the two particles is lesser than during the period of deformation. At the end of the period of restitution the two particles separate from each other. Fig.(d).



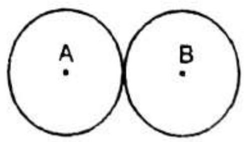
(a) Before Impact



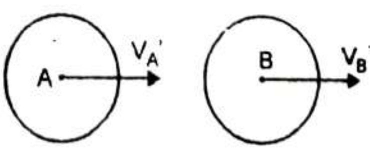
(b) On Impact Deformation Begins



(c) At Max. Deformation Restitution period Begins



(d) Restitution period ends



(e) Particles separate and move with new velocities

Fig. 15.6

5) The two particles A and B now have new velocities v_A' and v_B' respectively. Fig. (e).

15.7.1 Coefficient of Restitution:

Let us apply Impulse Momentum Equation in the x direction to the particle A during the period of deformation. Let P be the impulsive force exerted by particle B on A. At the start of period of deformation the velocity of particle A is v_A and at the end of period of deformation its velocity changes to v, being the common velocity of A and B.

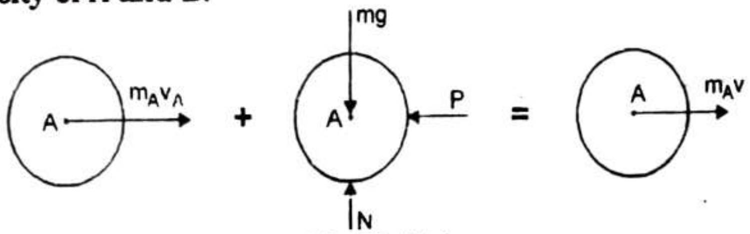


Fig. 15.7 (a)

$$mv_1 + \text{Impulse}_{1\text{-}2} = mv_2$$

$$m_A v_A - \int P\,dt = m_A v \qquad \text{........... (1)}$$

here $\int P\,dt$ is the impulse due to the impulsive force P acting during the period of deformation. Now let us apply Impulse Momentum Equation to the particle A during the period of restitution. During this period a smaller impulsive force say R be exerted by particle B on A. At the start of period of restitution the velocity of A is v and changes to v_A' at the end of the period of restitution.

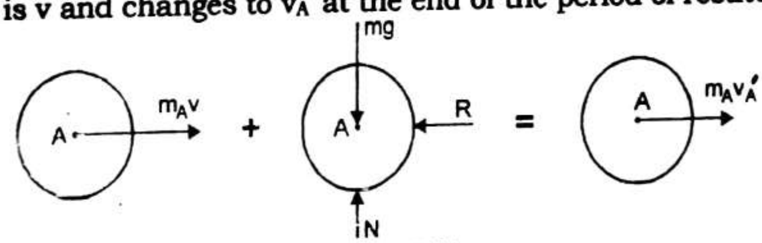


Fig. 15.7 (b)

$$mv_1 + \text{Impulse}_{1\text{-}2} = mv_2$$

$$m_A v - \int R\,dt = m_A v_A' \text{.......... (2)}$$

here $\int R\,dt$ is the impulse due to the impulsive force R acting during the period of restitution

From equations (1) and (2) we have

$$\frac{\int R\,dt}{\int P\,dt} = \frac{v - v_A'}{v_A - v}$$

If $$e = \frac{\int R\,dt}{\int P\,dt} \qquad \text{.......... (3)}$$

then $$e = \frac{v - v_A'}{v_A - v} \qquad \text{.......... (4)}$$

e is referred to as the *Coefficient of Restitution and is defined as the ratio of the impulse exerted between the colliding particles during the period of restitution to the impulse exerted during the period of deformation.*

Similarly if the Impulse Momentum Equation is applied to the particle **B,** we get

$$e = \frac{v_B' - v}{v - v_B} \qquad \text{.......... (5)}$$

From basic algebra if $A = \frac{B}{C} = \frac{D}{E}$, then $A = \frac{B+D}{C+E}$

From equations (4) and (5) we therefore have,

$$e = \frac{v - v_A' + v_B' - v}{v_A - v + v - v_B}$$

$$\therefore \qquad e = \frac{v_B' - v_A'}{v_A - v_B}$$

or $$\mathbf{v_B' - v_A' = e(v_A - v_B)} \qquad \text{...... [15.4]}$$

we will refer equation 15.4 as Coefficient of Restitution Equation.

