$\boxed{Q3}$

$$A = \begin{bmatrix} 2 & 3-i & 2+i \\ i & 0 & 1-i \\ 1+2i & 1 & 3i \end{bmatrix}, \quad A^{\theta} = \begin{bmatrix} 2 & -i & 1-2i \\ 3+i & 0 & 1 \\ 2-i & 1+i & -3i \end{bmatrix}$$

$$A = \frac{1}{2}(A + A^{\theta}) + i\,\frac{1}{2i}(A - A^{\theta})$$

$$P = \frac{1}{2}\begin{bmatrix} 4 & 3-2i & 3-i \\ 3+2i & 0 & 2-i \\ 3+i & 2+i & 0 \end{bmatrix}$$

$$Q = \frac{1}{2}\begin{bmatrix} 0 & 3 & 1+3i \\ -3 & 0 & 1-i \\ -1+3i & -i & 6i \end{bmatrix}$$

$a_{ij} = \overline{a_{ji}}$, Hence P & Q are Hermitian.

$$A = \begin{bmatrix} 2 & 3-i & 2+i \\ i & 0 & 1-i \\ 1+2i & 1 & 3i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 & 3-2i & 3-i \\ 3+2i & 0 & 2-i \\ 3+i & 2+i & 0 \end{bmatrix} + i\,\frac{1}{2i}\begin{bmatrix} 0 & 3 & 1+3i \\ -3 & 0 & -i \\ -1+3i & -i & 6i \end{bmatrix}$$

* Orthogonal Matrix

$AA^{T} = I = A^{T}A$

$|A| = \pm 1$

If A is orthogonal then A^{-1}, A^{T} are also orthogonal.

* Unitary Matrix

$$AA^{\theta} = A^{\theta}A = I$$

If A is unitary,

then $A^{-1} = A^{\theta}$

then A^{-1}, A^{θ}, AB, BA are also unitary.

2022

Q1] ii]

$\rightarrow$ $A = \begin{bmatrix} \sqrt{2} & 1 & \sqrt{3} \\ \sqrt{2} & -2 & 0 \\ \sqrt{2} & 1 & -\sqrt{3} \end{bmatrix}$ $\quad A^{T} = \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & -2 & 1 \\ \sqrt{3} & 0 & -\sqrt{3} \end{bmatrix}$

$$AA^{T} = \begin{bmatrix} 2+1+3 & 2-2+0 & 2+1-3 \\ 2-2+0 & 2+4+0 & 2-2+0 \\ 2+1-3 & 2-2+0 & 2+1+3 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$AA^{T} = 6I \neq I$$

Thus given matrix is not orthogonal

$\therefore \; AA^{T} = 6I$

$\therefore \; \frac{1}{6} AA^{T} = I$

$\therefore \; \left(\frac{1}{\sqrt{6}} A\right) \cdot \left(\frac{1}{\sqrt{6}} A^{T}\right) = I$

$\frac{1}{\sqrt{6}} A = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{2} & 1 & \sqrt{3} \\ \sqrt{2} & -2 & 0 \\ \sqrt{2} & 1 & \sqrt{3} \end{bmatrix}$ is the orthogonal matrix.

2022

Q4

i) $\begin{bmatrix} 2 & -4 & 3 & 1 & 0 \\ 1 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$ Reduce to Normal form & find it's rank.

$\rightarrow$ $R_2 \leftrightarrow R_1$

$$\begin{bmatrix} 1 & -2 & 1 & -4 & 2 \\ 2 & -4 & 3 & 1 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix} \xrightarrow[R_4 - 4R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & -2 & 1 & -4 & 2 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 1 & 0 & 12 & -3 \end{bmatrix} \xrightarrow[R_3 \leftrightarrow R_4]{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 1 & 0 & 12 & -3 \\ 0 & 0 & 1 & 9 & -4 \end{bmatrix}$$

