Chapter 14

Kinetics of Particles: Work Energy Method

14.1 Introduction:

In the first part of this chapter we shall use the Work Energy Principle method for analyzing kinetics of a moving particle or a system of particles. This is an alternate approach to Newton's Second Law. This method eliminates the determination of acceleration and thereby at times results in quicker solution.

In the second part of this chapter we will learn the Conservation of Energy method for solving certain special problems involving conservative forces.

14.2 Work of a Force:

Work is a scalar quantity. *It is defined as the product of the force and the displacement in the direction of the force.* It is denoted by letter U. Units of work are N.m or Joule (J).

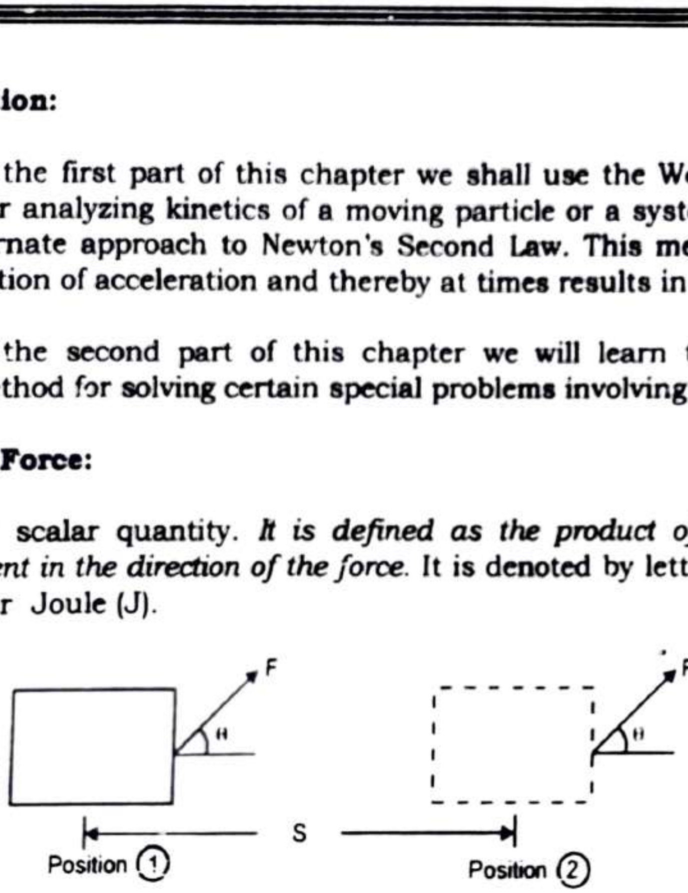


Fig.14.1

Consider a block acted upon by a constant force F acting at an angle θ as shown. Let this force cause the block to displace by s.

then, Work by force $\quad U = F \cos\theta \times s$ .... **[14.1 (a)]**

A special case, when $\theta = 0$ i.e force acts along the displacement, then

Work by force $\quad U = F \times s$ .... **[14.1 (b)]**

Also when $\theta = 90°$ i.e. force is $\perp$ to the displacement, then work by force = 0

14.3 Work of a Spring Force

Consider an undeformed spring of stiffness k as shown in Fig. 14.2 (a). Let the spring be deformed by some external agency, not shown in the figure, by an amount x_1, as shown in Fig. 14.2 (b).
Let this be position (1) of the spring. Now let the spring get further deformed such that its new deformation measured from the neutral position be x_2. This is position (2) of the spring. (Fig. 14.2 (c)).

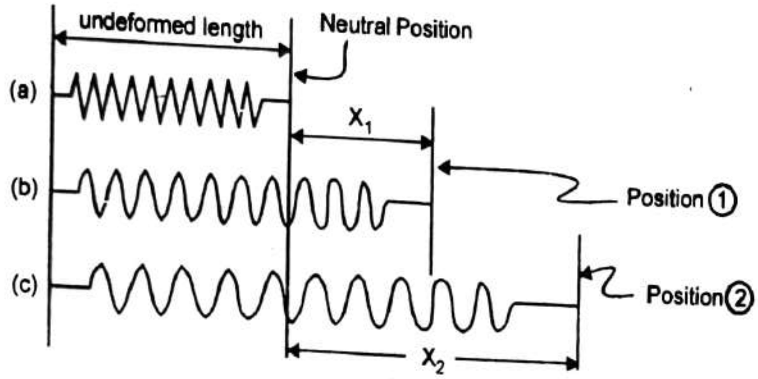


Fig.14.2

We know that the force in spring is variable as it is proportional to its deformation **x** and is directed towards the neutral position i.e. spring force

$$F = -k\,x$$

The work done by the spring between position (1) and position (2) is

$$U = \int_{x_1}^{x_2} -k\,x\,dx$$

$$= -\frac{1}{2}k\,(x_2^2 - x_1^2)$$

or work by spring $\mathbf{U = \frac{1}{2} k\,(x_1^2 - x_2^2)}$ .... **[14.2]**

here, k is the spring constant to be taken in N/m, x_1 and x_2 are the deformations in the spring in position (1) and position (2) respectively and should be taken in metre to get the work in the units of N.m (Joule).