The value of coefficient of restitution 'e' lies between 0 and 1. It mainly depends on the nature of the bodies of collision. For example, the rebound velocity of a rubber ball which falls freely from a certain height on to the ground is different from that of a tennis ball which falls freely from the same height.

A special case arises if e = 0 or if e = 1

If e = 0, the impact is referred to as a *perfectly plastic impact.* In such impact the two bodies move with same velocity after impact. The momentum is conserved but loss of energy takes place.

If e = 1, the impact is referred to as a *perfectly elastic impact.* In such impact momentum is conserved and there is also no loss of energy i.e. energy is also conserved.

For all other impacts e lies between 0 and 1. In such impacts the momentum is conserved, but there is loss of energy during impact, as heat and sound is generated.

15.7.2 *Procedure to solve Direct Central Impact Problems:*

Given velocities before impact i.e v_A, v_B and coefficient of restitution e and to find velocities after impact.

Step (1) During impact since there are no external forces acting on the colliding bodies, the momentum of the system is conserved. i.e. conservation of momentum is applicable, which is

Initial Momentum = Final Momentum

$$m_A v_A + m_B v_B = m_A v_A' + m_B v_B'$$

Step (2) Use Coefficient of Restitution Equation i.e.

$$v_B' - v_A' = e(v_A - v_B)$$

Step (3) solve the two equations to find the velocities v_A' and v_B' after impact.

Note: Take proper sign conventions for the direction of velocities while using the Conservation of Momentum and Coefficient of Restitution Equations.

15.7.3 *Special Case of Direct Central Impact*

A special case happens when a ball has a direct central impact with a rigid body of infinite mass. For example, a ball hitting a wall or striking the ground. Consider one such case of ball having a direct central impact with ground.

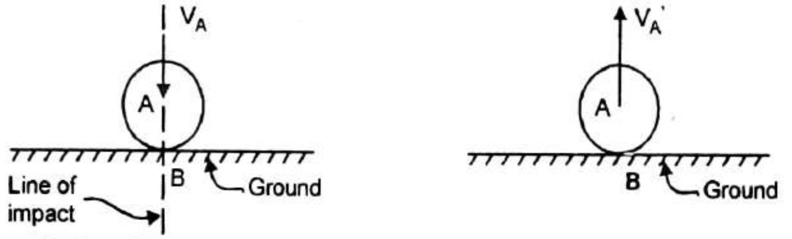


Fig. 15.8

Applying Coefficient of Restitution Equation, we get

$$v_B' - v_A' = e(v_A - v_B)$$

$\therefore \quad 0 - v_A' = e(v_A - 0) \qquad \because v_B = v_B' = 0$

$\therefore \quad v_A' = -e\, v_A$

The – ve sign indicates that the motion of ball after impact is in the opposite direction

$\therefore$ The magnitude of the velocity of a ball after a direct central impact with ground or wall is given by

$\mathbf{v_A' = e\, v_A}$ .... equation for special case of direct impact

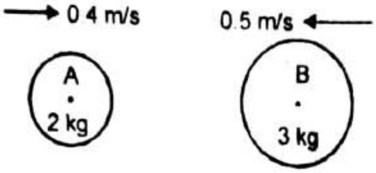
Ex. 15.7 A 2 kg ball moving with 0.4 m/s towards right, collides head on with another ball of mass 3 kg, moving with 0.5 m/s towards left. Determine the velocities of the balls after impact and the corresponding percentage loss of kinetic energy, when

i) the impact is perfectly elastic

ii) the impact is perfectly plastic

iii) the impact is such that $e = 0.7$

Solution:



This is a case of Direct Central Impact.

i) Impact is perfectly elastic i.e. $e = 1$

Using Conservation of Momentum Equation $\rightarrow +ve$

$$m_A v_A + m_B v_B = m_A v_A' + m_B v_B'$$

$$2 \times 0.4 + 3 \times (-0.5) = 2 v_A' + 3 v_B'$$

$$-0.7 = 2 v_A' + 3 v_B' \qquad \text{............ (1)}$$

Using Coefficient of Restitution Equation $\rightarrow +ve$

$$v_B' - v_A' = e[v_A - v_B]$$

$$v_B' - v_A' = 1[0.4 - (-0.5)]$$

$$v_B' = 0.9 + v_A' \qquad \text{............ (2)}$$

Solving equations (1) and (2), we get

$$v_A' = -0.68 \text{ m/s} = 0.68 \text{ m/s} \leftarrow \qquad \text{.......} \textbf{ Ans.}$$

$$v_B' = 0.22 \text{ m/s} = 0.22 \text{ m/s} \rightarrow \qquad \text{.......} \textbf{ Ans.}$$