$$\xrightarrow{R_2 - R_3} \begin{bmatrix} 1 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & -1 & -9 & 4 \\ 0 & 0 & 1 & 9 & -4 \end{bmatrix} \xrightarrow{R_3 + R_4} \begin{bmatrix} 1 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & -1 & -9 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2 \to C_2 + 2C_1$
$C_3 \to C_3 - C_1$
$C_4 \to C_4 + 4C_1$
$C_5 \to C_5 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & -1 & -9 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\substack{C_4 \to C_3 - 3C_2 \\ C_5 \to C_5 - C_2}]{C_3 \to C_3 + C_2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -9 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-R_3} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[C_5 \to C_5 + 4C_3]{C_4 \to C_4 - 9C_3} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rho(A) = 3$$

2021

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 1 \\ 4 & 3 & 2 \end{bmatrix}$$ Find Rank.

$\rightarrow$ $$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -5 & 1 \\ 0 & -5 & 2 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -5 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$ $\rho(A) = 3$

Q]

$\rightarrow$ $2x - 2y + z = Kx \Rightarrow (2-K)x - 2y + z = 0$

$2x - 3y + 2z = Ky \Rightarrow 2x + (-3-K)y + 2z = 0$

$-x + 2y - Kz = 0$

$$\begin{bmatrix} (2-K) & -2 & 1 \\ 2 & (-3-K) & 2 \\ -1 & 2 & -K \end{bmatrix} = A$$

$|A| = (2-K)\left[(-3-K)\cdot(-K) - 4\right] - (-2)\left[-2K + 2\right] + 1\left[4 - (3+K)\right] = 0$

$= (2-K)\left[3K + K^2 - 4\right] - 4K + 4 + 1 + K = 0$

$= -K^3 - K^2 + 5K - 3 = 0$

$\therefore K = 1, K = -3$

Q] $X_1 = [1, 1, 1]$, $X_2 = [1, 2, 4]$, $X_3 = [-2, 3, 8]$

Find LD or L.ID?

→ $a(1,1,1)$ $b(1,2,4)$ $c(-2,3,8)$

$a + b - 2c = 0$

$a + 2b + 3c = 0$

$a + 4b + 8c = 0$

$$\begin{bmatrix} 1 & 1 & -2 \\ 1 & 2 & 3 \\ 1 & 4 & 8 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 5 \\ 0 & 3 & 10 \end{bmatrix} \xrightarrow{R_3 - 3R_2} \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 5 \\ 0 & 0 & -5 \end{bmatrix}$$

Linearly Independent.

2022

Q3] i] A]

→ $A = \begin{bmatrix} p & 2 & p \\ p & p & 2 \\ 2 & p & p \end{bmatrix}$

$|A| = p(p^2 - 2p) - 2(p^2 - 4) + p(p^2 - 2p)$

$= 2p^3 - 6p^2 + 8$

$= p = -1, 2$

1]: Rank (A) = 3 if $p \neq -1$ or $p \neq 2$

2] If $p = -1$

$$\begin{bmatrix} -1 & -1 & 2 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \end{bmatrix} \qquad \begin{vmatrix} -1 & -1 \\ 2 & -1 \end{vmatrix} \neq 0$$

$\therefore$ Rank of matrix $= 2$

3] If $P = 2$,

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \qquad \begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix} = 0$$

Rank of matrix $= 1$

2022

Q5] iii] $20x + y - 2z = 17$, $3x + 20y - z = -18$, $2x - 3y + 20z = 25$ (Gauss - Seidel)