14.4 Work of a Weight Force:

Consider a particle of mass m i.e. weight = mg move along a curved path in a vertical plane from position (1) to position (2).

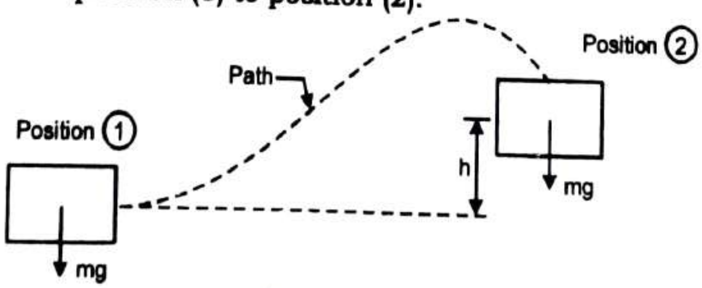


Fig.14.3

Let h be the vertical displacement between the two positions. Since the weight force which acts in the vertical direction has undergone a vertical displacement h, the work done by weight force,

$$U = \text{weight force} \times \text{vertical displacement}$$

or $\mathbf{U = -m\,g\,h}$ .... If displacement is upwards

also $\mathbf{U = m\,g\,h}$ .... If displacement is downwards

.... **[14.3]**

In simple words if the final position of the particle is above the initial position, work by weight is negative and is positive if the final position is below the initial position.

14.5 Work of a Friction Force:

Consider a block of mass m slide down a distance s on an inclined rough plane. If μ_s and μ_k are the coefficient of static and kinetic friction, the block's motion would be resisted by the frictional force = $\mu_k N$.

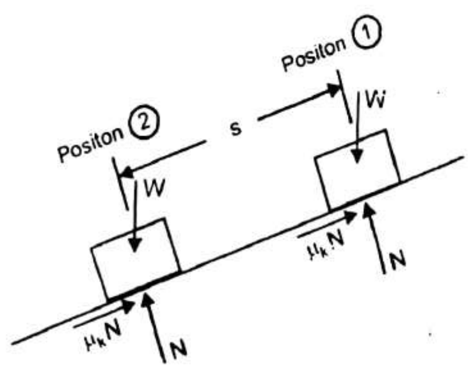


Fig.14.4

The work of a friction force is

$$\therefore \quad \mathbf{U = -\mu_k N.s} \qquad \text{..............} \ \mathbf{[14.4]}$$

here, s though implies displacement, would be equal to the actual distance moved by the particle since frictional force always orients itself so as to oppose motion. Due to this, work by friction is always negative.

14.6 Kinetic Energy:

It is defined as the energy possessed by the particle by virtue of its motion. If the particle is static i.e. not in motion it will not possess any kinetic energy. It is denoted by letter T.

If a particle of mass m has a velocity v at a given instant, its Kinetic Energy is

$$\mathbf{T = \tfrac{1}{2}\,mv^2} \qquad \text{...........} \ \mathbf{[14.5]}$$

Kinetic Energy is a scalar quantity and its S.I. unit is Newton-metre (N.m) or Joule (J).

14.7 Work Energy Principle

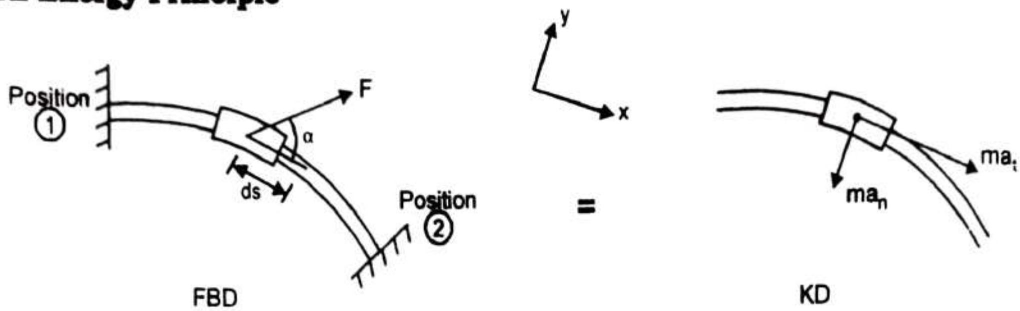


Fig.14.5

Consider a collar of mass m travel from position (1) with a velocity v_1 and reach position (2) with a new velocity v_2. The collar slides over a smooth curved guide kept in a horizontal plane, under the action of force F at angle α with the tangent to the path.
For an instant during its motion, applying equation of Newton's Second Law,

we get $\Sigma F_x = m.a_x$

$$F\cos\alpha = m.a_t$$

$$\therefore \quad F\cos\alpha = m\frac{dv}{dt} \qquad \text{since } a_t = \frac{dv}{dt}$$

$$\therefore \quad F\cos\alpha = m\frac{dv}{ds}\times\frac{ds}{dt}$$

here ds is the small arc length traveled by the collar in dt time.