Since impact is perfectly elastic, implies that the energy is conserved i.e. there will be no loss of kinetic energy.

ii) Impact is perfectly plastic i.e. $e = 0$

In this case, the particles move together with a common velocity after impact,

i.e $\quad v_A' = v_B' = v'$

Using Conservation of Momentum Equation

$$m_A v_A + m_B v_B = m_A v_A' + m_B v_B' \quad \rightarrow +ve$$

$$2 \times 0.4 + 3 \times (-0.5) = 2v' + 3v'$$

$\therefore$
$$v' = -0.14 \text{ m/s}$$

i.e.
$$v_A' = v_B' = 0.14 \text{ m/s} \leftarrow \qquad \text{.......} \textbf{ Ans.}$$

Kinetic energy of the system before impact

$$= \frac{1}{2} m v_A^{\,2} + \frac{1}{2} m v_B^{\,2}$$

$$= ½ \times 2 \times (0.4)^2 + ½ \times 3 \times (0.5)^2 = 0.535 \text{ J}$$

Kinetic energy of the system after impact

$$= ½ \times 2 \times (0.14)^2 + ½ \times 3 \times (0.14)^2 = 0.049 \text{ J}$$

$\therefore$ Percentage loss of kinetic energy

$$= \frac{0.535 - 0.049}{0.535} \times 100 = 90.84$$ .... **Ans.**

iii) Impact when $e = 0.7$

$$m_A v_A + m_B v_B = m_A v_A' + m_B v_B' \quad \rightarrow +ve$$

$$2 \times 0.4 + 3 \times (-0.5) = 2 v_A' + 3 v_B'$$

$$-0.7 = 2 v_A' + 3 v_B' \qquad \text{........... (1)}$$

Using Coefficient of Restitution Equation

$$v_B' - v_A' = e[v_A - v_B]$$

$$v_B' - v_A' = 0.7[0.4 - (-0.5)]$$

$$v_B' = 0.63 + v_A' \qquad \text{........... (2)}$$

Solving equations (1) and (2)

$$v_A' = -0.518 \text{ m/s} = 0.518 \text{ m/s} \leftarrow$$ .... **Ans.**

$$v_B' = 0.112 \text{ m/s} = 0.112 \text{ m/s} \rightarrow$$ .... **Ans.**

Kinetic energy of the system after impact

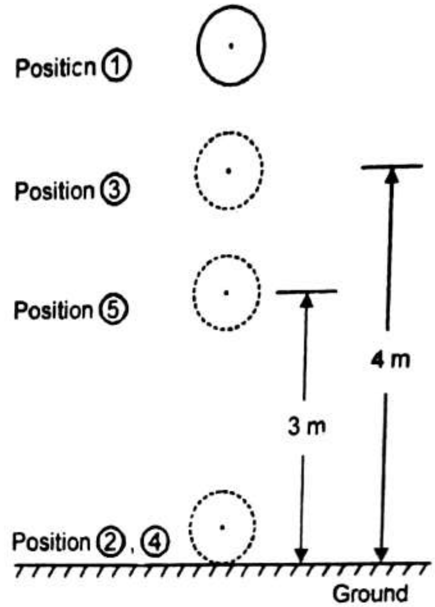
$$= ½ \times 2 \times (0.518)^2 + ½ \times 3 \times (0.112)^2 = 0.287 \text{ J}$$

$\therefore$ Percentage loss of kinetic energy

$$= \frac{0.535 - 0.287}{0.535} \times 100 = 46.33$$ .... **Ans.**

Ex. 15.8 A ball is dropped on a smooth horizontal floor from which it bounces to a height of 4 m. On the second bounce it attains a height of 3 m. Find the coefficient of restitution between the ball and the floor.

Solution: The ball is dropped from a certain height i.e position (1) and hits the ground i.e. position (2). Now it rises by 4 m to position (3) and then falls on the ground i.e. position (4)



Applying Motion Under Gravity (3)–(4) ↓ + ve

$u = 0$, $v = ?$, $s = 4$ m, $a = 9.81$ m/s^2

using $v^2 = u^2 + 2as$

$v^2 = 0 + 2 \times 9.81 \times 4$

$\therefore$ $v = 8.859$ m/s ...... velocity of ball just before impact

After impact with ground the ball rises to a height of 3 m i.e. position (5)

Applying Motion Under Gravity (4) – (5) ↑ + ve

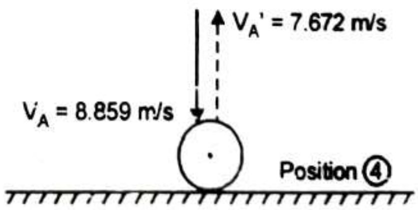
$u = ?$, $v = 0$, $s = 3$ m, $a = -9.81$ m/s^2

using $v^2 = u^2 + 2as$

$0 = u^2 + 2 \times (-9.81) \times 3$

$\therefore$ $u = 7.672$ m/s .... velocity of ball just after impact

Impact at position (4) is a special case of direct impact since the ball hits the ground which is a body of infinite mass.