$\rightarrow$ $x = \frac{1}{20}(17 - y + 2z)$

$y = \frac{1}{20}(-18 - 3x + z)$

$z = \frac{1}{20}(25 - 2x + 3y)$

Taking $x_0 = 0$, $y_0 = 0$, $z_0 = 0$

First Iteration: $x_1 = \frac{17}{20} = 0.85$

$y_1 = \frac{1}{20}(-18 - 3(0.85) + 0) = -1.0275$

$z = \frac{1}{20}(25 - 2(0.85) + 3(-1.0275)) = 1.0108$

Second Iterations:

$$x_2 = \frac{1}{20}(17 - (-1.0275) + 2(1.0108)) = 1.0024$$

$$y_2 = \frac{1}{20}(-18 - 3(1.0024) + 1.0108) = -0.9998$$

$$z_2 = \frac{1}{20}(25 - 2(1.0024) + 3(-0.9998)) = 0.9997$$

$\therefore \; x = 1, \; y = -1, \; z = 1$

2022

Q2B]

i]

$3x + y - Kz = 0, \; 4x - 2y - 3z = 0, \; 2kx + 4y + kz = 0$

$$A = \begin{bmatrix} 3 & 1 & -k \\ 4 & -2 & -3 \\ 2K & 4 & K \end{bmatrix}$$

$$|A| = 3(-2K + 12) - 1(4K + 6K) - k(16 + 4K) = 0$$

$$= -6K + 36 - 10k - 16K - 4K^2 = 0$$

$$= -4k^2 - 32K + 36 = 0$$

$$= 4K^2 + 32K - 36 = 0$$

$$= K^2 + 8K - 9 = 0$$

-9, 9, -1

$$= K^2 - 1K + 9K - 9 = 0$$

$$= K(K-1) + 9(K-1) = 0$$

$$= K = 1, -9$$

2018 Q2 A

$$A = \frac{1}{3}\begin{bmatrix} a & b & c \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$$

$$A^T = \frac{1}{3}\begin{bmatrix} a & -2 & 1 \\ b & 1 & -2 \\ c & 2 & 2 \end{bmatrix}$$

$AA^T = I$

$$AA^T = \frac{1}{9}\begin{bmatrix} a & -2 & 1 \\ b & 1 & -2 \\ c & 2 & 2 \end{bmatrix}\begin{bmatrix} a & b & c \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix} = \begin{bmatrix} a^2+5 & ab-4 & ac-2 \\ ab-4 & b^2+5 & \\ & & c^2+8 \end{bmatrix}$$

$a = \pm 2$, $b = \pm 2$, $c = \pm 1$

$ac - 2 = 0 \quad ac = 2 \qquad\qquad ab = 4$

$a = 2$, $c = 1$ or $a = -2$ & $c = -1 \qquad a = 2$, $b = 2$, $a = -2$, $b = -2$

$\therefore (2, 2, 1)$ or $(-2, -2, -1)$

2018 Q2 B ii

$$\begin{bmatrix} 1 & 1 & 4 & 6 \\ 2 & 3 & 2 & 1 \\ -1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \xrightarrow[R_3 + R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & 1 & -6 & -11 \\ 0 & 2 & 5 & 6 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

$R_3 - 2R_2$

$R_4 - 2R_2$

$$\rho(A) = 4 \qquad \begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & 1 & -6 & -11 \\ 0 & 0 & 17 & 28 \\ 0 & 0 & 0 & 55 \end{bmatrix} \xleftarrow{17R_4 - 12R_3} \begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & 1 & -6 & -11 \\ 0 & 0 & 17 & 28 \\ 0 & 0 & 12 & 23 \end{bmatrix}$$

∴ Linearly Independent

iii]

$$\begin{bmatrix} 1 & 2 & 1 & -1 \\ 1 & 2 & 0 & 2 \\ 0 & 4 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$\downarrow R_2 - R_1$

Rank of matrix = 3

$$\rightarrow \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 4 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

consistent & infinite solu^n

$\downarrow R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 4 & -1 & 3 \\ 0 & 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

$x + 2y + z - t = 2$

$4y - 1z + 3t = -1$

$-1z + 3t = -1$

$t = K$, $z = 1 + 3K$, $y = \frac{-2-6K}{4}$

$x = 2 + K$

2019

Q2B) i)

$$AA^T = \frac{1}{9}\begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix}\begin{bmatrix} -2 & 2 & 1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ It is orthogonal.