Since $v = \dfrac{ds}{dt}$, we have

$$\therefore \quad F\cos\alpha = m v\frac{dv}{ds}$$

or $F\cos\alpha\, ds = m\, v\, dv$

Integrating between position (1) where $s = s_1$ and $v = v_1$ and position (2) where $s = s_2$ and $v = v_2$

$$\int_{s_1}^{s_2} F\cos\alpha\, ds = m\int_{v_1}^{v_2} v\, dv$$

Sine the L.H.S. represents the work done by a force

We get $U_{1-2} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$

$$\therefore \quad \frac{1}{2}mv_1^2 + U_{1-2} = \frac{1}{2}mv_2^2$$

If $\frac{1}{2}mv_1^2 = T_1$ and $\frac{1}{2}mv_2^2 = T_2$, we get

$$T_1 + U_{1-2} = T_2 \qquad \text{............ [14.6(a)]}$$

Equation 14.6 (a) relates the change in kinetic energy of the particle to the work done by the force on it. This relation can be extended to a system of forces acting on a particle and hence Work Energy Principle states "*For a particle moving under the action of forces, the total work done by these forces is equal to the change in its kinetic energy.*" Equation 14.6 (a) is therefore expressed as

$$\mathbf{T_1 + \sum U_{1\text{-}2} = T_2} \qquad \text{.......... [14.6(b)]}$$

14.8 Application of Work Energy Principle:

Work Energy principle is a simpler approach to the kinetics of a moving particle or a system of particles. It involves the use of the scalar equation 14.6 (b) viz., $T_1 + \sum U_{1-2} = T_2$, where T_1 and T_2 represent the particle's kinetic energy in position (1) and (2) respectively and $U_{1\text{-}2}$ represents the work done by various forces acting on it. This concept does not involve the calculation of the particle's acceleration which is the case in Newton's Second Law method of solving kinetics. This principle is useful whenever the problem involves known or unknown parameters like forces, mass, velocity and displacement. Knowing the forces and the displacement of the particle, the particle's velocity in the new position can be found out, or in some cases knowing the particle's initial and final velocities and the active forces, the particle's displacement can be worked out.

If the system involves more than one particle, the principle can be applied to the system of particles also. In such cases, the total kinetic energy would be sum of the individual kinetic energies of different particles and the work done would be the summation of the work done by various forces on the individual particles.

14.9 Power and Efficiency

Consider a case of two persons in a race, set to climb the stairs and reach the top of a 10 storied building. Here both the persons would be doing an equal amount of work in reaching the top, but if one person reaches earlier than the other, he would be said to have exerted a greater power than the other one as he has completed the work in lesser time. Thus the rate at which the work is done is equally important.

Power is defined as the rate of doing work

$$\textbf{Power} = \frac{dU}{dt} \qquad \text{............ [14.7 (a)]}$$

If the rate of doing work is constant, then

$$\textbf{Power} = \frac{\textbf{Work}}{\textbf{Time}} \qquad \text{............ [14.7 (b)]}$$

Knowing that work $= F \times s$

$$\therefore \quad \text{Power} = \frac{F \times s}{t}$$

Since $v = \frac{s}{t}$, we get another relation for power as

$$\textbf{Power} = \mathbf{F} \times \mathbf{v} \qquad \text{............ [14.7 (c)]}$$

Power is a scalar quantity and its S.I unit is N.m/s or J/sec or Watt

$$1 \text{ N.m/s} = 1 \text{ J/s} = 1 \text{ W}$$

For any machine doing work, its efficiency is defined as

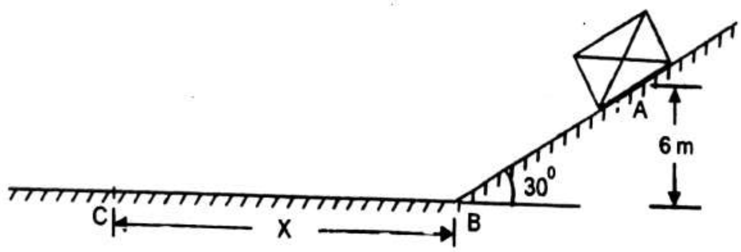
$$\eta = \frac{\text{Output work}}{\text{Input work}}$$

or $$\eta = \frac{\text{Power output}}{\text{Power input}}$$

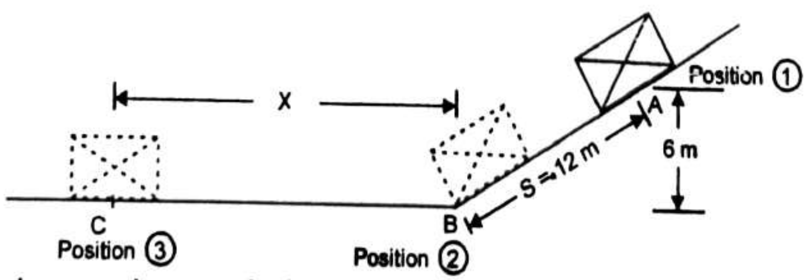
$$\% \ \eta = \frac{\textbf{Power output}}{\textbf{Power input}} \times \mathbf{100} \qquad \text{............ [14.8]}$$

All machines have efficiency less than 1 or 100 % because of various energy losses. Major energy losses are the frictional losses. Man has always strived to increase the efficiency of machines it uses by reducing the various energy losses it undergoes during its working.