Applying special case relation of direct impact, we get

$$v_A' = e\, v_A$$

$$7.672 = e \times 8.859$$

$\therefore$ $e = 0.866$ .... **Ans.**

Ex. 15.9 A 100 gm ball dropped vertically on a 500 gm steel plate resting on hard ground from a certain height h over the steel plate is found to rebound to a height of 800 mm. Now the ball when dropped from the same height h over the steel plate, rebounds to a height of 200 mm, when a spring is placed between the plate and ground. Determine; a) The coefficient of restitution between the ball and the plate.
b) The initial height h from which the ball was being dropped.

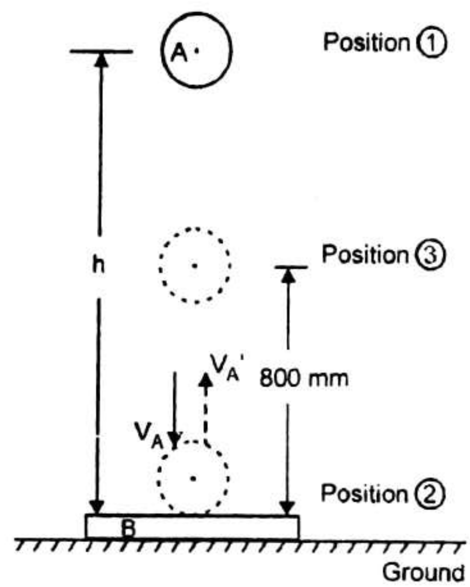
Solution:

Case (1) when the steel plate rests on hard ground.

Motion of ball position (1) to position (2)

It is M.U.G. ↓ + ve

$u = 0$, $v = v_A$, $s = h$, $a = 9.81\ m/s^2$



using $v^2 = u^2 + 2as$

$$v_A^2 = 0 + 2 \times 9.81 \times h \quad \text{........ (1)}$$

$$\therefore \quad v_A = 4.429\sqrt{h}\ \ m/s$$

At position (2) impact occurs. It is a direct central impact. Since steel plate is resting on hard ground, it will have no velocity after impact, i.e. $v_B' = 0$

Using Coefficient of Restitution Equation

$$v_B' - v_A' = e[v_A - v_B] \quad \downarrow + ve$$

$$0 - v_A' = e[4.429\sqrt{h} - 0]$$

$$v_A' = -4.429\, e\sqrt{h}$$

or $$v_A = 4.429\, e\sqrt{h} \quad \uparrow \quad \text{........ (2)}$$

After impact at position (2) the ball rebounds with a velocity $v_A' = 4.429\, e\sqrt{h}$ and rises 800 mm to reach position (3)

Applying M.U.G. position (2) – (3) ↑ + ve

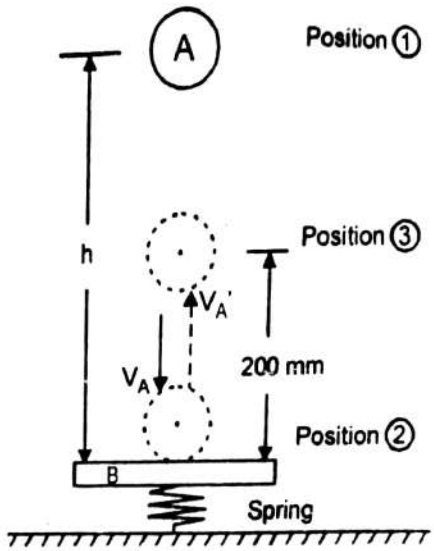
$u = v_A' = 4.429\, e\sqrt{h}$, $v = 0$, $s = 0.8$ m, $a = -9.81\ m/s^2$

using $v^2 = u^2 + 2as$

$$0 = (4.429\, e\sqrt{h}\,)^2 + 2 \times (-9.81) \times 0.8$$

$$\therefore \quad e\sqrt{h} = 0.8944 \quad \text{....... (3)}$$

Case (2) when the steel plate rests on a spring.