$\therefore A^{-1} = A^T$

ii)

$$V_1(2,-1,3,2) + V_2(1,3,42) + V_3(3,-5,22) = 0$$

$$\begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & -5 \\ 3 & 4 & 2 \\ 2 & 2 & 2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow -R_2} \begin{bmatrix} 1 & -3 & 5 \\ 2 & 1 & 3 \\ 3 & 4 & 2 \\ 2 & 2 & 2 \end{bmatrix} \xrightarrow[R_3-3R_1,\ R_2-2R_1]{R_2-2R_1} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 7 & -7 \\ 0 & 13 & -13 \\ 0 & 8 & -8 \end{bmatrix} \xrightarrow{\frac{R_2}{7}, \frac{R_3}{13}, \frac{R_4}{8}} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow[R_4-R_2]{R_3-R_2} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x + 3y + z = 0$$

$$y - z = 0$$

Let $z = t$, $y = t$, $x = 4t$

$$4t\,V_1 + tV_2 + tV_3 = 0$$

$$\boxed{4V_1 + V_2 + V_3 = 0}$$

iii)

$$A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 3 & -1 & 2 & 2 \\ 4 & 1 & 0 & -1 \\ 9 & 1 & 5 & 6 \end{bmatrix} \xrightarrow[\substack{R_3 - 2R_1 \\ 2R_4 - 9R_1}]{2R_2 - 3R_1} \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -5 & -5 & -8 \\ 0 & -1 & -6 & -9 \\ 0 & -7 & -17 & -24 \end{bmatrix}$$

$$\xrightarrow{-R_3;\, -R_2;\, R_4} \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 1 & 6 & 9 \\ 0 & 5 & 5 & 8 \\ 0 & 7 & 17 & 24 \end{bmatrix} \xrightarrow[R_4 - 7R_2]{R_3 - 5R_2} \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 1 & 6 & 9 \\ 0 & 0 & -25 & -37 \\ 0 & 0 & -25 & -37 \end{bmatrix}$$

$$\xrightarrow{-(R_4 - R_3)} \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 1 & 6 & 9 \\ 0 & 0 & 25 & 37 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore\ \rho(A) = 3$

iv)

$$\begin{bmatrix} 1 & 2 & 4 & 1 \\ 2 & 1 & 5 & 8 \\ 1 & 4 & 6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$\xrightarrow[R_3 - R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & -3 & -3 & 6 \\ 0 & 2 & 2 & -4 \end{bmatrix} \xrightarrow{\frac{R_3}{2},\, \frac{R_2}{3}} \begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & -1 & -1 & 2 \\ 0 & 1 & 1 & -2 \end{bmatrix} \xrightarrow{R_2 + R_3} \begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Non-Trivial solu^{n}.

V

$$x = \frac{1}{15}(14 - y + z)$$

$$y = \frac{1}{20}(23 - x - z)$$

$$z = \frac{1}{18}(37 - 2x + 3y)$$

Taking $x_0 = 0$, $y_0 = 0$, $z_0 = 0$

I^st Iteration:

$$x_1 = \frac{14}{15} = 0.93$$

$$y_1 = \frac{23}{20} = 1.15$$

$$z_1 = \frac{37}{18} = 2.05$$

II^nd Iteration:

$$x_2 = \frac{1}{15}(14 - (1.15) + 2.05) = 0.993$$

$$y_2 = \frac{1}{20}(23 - (0.93) - (2.05) = 1.001$$

$$z_2 = \frac{1}{18}(37 - 2(0.93) + 3(1.15) = 2.14$$

III^rd Iteration:

$$x_3 = \frac{1}{15}(14 - (1.001) + 2.14 = 1.009$$

$$y_3 = \frac{1}{20}(23 - (0.993) - (2.14) = 0.993$$

$$z_3 = \frac{1}{18}(37 - 2(0.993) + 3(1.001) = 2.11$$