Ex. 14.1 A 20 kg crate is released from rest on the top of incline at A. It travels on the incline and finally comes to rest on the horizontal surface at C. Find the distance x it travels on the horizontal surface and also the maximum velocity it attains during the motion. Take $\mu_k = 0.3$.



Solution:



The crate acquires maximum velocity at the lower most point B on the incline. Applying Work Energy Principle from position (1) to (2).

$T_1 = 0$ ....since block starts from rest.

$T_2 = \frac{1}{2} mv^2 = \frac{1}{2} \times 20 \times v^2 = 10\, v^2$ J

U_{1-2} 1) work by weight force

$$U = m\,g\,h$$
$$= 20 \times 9.81 \times 6 = 1177.2 \text{ J}$$

2) Work by frictional force

$$U = -\,u_k N.s$$

here, for the inclined surface, normal reaction,

$$N = W \cos 30 = 20 \times 9.81 \cos 30 = 169.9 \text{ N}$$

and distance traveled by block = $s = \dfrac{6}{\sin 30} = 12$ m

$$\therefore \quad U = -\,0.3 \times 169.9 \times 12 = -\,611.64 \text{ J}$$

Using $T_1 + \sum U_{1-2} = T_2$

$$0 + [\,1177.2 - 611.64\,] = 10\,v^2$$

$$\therefore \quad v = 7.52 \text{ m/s}$$

i.e $v_{max} = 7.52$ m/s at B .... **Ans.**

To find the distance x traveled on the horizontal surface, we will apply work energy principle from position (2) to position (3)

$$T_2 = \tfrac{1}{2} mv^2 = \tfrac{1}{2} \times 20 \times (7.52)^2 = 565.56 \text{ J}$$
$$T_3 = 0$$

U_{2-3} = only by frictional force

$$= -\,u_k N.s$$

For the horizontal surface, $N = W = 20 \times 9.81 = 196.2$ N

Also the distance traveled by block, $s = x$ meters.

$$\therefore \quad U_{2-3} = -\,0.3 \times 196.2 \times x$$
$$= -\,58.86\,x \text{ J}$$

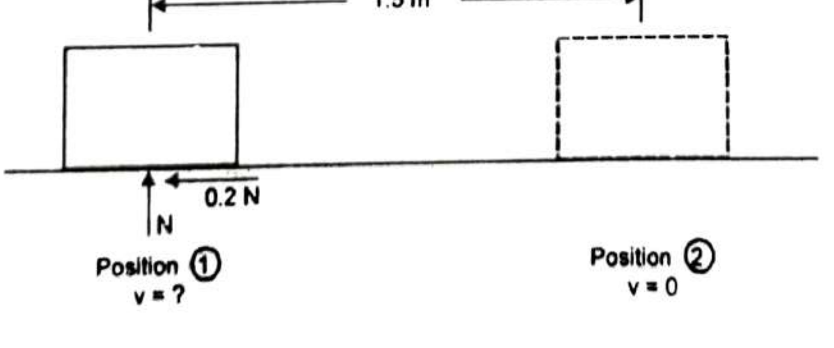
Using $T_2 + \Sigma U_{2-3} = T_3$

$$565.56 + [\,-\,58.86\,x\,] = 0$$

$$\therefore \quad x = 9.608 \text{ m}$$ .... **Ans.**

Ex. 14.2 A block is pushed with an initial velocity on a horizontal surface such that it travels 1500 mm before coming to rest. If $\mu_s = 0.25$ and $\mu_k = 0.2$ find the time of travel.

Solution:



Applying Work Energy Principle from position (1) to position (2)

$T_1 = ½\, mv^2 = 0.5\, mv^2$ J
$T_2 = 0$

U_{1-2} 1) by frictional force
$U = -\mu_k N.s$

here, the normal reaction, $N = m \times 9.81$

$\therefore \quad U = -0.2\,(m \times 9.81) \times 1.5$
$= -2.943\, m$ J

$$T_1 + \sum U_{1-2} = T_2$$
$$0.5\, mv^2 - 2.943\, m = 0$$

$\therefore \quad v = 2.426$ m/s .... Initial velocity of block

Kinematics

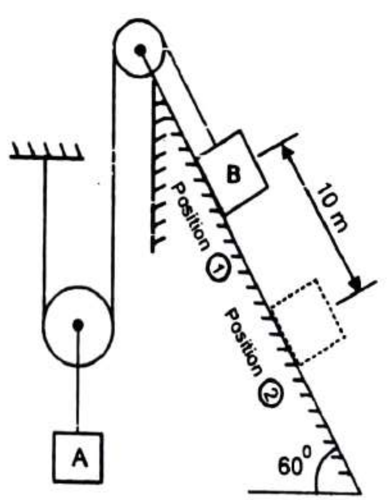
Block performs rectilinear motion with uniform acceleration (since forces remain constant)

$u = 2.426$ m/s , $v = 0$, $s = 1.5$ m, $a = ?$, $t = t$ sec.