Velocity of ball before impact will be the same as for case (1)

i.e $v_A = 4.429\sqrt{h}$ m/s

At position (2) impact occurs.

It is again a direct central impact. Since steel plate is now resting on a spring, the plate will move downwards after impact with a velocity v_B'. The ball rebounds upwards with a velocity v_A'.

Using Conservation of Momentum Equation $\downarrow$ + ve

$$m_A v_A + m_B v_B = m_A v_A' + m_B v_B'$$

$$0.1 \times 4.429\sqrt{h} + 0 = 0.1 \times (-v_A') + 0.5 \times v_B'$$

$$0.4429\sqrt{h} = -0.1 v_A' + 0.5 v_B' \quad \text{............ (4)}$$

Using Coefficient of Restitution Equation $\downarrow$ + ve

$$v_B' - v_A' = e[v_A - v_B]$$

$$v_B' - (-v_A') = e[4.429\sqrt{h} - 0]$$

$$v_B' = 4.429\, e\sqrt{h} - v_A' \quad \text{............. (5)}$$

Substituting for v_B' from equation (5) in equation (4)

$$0.4429\sqrt{h} = -0.1 v_A' + 0.5(4.429\, e\sqrt{h} - v_A')$$

$$0.6 v_A' = 2.2145\, e\sqrt{h} - 0.4429\sqrt{h} \quad \text{.............. (6)}$$

After impact at position (2), the ball rises 200 mm to position (3)

Applying M. U. G. position (2) to (3) $\uparrow$ + ve

$u = v_A'$, $v = 0$, $s = 0.2$ m, $a = -9.81$ m/s^2

using $v^2 = u^2 + 2as$

$$0 = (v_A')^2 + 2 \times (-9.81) \times 0.2$$

$$v_A' = 1.98 \text{ m/s}$$

Substituting value of v_A' and $e\sqrt{h}$ in equation (6), we get

$$0.6 \times 1.98 = 2.2145 \times 0.8944 - 0.4429\sqrt{h}$$

$\therefore$ $h = 3.2$ m .... **Ans.**

Substituting value of h in equation (3), we get

$e = 0.5$ .... **Ans.**

15.8 Oblique Central Impact

When the velocities of either one or both colliding particles is not directed along the line of impact, the impact is said to be Oblique Central Impact. In such an impact not only the magnitudes of velocities after impact are unknown, but the new direction of travel is also unknown and needs to be worked out.

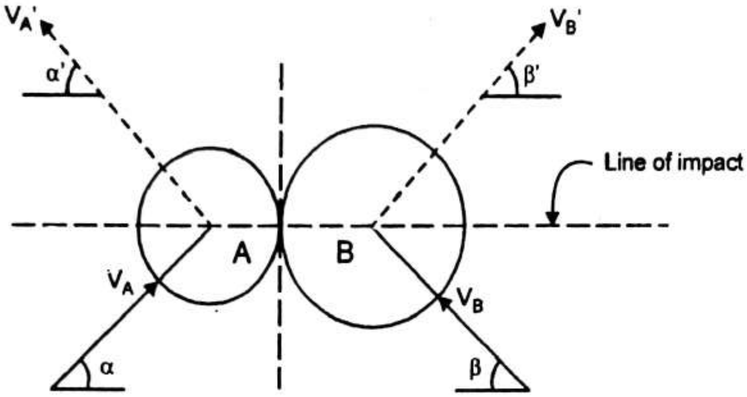


Fig. 15.9

In an Oblique Central Impact the impulsive force acts along the line of impact. Thus velocity changes occur only along the line of impact and no change in velocity takes place in a direction perpendicular to the line of impact.