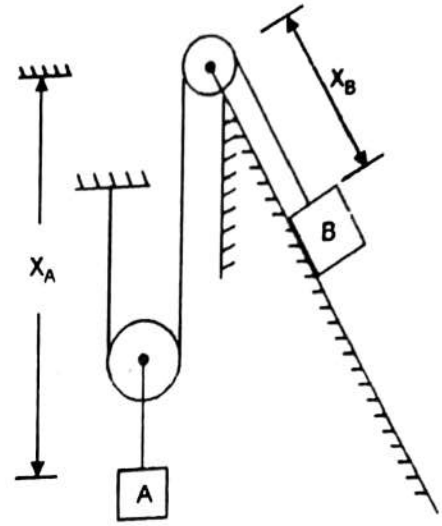
using $v^2 = u^2 + 2as$
$0 = (2.426)^2 + 2\, a \times 1.5$
$\therefore \quad a = -1.962$ m/s^2

using $v = u + at$
$0 = 2.426 - 1.962\, t$
$\therefore \quad t = 1.236$ sec .... **Ans.**

Ex. 14.3 Block B of weight 3000 N having a speed of 2 m/s in position (1) travels 10 m along and down the slope. Block A of weight 1000 N is connected to it by an inextensible string. Find the velocities of the blocks in the new position.
Take $\mu_s = 0.35$ and $\mu_k = 0.3$ at the inclined surface.



Solution: We shall first find the relation between the velocities and the distance traveled by the two connected blocks.



Using constant string length method.
If x_A and x_B are the variable positions of A and B measured from a fixed reference point, we have the length L of string in terms of x_A and x_B as
$L = (-2\,x_A) + x_B \pm$ constants
[x_A is – ve since with increase in x_B, x_A would decrease]

Differentiating w. r. to time
$$0 = -2\,v_A + v_B$$

or $v_B = 2\,v_A$ ....relation between the velocities

since the time interval is the same, we have

$x_B = 2\,x_A$ ....relation between distance traveled

Applying Work Energy Principle to the system of A and B from position (1) to (2).

$$T_1 = \tfrac{1}{2} m_A v_A^2 + \tfrac{1}{2} m_B v_B^2$$
$$= \tfrac{1}{2}(101.94) \times (0.5\,v_B)^2 + \tfrac{1}{2}(305.81)\,v_B^2$$
$$= 165.65\,v_B^2$$
$$= 165.65\,(2)^2 = 662.6 \text{ J}$$

$$T_2 = \tfrac{1}{2} m_A v_A^2 + \tfrac{1}{2} m_B v_B^2$$
$$= 165.65\,v_B^2 \text{ J}$$

U_{1-2}

1) by weight of block B
Block B moves vertically down by h = 10 sin 60 = 8.66 m
U = m g h = 3000 × 8.66 = 25981 J
= + 25981 J (+ ve since displacement is downwards)

2) by weight of block A
since $x_A = 0.5\,x_B$, we have,

vertical upward displacement of block A = 0.5 × 10 = 5 m

U = m g h = 1000 × 5 = 5000
= – 5000 J (since displacement is upwards)

3) by frictional force at the inclined surface
$U = -\mu k N.s$

block B travels a distance s = 10 m

and the normal reaction on the inclined surface

$$N = W \cos\theta = 3000 \cos 60 = 1500 \text{ Newton}$$

$$U = -0.3 \times 1500 \times 10 = -4500$$
$$= -4500 \text{ J}$$

Using $T_1 + \sum U_{1-2} = T_2$

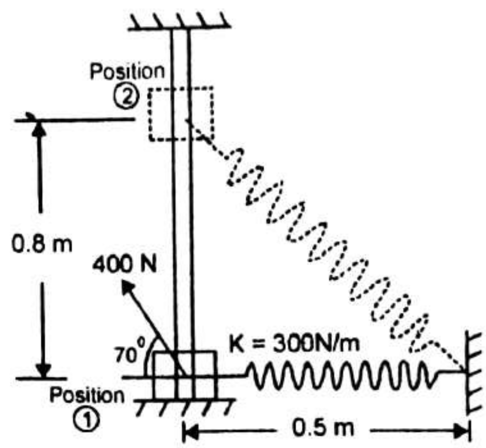
$662.6 + [25981 - 5000 - 4500] = 165.65\ v_B^2$

$\therefore \quad v_B = 10.17\ m/s$ .... **Ans.**

also $v_A = 0.5\ v_B$

$\therefore \quad v_A = 0.5 \times 10.17 = 5.086\ m/s$ .... **Ans.**

Ex. 14.4 A 25 kg steel collar is being raised from rest at position (1) by a 400 N force applied as shown. The collar is guided by a smooth rod and a spring whose free length is 0.3 m. Find the speed of the collar as it reaches position (2).



Solution: Applying Work Energy Principle to the moving collar from position (1) to (2).

$T_1 = 0$ .... since it starts from rest.

$T_2 = \frac{1}{2} mv^2 = \frac{1}{2}\ 25 \times v^2 = 12.5\ v^2$ J

U_{1-2}

1) by applied force

$= F \times s$

$= 400 \times \sin 70 \times 0.8 = 300.7$ J

$= +300.7$ J (+ ve since force acts in the direction of displacement)

2) by spring force

$U = \frac{1}{2} k\ (x_1^2 - x_2^2)$

here, deformation of spring in position (1)

x_1 = spring length – free length

$= 0.5 - 0.3 = 02$ m

also deformation of spring in position (2)

x_2 = spring length – free length

$= \sqrt{0.5^2 + 0.8^2} - 0.3$

$= 0.643$ m

$U = \frac{1}{2} \times 300\ [\ (0.2)^2 - (0.643)^2\ ]$

$= -56$ J

3) by weight force

$U = -m\ g\ h$ – ve because displacement is upwards

$U = -25 \times 9.81 \times 0.8 = -19.62$ J

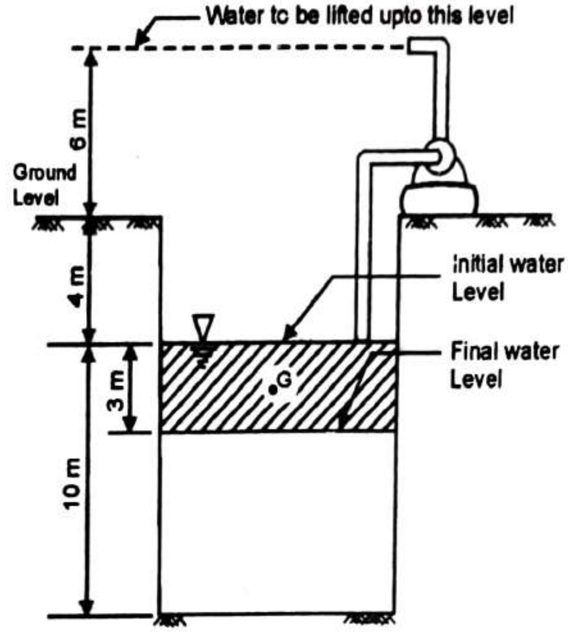
Using $T_1 + \sum U_{1-2} = T_2$

$0 + [\ 300.7 - 56 - 196.2\ ] = 12.5\ v^2$

$\therefore \quad v = 1.97\ m/s$

$\therefore$ velocity of the collar in position (2) is $v = 1.97$ m/s .... **Ans.**

Ex. 14.5 A well is 10 m in diameter and contains water 10 m from the bottom. The top of the water is 4 m below ground level. Find the work done in lifting the water to a level 6 m above ground. Due to lifting the water level reduces by 3 m and it takes 15 minutes to do so. If the efficiency of the pump is 70 %, find the power of the pump.



Solution: Volume of water lifted

$$= \frac{\pi d^2}{4} \times h = \frac{\pi \times 10^2}{4} \times 3$$

$$= 235.62 \text{ m}^3$$

Mass of water lifted = Density × volume
= 1000 × 235.62
= 235620 kg

Weight of water lifted = 235620 × 9.81
= 2311427 N

The weight of water acts through its G.
The average height through which the water is lifted is the vertical displacement of G,

i.e $$h = \frac{3}{2} + 4 + 6 = 11.5 \text{ m}$$

Work done in lifting the water through 11.5 m height

$= W \times h$

$= 2311427 \times 11.5$

$= 26581408$ J .... **Ans.**

This work is done in 15 minutes = 900 sec

$$\therefore \text{ Power output} = \frac{\text{Work done}}{\text{Time}}$$

$$= \frac{26581408}{900} = 29535 \text{ W}$$

Now $$\% \eta = \frac{\text{Power output}}{\text{Power Input}} \times 100$$

$$70 = \frac{29535}{\text{Power input}} \times 100$$

$\therefore$ power input = 42193 W .... **Ans.**

14.10 Energy

The capacity of particle to do work is defined as the energy possessed by the particle. Energy exists in nature in various forms viz.,

i) Mechanical
ii) Electrical
iii) Heat
iv) Sound
v) Light
vi) Chemical etc.

Mechanical energy, which is the sum of the potential energy (V) and kinetic energy (T), is of interest in study of kinetics.

Potential Energy:

It is defined as the energy possessed by a particle by virtue of its position with respect to a datum, or by virtue of elastic forces acting on it. It is denoted by letter V. Potential energy is a scalar quantity and its S.I. Units are Newton-metre (N.m) or Joule (J).