15.8.1 *Procedure to Solve Oblique Central Impact Problems:*

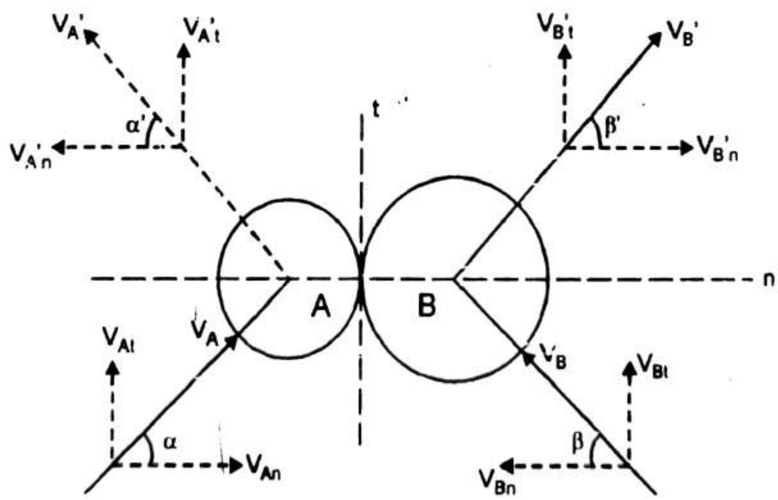


Fig. 15.10

Given: Initial velocities of particles A and B to be v_A at an angle α and v_B at an angle β, coefficient of restitution e.

To find: Velocities v_A' and v_B' and new angles α' and β after impact.

Step 1) **Let the line of impact be now called as n direction of impact. Let t be the direction perpendicular to the n direction.**

Step 2) Resolve the initial velocities v_A and v_B along the n and t direction so as to get their components v_{An}, v_{At} and v_{Bn}, v_{Bt}.

Step 3) Work as a direct impact problem in the n direction, taking v_{An} and v_{Bn} as initial velocities. Using Conservation of Momentum and Coefficient of Restitution Equations, find velocity components $v_{A'n}$ and $v_{B'n}$ after impact.

Step 4) Work in the t direction. Since velocities don't change in the t direction, we have

$$v_{A't} = v_{At}$$

and $$v_{B't} = v_{Bt}$$

Step 5) The magnitudes of velocities after impact are therefore given as

$$v_A' = \sqrt{(v_{A'n})^2 + (v_{A't})^2}$$

and $$v_B' = \sqrt{(v_{B'n})^2 + (v_{B't})^2}$$

The new direction of velocity is given as

$$\alpha' = \tan^{-1}\frac{v_{A't}}{v_{A'n}} \qquad \text{and} \qquad \beta' = \tan^{-1}\frac{v_{B't}}{v_{B'n}}$$

15.8.2 *Special Case of Oblique Central Impact:*

A special case happens when a ball has oblique central impact with a rigid infinite mass such as a rigid wall or ground. Consider one such case of a ball having an oblique impact with the ground.

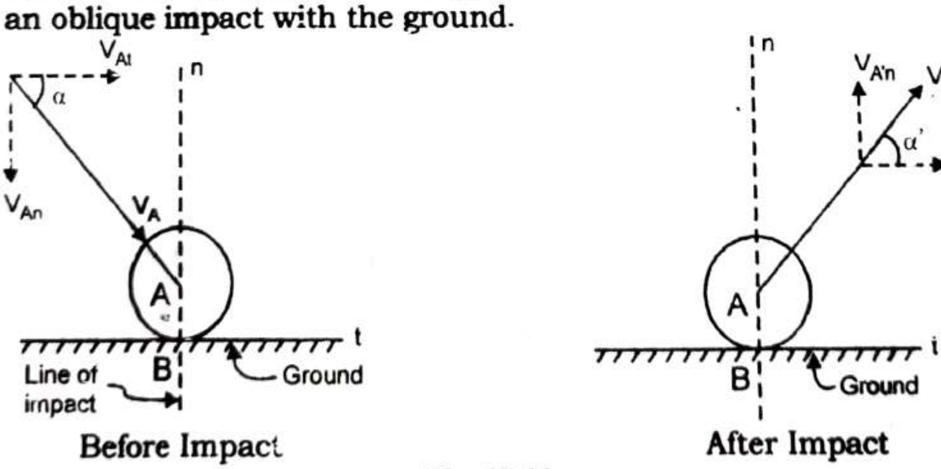


Fig. 15.11

Applying Coefficient of Restitution Equation in the n direction of impact, we get

$$v_{B'n} - v_{A'n} = e\,[v_{An} - v_{Bn}]$$

$$0 - v_{A'n} = e\,[v_{An} - 0] \qquad \because v_{Bn} = v_{B'n} = 0$$

$\therefore$ $$v_{A'n} = -e\,v_{An}$$

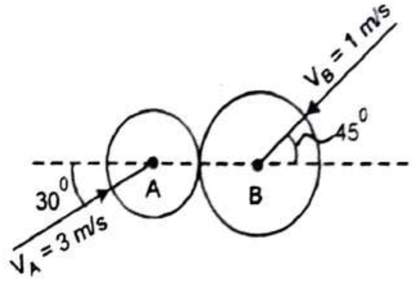
since velocities don't change in the t direction, we get

$$v_{A't} = v_{At}$$

$\therefore$ The magnitude of the velocity components of a ball after oblique impact with ground or rigid wall is given by