Consider a particle of mass m moving on a vertical curved surface as shown. If the ground is chosen as the datum, the potential energy due to its position is given by

$$\mathbf{V = m\,g\,h} \quad .... [14.9\ (a)]$$

here, $V_1 = m\,g\,h_1$
$V_2 = m\,g\,h_2$
or $V_3 = m\,g\,h_3$

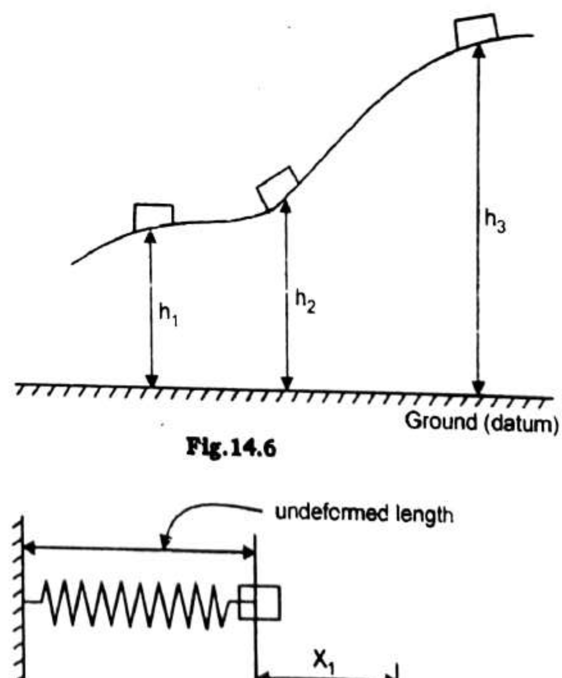


Fig.14.6

Consider the same block now acted upon by spring force. If **x** is the deformation of the spring from the neutral position, the potential energy due to elastic force is given by

$$\mathbf{V = \tfrac{1}{2} \times k\,x^2} \quad ......[14.9\ (b)]$$

here, $V_1 = \tfrac{1}{2} \times k\,x_1^2$
or $V_2 = \tfrac{1}{2} \times k\,x_2^2$

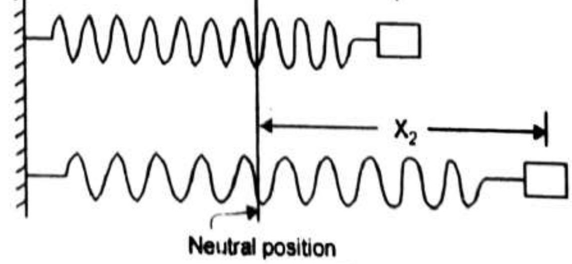


Fig.14.7

Kinetic Energy

This form of mechanical energy has been explained in the earlier part of this chapter in article 14.6.

14.11 Conservation of Energy:

In mechanics we define *conservative forces* to be those forces who do work independent of the path followed by the particle on which they act. Work of a spring force and a weight force come under the category of the conservative forces. *Non-conservative forces* are those forces whose work depends on the path followed by the particle on which they act. Frictional force is a non-conservative force.

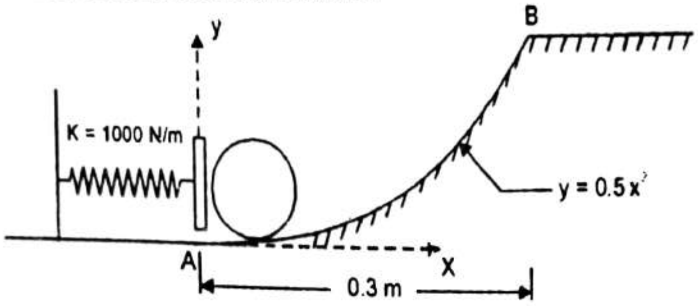
When a particle is acted upon by only conservative forces, the mechanical energy of the particle remains constants

i.e. $\mathbf{T + V = constant}$ .... **14.10 (a)**

or $\mathbf{T_1 + V_1 = T_2 + V_2}$ .... **14.10 (b)**

Equation 14.10 (b) is referred to as Conservation of Energy Equation and can be alternatively used to solve problems in kinetics involving conservative forces only.

Ex. 14.6 Determine the smallest amount by which the ball at A must be compressed against the spring so that when it is released from A, it reaches point B. Weight of ball is 2.5 N. Neglect friction on the curved surface.



Solution: From the equation of the curve

$$y = 0.5\, x^2$$

at $x = 0.3$ m, $y = 0.5\,(0.3)^2$

$$y = 0.045 \text{ m}$$

Since only conservative forces are involved, we will use Conservation of Energy Principle.

Let the spring be compressed by x meters.

Taking the datum at A i.e. at position (1) for potential energy calculations.