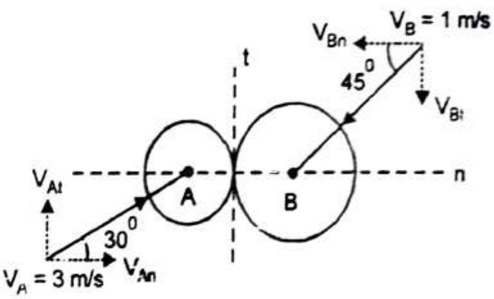
$$\left.\begin{aligned} v_{A'n} &= e\,v_{An} \\ \text{and} \quad v_{A't} &= v_{At} \end{aligned}\right\} \text{...... equations for special case of oblique impact}$$

Ex. 15.10 Two smooth balls collide as shown. Find the velocities after impact.
Take $m_A = 1$ kg, $m_B = 2$ kg and $e = 0.75$



Solution: This is a case of Oblique Central Impact
Let the line of impact be the n direction and a perpendicular to it be the t direction.
Resolving the velocities along n and t direction.

$v_{An} = 2.6$ m/s $\rightarrow$, $\quad v_{At} = 1.5$ m/s $\uparrow$
$v_{Bn} = 0.707$ m/s $\leftarrow$, $\quad v_{Bt} = 0.707$ m/s $\downarrow$



Working in n direction

Using Conservation of Momentum Equation $\rightarrow$ + ve

$$m_A v_{An} + m_B v_{Bn} = m_A v_{A'n} + m_B v_{B'n}$$
$$1 \times 2.6 + 2 \times (-0.707) = 1 \times v_{A'n} + 2\,v_{B'n}$$
$$1.186 = v_{A'n} + 2\,v_{B'n} \quad \text{............. (1)}$$

Using Coefficient of Restitution Equation $\rightarrow$ + ve

$$v_{B'n} - v_{A'n} = e\left[v_{An} - v_{Bn}\right]$$
$$v_{B'n} - v_{A'n} = 0.75\left[2.6 - (-0.707)\right]$$
$$v_{B'n} = v_{A'n} + 2.48 \quad \text{............. (2)}$$

Solving equations (1) and (2), we get

$$v_{A'n} = -\ 1.26 \text{ m/s} = 1.26 \text{ m/s} \leftarrow$$
$$v_{B'n} = \ 1.22 \text{ m/s} = 1.22 \text{ m/s} \rightarrow$$

Working in t direction

Since velocities don't change in t direction

$$v_{A't} = v_{At} = 1.5 \text{ m/s} \uparrow$$
$$v_{B't} = v_{Bt} = 0.707 \text{ m/s} \downarrow$$

$\therefore$ Total velocity $\qquad v_A' = \sqrt{(v_{A'n})^2 + (v_{A't})^2}$

$$= \sqrt{(1.26)^2 + (1.5)^2} = 1.96 \text{ m/s}$$

at angle $\qquad \alpha' = \tan^{-1}\left(\frac{v_{A't}}{v_{A'n}}\right) = \tan^{-1}\left(\frac{1.5}{1.26}\right) = 50°$

$\therefore \qquad v_A' = 1.96 \text{ m/s}, \ \alpha' = 50°$ .... **Ans.**

Similarly total velocity $\qquad v_B' = \sqrt{(v_{B'n})^2 + (v_{B't})^2}$

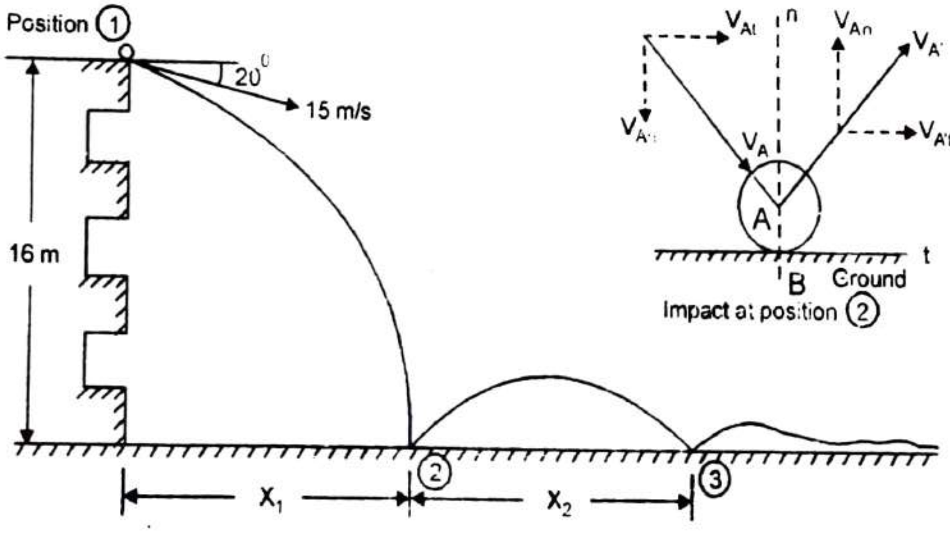
$$= \sqrt{(1.22)^2 + (0.707)^2} = 1.41 \text{ m/s}$$

at angle $\qquad \beta' = \tan^{-1}\left(\frac{v_{B't}}{v_{B'n}}\right) = \tan^{-1}\left(\frac{0.707}{1.22}\right) = 30.1°$

$\therefore \qquad v_B' = 1.41 \text{ m/s}, \ \beta' = 30.1°$ .... **Ans.**

Ex. 15.11 A ball is thrown downwards with a velocity of 15 m/s at an angle of 20° with the horizontal from the top of a building 16 m high. Find where the ball strikes the ground on its second bounce from the foot of the building. Take e = 0.8.

Solution: The ball projected from position (1) has its first bounce at position (2) on ground. The exagerrated view of impact at position (2) is shown. It is a case of oblique impact. Let v_A and v_A' be the velocities of the ball before and after impact. Resolving them along the n and t directions of impact.



First we will work with the projectile motion of the ball between position (1) – (2)
Let x_1 be the horizontal range.

Projectile Motion (1) – (2)

Horizontal Motion

$v = 15\cos 20 = 14.1$ m/s

$s = x_1$

$t = t$ sec

using $v = \frac{s}{t}$

$$14.1 = \frac{x_1}{t} \quad \ldots\ldots\ldots (1)$$

Substituting $t = 1.357$ sec obtained from vertical motion

$\therefore\ x_1 = 19.13$ m

also since velocity remains constant in the horizontal direction

$v_{At} = 14.1$ m/s $\rightarrow$

Vertical Motion $\downarrow$ + ve

$u = 15\sin 20 = 5.13$ m/s

$v = v_{An}$

$s = 16$ m

$a = 9.81$ m/s^2

$t = t$ sec

Using $s = ut + \frac{1}{2}at^2$

$$16 = 5.13\,t + \tfrac{1}{2} \times 9.81 \times t^2$$

$4.905\,t^2 + 5.13\,t - 16 = 0$

$\therefore\quad t = 1.357$ sec

Using $v = u + at$

$$v_{An} = 5.13 + 9.81 \times 1.357$$

$\therefore\quad v_{An} = 18.44$ m/s $\downarrow$

Analysing impact at position (2)

It is a special case of oblique impact, since the ball hits the ground, which is a body of infinite mass.

Having found $v_{At} = 14.1$ m/s $\rightarrow$, $v_{An} = 18.44$ m/s $\downarrow$

Using relations of special case of oblique impact

$\therefore\qquad v_{A'n} = e.\,v_{An} = 0.8 \times 18.44 = 14.752$ m/s $\uparrow$

and $\qquad v_{A't} = v_{At} \quad = 14.1$ m/s $\rightarrow$

After impact at position (2), the ball rebounds and performs projectile motion. It lands again on the ground at position (3). Let x_2 be the horizontal range between position (2) and (3).

Projectile Motion (2) – (3)

Horizontal Motion

$v = 14.1$ m/s

$s = x_2$

$t = t$ sec

using $v = \frac{s}{t}$

$$14.1 = \frac{x_2}{t}$$

Substituting $t = 3$ sec obtained from vertical motion

$$14.1 = \frac{x_2}{3} \quad \therefore\ x_2 = 42.4 \text{ m}$$

Vertical Motion $\uparrow$ + ve

$u = 14.752$ m/s

$v = -$

$s = 0$

$a = -9.81$ m/s^2

$t = t$ sec

Using $s = ut + \frac{1}{2}at^2$

$0 = 14.752 \times t + \frac{1}{2} \times (-9.81) \times t^2$

$\therefore\ t = 3$ sec

Total horizontal distance from the foot of tower

$$x = x_1 + x_2$$
$$= 19.13 + 42.4$$
$$= 61.53 \text{ m} \quad \ldots\ldots\ldots\ldots \textbf{Ans.}$$