<u>Potential Energy [V] calculations</u>

of the spring at position (1) $= \frac{1}{2} k x^2 = \frac{1}{2} \times 1000\, x^2$

$= 500\, x^2$ J

at position (2) $= 0$

of the ball at position (1) $= 0$

at position (2) $= m g h = 2.5 \times 0.045$

$= 0.1125$ J

<u>Kinetic Energy [T] calculations</u>

of the ball at position (1) $= 0$

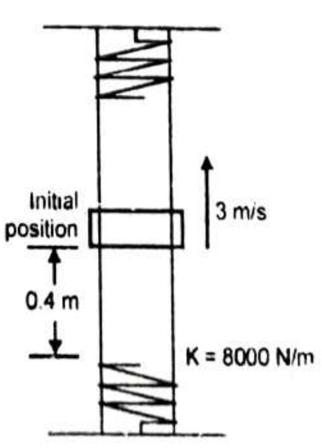
at position (2) $= 0$

Using $T_1 + V_1 = T_2 + V_2$

$0 + 500\, x^2 = 0 + 0.1125$

$\therefore$ $x = 0.015$ m .... **Ans.**

Ex. 14.7 A 10 kg collar slides freely on a vertical rod as shown. The collar is projected upwards with a velocity of 3 m/s. It hits the upper spring, compresses it and is then directed downwards. The collar now hits the lower spring. Find the maximum deformation of the lower spring.



Solution: Since the problem only involves conservative forces, we shall use Conservation of Energy Equation.

In this problem the upper spring has no role to play since the work by weight of collar in going up cancels with the work by weight of the collar in coming down. The work in spring compression cancels with the spring decompression. Hence the collar now has a speed of 3 m/s downwards at the given position.

Taking the initial position of the collar as datum.

Let the lower spring compress by x meters.

Kinetic Energy [T] calculations:

of the collar at position (1) $= \frac{1}{2} mv^2 = \frac{1}{2} \times 10 \times 3^2 = 45$ J
at position (2) $= 0$

Potential Energy [V] calculations

of the collar in position (1) $= 0$
in position (2) $= m\,g\,h$

$$= 10 \times 9.81 \times (-0.4 - x)$$
$$= -39.24 - 98.1\,x \quad \text{J}$$

of the spring in position (1) $= 0$
in position (2) $= \frac{1}{2} k\,x^2$

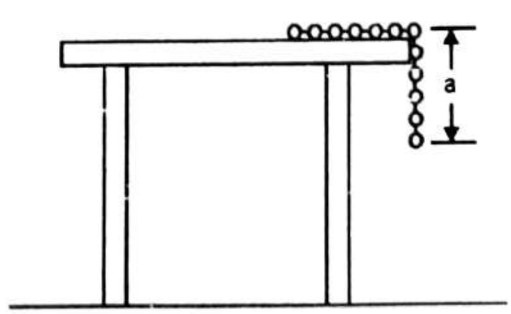
$$= \frac{1}{2} \times 8000\,x^2$$
$$= 4000\,x^2 \quad \text{J}$$

Using $T_1 + V_1 = T_2 + V_2$.

$$45 + 0 = 0 + [(-39.24 - 98.1\,x) + 4000\,x^2]$$
$$4000\,x^2 - 98.1\,x - 84.24 = 0$$

or $x = 0.1579$ m .... **Ans.**

Ex. 14.8 A chain of total length L is held at rest on a smooth table such that portion of length a of the chain hangs from the edge of the table as shown. The chain is now released from rest. Determine the speed of the last link of the chain as it leaves the table.



Solution: Let the weight of the chain per unit length be w kg
Let us use Conservation of Energy method, since only conservative forces are involved.

Let the table top be the datum for potential energy calculation. Let v be the velocity of the chain as it leaves the table.

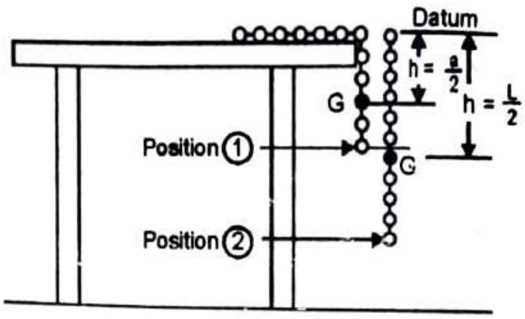
Kinetic Energy (T) Calculation

of the chain at position (1) $= 0$

at position (2) $= \frac{1}{2} m v^2$

$$= \frac{1}{2} \times \frac{w \times L}{g} \times v^2$$

$$= \frac{w L v^2}{2 g} \quad J$$



Potential Energy (V) Calculation

of the chain at position (1) $= m g h$

$$= \frac{w a}{g} \times g \times \left(\frac{-a}{2}\right)$$

$$= \frac{-w a^2}{2} \quad J$$

at position (2) $= m g h$

$$= \frac{w L}{g} \times g \times \left(\frac{-L}{2}\right)$$

$$= \frac{-w L^2}{2} \quad J$$

Using equation of Conservation of Energy

$$T_1 + V_1 = T_2 + V_2$$

$$0 + \left(\frac{-w a^2}{2}\right) = \frac{w L v^2}{2 g} + \left(\frac{-w L^2}{2}\right)$$

$$-a^2 = \frac{L v^2}{g} - L^2$$

$$L^2 - a^2 = \frac{L v^2}{g}$$

or $$v = \sqrt{\frac{g}{L}(L^2 - a^2)}$$ ....**Ans.**